1. Induction on strings.
   On page 302 of the book, the operation **rev** on strings is defined as follows:

$$\mathbf{rev}\ (\epsilon) := \epsilon$$
$$\mathbf{rev}\ (xa) := a\mathbf{rev}(x)$$

   for any $x \in \Sigma^*$ and $a \in \Sigma$.

   Show that, for any strings $x, y \in \Sigma^*$, we have $\mathbf{rev}(xy) = \mathbf{rev}(y)\ \mathbf{rev}(x)$. In other words, the reverse of $xy$ is the reverse of $y$ followed by the reverse of $x$.

   **Base Case:**
   Consider the case where $|y| = 0$. This is equivalent to $y = \epsilon$.
   By definition, the null string $\epsilon$ serves as an identity for string concatenation.

   *Proof.*

$$\begin{aligned}
\mathbf{rev}(xy) &= \mathbf{rev}(x\epsilon) && \text{by substituting } \epsilon \text{ in for } y \\
&= \mathbf{rev}(x) && \text{by definition of } \epsilon \\
&= \epsilon\ \mathbf{rev}(x) && \text{by definition of } \epsilon \\
&= \mathbf{rev}(\epsilon)\ \mathbf{rev}(x) && \text{by definition of } \mathbf{rev} \\
&= \mathbf{rev}(y)\ \mathbf{rev}(x) && \text{by substituting } y \text{ in for } \epsilon
\end{aligned}$$

   □

   **Induction Hypothesis:**
   Assume that for $|y| = k$, where $k \geq 1$, $\mathbf{rev}(xy) = \mathbf{rev}(y)\ \mathbf{rev}(x)$.

   **Inductive Step:**
   Let $y \neq \epsilon$. Therefore, $y$ can be represented as the concatenation $bc$ where $b \in \Sigma^*$ and $c \in \Sigma$.

   *Proof.*

$$\begin{aligned}
\mathbf{rev}(xy) &= \mathbf{rev}(x(bc)) && \text{by substituting } bc \text{ in for } y \\
&= \mathbf{rev}((xb)c) && \text{by associativity of concatenation} \\
&= c\ \mathbf{rev}(xb) && \text{by definition of } \mathbf{rev} \\
&= c\ \mathbf{rev}(b)\mathbf{rev}(x) && \text{by Induction Hypothesis} \\
&= \mathbf{rev}(bc)\ \mathbf{rev}(x) && \text{by definition of } \mathbf{rev} \\
&= \mathbf{rev}(y)\ \mathbf{rev}(x) && \text{by substituting } y \text{ in for } bc
\end{aligned}$$

   □