CSCI 5352 Network Analysis and Modeling Fall 2020
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Problem Set 1
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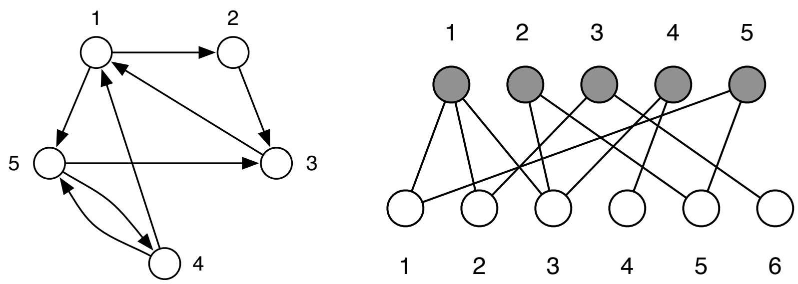
1. (12 pts) Consider the following two networks:



(A) (B)

(a) (3 pts) Give the adjacency matrix for network (A).

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(b) (3 pts) Give adjacency list for network (A).

1	2, 5
2	3
3	1
4	1, 5
5	3, 4

(c) (6 pts) Give adjacency matrices for both one-mode projections of network (B).

B-Top:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

B-Bottom:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. (15 pts) Let $\mathbf{A}$ be the adjacency matrix of a simple graph (unweighted, undirected edges with no self-loops) and $\mathbf{1}$ be the column vector whose elements are all 1. In terms of these quantities, multiplicative constants and simple matrix operations like transpose and trace, write expressions for

   (a) (3 pts) the vector $\mathbf{k}$ whose elements are the degrees k_i of the vertices
   $\mathbf{A1}$
   You need to sum all rows, so multiply $\mathbf{A}$ and $\mathbf{1}$.

   (b) (3 pts) the number m of edges in the network
   $\frac{\mathbf{1}^T\mathbf{A1}}{2}$
   You need the sum of all matrix entries divided by 2 (because each edge e_{ij} is repeated as edge e_{ji}), so multiply $\mathbf{1}^T$ and $\mathbf{A}$ to sum the columns, then multiply the resultant row vector by $\mathbf{1}$ to sum it. Finally divide by 2 to take care of the duplication.

   (c) (5 pts) the matrix $\mathbf{N}$ whose elements $\mathbf{N}_{ij}$ is equal to the number of common neighbors of vertices i and j
   $\mathbf{A}^2$
   Book section 6.11 talks about walks and "common neighbors" can be thought of as walks of length two (i $\rightarrow$ k $\rightarrow$ j, nodes i and j have common neighbor k). It is explained that $\mathbf{A}_{ik}$ $\mathbf{A}_{kj}$ is equal to 1 if there is a walk along those edges. To get the total paths from i to j (different k's), we need to sum over all values of k: $\sum_{k=1}^{n}(\mathbf{A}_{ik} * \mathbf{A}_{kj}) = [\mathbf{A}^2]_{ij}$. So the matrix $\mathbf{A}^2$ holds this information for all node combinations.

   (d) (4 pts) the total number of triangles in the network, where a triangle means three vertices, each connected by edges to both of the others.
   $\frac{\mathrm{Tr}(\mathbf{A}^3)}{6}$
   Section 6.11 generalizes r-length walks as being held by $\mathbf{A}^r$. Triangles are represented by walks of length three that start and end at the same point. So for triangles starting/ending on vertex i, look at $[\mathbf{A}^3]_{ii}$. To get all triangles you would sum $[\mathbf{A}^3]_{ii}$ across all values of i, which is equal to $\mathrm{Tr}(\mathbf{A}^3)$. However, this counts each triangle 6 times (i.e. Triangle a-b-c is counted as: a-b-c, b-c-a, c-a-b, c-b-a, a-c-b, and b-a-c, once for each starting vertex and each direction around the triangle), so the quantity must be divided by 6 to give us the desired answer.

3. (10 pts) Consider a bipartite network, with its two types of vertices, and suppose there are n_1 vertices of type 1 and n_2 vertices of type 2. Show that the mean degrees c_1 and c_2 of the two types are given by

$$c_2 = \frac{n_1}{n_2} c_1 \ .$$

The mean degree of a set of nodes multiplied by the number of nodes in the set gives the sum of degrees in the set. In a bipartite graph, all edges connect nodes from opposite sets. Therefore, the sum of degrees of one set must equal the sum of degrees of the other. We have the following:

$$c_1 n_1 = c_2 n_2 \rightarrow c_2 = \frac{n_1}{n_2} c_1$$