

1. (20 pts) Consider the following simple and rather unrealistic model of a network: each of n vertices belongs to one of g groups. The m th group has n_m vertices and each vertex in that group is connected to others in the group with independent probability $p_m = A(n_m - 1)^{-\beta}$, where A and β are constants, but not to any vertices in other groups. Thus, this network takes the form of a set of disjoint groups of communities.

- (a) Calculate the expected degree $\langle k \rangle$ of a vertex in group m .

$$\langle k \rangle = (n_m - 1)p_m$$

$$\langle k \rangle = (n_m - 1)A(n_m - 1)^{-\beta}$$

$$\langle k \rangle = A(n_m - 1)^{1-\beta}$$

- (b) Calculate the expected value $\langle C_m \rangle$ of the local clustering coefficient for vertices in group m .

$$\langle C_m \rangle = \frac{\binom{A(n_m-1)^{1-\beta}}{2} p_m}{\binom{A(n_m-1)^{1-\beta}}{2}} = p_m = A(n_m - 1)^{-\beta}$$

- (c) Hence show that $\langle C_m \rangle \propto \langle k \rangle^{-\beta/(1-\beta)}$. What value would β have to assume for the expected value of the local clustering coefficient to fall off as $\langle k \rangle^{-0.75}$, as has been conjectured by some researchers?

$$\lim_{n_m \rightarrow \infty} \frac{\langle C_m \rangle}{\langle k \rangle^{-\beta/(1-\beta)}} = \lim_{n_m \rightarrow \infty} \frac{A(n_m-1)^{-\beta}}{(A(n_m-1)^{1-\beta})^{-\beta/(1-\beta)}} = \lim_{n_m \rightarrow \infty} \frac{A(n_m-1)^{-\beta}}{A^{-\beta/(1-\beta)} A(n_m-1)^{-\beta}} = \lim_{n_m \rightarrow \infty} \frac{1}{A^{-\beta/(1-\beta)}} = 1$$

$$\langle k \rangle^{-\beta/(1-\beta)} = \langle k \rangle^{-0.75}$$

$$-\beta/(1-\beta) = -0.75$$

$$\beta = \frac{4}{7}$$

2. (20 pts) Consider the random graph $G(n, p)$ with average degree c .

- (a) Show that in the limit of large n the expected number of triangles in the network is $\frac{1}{6}c^3$. In other words, show that the number of triangles is constant, neither growing nor vanishing in the limit of large n .

$$\binom{n}{3} p^3 = \binom{n}{3} \left(\frac{c}{n-1}\right)^3 = \frac{n!}{3!(n-3)!} \left(\frac{c}{n-1}\right)^3 = \frac{n(n-1)(n-2)}{6} \left(\frac{c}{n-1}\right)^3$$

$$\lim_{n \rightarrow \infty} \frac{n(n-1)(n-2)}{6} \left(\frac{c}{n-1}\right)^3 = \frac{1}{6}c^3$$

- (b) Show that the expected number of connected triples in the network, as in Eq. (7.28) in *Networks* [v1: 7.41], is $\frac{1}{2}nc^2$.

$$3\binom{n}{3} p^2 = 3\binom{n}{3} \left(\frac{c}{n-1}\right)^2 = \frac{3n!}{3!(n-3)!} \left(\frac{c}{n-1}\right)^2 = \frac{n(n-1)(n-2)}{2} \left(\frac{c}{n-1}\right)^2$$

$$\lim_{n \rightarrow \infty} \frac{n(n-1)(n-2)}{2} \left(\frac{c}{n-1}\right)^2 = \frac{1}{2}nc^2$$

- (c) Hence, calculate the clustering coefficient C , as defined in Eq. (7.28) in *Networks* [v1: 7.41], and confirm that it agrees for large n with the value given in Eq. (11.11) in *Networks* [v1: 12.11].

$$C = \frac{\text{number of triangles}}{\text{number of connected triples}} = \frac{\frac{1}{6}c^3}{\frac{1}{2}nc^2} = \frac{c}{3n}$$

$$\lim_{n \rightarrow \infty} \frac{c}{3n} = \lim_{n \rightarrow \infty} \frac{c}{n-1} \xrightarrow{\text{therefore}} \frac{c}{3n} \propto \frac{c}{n-1}$$