CSCI 5352 Network Analysis and Modeling Fall 2020
Prof. Daniel Larremore
Problem Set 3
Student: Alex Book; Collaborators: None

1. (20 pts) Consider the following simple and rather unrealistic model of a network: each of n vertices belongs to one of g groups. The mth group has n_m vertices and each vertex in that group is connected to others in the group with independent probability $p_m = A(n_m - 1)^{-\beta}$, where A and β are constants, but not to any vertices in other groups. Thus, this network takes the form of a set of disjoint groups of communities.

   (a) Calculate the expected degree $\langle k \rangle$ of a vertex in group m.
   $\langle k \rangle = (n_m - 1)p_m$
   $\langle k \rangle = (n_m - 1)A(n_m - 1)^{-\beta}$
   $\langle k \rangle = A(n_m - 1)^{1-\beta}$

   (b) Calculate the expected value $\langle C_m \rangle$ of the local clustering coefficient for vertices in group m.
   $\langle C_m \rangle = \frac{\binom{A(n_m-1)^{1-\beta}}{2} p_m}{\binom{A(n_m-1)^{1-\beta}}{2}} = p_m = A(n_m - 1)^{-\beta}$

   (c) Hence show that $\langle C_m \rangle \propto \langle k \rangle^{-\beta/(1-\beta)}$. What value would β have to assume for the expected value of the local clustering coefficient to fall off as $\langle k \rangle^{-0.75}$, as has been conjectured by some researchers?
   $\lim_{n_m \to \infty} \frac{\langle C_m \rangle}{\langle k \rangle^{-\beta/(1-\beta)}} = \lim_{n_m \to \infty} \frac{A(n_m-1)^{-\beta}}{(A(n_m-1)^{1-\beta})^{-\beta/(1-\beta)}} = \lim_{n_m \to \infty} \frac{A(n_m-1)^{-\beta}}{A^{-\beta/(1-\beta)} A(n_m-1)^{-\beta}} = \lim_{n_m \to \infty} \frac{1}{A^{-\beta/(1-\beta)}} = 1$

   $\langle k \rangle^{-\beta/(1-\beta)} = \langle k \rangle^{-0.75}$
   $-\beta/(1-\beta) = -\frac{3}{4}$
   $-\beta = \frac{3}{4}\beta - \frac{3}{4}$
   $\beta = \frac{3}{7}$

2. (20 pts) Consider the random graph $G(n, p)$ with average degree c.

   (a) Show that in the limit of large n the expected number of triangles in the network is $\frac{1}{6}c^3$. In other words, show that the number of triangles is constant, neither growing nor vanishing in the limit of large n.
   $\binom{n}{3}p^3 = \binom{n}{3}(\frac{c}{n-1})^3 = \frac{n!}{3!(n-3)!}(\frac{c}{n-1})^3 = \frac{n(n-1)(n-2)}{6}(\frac{c}{n-1})^3$
   $\lim_{n \to \infty} \frac{n(n-1)(n-2)}{6}(\frac{c}{n-1})^3 = \frac{1}{6}c^3$

   (b) Show that the expected number of connected triples in the network, as in Eq. (7.28) in *Networks* [v1: 7.41], is $\frac{1}{2}nc^2$.
   $3\binom{n}{3}p^2 = 3\binom{n}{3}(\frac{c}{n-1})^2 = \frac{3n!}{3!(n-3)!}(\frac{c}{n-1})^2 = \frac{n(n-1)(n-2)}{2}(\frac{c}{n-1})^2$
   $\lim_{n \to \infty} \frac{n(n-1)(n-2)}{2}(\frac{c}{n-1})^2 = \frac{1}{2}nc^2$

(c) Hence, calculate the clustering coefficient C, as defined in Eq. (7.28) in *Networks* [v1: 7.41], and confirm that it agrees for large n with the value given in Eq. (11.11) in *Networks* [v1: 12.11].

$C = \frac{\text{number of triangles}}{\text{number of connected triples}} = \frac{\frac{1}{6}c^3}{\frac{1}{2}nc^2} = \frac{c}{3n}$

$lim_{n\to\infty}\frac{c}{3n} = lim_{n\to\infty}\frac{c}{n-1} \xrightarrow{\text{therefore}} \frac{c}{3n} \propto \frac{c}{n-1}$