Homework 6
Colorado CSCI 5454

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Other external resources used: N/A

Problem 1

Part a

We will use the idea presented in lecture videos, and randomly select (with equal probability) either **True** or **False** for each of the k variables. We then find the expected value for any given clause.

Each literal of a given clause has a probability of evaluating to **False** of $\frac{1}{2}$. Taken across k terms in the given clause, the probability of a given clause evaluating to **False** is $\frac{1}{2^k}$. So, the probability of a given clause evaluating to **True** is $1 - \frac{1}{2^k}$. Therefore, the expected value across m clauses $\mathbb{E}[\text{ALG}] = m - \frac{m}{2^k}$.

Because there are m total clauses, we can say that $\mathbb{E}[\text{OPT}] \leq m$.
So, $\mathbb{E}[\text{ALG}] \geq (1 - \frac{1}{2^k})\mathbb{E}[\text{OPT}]$. The approximation factor is $1 - \frac{1}{2^k}$.

Part b

When the randomized algorithm described above is applied to the Max-at-most-k-SAT problem, the same approximation factor is not guaranteed to be achieved. The expected value for each clause is *at best* $1 - \frac{1}{2^k}$. We say "*at best*" because clauses can have less than k literals, and the expected value of such clauses would decrease (as k decreases, the value of $1 - \frac{1}{2^k}$ also decreases). Therefore, the same approximation factor is not upheld for the Max-at-most-k-SAT problem.

Problem 2

Part a

Let OPT be the optimal solution to Unweighted Min Vertex Cover. Let ALG be the output from the given algorithm.

First, we will prove that ALG is an Unweighted Vertex Cover.
Suppose M is a maximal matching. Suppose there exists some edge $\{u, v\} \in E$ where $u \notin \text{ALG}$ and $v \notin \text{ALG}$. This means that M must not be maximal, as you could then add $\{u, v\}$ to M and still have a matching. By contradiction, no such edge can exist, and ALG is indeed a vertex cover.

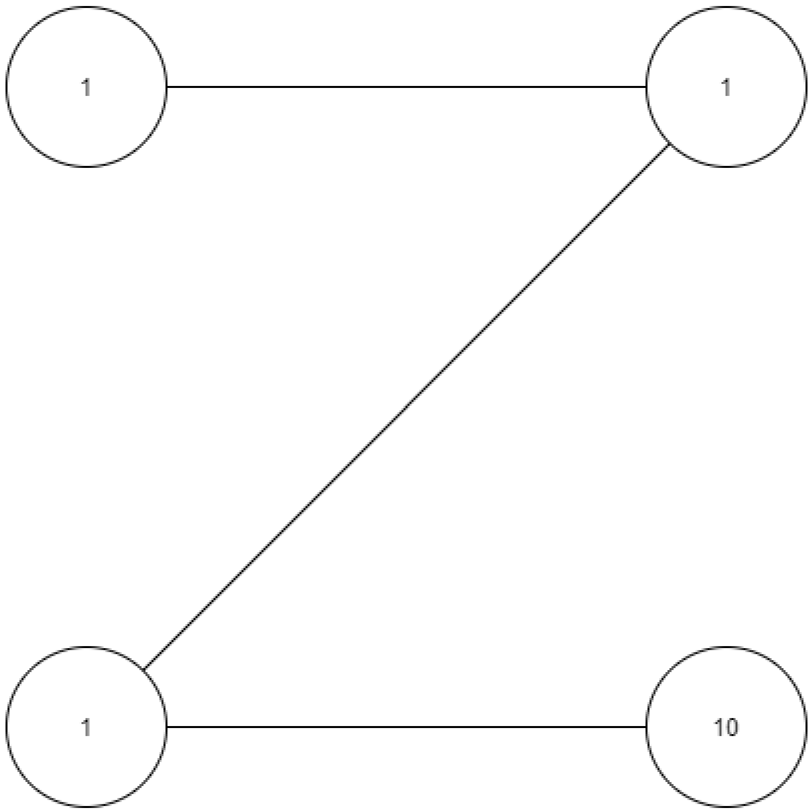
Next, we will prove the approximation factor of ALG. We can say with certainty that $|\text{OPT}| \geq |M|$, because for each edge in M, at least one of the endpoint vertices must be included in OPT. We can also say that $|\text{ALG}| = 2|M|$, as each edge in M has two vertices, each of which will be added to ALG. This leads us to the following:

$$
\begin{aligned}
|\text{OPT}| &\geq |M| \\
|\text{ALG}| &= 2|M| \\
2|\text{OPT}| &\geq 2|M| = |\text{ALG}| \\
|\text{ALG}| &\leq 2|\text{OPT}|
\end{aligned}
$$

Therefore, ALG is a 2-approximation of the optimal solution to Unweighted Min Vertex Cover (OPT).

Part b

We contend that the above algorithm does not always have the same approximation ratio when applied to Weighted Min Vertex Cover. Consider the following graph:



One maximal matching on this graph is the top edge and the bottom edge. Applying our algorithm to this matching would yield a total weight of 13. However, the optimal Weighted Min Vertex Cover would include only the top-right and bottom-left vertices, and have a weight of 2. So, the inequality proven in **part a** ($|\text{ALG}| \leq 2|\text{OPT}|$) does not hold true, as $13 \nleq 2 * 2$. Therefore, the approximation ratio of ALG is not always upheld when applied to the Weighted Min Vertex Cover Problem.

Problem 3

Part a

$$P(\text{all tails}) = (1-p)^k$$

Part b

$$\begin{aligned} P(\text{at least one heads}) &= 1 - P(\text{all tails}) \\ &= 1 - (1-p)^k \end{aligned}$$

Part c

$$\begin{aligned} P(\text{at least one heads}) &= 1 - (1-p)^k \\ &= 1 - (1-p)^{\frac{1}{p}\ln\left(\frac{1}{\delta}\right)} \\ &\geq 1 - (1 - (1 - e^{-p}))^{\frac{1}{p}\ln\left(\frac{1}{\delta}\right)} \\ &\geq 1 - (e^{-p})^{\frac{1}{p}\ln\left(\frac{1}{\delta}\right)} \\ &\geq 1 - e^{-\ln\left(\frac{1}{\delta}\right)} \\ &\geq 1 - e^{-(\ln(1) - \ln(\delta))} \\ &\geq 1 - e^{\ln(\delta)} \\ &\geq 1 - \delta \end{aligned}$$