Homework 1
Colorado CSCI 5454

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People I studied with for this homework: Cole Sturza
Other external resources used: N/A

Problem 1

$f(n) = 7n^4 + 20n^2$

Part a

Prove $f(n) \in \Theta(n^4)$

$7n^4 + 20n^2 \leq 7n^4 + 20n^4 = 27n^4 (\text{ for } n \geq 1)$
$f(n) \in \Omega(n^4), c = 27, N = 1$

$7n^4 + 20n^2 \geq n^4 (\text{ for } n \geq 1)$
$f(n) \in O(n^4), c = 1, N = 1$

$f(n) \in \Omega(n^4), f(n) \in O(n^4)$
$\therefore f(n) \in \Theta(n^4)$

Part b

Prove $f(n) \in \omega(n^2)$

$\lim\limits_{n \to \infty} \frac{7n^4+20n^2}{n^2} = \lim\limits_{n \to \infty} 7n^2 + 20 = \infty$
$\therefore f(n) \in \omega(n^2)$

Part c

Prove $f(n) \in o(n^5)$

$$\begin{aligned}
\lim_{n\to\infty} \frac{7n^4 + 20n^2}{n^5} &= \lim_{n\to\infty} \frac{7n^2+20}{n^3} = \frac{\infty}{\infty} \\
&\overset{\text{LH}}{=} \lim_{n\to\infty} \frac{14n}{3n^2} \\
&= \lim_{n\to\infty} \frac{14}{3n} = 0
\end{aligned}$$

$\therefore f(n) \in o(n^5)$

Problem 2

$f(n) = \sum_{j=1}^{n} j^{57}$. Prove that $f(n) \in \Theta(n^{58})$.

$$\begin{aligned}
f(n) &= 1^{57} + 2^{57} + 3^{57} + \cdots + n^{57} \\
&\leq n^{57} + n^{57} + n^{57} + \cdots + n^{57} \\
&= n(n^{57}) \\
&= n^{58}
\end{aligned}$$

$f(n) \leq n^{58}$ for $n \geq 1$
$\therefore f(n) \in O(n^{58}), c = 1, N = 1$

$$\begin{aligned}
f(n) &= 1^{57} + 2^{57} + 3^{57} + \cdots + n^{57} \\
&\geq \sum_{j=\frac{n}{2}}^{n} (\frac{n}{2})^{57} \\
&= (\frac{n}{2})^{57} + (\frac{n}{2})^{57} + (\frac{n}{2})^{57} + \cdots + (\frac{n}{2})^{57} \\
&= \frac{n}{2}(\frac{n}{2})^{57} \\
&= (\frac{n}{2})^{58}
\end{aligned}$$

$2^{58} f(n) \geq n^{58}$ for $n \geq 1$
$\therefore f(n) \in \Omega(n^{58}), c = 2^{58}, N = 1$

$\therefore f(n) \in \Theta(n^{58})$

Problem 3

$f(n) = \sum_{j=1}^{n} \sqrt{j}$. Show that $f(n) \in \Theta(n^{\frac{3}{2}})$.

$$
\begin{aligned}
f(n) &= \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \\
&\leq \sqrt{n} + \sqrt{n} + \sqrt{n} + \cdots + \sqrt{n} \\
&= n\sqrt{n} \\
&= n^{\frac{3}{2}}
\end{aligned}
$$

$f(n) \leq n^{\frac{3}{2}}$ for $n \geq 1$
$\therefore f(n) \in O(n^{\frac{3}{2}}), c = 1, N = 1$

$$
\begin{aligned}
f(n) &= \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \\
&\geq \sum_{j=\frac{n}{2}}^{n} \sqrt{\frac{n}{2}} \\
&= \frac{n}{2}\sqrt{\frac{n}{2}} \\
&= (\frac{n}{2})^{\frac{3}{2}}
\end{aligned}
$$

$2^{\frac{3}{2}} f(n) \geq n^{\frac{3}{2}}$ for $n \geq 1$
$\therefore f(n) \in \Omega(n^{\frac{3}{2}}), c = 2^{\frac{3}{2}}, N = 1$

$\therefore f(n) \in \Theta(n^{\frac{3}{2}})$

Problem 4

DAG $G = (V, E)$

Part a

Prove that if G is order-respecting, then it is a DAG.

Assume there exists a cycle in G containing some vertices $\{v_1, \ldots, v_k\}$. Since G is order-respecting, every edge (u, v) from v_1 to its neighbor(s), from those neighbor(s) to theirs, and eventually to v_k must all be such that $A[u] < A[v]$. As follows, it must be true that $A[1] < A[k]$. However, to complete the cycle there must also exist an edge (v_k, v_1), which leads to $A[k] > A[1]$. This is contradictory, and such an edge would mean G is no longer order-respecting.
$\therefore$ by contradiction, G must be a DAG.

Part b

Prove that if G is a DAG, there exists some array A for which G is order-respecting.

Take the output from a topological sorting algorithm (e.g. the algo given in lecture notes), let it be represented as T. Loop through T to create A, doing the following: Assign $A[u]$ (where u is the current vertex in T) the value i (where i is the current index in T). This will result in A being an array that stores the ordering of each vertex in T.

Take any edge (u, v) in G. In a topological sorting of the vertices in G (such as the aforementioned T), u must come before v (per the given definition of topological sort). Since we are looping through such a sorting from beginning to end while assigning increasing values, we can say with certainty that $A[u] < A[v]$ (u comes before v in the sorting, so $A[u]$ is assigned a value before $A[v]$, thus $A[u] < A[v]$).

$\therefore$ if G is a DAG, there must exist some array A for which G is order-respecting.