Homework 2
Colorado CSCI 5654

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Problem 1

Entering	Leaving	Objective Fn. value in next dictionary	Next Dictionary Degenerate (Y/N)?
x_2	w_1, w_5	15	Y
x_5	w_5	12	N
x_6	w_2, x_3, x_4	13	Y
w_3	w_2, x_3	17	Y

Problem 2

(A)

$x_{N,j} = -\frac{b_i + a_{i1}x_{N,1} + \cdots + a_{ij-1}x_{N,j-1} + a_{ij+1}x_{N,j+1} + \cdots + a_{in}x_{N,n} - x_{B,i}}{a_{ij}}$
$x_{N,j}$ is multiplied by c_j during one pivot step of simplex. The negative in front of the entire fraction cancels with the negative in front of $x_{B,i}$ in the numerator. This leaves the objective row coefficient for $x_{B,i}$ to be $\frac{c_j}{a_{ij}}$.

(B)

c_j must be positive before the pivot step of simplex, as it is the coefficient of the entering variable. a_{ij} must be negative before the pivot step of simplex, as it is the coefficient of the leaving variable. Therefore the entire objective row coefficient of $x_{B,i}$ after the pivot step of simplex must be negative. This means $x_{B,i}$ cannot be an entering variable in the next dictionary (entering variables must have a positive coefficient).

Problem 3

(A)

$$
\begin{array}{r|rrr}
w_1 & & x_1 & +x_2 \\
w_2 & 1 & +x_1 & +x_2 \\
\hline
z & 1 & +x_1 & +x_2
\end{array}
$$

(B)

$$
\begin{array}{r|rrr}
w_1 & 1 & -x_1 & +x_2 \\
w_2 & & & +x_2 \\
\hline
z & 1 & +x_1 & +x_2
\end{array}
$$

(C)

This is not possible, as a non-degenerate dictionary must increase the value of the objective function upon pivoting (degenerate dictionaries are by definition not increasing the objective function).

(D)

This is not possible, as the simplex method (by which a pivot is employed) transitions from one feasible dictionary to another. So if the initial dictionary is feasible, so must be the resulting dictionary after a pivot occurs.

(E)

$$
\begin{array}{r|rrr}
w_1 & 1 & +x_1 & +x_2 \\
w_2 & 1 & -x_1 & +x_2 \\
\hline
z & 1 & +x_1 & +x_2
\end{array}
$$

Problem 4

(A)

First we will check that $x_0 + \lambda r$ is feasible for any $\lambda \geq 0$:

$$\begin{aligned} A(x_0 + \lambda r) &\leq b \\ Ax_0 + A\lambda r &\leq b \\ b &\leq b \end{aligned}$$

The transition from line 2 to line 3 is valid because we are given that $Ax_0 \leq b$ and $Ar \leq 0$ (so $A\lambda r \leq 0$ for all $\lambda \geq 0$).

$$\begin{aligned} x_0 + \lambda r &\geq 0 \\ 0 &\geq 0 \end{aligned}$$

The transition from line 1 to line 2 is valid because we are given that $x_0 \geq 0$ and $r \geq 0$ (so $\lambda r \geq 0$ for all $\lambda \geq 0$).

We will then see what value it achieves for the objective:

$$c^T(x_0 + \lambda r) = c^T x_0 + c^T \lambda r$$

We are given that $c^T r > 0$, so $c^T \lambda r$ (and thus the entire objective function) approaches infinity as we push $\lambda \to \infty$. Therefore the given LP is unbounded from above.

(B)

We know a few certainties: $b_1, ..., b_m \geq 0$ (since the dictionary is feasible), $a_{1j}, ..., a_{mj} \geq 0$ (since there is no corresponding leaving variable for $X_{N,j}$), and $c_j > 0$ (since $X_{N,j}$ is the entering variable). Thus we can set $X_{N,j}$ to any arbitrary positive number λ and none of the basic variables will become negative, so the resulting solution is feasible. As we push $\lambda \to \infty$, the value of the objective function increases $\to \infty$. Therefore the given LP must be unbounded from above.

Problem 5

(A)

Dual problem:

$$
\begin{array}{llllll}
\text{max.} & -5y_1 & -6y_2 & -4y_3 & -2y_4 & \\
\text{s.t.} & -y_1 & -2y_2 & +y_3 & +y_4 & \leq -2 \\
 & y_1 & & -y_3 & & \leq 3 \\
 & y_1 & -y_2 & & & \leq -1 \\
 & & y_2 & +y_3 & & \leq 1 \\
 & & & & -y_4 & \leq -1 \\
 & y_1, & y_2, & y_3, & y_4, & \geq 0
\end{array}
$$

(B)

$x_1 = \frac{10}{3}, x_2 = 8, x_3 = 0, x_4 = \frac{2}{3}, x_5 = \frac{16}{3}$

Complementary slackness tells us the following must be true:

$x_j = 0$ **or** $c_j = \sum_i y_i a_{ij}$ for all $x_j \in \bar{x}, c_j \in \bar{c}$
$y_i = 0$ **or** $b_i = \sum_j a_{ij} x_j$ for all $y_i \in \bar{y}, b_i \in \bar{b}$

We now have the following system of equations to satisfy $c_j = \sum_i y_i a_{ij}$ for all $c_j \in \bar{c}$ ($x_3 = 0$, so we skip checking the third constraint):

$$
\begin{array}{lllll}
-y_1 & -2y_2 & +y_3 & +y_4 & = -2 \\
y_1 & & -y_3 & & = 3 \\
 & y_2 & +y_3 & & = 1 \\
 & & & -y_4 & = -1
\end{array}
$$

Solving the system, we get for the dual solution: $y_1 = 4, y_2 = 0, y_3 = 1, y_4 = 1$. We then check the dual solution to satisfy $b_i = \sum_j a_{ij} x_j$ for all $b_i \in \bar{b}$ (we can skip y_2, since it satisfies $y_i = 0$).

Checking y_1, it must be true that $x_1 - x_2 - x_3 = 5$. However, $\frac{10}{3} - 8 - 0 = -\frac{14}{3} \neq 5$.

The dual solution cannot be used to certify the given primal solution. Therefore, the given primal solution is not optimal.

Problem 6

Dual problem:

$$
\begin{array}{rlll}
\text{min.} & b^T y & +y_0 & \\
\text{s.t.} & A^T y & +\bar{1} y_0 & \geq c \\
 & y, & y_0 & \geq 0
\end{array}
$$

We can set $y = 0$, which is feasible, and equivalently leaves us with the following problem:

$$
\begin{array}{rll}
\text{min.} & y_0 & \\
\text{s.t.} & \bar{1} y_0 & \geq c \\
 & y_0 & \geq 0
\end{array}
$$

If we then set $y_0 = \max\{c, 0\}$, we have a feasible solution. Therefore, setting $y = 0, y_0 = \max\{c, 0\}$, the dual of the given problem is always feasible. Therefore the primal must also always be feasible (strong duality theorem).