Homework 4
Colorado CSCI 5654

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Note: All code (and thus, worked-out solution) for this assignment (namely, problems 3 and 4) can be found here.

Problem 1

(A)

$\min_{x_1,x_2,x_3 \in \mathbb{R}^3} \max(2x_1 + 3x_2 - 5x_3, x_1, x_2, 2x_1 - x_2 + x_3)$

Let $t = \max(2x_1 + 3x_2 - 5x_3, x_1, x_2, 2x_1 - x_2 + x_3)$

$$\begin{aligned}
\text{min.} \quad & t \\
\text{s.t.} \quad & 2x_1 + 3x_2 - 5x_3 \leq t \\
& x_1 \leq t \\
& x_2 \leq t \\
& 2x_1 - x_2 + x_3 \leq t
\end{aligned}$$

(B)

$\min_{x_1,x_2,x_3\in\mathbb{R}^3}(|x_1+x_2|+|x_2-x_3|+|x_3-x_1|+|x_1+x_2+x_3|)$

Let $t_1=|x_1+x_2|, t_2=|x_2-x_3|, t_3=|x_3-x_1|, t_4=|x_1+x_2+x_3|$

$$\begin{aligned}
\text{min. } & t_1+t_2+t_3+t_4 \\
\text{s.t. } & x_1+x_2 \le t_1 \\
& -(x_1+x_2) \le t_1 \\
& x_2-x_3 \le t_2 \\
& -(x_2-x_3) \le t_2 \\
& x_3-x_1 \le t_3 \\
& -(x_3-x_1) \le t_3 \\
& x_1+x_2+x_3 \le t_4 \\
& -(x_1+x_2+x_3) \le t_4
\end{aligned}$$

(C)

$$
\begin{aligned}
\text{min. } & \max(|x_1|, |x_2|, |x_3|, |x_1 + x_2|) \\
\text{s.t. } & x_1 - x_2 \leq 5 \\
& x_2 \leq 3
\end{aligned}
$$

$$\downarrow \text{Let } t = \max(|x_1|, |x_2|, |x_3|, |x_1 + x_2|)$$

$$
\begin{aligned}
\text{min. } & t \\
\text{s.t. } & x_1 - x_2 \leq 5 \\
& x_2 \leq 3 \\
& |x_1| \leq t \\
& |x_2| \leq t \\
& |x_3| \leq t \\
& |x_1 + x_2| \leq t
\end{aligned}
$$

$$\downarrow$$

$$
\begin{aligned}
\text{min. } & t \\
\text{s.t. } & x_1 - x_2 \leq 5 \\
& x_2 \leq 3 \\
& x_1 \leq t \\
& -x_1 \leq t \\
& x_2 \leq t \\
& -x_2 \leq t \\
& x_3 \leq t \\
& -x_3 \leq t \\
& x_1 + x_2 \leq t \\
& -(x_1 + x_2) \leq t
\end{aligned}
$$

Problem 2

(A)

$f_2(x) = \sqrt{\sum_{j=1}^{2k+1}(x_j - x)^2}$

$$\frac{d}{dx}\sqrt{\sum_{j=1}^{2k+1}(x_j - x)^2} = \frac{1}{2}[\sum_{j=1}^{2k+1}(x_j - x)^2]^{-\frac{1}{2}} \cdot -2\sum_{j=1}^{2k+1}(x_j - x)$$
$$0 = -\frac{\sum_{j=1}^{2k+1}(x_j - x)}{\sum_{j=1}^{2k+1}(x_j - x)^2}$$
$$0 = \sum_{j=1}^{2k+1}(x_j - x)$$
$$0 = -x(2k+1) + \sum_{j=1}^{2k+1} x_j$$
$$x = \frac{\sum_{j=1}^{2k+1} x_j}{2k+1} = \text{mean}(\{x_1, x_2, \ldots, x_{2k+1}\})$$

(B)

$f_1(x) = \sum_{j=1}^{2k+1} |x_j - x|$

The median of $\{x_1, x_2, \ldots, x_{2k+1}\}$ is x_{k+1}. Let us consider $f_1(x_{k+1})$ against $f_1(x_{k+1} + \epsilon)$ for tiny values $\epsilon > 0, \epsilon < 0$.

$f_1(x_{k+1} + \epsilon) = \sum_{j=1}^{2k+1} |x_j - x - \epsilon|$

When $\epsilon > 0$, the k values greater than the median will see a change in the value of $|x_j - x - \epsilon|$ of $-\epsilon$, the k values less than the median will see in a change in the value of the same quantity of ϵ, and the median value will see a change in the the value of the same quantity of ϵ. Therefore the total change is $-\epsilon k + \epsilon(k+1) = \epsilon > 0$. So $f_1(x_{k+1}) < f_1(x_{k+1} + \epsilon)$.

When $\epsilon < 0$, the k values greater than the median will see a change in value to $|x_j - x - \epsilon|$ of $-\epsilon$, the k values less than the median will see in a change in value to the same quantity of ϵ, and the median value will see a change in the value to the same quantity of $-\epsilon$. Therefore the total change is $-\epsilon(k+1) + \epsilon k = -\epsilon > 0$. So $f_1(x_{k+1}) < f_1(x_{k+1} + \epsilon)$.

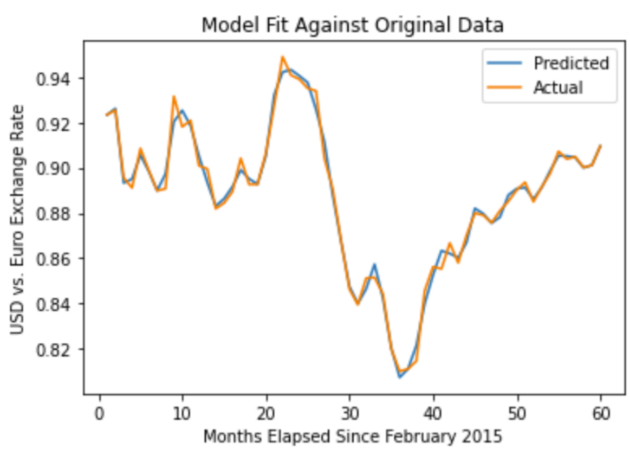
Therefore the value x_{k+1} minimizes $f_1(x)$.

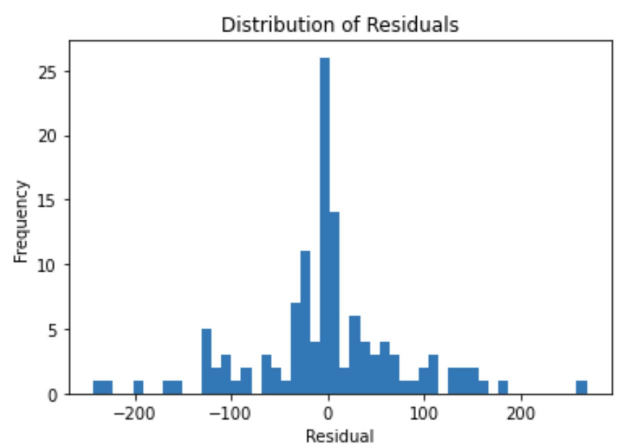
Problem 3

(A)

Coefficients:

```
[ -17.75 ,   63.881,   -4.305,    0.06 ,  -17.75 ,   -0.001,
    0.001,   -0.001,   -0.001,   -0.008,   -0.063,   -0.792,
   -4.352,   24.219,  156.829, -201.597,  -94.755, -112.882,
   28.312,   -9.874,  -26.783,   34.072,   71.124,   55.257,
   24.885,   12.364,   20.839,   37.646,   50.404,   53.612,
   47.943,   37.074,   24.992,   14.578,    7.27 ,    3.315,
    2.201,    3.076,    5.039,    7.313,    9.321,   10.693,
   11.242,   10.922,    9.786,    7.948,    5.552,    2.752,
   -0.306,   -3.491,   -6.69 ,   -9.814,  -12.792,  -15.573,
  -18.125,  -20.426,  -22.467,  -24.249,  -25.779,  -27.067,
  -28.13 ,  -28.984,  -29.647,  -30.139,    0.   ,   -0.   ,
    0.   ,    0.003,    0.001,   -0.01 ,   -0.027,   -0.319,
   -6.754,  -41.521,   72.674,  268.807, -231.363,   59.368,
  -52.236, -242.753,  -21.88 ,  106.577,  -22.345, -165.553,
 -155.209,  -28.183,   94.225,  139.922,  105.191,   25.902,
  -55.685, -110.155, -125.86 , -105.471,  -59.816,   -2.294,
   54.916,  102.603,  134.768,  149.953,  148.909,  134.017,
  108.527,   75.975,   39.768,    2.929,  -32.028,  -63.147,
  -89.009, -108.683, -121.658, -127.767, -127.112, -119.997,
 -106.869,  -88.269,  -64.795,  -37.064,   -5.691,   28.724,
   65.619,  104.466,  144.786,  186.147]
```



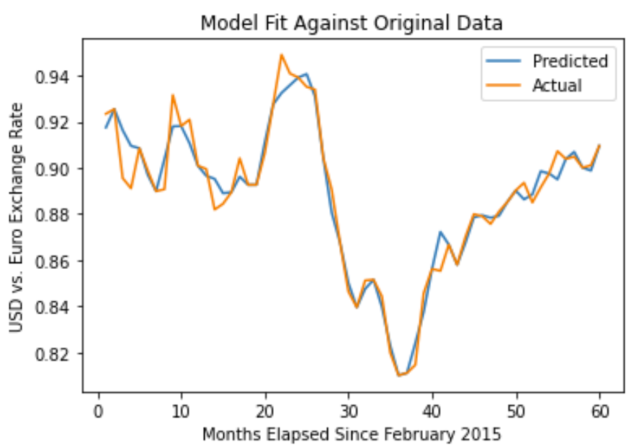


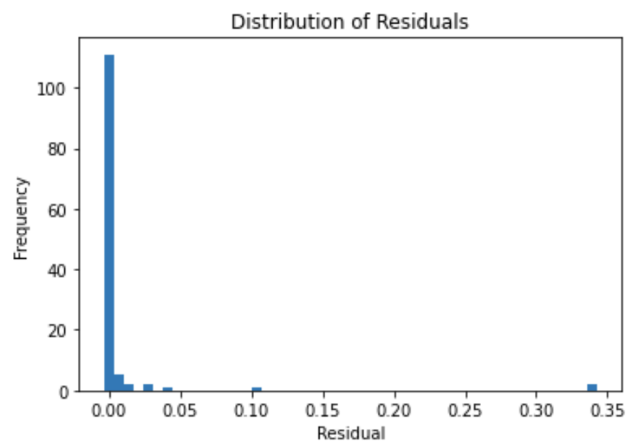
This model sees quite a bit of overfitting, as can be seen if one extends the predictions past 60 months, where the exchange rate shoots off upwards very quickly.

(B)

Coefficients:

```
[ 0.343,  0.029, -0.001,  0.   ,  0.343, -0.   , -0.   , -0.002,
  0.001, -0.004,  0.   ,  0.006,  0.   ,  0.   ,  0.012,  0.001,
  0.013,  0.   , -0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.038,  0.004,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.025,  0.008,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.105,
  0.   , -0.   , -0.002,  0.004,  0.001, -0.002, -0.003,  0.005,
 -0.001,  0.   , -0.   ,  0.   , -0.   , -0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,  0.   ,
  0.   ,  0.   ,  0.   ,  0.   ]
```





This model sees a slight bit of overfitting, as can be seen if one extends the predictions past 60 months, where the exchange rate shoots off upwards at a fair rate (not nearly as quickly as the model in part **(A)**, but still noticeable).

Problem 4

(A)

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -2 & -2 & 3 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} .2 & .3 & .2 & .2 & .1 \end{bmatrix}^T$$

Row player utility $= x_0^T A y$

$$\begin{aligned} \text{min.} \ & x_0^T A y \\ \text{s.t.} \ & e^T y = 1 \\ & y \geq 0 \end{aligned}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This means that player B will always choose the fourth option, as such a choice minimizes player A's score/utility.

(B)

$$\begin{aligned} \text{min.} \ & \mathbf{c}^T \mathbf{y} \\ \text{s.t.} \ & \mathbf{y} \text{ is a stochastic vector} \end{aligned}$$

Let $c_j = \min(\mathbf{c})$

$$\begin{aligned} \mathbf{c}^T \mathbf{y} &= c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \\ &\geq c_j y_1 + c_j y_2 + \cdots + c_j y_n \\ &= c_j \mathbf{y} \\ &= c_j && \textbf{(since y is stochastic)} \end{aligned}$$

Therefore, $c_j = \min\{c_1, c_2, \ldots, c_n\}$ is the optimal solution to the given linear program.

(C)

$\mathbf{x}^T A = c \rightarrow c_j = \mathbf{x}^T A_{(*,j)}$

$$
\begin{aligned}
\min_{\mathbf{y}} \mathbf{x}^T A \mathbf{y} &= \min_{\mathbf{y}} \mathbf{c} \cdot \mathbf{y} \\
\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}^T A \mathbf{y} &= \max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{c} \cdot \mathbf{y} \\
&= \max_{\mathbf{x}} \min\{\mathbf{x}^T A_{(*,1)}, \mathbf{x}^T A_{(*,2)}, \ldots, \mathbf{x}^T A_{(*,n)}\} \qquad \text{using part } \textbf{(B)}
\end{aligned}
$$

(D)

$$
\begin{aligned}
\max\nolimits_x \min(&\mathbf{c}_1^T \mathbf{x}, \mathbf{c}_2^T \mathbf{x}, \ldots, \mathbf{c}_n^T \mathbf{x}) \\
\text{s.t. } & \mathbf{1}^T \mathbf{x} = 1 \\
& \mathbf{x} \geq 0
\end{aligned}
$$

$$\downarrow \text{Let } t = \min(\mathbf{c}_1^T \mathbf{x}, \mathbf{c}_2^T \mathbf{x}, \ldots, \mathbf{c}_n^T \mathbf{x})$$

$$
\begin{aligned}
\text{max. } & t \\
\text{s.t. } & \mathbf{1}^T \mathbf{x} = 1 \\
& \mathbf{c}_1^T x \geq t \\
& \mathbf{c}_2^T x \geq t \\
& \vdots \\
& \mathbf{c}_n^T x \geq t \\
& \mathbf{x} \geq 0
\end{aligned}
$$

$$\downarrow$$

$$
\begin{aligned}
\text{max. } & t \\
\text{s.t. } & \mathbf{1}^T \mathbf{x} = 1 \\
& -\mathbf{c}_1^T x + t \leq 0 \\
& -\mathbf{c}_2^T x + t \leq 0 \\
& \vdots \\
& -\mathbf{c}_n^T x + t \leq 0 \\
& \mathbf{x} \geq 0
\end{aligned}
$$

(E)

$$\begin{aligned} \max.\ & t \\ \text{s.t.}\ & -A^T\mathbf{x} + \mathbf{1}t \le 0 \\ & \mathbf{1}^T\mathbf{x} = 1 \\ & \mathbf{x} \ge 0 \end{aligned}$$

$\downarrow$ Let $t = t^+ - t^-$

$$\begin{aligned} \max.\ & t^+ - t^- \\ \text{s.t.}\ & -A^T\mathbf{x} + \mathbf{1}(t^+ - t^-) \le 0 \\ & \mathbf{1}^T\mathbf{x} \le 1 \\ & -\mathbf{1}^T\mathbf{x} \le -1 \\ & \mathbf{x}, t^+, t^- \ge 0 \end{aligned}$$

$\downarrow$ Convert to block matrix

$$\begin{aligned} \max.\ & \begin{bmatrix} 0_{n\times1} \\ 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ t^+ \\ t^- \end{bmatrix} \\ \text{s.t.}\ & \begin{bmatrix} -A^T_{n\times n} & 1_{n\times1} & 1_{n\times1} \\ 1_{1\times n} & 0 & 0 \\ -1_{1\times n} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ t^+ \\ t^- \end{bmatrix} \le \begin{bmatrix} 0_{n\times1} \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

$\downarrow$ Switch to dual block matrix

$$\begin{aligned} \min.\ & \begin{bmatrix} 0_{n\times1} \\ 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} y \\ z^+ \\ z^- \end{bmatrix} \\ \text{s.t.}\ & \begin{bmatrix} -A_{n\times n} & 1_{n\times1} & 1_{n\times1} \\ 1_{1\times n} & 0 & 0 \\ -1_{1\times n} & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z^+ \\ z^- \end{bmatrix} \le \begin{bmatrix} 0_{n\times1} \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

$\downarrow$ Convert to dual, and $z = z^+ - z^-$

$$\begin{aligned} \min.\ & z \\ \text{s.t.}\ & -A\mathbf{y} + \mathbf{1}z \le 0 \\ & \mathbf{1}^T\mathbf{y} = 1 \\ & \mathbf{y} \ge 0 \end{aligned}$$

(F)

See work in code linked at top of this document
Row player equilibrium strategy: $\begin{bmatrix} 0 & 0 & .5 & .5 & 0 \end{bmatrix}$
Column player equilibrium strategy: $\begin{bmatrix} .387 & .105 & .122 & .387 & 0 \end{bmatrix}$
Value of the game: 0