Homework 3
Colorado CSCI 5654

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Note: All code for this assignment (namely, problems 1, 3c, and 4) can be found here.

Problem 1

(A)

1.

solution: $x_1 = 2, x_2 = -8, x_3 = 4, w_1 = -7, w_2 = -3, w_3 = 2$
objective row: $16 + 0x_4 - 6x_5 + 0x_6 - 3w_4 - 2w_5 - 4w_6$
objective value: 16

2.

solution: $x_1 = -1, x_2 = -4, x_5 = 0, w_3 = -2, w_5 = 7, w_6 = -3$
objective row: $14 + 4x_3 - 6x_4 - 6x_6 + 2w_1 + 0w_2 - 5w_4$
objective value: 14

3.

solution: $x_1 = -1, x_2 = -6, x_6 = 2, w_4 = -4, w_5 = 5, w_6 = -1$
objective row: $22 - 5x_3 - 6x_4 - 5x_5 - 3w_1 + 5w_2 + 4w_3$
objective value: 22

(B)

1.

constant column: [2 6 4 3 9 0]^T
objective row: $-8 - 2x_1 - 4x_2 + 4x_6 - 2w_3 + 1w_4 + 0w_5$

2.

entering variable: x_6

3.

column corresponding to entering variable: [-1 -2 0 1 -2 -1]^T

4.

leaving variable: w_6

5.

basic variable in the next dictionary: $\{x_3, x_4, x_5, w_1, w_2, x_6\}$

(C)

x = [0 0 0 0 1 4]

$$\begin{aligned}
w_1 &= 3 - x_1 + x_2 + x_6 && = 7 \\
w_2 &= -1 - x_1 + x_4 + x_5 && = 0 \\
w_3 &= -2 + x_3 + x_6 && = 2 \\
w_4 &= 4 + x_2 + x_3 - x_4 - x_6 && = 0 \\
w_5 &= 6 - x_1 - x_3 - x_5 - x_6 && = 1 \\
w_6 &= -2 + x_1 + x_4 - x_5 + x_6 && = 1
\end{aligned}$$

basic variables: $x_5, x_6, w_1, w_3, w_5, w_6$
non-basic variables: $x_1, x_2, x_3, x_4, w_2, w_4$

complete final dictionary:

$$\begin{array}{r|lllllll}
x_5 & 1 & +1x_1 & & & -1x_4 & +1w_2 & \\
x_6 & 4 & & +1x_2 & +1x_3 & -1x_4 & & -1w_4 \\
w_1 & 7 & -1x_1 & +2x_2 & +1x_3 & -1x_4 & & -1w_4 \\
w_3 & 2 & & +1x_2 & +2x_3 & -1x_4 & & -1w_4 \\
w_5 & 1 & -2x_1 & -1x_2 & -2x_3 & +2x_4 & -1w_2 & +1w_4 \\
w_6 & 1 & & +1x_2 & +1x_3 & +1x_4 & -1w_2 & -1w_4 \\
\hline
z & 4 & -2x_1 & -2x_2 & & -2x_4 & & -1w_4
\end{array}$$

Problem 2

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax \leq b \end{aligned}$$

$\downarrow$ replace x with the difference of two positive numbers x^+ and x^-

$$\begin{aligned} &\max\ c^T(x^+ - x^-) \\ &\text{s.t. } A(x^+ - x^-) \leq b \\ &\quad\ x^+, x^- \geq 0 \end{aligned}$$

$\downarrow$ use block matrices

$$\begin{aligned} &\max\ \begin{bmatrix} c & -c \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \\ &\text{s.t. } \begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \leq b \\ &\quad\ \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq \bar{0} \end{aligned}$$

$\downarrow$ primal to dual as shown in class

$$\begin{aligned} &\min\ b^{\mathrm{T}} y \\ &\text{s.t. } \begin{bmatrix} A & -A \end{bmatrix}^{\mathrm{T}} y \geq \begin{bmatrix} c \\ -c \end{bmatrix} \\ &\quad\ y \geq 0 \end{aligned}$$

$\downarrow$

$$\begin{aligned} &\min\ b^{\mathrm{T}} y \\ &\text{s.t. } A^{\mathrm{T}} y \geq c \\ &\quad\ -A^{\mathrm{T}} y \geq -c \\ &\quad\ y \geq 0 \end{aligned}$$

$\downarrow$

$$\begin{aligned} &\min\ b^{\mathrm{T}} y \\ &\text{s.t. } A^{\mathrm{T}} y = c \\ &\quad\ y \geq 0 \end{aligned}$$

$\square$

Problem 3

(A)

First we certify that the primal and dual solutions are feasible and dual-feasible, respectively.
Primal:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \leq 0 \\ x_1 + 2x_3 &= 3 \leq 3 \\ -x_1 + x_2 &= -1.5 \leq 0 \\ 2x_2 + x_3 &= 3 \leq 3 \\ x_1 + x_3 &= 3 \leq 5 \\ x_1, x_2, x_3 &= 3, 1.5, 0 \geq 0 \end{aligned}$$

Dual:

$$\begin{aligned} \min\ & 3y_2 + 3y_4 + 5y_5 \\ \text{s.t.}\ & y_1 + y_2 - y_3 + y_5 \geq 3 \\ & -2y_1 + y_3 + 2y_4 \geq 4 \\ & y_1 + 2y_2 + y_4 + y_5 \geq -1 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

$$\begin{aligned} y_1 + y_2 - y_3 + y_5 &= 3 \geq 3 \\ -2y_1 + y_3 + 2y_4 &= 4 \geq 4 \\ y_1 + 2y_2 + y_4 + y_5 &= 8 \geq -1 \\ y_1, y_2, y_3, y_4, y_5 &= 3, 0, 0, 5, 0 \geq 0 \end{aligned}$$

All constraints are satisfied, so the solutions are feasible. The strong duality theorem tells us that if the primal has an optimal solution x then there exists a dual-feasible (and optimal) y such that $c^{\mathrm{T}}x = b^{\mathrm{T}}y$.

$c^{\mathrm{T}}x = 3(3) + 4(1.5) - 1(0) = 15$
$b^{\mathrm{T}}y = 0(3) + 3(0) + 0(0) + 3(5) + 5(0) = 15$

Therefore, x is optimal with the corresponding dual solution y. $\square$

(B)

The non-zero values in the primal solution tell us that x_1 and x_2 must both be basic variables. The non-zero values in the dual solution tell us that y_1 and y_4 must both be basic variables. We let w and p be the slack variables for the primal and dual, respectively. Complementary slackness tells us that w_2, w_3, w_5, and p_3 must also be basic variables. All other variables (x_3, w_1, w_4, y_2, y_3, y_5, p_1, and p_2) must be non-basic.

Primal
Basic: $\{x_1, x_2, w_2, w_3, w_5\}$
Non-basic: $\{x_3, w_1, w_4\}$

Dual
Basic: $\{y_1, y_4, p_3\}$
Non-basic: $\{y_2, y_3, y_5, p_1, p_2\}$

(C)

For the final dictionary to be feasible, the constant column must remain non-negative ($A_B^{-1}(b + \Delta b) \geq 0$). With the basis $\{x_1, x_2, w_2, w_3, w_5\}$, we can find:

$$A_B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & .5 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & .5 & 0 \\ -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$A_B^{-1}(b + \Delta b) \geq 0 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & .5 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & .5 & 0 \\ -1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3+t \\ 5 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3+t \\ .5(3+t) \\ 3-(3+t) \\ .5(3+t) \\ -(3+t)+5 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} t \\ t \\ -t \\ t \\ -t \end{bmatrix} \geq \begin{bmatrix} -3 \\ -3 \\ 0 \\ -3 \\ -2 \end{bmatrix}$$

Therefore we have that $-3 \leq t \leq 0$.

From lecture, we have $z' = z^* + \Delta b y^*$. $z^* = 15$, $\Delta b = [0\ 0\ 0\ t\ 0]^{\mathrm{T}}$, $y^* = [3\ 0\ 0\ 5\ 0]$. Therefore, $z' = z^* + \Delta b y^* = 15 + 5t$.

Problem 4

Vertex	Constraints Saturated	Degenerate?
(0, 0, 0)	{1, 2, 3, 4}	yes
(0, 2, 0)	{2, 4, 7, 8}	yes
(1, 1, -2)	{3, 4, 5, 8}	yes
(1, 1, 2)	{1, 2, 6, 7}	yes
(2, 0, 0)	{1, 3, 5, 6}	yes
(2, 2, 0)	{5, 6, 7, 8}	yes