The role of asset payouts in the estimation of default barriers

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Abstract

In the barrier option model of corporate security valuation, the firm's creditors impose a default-triggering barrier on the firm value to protect their claim. Two disputed issues in the literature are whether the implied default barrier is positive, and whether it is above or below the book value of the firm's liabilities. We extend the model of Brockman and Turtle (2003, *Journal of Financial Economics* 67, 511–529) by embedding asset payouts in the valuation of shareholders' equity. Using a sample of US stocks from the NYSE, AMEX, and NASDAQ exchanges, our paper exploits market and firm information to compute the implied default barrier for thirty 2-digit SIC groups, including industrials and banks. Our results show that the implied default barrier is lower than it is in the received literature, and it can be less than total liabilities, even zero for some firms. The implied physical default probabilities are significantly lower in the presence of payouts, providing a closer fit to the historical corporate default rates, particularly for issuers of speculative-grade bonds.

JEL Classification: G12, G33, G35

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1 Introduction

The contingent claims analysis of corporate securities, pioneered by Black and Scholes (1973) and Merton (1974), hereafter BSM, has spawned a vast literature that can be classified in two strands: Papers in which the market value of the firm does not change with financing although debt is defaultable (Black and Cox, 1976; Longstaff and Schwartz, 1995; Brockman and Turtle, 2003), and papers in which a unique optimal capital structure exists (Leland, 1994; Leland and Toft, 1996; Leland, 1998).[1]

A common feature in these two strands is the presence of path-dependence in the valuation of the contingent claims, since firm default is declared when the asset value reaches a lower critical threshold. In the models of optimal capital structure, the default barrier is endogenous, and reflects the costs of bankruptcy, the maturity of debt, the tax-shields of coupon payments, and the level of leverage (Leland and Toft, 1996). When capital structure is irrelevant for the valuation of the firm's assets, the default barrier is exogenous and has to be estimated. Two disputed issues in the literature are whether the probability of early default is priced by the market, and whether the implied default barrier is above or below the book value of liabilities.

A case in point is the paper of Brockman and Turtle (2003), hereafter BT, who develop a structural model in which corporate securities are expressed via barrier options. The creditors impose a lower barrier which triggers default, to protect their claim from the strategic depletion of the firm's assets by shareholders. Therefore, equity can be viewed as a down-and-out call option (DOC) written on the firm's asset value. The creditor's claim is enhanced by a down-and-in call option, implying that debt value is not bounded by the amount lent, but can be as large as the asset value, at least theoretically.

In the BT paper, the book value of non-equity liabilities proxies for the market value of debt to arrive at an estimate of the market value of the firm. Using quarterly observations for the asset value, an estimate of the asset volatility is computed, and the default barrier is derived from the DOC formula. BT show that the barrier is economically and statistically significant, implying that the barrier affects stock and bond prices, as investors translate the various debt covenants to restrictions on firm asset value. Furthermore, BT find that the barrier, as a proportion of the firm's market value, is always higher than leverage.[2]

[1] See Sundaresan (2013) for a literature review of structural models.

[2] In the remainder of the paper, and depending on the context, we shall interchangeably compare the barrier as a proportion of the firm's market value with the leverage ratio, and the dollar barrier with the book value of liabilities.

According to Reisz and Perlich (2007) and Wong and Choi (2009), the proxy used by BT for the value of debt is a shortcoming of the empirical application of the BT model, because it always results in positive and higher-than-leverage levels of the default barrier. This bias leads to an overestimation of the firm default probabilities and an underestimation of the debt yield spreads. Using the maximum likelihood methodology of Duan (1994, 2000), Wong and Choi (2009) show that the barrier is positive but smaller than outstanding liabilities for the majority of their sample firms. The same conclusion is drawn by Reisz and Perlich (2007), who calibrate the BT model using a system of equations that specifies simultaneously the asset value, the firm volatility, and the default barrier. However, the presence of below-leverage barriers in these papers is not strictly attributed to the absence of arbitrage opportunities within the structural model framework.

The purpose of this paper is to re-examine the original BT model, and correct the aforementioned shortcomings through the explicit modelling of asset payouts. While asset payouts have a direct effect on the valuation of equity, they were overlooked in BT. In our model, we assume that the firm's assets provide a continuous payout that is fully distributed between shareholders and bondholders. The market value of equity is defined as the sum of the DOC option value and the dividend present value, and is significantly lower than the equity value in the original BT model. This is exactly the new feature that leads to lower barrier estimates and drives our results. We'll be referring to this setup as the Payout BT (PBT) model.

Applying the new model on a sample of US firms listed on NYSE, AMEX, and NASDAQ, including banks, shows that the implied barrier is lower than the book value of total liabilities for approximately two-thirds of the firms in thirty different industry sectors. Introducing asset payouts resolves the inherent bias of the BT model in extracting empirically the firm default barrier, without resorting to maximum-likelihood or simultaneous-equation methods. The estimated barrier is statistically significant across all different industry groups, suggesting that the market prices the probability of early default. Regarding banks, the implied barriers can be both above and below the total liabilities, thus capturing different regulatory policy regimes with stringent, lax, or even negative minimum capital requirements.[3]

Furthermore, the new model improves the estimation of default probabilities. The implied physical default probabilities provide a closer fit to the historical corporate default

[3] Negative capital requirements can arise when: (i) the bank capital is reported on an inappropriate accounting basis (e.g. historical cost accounting), even if the regulator imposes positive minimum capital requirements (Kane, 1989; Harding et al., 2013), and (ii) the regulator follows a capital forbearance policy (Ronn and Verma, 1986).

rates, compared to the BT model. Noteworthy, the fit is better for issuers of speculative grade bonds, whose exposure to default risk is higher. The implied default probabilities are positively related to the credit rating quality for industrial firms, aligning with the view of the main rating agencies.

The contribution of our paper in the structural model literature is threefold. First, it extends the barrier option model of corporate securities to the case of asset payouts. Second, it shows that the barrier option model is robust, and can produce lower than leverage barriers without resorting to complex estimation procedures. Third, it demonstrates that the barrier is related to firm characteristics and market information such as credit ratings, and provides a better fit to the historical default rates.

The paper is related to Brockman and Turtle (2003) as well as the associated models (Black and Cox, 1976; Ericsson and Reneby, 2005; Forte and Lovreta, 2012; Sokolinskiy, 2019). From a technical perspective, our work is related to Mjøs and Persson (2010) and Mjøs et al. (2013), since the present value of dividends is approximated as the limit of a finite multi-level annuity with bankruptcy risk, where the model of Ingersoll (1976) for dual purpose funds occurs as a special solution when the barrier is zero. The paper is also related to structural model applications in banking (Episcopos, 2008; Chang, 2014), due to the direct influence of the regulators on the default barrier of financial institutions.

The rest of the paper is structured as follows. Section 2 describes the new model, and Section 3 provides numerical comparisons with BT, based on reasonable parameter values as a base case. Section 4 specifies the PBT model for various industries in the US, after giving a detailed justification of the empirical application of the model. Section 5 presents the empirical results, and examines the relation of the implied default barriers with the default probabilities and credit ratings. Section 6 concludes.

2 The structural framework

2.1 Model assumptions and equity valuation

The firm has assets in place, V_t, with a market value following a log-normal diffusion process. We assume that the firm's assets provide a continuous stream of payments, fully distributed between the bondholders and the shareholders. Thus, the dynamics of V_t are described by the following equation:

$$\frac{dV_t}{V_t} = (r + \pi - q)dt + \sigma dW_t^{\mathbb{P}}, \quad V_0 = V > 0 \tag{1}$$

where r denotes the risk-free rate, which is constant and the same across all maturities. The parameter π denotes the asset risk premium, q denotes the payout rate and σ the constant asset return volatility. The term $dW_t^{\mathbb{P}}$ denotes the increments of a standard Wiener process, defined in the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the valuation of the contingent claims, we shall work with the risk-neutral measure $\mathbb{Q}$, under which the asset risk premium is zero. Therefore, the dynamics of V_t under the $\mathbb{Q}$-measure are given by:

$$\frac{dV_t}{V_t} = (r - q)dt + \sigma dW_t^{\mathbb{Q}}, \quad V_0 = V > 0 \tag{2}$$

The firm owes a single amount, X, to its creditors payable at time T, and there is an exogenously given lower barrier, H, which triggers default if crossed by the asset value. The early default time τ is a random variable, defined as $\tau = \inf\{t > 0 : V_t = H\}$. There are no bankruptcy costs, taxes or any agency costs that would make the value of the firm different from the value of its assets. Dividends and interest are paid continuously to shareholders and to creditors at the constant rates of δ and c, respectively. In other words, $q = \delta + c$. The latter assumption is made in order to be able to get tractable solutions, because it is known (Merton, 1974) that the general problem of arbitrary function payouts does not have known closed-form solutions.

Cox and Ross (1976) suggest that the value of any security on the assets of the firm, can be considered as the value of the security with only the terminal return, plus the value of potential payouts received. In our case, the shareholders' claim has two components, namely, the down-and-out call option written on V, $DOC(H, q)$, and the dividends to be received until bankruptcy takes place, if at all, in the $(0, T]$ interval. Thus, the value of equity is

$$V_E(H, q) = DOC(H, q) + PVD(H, q) \tag{3}$$

where

$$PVD(H, q) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{min\{\tau,T\}} \delta V_t e^{-rt} dt\right] \tag{4}$$

PVD denotes the expected value of the present value of all dividends received by the shareholders until early bankruptcy or debt maturity.[4] To the best of our knowledge, the value of dividends in Eq. 4 does not have a closed-form solution. However, we tackle this problem in Section 2.2 by deriving a tractable approximation based on the theory of multilevel annuities (Mjøs and Persson, 2010). The formula for DOC when the underlying is paying continuous payouts (Clewlow et al., 1994) is reported in Appendix A. The original BT model is a special case of PBT, if we set $q = 0$. If we set $H = 0$ and $q = 0$, we get the standard BSM model.[5]

If the firm pays no dividends, then $\delta = 0$, and V_E in Eq. 3 will have only one component, $DOC(H, q)$, which is strictly less than $DOC(H, 0)$ under the BT assumptions. If the firm pays dividends and the barrier is equal to the asset value, that is, $\delta \neq 0$ and $V_t \to H$, then default is imminent ($\tau \to 0$) and the present value of dividends is zero along with the value of the down-and-out call. If the firm pays dividends and the barrier is zero, that is, $\delta \neq 0$ and $H = 0$, then PVD is the value of an income share in the Ingersoll (1976) dual purpose fund model, with

$$PVD(0, q) = \frac{\delta V}{q}\left(1 - e^{-qT}\right) \tag{5}$$

The term $V(1 - e^{-qT})$ is the value of the total dollar asset payout in the interval $(0, T]$, of which the shareholders receive only a proportion δ/q. A positive barrier reduces that amount.

In Eq. 3, the equity value has to also obey the condition $V_E = V - X$ to make the model internally consistent, as X is both the exercise price and the total liabilities in the BT model and ours. Thus, the implied barrier in the PBT model is given by the solution of the following equation:

$$H^{PBT} = \left\{H \in \mathbb{R}^+ : V_E(H, q) = V - X\right\} \tag{6}$$

Similarly, the implied barrier in the BT model is:

$$H^{BT} = \left\{H \in \mathbb{R}^+ : V_E(H, 0) = V - X\right\} \tag{7}$$

[4]For easier reading, we'll be omitting some or all the arguments of the PVD and DOC functions.

[5]It can be verified that $DOC(H, q) + DIC(H, q) = c(q)$, where $DIC(H, q)$ is a down-and-in call, and $c(q)$ is a standard call option.

2.2 Approximation of the present value of dividends

Before proceeding to the numerical analysis of the model, we have to determine an analytical expression for the present value of dividends. While the expected value in the right hand side of Eq. 4, can be estimated via a Monte Carlo method through simulation of discrete trajectories of the diffusion process in Eq. 2, such task would be time-consuming and would make the empirical validation of the PBT model extremely difficult.

Therefore, we approximate PVD as the limit of a finite multi-level annuity with bankruptcy risk, based on the results of Mjøs and Persson (2010) and Mjøs et al. (2013). The present value of dividends is treated as a portfolio of primitive Arrow-Debreu securities, whose value is mathematically expressed as

$$PVD(H,q) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{min\{\tau,T\}} \delta V_t e^{-rt} dt\right] \tag{8}$$

$$\approx \mathbb{E}^{\mathbb{Q}}\left[\int_0^{min\{\tau,T\}} \sum_{i=0}^{n+m} c_{i+1} e^{-rt} \mathbf{1}_{H_{i+1}<V_t<H_i} dt\right] \quad \text{for large } n,m \tag{9}$$

where $H_i, i \in \{1,2,...,n+m\}$ denotes a sequence of non-absorbing barriers that are greater than the default barrier H. Given the initial asset value V_0, we have n non-absorbing barriers that are greater than V_0, and m that are below V_0. Between two consecutive barriers, we treat the instantaneous dividends as constant. Thus for each region (H_{i+1}, H_i) the dividend is equal to c_{i+1}. We fix c_{i+1} at the average dividend of the interval endpoints, that is, $c_{i+1} = \delta(H_i + H_{i+1})/2$. By letting n and m get very large, we achieve a good approximation of PVD, in which the dividend stream is stochastic and state-dependent. In Appendix B, we provide the technical details and the formulas regarding the approximation of PVD. In Appendix C, we compare this approximation with a Monte-Carlo valuation, considering a wide range of parameter values.

3 Numerical comparison between BT and PBT

We now turn to some numerical examples to fix the above ideas. As a base case for the parameters, we set $V = 100$, $\sigma = 0.3$, $X = 45$, $T = 10$ years and $r = 0.06$. These parameters are very close to the estimates in the BT paper. The payout and dividend parameters, are $q = 0.04$ and $\delta = 0.02$. This is done in order to arrive at non-zero barriers in our illustrations, although as reported in Huang and Huang (2012), a q of about 0.06 is roughly the historical payout rate of US companies adjusted for leverage. Regarding the approximation of the

Table 1: Equity, dividends and probability of survival

Barrier	Equity value (BT)	Equity value (PBT)	PV of dividends	Survival probability
0.00	76.6550	61.5236	16.4839	1.0000
10.00	76.6549	61.5166	16.4771	0.9719
20.00	76.6019	61.3588	16.3751	0.8629
30.00	76.0066	60.4382	16.0298	0.7203
40.00	73.8787	57.8568	15.3204	0.5766
50.00	69.3427	53.0901	14.1610	0.4445
60.00	61.9257	46.1254	12.4910	0.3277
70.00	51.4015	37.1540	10.2673	0.2263
80.00	37.6312	26.3683	7.4585	0.1389
90.00	20.5208	13.9367	4.0418	0.0641
99.99	0.0222	0.0147	0.0044	0.0001

Notes: The table shows the value of equity, under the BT and PBT model, respectively, as functions of the default barrier. PV of dividends is the part of equity in PBT, which is due to the present value of dividends, that is $PVD(H, q)$. Survival probability denotes the risk-neutral probability of firm survival in the interval $(0, T)$ in the PBT model. Base case parameters: Firm asset value $V = 100$, asset volatility $\sigma = 0.3$, book value of total liabilities $X = 45$, debt maturity $T = 10$ years, risk-free rate $r = 0.06$, payout rate $q = 0.04$, and dividend rate $\delta = 0.02$.

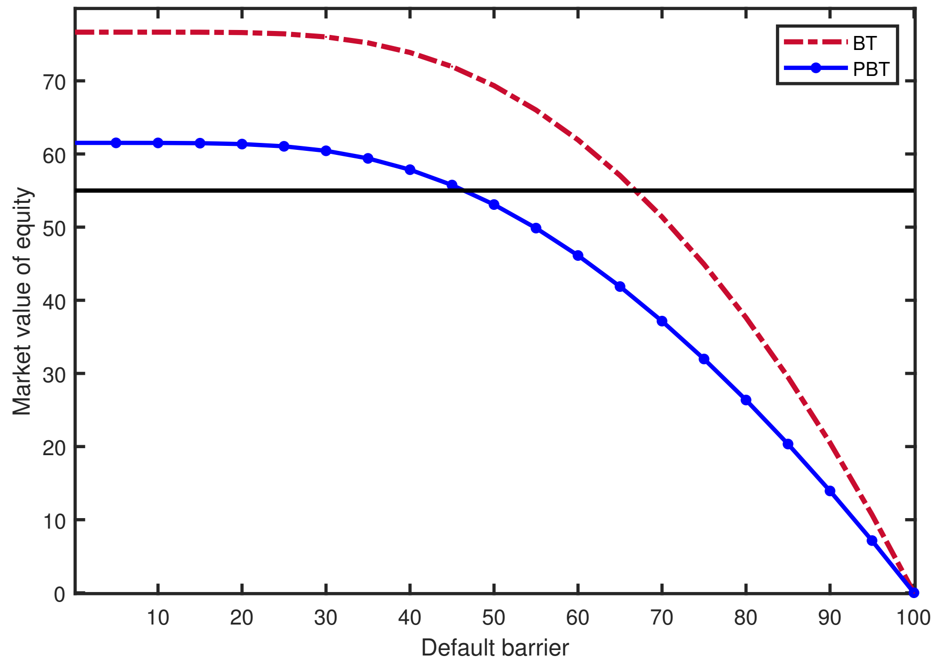
present value of dividends, we consider the following rule. For a given default barrier H, we fix the higher non-absorbing barrier H_1 to a constant value H^{max}. Then, we specify a step ΔV, and define the terms of the sequence $H_i, i \in \{1, 2, ..., n+m\}$ as $H_i = H_{i-1} - \Delta V$. The last non-absorbing barrier is $H_{n+m} = H + \Delta V$. Regarding the non-absorbing barrier H_0, we use the convention that $H_0 = +\infty$. For the remainder of the paper, we shall set $H^{max} = 10000$ and $\Delta V = 10$. In Appendix C, we show that these values yield a good approximation.

3.1 Equity as a function of the barrier

Table 1 shows the value of equity in the BT and the PBT models (second and third column, respectively), as a function of the barrier (first column). The next two columns show the present value of dividends and the risk-neutral probability of survival in the PBT model.[6] We observe that the BT equity function is higher than the PBT equity function at all barrier

[6]The probability of survival when the underlying asset pays proportional payouts at the rate of q is given in Appendix A. The BT formula is a special case with $q = 0$.

Figure 1: Equity value as a function of the barrier



Notes: The dashed and solid curves show the value of equity as a function of the barrier in the BT and PBT models, respectively. Base case parameters: Firm asset value $V = 100$, asset volatility $\sigma = 0.3$, book value of total liabilities $X = 45$, debt maturity $T = 10$ years, risk-free rate $r = 0.06$, payout rate $q = 0.04$, and dividend rate $\delta = 0.02$. The horizontal line represents the book equity value of $V - X = 55$, and its intersection with the two curves corresponds to solutions for the barrier $H^{BT} = 66.9053$ and $H^{PBT} = 46.5330$ in the BT and PBT models, respectively.

levels. When the barrier is zero, the two equity values in the BT and PBT model are equal to the standard BSM call options, i.e., $c(0) = 76.6550$ and $c(q) = 61.5236$, respectively.

As shown also in Fig. 1, the equity functions in terms of the barrier are downward sloping and concave, converging to zero when the barrier equals the asset value. The single equity value that fits any of the two models is $V - X = 55$ (the horizontal line in Fig. 1). The corresponding solution on the barrier axis is unique and equals 66.9052 and 46.5330 in the BT and PBT model, respectively. Therefore, our first result is that, when we allow for asset payouts, the implied default barrier is lower than it is in the original BT formulation.

We also confirm the Wong and Choi (2009) result that the BT barrier is higher than the book value of corporate liabilities. Furthermore, we note that PVD is a declining and concave function of the barrier, starting from the Ingersoll (1976) value of 16.4839 and declining to zero when the barrier equals asset value, as the corresponding survival probability goes from

1 to 0. Obviously $DOC(H, q) < DOC(H, 0)$, but it turns out that adding the value of dividends to $DOC(H, q)$ does not make the value of equity in the PBT model exceed the value of equity in the BT model. This is confirmed by an extensive numerical search for reasonable parameter values.

3.2 Barrier sensitivity to parameter changes

The sensitivity of the barrier to changes in other parameters is shown in Table 2. The middle column of the table shows the barriers of the BT and PBT models for the base case, and the other two columns show the barrier computed after changing the respective parameters by 10%. In this example, the barrier is generally more sensitive to parameter changes in the PBT model than it is in the BT model. This is attributed to two factors. First, the BT barrier is bounded from below by the book value of firm's liabilities by construction. Second, as shown in Fig. 1, the equity function is less steep in the PBT model than it is in the BT model.

The book value of total liabilities (after adjusting for equity to sum up to a value of the firm of 100) is positively related to the barrier in both models. Debt maturity has only a slightly positive effect on the barrier in the BT model, thereby confirming the result in BT that debt maturity does not matter much for the computation of the barrier. However, in the PBT model, debt maturity has a negative effect on the barrier.

Asset volatility lowers the barrier in both models. The result makes sense, because a higher volatility leads to a lower equity value by reducing the value of the down-and-out call and the present value of the dividends. Thus, the barrier has to be lower to maintain the same level of equity value.

The risk-free rate is positively related to the barrier in both models. The payout rate (while keeping the proportion of the dividend to total payout constant) has a negative effect on the barrier, because it lowers the value of the down-and-out option and the dividend component of the equity in the PBT model. The BT model does not have such a parameter. In the same spirit, increasing the dividend parameter, while keeping the total payout rate constant, leads to a higher value of the dividend component and a higher value of equity. This shifts the equity function upwards, leading to higher implied barrier solutions.

The lower barrier in the PBT model leads to a lower risk-neutral probability of default, despite the fact that the asset drift is smaller due to the presence of the payout. The probability of default is 0.5113, i.e., 18.34% lower than the corresponding probability of 0.6261 in the BT model. This result is important, considering that the original BT model

Table 2: Numerical comparisons between BT and PBT for a positive barrier case

Book value of total liabilities			
	$X = 40.5$	$X = 45$	$X = 49.5$
BT	63.6271	66.9052	70.0607
PBT	39.4252	46.5330	52.5417
Debt maturity			
	$T = 9$	$T = 10$	$T = 11$
BT	66.5437	66.9052	67.2324
PBT	47.2061	46.5330	45.8451
Asset volatility			
	$\sigma = 0.27$	$\sigma = 0.3$	$\sigma = 0.33$
BT	69.4009	66.9052	64.7367
PBT	48.1744	46.5330	45.2081
Risk-free rate			
	$r = 0.054$	$r = 0.06$	$r = 0.066$
BT	65.3494	66.9052	68.3631
PBT	43.8189	46.5330	49.0019
Payout rate[†]			
	$q = 0.036$	$q = 0.04$	$q = 0.044$
BT	66.9052	66.9052	66.9052
PBT	48.9872	46.5330	43.9193
Dividend rate			
	$\delta = 0.018$	$\delta = 0.02$	$\delta = 0.022$
BT	66.9052	66.9052	66.9052
PBT	43.3838	46.5330	49.1692
Risk-neutral default probability			
BT		0.6261	
PBT		0.5113	

[†]Changes in q are made so that the ratio δ/q remains constant.

Notes: The table shows the implied barriers in the BT and PBT model, when the parameters change by 10% from their base case levels (middle column), provided the barriers are positive. Base case parameters: Firm asset value $V = 100$, asset volatility $\sigma = 0.3$, book value of total liabilities $X = 45$, debt maturity $T = 10$ years, risk-free rate $r = 0.06$, payout rate $q = 0.04$, and dividend rate $\delta = 0.02$.

has been criticized on the high estimated probability of default (Reisz and Perlich, 2007). It is an empirical issue whether the PBT default probability gives more accurate predictions than the BT probability does, although we shall not pursue this issue further.

3.3 Conditions for extreme barriers, and BT average estimates

If the market value of equity is higher than the sum of $DOC(0,q)$ and $PVD(0,q)$, the barrier solution in the PBT model is zero. In contrast, the market value of equity in the BT model cannot be higher than the BSM option value, as shown by Wong and Choi (2009). It is interesting to examine numerically the range of parameters for which the barrier is zero or higher than total liabilities for the base case.

Zero barrier outcomes are likely to be observed when the value of total liabilities, the risk-free rate or the dividend rate are low ($X < 33.70$, $r < 2.59\%$, $\delta < 1.20\%$), and when debt maturity and payout rate are high ($T > 46.75$ years, $q > 4.98\%$). The volatility parameter by itself did not produce zero barriers in the base case, despite increasing it to very high levels.

Above leverage barriers are likely to be observed when the face value of total liabilities, volatility, the risk-free rate or the dividend rate are high ($X > 42.05$, $\sigma > 33.53\%$, $r > 5.65\%$, $\delta > 1.90\%$), and when debt maturity and payout rate are low ($T < 12.21$ years, $q < 4.11\%$).

Although this exercise cannot claim generality, in the empirical section of the paper we expect to find firms with barriers below firm's liabilities, not as aberrations as noted by Wong and Choi (2009) referring to the BT model, but as consistent outcomes.

Extreme allocations of dividends and coupons in the total payout need special mention. As δ approaches zero and more asset payout is distributed to creditors, we are likely to observe smaller and even zero implied barriers. The converse is true for the coupon rate, that is, as c approaches zero, the barrier is higher and more likely to be above the book value of total liabilities. Both of these results make financial sense, because they relate to the creditors' protection, other things equal.

Out of curiosity, let us also consider the average values presented in Brockman and Turtle (2003, p.p. 520-21, Tables 1-2). The inputs to the BT model are market value of assets $V = 100$, $X = 44.72$, $\sigma = 0.2904$, $T = 10$ years, and $r = 0.0581$. If we consider these as inputs of a hypothetical average firm, we find an implied barrier of 66.9781, which is higher than X. This is not far from the average implied barrier reported in BT, which is 69.2. In contrast, assuming $q = 0.06$ and $\delta = 0.03$ (roughly close to historical averages), the PBT model would yield a significantly lower barrier of 25.2734, which is below X.

4 Data and model justification

Our sample includes companies traded on the NYSE, AMEX, and NASDAQ exchanges from the industries in the BT paper (two-digit SIC codes 20-59) and banks (two-digit SIC code 60). The data cover the period 2013-2017 for a total of 5795 firm-years from the industrial sectors and 1366 firm-years from the banking sector, and include firms whose long-term debt was rated by the largest three rating agencies, as well as firms without such credit rating. The source of data was the Thomson-Reuters Eikon database.

The original BT sample did not include banks. However, banking is a perfect application area for the barrier option model. Unlike non-financial corporations where lenders might be diverse and with different objectives in case of default, in banking the Federal Deposit Insurance Corporation (FDIC) acts on behalf of the depositors according to the depositor preference law. Thus, in effect, the FDIC and other regulators have a direct influence on the default barrier of depository institutions.

To specify the parameters of the model, the market value of each firm, V, is found as the market value of equity V_E plus the book value of total liabilities, X. The latter also proxies for the amount due to creditors at time T. Following BT, the expiration horizon is set to 10 years for industrial firms. For banks, the time to expiration is set to 3 years. Asset volatility, σ, is measured by the standard deviation of the firm asset returns for 20 to 40 preceding quarters (depending on data availability) times the square root of 4.[7] The payout rate, q, is computed as the sum of the total cash dividends and cash interest paid divided by the market value of the firm. The dividend rate, δ, is the total cash dividends as a fraction of the firm's asset value.

Because of the broader importance of the above parameters to structural models, a further discussion of their use is necessary. Early papers testing the contingent claims models, such as Jones et al. (1984) have used the market value of traded debt plus the book value of non-traded debt to estimate the total value of the firm's debt. Other papers have taken the book value of all liabilities, considering that the firm can default not only by missing an interest payment, but also by being unable to meet other short-term obligations. Examples include Jung et al. (1996), Eom et al. (2004), Ericsson and Reneby (2005), and Schaefer and Strebulaev (2008).

We follow the literature and use the BT formulation, although we must highlight the

[7] The following iterative procedure is used. First, we check whether in the 40 preceding quarters all values are reported. If this holds, we calculate the asset volatility. Otherwise, we check the 39 preceding quarters and check again for data availability. This algorithm is repeated until we have zero missing values.

benefits and limitations of such a choice. Firms may issue short-term and long-term debt, and at any given time there are several debt instruments with different coupons, face values, maturities, and other characteristics such as the type and size of coupons, seniority, convertibility, callability, embedded options, specific covenants, etc. Accountants are very meticulous in recording the correct value of a debt issue and in factoring-in the time value of money, as debt is gradually paid off. In general, however, the market value of a debt instrument issued in the past will deviate from its book value due to changes in the interest rates and changes in the probability of default, among other reasons. Sweeney et al. (1997) vividly show how the two values of debt, i.e., book and market, differed significantly in the late 1970s and early 1980s, at a time when interest rates rose to unusually high levels. Nevertheless, when focusing on firms with a steady credit rating, it is reasonable to estimate the market value of debt using its book value. Besides, interest rates tend to be mean-reverting (Wu and Zhang, 1996), and can be above or below the coupon rate of some part of the firm's debt. Thus, the total liabilities proxy is satisfactory as a first guess.[8]

Regarding the time to the option expiration, T, an average duration of the firm's outstanding debt would likely be more appropriate. Eberhart (2005) uses the weighted average of the maturity of long-term debt and its current liabilities as an estimate, while Brockman and Turtle (2003) assume $T = 10$ years, as an approximation of a firm's operating life. Although in principle one can compute a weighted average duration of a firm's bonds, we set $T = 10$ years for industrial firms to make our results directly comparable to the BT model. We realize however, that the maturity of debt is a result of an optimizing process characterizing a particular firm (Jun and Jen, 2003; Nengjiu and Hui, 2006). Stohs and Mauer (1996) report that 95% of firms have a debt maturity of less than 10 years in their sample. Therefore, very high values of T, as done in BT's robustness tests, should be treated with caution.

For the maturity of bank liabilities, we consider a lower time to expiration due to the different composition of the bank funding base. Banks are primarily financed with deposits and short-term wholesale funds, and secondarily with contingent liabilities and subordinated bonds. Earlier structural models in banking that focus on the deposit insurance pricing decision of the FDIC, such as Ronn and Verma (1986) and Merton (1977), treat T as the time until next audit of the bank's assets by the regulator, and set it to 1 year. Other researchers, such as Nagel and Purnanandam (2020), assume that bank liabilities have a

[8] Another alternative is to directly estimate each individual component of the firm's liability structure (Bowman, 1980). However, this approach requires detailed information of all the liability components.

higher maturity of 5 years. Imerman (2020) builds on the compound option structural model of Geske (1977) for analysing the capital adequacy of the US banks in the aftermath of the subprime crisis, and assumes that bank's liabilities consist of short-term and long-term debt that matures in 1 year and 20 years, respectively. Here, we treat the bank liabilities as homogenous, and therefore, we set $T = 3$.

Asset payout was not included in the original BT model. The payout can be found either by using dividend yields and coupon rates and adjusting for leverage (Eom et al., 2004; Huang and Huang, 2012), or directly summing up payouts to shareholders and creditors and dividing by the market value of the firm (Forte and Lovreta, 2012). We opt for the latter approach.

Following a common practice in the literature, we use the book value of total liabilities to proxy also for the face value of debt owed to creditors, i.e., the exercise price of the option. In contrast, Eberhart (2005) estimates the exercise price in the Merton (1974) model by compounding the book value of debt to the estimated maturity of the firm's debt at the coupon rate of the firm's bonds. To an extent, the coupon rate could play the role of the interest payout rate in our model, in terms of reducing the value of the DOC option and leading to lower barriers. However, the approach of Eberhart (2005) creates additional challenges, because coupon rates may not be applicable to some types of debt (e.g., zero-coupon bonds), the data for that rate may not be available (e.g., for bank loans), or the stated rate may obscure important information for valuation (such as in convertible or callable bonds).

A popular, short-cut alternative is found in the KMV model (Crosbie and Bohn, 2002), where the exercise price is set to the sum of the current liabilities and half of the book value of long-run debt of the firm. Although this assumption is practical if the aim is to predict default in the immediate future, it relies heavily on a proprietary database for calibration (Charitou et al., 2013). In applying the KMV model to the BT model, Reisz and Perlich (2007) use the barrier option delta in one of the two equations needed for the KMV model to compute V and σ and force H to be lower than the book value of debt. Their model, however, did not include asset payouts.

Finally, we need to specify the asset risk premium π. According to Reisz and Perlich (2007), the estimates of the asset risk premium are very unstable and dependent heavily on the stock performance in the period examined. Thus, they consider two different approaches for calibrating the asset risk premium. In the first one, they assume a constant asset premium of $\pi = 4\%$, influenced by the choice of Leland (2002). In the second approach, they express

Table 3: Descriptive sample statistics for the pooled time-series cross section of firms

	Mean	Median	Std. Dev.	Min.	Max.
Panel A: Industrial firms					
Market value of assets ($M)	12,416.09	1,570.39	42,271.25	0.8566	809,308.60
Leverage (%)	30.9211	27.3862	20.8451	0.0081	98.8100
Asset volatility	0.3603	0.2980	0.2297	0.0409	2.3625
Dividend rate	0.0082	0.0000	0.0184	0.0000	0.5527
Interest rate	0.0064	0.0030	0.0118	0.0000	0.5883
Payout rate	0.0146	0.0107	0.0217	0.0000	0.5883
Risk-free rate	0.0071	0.0063	0.0056	0.0013	0.0165
Panel B: Banking firms					
Market value of assets ($M)	36,004.30	2,521.33	228,395.10	36.8842	2,675,027.50
Leverage (%)	85.3252	86.0654	6.5895	24.9716	99.2909
Asset volatility	0.1070	0.0851	0.0757	0.0288	0.7348
Dividend rate	0.0030	0.0027	0.0036	0.0000	0.0903
Interest rate	0.0040	0.0036	0.0037	0.0000	0.0570
Payout rate	0.0070	0.0065	0.0056	0.0000	0.1202
Risk-free rate	0.0071	0.0063	0.0056	0.0013	0.0165

Notes: The table shows sample descriptive statistics from 5795 firm-years of industrial firms (2-digit SIC codes from 20 to 59) and 1366 firm-years of banking firms (2-digit SIC code 60). The market value of the firm is measured in million dollars, and is computed as the sum of the market value of equity and book value of total liabilities. Leverage is defined as the book value of total liabilities divided by the firm asset value. Asset volatility, σ, is measured by the standard deviation of the firm asset returns for 20 to 40 preceding quarters (depending on data availability) times the square root of 4. The dividend rate and interest rate, denoted with δ and c, are defined as dividends and interest paid as a proportion of firm value, respectively. The total payout rate q is measured as the ratio of total dividends plus interest paid to the value of the firm ($q = c + \delta$). The risk-free rate r is proxied by the yield on the one-year Treasury bill.

the asset premium as a function of the market price of risk $(r + \pi - q)/\sigma_A$, which is fixed at 0.15 to roughly match the estimates of Huang and Huang (2012). We opt for the second approach, and set $\pi = 0.15\sigma_A$.

Returning to our data, Panel A of Table 3 presents the summary statistics for 5795 firm-year observations, for industrial companies. The average market value of the firm is about 12.42 billion dollars with average volatility of 36.03%. The average firm leverage is 30.92%. The average dividends paid, interest paid, and payout are 0.82%, 0.64% and 1.46% of firm value, respectively, while the average risk-free rate is 0.71%. The sample firm value ranges from about 0.86 million dollars to 809 billion dollars, although the median is 1.57 billion dollars. Leverage ranges from 0.01% to 98.81% with the median being 27.39%. Asset volatility ranges from 4.09% to 236.25%, while the median volatility is 29.80%. Dividends, interest and total payout range from 0% to 55.27%, 0% to 58.83% and 0% to 58.83% of firm value, respectively, while their median values are 0.00%, 0.30% and 1.07%, respectively. The risk-free rate ranges from 0.13% to 1.65%, with a median of 0.63%.

Banks exhibit quite different characteristics, as expected. Panel B of Table 3 presents summary statistics for 1366 firm-years. The average bank market value is about 36 billion dollars with an average volatility of 10.70%. The average leverage is 85.33%, while the average dividend, interest and payout is 0.30%, 0.40%, and 0.70% of firm value, respectively. The bank asset value ranges from about 36.88 million dollars to 2.68 trillion dollars, although the median is 2.52 billion dollars. Leverage ranges from 24.97% to 99.29% with the median being 86.06%. Bank asset volatility ranges from 2.88% to 73.48%, while the median volatility is 8.51%. Bank dividends, interest and total payout range from 0% to 9.03%, 0% to 5.70% and 0% to 12.02% of firm value, respectively, while their median values are 0.27%, 0.36% and 0.65%, respectively.

5 Empirical results

5.1 Implied barrier estimates

The pooled sample results for 5795 industrial firm-years is shown in Panel A of Table 4. To be able to compare results, the barriers are expressed as proportions of firm value. The average barrier is 17.56% of firm value, with a standard deviation of 20.59% and a t-statistic of 64.93, which is statistically significant at the 1% level. Thus, the barrier is statistically different from zero. We also observe that the average barrier is much smaller than the average leverage of 30.92%. The result agrees with the BT conclusion that the barrier is economically and statistically significant. However, the BT estimates are above the debt by construction as shown in Wong and Choi (2009).

In Panels B and C of Table 4, the same results are confirmed, namely, that the average barrier is different from zero and statistically significant at the 1% level in all years, and all industry groups. Panel D of Table 4 shows that the average barrier is lower than leverage.

Table 4: Industrial firms: Barriers by year, industry, leverage and credit rating status

	Obs	Mean	PZ	PAL	Std. Dev.	t-stat
Panel A: Pooled sample						
	5795	17.5585	0.3695	0.3517	20.5851	64.9324
Panel B: By year						
2013	1172	11.3160	0.4744	0.2679	16.3635	23.6745
2014	1073	13.0996	0.4334	0.2740	17.4850	24.5409
2015	1117	19.1885	0.3527	0.3500	21.8413	29.3622
2016	1188	18.1877	0.3737	0.3418	20.9444	29.9307
2017	1245	25.2151	0.2265	0.5084	22.2776	39.9372
Panel C: By industry						
1. Food and beverages (20)	192	13.7608	0.5156	0.3073	19.2871	9.8862
2. Textile and apparel (22, 23)	70	19.0640	0.2857	0.4000	17.9413	8.8901
3. Paper products (26)	65	12.1640	0.6923	0.1077	19.7896	4.9556
4. Chemical (28)	977	14.6672	0.2712	0.4442	17.7340	25.8515
5. Petroleum (29)	61	25.4034	0.2131	0.2131	19.0606	10.4093
6. Stone, clay, glass (32)	33	10.6985	0.6061	0.1818	18.6830	3.2895
7. Primary metals (33)	116	21.1278	0.3448	0.2241	21.2123	10.7274
8. Fabricated metals (34)	158	15.6615	0.4873	0.3228	21.3410	9.2246
9. Machinery (35)	429	18.6065	0.3287	0.3800	19.3424	19.9243
10. Appliances, electrical equipment (36)	699	17.6806	0.2546	0.4535	18.7965	24.8690
11. Transportation equipment (37)	235	19.6089	0.3447	0.3702	21.1796	14.1928
12. Miscellaneous manufacturing (38, 39)	738	14.5471	0.3035	0.4051	16.5138	23.9308
13. Railroads (40)	16	21.4115	0.5625	0.2500	25.5209	3.3559
14. Other transportation (42, 44, 45, 47)	219	22.3692	0.3881	0.2968	22.6322	14.6267
15. Utilities (49)	370	14.1859	0.6919	0.1162	24.6609	11.0649
16. Department stores (53)	46	17.4091	0.6304	0.2826	26.7992	4.4059
17. Other retail trade (50-52, 54-59)	817	21.7875	0.3831	0.3317	22.5465	27.6211
18. Miscellaneous (21, 24, 25, 27, 30, 31, 46, 48)	554	19.9464	0.4440	0.2744	23.7622	19.7576
Panel D: By leverage						
Leverage ≤ 0.1	1007	6.3884	0.1569	0.7080	5.4779	37.0078
$0.1 <$ Leverage ≤ 0.2	1134	10.4747	0.3651	0.4550	9.3845	37.5872
$0.2 <$ Leverage ≤ 0.3	1031	14.2628	0.4394	0.3220	14.0294	32.6433
$0.3 <$ Leverage ≤ 0.4	828	19.0353	0.4372	0.2717	17.7979	30.7755
$0.4 <$ Leverage ≤ 0.5	672	22.6232	0.4241	0.1771	20.9414	28.0049
$0.5 <$ Leverage ≤ 0.6	528	23.8809	0.5000	0.1155	24.7934	22.1326
$0.6 <$ Leverage ≤ 0.7	303	31.4076	0.4323	0.1056	28.7481	19.0172
$0.7 <$ Leverage ≤ 0.8	175	46.3000	0.2914	0.1486	31.1879	19.6388
$0.8 <$ Leverage ≤ 0.9	74	60.1137	0.2162	0.1351	32.9253	15.7058
$0.9 <$ Leverage ≤ 1	43	71.9691	0.1628	0.0930	33.2556	14.1911
Panel E: By credit rating						
Rated	2112	15.3868	0.6056	0.1776	22.4769	31.4601
Non-rated	3683	18.8039	0.2340	0.4515	19.3106	59.0952

Notes: The barrier mean is computed based on the sample of firm-year observations of industrial firms by solving numerically the equity function in our model, PBT. Equity value is the sum of the DOC option value on the firm assets and the present value of dividends. PZ and PAL denote the proportion of zero barriers and above-leverage barriers found. Std. Dev. is the standard deviation of the barrier in each category. The t-statistic tests the hypothesis that the barrier is zero and is statistically significant at the 1% level. Two-digit SIC groups are reported in parentheses. Leverage is defined as the book value of total liabilities divided by the firm asset value. Credit rating is the long-term debt credit rating by the Standard & Poors, Moody's and Fitch rating agencies.

Table 5: Banking firms: Barriers by year and credit rating status

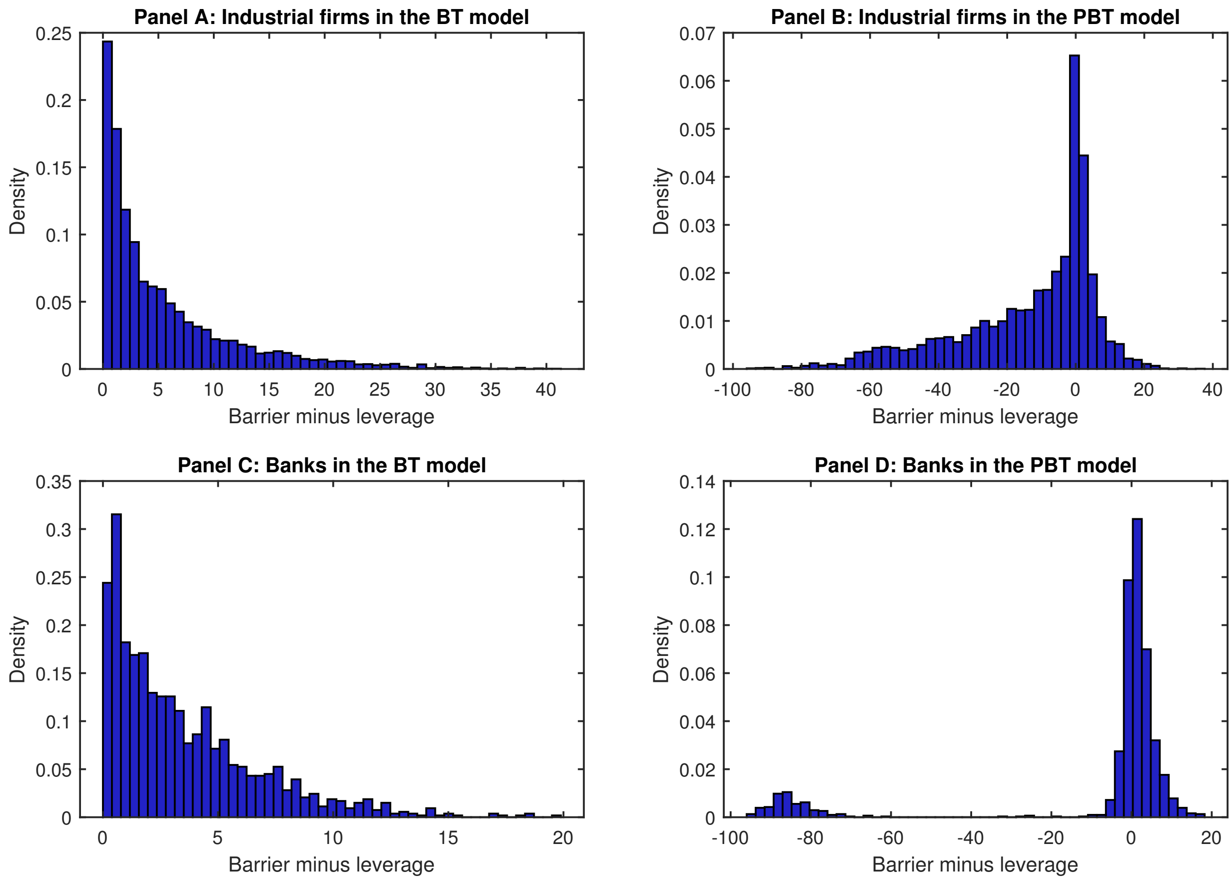
	Obs	Mean	PZ	PAL	Std. Dev.	t-stat
Panel A: Pooled sample						
	1366	77.2393	0.1054	0.6259	28.5187	100.1000
Panel B: By year						
2013	269	60.3843	0.2788	0.1970	39.9414	24.7957
2014	255	67.0628	0.1922	0.1922	36.3261	29.4804
2015	270	83.8405	0.0444	0.7815	19.4556	70.8093
2016	278	84.0955	0.0288	0.8993	15.4927	90.5038
2017	294	88.9422	0.0000	0.9932	6.1878	246.4574
Panel C: By credit rating						
Rated	550	81.9408	0.0527	0.8436	20.5771	93.3892
Non-rated	816	74.0704	0.1409	0.4792	32.4343	65.2357

Notes: The barrier mean is computed based on the sample of firm-year observations of banking firms by solving numerically the equity function in our model, PBT. Equity value is the sum of the DOC option value on the firm assets and the present value of dividends. PZ and PAL denote the proportion of zero barriers and above-leverage barriers found. Std. Dev. is the standard deviation of the barrier in each category. The t-statistic tests the hypothesis that the barrier is zero and is statistically significant at the 1% level. Credit rating is the long-term debt credit rating by the Standard & Poors, Moody's and Fitch rating agencies.

In the pooled sample, the proportion of zero barrier outcomes (PZ) is 36.95% and the proportion of barriers above leverage (PAL) is 35.17%. The proportion of below-leverage barriers is 64.83%, which is close to the proportion of 55.70% found by Wong and Choi (2009). PZ seems to decrease from year to year in this specific sample, while PAL increases (Panel B). When it comes to specific industry groups in Panel C, PZ is negatively correlated with the average barrier in a particular group, and PAL positively correlated (correlation coefficient -0.5312 and 0.1424, respectively). Finally, at the low- and high-ends of the firms' leverage distribution, PZ tends to be smaller, than it is for the middle of the distribution. Surprisingly, PAL declines as leverage increases.

The pooled sample results of 1366 firm-year observations for banks, is shown in Panel A of Table 5. The average barrier is 77.24% of firm value, with a standard deviation of 28.51% and a t-statistic of 100.10, which is statistically significant at the 1% level. Thus, the barrier for banks is statistically different from zero. Moreover, the average barrier is smaller than the average leverage of 85.33%. In contrast to industrial firms, the proportion of zero barriers is very small on average (10.54%) and declines over the five years examined (Panel B), while

Figure 2: Barrier-minus-leverage histograms: Industrial and banking firms



Notes: The figures show the difference between the implied default barriers and the corresponding leverage in the BT and PBT models. The top two graphs show the histogram of the computed barriers for 5795 firm-year observations in industrial firms. The bottom two graphs show the histogram of the computed barriers for 1366 firm-year observations in banks.

the proportion of barriers above leverage is 62.59% and increases dramatically over the years. The presence of a high portion of below-leverage barriers in the first years of our sample is consistent with the lax regulatory policy that the FDIC followed, in the aftermath of the subprime crisis (Lai and Ye, 2021). Thus, the PBT model accounts not only for stringent regulatory policies with positive net worth covenants, but also for regulatory policies with capital forbearance.[9]

[9]Capital forbearance has been explicitly modelled in the structural model literature (Ronn and Verma, 1986; Allen and Saunders, 1993; Duan and Yu, 1999; Lai and Ye, 2021).

Figure 2 compares the histogram distributions of the difference between the estimated barrier and leverage in industrial firms and banks (top section and lower section of Fig. 2, respectively). It is clear that in the BT model, the differences are always positive, while in the PBT model are mostly negative for the industrial firms. Nevertheless, the BT and PBT barriers are positively related, and actually have a correlation coefficient of 0.4699.

Regarding banks, the barrier-minus-leverage histogram retains its basic characteristics in the BT model, but the picture is quite different in the PBT model, in which the histogram is concentrated around zero with some exceptions. The bank barriers in the BT and PBT model are also positively correlated, with a correlation coefficient of 0.2922.

5.2 The relationship between barriers, default probabilities, and credit ratings

Dionne and Laajimi (2012) have examined the determinants of the barrier computed via the maximum likelihood method, such as asset volatility and other firm-specific factors, although they have not looked at credit ratings. However, credit ratings are a key factor for any barrier model, because they directly connect the implied decisions of the creditors to the market valuation of a firm's ability to repay its debt. A borrower's credit rating is related to the barrier in obvious and subtle ways, and it is not clear which variables of the model are affected. For instance, we expect a better credit rating to be related to lower asset volatility which increases the barrier. However, a better credit rating may be associated with higher equity values (implying lower leverage) and a higher payout rate, thus leading to lower barriers. The combined effect of such channels of influence on the default barrier cannot be ascertained a priori for every firm, and remains an empirical issue. Nonetheless, we expect to find that lower credit ratings are associated with a higher default probability.

The Thomson-Reuters Eikon database provides ratings of the long-term debt of issuing firms, as calculated by various independent agencies. In any firm-year, we consider only the rating by any of the top three agencies, namely, Standard and Poors, Moody's and Fitch, and we classify those firm-years as rated. When there is no rating by the top three agencies or no rating at all, we consider the firm as non-rated. Regarding industrial firms, as shown in Panel E of Table 4, the average barrier for rated firms (2112 firm-year observations) is 15.39% of firm value, with a standard deviation of 22.48% and a t-statistic of 31.46, which is statistically significant at the 1% level. In contrast, the average barrier for non-rated firms (3683 firm-year observations) is 18.80% with a standard deviation of 19.31% and a t-statistic of 59.10, significant at the 1% level. Thus, we see that the implied barrier is

Table 6: Industrial firms: Default probabilities by year and credit rating status

	Obs	BT model	PBT model
Panel A: Pooled sample			
	5795	30.44%	21.71%
Panel B: By year			
2013	1172	25.36%	16.83%
2014	1073	26.61%	18.86%
2015	1117	32.57%	23.85%
2016	1188	32.61%	22.83%
2017	1245	34.54%	25.78%
Panel C: By credit rating			
Rated	2112	24.96%	11.87%
Non-rated	3683	33.58%	27.35%
Panel D: By rating grade			
Investment-grade	1379	17.22%	7.43%
Speculative-grade	733	39.51%	20.22%

Notes: The table shows the average 10-year cumulative physical default probabilities based on the sample of firm-year observations of industrial firms. BT model and PBT model, denote the average default probability in each model, respectively. Credit rating is the long-term debt credit rating by the Standard & Poors, Moody's and Fitch rating agencies. Firms are classified as investment grade if their letter rating is BBB or higher.

higher for non-rated firms. The result agrees with common sense, because it means that the creditors impose a higher default barrier to protect their claim, as a precautionary measure against the absence of credit information from the three rating agencies.

Regarding banks, as shown in Panel C of Table 5, the average barrier for rated banks (550 observations) is 81.94% of firm value and is statistically different from zero (t-statistic 93.39). The average barrier for non-rated banks is 74.07% of firm value and is also statistically different from zero (t-statistic 65.24).[10] The proportion of zero barriers is higher for rated industrial firms than it is for non-rated firms (60.56% compared to 23.40%), while the corresponding proportion of above-leverage barriers is lower for rated than for non-rated

[10]This result seems counter-intuitive, since we expect non-rated banks to be assigned a higher implied default barrier. When focusing on a specific year, we find that the default barrier is on average lower for rated-banks in all years except 2014. This discrepancy between the results of Panel C and the year-specific estimates for banks, is attributed to the presence of a low number of rated banks in 2013 and 2014.

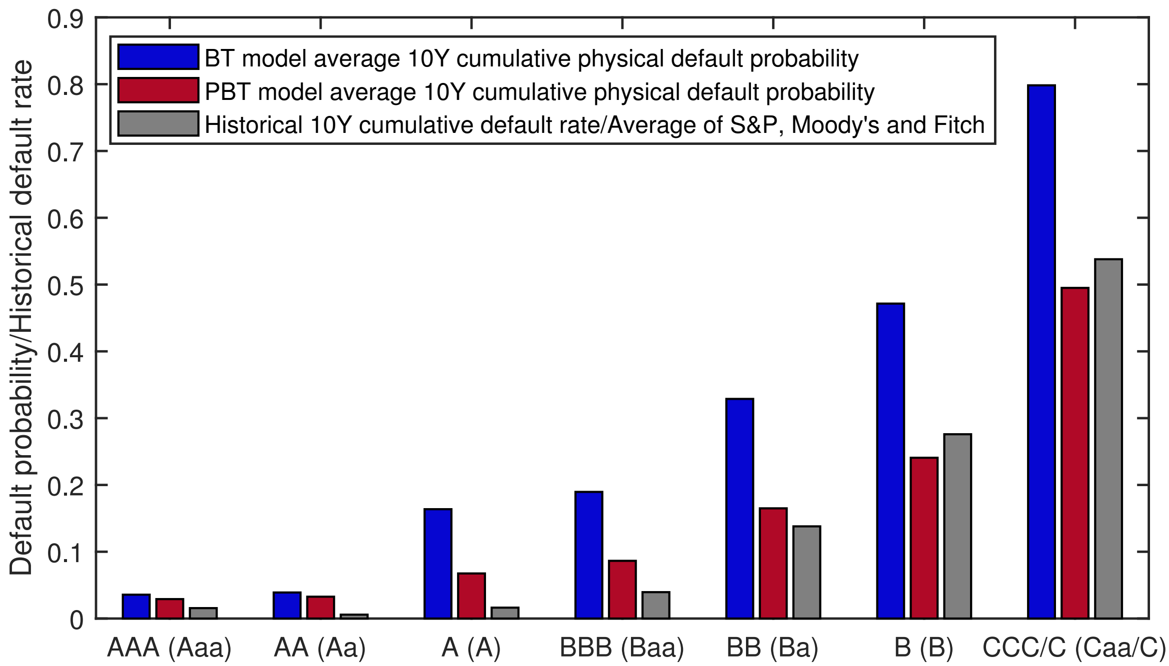
firms (17.76% compared to 45.15%). In banks, the proportion of zero barriers is 5.27% for rated banks and 14.09% for non-rated banks, and the proportion of above-leverage barriers is 84.36% for rated banks and 47.92% for non-rated banks.

Next, we turn to the relationship of the firm credit rating status with the corresponding probability of default for industrial firms.[11] Table 6 provides the average physical cumulative default probabilities across different years and rating profiles. Panel A shows that the average default probability in the PBT model is 21.71%, which is significantly lower than the average BT probability of 30.44%. As expected, the lower implied default barriers are mapped to lower default probabilities. In addition, the default probabilities have a positive trend and increase from year to year in the specific sample (Panel B). Regarding the firm's credit rating, as shown in Panel C, the default probability is higher when the firm is classified as non-rated. Specifically, in the PBT model, the average default probability is 27.35% for non-rated firms, and 11.87% when firms are rated. Similar results are drawn from Panel D, which displays the default probabilities for different rating grades. In our model, the average default probability of speculative-bond issuers is 20.22%, and is higher than the corresponding default probability of investment-bond issuers, which equals to 7.43%. The PBT model successfully ranks firms according to their default risk, when we proxy the firm creditworthiness with the presence of a credit rating, or the assignment of an investment/speculative rating-grade by the three agencies.

We complete our analysis by focusing on the letter rating of each firm. Given the fact that the BT model overestimates the firm default risk (Reisz and Perlich, 2007), we compare the implied default probabilities with the average historical cumulative default rates. The latter are retrieved from each rating agency (Moody's Investors Service, 2018; Standard & Poors Ratings Direct, 2019; Fitch Ratings, 2020) and refer to the historical corporate default rates in the US or North America. From Fig. 3, we observe that in both the BT and PBT models, the default probabilities monotonically increase with the letter rating. In the PBT model, the cross-section of the default probabilities is much closer to the historical default rates, particularly for speculative bond issuers whose exposure to default risk is higher. On average, the presence of payouts resolves the BT shortcoming of retrieving default probabilities. This result is further justified by Fig. 4, which provides the term-structure of the cumulative default probabilities. The PBT model provides almost a perfect fit to the term-structure of speculative-grade default rates, both form short-term and long-term tenors.

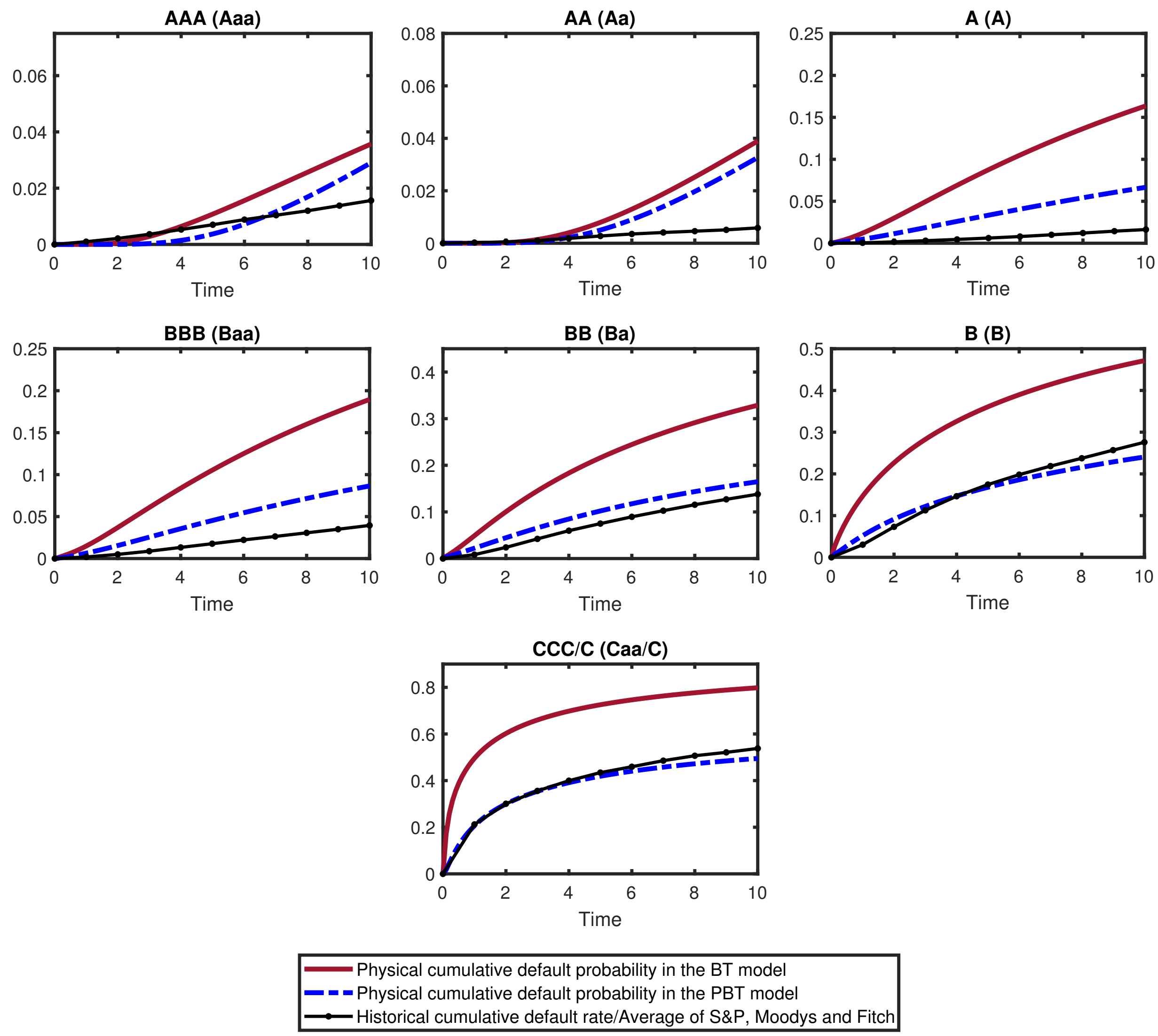
[11]We do not examine the relationship between the default probability and the credit rating of banking firms because the firm-year observations of bank ratings are clustered around A.

Figure 3: Model-implied default probabilities and historical default rates



Notes: The figure shows the average 10-year physical default probabilities across the different rating categories, both for the BT and PBT models. The asset risk premium is set at $\pi = 0.15\sigma_A$. In addition, we provide the 10-year historical default rates at each rating class, defined as the average of (i) the Standard and Poor's average cumulative default rates for corporates in US (1981-2018) (ii) the Fitch North American corporate finance average 10 year cumulative default rates (1990-2019) and (iii) the Moody's average issuer-weighted 10 year cumulative default rates by broad rating categories in North America (1983-2018). The default rates were retrieved from Standard & Poors Ratings Direct (2019), Fitch Ratings (2020) and Moody's Investors Service (2018), respectively. The horizontal axis displays the corresponding rating class from Standard and Poor's and Fitch, and in parenthesis the equivalent Moody's rating grade.

Figure 4: The term structure of default probabilities



Notes: The figure shows the term structure of the cumulative physical default probabilities across different rating categories, both for the BT and PBT models. The asset risk premium is set at $\pi = 0.15\sigma_A$. In addition, we provide the cumulative historical default rates at each letter rating, defined as the average of (i) the Standard and Poor's average cumulative default rates for corporates in US (1981-2018) (ii) the Fitch North American corporate finance average cumulative default rates (1990-2019) and (iii) the Moody's average issuer-weighted cumulative default rates by broad rating categories in North America (1983-2018). The default rates were retrieved from Standard & Poors Ratings Direct (2019), Fitch Ratings (2020) and Moody's Investors Service (2018), respectively. The term-structure of default rates is derived by linearly interpolating the defaults rates of each year i with $i = 1, 2, ..., 10$. The title of each subplot displays the corresponding letter rating from Standard and Poor's and Fitch, and in parenthesis the equivalent Moody's rating grade.

6 Concluding comments

The paper extends the barrier option model of corporate security valuation (Brockman and Turtle, 2003) by including asset payouts which are fully distributed to shareholders and creditors. The addition of payouts leads to lower equity values than in the original BT model, yielding implied default barriers that can be positive or zero, and above or below the firm's leverage. In contrast, the original BT model led to strictly positive and above-leverage barriers.

Numerical examples show that the implied barrier in the new model is positively related to the book value of total liabilities, the risk-free rate and the dividend rate, and negatively related to asset volatility and debt maturity.

Actual data estimates from a large sample of industrial and banking US firms show that the PBT barriers are statistically significant. This result is very clear and holds for a variety of firms, years, industry groups, and firm characteristics such as indebtedness and credit ratings. Compared to BT, the proposed barrier model yields empirical estimates of default probabilities that are closer to the cross-section of the historical default rates, and, therefore, it is relevant to the real world of credit.

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Appendix A: DOC formula and default probability functions

DOC formula: Using the same notation as in BT for ease of comparison, the value of the $DOC(H, q)$ is expressed by the following equation (Clewlow et al., 1994)

$$DOC(H,q) = Ve^{-qT}\mathcal{N}(\eta_1) - Xe^{-rT}\mathcal{N}(\eta_1 - \sigma\sqrt{T}) \tag{10}$$

$$- Ve^{-qT}\left(\frac{H}{V}\right)^{2\eta}\mathcal{N}(\eta_2) + Xe^{-rT}\left(\frac{H}{V}\right)^{2\eta-2}\mathcal{N}(\eta_2 - \sigma\sqrt{T}) \tag{11}$$

where $\mathcal{N}(\cdot)$ denotes the cumulative standard normal distribution function. The parameters η_1, η_2 and η are:

$$\eta_1 = \begin{cases} \dfrac{ln(V/X) + (r - q + 0.5\sigma^2)T}{\sigma\sqrt{T}}, & X \geq H \\ \dfrac{ln(V/H) + (r - q + 0.5\sigma^2)T}{\sigma\sqrt{T}}, & X < H \end{cases} \tag{12}$$

$$\eta_2 = \begin{cases} \dfrac{ln(H^2/VX) + (r - q + 0.5\sigma^2)T}{\sigma\sqrt{T}}, & X \geq H \\ \dfrac{ln(H/V) + (r - q + 0.5\sigma^2)T}{\sigma\sqrt{T}}, & X < H \end{cases} \tag{13}$$

$$\eta = \frac{r-q}{\sigma^2} + \frac{1}{2} \tag{14}$$

Risk-neutral default probability: From BT, the risk-neutral default probability is given by:

$$\begin{aligned} \mathbb{Q}\left(\inf_{t\in[0,T]} V_t \leq H \middle| V = V_0\right) &= \mathcal{N}\left(\frac{ln\,(H/V) - (r - q - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &+ exp\left(\frac{2(r - q - 0.5\sigma^2)ln(H/V)}{\sigma^2}\right) \\ &\times \left[1 - \mathcal{N}\left(\frac{-ln\,(H/V) - (r - q - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right)\right] \end{aligned} \tag{15}$$

The survival probability is calculated as $1 - \mathbb{Q}\left(\inf_{t\in[0,T]} V_t \leq H \middle| V = V_0\right)$

Physical default probability: The default probability under the physical measure $\mathbb{P}$ is given by:

$$
\begin{aligned}
\mathbb{P}\left(\underset{t\in[0,T]}{inf} V_t \le H \middle| V = V_0\right) &= \mathcal{N}\left(\frac{ln\,(H/V) - (r+\pi-q-0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&+ exp\left(\frac{2(r+\pi-q-0.5\sigma^2)ln(H/V)}{\sigma^2}\right) \\
&\times\left[1-\mathcal{N}\left(\frac{-ln\,(H/V)-(r+\pi-q-0.5\sigma^2)T}{\sigma\sqrt{T}}\right)\right]
\end{aligned} \tag{16}
$$

The survival probability is calculated as $1 - \mathbb{P}\Big(\underset{t\in[0,T]}{inf} V_t \le H \Big| V = V_0\Big)$

Appendix B: Valuation of the present value of dividends

We approximate the claim $PVD(H,q)$ as the limit of a finite multi-level annuity with bankruptcy risk, based on the results of Mjøs and Persson (2010) and Mjøs et al. (2013). Let us denote with $H_i, i \in \{1, 2, ..., n+m\}$ a sequence of non-absorbing barriers that are greater than the default barrier H. Given the initial asset value V_0, we have n non-absorbing barriers that are greater than V_0, and m that are below V_0. For $i = 0$, we use the convention that $H_0 = +\infty$. Between two consecutive barriers, we treat the instantaneous dividends as constant. Thus for each region (H_{i+1}, H_i) the dividend is equal to c_{i+1}. In analogy to the midpoint rule used for the approximation of the Riemann integral, we fix c_{i+1} at the average dividend of the interval endpoints, with $c_{i+1} = \delta(H_i + H_{i+1})/2$. By letting n and m get very large, we achieve a good approximation of $PVD(H,q)$, in which the dividend stream is stochastic and state-dependent. Mathematically, we have

$$
PVD(H,q) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{min\{\tau,T\}} \delta V_t e^{-rt} dt\right] \tag{17}
$$

$$
\approx \mathbb{E}^{\mathbb{Q}}\left[\int_0^{min\{\tau,T\}} \sum_{i=0}^{n+m} c_{i+1} e^{-rt} \mathbf{1}_{H_{i+1}<V_t<H_i} dt\right] \quad \text{for large } n, m \tag{18}
$$

If we denote with $\overline{PVD}(H,q)$ the approximate present value of dividends described by Eq. 18, then from Mjøs et al. (2013) we can derive the following solution

$$
\overline{PVD}(H,q) = c_{n+m+1} Z^c(V) - \sum_{i=1}^{n} \mathcal{A}_b^c(V, H_i)(c_{i+1} - c_i) - \sum_{i=n+1}^{n+m} \mathcal{A}_a^c(V, H_i)(c_{i+1} - c_i) \tag{19}
$$

where $Z^c(V)$ denotes the present value of a claim that pays one dollar contingent on the firm being solvent. The terms $\mathcal{A}^c_b(V, H_i)$ and $\mathcal{A}^c_a(V, H_i)$ denote the value of an above annuity that pays 1 dollar contingent on the asset value exceeding the barrier H_i, with $V_0 < H_i$ and $V_0 > H_i$, respectively. The functional form of $Z^c(V)$, $\mathcal{A}^c_b(V, H_i)$ and $\mathcal{A}^c_a(V, H_i)$ is derived analytically. First we define,

$$\alpha = \frac{\left(\frac{1}{2}\sigma^2 - \mu^{\mathbb{Q}} + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu^{\mathbb{Q}}\right)^2 + 2\sigma^2 r}\right)}{\sigma^2} \quad \text{and} \quad \beta = \frac{\left(\mu^{\mathbb{Q}} - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu^{\mathbb{Q}}\right)^2 + 2\sigma^2 r}\right)}{\sigma^2} \tag{20}$$

where $\mu^{\mathbb{Q}}$ denotes the risk-neutral drift of the asset process, with $\mu^{\mathbb{Q}} = r - q$. The approximation $\overline{PVD}(H, q)$ is expressed via some risk-neutral probability functions. These functions are also given by Mjøs and Persson (2010), who provide the theoretical framework for pricing level-dependent annuities. Along with the risk-neutral measure $\mathbb{Q}$, they introduce two supplementary probability measures $\mathbb{Q}^\alpha$ and $\mathbb{Q}^\beta$, which are equivalent to $\mathbb{Q}$. These two measures are also required for the valuation of the level-dependent claims. Thus, we have

1. **Under the $\mathbb{Q}$-measure:** The survival probability of the firm is

$$\mathbb{Q}(V) = \mathbb{Q}(\tau > T) = \mathcal{N}(d_1) - \left(\frac{V}{H}\right)^{\alpha-\beta} \mathcal{N}(-d_2) \tag{21}$$

with

$$d_1 = \frac{ln\left(\frac{V}{H}\right) + (\mu^{\mathbb{Q}} - 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{ln\left(\frac{V}{H}\right) - (\mu^{\mathbb{Q}} - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{22}$$

In addition, we have the probability of the terminal asset value exceeding the non-absorbing barrier H_i, conditional on the firm being solvent, that is

$$\mathbb{Q}_{gg}(H_i) = \mathbb{Q}(V_T > H_i, \tau > T) = \mathcal{N}(d_3) - \left(\frac{V}{H}\right)^{\alpha-\beta} \mathcal{N}(-d_4) \tag{23}$$

where

$$d_3 = \frac{ln\left(\frac{V}{H_i}\right) + (\mu^{\mathbb{Q}} - 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_4 = \frac{ln\left(\frac{VH_i}{H^2}\right) - (\mu^{\mathbb{Q}} - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{24}$$

2. **Under the $\mathbb{Q}^{\beta}$-measure:** The survival probability of the firm is

$$\mathbb{Q}_g^{\beta}(V) = \mathbb{Q}^{\beta}(\tau > T) = \mathcal{N}(d_1^{\beta}) - \left(\frac{V}{H}\right)^{\alpha+\beta} \mathcal{N}(-d_2^{\beta}) \tag{25}$$

with

$$d_1^{\beta} = \frac{ln\left(\frac{V}{H}\right) + (\mu^{\mathbb{Q}} - \sigma^2\beta - 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2^{\beta} = \frac{ln\left(\frac{V}{H}\right) - (\mu^{\mathbb{Q}} - \sigma^2\beta - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{26}$$

The probability of default is $\mathbb{Q}_l^{\beta}(V) = \mathbb{Q}^{\beta}(\tau < T) = 1 - \mathbb{Q}_g^{\beta}(V)$. In addition, we have the probability of the terminal asset value exceeding the non-absorbing barrier H_i, conditional on the firm being solvent, that is

$$\mathbb{Q}_{gg}^{\beta}(H_i) = \mathbb{Q}^{\beta}(V_T > H_i, \tau > T) = \mathcal{N}(d_3^{\beta}) - \left(\frac{V}{H}\right)^{\alpha+\beta} \mathcal{N}(-d_4^{\beta}) \tag{27}$$

where

$$d_3^{\beta} = \frac{ln\left(\frac{V}{H_i}\right) + (\mu^{\mathbb{Q}} - \sigma^2\beta - 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_4^{\beta} = \frac{ln\left(\frac{VH_i}{H^2}\right) - (\mu^{\mathbb{Q}} - \sigma^2\beta - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{28}$$

3. **Under the $\mathbb{Q}^{\alpha}$-measure:** The probability of the terminal asset value not exceeding the non-absorbing barrier, conditional on the firm being solvent is

$$\begin{aligned} \mathbb{Q}_{lg}^{\alpha}(H_i) &= \mathbb{Q}^{\alpha}(V_T < H_i, \tau > T) \\ &= \mathcal{N}(d_1^{\alpha}) - \mathcal{N}(d_3^{\alpha}) + \left(\frac{V}{H}\right)^{-(\alpha+\beta)} [\mathcal{N}(-d_4^{\alpha}) - \mathcal{N}(-d_2^{\alpha})] \end{aligned} \tag{29}$$

where

$$d_1^{\alpha} = \frac{ln\left(\frac{V}{H}\right) + (\mu^{\mathbb{Q}} + \sigma^2\alpha - 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2^{\alpha} = \frac{ln\left(\frac{V}{H}\right) - (\mu^{\mathbb{Q}} + \sigma^2\alpha - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{30}$$

$$d_3^\alpha = \frac{ln\left(\frac{V}{H_i}\right) + (\mu^{\mathbb{Q}} + \sigma^2\alpha - 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_4^\alpha = \frac{ln\left(\frac{VH_i}{H^2}\right) - (\mu^{\mathbb{Q}} + \sigma^2\alpha - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{31}$$

Given these probability functions, we can value the components of $\overline{PVD}(H, q)$.

1. The claim $Z^c(V)$ is

$$Z^c(V) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{min(\tau,T)} e^{-rs}ds\right] = \frac{1}{r}\left[1 - e^{-rT}\mathbb{Q}(V) - \mathbb{Q}_l^\beta(V)\left(\frac{V}{H}\right)^{-\beta}\right] \tag{32}$$

2. The above annuity $\mathcal{A}_a^c(V, H_i)$ is given by

$$\mathcal{A}_a^c(V, H_i) = \frac{\gamma_\alpha(V, H_i)}{r} \tag{33}$$

where

$$\begin{aligned}\gamma_\alpha(V, H_i) = 1 &- x\left(\frac{V}{H}\right)^{-\beta}\left[1 - \mathbb{Q}_{gg}^\beta(H_i)\right] \\ &- y\left[\left(\frac{V}{H_i}\right)^\alpha \mathbb{Q}_{lg}^\alpha(H_i) + \left(\frac{H}{H_i}\right)^\alpha\left(\frac{V}{H}\right)^{-\beta}\mathbb{Q}_l^\beta(V)\right] - e^{-rT}\mathbb{Q}_{gg}(H_i)\end{aligned} \tag{34}$$

with

$$x = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad y = \frac{\beta}{\alpha+\beta} \tag{35}$$

3. The above annuity $\mathcal{A}_b^c(V, H_i)$ is given by

$$\mathcal{A}_b^c(V, H_i) = \frac{\gamma_\beta(V, H_i)}{r} \tag{36}$$

where

$$\begin{aligned}\gamma_\beta(V, H_i) = x&\left(\frac{V}{H_i}\right)^{-\beta}\mathbb{Q}_{gg}^\beta(H_i) \\ &+ y\left[\left(\frac{V}{H_i}\right)^\alpha\left(1 - \mathbb{Q}_{lg}^\alpha(H_i)\right) - \left(\frac{H}{H_i}\right)^\alpha\left(\frac{V}{H}\right)^{-\beta}\mathbb{Q}_l^\beta(V)\right] \\ &- e^{-rT}\mathbb{Q}_{gg}(H_i)\end{aligned} \tag{37}$$

Appendix C: Benchmarking the approximation with a Monte Carlo valuation

In this section we benchmark the approximation $\overline{PVD}$, that uses the theory of multi-level annuities, with a Monte Carlo valuation, that is based on the simulation of discrete trajectories of the geometric Brownian motion. We generate N different random paths of the asset process $\{V_t : t \in (0,T]\}$, with a time step of Δt. Thus, under the exact solution of the SDE described in Eq. 2, we have:

$$V_{t+\Delta t} = V_t \; exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_t\right\} \quad \text{with} \;\; V_0 = 100 \tag{38}$$

where $\epsilon_t \sim \mathcal{N}(0,1)$. The PBT model is concerned with path-dependent contingent claims, so we have to retain the whole path. The present value of dividends under the Monte-Carlo valuation equals

$$PVD^{MC} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=0}^{min\{\tau_i,T\}} e^{-rt}\delta V_t(\omega_i)dt \tag{39}$$

where ω_i denotes the realization of the $i-th$ path and τ_i the early default time of the $i-th$ asset path. The monitoring points of the process are discrete, thus, the probability of default is going to be downward biased. Between two consecutive non-defaulting observations $V_{t-\Delta t}$ and V_t, there is a positive probability π_t that the firm has defaulted, or equivalently, that the DOC option has been knocked-out, in the region $(t-\Delta t, t)$. Therefore, we condition for the so-called *survivorship bias*, using the Brownian-Bridge technique (Gobet, 2009). At each monitoring point $t \in \{0, \Delta t, 2\Delta t, ...T\}$ we calculate the probability π_t as

$$\pi_t = \mathbb{Q}\left[\inf_{s\in(t-\Delta t,t)} V_s \leq H | V_{t-\Delta t}, V_t > H\right] = exp\left(\frac{-2ln(V_{t-\Delta t}/H)ln(V_t/H)}{\sigma^2\Delta t}\right) \tag{40}$$

and compare it with a random number $z_t \sim U(0,1)$. If $\pi_t > z_t$, then we assume that firm default has been declared in $t-0.5\Delta t$. In our Monte-Carlo simulation exercise, the early default time τ_i is defined as

$$\tau_i = \inf\{\mathcal{V}_i \cup \mathcal{P}_i\} \tag{41}$$

where

$$\mathcal{V}_i = \{t \in \{0, \Delta t, 2\Delta t, ..., T\} : V_t(\omega_i) \leq H\} \tag{42}$$

Table 7: Valuation of PVD with multi-level annuities versus Monte-Carlo

	Multi-level annuity with $H^{max} = 3000$	Multi-level annuity with $H^{max} = 5000$	Multi-level annuity with $H^{max} = 10000$	Monte-Carlo pricing with $N = 1 \times 10^6$ and $dt = \frac{1}{252}$
Panel A: $\sigma = 0.30$				
$H = 30$	16.0292	16.0297	16.0298	**16.0150**
$H = 50$	14.1604	14.1610	14.1610	**14.1835**
$H = 70$	10.2667	10.2672	10.2673	**10.2660**
Panel B: $\sigma = 0.40$				
$H = 30$	15.3960	15.4115	15.4167	**15.4271**
$H = 50$	12.8584	12.8738	12.8789	**12.8965**
$H = 70$	8.7930	8.8068	8.8115	**8.8337**
Panel C: $\sigma = 0.50$				
$H = 30$	14.6582	14.7396	14.7843	**14.7772**
$H = 50$	11.7662	11.8442	11.8874	**11.9138**
$H = 70$	7.7521	7.8160	7.8522	**7.8701**

and

$$\mathcal{P}_i = \{t \in \{0, 0.5\Delta t, \Delta t, ..., T - 0.5\Delta t\} : \pi_t(\omega_i) > z_t(\omega_i)\} \quad (43)$$

For the comparison of the two approaches, we consider the base case parameters. We set $V = 100$, $X = 45$, $T = 10$, $r = 0.06$, $q = 0.04$, $\delta = 0.02$ and $\Delta V = 10$. We let the asset volatility range between very high values, because we want to know whether our approximation of PVD is prone to large deviations of the asset process. In addition, we consider different values of the default barriers. Thus, we set $\sigma \in \{0.30, 0.40, 0.50\}$ and $H \in \{30, 50, 70\}$. Regarding H^{max}, we try various values, since it is an arbitrary upper bound that needs to be calibrated. This bound is critical, because dividends are treated as constant at the region $(H^{max}, +\infty)$.

In Table 7, we provide the present value of dividends, under the two approaches. We observe that when the upper bound H^{max} is low (Panel A), the approximation is downward biased, especially when the asset volatility is high. An increase of H^{max} to 5000 (Panel B), leads to a closer approximation of the true price of PVD, that is the price under the Monte-Carlo valuation. When setting H^{max} at 10000, which is the value used in the empirical analysis of our model, we achieve a very good accuracy, across all values of the asset volatilities and the default barriers.