CONICHE

PROF.
MARCO
COMPAGNONI

Classificazione delle coniche

Esempi

SEZIONE 11.3

Invarianti Metrici (Definizione 11.8)

- $I_i = (-1)^{n-i} c_{n-i} \quad 1 \leq i \leq n$, dove $P_A(\lambda) = \sum_{i=0}^{n} c_i \lambda^i$;
- $I_{n+1} = |C|$.

$n=2$: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \Rightarrow \quad P_A(\lambda) = \underbrace{\overbrace{|A|}^{I_2}}_{c_0} \underbrace{- \overbrace{Tr(A)}^{I_1}}_{c_1} \lambda + \lambda^2$

$n=3$: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \Rightarrow$

$$P_A(\lambda) = \underbrace{\overbrace{|A|}^{I_3}}_{c_0} \underbrace{- \overbrace{\left(\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} \right)}^{I_2}}_{c_1} \lambda \underbrace{+ \overbrace{Tr(A)}^{I_1}}_{c_2} \lambda^2 - \lambda^3$$

OSS: gli invarianti non cambiano a seguito del cambio di coordinate, ma moltiplicando l'equazione di Q per k si ottiene $I_i' = k^i \cdot I_i \quad 1 \leq i \leq n+1$.

CLASSIFICAZIONE CURVE CONICHE (TEOREMA 11.13)

In $\mathbb{E}^2$ l'equazione di ogni curva conica può essere ridotta ad una ed una sola delle seguenti forme canoniche:

$r(C)$	$r(A)$	I_3	I_2	$I_1 I_3$	Forma canonica	Moduli	Nome
3	2	$\neq 0$	> 0	< 0	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} = 1$	$\alpha, \beta > 0$	*Ellisse con punti reali*
3	2	$\neq 0$	> 0	> 0	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} = -1$	$\alpha, \beta > 0$	*Ellisse privo di punti reali*
3	2	$\neq 0$	< 0		$\frac{\tilde{x}^2}{\alpha^2} - \frac{\tilde{y}^2}{\beta^2} = 1$	$\alpha, \beta > 0$	*Iperbole*
3	1	$\neq 0$	$= 0$		$\tilde{x}^2 - 2\rho\tilde{y} = 0$	$\rho > 0$	*Parabola*
2	2	$= 0$	> 0	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} + \tilde{y}^2 = 0$	$\alpha > 0$	*Coppia di rette incidenti con un unico punto reale*
2	2	$= 0$	< 0	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} - \tilde{y}^2 = 0$	$\alpha > 0$	*Coppia di rette incidenti con infiniti punti reali*
2	1	$= 0$	$= 0$	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} = 1$	$\alpha > 0$	*Coppia di rette parallele con infiniti punti reali*
2	1	$= 0$	$= 0$	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} = -1$	$\alpha > 0$	*Coppia di rette parallele prive di punti reali*
1	1	$= 0$	$= 0$	$= 0$	$\tilde{x}^2 = 0$		*Retta doppia*

DIM: studiamo il primo caso, gli altri sono analoghi.

$r(C) = 3$, $r(A) = 2$ $\Rightarrow$ $\mathcal{Q}|\tilde{\mathcal{B}}_0$: $\lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 + \tilde{c} = 0$.

$$\tilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad C = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \tilde{c} \end{bmatrix} \Rightarrow \begin{cases} I_1 = \lambda_1 + \lambda_2 , \\ I_2 = \lambda_1 \lambda_2 , \; I_3 = \lambda_1 \lambda_2 \tilde{c} \neq 0 \end{cases}$$

Supponiamo $I_2 > 0$ $\Rightarrow$ λ_1, λ_2 concordi in segno $\Rightarrow$

$I_1 I_3 = (\lambda_1 + \lambda_2) \lambda_1 \lambda_2 \tilde{c} < 0$ implica che $\tilde{c}$ ha segno discorde da λ_1, λ_2 $\Rightarrow$ $\alpha = \sqrt{-\frac{\tilde{c}}{\lambda_1}}$, $\beta = \sqrt{-\frac{\tilde{c}}{\lambda_2}}$ > 0

$$\Rightarrow \frac{\tilde{x}^2}{-\frac{\tilde{c}}{\lambda_1}} + \frac{\tilde{y}^2}{-\frac{\tilde{c}}{\lambda_2}} = 1 \quad \Rightarrow \quad \frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} = 1$$

▨

Esempio 11.14

$$\mathcal{Q}: \; x^2 + 3y^2 - 2y - 1 = 0$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & -1 \end{bmatrix} \qquad I_1 = 4 \quad I_2 = 3 \quad I_3 = -4$$

$\Rightarrow \mathcal{Q}$ è un'ellisse $\quad (I_2 > 0, \; I_1 I_3 < 0)$

$$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \tilde{c} = \frac{I_3}{I_2} = -\frac{4}{3} \quad \Rightarrow \quad \tilde{x}^2 + 3\tilde{y}^2 - \frac{4}{3} = 0$$

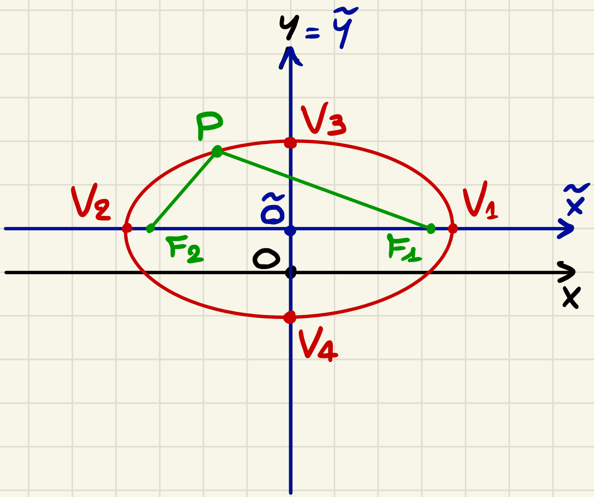
$$\alpha = \sqrt{-\frac{\tilde{c}}{\lambda_1}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \qquad \beta = \sqrt{-\frac{\tilde{c}}{\lambda_2}} = \sqrt{\frac{4}{9}} = \frac{2}{3} \quad \Rightarrow$$

$$\mathcal{Q}|_{\tilde{\mathcal{B}}_0}: \; \frac{\tilde{x}^2}{\left(\frac{2}{\sqrt{3}}\right)^2} + \frac{\tilde{y}^2}{\left(\frac{2}{3}\right)^2} = 1$$

$$X = Q\tilde{X} + T \qquad Q = I \qquad T: [A|-B] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 3 & 1 \end{array}\right] \quad \Rightarrow \quad T = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$$

$$\begin{cases} x = \tilde{x} \\ y = \tilde{y} + 1/3 \end{cases} \qquad \frac{\tilde{x}^2}{(2/\sqrt{3})^2} + \frac{\tilde{y}^2}{(2/3)^2} = 1$$

Studiamo le proprietà geometriche dell'ellisse.



- Vertici:

$$V_1|_{\tilde{B}_0} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ 0 \end{bmatrix}, \qquad V_2|_{\tilde{B}_0} = \begin{bmatrix} -\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{3} \\ 0 \end{bmatrix}$$

$$V_3|_{\tilde{B}_0} = \begin{bmatrix} 0 \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}, \qquad V_4|_{\tilde{B}_0} = \begin{bmatrix} 0 \\ -\beta \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \end{bmatrix}$$

- Fuochi: $\varphi = \sqrt{\alpha^2 - \beta^2} = \sqrt{11}/3 \Rightarrow$

$$F_1|_{\tilde{B}_0} = \begin{bmatrix} \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{11}/3 \\ 0 \end{bmatrix}, \qquad F_2|_{\tilde{B}_0} = \begin{bmatrix} -\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{11}/3 \\ 0 \end{bmatrix}$$

$$d(P,F_1) + d(P,F_2) = 2\alpha = 4/\sqrt{3}$$

- $\tilde{O}$ è centro di simmetria: $\tilde{O}|_{B_0} = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} = T$.
- $\tilde{x}, \tilde{y}$ sono assi di simmetria:

$$r_1|_{B_0}: \; P|_{B_0} = \tilde{O}|_{B_0} + t\, v_1|_{B} = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$r_2|_{B_0}: \; P|_{B_0} = \tilde{O}|_{B_0} + t\, v_2|_{B} = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Autovettori di A

ESEMPIO 11.15

$$\mathcal{Q}: \quad 3x^2-3y^2-8xy+2x+4y-2=0$$

$$A=\begin{bmatrix}3 & -4\\ -4 & -3\end{bmatrix}, \quad B=\begin{bmatrix}1\\ 2\end{bmatrix}, \quad C=\begin{bmatrix}3 & -4 & 1\\ -4 & -3 & 2\\ 1 & 2 & -2\end{bmatrix}$$

$I_1=0$ $\quad I_2=-25$

$I_3=25$

Iperbole equilatera

$\Rightarrow \mathcal{Q}$ è un'iperbole $(I_3 \neq 0,\ I_2<0)$

$$P_A(\lambda)=(3-\lambda)(-3-\lambda)-16=\lambda^2-25=(\lambda-5)(\lambda+5) \Rightarrow$$

$$\lambda_1=5, \quad \lambda_2=-5, \quad \tilde{c}=\frac{I_3}{I_2}=-1 \Rightarrow \mathcal{Q}|\tilde{B}_0: \quad 5\tilde{x}^2-5\tilde{y}^2-1=0$$

$$\alpha=\sqrt{-\frac{\tilde{c}}{\lambda_1}}=\sqrt{\frac{1}{5}}, \quad \beta=\sqrt{\frac{\tilde{c}}{\lambda_2}}=\sqrt{\frac{1}{5}} \Rightarrow$$

$$\mathcal{Q}|\tilde{B}_0: \quad \frac{\tilde{x}^2}{\alpha^2}-\frac{\tilde{y}^2}{\beta^2}=1 \Rightarrow \frac{\tilde{x}^2}{\left(\sqrt{\frac{1}{5}}\right)^2}-\frac{\tilde{y}^2}{\left(\sqrt{\frac{1}{5}}\right)^2}=1$$

$$V_5 = \operatorname{Ker}(A - 5I_2) = \operatorname{Ker}\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = \operatorname{Ker}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \mathcal{L}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$$

$$V_{-5} = \operatorname{Ker}(A + 5I_2) = \operatorname{Ker}\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = \operatorname{Ker}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \mathcal{L}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$\Rightarrow \quad Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \in SO(2;\mathbb{R})$$

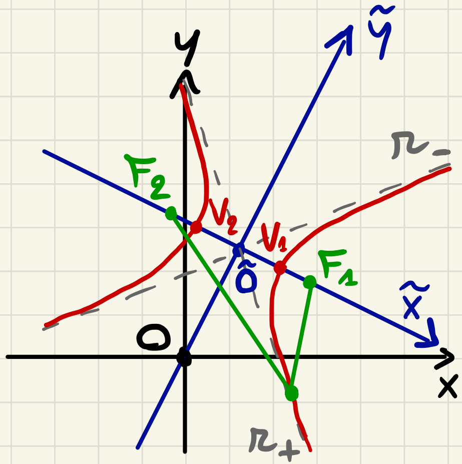
$$T: \quad [A \mid -B] = \left[\begin{array}{cc|c} 3 & -4 & -1 \\ -4 & -3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -4 & -1 \\ 0 & -25 & -10 \end{array}\right] \quad \Rightarrow \quad T = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} \Rightarrow$$

$$\begin{cases} x = 2/\sqrt{5}\,\tilde{x} + 1/\sqrt{5}\,\tilde{y} + 1/5 \\ y = -1/\sqrt{5}\,\tilde{x} + 2/\sqrt{5}\,\tilde{y} + 2/5 \end{cases}$$

Studiamo le proprietà geometriche dell'iperbole.

$$\begin{cases} x = 2/\sqrt{5}\,\tilde{x} + 1/\sqrt{5}\,\tilde{y} + 1/5 \\ y = -1/\sqrt{5}\,\tilde{x} + 2/\sqrt{5}\,\tilde{y} + 2/5 \end{cases}$$

$$\mathcal{I}|_{\tilde{B}_0} : \frac{\tilde{x}^2}{(1/\sqrt{5})^2} - \frac{\tilde{y}^2}{(1/\sqrt{5})^2} = 1$$



- Vertici :

$$V_1|_{\tilde{B}_0} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 0 \end{bmatrix}, \qquad V_2|_{\tilde{B}_0} = \begin{bmatrix} -\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 0 \end{bmatrix}$$

- Fuochi : $\varphi = \sqrt{\alpha^2 + B^2} = \sqrt{2/5} \Rightarrow$

$$F_1|_{\tilde{B}_0} = \begin{bmatrix} \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2/5} \\ 0 \end{bmatrix}, \qquad F_2|_{\tilde{B}_0} = \begin{bmatrix} -\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2/5} \\ 0 \end{bmatrix}$$

$$|d(P, F_1) - d(P, F_2)| = 2\alpha$$

- $\tilde{O}$ è centro di simmetria : $\tilde{O}|_{B_0} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} = T$.

- $\tilde{x}, \hat{y}$ sono assi di simmetria :

$$r_1|_{B_0} : \quad P|_{B_0} = \tilde{O}|_{B_0} + t \cdot v_1|_B = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$r_2|_{B_0} : \quad P|_{B_0} = \tilde{O}|_{B_0} + t \cdot v_2|_B = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Asintoti : $r_\pm|_{\tilde{B}_0} : B\tilde{x} \pm \alpha\tilde{y} = 0 \Rightarrow \tilde{x} \pm \tilde{y} = 0 \Rightarrow$ Ip. equilatera $(\lambda_1 = -\lambda_2)$