1) Determinare la proiezione di $\underline{v}$ su $\underline{w}$ essendo $\underline{v}=\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$ e $\underline{w}=\begin{bmatrix}-1\\2\\0\\2\end{bmatrix}$ e scrivere la matrice che rappresenta P_W rispetto alla base canonica di $\mathbb{R}^4$, essendo $W=\mathcal{L}(\underline{w})$

$$P_W(\underline{v})=\frac{\langle \underline{v},\underline{w}\rangle}{\|w\|^2}\,\underline{w}=-\frac{2}{9}\underline{w}=\frac{1}{9}\begin{bmatrix}2\\-4\\0\\4\end{bmatrix}$$

$$F_{|_E}=[P_W(e_1)\,|\,P_W(e_2)\,|\,P_W(e_3)\,|\,P_W(e_4)]=\frac{1}{9}\begin{bmatrix}1&-2&0&-2\\-2&4&0&4\\0&0&0&0\\-2&4&0&4\end{bmatrix}$$

2) Sia $V=\mathbb{R}^4$, $U=\{(x,y,z,w)\in\mathbb{R}^4 \mid y-z-w=0\}$. Trovare una base ortonormale di U. Verifica se $\underline{v}=\begin{bmatrix}2\\1\\-3\\1\end{bmatrix}\in U$ e calcola $P_U(\underline{v})$.

1) $y-z-w=0 \Rightarrow \begin{cases}x=\alpha\\y=z+w\\z=\beta\\w=\gamma\end{cases} \rightarrow \begin{cases}x=\alpha\\y=\beta+\gamma\\z=\beta\\w=\gamma\end{cases} \rightarrow \alpha\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\\1\\0\end{bmatrix}+\gamma\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \Rightarrow \mathcal{B}_U=\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$. Ortogonalizzo $\mathcal{B}_U$ con

Gram-Schmidt:

$$\left.\begin{aligned}
\underline{v}_1&=\underline{u}_1\\
\underline{v}_2&=\underline{u}_2-P_{\underline{v}_1}(\underline{u}_2)=\underline{u}_2-\frac{\langle \underline{v}_1,\underline{u}_2\rangle}{\|\underline{u}_2\|}\underline{v}_1=\underline{u}_2\\
\underline{v}_3&=\underline{u}_3-P_{\underline{v}_1}(\underline{u}_3)-P_{\underline{v}_2}(\underline{u}_3)=\underline{u}_3-\frac{\langle \underline{v}_1,\underline{u}_3\rangle}{\|\underline{v}_1\|}\underline{v}_1-\frac{\langle \underline{v}_2,\underline{u}_3\rangle}{\|\underline{v}_2\|}\underline{v}_2=\underline{u}_3-\frac{\langle \underline{v}_2,\underline{u}_3\rangle}{\|\underline{v}_2\|}\underline{v}_2=\begin{bmatrix}0\\1/2\\-1/2\\1\end{bmatrix}
\end{aligned}\right\}\mathcal{B}_\perp=\{v_1,v_2,v_3\}\Rightarrow \mathcal{B}_\perp'=\left\{\frac{v_1}{\|v_1\|},\frac{v_2}{\|v_2\|},\frac{v_3}{\|v_3\|}\right\}$$

2) $v\notin U$; $P_U(v)=\frac{\langle v,v_1\rangle}{\|v_1\|}v_1+\frac{\langle v,v_2\rangle}{\|v_2\|}v_2+\frac{\langle v,v_3\rangle}{\|v_3\|}v_3=\begin{bmatrix}2\\0\\-2\\2\end{bmatrix}$

3) Siano $\underline{v}_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$ e $\underline{v}_2=\begin{bmatrix}2\\1\\-3\end{bmatrix}$ e $V=\mathcal{L}(\underline{v}_1,\underline{v}_2)$. Determina il complemento ortogonale di V.

v_1, v_2 sono indipendenti $\Rightarrow \mathcal{B}_V=\{\underline{v}_1,\underline{v}_2\}$. $V^\perp=\{(x,y,z)\in\mathbb{R}_3 \mid \langle v,\underline{v}_1\rangle=0 \wedge \langle v,v_2\rangle=0\} \Rightarrow \begin{cases}x+z=0\\2x+y-3z=0\end{cases} \rightarrow \mathrm{Ker}\left(\overbrace{\begin{bmatrix}1&0&1\\2&1&-3\end{bmatrix}}^{\text{matrice le cui righe sono i vett della base di V}}\right) \rightarrow \begin{cases}x=-\alpha\\y=5\alpha\\z=\alpha\end{cases} \rightarrow \alpha\begin{bmatrix}-1\\5\\1\end{bmatrix} \Rightarrow$

$\Rightarrow V^\perp=\alpha\begin{bmatrix}-1\\5\\1\end{bmatrix}$ (in questo caso si poteva anche usare il duale di Hodge ($\dim(V^\perp)=1$))

4) Sia $W=\mathcal{L}\left(\begin{bmatrix}1\\1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-2\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}\right)$. Determina una base di $W^\perp$.

$\mathrm{Rg}\left(\begin{bmatrix}1&1&1\\1&-2&0\\0&0&-1\\1&0&2\end{bmatrix}\right)=3 \Rightarrow \mathcal{B}_V=\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-2\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}\right\} \Rightarrow \dim(W^\perp)=1$. Usiamo il duale di Hodge: $\begin{vmatrix}1&1&1&e_1\\1&-2&0&e_2\\0&0&-1&e_3\\1&0&2&e_4\end{vmatrix}=\dots=-e_2-2e_1+3e_4+4e_3=\begin{bmatrix}-2\\-1\\4\\3\end{bmatrix} \Rightarrow V^\perp=\mathcal{L}\left(\begin{bmatrix}-2\\-1\\4\\3\end{bmatrix}\right)$

5) Sia V uno s.v con $\dim(V)=25$ e U un suo sottospazio con $\dim(U)=9$. Sia R_U la riflessione ortogonale rispetto a U. R_U è diagonalizzabile?
Calcola $p_{R_U}(\lambda)$, $\det(R_U)$, $\mathrm{Tr}(U)$

1) $V=U\oplus U^\perp \Rightarrow \mathcal{B}_V=\mathcal{B}_U\cup\mathcal{B}_{U^\perp}$. $\mathcal{B}_U=\{\underline{u}_1,\dots,\underline{u}_9\}$, $\mathcal{B}_{U^\perp}=\{\underline{u}_1',\dots,\underline{u}_{16}'\}$. $R_U(u_i)=1\cdot\underline{u}_i \Rightarrow 1$ autovalore ; $R_U(\underline{u}_j)=-1\cdot u_j \Rightarrow -1$ autovalore
$\mathcal{B}_V$ è quindi una base di autovettori di $R_U \Rightarrow R_U$ è diagon.

2) $$\left.\begin{aligned}\lambda_1&=1 \quad m.a.(1)=m.g.(1)=9\\ \lambda_2&=-1 \quad m.a(-1)=m.g.(-1)=16\end{aligned}\right\}\Rightarrow p_{R_U}(\lambda)=(-1)^{25}(\lambda-1)^9(\lambda+1)^{16};\ \det(R_U)=\lambda_1^9\cdot\lambda_2^{16}=1;\ \mathrm{Tr}(R_U)=9-16=-7$$

6) Siano $V=\{(x,y,z,w)\in\mathbb{R}^4 \mid y-z+w=0,\ y-w=0\}$ e $W=\mathcal{L}\left(\begin{bmatrix}1\\1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\2\\0\end{bmatrix}\right)$. Trova basi e dim. di V e W. Prova che $V\oplus W=\mathbb{R}^4$. Trovare una base ortonormale di $V^\perp$. Provare che W e $V^\perp$ sono isomorfi

1) ... $\mathcal{B}_V=\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\2\\1\end{bmatrix}\right\}$, $\mathcal{B}_W=\left\{\begin{bmatrix}1\\0\\1\\-1\end{bmatrix},\begin{bmatrix}0\\1\\2\\0\end{bmatrix}\right\}$ $\dim V=\dim W=2$

2) $V\oplus W=\mathbb{R} \Leftrightarrow \dim V+\dim W=4 \wedge V\cap W=\{0\} \Rightarrow \mathcal{B}_V\cup\mathcal{B}_W$ è base di $\mathbb{R}_4$. [...] $V\oplus W=\mathbb{R}^4$

3) $\dim V^\perp=2 \Rightarrow \begin{cases}\langle v,\underline{v}_1\rangle=0\\\langle v,\underline{v}_2\rangle=0\end{cases} \rightarrow \begin{cases}x=0\\y+2z+w=0\end{cases}$ ... HOMEWORK ($\mathcal{B}_{V^\perp}=\left\{\begin{bmatrix}0\\-2\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}$ da ort. con Gram-Schmidt

4) V e W sono isomorfi poiché $\dim V=\dim W=2$