Angoli tra Vettori (8.5)

Disuguaglianze di Schwarz e Triangolare (Proposizione 8.34)

i) $|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \cdot \|\underline{v}\|$, l'uguaglianza vale sse $\underline{v} = t \cdot \underline{u}$ con $t \in \mathbb{R}$, o $\underline{u} = \underline{0}$;

ii) $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$, l'uguaglianza vale sse $\underline{v} = t \cdot \underline{u}$ con $t \in [0, +\infty)$, o $\underline{u} = \underline{0}$.

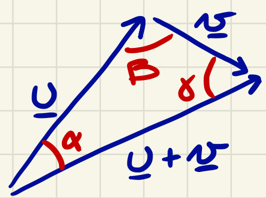
Dim: i) . $\underline{u} = \underline{0}$: è banale verificare l'uguaglianza.

. $\underline{u} \neq \underline{0}$: chiamiamo $U = \mathcal{L}(\underline{u}) \Rightarrow$

$$\|\underline{v}\|^2 = \|\underbrace{P_U(\underline{v})}_{\in U} + \underbrace{(\underline{v} - P_U(\underline{v}))}_{\in U^\perp}\|^2 = \|P_U(\underline{v})\|^2 + \underbrace{\|\underline{v} - P_U(\underline{v})\|^2}_{\geq 0} \geq \|P_U(\underline{v})\|^2 \Rightarrow$$

$$\|\underline{v}\|^2 \geq \left\| \frac{\langle \underline{v}, \underline{u} \rangle}{\|\underline{u}\|^2} \cdot \underline{u} \right\|^2 = \frac{\langle \underline{v}, \underline{u} \rangle^2}{\|\underline{u}\|^4} \cdot \|\underline{u}\|^2 = \frac{\langle \underline{v}, \underline{u} \rangle^2}{\|\underline{u}\|^2} \Rightarrow \|\underline{u}\|^2 \|\underline{v}\|^2 \geq \langle \underline{v}, \underline{u} \rangle^2.$$

Vale l'uguaglianza sse $\underline{v} - P_U(\underline{v}) = \underline{0}$ sse $\underline{v} \in U$ sse $\underline{v} = t \cdot \underline{u}$. ☒



$\|\underline{u} + \underline{v}\| \geq \|\underline{u}\| + \|\underline{v}\|$

$\alpha, \beta, \gamma = ?$

Vogliamo ricavare gli angoli a partire dalle lunghezze

OSS: se $\underline{u}, \underline{v} \neq \underline{0} \Rightarrow \dfrac{|\langle \underline{u}, \underline{v} \rangle|}{\|\underline{u}\| \cdot \|\underline{v}\|} \leq 1 \Rightarrow -1 \leq \dfrac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \cdot \|\underline{v}\|} \leq 1$

ANGOLO TRA DUE VETTORI (DEFINIZIONE 8.35)

$$\underline{u}, \underline{v} \in V \setminus \{\underline{0}\} \Rightarrow \widehat{\underline{u}\,\underline{v}} = \arccos\left(\frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \cdot \|\underline{v}\|}\right).$$

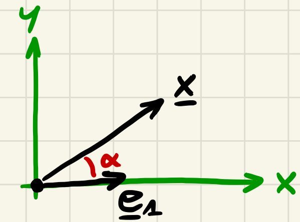
OSSERVAZIONI

- $\widehat{\underline{u}\,\underline{v}}$ è ben definito e appartiene a $[0, \pi]$;
- $\widehat{\underline{u}\,\underline{v}} = 0$ sse $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \cdot \|\underline{v}\|$ sse $\underline{v} = t \cdot \underline{u}$ con $t > 0$;
- $\widehat{\underline{u}\,\underline{v}} = \pi$ sse $\langle \underline{u}, \underline{v} \rangle = -\|\underline{u}\| \cdot \|\underline{v}\|$ sse $\underline{v} = t \cdot \underline{u}$ con $t < 0$;
- $\widehat{\underline{u}\,\underline{v}} = \pi/2$ sse $\langle \underline{u}, \underline{v} \rangle = 0$ sse $\underline{u} \perp \underline{v}$.

Esempi

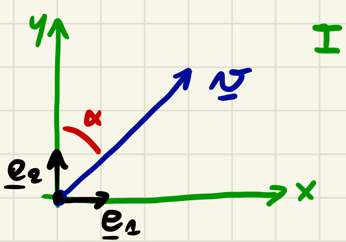
- $V = \mathbb{R}^2 \qquad \underline{e}_1 = (1,0) \qquad \underline{x} = (x,y)$

$$\alpha = \widehat{\underline{e}_1\, \underline{x}}_E = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

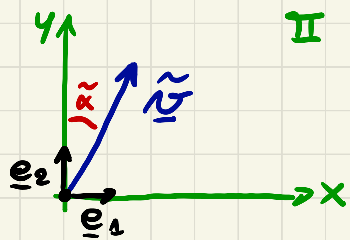


OSS: l'angolo dipende dalla struttura euclidea

Supponiamo di riscalare un'immagine di un fattore 1/2 lungo la direzione x.



$\longrightarrow$



Nella seconda immagine, se si usa $\langle\, ,\rangle_E$ si ottiene un risultato errato

$$\underline{v} = (3,3) \longrightarrow \tilde{\underline{v}} = \left(\tfrac{3}{2}, 3\right) \qquad G_{II} = \begin{bmatrix} \overset{\|e_1\|^2 \ (\|v\| = \sqrt{\langle v,v\rangle})}{\langle \underline{e}_1, \underline{e}_1\rangle_{II}} & \langle \underline{e}_1, \underline{e}_2\rangle_{II} \\ \langle \underline{e}_2, \underline{e}_1\rangle_{II} & \langle \underline{e}_2, \underline{e}_2\rangle_{II} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

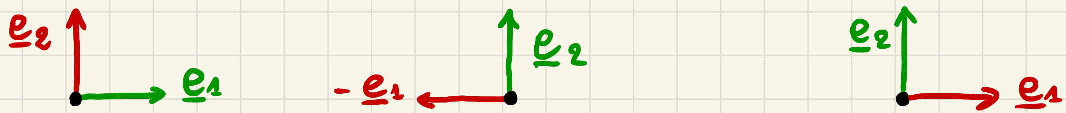
$$\Rightarrow \quad \tilde{\alpha} = \arccos\left(\frac{\langle \tilde{\underline{v}}, \underline{e}_2\rangle_{II}}{\|\tilde{\underline{v}}\|_{II}\ \|\underline{e}_2\|_{II}}\right) = \arccos\left(\frac{3}{\sqrt{9}\cdot 1}\right) = \frac{\pi}{4} = \alpha \qquad \|\underline{v}\| = \sqrt{\underline{v}_{\mathbb{B}}^T \cdot G_{\mathbb{B}} \cdot \underline{v}_{\mathbb{B}}} = \sqrt{\langle v,v\rangle}$$

Spazi Vettoriali Orientati (8.6)

Definizione 8.37

V s.v. su $\mathbb{R}$. $\mathcal{B}$ è orientata come $\mathcal{B}'$ se $|M_{\mathcal{B}\mathcal{B}'}| > 0$.

$V = \mathbb{R}^2$ $\quad \mathcal{B}_2 = \{ \underline{e}_1, \underline{e}_2 \}$ $\quad \mathcal{B}' = \{ \underline{e}_2, -\underline{e}_1 \}$ $\quad \mathcal{B}'' = \{ \underline{e}_2, \underline{e}_1 \}$



$$M_{\mathcal{B}_2\mathcal{B}'} = \left[\underline{e}_1|_{\mathcal{B}'} \;\; \underline{e}_2|_{\mathcal{B}'} \right] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow |M_{\mathcal{B}_2\mathcal{B}'}| = 1 > 0$$

$$M_{\mathcal{B}_2\mathcal{B}''} = \left[\underline{e}_1|_{\mathcal{B}''} \;\; \underline{e}_2|_{\mathcal{B}''} \right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow |M_{\mathcal{B}_2\mathcal{B}''}| = -1 < 0$$

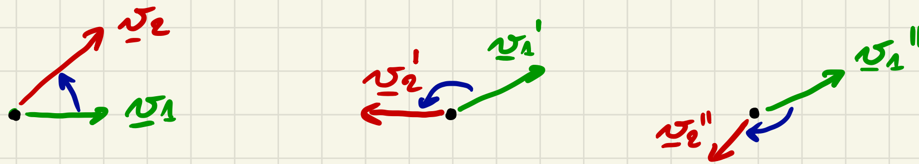
$\mathcal{B}_2$ e $\mathcal{B}'$ sono legate da una riflessione.

$\mathcal{B}_2$ e $\mathcal{B}''$ sono legate da una rotazione.

PROPOSIZIONE 8.39

- L'orientazione è una relazione di equivalenza in $\{ \tilde{B} \text{ basi di } V \}$.
- Esistono solo due classi di orientazione.

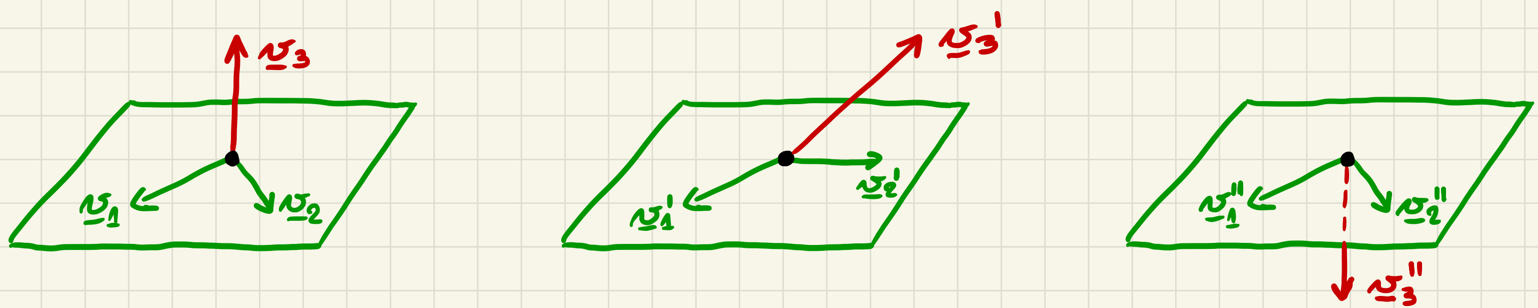
$\mathbb{R}^2$



- $[B] = [B']$
- $[B] \neq [B'']$

$$[B] = \{ \tilde{B} \text{ basi di } V \mid |M_{B\tilde{B}}| > 0 \}$$

$\mathbb{R}^3$



$$[B] = [B'] \neq [B'']$$

DEFINIZIONE 8.40 $\mathcal{B} = \{\text{classi di orientazione di } V\} = \{[\mathcal{B}], [\mathcal{B}']\}$.

i) Un'orientazione su V è una funzione biunivoca $\xi : \mathcal{B} \to \{-1, 1\}$;

ii) $\mathcal{B}$ si dice orientata positivamente se $\xi([\mathcal{B}]) = 1$;

iii) $\mathcal{B}$ si dice orientata negativamente se $\xi([\mathcal{B}]) = -1$;

iv) (V, ξ) si dice spazio vettoriale orientato.

ORIENTAZIONI CANONICHE

- $V = \mathbb{R}^2$ $\quad \mathcal{B}_2 = \{\underline{e}_1, \underline{e}_2\}$ $\quad \mathcal{B}' = \{\underline{e}_2, -\underline{e}_1\}$ $\quad \mathcal{B}'' = \{\underline{e}_2, \underline{e}_1\}$

  $\xi([\mathcal{B}_2]) = 1 = \xi([\mathcal{B}'])$ $\qquad \xi([\mathcal{B}'']) = -1$

  $\xi^{-1}(1) =$ basi antiorarie $\qquad \xi^{-1}(-1) =$ basi orarie

- $V = \mathbb{R}^3$ $\quad \mathcal{B}_3 = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ $\quad \mathcal{B}' = \{\underline{e}_2, \underline{e}_1, \underline{e}_3\}$

  $\xi([\mathcal{B}_3]) = 1$ $\qquad \xi([\mathcal{B}']) = -1$

  $\xi^{-1}(1) =$ terne destrorse $\qquad \xi^{-1}(-1) =$ terne sinistrorse

Duale di Hodge e Prodotto Vettoriale

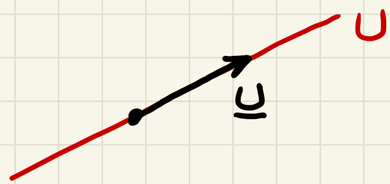
Caratterizzazione dei Vettori in Spazi Euclidei (Proposizione 8.43)

V s.v.e., $\underline{u} \in V$ è determinato da:

i) il modulo $\|\underline{u}\|$;

ii) se $\underline{u} \neq \underline{0}$, la sua direzione $U = \mathcal{L}(\underline{u})$;

iii) se $\underline{u} \neq \underline{0}$, la sua orientazione in U (verso di $\underline{u}$).



LEMMA 8.44

V s.v.e., $U = \{\underline{u}_1, \dots, \underline{u}_m\}$ l.i. $\Rightarrow$ $|G_U| = G(U) > 0$.

DUALE DI HODGE (DEFINIZIONE 8.45)

V s.v.e.o, $\dim(V) = n$ $\quad 2 \leq n < +\infty$, $\quad U = \{\underline{u}_1, \dots, \underline{u}_{n-1}\} \subset V$

$$* : \begin{array}{ccc} V^{n-1} & \longrightarrow & V \\ (\underline{u}_1, \dots, \underline{u}_{n-1}) & & \underline{u} = *(\underline{u}_1, \dots, \underline{u}_{n-1}) \end{array} \qquad \text{dove:}$$

i) $\|\underline{u}\| = \sqrt{G(U)}$;

ii) se $\|\underline{u}\| \neq 0$, la direzione di $\underline{u}$ è $U^\perp$;

iii) se $\|\underline{u}\| \neq 0$, il verso di $\underline{u}$ è tale che $\{\underline{u}_1, \dots, \underline{u}_{n-1}, \underline{u}\}$ è una base positivamente orientata di V.

OSS: i) $G(U) \geq 0 \Rightarrow \|\underline{u}\|$ è ben definita

$U^\perp = \underbrace{V_n}_{\cancel{n}} - \underbrace{V_{n-1}}_{\cancel{n}+1} = 1$

ii) se $\|\underline{u}\| \neq 0 \Rightarrow U$ è l.i. $(G(U) > 0)$ $\Rightarrow \dim(U^\perp) = 1$

iii) se $\|\underline{u}\| \neq 0 \Rightarrow \{\underline{u}_1, \dots, \underline{u}_{n-1}, \underline{u}\}$ è una base di V.

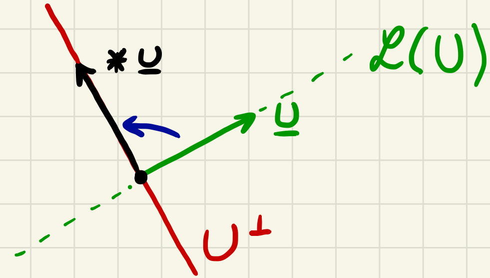
Esempio 8.46

• $n = 2$: $\quad U = \{\underline{u}\} \implies$

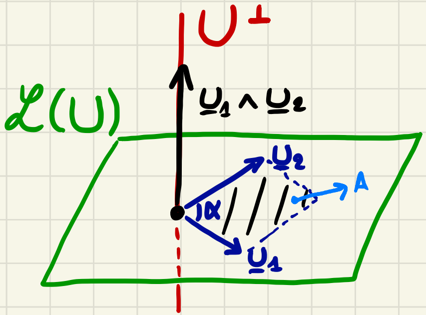
i) $\|*\underline{u}\| = \sqrt{\underbrace{\|\underline{u}\|^2}_{}} = \|\underline{u}\|$ $\quad [\langle \underline{u}, \underline{u}\rangle] = [\|\underline{u}\|^2]$

ii) $*\underline{u} \perp \underline{u}$

iii) $\{\underline{u}, *\underline{u}\}$ è orientata positivamente



• $n = 3$: $\quad U = \{\underline{u}_1, \underline{u}_2\} \implies *(\underline{u}_1, \underline{u}_2) = \underline{u}_1 \wedge \underline{u}_2$ $\quad$ prodotto vettoriale



$$\|\underline{u}_1 \wedge \underline{u}_2\|^2 = \begin{vmatrix} \|\underline{u}_1\|^2 & \langle \underline{u}_1, \underline{u}_2\rangle \\ \langle \underline{u}_1, \underline{u}_2\rangle & \|\underline{u}_2\|^2 \end{vmatrix} = \|\underline{u}_1\|^2 \|\underline{u}_2\|^2 - \langle \underline{u}_1, \underline{u}_2\rangle^2 =$$

$$= \|\underline{u}_1\|^2 \|\underline{u}_2\|^2 - \|\underline{u}_1\|^2 \|\underline{u}_2\|^2 \cos^2\alpha = \|\underline{u}_1\|^2 \|\underline{u}_2\|^2 \operatorname{sen}^2\alpha$$

$$\implies \|\underline{u}_1 \wedge \underline{u}_2\| = \|\underline{u}_1\| \, \|\underline{u}_2\| \operatorname{sen}\alpha = A$$

$$(\alpha \in [0, \pi] \implies \operatorname{sen}\alpha > 0)$$

Problema: come calcolare efficientemente $*(\underline{u}_1, \dots, \underline{u}_{n-1})$?

Matrice e determinante formale: $A = \begin{bmatrix} 1 & x \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \Rightarrow |A| = 1 \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - x \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -x & -1 \\ 1 & -x \end{bmatrix}$

Caratterizzazione di $*$ (Proposizione 8.47)

V s.v.e.o., $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$ base o.n. orientata positivamente $\Rightarrow$

$*(\underline{u}_1, \dots, \underline{u}_{n-1})$ è uguale a $\underline{u} = \begin{vmatrix} u_{11} & \cdots & u_{1\,n-1} & \underline{v}_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \cdots & u_{n\,n-1} & \underline{v}_n \end{vmatrix}$, con $\begin{bmatrix} u_{1i} \\ \vdots \\ u_{ni} \end{bmatrix} = \underline{u}_i|_{\mathcal{B}}$.

Esempi

- $\underline{u}_1 = (1, 0, 1)$, $\underline{u}_2 = (0, 1, 1)$, $\mathcal{B}_3 = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ o.n.

$$*(\underline{u}_1, \underline{u}_2) = \underline{u}_1 \wedge \underline{u}_2 = \begin{vmatrix} 1 & 0 & \underline{e}_1 \\ 0 & 1 & \underline{e}_2 \\ 1 & 1 & \underline{e}_3 \end{vmatrix} = \underline{e}_1(-1) + \underline{e}_2(-1) + \underline{e}_3(1) = (-1, -1, 1) \perp \underline{u}_1, \underline{u}_2$$

- $U = \mathcal{L}(\underline{u}_1 = (1, 0, -1, 0), \underline{u}_2 = (0, 1, 0, -1), \underline{u}_3 = (0, 0, 1, 0)) \Rightarrow$

$$\underline{u} = *(\underline{u}_1, \underline{u}_2, \underline{u}_3) = \begin{vmatrix} 1 & 0 & 0 & \underline{e}_1 \\ 0 & 1 & 0 & \underline{e}_2 \\ -1 & 0 & \boxed{1} & \underline{e}_3 \\ 0 & -1 & 0 & \underline{e}_4 \end{vmatrix} = \underline{e}_1 \cdot 0 + \underline{e}_2 \cdot 1 + \underline{e}_4 \cdot 1 = (0, 1, 0, 1)$$

$\dim(U) = 3 \quad \dim(U^\perp) = 4 - 3 = 1 \Rightarrow U^\perp = \mathcal{L}((0, 1, 0, 1))$.

PROPRIETÀ (OSSERVAZIONE 8.49 - PROPOSIZIONE 8.51 - 8.52)

- $|\langle \underline{v}, *(\underline{v}_1, \dots, \underline{v}_{n-1}) \rangle| = \sqrt{G(\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{v})}$ prodotto misto
- $(t_1 \underline{v}_1 + t_2 \underline{v}_2) \wedge \underline{v}_3 = t_1 (\underline{v}_1 \wedge \underline{v}_3) + t_2 (\underline{v}_2 \wedge \underline{v}_3)$
  $\underline{v}_1 \wedge (t_2 \underline{v}_2 + t_3 \underline{v}_3) = t_2 (\underline{v}_1 \wedge \underline{v}_2) + t_3 (\underline{v}_1 \wedge \underline{v}_3)$ bilinearità
- $\underline{v}_1 \wedge \underline{v}_2 = - \underline{v}_2 \wedge \underline{v}_1$ antisimmetria
- $\langle \underline{v}_1, \underline{v}_2 \wedge \underline{v}_3 \rangle = \langle \underline{v}_2, \underline{v}_3 \wedge \underline{v}_1 \rangle = \langle \underline{v}_3, \underline{v}_1 \wedge \underline{v}_2 \rangle$ ciclicità
- $\underline{v}_1 \wedge (\underline{v}_2 \wedge \underline{v}_3) = \langle \underline{v}_1, \underline{v}_3 \rangle \cdot \underline{v}_2 - \langle \underline{v}_1, \underline{v}_2 \rangle \cdot \underline{v}_3$ prodotto triplo
- $\underline{v}_1 \wedge (\underline{v}_2 \wedge \underline{v}_3) + \underline{v}_2 \wedge (\underline{v}_3 \wedge \underline{v}_1) + \underline{v}_3 \wedge (\underline{v}_1 \wedge \underline{v}_2) = \underline{0}$ identità Jacobi

OSS: $\underline{v}_1 \wedge (\underline{v}_2 \wedge \underline{v}_3) - (\underline{v}_1 \wedge \underline{v}_2) \wedge \underline{v}_3 = - \underline{v}_2 \wedge (\underline{v}_3 \wedge \underline{v}_1) \neq \underline{0}$ in generale

$\Rightarrow$ $\wedge$ non è associativo!