SVILUPPI NOTI

$$e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+n) = x - \frac{x^2}{2} + \dots + (-1)^{n+1}\frac{x^n}{n!} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(n+1)!} + o(x^{2n+2})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(n^{2n+1})$$

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1) $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4) \Rightarrow$ Serie geometrica

$f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2}$ $\quad f'(0) = 1$

$f''(x) = \frac{2}{(1-x)^3}$ $\quad f''(0) = 2$

$f'''(x) = \frac{6}{(1-x)^4}$ $\quad f'''(0) = 6$

$f''''(x) = \frac{24}{(1-x)^5}$ $\quad f''''(0) = 24$

2) $f(x) = (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + o(x^3) \Rightarrow$ assomiglio al binomiale $\Rightarrow \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$

$\hookrightarrow f(x) = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \dots + \binom{\alpha}{n}x^n \Rightarrow$ Generalizzazione dello sviluppo di Newton

$f(0) = 1$

$f'(x) = \alpha(1+x)^{\alpha-1}$ $\quad f'(0) = \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$ $\quad f''(0) = \alpha(\alpha-1)$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$ $\quad f'''(0) = \alpha(\alpha-1)(\alpha-2)$

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1) $f(x) = e^{\sin x} =$ $\qquad T_4^0(x) = ?$

Sappiamo che per $t \to 0$ $\quad e^t = 1 + t + \frac{t^2}{2} + \dots + o(t^n)$. $\Rightarrow f(x) = 1 + \sin x + \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{3!} + \frac{(\sin x)^4}{4!} + o((\sin x)^4)$. Adesso mettiamo nelle parentesi lo sviluppo di $\sin x$ a un ordine almeno pari di quello di e^t:

$$\sin x = x - \frac{x^3}{3!} + o(x^4) \Rightarrow e^{\sin x} = 1 + x - \frac{x^3}{3!} + o(x^4) + \frac{1}{2}\underbrace{\left(x - \frac{x^3}{3!} + o(x^4)\right)^2}_{\frac{1}{2}x^2 + \frac{x^6}{3!}\} o(x^4) + \dots} + \frac{1}{3!}\underbrace{\left(x - \frac{x^3}{3!} + o(x^4)\right)^3}_{\frac{1}{3!}x^3 - \frac{x^9}{3!}\} o(x^4) + \dots} + \frac{1}{4!}\left(x - \frac{x^3}{3!} + o(x^4)\right)^4 + o\Big(\underbrace{\left(x - \frac{x^3}{3!}\right)^4}_{\sin x \sim x \Rightarrow o(x^4)}\Big)$$

$$= 1 + x - \cancel{\frac{x^3}{3!}} + \frac{1}{2}x^2 - \frac{x^4}{3!} + o(x^4) + \cancel{\frac{1}{3!}x^3} + \frac{1}{4!}x^4 + o(x^4) + o(x^4) = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)$$

2) $f(x) = (1-2x^2)e^{-x^2}$ $\qquad f^{(8)}(0) = ?$

$$(1-2x^2)\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + o(x^8)\right) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + o(x^8) - 2x^2 + 2x^4 - x^6 + \frac{x^8}{3} - \frac{x^{10}}{12} - 2x^2 o(x^8) = 1 - 3x^2 + \frac{5}{2}x^4 - \frac{7}{6}x^6 + \frac{3}{8}x^8 + o(x^8)$$

MOLTIPLICO

$\hookrightarrow \frac{3}{8} = \frac{f^{(8)}(0)}{8!} \Rightarrow f^{(8)} = \frac{3}{8} \cdot 8!$

3) $f(x) = x^3 + x^2 - x - 1$ $\qquad T_2^0(x), T_4^0(x), T_6^0(x)$

$T_2^0(x) = -1 - x + x^2 \rightarrow x^3 + [x^2 - x - 1]$

$T_4^0(x) = f(x) = -1 - x + x^2 + x^3 = T_6^0(x)$

4) $f(x) = x^3 + x^2 - x - 1$ $\qquad T_1^1(x) = ?$

$f(x) = (x-1)(x+1)^2$ $\qquad f(x) \overset{x\to1}{\sim} 4(x-1) \Rightarrow f(x) = 4x - 4 + o(x-1) = T_1^1(x)$

5) $f(x) = x^3 + x^2 - x - 1$ $\qquad T_2^{-1}(x) = ?$

$f(x) = (x-1)(x+1)^2$ $\qquad f(x) \overset{x\to-1}{\sim} -2(x+1)^2 \Rightarrow f(x) = -2(x+1)^2 + o((x+1)^2)$

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1) $$\lim_{x\to0} \frac{\ln(1+3x) - 3\sin(x+x^2)}{x^2} = \lim_{x\to0} \frac{\cancel{3x} - \frac{9}{2}x^2 + 9x^3 + o(x^3) \cancel{-3x} - 3x^2 + \frac{x^3}{2} + o(x^3)}{x^2} = \lim_{x\to0} \frac{-\frac{15}{2}\cancel{x^2}}{\cancel{x^2}} + o(x^2) = -\frac{15}{2}$$

$\ln(1+3x) = 3x - \frac{9x^2}{2} + \frac{\overset{9}{\cancel{27}}x^3}{\cancel{3}} + o(x^3)$

$3\sin(x+x^2) = 3\left[x + x^2 - \frac{(x+x^2)^3}{3!} + \underbrace{o((x+x^2)^3)}_{[illegible]}\right] = 3\left[x + x^2 - \frac{x^3 + o(x^3)}{3!} + o(x^3)\right]$

2) $$\lim_{x\to+\infty}\left(\sqrt[3]{x^3 + x^2 + 1} - \sqrt{x^2 + 2x}\right) = \lim_{x\to+\infty}\left[x\left(\sqrt[3]{1 + \frac{1}{x} + \frac{1}{x^3}} - \sqrt{1 + \frac{2}{x}}\right)\right] = \lim_{x\to+\infty} x\left(\cancel{1} + \cancel{\frac{1}{x}} - \frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \cancel{-1} \cancel{-\frac{1}{x}} + \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right) = \lim_{x\to+\infty} x \cdot \frac{K_1}{K_2 x^2} = 0$$

$$(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2}t^2 + o(t^2) = 1 + \frac{1}{3}\left(\frac{3}{x} + \frac{1}{x^3}\right) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2}\left(\frac{3}{x} + \frac{1}{x^3}\right)^2 + o\left(\left(\frac{3}{x} + \frac{1}{x^3}\right)^2\right) = 1 + \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{\cancel{9}}\frac{\cancel{9}}{x^2} + o\left(\frac{1}{x^2}\right) = 1 + \frac{1}{x} + \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)$$

$t = \frac{3}{x} + \frac{1}{x^3}$; $\alpha = \frac{1}{3}$ $\nearrow$

$t = \frac{2}{x}$; $\alpha = \frac{1}{2}$ $\downarrow$

HOMEWORK $\left[1 + \frac{1}{x} + \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right]$

3) $$\lim_{x\to0^+} \frac{\sqrt{1+x} - e^{x/2}}{(\ln(1+x))^\alpha} = \lim_{x\to0^+} \frac{-\frac{1}{4}x^2 + o(x^2)}{x^\alpha} = \begin{cases} 0 & \alpha < 2 \\ -\frac{1}{4} & \alpha = 2 \\ -\infty & \alpha > 2 \end{cases}$$

$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$

$e^{x/2} = 1 + \frac{x}{2} + \frac{x^2}{8} + o(x^2)$

H:

$T_4^0(x)$ di: $\frac{1}{\sqrt{1+x^2+x^3}}$

$\lim_{x\to+\infty}\left(\pi x^2 - 2x^2\arctan(x) - 2x\right)$

$\lim_{x\to0^+} \frac{\arctan x + \ln(1+x) + \frac{1}{2}x^2 - 2x}{x^{2\alpha}(e^{x^2} - 1)}$