Basi Ortonormali

Prof.
Marco
Compagnoni

BASI ORTONORMALI:

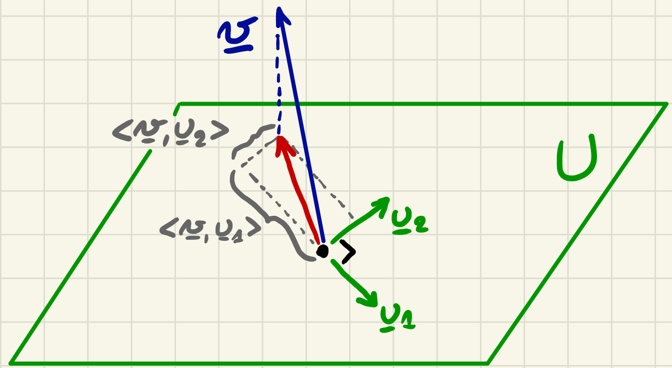
- Proiezione ortogonale
- Esistenza delle basi ortonormali
- Algoritmo di Gram-Schmidt

SEZIONE 8.4

Caratterizzazione della Proiezione (Proposizione 8.26)

V s.v.e. f.g., U sottospazio con $B_U=\{\underline{u}_1,\dots,\underline{u}_m\}$ base o.n. $\Rightarrow$

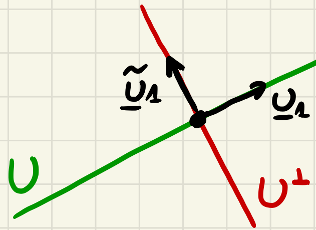
$$P_U(\underline{v})=\sum_{i=1}^{m}\langle \underline{v},\underline{u}_i\rangle\cdot\underline{u}_i \quad \text{per ogni } \underline{v}\in V.$$



$V=\mathbb{R}^3 \qquad U=\{(x,y,z)\in\mathbb{R}^3 \mid z=0\}$

$B_U=\{(1/\sqrt{2},1/\sqrt{2},0),(1/\sqrt{2},-1/\sqrt{2},0)\} \qquad \underline{v}=(1,1,1)$

$$P_U(\underline{v})=\langle \underline{v},\underline{u}_1\rangle\,\underline{u}_1+\cancel{\langle \underline{v},\underline{u}_2\rangle}\,\underline{u}_2=\frac{2}{\sqrt{2}}(1/\sqrt{2},1/\sqrt{2},0)=(1,1,0)$$



$B_U = \{\underline{u}_1, \dots, \underline{u}_m\}$ o.n.

DIM: consideriamo $U^\perp$ ed una sua base $B_U^\perp = \{\tilde{\underline{u}}_1, \dots, \tilde{\underline{u}}_n\}$, che esiste perché $\dim(U^\perp) = \dim(V) - \dim(U) < \infty$ $\Rightarrow$

$B = B_U \cup B_U^\perp$ è base di V, su cui la proiezione agisce come

$P_U(\underline{u}_i) = \underline{u}_i$, $P_U(\tilde{\underline{u}}_j) = \underline{0}$ per ogni i, j.

Per il teorema di interpolazione, P_U è l'unica funzione che si comporta in questo modo su B.

Consideriamo la funzione $f \in \mathrm{Hom}(V, U)$ definita da $f(\underline{v}) = \sum_{k=1}^{m} \langle \underline{v}, \underline{u}_k \rangle \cdot \underline{u}_k$

- $f(\underline{u}_i) = \sum_{k=1}^{m} \langle \underline{u}_i, \underline{u}_k \rangle \cdot \underline{u}_k \overset{o.n.}{=} \sum_{k=1}^{m} \delta_{ik} \cdot \underline{u}_k = \underline{u}_i = P_U(\underline{u}_i)$
- $f(\tilde{\underline{u}}_j) = \sum_{k=1}^{m} \langle \tilde{\underline{u}}_j, \underline{u}_k \rangle \cdot \underline{u}_k \overset{o.n.}{=} \sum_{k=1}^{m} 0 \cdot \underline{u}_k = \underline{0} = P_U(\tilde{\underline{u}}_j)$

$\Rightarrow f = P_U$ $\boxed{}$

COROLLARIO 8.27

OSS: $P_V = Id_V$

V s.v.e. f.g., $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$ base o.n. $\Rightarrow$ per ogni $\underline{v} \in V$ vale

$$\underline{v} = P_V(\underline{v}) = \sum_{i=1}^{n} \langle \underline{v}, \underline{v}_i \rangle \cdot \underline{v}_i .$$

$\langle \underline{v}, \underline{v}_i \rangle \to$ componenti $\underline{v}|_{\mathcal{B}}$

$V = Mat(2,1;\mathbb{R})$ $\quad G = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ $\quad \mathcal{B} = \left\{ A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$$\langle A,A \rangle_G = [1 \ \ 1] \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1 \ \ 1] \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 1$$

$$\langle B,B \rangle_G = [1 \ \ -1] \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [1 \ -1] \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = 1$$

$$\langle A,B \rangle_G = [1 \ \ 1] \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [1 \ 1] \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = 0$$

$\Rightarrow$ $\mathcal{B}$ è base o.n. di V rispetto $\langle , \rangle_G$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in V \Rightarrow X = \langle X, A \rangle_G \cdot A + \langle X, B \rangle_G \cdot B =$$

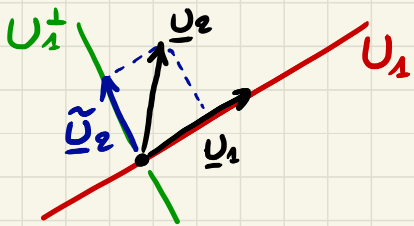
$$= \left([x_1 \ \ x_2] \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left([x_1 \ x_2] \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$= \frac{x_1 + x_2}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x_1 - x_2}{2} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow X|_{\mathcal{B}} = \begin{bmatrix} \frac{x_1 + x_2}{2} \\ \frac{x_1 - x_2}{2} \end{bmatrix}$$

Problemi: i) esistono sempre le basi ortonormali?

ii) come facciamo a costruirle?

IDEA: 

$U = \{\underline{v}_1, \underline{v}_2\}$ non è ortogonale

$\tilde{\underline{v}}_2 = \underline{v}_2 - P_{U_1}(\underline{v}_2) \perp U_1$

$\tilde{U} = \{\underline{v}_1, \tilde{\underline{v}}_2\}$ è ortogonale

TEOREMA 8.30

V s.v.e. f.g., $U = \{\underline{u}_1, \dots, \underline{u}_m\}$ l.i., $U_i = \mathcal{L}(\underline{u}_1, \dots, \underline{u}_i)$ $1 \le i \le m$.

Definiamo: $\begin{cases} \tilde{\underline{u}}_1 = \underline{u}_1 \\ \tilde{\underline{u}}_i = \underline{u}_i - P_{U_{i-1}}(\underline{u}_i) & 2 \le i \le m \end{cases} \Longrightarrow$ $\tilde{U} = \{\tilde{\underline{u}}_1, \dots, \tilde{\underline{u}}_m\}$ è l.i. ed ortogonale.

ESISTENZA DELLE BASI ORTONORMALE (COROLLARIO 8.31)

V s.v.e. f.g. $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$ base. Allora

$\tilde{\mathcal{B}} = \{ \tilde{\underline{v}}_1 / \|\tilde{\underline{v}}_1\|, \dots, \tilde{\underline{v}}_n / \|\tilde{\underline{v}}_n\| \}$ è una base o.n. di V.

OSS: $B_U = \{ \underline{u}_1, \dots, \underline{u}_m \}$ ortogonale $\Rightarrow$ $P_U(\underline{v}) = \sum_{i=1}^{m} \langle \underline{v}, \frac{\underline{u}_i}{\|\underline{u}_i\|} \rangle \cdot \frac{\underline{u}_i}{\|\underline{u}_i\|} =$

$$= \sum_{i=1}^{m} \frac{\langle \underline{v}, \underline{u}_i \rangle}{\|\underline{u}_i\|^2} \cdot \underline{u}_i$$

ALGORITMO DI GRAM-SCHMIDT

$$\mathcal{B} = \{ \underline{v}_1, \dots, \underline{v}_m \} \Rightarrow \begin{cases} \tilde{\underline{v}}_1 = \underline{v}_1 , \\ \tilde{\underline{v}}_i = \underline{v}_i - \sum_{j=1}^{i-1} \frac{\langle \underline{v}_i, \tilde{\underline{v}}_j \rangle}{\|\tilde{\underline{v}}_j\|^2} \cdot \tilde{\underline{v}}_j \qquad 2 \leq i \leq m . \end{cases}$$

$V = \mathbb{R}^3$, $\mathcal{B} = \{ \overbrace{(1,0,1)}^{\underline{v}_1}, \overbrace{(0,1,1)}^{\underline{v}_2}, \overbrace{(1,0,-1)}^{\underline{v}_3} \}$, prodotto euclideo

$$\tilde{\underline{v}}_1 = \underline{v}_1 = (1,0,1)$$

$$\tilde{\underline{v}}_2 = \underline{v}_2 - \underbrace{\frac{\langle \underline{v}_2, \tilde{\underline{v}}_1 \rangle}{\|\tilde{\underline{v}}_1\|^2} \cdot \tilde{\underline{v}}_1}_{P_{U_1}(\underline{v}_2)} = (0,1,1) - \frac{1}{2}(1,0,1) = (-1/2, 1, 1/2) =$$

$$\tilde{\underline{v}}_3 = \underline{v}_3 - \underbrace{\left(\cancel{\frac{\langle \underline{v}_3, \tilde{\underline{v}}_1 \rangle}{\|\tilde{\underline{v}}_1\|^2}} \cdot \tilde{\underline{v}}_1 + \frac{\langle \underline{v}_3, \tilde{\underline{v}}_2 \rangle}{\|\tilde{\underline{v}}_2\|^2} \cdot \tilde{\underline{v}}_2 \right)}_{P_{U_2}(\underline{v}_3)} = (1,0,-1) + \frac{1}{3/2}(-1/2, 1, 1/2) =$$

$$= (2/3, 2/3, -2/3)$$

OSS. 8.33: è possibile rilassare il vincolo che V debba essere finitamente generato per definire $P_U(V)$.

Infatti, richiedendo che U sia finitamente generato, ora sappiamo che esiste $B_U = \{\underline{u}_1, \dots, \underline{u}_m\}$ ortonormale.

Quindi possiamo definire

$$P_U(\underline{v}) = \sum_{i=1}^{m} \langle \underline{v}, \underline{u}_i \rangle \cdot \underline{u}_i .$$