COMPLEMENTO ORTOGONALE (DEFINIZIONE 8.19)

V s.v.e., $U \subseteq V$ sottoinsieme. $U^\perp = \{ \underline{v} \in V \mid < \underline{v}, \underline{u} > = 0$ per ogni $\underline{u} \in U \}$.

$V = \mathbb{R}[x]_2 \quad I = [0,1] \quad \mathcal{B}_2 = \{1, x, x^2\} \quad G_I = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$

$U = \{ P(x) = 1 \} \Rightarrow U^\perp = \{ Q(x) = b_0 + b_1 x + b_2 x^2 \mid \int_0^1 Q(x) \cdot 1 \, dx = 0 \}$

$\int_0^1 Q(x) \cdot 1 \, dx = \int_0^1 b_0 + b_1 x + b_2 x^2 = b_0 x + \frac{b_1}{2} x^2 + \frac{b_2}{3} x^3 \Big|_0^1 = b_0 + \frac{b_1}{2} + \frac{b_2}{3} = 0$

$\Rightarrow U^\perp = \{ Q(x) = b_0 + b_1 x + b_2 x^2 \mid b_0 + \frac{b_1}{2} + \frac{b_2}{3} = 0 \} = \{ Q(x) = b_1 (x - \frac{1}{2}) + b_2 (x^2 - \frac{1}{3}) \}$

$< Q, 1 >_I = Q|_{\mathcal{B}_2}^T \, G_I \, 1|_{\mathcal{B}_2} = [b_0 \; b_1 \; b_2] \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [b_0 \; b_1 \; b_2] \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} =$

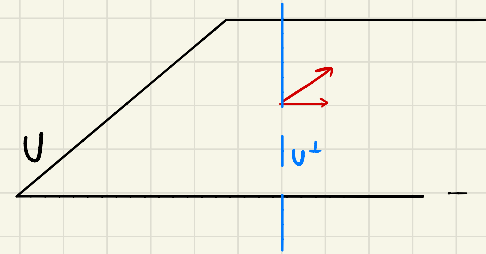
$= b_0 + \frac{b_1}{2} + \frac{b_2}{3} = 0$

OSS: sia V s.v.e. e $\mathcal{B}$ base ortonormale $\Rightarrow G_{|\mathcal{B}} = I_n$.

$U = \{ \underline{u} \} \Rightarrow U^\perp = \{ \underline{v} \in V \mid \underline{v}|_{\mathcal{B}}^T * \underline{u}|_{\mathcal{B}} = 0 \}$

$V = \mathbb{R}^3 \quad U = \mathcal{L}(\underline{u}_1 = (1,0,1), \underline{u}_2 = (1,1,0)) \quad \mathcal{B}_3 = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$\underline{u}_1, \underline{u}_2$ sono indipendenti e generano il piano



$$\begin{cases} x = t_1 + t_2 \\ y = t_2 \\ z = t_1 \end{cases} \Rightarrow \quad x - y - z = 0$$

$U^\perp = \{ \underline{v} \in V \mid \langle \underline{v}, t_1 \underline{u}_1 + t_2 \underline{u}_2 \rangle_E = 0 \}$ $\quad \mathcal{B}_3$ è ortonormale

$\langle \underline{v}, t_1 \underline{u}_1 + t_2 \underline{u}_2 \rangle_E = t_1 \langle \underline{v}, \underline{u}_1 \rangle_E + t_2 \langle \underline{v}, \underline{u}_2 \rangle_E = 0$ per ogni t_1, t_2

sse $\langle \underline{v}, \underline{u}_1 \rangle_E = \langle \underline{v}, \underline{u}_2 \rangle_E = 0$

$$\begin{cases} \underline{u}_{1|\mathcal{B}}^T \cdot G \cdot \underline{v}_{|\mathcal{B}} = 0 \\ \underline{u}_{2|\mathcal{B}}^T \cdot G \cdot \underline{v}_{|\mathcal{B}} = 0 \end{cases} \Rightarrow \operatorname{Ker}\left(\begin{bmatrix} U_{1|\mathcal{B}}^T \cdot \underline{v}_{|\mathcal{B}} \\ U_{2|\mathcal{B}}^T \cdot \underline{v}_{|\mathcal{B}} \end{bmatrix} \right)$$

$\hookrightarrow G = I_n$ perché usiamo basi ortonormali

Definiamo $A = [\underline{u}_{1|\mathcal{B}_3} \quad \underline{u}_{2|\mathcal{B}_3}] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow$

$\underline{v} \in U^\perp$ sse $\underline{v}_{|\mathcal{B}_3}^T * A = 0_{12}$ sse $A^T * \underline{v}_{|\mathcal{B}} = 0_{21}$

sse $\underline{v}_{|\mathcal{B}_3} \in \operatorname{Ker}(A^T) = \operatorname{Ker}\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \right) = \operatorname{Ker}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \right)$

$\Rightarrow U^\perp : \begin{cases} x + z = 0 \\ y - z = 0 \end{cases}$ retta $\perp$ a U $\quad \begin{cases} x = -t \\ y = t \\ z = t \end{cases}$

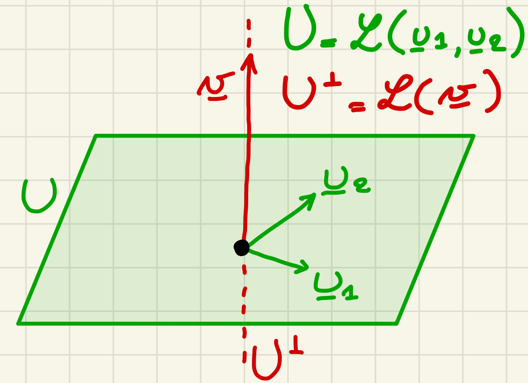
Proposizione 8.21

V s.v.e., $U \subseteq V$ sottoinsieme e $\overline{U} = \mathcal{L}(U)$.

i) $U^\perp$ è sottospazio di V;

ii) $U^\perp = \overline{U}^\perp$;

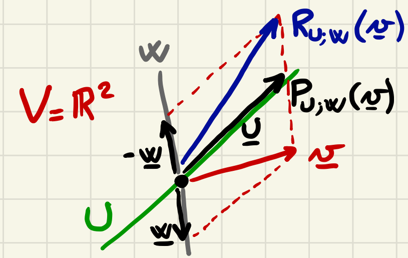
iii) se V è f.g. allora $\boxed{V = \overline{U} \oplus U^\perp}$.



Remind: sia $V = U \oplus W$

$\Rightarrow$ per ogni $\underline{v}$ esiste unica $\underline{v} = \underline{u} + \underline{w}$

$\Rightarrow P_{U;W}(\underline{v}) = \underline{u}$, $R_{U;W}(\underline{v}) = \underline{u} - \underline{w}$

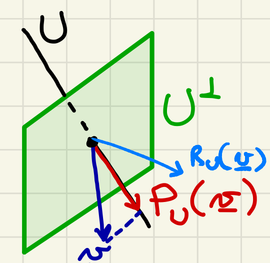


Proiezione e Riflessione Ortogonale (Definizione 8.23)

V s.v.e. f.g. $U \subseteq V$ sottospazio. $\underline{v} \in V$

i) la proiezione ortogonale su U è $P_U = P_{U;U^\perp}$; $P_U(\underline{v})$

ii) la riflessione ortogonale rispetto ad U è $R_U = R_{U;U^\perp}$. $R_U(\underline{v})$



$V = \mathbb{R}^3$, $U = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$, $\underline{v} = (h,1,1)$ $h \in \mathbb{R}$

$U: \ x+y+z=0 \implies \begin{cases} x = -t_1 - t_2 \\ y = t_1 \\ z = t_2 \end{cases} \implies (x,y,z) = t_1 \underbrace{(-1,1,0)}_{\underline{v}_1} + t_2 \underbrace{(-1,0,1)}_{\underline{v}_2}$

$\implies B_U = \{\underline{v}_1, \underline{v}_2\}$ è base di U

$U^\perp \sim \operatorname{Ker} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \operatorname{Ker} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \mathcal{L}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \implies B_{U^\perp} = \{\overbrace{(1,1,1)}^{\underline{v}_3}\}$

$B = B_U \cup B_{U^\perp} = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ è base di V

$\underline{v} = \alpha \underline{v}_1 + \beta \underline{v}_2 + \gamma \underline{v}_3 \implies \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} h \\ 1 \\ 1 \end{bmatrix} \implies$

$\underline{v} = \frac{1-h}{3} \underline{v}_1 + \frac{1-h}{3} \underline{v}_2 + \frac{h+2}{3} \underline{v}_3 \implies P_U(\underline{v}) = \frac{1-h}{3}(-2,1,1)$

- $h = 1$: $P_U(\underline{v}) = 0 \implies \underline{v} \in U^\perp$
- $h = -2$: $P_U(\underline{v}) = \underline{v} \implies \underline{v} \in U$

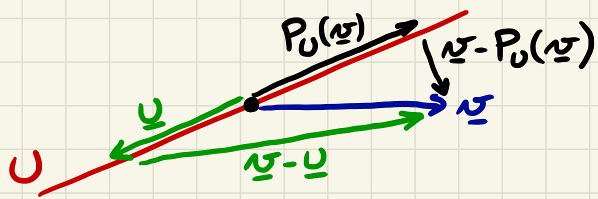
Caratterizzazione Metrica della Proiezione (Proposizione 8.24)

V s.v.e. f.g., $U \subseteq V$ sottospazio $\Rightarrow$ $P_U(\underline{v}) = \underset{\underline{u} \in U}{\operatorname{argmin}} \|\underline{v} - \underline{u}\|$.

Un vettore $\underline{u}$ è proiez. di $\underline{v}$ se la sua distanza è la minima possibile.

↳ "argomento del minimo della funzione"

es.: $\operatorname{argmin}(x^2+1) \Rightarrow f'(x)=0 \quad 2x=0 \quad x=0 \Rightarrow$
$\operatorname{argmin}(x^2+1)=0$ (valore di x affinché $f(x)$ abbia valore minimo)



Dim: dimostriamo che $\|\underline{v} - \underline{u}\| > \|\underline{v} - P_U(\underline{v})\|$ per ogni $\underline{u} \neq P_U(\underline{v})$.

$$\|\underline{v} - \underline{u}\|^2 = \|\underbrace{(\underline{v} - P_U(\underline{v}))}_{\in U^\perp} + \underbrace{(P_U(\underline{v}) - \underline{u})}_{\in U}\|^2 = \|\underline{v} - P_U(\underline{v})\|^2 + \underbrace{\|P_U(\underline{v}) - \underline{u}\|^2}_{>0} >$$

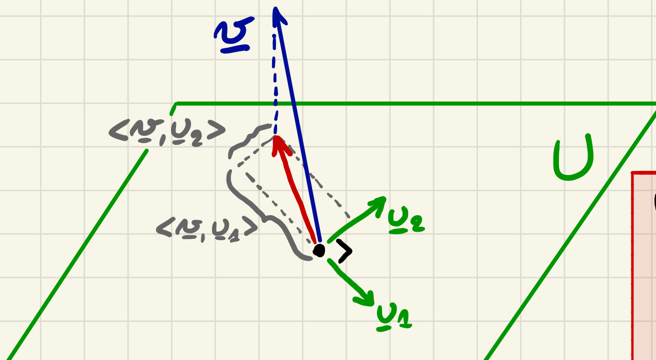
$$> \|\underline{v} - P_U(\underline{v})\|^2$$

$$\Rightarrow \|\underline{v} - \underline{u}\| > \|\underline{v} - P_U(\underline{v})\|. \qquad \boxtimes$$

CARATTERIZZAZIONE DELLA PROIEZIONE (PROPOSIZIONE 8.26)

V s.v.e. f.g., U sottospazio con $B_U = \{\underline{u}_1, \dots, \underline{u}_m\}$ base o.n. (ORTONORMALE) $\Rightarrow$

$$P_U(\underline{v}) = \sum_{i=1}^{m} \langle \underline{v}, \underline{u}_i \rangle \cdot \underline{u}_i \quad \text{per ogni } \underline{v} \in V.$$



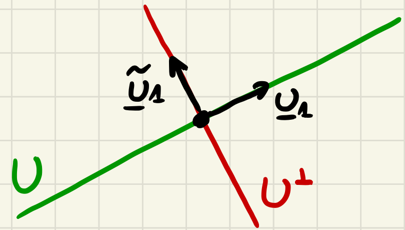
$$P_U(\underline{v}) = \sum_{i=1}^{m} \left\langle \underline{v}, \frac{\underline{u}_i}{\|\underline{u}_i\|} \right\rangle \cdot \frac{\underline{u}_i}{\|\underline{u}_i\|} = \sum_{i=1}^{m} \frac{\langle \underline{v}, \underline{u}_i \rangle}{\|\underline{u}_i\|^2} \underline{u}_i$$

!! nel caso B sia ORTOGONALE !!

$V = \mathbb{R}^3 \qquad U = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$

$B_U = \{(1/\sqrt{2}, 1/\sqrt{2}, 0), (1/\sqrt{2}, -1/\sqrt{2}, 0)\} \qquad \underline{v} = (1,1,1)$

$P_U(\underline{v}) = \langle \underline{v}, \underline{u}_1 \rangle \underline{u}_1 + \cancel{\langle \underline{v}, \underline{u}_2 \rangle} \underline{u}_2 = \frac{2}{\sqrt{2}} (1/\sqrt{2}, 1/\sqrt{2}, 0) = (1,1,0)$



$B_U = \{\underline{u}_1, \dots, \underline{u}_m\}$ o. n.

DIM: consideriamo $U^\perp$ ed una sua base $B_U^\perp = \{\underline{\tilde{u}}_1, \dots, \underline{\tilde{u}}_n\}$, che esiste perché $\dim(U^\perp) = \dim(V) - \dim(U) < \infty \implies$

$B = B_U \cup B_U^\perp$ è base di V, su cui la proiezione agisce come

$P_U(\underline{u}_i) = \underline{u}_i$, $P_U(\underline{\tilde{u}}_j) = \underline{0}$ per ogni i, j.

Per il teorema di interpolazione, P_U è l'unica funzione che si comporta in questo modo su B.

Consideriamo la funzione $f \in \mathrm{Hom}(V, U)$ definita da $f(\underline{v}) = \sum_{k=1}^{m} \langle \underline{v}, \underline{u}_k \rangle \cdot \underline{u}_k$

- $f(\underline{u}_i) = \sum_{k=1}^{m} \langle \underline{u}_i, \underline{u}_k \rangle \cdot \underline{u}_k \overset{o.n.}{=} \sum_{k=1}^{m} \delta_{ik} \cdot \underline{u}_k = \underline{u}_i = P_U(\underline{u}_i)$
- $f(\underline{\tilde{u}}_j) = \sum_{k=1}^{m} \langle \underline{\tilde{u}}_j, \underline{u}_k \rangle \cdot \underline{u}_k \overset{o.n.}{=} \sum_{k=1}^{m} 0 \cdot \underline{u}_k = \underline{0} = P_U(\underline{\tilde{u}}_j)$

$\implies f = P_U$ ▨

COROLLARIO 8.27

OSS: $P_V = Id_V$ (Proiezione su V di $\underline{v} \in V$ è $\underline{v}$)

V s.v.e. f.g., $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$ base o.n. $\Rightarrow$ per ogni $\underline{v} \in V$ vale

$$\underline{v} = P_V(\underline{v}) = \sum_{i=1}^{n} \langle \underline{v}, \underline{v}_i \rangle \cdot \underline{v}_i .$$

$\langle \underline{v}, \underline{v}_i \rangle \to$ componenti $\underline{v}|_{\mathcal{B}}$

$V = Mat(2,1;\mathbb{R})$ $\quad G = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ $\quad \mathcal{B} = \left\{ A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$$\langle A, A \rangle_G = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 1$$

$$\langle B, B \rangle_G = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = 1$$

$$\langle A, B \rangle_G = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = 0$$

$\Rightarrow$ $\mathcal{B}$ è base o.n. di V rispetto $\langle , \rangle_G$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in V \Rightarrow X = \langle X, A \rangle_G \cdot A + \langle X, B \rangle_G \cdot B =$$

$$= \left(\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

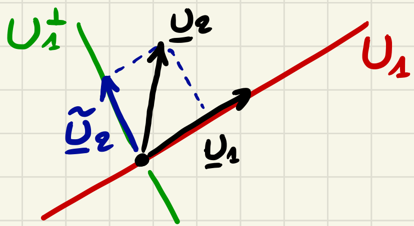
$$= \frac{x_1 + x_2}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x_1 - x_2}{2} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow X|_{\mathcal{B}} = \begin{bmatrix} \frac{x_1 + x_2}{2} \\ \frac{x_1 - x_2}{2} \end{bmatrix}$$

Problemi: i) esistono sempre le basi ortonormali?

ii) come facciamo a costruirle?

IDEA:



$U = \{\underline{u}_1, \underline{u}_2\}$ non è ortogonale

$\tilde{\underline{u}}_2 = \underline{u}_2 - P_{U_1}(\underline{u}_2) \perp U_1$

$\tilde{U} = \{\underline{u}_1, \tilde{\underline{u}}_2\}$ è ortogonale

Teorema 8.30

V s.v.e. f.g., $U = \{\underline{u}_1, \dots, \underline{u}_m\}$ l.i., $U_i = \mathcal{L}(\underline{u}_1, \dots, \underline{u}_i)$ $1 \leq i \leq m$.

Definiamo: $\begin{cases} \tilde{\underline{u}}_1 = \underline{u}_1 \\ \tilde{\underline{u}}_i = \underline{u}_i - P_{U_{i-1}}(\underline{u}_i) \quad 2 \leq i \leq m \end{cases}$ $\Longrightarrow$ $\tilde{U} = \{\tilde{\underline{u}}_1, \dots, \tilde{\underline{u}}_m\}$ è l.i. ed ortogonale.

Esistenza delle basi ortonormale (Corollario 8.31)

V s.v.e. f.g. $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_m\}$ base. Allora

$\tilde{\mathcal{B}} = \{ \tilde{\underline{v}}_1 / \|\tilde{\underline{v}}_1\|, \dots, \tilde{\underline{v}}_m / \|\tilde{\underline{v}}_m\| \}$ è una base o.n. di V.

OSS: $\mathcal{B}_U = \{\underline{u}_1, \dots, \underline{u}_m\}$ ortogonale $\Rightarrow \; P_U(\underline{v}) = \sum_{i=1}^{m} \left\langle \underline{v}, \frac{\underline{u}_i}{\|\underline{u}_i\|} \right\rangle \cdot \frac{\underline{u}_i}{\|\underline{u}_i\|} = \sum_{i=1}^{m} \frac{\langle \underline{v}, \underline{u}_i \rangle}{\|\underline{u}_i\|^2} \cdot \underline{u}_i$

Algoritmo di Gram-Schmidt (8.30 u 8.31)

$$\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_m\} \Rightarrow \begin{cases} \underline{\tilde{v}}_1 = \underline{v}_1, \\ \underline{\tilde{v}}_i = \underline{v}_i - \sum_{j=1}^{i-1} \frac{\langle \underline{v}_i, \underline{\tilde{v}}_j \rangle}{\|\underline{\tilde{v}}_j\|^2} \cdot \underline{\tilde{v}}_j \qquad 2 \le i \le m. \end{cases}$$

$V = \mathbb{R}^3$, $\mathcal{B} = \{\overbrace{(1,0\ 1)}^{\underline{v}_1}, \overbrace{(0,1,1)}^{\underline{v}_2}, \overbrace{(1,0,-1)}^{\underline{v}_3}\}$, prodotto euclideo

$$\underline{\tilde{v}}_1 = \underline{v}_1 = (1,0,1)$$

$$\underline{\tilde{v}}_2 = \underline{v}_2 - \underbrace{\frac{\langle \underline{v}_2, \underline{\tilde{v}}_1 \rangle}{\|\underline{\tilde{v}}_1\|^2} \cdot \underline{\tilde{v}}_1}_{P_{U_1}(\underline{v}_2)} = (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right) =$$

$$\underline{\tilde{v}}_3 = \underline{v}_3 - \underbrace{\left(\frac{\cancel{\langle \underline{v}_3, \underline{\tilde{v}}_1 \rangle}}{\|\underline{\tilde{v}}_1\|^2}^{= 1-1=0} \cdot \underline{\tilde{v}}_1 + \frac{\langle \underline{v}_3, \underline{\tilde{v}}_2 \rangle}{\|\underline{\tilde{v}}_2\|^2} \cdot \underline{\tilde{v}}_2 \right)}_{P_{U_2}(\underline{v}_3)} = (1,0,-1) + \frac{1}{3/2}\left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right) =$$

$$= \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\right) \Rightarrow \mathcal{B}_{vn} = \left\{ \frac{\tilde{v}_1}{\|\tilde{v}_1\|}, \dots \right\}$$

$\to$ DA NORM.

OSS. 833: è possibile rilassare il vincolo che V debba essere finitamente generato per definire $P_U(V)$.

Infatti, richiedendo che U sia finitamente generato, ora sappiamo che esiste $B_U = \{\underline{u}_1, \dots, \underline{u}_m\}$ ortonormale.

Quindi possiamo definire

$$P_U(\underline{v}) = \sum_{i=1}^{m} \langle \underline{v}, \underline{u}_i \rangle \cdot \underline{u}_i \, .$$