Funzione continua

Dato $f: A \subseteq \mathbb{R} \to \mathbb{R}$ e preso x_0, f è continua in x_0 se:

1) esiste finito $\lim_{x \to x_0} f(x) = l$
2) $l = f(x_0)$

Tutte le funzioni elementari sono continue nel loro dominio

Punti di Discontinuità

1) $\lim_{x \to x_0} f(x) = l$ finito, $f(x_0) \neq l \Rightarrow x_0$ punto di discontinuità eliminabile (3° sp.) $\Rightarrow g(x) = \begin{cases} f(x) & x \neq x_0 \\ l & x = x_0 \end{cases} \Rightarrow$ prolungamento per continuità
2) $\nexists \lim_{x \to x_0} f(x) \to \lim_{x \to x_0^+} f(x) = l_1$, $\lim_{x \to x_0^-} f(x) = l_2$ finiti $\Rightarrow$ discontinuità di salto (1° specie)
3) $\nexists \lim_{x \to x_0} f(x) \; / \; \lim_{x \to x_0^+} f(x) = \pm\infty$, $\lim_{x \to x_0^-} f(x) = \pm\infty$ o l $\Rightarrow$ discontinuità di 2° specie

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1) $f(x) = \sqrt{\dfrac{\ln(x+5)}{x+2}}$ $\quad \begin{cases} \dfrac{\ln(x+5)}{x+2} \geq 0 \\ x+5 > 0 \end{cases}$ $\quad \begin{cases} x \leq -4 \vee \boxed{x > -2} \\ x > -5 \end{cases}$ $\Rightarrow$ D: $\boxed{(-5; -4] \cup (-2; +\infty)}$

include $x \neq -2$

2) $f(x) = \dfrac{\ln(|2 - |x-2|| - 2x - 1)}{x+2}$ $\quad \begin{cases} |2 - |x-2|| - 2x - 1 > 0 \\ x \neq -2 \end{cases} \to \begin{cases} |2 - x + 2| & x - 2 \geq 0 \\ |2 + x - 2| & x - 2 < 0 \end{cases}$

$\begin{cases} 4 - x & 2 \leq x \leq 4 \\ 4 + x & x > 4 \\ x & 0 \leq x < 2 \\ -x & x < 0 \end{cases}$

$\begin{cases} 4 - x - 2x > 0 \\ 2 \leq x \leq 4 \\ x + 2 \neq 0 \end{cases}$ $\nexists x$
$\cup$
$\begin{cases} x - 4 - 2x + 1 > 0 \\ x > 4 \\ x \neq -2 \end{cases}$ $\nexists x$
$\cup$
$\begin{cases} x - 2x + 1 > 0 \\ 0 \leq x < 2 \\ x \neq -2 \end{cases}$ $x < 1$
$\cup$
$\begin{cases} -x - 2x + 1 > 0 \\ x < 0 \\ x \neq -2 \end{cases}$ $(-\infty; -2) \cup (-2; 0)$

$\Rightarrow \boxed{(-\infty; 1)}$

3) $f(x) = \ln(\sqrt{x^2 + 2x} + 7x - 1)$ $\quad \sqrt{x^2+2x} > -7x + 1$ $\quad \begin{cases} -7x + 1 < 0 \\ x^2 + 2x > 0 \end{cases} \cup \begin{cases} -7x + 1 > 0 \\ x^2 + 2x > 0 \\ x^2 + 2x > 49x^2 + 1 - 14x \end{cases} \to [\ldots] \to \boxed{\left(\tfrac{1}{12}; +\infty\right)}$

4) $f(x) = \dfrac{\ln(x+1)}{\sqrt{e^{x^2-2x} - 1}}$ $\quad \begin{cases} x + 1 > 0 \\ e^{x^2-2x} > 1 \end{cases}$ $\quad \begin{cases} x > -1 \\ x^2 - 2x > 0 \end{cases}$ $\quad \begin{cases} x > -1 \\ x < 0 \vee x > 2 \end{cases} \to \boxed{(-1; 0) \cup (2; +\infty)}$

5) $\lim_{x \to 0} x \cdot \arctan\left(\frac{1}{x}\right) = \begin{cases} x \cdot \frac{\pi}{2} = 0 \\ x \cdot \left(-\frac{\pi}{2}\right) = 0 \end{cases} \Rightarrow$ per confronto $\lim_{x \to 0} x \cdot \arctan\left(\frac{1}{x}\right) = 0$. Usando la definizione: $\forall \varepsilon > 0 \; \exists \delta \in \mathbb{R}: \forall x \in (x_0 - \delta; x_0 + \delta) \; |x \arctan(\frac{1}{x})| < \varepsilon$ $\quad |x \arctan(\frac{1}{x})| < \frac{\pi}{2}|x| \Rightarrow \frac{\pi}{2}|x| < \varepsilon \quad |x| < \frac{2\varepsilon}{\pi} \quad -\frac{2\varepsilon}{\pi} < x < \frac{2\varepsilon}{\pi} \Rightarrow \delta = \min\left(-\frac{2\varepsilon}{\pi}, \frac{2\varepsilon}{\pi}\right) \Rightarrow$ ✓

6) $\lim_{x \to 0^-} \sqrt{\dfrac{x+4}{1-e^x}}$ $\quad e^x - 1 \sim x$ per $x \to 0 \Rightarrow 1 - e^x \sim -x$ per $x \to 0 \Rightarrow \lim_{x \to 0^-} \sqrt{\dfrac{x-4}{-x}} = \sqrt{\dfrac{4}{0^+}} = +\infty$

7) $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} \dfrac{\cancel{(x-2)}(x-1)}{\cancel{x-2}} = 1$

8) $\lim_{x \to +\infty} x \underbrace{(7 + \cos(x))}_{\text{positiva limitata}} = +\infty$

9) $\lim_{x \to 2} \dfrac{|4x-3| - |1-3x|}{|2x-3| - |x-3|} = \lim_{x \to 2} \dfrac{4x - 3 + 1 - 3x}{2x - 3 + x - 3} = \lim_{x \to 2} \dfrac{x-2}{3x-6} = \dfrac{1}{3}$

10) $\lim_{x \to 0^+} x^2 \, 3^{\frac{1}{x}}$ $\quad t = \frac{1}{\sqrt{x}}$ $\quad \lim_{t \to +\infty} \frac{1}{t^4} 3^{t^2} = +\infty$

11) $\lim_{x \to 2} \dfrac{x-2}{\sqrt{4 + x^2 - 4x}} = \lim_{x \to 2} \dfrac{x-2}{\sqrt{(x-2)^2}} = \lim_{x \to 2} \dfrac{x-2}{|x-2|}$ $\quad \nexists$ limite: $\lim_{x \to 2^-} \frac{x-2}{|x-2|} = -1$, $\lim_{x \to 2^+} \frac{x-2}{|x-2|} = 1$

12) $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \log x}$ $\quad t = \frac{1}{x}$ $\quad \lim_{t \to +\infty} e^{\frac{1}{t} \log \frac{1}{t}} = \lim_{t \to +\infty} e^{\frac{\ln \frac{1}{t}}{t}} = \lim_{t \to +\infty} e^{-\frac{\ln t}{t}} = e^0 = 1$

13) $\lim_{x \to \frac{\pi}{2}} \dfrac{\sin x - 1}{\left(x - \frac{\pi}{2}\right)^2}$ $\quad t = x - \frac{\pi}{2}$ $\quad \lim_{t \to 0} \dfrac{\sin\left(t + \frac{\pi}{2}\right) - 1}{t^2} = \lim_{t \to 0} \dfrac{\cos t - 1}{t^2} = -\dfrac{1}{2}$

14) $\lim_{x \to 0} \dfrac{\sqrt[5]{1+3x} - 1}{x} = 3 \cdot \dfrac{1}{5} = \dfrac{3}{5}$

15) $\lim_{x \to -1} \dfrac{x+1}{\sqrt[4]{x+17} - 2}$ $\quad$ HOMEWORK [32]

16) $\lim_{x \to 0} \dfrac{x \tan x}{1 - \cos x}$ $\quad x \sim \tan x$ $\quad \lim_{x \to 0} \dfrac{x^2}{1 - \cos x} = 2$

$\lim_{x \to +\infty} \left(\dfrac{x-1}{x+3}\right)^{x-2} \overset{\frac{N(x)}{D(x)} = Q(x) - \frac{R(x)}{D(x)}}{=} \lim_{x \to +\infty} \left(1 - \dfrac{4}{x+3}\right)^{x-2}$ $\quad t = -\dfrac{x+3}{4}$ $\quad \lim_{t \to -\infty} \left(1 + \dfrac{1}{t}\right)^{-4t-5} = \lim_{t \to -\infty} \dfrac{\left[\left(1 + \frac{1}{t}\right)^t\right]^{-4}}{\left(1 + \frac{1}{t}\right)^5} = \lim_{t \to -\infty} \dfrac{1}{e^4} \cdot 1 = \dfrac{1}{e^4}$

17) $\lim_{x \to 1} \dfrac{\ln x}{e^x - e} =$ $\quad t = x - 1$ $\quad \lim_{t \to 0} \dfrac{\ln(t+1)}{e(e^t - 1)} \overset{e^t - 1 \sim t \;\; t \to 0}{=} \lim_{t \to 0} \dfrac{\ln(t+1)}{ex} \overset{\ln(t+1) \sim t \;\; t \to 0}{=} \lim_{t \to 0} \dfrac{x}{ex} = \dfrac{1}{e}$