Grassmann

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GRASSMANN:
- Dimensione sottospazi
- Formula di Grassmann

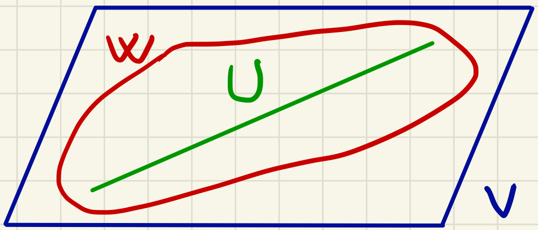
SEZIONE 4.6

OSS. : parliamo di spazi finitamente generati.

PROPOSIZIONE 4.46 Sia $U \subseteq V \Rightarrow$

i) $\dim(U) < \dim(V)$ sse $U \subset V$;

ii) $\dim(U) = \dim(V)$ sse $U = V$.



$U \subseteq V$, $\dim(U) = 1 < 2 = \dim(V)$

La proposizione afferma che non può esistere un sottospazio W tale che $U \subset W \subset V$:

- $\dim(W) = 1 \Rightarrow W = U$; $\cdot$ $\dim(W) = 2 \Rightarrow W = V$.

FORMULA DI GRASSMANN (LEMMA 4.48, TEOREMA 4.49)

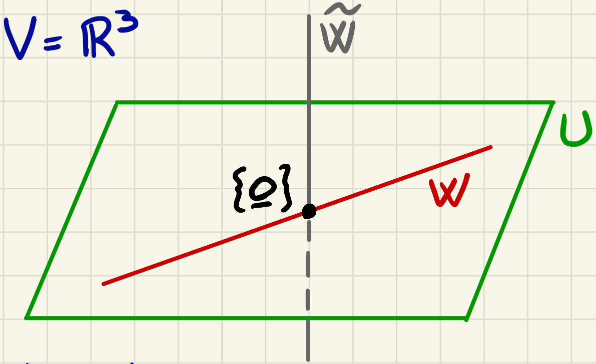
$U, W \subseteq V$ sottospazi finitamente generati. Allora:

i) $\dim(U \cap W) \leq \min(\dim(U), \dim(W))$;

ii) $\mathcal{B}_U \cup \mathcal{B}_W$ = insieme di generatori di $U + W$;

iii) $\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.

$V = \mathbb{R}^3$



- $U + W = U \quad U \cap W = W \implies$

  $\dim(U+W) + \dim(U \cap W) = 2 + 1 = \dim(U) + \dim(W)$

- $U + \widetilde{W} = V \quad U \cap \widetilde{W} = \{\underline{0}\} \implies$

  $\dim(U+\widetilde{W}) + \dim(U \cap \widetilde{W}) = 3 + 0 = \dim(U) + \dim(\widetilde{W})$

- $W + \widetilde{W} =$ piano $\quad W \cap \widetilde{W} = \{\underline{0}\} \implies$

  $\dim(W+\widetilde{W}) + \dim(W \cap \widetilde{W}) = 2 + 0 = \dim(W) + \dim(\widetilde{W})$

$\dim(V) = 3 \quad \dim(U) = 2$

$\dim(W) = 1 \quad \dim(\widetilde{W}) = 1$

$\dim(\{\underline{0}\}) = 0$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad V = \mathrm{Mat}(4,1;\mathbb{Q}).$$

$$U = \mathrm{Ker}(A) = \left\{ s_1 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_2 \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}, \quad W = \mathrm{Ker}(B) = \left\{ t_1 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t_2 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \implies$$

$$\Rightarrow \mathcal{B}_U = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}, \quad \mathcal{B}_W = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow \dim(U) = \dim(W) = 2.$$

$$U \cap W = \operatorname{Ker}\left[\frac{A}{B}\right] = \operatorname{Ker}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \operatorname{Ker}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left\{ \alpha \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$U + W = \mathcal{L}(\mathcal{B}_U \cup \mathcal{B}_W) = \mathcal{L}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \subset V \Rightarrow$$

$$x \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -x - z = 0 \\ -y = 0 \\ x = 0 \\ 2y + z = 0 \end{cases} \Rightarrow x = y = z = 0 \Rightarrow$$

$\Rightarrow \mathcal{B}_U \cup \mathcal{B}_W$ è una base di $U + W \Rightarrow \dim(U+W) = 3$.

$$\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W).$$

COROLLARIO 4.52

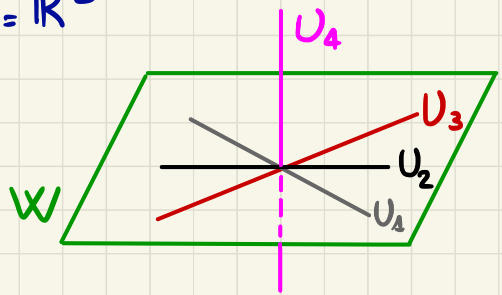
$U_1, \dots, U_m \subseteq V$ sottospazi, $U_{\hat{k}} = \sum_{i=1}^{m} U_i \quad i \neq k$.

$V = U_1 \oplus \dots \oplus U_m$ se e solo se:

i) $\dim(V) = \sum_{i=1}^{m} \dim(U_i)$;

ii) $\dim(U_k \cap U_{\hat{k}}) = 0$ per ogni $k = 1, \dots, m$.

$V = \mathbb{R}^3$



- $\dim(U_1) + \dim(U_2) + \dim(U_3) = 3$ ma
  $V \neq U_1 \oplus U_2 \oplus U_3$. Infatti
  $U_3 \cap U_{\hat{3}} = U_3 \cap W = U_3 \Rightarrow \dim(U_3 \cap U_{\hat{3}}) = 1$
- $V = U_1 \oplus U_2 \oplus U_4$