Homework 3

PSTAT 223A

Alex Bernstein

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Problem 1 (5.1)

Verify the given processes solve the given stochastic differential equations: (B_t denotes the 1-dimensional Brownian Motion

(i) $X_t = e^{B_t}$ solves $dX_t = \frac{1}{2}X_t dt + X_t dB_t$

Proof. We apply Ito's Lemma to $X_t = e^{B_t}$:

$$\begin{aligned} dX_t &= 0dt + e^{B_t}dB_t + \frac{1}{2}e^{B_t}dB_t^2 \\ &= e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt \\ &= X_t dB_t + \frac{1}{2}X_t dt \end{aligned}$$

Therefore e^{B_t} solves the given SDE. □

(ii) $X_t = \frac{B_t}{1+t}; B_0 = 0$ solves

$$dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dB_t; \quad X_0 = 0$$

Proof. We apply Ito's Lemma to $X_t = \frac{B_t}{1+t}$, $B_0 = 0$

$$\begin{aligned} dX_t &= -\frac{B_t}{(1+t)^2}dt + \frac{1}{1+t}dB_t + \frac{1}{2}0dB_t^2 \\ &= -\frac{X_t}{1+t}dt + \frac{1}{1+t}dB_t \\ X_0 &= \frac{B_0}{1+0} = 0 \end{aligned}$$

Therefore $X_t = \frac{B_t}{1+t}$ solves the given SDE. □

(iii) $X_t = \sin B_t$ with $B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ solves

$$dX_t = -\frac{1}{2}X_t dt + \sqrt{1 - X_t^2}dB_t \text{ for } t < \inf\left\{s > 0; B_s \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$$

Proof. Applying Itô's Lemma to $X_t = \sin B_t$:

$$\begin{aligned} dX_t &= 0dt + \cos B_t dB_t + \frac{1}{2}(-\sin B_t)dB_t^2 \\ &= \cos B_t dB_t - \frac{1}{2}\sin B_t dt \\ &= \frac{1}{2}\sin B_t dt + \sqrt{1-\sin^2(B_t)}dB_t \\ &= \frac{1}{2}X_t + \sqrt{1-X_t^2}dB_t \end{aligned}$$

where the condition on t ensures the function is 1-1. $X_0 = \sin B_0 = \sin a$, where $\pi/2 < a < \pi/2$. Therefore $X_t = \sin B_t$ solves the SDE given. □

(iv) $(X_1(t), X_2(t)) = \left(t, e^t B_t\right)$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$$

Proof. Applying Itô's formula to $(X_1(t), X_2(t))$, we get:

$$\begin{aligned} dX_t &= \begin{pmatrix} 1 \\ e^t B_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^t \end{pmatrix} dB_t + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} dt \\ &= \begin{pmatrix} 1 \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_1(t)} \end{pmatrix} dB_t \end{aligned}$$

We therefore have that $(X_1(t), X_2(t))$ solves the given SDE. □

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t$$

Proof. Applying Ito's Formula to $(X_1(t), X_2(t))$, we get:

$$\begin{aligned} dX_t &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} \sinh(B_t) \\ \cosh(B_t) \end{pmatrix} dB_t + \frac{1}{2}\begin{pmatrix} \cosh(B_t) \\ \sinh(B_t) \end{pmatrix} dt \\ &= \frac{1}{2}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} X_2(t) \\ X_1(t) \end{pmatrix} dB_t \end{aligned}$$

We therefore have that $(X_1(t), X_2(t))$ solves the given SDE. □

Problem 2 (5.3)

Let $(B_1, \ldots, B_n)$ be a Brownian Motion in $\mathbb{R}^n$, $\alpha_1, \ldots, \alpha_n$ constants. Solve the stochastic differential equation:

$$dX_t = rX_t dt + X_t \left(\sum_{k=1}^{n} \alpha_k dB_k(t) \right); \quad X_0 > 0$$

Proof. Let $B_0 = 0$. Dividing the original SDE by X_t, we get:

$$\frac{dX_t}{X_t} = rdt + \sum_{k=1}^{n} \alpha_k dB_k(t)$$

Let $g(t, x) = \log x$. Then $d_x g(t, x) = \frac{1}{x}$, $d_{xx} g(t, x) = \frac{-1}{x^2}$, and $d_t g(t, x) = 0$. Applying Ito's Lemma to $d \log X_t$, we get:

$$d \log X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} d\langle X, X \rangle_t$$

Solving for $d\langle X, X \rangle_t = (dX_t)^2$, note that all the cross terms and the first term are 0. We therefore get:

$$d\langle X, X \rangle_t = (dX_t)^2 = X_t^2 \sum_{k=1}^{n} \alpha_k^2 dt$$

Therefore

$$\begin{aligned}
d \log X_t &= rdt + \sum_{k=1}^{n} \alpha_k dB_k(t) - \frac{1}{2X_t^2} X_t^2 \sum_{k=1}^{n} \alpha_k^2 dt \\
&= \left(r - \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \right) dt + \sum_{k=1}^{n} \alpha_k dB_k(t) \\
\int_0^t d \log X_s &= \int_0^t \left(r - \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \right) ds + \sum_{k=1}^{n} \int_0^t \alpha_k dB_k(s) \\
\log(X_t) - \log(X_0) = \log\left(\frac{X_t}{X_0} \right) &= rt - \frac{t}{2} \sum_{k=1}^{n} \alpha_k^2 + \sum_{k=1}^{n} \alpha_k B_k(t), \text{ therefore} \\
X_t &= X_0 \exp\left\{ rt - \frac{t}{2} \sum_{k=1}^{n} \alpha_k^2 + \sum_{k=1}^{n} \alpha_k B_k(t) \right\}
\end{aligned}$$

□

Problem 3 (5.5)

(i) Solve the Ornstein-Uhlenbeck equation (or Langevin equation)

$$dX_t = \mu X_t dt + \sigma dB_t$$

where μ, σ are real constants and $B_t \in \mathbb{R}$.

Proof. We will use $e^{-\mu t}$ as an integrating factor. Applying Itô's Lemma to $d(e^{-\mu t}X_t)$ we get:

$$\begin{aligned} d(e^{-\mu t}X_t) &= -\mu e^{-\mu t}X_t dt + e^{-\mu t}dX_t \\ &= -\mu e^{-\mu t}X_t dt + e^{-\mu t}(\mu X_t dt + \sigma dB_t) \\ &= e^{-\mu t}\sigma dB_t \\ \int_0^t d(e^{-\mu s}X_s) &= \int_0^t e^{-\mu s}\sigma dB_s \\ e^{-\mu t}X_t - X_0 &= \sigma \int_0^t e^{-\mu s} dB_s \\ X_t &= X_0 e^{\mu t} + \sigma e^{\mu t} \int_0^t e^{-\mu s} dB_s \\ &= X_0 e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} dB_s \end{aligned}$$

□

(ii) Find $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t)$:

(a) $\mathbb{E}[X_t] = e^{\mu t}\mathbb{E}[X_0]$

Proof.

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}\Big\{X_0 e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} dB_s\Big\} \\ &= e^{\mu t}\mathbb{E}[X_0] + \sigma \mathbb{E}\Big(\int_0^t e^{\mu(t-s)} dB_s\Big) \\ &= e^{\mu t}\mathbb{E}[X_0] + 0 \\ &= e^{\mu t}X_0, \quad \text{if } X_0 \text{ is known} \end{aligned}$$

□

(b) $\mathrm{Var}(X_t) = e^{2\mu t}\mathrm{Var}(X_0) + \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)$

Proof.

$$
\begin{aligned}
\mathrm{Var}(X_t) &= \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = \mathbb{E}[X_t^2] \\
\mathbb{E}[X_t^2] &= \mathbb{E}\Big\{\Big(X_0 e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} dB_s\Big)^2\Big\} \\
&= \mathbb{E}\Big\{X_0^2 e^{2\mu t} + 2X_0 e^{\mu t}\sigma \int_0^t e^{\mu(t-s)} dB_s + \sigma^2 \Big(\int_0^t e^{\mu(t-s)} dB_s\Big)^2\Big\} \\
&= e^{2\mu t}\mathbb{E}(X_0^2) + 0 + \sigma^2 \mathbb{E}\Big\{\Big(\int_0^t e^{\mu(t-s)} dB_s\Big)^2\Big\} \\
\text{(Itô's Isometry)} &= e^{2\mu t}\mathbb{E}(X_0^2) + \sigma^2 \mathbb{E}\Big\{\int_0^t e^{2\mu(t-s)} ds\Big\} \\
&= e^{2\mu t}\mathbb{E}(X_0^2) + \sigma^2 e^{2\mu t}\mathbb{E}\Big\{\int_0^t e^{-2\mu s} ds\Big\} \\
&= e^{2\mu t}\mathbb{E}(X_0^2) + \sigma^2 e^{2\mu t}\mathbb{E}\Big\{\Big(\frac{1}{2\mu} - \frac{e^{-2\mu t}}{2\mu}\Big)\Big\} \\
&= e^{2\mu t}\mathbb{E}(X_0^2) + \sigma^2 \frac{e^{2\mu t}}{2\mu}\Big(1 - e^{-2\mu t}\Big) \\
&= e^{2\mu t}\mathbb{E}(X_0^2) + \frac{\sigma^2}{2\mu}\Big(e^{2\mu t} - 1\Big)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\mathrm{Var}(X_t) &= \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 \\
&= e^{2\mu t}\big\{\mathbb{E}[X_0^2] - \mathbb{E}[X_0]^2\big\} + \frac{\sigma^2}{2\mu}\Big(e^{2\mu t} - 1\Big) \\
&= e^{2\mu t}\mathrm{Var}(X_0) + \frac{\sigma^2}{2\mu}\Big(e^{2\mu t} - 1\Big)
\end{aligned}
$$

If X_0 is known, then $\mathrm{Var}(X_0) = 0$ and:

$$
\mathrm{Var}(X_t) = \frac{\sigma^2}{2\mu}\Big(e^{2\mu t} - 1\Big)
$$

□

Problem 4 (5.7)

The mean-reverting Ornstein-Uhlenbeck process is the solution X_t of the Stochastic Differential Equation

$$dX_t = (m - X_t)\, dt + \sigma dB_t$$

where m, σ are real constants and $B_t \in \mathbb{R}$.

(i) Solve this equation

Proof. We will use the integrating factor e^t. Applying Itô's Lemma to $d(e^t X_t)$ we get:

$$\begin{aligned}
d(e^t X_t) &= X_t e^t dt + e^t dX_t \\
&= X_t e^t dt + e^t \Big\{ (m - X_t) dt + \sigma e^t dB_t \Big\} \\
&= m e^t dt + \sigma dB_t \\
\int_0^t d(e^s X_s) &= \int_0^t m e^s ds + \int_0^t \sigma e^s dB_s \\
e^t X_t - X_0 &= m(e^t - 1) + \sigma \int_0^t e^s dB_s \\
X_t &= e^{-t} X_0 + m(1 - e^{-t}) + \sigma \int_0^t e^{s-t} dB_s
\end{aligned}$$

□

(ii) Find $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t)$:

(a) $\mathbb{E}[X_t] = e^{-t}\mathbb{E}[X_0] + m(1 - e^{-t})$

Proof. This is trivial; m and e^{-t} are constant and the expectation of an Itô integral is 0. □

(b) $\mathrm{Var}(X_t) = e^{-2t}\mathrm{Var}(X_0) + \frac{\sigma^2}{2}(1 - e^{-2t})$

Proof.

$$\begin{aligned}
\mathbb{E}[X_t^2] = \mathbb{E}\Big\{ & X_0^2 e^{-2t} + m^2(1 - e^{-t})^2 + \sigma^2 \Big(\int_0^t e^{s-t} dB_s \Big)^2 \\
& + 2X_0 e^{-t} m(1 - e^{-t}) + 2X_0 e^{-t} \int_0^t e^{s-t} dB_s + 2m(1 - e^{-t}) \int_0^t e^{s-t} dB_s \Big\} \\
= \mathbb{E}\Big\{ & X_0^2 e^{-2t} + m^2(1 - e^{-t})^2 + \sigma^2 \Big(\int_0^t e^{s-t} dB_s \Big)^2 \Big\} + 2e^{-t} m(1 - e^{-t}) \mathbb{E}[X_0] \\
= 2e^{-t} & m(1 - e^{-t}) \mathbb{E}[X_0] + e^{-2t}\mathbb{E}[X_0^2] + m^2(1 - e^{-t})^2 + \sigma^2 \mathbb{E}[\Big(\int_0^t e^{s-t} dB_s \Big)^2] \\
= 2e^{-t} & m(1 - e^{-t}) \mathbb{E}[X_0] + e^{-2t}\mathbb{E}[X_0^2] + m^2(1 - e^{-t})^2 + \sigma^2 \mathbb{E}[\int_0^t e^{2(s-t)} ds] \\
= 2e^{-t} & m(1 - e^{-t}) \mathbb{E}[X_0] + e^{-2t}\mathbb{E}[X_0^2] + m^2(1 - e^{-t})^2 + \sigma^2 e^{-2t} \frac{1}{2} \Big(e^{2t} - 1 \Big) \\
= 2e^{-t} & m(1 - e^{-t}) \mathbb{E}[X_0] + e^{-2t}\mathbb{E}[X_0^2] + m^2(1 - e^{-t})^2 + \sigma^2 \frac{1}{2} (1 - e^{-2t})
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Var}(X_t) &= \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 \\
&= 2e^{-t}m(1-e^{-t})\mathbb{E}[X_0] + e^{-2t}\mathbb{E}[X_0^2] + m^2(1-e^{-t})^2 + \sigma^2\frac{1}{2}(1-e^{-2t}) \\
&- (e^{-t}\mathbb{E}[X_0] + m(1-e^{-t}))^2 \\
&= 2m\mathbb{E}[X_0]e^{-t}(1-e^{-t}) + e^{-2t}\mathbb{E}[X_0^2] + m^2(1-e^{-t})^2 + \frac{\sigma^2}{2}(1-e^{-2t}) \\
&- e^{-2t}\mathbb{E}[X_0]^2 - 2m\mathbb{E}[X_0]e^{-t}(1-e^{-t}) - m^2(1-e^{-t})^2 \\
&= e^{-2t}\big(\mathbb{E}[X_0^2] - \mathbb{E}[X_0]^2\big) - \frac{\sigma^2}{2}(1-e^{-2t}) \\
&= e^{-2t}\mathrm{Var}(X_0) - \frac{\sigma^2}{2}(1-e^{-2t})
\end{aligned}$$

as expected. $\square$

Problem 5 (5.11)

For a fixed $a, b \in \mathbb{R}$ consider the following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t}dt + dB_t; \quad 0 \leq t < 1, Y_0 = a.$$

Verify that

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}; \quad 0 \leq t < 1$$

solves the equation and prove that $\lim_{t \longrightarrow 1} Y_t = b$ a.s. The process Y_t is called the Brownian Bridge (from a to b).

Proof. Using the given value of Y_t and applying Itô's Lemma:

$$\begin{aligned}
Y_t &= bt + \Big(a + \int_0^t \frac{dB_s}{1-s}\Big)(1-t) \\
dY_t &= \Big(b - a - \int_0^t \frac{dB_s}{1-s}\Big)dt + \frac{dB_t}{1-t}(1-t) \\
&= \Big(b - a - \int_0^t \frac{dB_s}{1-s}\Big)dt + dB_t \\
&= dB_t + \frac{1-t}{1-t}\Big(b - a - \int_0^t \frac{dB_s}{1-s}\Big)dt \\
&= dB_t + \frac{1}{1-t}\Big(b - bt - a(1-t) - (1-t)\int_0^t \frac{dB_s}{1-s}\Big)dt \\
&= dB_t + \frac{1}{1-t}\Big\{b - \Big(bt + a(1-t) + (1-t)\int_0^t \frac{dB_s}{1-s}\Big)\Big\}dt \\
&= dB_t + \frac{b - Y_t}{1-t}dt
\end{aligned}$$

as expected, where the regularity conditions to ensure existence (i.e. $0 \leq t < 1$) apply. It suffices to show that

$$\sup\Big\{\mathbb{E}\Big|\Big(\int_0^t \frac{dB_s}{1-s}ds\Big)\Big| : 0 \leq t < 1\Big\} < \infty$$

This is trivially true, because the expectation of the Itô integral is 0 for $0 \leq t < 1$. Therefore, we have that $\mathbb{E}[Y_t] = bt + a(1-t)$ and, by Martingale Convergence,

$$Y_t \xrightarrow{t \nearrow 1} b \text{ a.s.}$$

This limit exists almost surely, $\forall t < 1$. $\square$

Problem 6 (5.17) The Grönwall Inequality

Let $v(t)$ be a nonlinear function such that

$$v(t) \leq C + A\int_0^t v(s)ds \quad \text{for } 0 \leq t \leq T$$

for some constants C, A. Prove that

$$v(t) \leq C\exp(At) \quad \text{for } 0 \leq t \leq T.$$

Proof. Following the hint provided, let $w(t) = \int_0^t v(s)ds$, and therefore $w'(t) = v(t)$ and $w(0) = 0$. Therefore $w'(t) = C + Aw(t)$. Therefore:

$$\frac{w'(t)}{C + Aw(t)} \leq 1$$
$$\frac{Aw'(t)}{C + Aw(t)} \leq A$$

Note that:

$$\begin{aligned}\frac{d}{dt}\log(C + Aw(t)) = \frac{Aw'(t)}{C + Aw(t)} &\leq A, \text{ so} \\ \log(C + Aw(t)) - \log(C + Aw(0)) &\leq At \\ \log\left(\frac{C + Aw(t)}{C}\right) &\leq At \\ C + Aw(t) &\leq Ce^{At}\end{aligned}$$

Therefore,

$$\begin{aligned}w(t) &\leq \frac{C}{A}(e^{At} - 1) \\ \int_0^t v(s)ds &\leq \frac{C}{A}(e^{At} - 1)\end{aligned}$$

Therefore,

$$\begin{aligned}v(s) &\leq C + A\int_0^t v(s)ds \\ &\leq C + C(e^{At} - 1) \\ &= Ce^{At}\end{aligned}$$

Therefore $v(s) \leq Ce^{At}$, and this proves Grönwall's inequality. $\square$

Problem 7 (5.18)

The geometric mean-reversion process X_t is defined as the solution of the stochastic differential equation:

$$dX_t = \kappa\left(\alpha - \log X_t\right) X_t dt + \sigma X_t dB_t; \quad X_0 = x > 0$$

where $\kappa, \alpha, \sigma,$ and x are positive constants.

(i) Show that the SDE above is solved by

$$X_t := \exp\left\{e^{-\kappa t}\ln x + \left(\alpha - \frac{\sigma^2}{2\kappa}\right)\left(1 - e^{\kappa t}\right) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s} dB_s\right\}$$

Proof. We first use the substitution that $Y_t = \log X_t$. Then, applying Itô's formula to dY_t, we get:

$$\begin{aligned}
dY_t &= 0dt + \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}d\langle X, X\rangle_t \\
&= \frac{1}{X_t}\kappa(\alpha - \log X_t)X_t dt + \sigma X_t dB_t - \frac{1}{2X_t^2}\sigma^2 X_t^2 dt \\
&= \kappa(\alpha - Y_t)dt + \sigma dB_t - \frac{\sigma^2}{2}dt
\end{aligned}$$

This is now a linear SDE in Y_t. We will use the integrating factor $e^{\kappa t}$ to cancel the Y_t term. Applying Ito's Lemma to $e^{\kappa t}Y_t$:

$$\begin{aligned}
d(e^{\kappa t}Y_t) &= \kappa e^{\kappa t}Y_t dt + e^{\kappa t}dY_t + 0\frac{1}{2}d\langle Y, Y\rangle_t \\
&= \kappa e^{\kappa t}Y_t dt + e^{\kappa t}\left(\kappa\alpha dt - \kappa Y_t dt\right) + \sigma dB_t - \frac{\sigma^2}{2}dt \\
&= e^{\kappa t}\kappa\alpha dt - \frac{\sigma^2}{2}dt + \sigma dB_t
\end{aligned}$$

We can now integrate both sides:

$$\begin{aligned}
\int_0^t d(e^{\kappa s}Y_s) &= \kappa\alpha\int_0^t e^{\kappa s} - \frac{\sigma^2}{2}ds + \sigma\int_0^t dB_s \\
e^{\kappa t}Y_t - Y_0 &= (\kappa\alpha - \frac{\sigma^2}{2})\frac{1}{\kappa}\left(e^{\kappa t} - 1\right) + \int_0^t \sigma dB_s \\
Y_t &= e^{-\kappa t}Y_0 + (\alpha - \frac{\sigma^2}{2\kappa})(1 - e^{-\kappa t}) + \sigma e^{-\kappa t}\int_0^t dB_s
\end{aligned}$$

Taking our initial condition $X_0 = x > 0$, we get $Y_0 = \log X_0 = \log x$. Therefore, substituting $Y_t = e^{X_t}$ back into the equation, we get:

$$\begin{aligned}
\log X_t &= e^{\kappa t}\log x + (\alpha - \frac{\sigma^2}{2\kappa})(1 - e^{-\kappa t}) + \sigma e^{-\kappa t}\int_0^t dB_s \\
X_t &= \exp\left\{e^{\kappa t}\log x + (\alpha - \frac{\sigma^2}{2\kappa})(1 - e^{-\kappa t}) + \sigma e^{-\kappa t}\int_0^t dB_s\right\}
\end{aligned}$$

as expected. □

(ii) Show that

$$E\left[X_t\right] = \exp\left(e^{-\kappa t}\ln x + \left(\alpha - \frac{\sigma^2}{2\kappa}\right)\left(1-e^{-\kappa t}\right) + \frac{\sigma^2\left(1-e^{-2\kappa t}\right)}{2\kappa}\right)$$

(note, book is wrong; last term should have a 4κ not a 2κ in the denominator)

Proof.

$$\mathbb{E}X_t = \exp\Big\{e^{-\kappa t}\log x + (\alpha - \frac{\sigma^2}{2\kappa})(1-e^{-\kappa t})\Big\}\mathbb{E}\Big\{\exp\big(\sigma e^{-kt}\int_0^t e^{ks}dB_s\big)\Big\} \tag{1}$$

We wish to show that

$$\sigma e^{-kt}\int_0^t e^{ks}dB_s$$

is normally distributed. It suffices to show that $\int_0^t e^{ks}dB_s$ is normally distributed.

$$\begin{aligned} d(e^{kt}B_t) &= ke^{kt}B_t dt + e^{kt}dB_t \\ e^{kt}B_t &= \int_0^t ke^{ks}B_s ds + \int_0^t e^{ks}dB_s \\ \int_0^t e^{ks}dB_s &= e^{kt}B_t - \int_0^t ke^{ks}B_s ds \end{aligned}$$

We know the right-hand side terms are normally distributed, so their difference is also normally distributed. Further, it is mean 0. We now know that (ii) is normally distributed with 0 mean. We can calculate the expectation term in (1) by calculating the MGF of (ii) where the argument of the MGF is 1. In order to do that, we need the its variance. We will now calculate $\mathrm{Var}(\sigma e^{-kt}\int_0^t e^{ks}dB_s)$.

$$\begin{aligned} \mathrm{Var}(\sigma e^{-kt}\int_0^t e^{ks}dB_s) &= \mathbb{E}\Big\{\big(\sigma e^{-kt}\int_0^t e^{ks}dB_s\big)^2\Big\} \\ \text{(Itô's Isometry)} &= \mathbb{E}\Big\{\sigma^2 e^{-2kt}\int_0^t e^{2ks}ds\Big\} \\ &= \sigma^2 e^{-2kt}\int_0^t e^{2ks}ds \\ &= \frac{\sigma^2 e^{-2kt}}{2k}\Big(e^{2kt}-1\Big) \\ &= \frac{\sigma^2}{2k}\Big(1-e^{-2kt}\Big) \end{aligned}$$

Therefore the MGF of $\sigma e^{-kt}\int_0^t e^{ks}dB_s$ with an argument of 1 is

$$e^{\frac{1}{2}\frac{\sigma^2}{2k}(1-e^{-2kt})} = e^{\frac{\sigma^2(1-e^{-2kt})}{4k}}$$

so we therefore have

$$\mathbb{E}X_t = \exp\Big\{e^{-\kappa t}\log x + (\alpha - \frac{\sigma^2}{2\kappa})(1-e^{-\kappa t}) + \frac{\sigma^2(1-e^{-2kt})}{4k}\Big\}$$

$\square$