Homework 4

PSTAT 223A
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Problem 7.1

Find the generator of the following Itô diffusions. Note that $f \in \mathcal{C}_b^2$ in all cases (twice continuously differentiable and bounded).

(i) $dX_t = \mu X_t dt + \sigma dB_t$

Proof. We know $b(X_t, t) = \mu X_t$ and $\sigma(X_t, t) = \sigma$, so our generator is:

$$Af(x) = \mu x \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{1}{2}\sigma^2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$$

$\square$

(ii) $dX_t = rX_t dt + \alpha X_t dB_t$

Proof. We have $b(X_t, t) = rX_t$ and $\sigma(X_t, t) = \alpha X_t$ so

$$Af(x) = rx \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{x^2\alpha^2}{2} \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$$

$\square$

(iii) $dY_t = rdt + \alpha Y_t dB_t$

Proof. We have $b(t, X_t) = r$ and $\sigma(t, X_t) = \alpha Y_t$ so

$$Af(x) = r \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\alpha^2 x^2}{2} \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$$

$\square$

(iv) $dY_t = \begin{bmatrix} dt \\ dX_t \end{bmatrix}$ where X_t is as in (i)

Proof. Note that

$$\begin{bmatrix} dt \\ dX_t \end{bmatrix} = \begin{bmatrix} 1 \\ \mu X_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} dB_t$$

so letting $x_2 = x$ and $x_1 = t$:

$$\begin{aligned} Af(x) &= \frac{\mathrm{d}f}{\mathrm{d}x_1} + \mu x \frac{\mathrm{d}f}{\mathrm{d}x_2} + \frac{1}{2}\sigma^2 \frac{\mathrm{d}^2 f}{\mathrm{d}{x_2}^2} \\ &= \frac{\mathrm{d}f}{\mathrm{d}t} + \mu x \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{1}{2}\sigma^2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \end{aligned}$$

□

(v) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$

Proof.

$$Af(x) = \frac{\mathrm{d}f}{\mathrm{d}X_1} + X_2 \frac{\mathrm{d}f}{\mathrm{d}X_2} + \frac{1}{2} e^{X_1} \frac{\mathrm{d}^2 f}{\mathrm{d}{X_2}^2}$$

□

(vi) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$

Proof.

$$Af(x) = \frac{\mathrm{d}f}{\mathrm{d}X_1} + \frac{1}{2} \frac{\mathrm{d}^2 f}{\mathrm{d}{X_1}^2} + \frac{1}{2} X_1^2 \frac{\mathrm{d}^2 f}{\mathrm{d}{X_2}^2}$$

□

(vii) $X_t = (X_1, X_2, \ldots, X_n)$ where

$$dX_k(t) = r_k X_k dt + X_k \cdot \sum_{j=1}^{n} \alpha_{kj} dB_j; \quad 1 \le k \le n$$

Proof.

$$Af(x) = \sum_{k=1}^{n} r_k X_k \frac{\mathrm{d}f}{\mathrm{d}X_k} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j \big(\sum_{k=1}^{n} \alpha_{ik} \alpha_{jk} \big) \frac{\mathrm{d}^2 f}{\mathrm{d}x_i \mathrm{d}x_j}$$

□

Problem 7.2

Find the Itô diffusion whose generator is the following:

(i) $Af(x) = f'(x) + f''(x); f \in C_0^2(\mathbf{R})$

Proof.

$$Af(x) = b(X_t)\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{1}{2}\sigma^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$$

so $b(X_t) = 1$ and $\sigma(X_t)^2 = 2$, so

$$dX_t = dt + \sqrt{2}dB_t$$

□

(ii) $Af(t,x) = \frac{\partial f}{\partial t} + cx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2}; f \in C_0^2\left(\mathbf{R}^2\right)$ where c, α are constants.

Proof. let $b(x_1) = 1$ and $b(x_2) = cx$, and $\sigma = \alpha x$ where $x_1 = t$ and $x_2 = x$. Then:

$$\begin{bmatrix}\mathrm{d}X_1\\ \mathrm{d}X_2\end{bmatrix} = \begin{bmatrix}1\\ cx_2\end{bmatrix} dt + \begin{bmatrix}0\\ \alpha x_2\end{bmatrix} dB_t$$

□

(iii) $Af(x_1,x_2) = 2x_2\frac{\partial f}{\partial x_1} + \ln\left(1+x_1^2+x_2^2\right)\frac{\partial f}{\partial x_2} + \frac{1}{2}\left(1+x_1^2\right)\frac{\partial^2 f}{\partial x_1^2} + x_1\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x_2^2}; \quad f \in C_0^2\left(\mathbf{R}^2\right)$

Proof. Translating the above two-dimensional process into the Itô diffusion gives us:

$$\begin{bmatrix}\mathrm{d}X_1\\ \mathrm{d}X_2\end{bmatrix} = \begin{bmatrix}2X_2\\ \log(1+X_1^2+X_2^2)\end{bmatrix} dt + \begin{bmatrix}X_1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}dB_1\\ dB_2\end{bmatrix}$$

□

Problem 7.4

Let B_t^x be a 1-dimensional Brownian Motion starting at $x \in \mathbb{R}_+$. Put $\tau = \inf\{t > 0; B_t^x = 0\}$.

(i) Prove $\tau < \infty$ a.s. $\mathcal{P}^x$ for all $x > 0$.

Proof. Let $0 < x < k$ for some k. Let $\tau_k = \inf\{t > 0; B_t = 0 \text{ or } B_t = k\}$. τ_k is an exit time, so $\mathcal{P}^x(\tau_k < \infty) = 1$. We apply Dynkin's formula to $f(x) = x$, have $Af(x) = 0$ and let $\mathcal{P}^x(X_{\tau_k} = k) = p_k$:

$$\begin{aligned}\mathbb{E}^x(X_{\tau_k}) &= x\\ X_{\tau_k}p_k + 0(1-p_k) &= x\\ p_k &= \frac{x}{X_{\tau_k}} = \frac{x}{k}\end{aligned}$$

So

$$\begin{aligned}\mathcal{P}^x(\tau < \infty) &= \lim_{k\longrightarrow\infty}\left(1 - \mathcal{P}^x(\tau_k = k)\right) = \lim_{k\longrightarrow\infty}\left(1 - p_k\right)\\ &= \lim_{k\longrightarrow\infty}\left(1 - \frac{x}{k}\right) = 1\end{aligned}$$

So $\tau < \infty$ a.s. □

(ii) Prove that $\mathbb{E}^x(\tau_k) = \infty$ for all $x > 0$

Proof. We use the same exit time formulation as in the previous part and apply Dynkin's formula to $f(x) = x^2$, and $Af(x) = \frac{1}{2}2 = 1$:

$$\begin{aligned} \mathbb{E}^x(X_{\tau_k}^2) &= x^2 + \mathbb{E}^x\Big(\int_0^{\tau_k} Af(X_s)ds\Big) \\ &= x^2 + \mathbb{E}^x(\tau_k) \\ 0\mathcal{P}^x(X_{\tau_k} = 0) + k^2\mathcal{P}^x(X_{\tau_k} = k) &= x^2 + \mathbb{E}^x(\tau_k) \\ \text{(letting } \mathcal{P}^x(X_{\tau_k} = k) = p_k) \qquad \mathbb{E}^x(\tau_k) &= k^2 p_k - x^2 \end{aligned}$$

Combining with p_k derived in the previous part:

$$\begin{aligned} \mathbb{E}^x(\tau_k) &= k^2\frac{x}{k} - x^2 = kx - x^2 \\ \mathbb{E}^x(\tau) &= \lim_{k\longrightarrow\infty} \mathbb{E}^x(\tau_k) = \infty \end{aligned}$$

as expected. □

Problem 7.9

Let X_t be a geometric Brownian Motion, i.e.

$$dX_t = rX_t dt + \alpha X_t dB_t, \quad X_0 = x > 0,$$

$B_t \in \mathbb{R}$; r, α are constants.

(i) Find the generator A of X_t and compute $Af(x)$ when $f(x) = x^\gamma$; $x > 0$, γ constant.

Proof.

$$\begin{aligned} Af(x) &= rx\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{1}{2}\alpha^2 x^2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \\ A(x^\gamma) &= rx\gamma x^{\gamma-1} + \frac{1}{2}ga^2x^2\gamma(\gamma-1)x^{\gamma-2} \\ &= x^\gamma\Big(r\gamma + \frac{1}{2}\alpha^2(\gamma^2 - \gamma)\Big) \end{aligned}$$

□

(ii) If $r < \frac{1}{2}\alpha^2$ then $X_t \longrightarrow 0$ as $t \longrightarrow \infty$, a.s. Q^x, but what is the probability p that X_t, when starting from $x < R$ ever hits R?

Proof. We will apply Dynkin's Formula with $f(x) = x^{\gamma_1}$ where $\gamma_1 = 1 - \frac{2r}{\alpha^2}$. Note that solving the above SDE defining the Geometric Brownian Motion with $X_0 = x$ gives us:

$$X_t = xe^{\left(r - \frac{\alpha^2}{2}\right)t + \alpha B_t}$$

Applying our known value of γ_1 to the generator for x^{γ_1} gives us:

$$\begin{aligned}
A(x^{\gamma_1}) &= x^{\gamma_1}\Big(r(1-\frac{2r}{\alpha^2}) + \frac{1}{2}\alpha^2(1-\frac{2r}{\alpha^2})(-\frac{2r}{\alpha^2})\Big) \\
&= x^{\gamma_1}\Big(r(1-\frac{2r}{\alpha^2}) + \frac{1}{2}\alpha^2(-\frac{2r}{\alpha^2}+\frac{4r^2}{\alpha^4})\Big) \\
&= x^{\gamma_1}0 = 0
\end{aligned}$$

Now, define $\tau_R = \inf\{t > 0; X_t = 0 \text{ or } X_t = R\}$. This is an exit time, and $0 < X_0 = x < R$, so $\mathcal{P}^x(\tau_R < \infty) = 1$. Putting this together, we get:

$$\begin{aligned}
\mathbb{E}^x(X_{\tau_R}^{\gamma_1}) &= x^{\gamma_1} \\
0\mathcal{P}^x(X_{\tau_R} = 0) + R^{\gamma_1}\mathcal{P}^x(X_{\tau_R} = R) &= x^{\gamma_1} \\
p_R = \mathcal{P}^x(X_{\tau_R} = R) &= \Big(\frac{x}{R}\Big)^{\gamma_1}
\end{aligned}$$

as expected. □

(iii) If $r > \frac{1}{2}\alpha^2$ then $X_t \xrightarrow{t\longrightarrow\infty} \infty$. Let $\tau = \inf\{t > 0; X_t \geq R\}$. Use Dynkin's formula with $f(x) = \log x$, $x > 0$ to prove that

$$E^x[\tau] = \frac{\ln\frac{R}{x}}{r - \frac{1}{2}\alpha^2}$$

Proof. Let $\tau_\rho = \inf\{t > 0 X_t = R \text{ or } X_t = \rho\}$ where $0 < \rho < x = X_0 < R$. We therefore have that τ_ρ is an exit time and $\mathcal{P}^x(\tau_\rho < \infty) = 1$. Applying the the diffusion generator to $\log x$ and integrating that from 0 to τ_ρ gives us:

$$\begin{aligned}
A\log(X_s) &= r - \frac{1}{2}a^2 \\
\int_0^{\tau_\rho} A\log(X_s)ds &= \int_0^{\tau_\rho}\Big(r - \frac{1}{2}a^2\Big)ds = \tau_\rho\big(r - \frac{1}{2}a^2\big)
\end{aligned}$$

Applying Dynkin's formula gives to $\log X_{\tau_\rho}$ gives us:

$$\begin{aligned}
\mathbb{E}^x(\log(X_{\tau_\rho})) &= \log(x) + \mathbb{E}^x\Big(\int_0^{\tau_\rho} A\log(X_s)ds\Big) \\
&= \log(x) + \big(r - \frac{1}{2}a^2\big)\mathbb{E}^x(\tau_\rho) \\
\mathcal{P}^x(\log(X_{\tau_\rho}) = \rho)\log\rho + \mathcal{P}^x(\log(X_{\tau_\rho}) = R)\log R &= \log(x) + \big(r - \frac{1}{2}a^2\big)\mathbb{E}^x(\tau_\rho).
\end{aligned}$$

Let $p_R = \mathcal{P}^x(X_{\tau_\rho}) = R)$ so $\mathcal{P}^x(\log(X_{\tau_\rho}) = \rho) = 1 - p_R$. We now have:

$$\begin{aligned}
(1-p_R)\log\rho + p_R\log R &= \log(x) + \big(r - \frac{1}{2}\alpha^2\big)\mathbb{E}^x(\tau_\rho) \\
\mathbb{E}^x(\tau_\rho) &= \frac{p_R(\log R - \log\rho) + \log\rho - \log x}{r - \frac{1}{2}\alpha^2} \\
&= \frac{p_R\log R + (1-p_R)\log\rho - \log x}{r - \frac{1}{2}\alpha^2}
\end{aligned}$$

From the previous part, we know:

$$\mathbb{E}(X_{\tau_\rho}^{\gamma_1}) = p_R R^{\gamma_1} + (1-p_R)\rho^{\gamma_1} = x^{\gamma_1}$$

so

$$p_R = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}$$

and

$$1 - p_R = \frac{R^{\gamma_1} - x^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}$$

Taking

$$\begin{aligned}
\lim_{\rho\longrightarrow 0}(1-p_R)\log\rho &= \lim_{\rho\longrightarrow 0}\frac{R^{\gamma_1} - x^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}\log\rho \\
(\text{ L'Hopital's Rule}) &= \lim_{\rho\longrightarrow 0}\frac{R^{\gamma_1} - x^{\gamma_1}}{-\gamma_1\rho^{\gamma_1-1}}\rho^{-1} \\
&= \lim_{\rho\longrightarrow 0} -\frac{R^{\gamma_1} - x^{\gamma_1}}{\gamma_1\rho^{\gamma_1}} \\
(\text{Note that } \gamma_1 < 0 \text{ so } -\gamma_1 > 0) &= \lim_{\rho\longrightarrow 0}\frac{-\rho^{-\gamma_1}}{\gamma_1} = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{\rho\longrightarrow 0} p_R &= \lim_{\rho\longrightarrow 0}\frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}} \\
&= \lim_{\rho\longrightarrow 0}\frac{x^{\gamma_1}\rho^{-\gamma_1} - 1}{R^{\gamma_1}\rho^{-\gamma_1} - 1} = 1,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^x(\tau_\rho) &= \frac{p_R\log R + (1-p_R)\log\rho - \log x}{r - \frac{1}{2}\alpha^2} \\
&\xrightarrow{\rho\longrightarrow 0} \frac{\log R - \log x}{r - \frac{1}{2}\alpha^2} \\
&= \frac{\log\left(\frac{R}{x}\right)}{r - \frac{1}{2}\alpha^2} = \mathbb{E}^x(\tau)
\end{aligned}$$

as expected. $\square$

Problem 7.18

(i) Let

$$dX_t = b\left(X_t\right)dt + \sigma\left(X_t\right)dB_t; \quad X_0 = x$$

be a 1-dimensional Itô diffusion witch characteristic operator $\mathcal{A}$. Let $f \in \mathcal{C}^2(\mathbb{R})$ be a solution to the differential equation

$$\mathcal{A}f(x) = b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) = 0; \quad x \in \mathbf{R}$$

let $(a,b) \subset \mathbb{R}$ be an open interval such that $x \in (a,b)$ and put

$$\tau = \inf\{t > 0; X_t \notin (a,b)\}.$$

Assume that $\tau < \infty$ a.s. $\mathcal{Q}^x$ and define

$$p = \mathcal{P}^x[X_\tau = b].$$

Use Dynkin's formula to prove that

$$p = \frac{f(x) - f(a)}{f(b) - f(a)}$$

Proof. We know by definition, $\mathcal{A}f(x) = 0$, and τ is an exit time, so $\mathcal{P}^x(\tau < \infty) = 1$. Applying Dynkin's formula, and noting the expectation on the RHS is 0:

$$\begin{aligned}
\mathbb{E}^x(f(X_\tau)) &= f(x) + 0 \\
f(a)\mathcal{P}^x(X_\tau = a) + f(b)\mathcal{P}^x(X_\tau = b) &= f(x) \\
f(a)(1-p) + f(b)p &= f(x) \\
p &= \frac{f(x) - f(a)}{f(b) - f(a)}
\end{aligned}$$

as expected. □

(ii) Now specialize to the process

$$X_t = x + B_t; \quad t \geq 0.$$

Prove that

$$p = \frac{x-a}{b-a}$$

Proof. $X_t = x + B_t$. Note that $dX_t = dB_t$, so this is a Brownian Motion (that starts at $x \neq 0$). We therefore have $\mathcal{A}f(x) = \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$. Letting $f(x) = x$ gives us $\mathcal{A}f(x) = \mathcal{A}x = 0$. We therefore have:

$$\begin{aligned}
x &= \mathbb{E}^x(X_\tau) = bp + a(1-p) \\
p &= \frac{x-a}{b-a}
\end{aligned}$$

as expected □

(iii) Find p if

$$X_t = x + \mu t + \sigma B_t; \quad t \geq 0$$

where $\mu, \sigma \in \mathbb{R}$ are nonzero constants.

Proof. $X_t = x + \mu t + B_t$ so $dX_t = \mu dt + dB_t$. We therefore have $\mathcal{A}f(x) = \mu\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$. Finding a solution to $\mathcal{A}f(x) = 0$ gives us the straightforward solution to a homogeneous ODE given by:

$$f(x) = e^{\left(\frac{-2\mu x}{\sigma^2}\right)} + c$$

(We let $c = 0$ w.l.o.g.) Applying Dynkin's formula gives us:

$$\mathbb{E}^x(f(X_\tau)) = f(x) + 0$$
$$(1-p)e^{\left(\frac{-2\mu a}{\sigma^2}\right)} + pe^{\left(\frac{-2\mu b}{\sigma^2}\right)} = e^{\left(\frac{-2\mu x}{\sigma^2}\right)}$$

solving for p gives us:

$$p = \frac{e^{\left(\frac{-2\mu x}{\sigma^2}\right)} - e^{\left(\frac{-2\mu a}{\sigma^2}\right)}}{e^{\left(\frac{-2\mu b}{\sigma^2}\right)} - e^{\left(\frac{-2\mu a}{\sigma^2}\right)}}$$

□