Homework 4

PSTAT 223A

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Problem 5.16

Consider the general nonlinear SDE of the form

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t \quad X_0 = x$$

where $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $c : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous deterministic functions.

(i) Define the integrating factor as:

$$F_t = F_t(\omega) = \exp\left(-\int_0^t c(s)dB_s + \frac{1}{2}\int_0^t c^2(s)ds\right)$$

and show that the SDE can be written as:

$$d(F_t X_t) = F_t f(t, X_t)dt$$

Proof. Use the ansatz the F_t is an ıto process, i.e. $dF_t = a(t, F_t)dt + b(t, F_t)dB_t$. Applying the integration by parts formula:

$$\begin{aligned} d(F_t X_t) &= X_t dF_t + F_t dX_t + dX_t dF_t \\ &= X_t(a(t, F_t)dt + b(t, F_t)dB_t + F(t)(f(t, X_t)dt + +c(t)X_t dB_t + b(t, F_t)c(t)X_t dt \end{aligned}$$

by assumption, $a(t, F_t)X_t + b(t, F_t)c(t)X_t = 0$ and $b(t, F_t)X_t + c(t)X_t F_t = 0$. Therefore, $b(t, F_t) = -c(t)F_t$ and $a(t, F_t) = c^2(t)F_t$, so

$$dF_t = c^2(t)F_t dt - c(t)F_t dB_t$$

. Solving by taking $d(\log F_t)$, we get

$$F_t = e^{-\int_0^t c(s)dB_s + \int_0^t \frac{c^2(s)}{2}ds}.$$

because $F_0 = 1$. □

(ii) Define $Y_t = F_t X_t$ so that $X_t = F_t^{-1} Y_t$. Note that the previous deterministic differential equation gives us the form

$$\frac{dY_t}{dt} = F_t f(t, F_t^{-1} Y_t)$$

.

Proof. Note that $Y_t = F_t X_t$, and so $X_t = F_t^{-1} Y_t$. By the previous part,

$$\frac{dY_t}{dt} = F_t f(t, X_t) = F_t f(t, F_t^{-1} Y_t)$$

Note that $Y_0 = F_0 X_0$ and $F_0 = 1$, so $Y_0 = X_0 = x$ as expected. $\square$

(iii) Solve the SDE

$$dX_t = \frac{1}{X_t} dt + \alpha X_t dB_t; \quad X_0 = x > 0$$

where α is a constant.

Proof. Let

$$\begin{aligned} Y_t = F_t X_t &= X_t e^{-\int_0^t \alpha dB_s + \frac{1}{2}\int_0^2 \alpha^2 ds} \\ &= X_t e^{-\alpha B_t + \frac{\alpha^2 t}{2}} \end{aligned}$$

Separating variables and integrating, we get

$$Y_t = \left(2 \int_0^t e^{-2\alpha B_s + a^2 s} ds + Y_0^2\right)^{\frac{1}{2}}$$

so, solving for X_t we get:

$$X_t = e^{\alpha B_t - \frac{\alpha^2 t}{2}} \left(x^2 + 2 \int_0^t e^{-2\alpha B_s + \alpha^2 s} ds\right)^{\frac{1}{2}}$$

$\square$

(iv) Apply this method to study the solutions of the SDE:

$$dX_t = X_t^\gamma dt + \alpha X_t dB_t; \quad X_0 = x > 0$$

where α and γ are constants. What what value of γ do we get explosions?

Proof. Using the same method as previously, define $F_t = e^{-\alpha B_t + \frac{\alpha^2 t}{2}}$ so

$$f(t, F_t^{-1} Y_t) = \left(e^{\alpha B_t - \frac{\alpha^2 t}{2}} Y_t\right)^\gamma$$

We therefore have:

$$\int_0^t \frac{1}{Y_s^\gamma} dY_t = \frac{1}{-\gamma + 1}\left(Y_t^{-\gamma+1} - Y_0^{\gamma+1}\right) = \int_0^t e^{\alpha B_g - \frac{\alpha^2 t}{2}} dt$$

which blows up if $\gamma = 1$. However, if $\gamma = 1$, this is a known Geometric Brownian Motion that can be solved by taking $d \log X_t$ in a more traditional solution method, yielding

$$X_t = X_0 e^{-(\frac{\alpha^2}{2} - 1)dt + \alpha \int_0^t dB_s}.$$

$\square$

Problem 6.2

In the linear filtering problem, with $C(t) = 0$ and $S_t = S(t) = \mathbb{E}[(X_t - \hat{X}_t)^2]$ and $S(0) > 0$

(i) Show that:

$$R(t) := \frac{1}{S(t)}$$

satisfies the *linear* differential equation

$$R'(t) = -2F(t)R(t) + \frac{G^2(t)}{D^2(t)}; \quad R(0) = \frac{1}{S(0)}$$

Proof. We know that

$$\begin{aligned}
\frac{dS_t}{dt} &= 2F(t)S_t - \left(\frac{G(t)S_t}{D(t)}\right)^2 \qquad \text{so} \\
\frac{dR_t}{dt} &= \frac{d}{dt}\frac{1}{S_t} = -\frac{1}{S_t^2}\frac{d}{dt}S_t = -\frac{1}{S_t^2}(2F(t)S_t - \frac{G(t)^2S_t^2}{D_t^2} \\
&= -2F(t)R(t) - \frac{G(t)^2}{D(t)^2}
\end{aligned}$$

and $R_0 = \frac{1}{S_0} > 0$. $\square$

(ii) Prove that for the filtering problem given by Øksendal 6.3.8 and 6.3.9 we have

$$\frac{1}{S(t)} = \frac{1}{S(0)}\exp\left(-2\int_0^t F(s)ds\right) + \int_0^t \exp\left(-2\int_s^t F(u)du\right)\frac{G^2(s)}{D^2(s)}ds$$

Proof. We introduce

$$e^{\left(2\int_0^t F(s)ds\right)}$$

as an integrating factor. Multiplying the above equation by that and moving around factors, we get:

$$\begin{aligned}
\frac{dR(t)}{dt}e^{\left(2\int_0^t F(s)ds\right)} + 2F(t)R(t)e^{\left(2\int_0^t F(s)ds\right)} &= \frac{G^2(t)}{D^2(t)}e^{\left(2\int_0^t F(s)ds\right)} \\
\frac{d}{dt}\left(R(t)e^{\left(2\int_0^t F(s)ds\right)}\right) &= \frac{G^2(t)}{D^2(t)}e^{\left(2\int_0^t F(s)ds\right)}
\end{aligned}$$

Integrating both sides, we get:

$$\begin{aligned}
R(t)e^{\left(2\int_0^t F(s)ds\right)} - R(0) &= \int_0^t \frac{G^2(s)}{D^2(s)}e^{\left(2\int_0^s F(r)dr\right)}ds \\
R(t) &= e^{\left(-2\int_0^t F(s)ds\right)}\left(R(0) + \int_0^t \frac{G^2(s)}{D^2(s)}e^{\left(2\int_0^s F(r)dr\right)}ds\right) \\
&= e^{\left(-2\int_0^t F(s)ds\right)}R(0) + \int_0^t e^{-2\left(\int_0^t F(r)dr - \int_0^s F(r)dr\right)}\frac{G^2(s)}{D^2(s)}ds \\
&= e^{\left(-2\int_0^t F(s)ds\right)}R(0) + \int_0^t e^{-2\left(\int_s^t F(r)dr\right)}\frac{G^2(s)}{D^2(s)}ds \\
\frac{1}{S(t)} &= e^{\left(-2\int_0^t F(s)ds\right)}\frac{1}{S(0)} + \int_0^t e^{-2\left(\int_s^t F(r)dr\right)}\frac{G^2(s)}{D^2(s)}ds
\end{aligned}$$

as expected. ∎

Problem 6.5

Prove that in the linear setup (6.2.3), (6.2.4) the predicted value

$$\mathbb{E}[X_T|\mathcal{G}_t], \qquad T > t$$

is given by

$$E\left[X_T|\mathcal{G}_t\right] = \exp\left(\int_t^T F(s)ds\right) \cdot \widehat{X}_t$$

Proof.

$$\begin{aligned}
\mathbb{E}[X_T|\mathcal{G}_t] &= \mathbb{E}\Big\{ \exp\big(\int_t^T F(s)ds\big)X_t + \int_t^T \exp\big(\int_s^T F(r)dr\big)C(s)dU_s\Big|\mathcal{G}_t\Big\} \\
&= \mathbb{E}\Big\{ \exp\big(\int_t^T F(s)ds\big)X_t\Big|\mathcal{G}_t\Big\} \\
&= \mathbb{E}\{X_t|\mathcal{G}_t\} \exp\big(\int_t^T F(s)ds\big) \\
&= \hat{X}_t \exp\big(\int_t^T F(s)ds\big)
\end{aligned}$$

as expected. ∎

Problem 6.8b

Transform the following Stratonovich equation into the Itô version.

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \circ dB_t \quad (B_t \in \mathbf{R})$$

Proof. We need to derive $\tilde{b}_1(t,x)$ and $\tilde{b}_1(t,x)$.

$$\begin{aligned}
\tilde{b}_1(t,x) &= b_1(t,x) + \frac{1}{2}\Big(\frac{d\sigma_{11}}{dx_1}\sigma_{11} + \frac{d\sigma_{12}}{dx_1}\sigma_{12} + \frac{d\sigma_{11}}{dx_2}\sigma_{21} + \frac{d\sigma_{12}}{dx_2}\sigma_{22}\Big) \\
&= b_1(t,x) + \frac{1}{2}X_1
\end{aligned}$$

Similarly,

$$\tilde{b}_2(t,x) = b_2(t,x) + \frac{X_2}{2}.$$

We therefore have:

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} dB_t$$

∎

Problem 6.9b

Transform the following Itô equation into the Stratonovich version.

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} X_1 & -X_2 \\ X_2 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$$

Proof. Note that $\tilde{b}_i(t, x) = 0$. We need to derive $b_1(t, x)$ and $b_2(t, x)$.

$$\begin{aligned} b_1(t, x) &= -\frac{1}{2}\Big(\frac{d\sigma_{11}}{dx_1}\sigma_{11} + \frac{d\sigma_{12}}{dx_1}\sigma_{12} + \frac{d\sigma_{11}}{dx_2}\sigma_{21} + \frac{d\sigma_{12}}{dx_2}\sigma_{22}\Big) \\ &= -\frac{1}{2}(X_1 + 0 + 0 - X_1) = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} b_2(t, x) &= -\frac{1}{2}\Big(\frac{d\sigma_{21}}{dx_1}\sigma_{11} + \frac{d\sigma_{22}}{dx_1}\sigma_{12} + \frac{d\sigma_{21}}{dx_2}\sigma_{21} + \frac{d\sigma_{22}}{dx_2}\sigma_{22}\Big) \\ &= -\frac{1}{2}(-X_2 + X_2) = 0, \end{aligned}$$

so our ultimate result is

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} X_1 & -X_2 \\ X_2 & X_1 \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}.$$

□

Problem 6.15

Suppose $X_t \in \mathbb{R}$ at time t is a geometric Brownian Motion given by the equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t; \quad X_0 = x > 0$$

where $\sigma \neq 0$ and x are known constants. The parameter μ is also constant, but we do not know its value, only its probability distribution, which is assumed to be normal with mean $\bar{\mu}$ and variance a^2. We assume that μ is independent of $\{B_s\}_{s\geq 0}$ and $\mathbb{E}[\mu^2] < \infty$.
We assume that we can observe the value of X_t for all t. Thus we have access to the information "(σ -algebra)" $\mathcal{M}_t$ generated by X_s; $s \leq t$. Let $\mathcal{N}_t$ be the σ-algebra generated by ξ_t $s \leq t$. where

$$d\xi_t = \mu dt + \sigma dB_t; \quad \xi_0 = x$$

(i) Prove $\mathcal{M}_t = \mathcal{N}_t$

Proof.

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dB_t \\ \frac{dX_t}{X_t} &= \mu dt + \sigma dB_t \end{aligned}$$

Let

$$\begin{aligned}
F_t &= e^{-\int_0^t \sigma dB_s + \frac{1}{2}\int_0^t \sigma^2 ds} \\
Y_t &= F_t X_t \\
X_t &= \frac{Y_T}{F_t} \quad \text{so } \frac{dY_t}{dt} \qquad\qquad\qquad = \mu Y_t e^{-\int_0^t \sigma dB_s + \frac{1}{2}\int_0^t \sigma^2 ds + \int_0^t \sigma dB_s - \frac{1}{2}\int_0^t \sigma^2 ds} \\
&= \mu Y_t
\end{aligned}$$

Solving and substituting X_t back in:

$$X_t = X_0 e^{\int_0^t \sigma dB_s + \mu t - \frac{1}{2}\sigma^2 t}$$

solving for ξ_t:

$$\xi_t = \xi_0 + \mu_t + \sigma \int_0^t dB_s$$

where ξ_0 is a known constant. We therefore have that

$$X_t = X_0 e^{\xi_t - \frac{\sigma^2 t}{2} - \xi_0}$$

Similarly, inverting this, we get that $\xi_t = \frac{\log X_t}{X_0} + \xi_0 \frac{\sigma^2 t}{2}$ and because these variables are both measurable functions of each other, their generated Sigma Algebras are the same, i.e. $\mathcal{M}_t \subseteq \mathcal{N}_t$ and $\mathcal{N}_t \subseteq \mathcal{M}_t$, so $\mathcal{M}_t = \mathcal{N}_t$. □

(ii) Prove that

$$E[\mu|\mathcal{N}_t] = \left(\theta + \sigma^{-2}t\right)^{-1}\left(\overline{\mu}\theta + \sigma^{-2}\xi_t\right)$$

where

$$\theta = E\left[(\mu - \overline{\mu})^2\right]^{-1}, \quad \overline{\mu} = E[\mu]$$

Proof. Following example 6.2.9 from Øksendal, we know that $\mu_0 = \bar{\mu}$ because the distribution of the dynamics is constant. Also, note that $a^2 = \frac{1}{\theta}$ We therefore have:

$$\begin{aligned}
\hat{\mu}_t = \mathbb{E}[\mu|\mathcal{N}_t] &= \frac{\sigma^2 \mu_0}{\sigma^2 + a^2 t} + \frac{a^2 \xi_t}{\sigma^2 + a^2 t} \\
&= \frac{\sigma^2 \bar{\mu}}{\sigma^2 + a^2 t} + \frac{a^2 \xi_t}{\sigma^2 + a^2 t} \\
&= (\bar{\mu}\theta + \xi_t \sigma^{-2})(\theta + \sigma^{-2}t)^{-1}
\end{aligned}$$

as expected. □

(iii) Define

$$\widetilde{B}_t = \int_0^t \sigma^{-1}(\mu - E[\mu|\mathcal{M}_s])ds + B_t$$

Prove that $\widetilde{B}_t$ is a Brownian motion.

Proof. Define $N_t = \xi_t - \int_0^t \hat{\mu}_s ds$. Then

$$dN_t = d\xi_t - \hat{\mu}_t dt.$$

Define

$$\begin{aligned} dR_t &= \frac{1}{\sigma} dN_t = \frac{1}{\sigma}[\mu dt + \sigma dB_t - \hat{\mu}_t dt] \\ \Rightarrow R_t &= \frac{1}{\sigma}\int_0^t \mu - \mathbb{E}[\mu | M_s] ds + B_t & = \widetilde{B}_t \end{aligned}$$

By Øksendal 6.2.6, $R_t = \widetilde{B}_t$ is a Brownian Motion. $\square$

(iv) Prove that $\widetilde{B}_t$ is $\mathcal{M}_t$-measurable for all t. Hence

$$\widetilde{\mathcal{F}}_t \subseteq \mathcal{M}_t$$

where $\widetilde{\mathcal{F}}_t$ is the σ-algebra generated by $\widetilde{B}_s$ $s \leq t$.

Proof. Note that

$$\widetilde{B}_t = \frac{1}{\sigma}\Big(\xi_t - \int_0^t \hat{\mu}_s ds\Big)$$

which is a measurable function of ξ_t and therefore $\mathcal{N}_t$ measurable, which, by **a)** means that it is also $\mathcal{M}_t$ measurable. $\square$

(v) Prove that ξ_t is $\widetilde{\mathcal{F}}_t$- measurable for all t. Combined with **d)** and **a)** this gives that

$$\widetilde{\mathcal{F}}_t = \mathcal{M}_t = \mathcal{N}_t = \mathcal{F}_t$$

Proof. Note that

$$d\xi_t = \sigma d\widetilde{B}_t + \frac{\sigma^2 \bar{\mu}\theta}{\sigma^2\theta + t} dt + \frac{\xi_t}{\sigma^2\theta + t} dt$$

Taking $\frac{1}{\sigma^2\theta + t}$ as an integrating factor, we get:

$$\begin{aligned} d\Big(\frac{\xi_t}{\sigma^2\theta + t}\Big) &= -\xi_t\Big(\frac{\sigma^2\theta}{(\sigma^2\theta + t)^2}\Big)dt + \frac{1}{\sigma^2\theta + t} d\xi_t \\ &= \frac{1}{\sigma^2\theta + t}\Big(\sigma d\widetilde{B}_t + \frac{\bar{\mu}\sigma^2}{\sigma^2\theta + t}\Big) \end{aligned}$$

Note Solving for ξ_t, we get:

$$\begin{aligned} \xi_t &= \frac{\xi_0}{\sigma^2\theta} + \int_0^t \frac{\bar{\mu}\sigma^2\theta}{(\sigma^2\theta + s)^2} ds + \int_0^t \frac{1}{\sigma\theta + s} dB_s \\ &= -\bar{\mu}\sigma^2\theta\Big(\frac{1}{\sigma^2\theta + t} - \frac{1}{\sigma^2\theta}\Big) + \sigma\int_0^t \frac{d\widetilde{B}_s}{\sigma^2\theta + s} \\ &= \bar{\mu} - \frac{\bar{\mu}\sigma^2\theta}{\sigma^2\theta + t} + \sigma\int_0^t \frac{d\widetilde{B}_s}{\sigma^2\theta + s} \end{aligned}$$

which is an $\widetilde{\mathcal{F}}_t$-measurable function (everything is either constant or a stochastic integral of $\widetilde{B}_t$). Therefore ξ_t is $\widetilde{\mathcal{F}}_t$-measurable, and the given equality of filtrations holds. $\square$

(vi) Prove that

$$dX_t = E[\mu|\mathcal{M}_t]X_t dt + \sigma X_t d\widetilde{B}_t$$

Proof. We have:

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dB_t \\ &= X_t d\xi_t \\ &= X_t(\sigma d\widetilde{B}_t + \hat{\mu}_t dt) \\ &= \hat{\mu}_t X_t dt + \sigma X_t d\widetilde{B}_t \\ &= \mathbb{E}(\mu|\mathcal{M}_t) X_t dt + \sigma X_t d\widetilde{B}_t \end{aligned}$$

as expected. □