Homework 2

PSTAT 223A
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Problem 1 (3.1)

Prove directly from the definition of Itô Integrals that

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds.$$

Proof. Letting $\Delta B_j = B_{j+1} - B_j$ and $\Delta t_j = t_{j+1} - t_j$.
Using the hint and our in-class notation:

$$\sum_{j=0}^{t} \Delta(s_j B_j) = \sum_{j=0}^{t} \Delta s_j \Delta B_j = \sum_{j=0}^{t} s_j \Delta B_j + \sum_{j=0}^{t} B_{j+1} \Delta s_j$$

Taking the limit $\Delta s \longrightarrow 0$, first note:

$$\lim_{\Delta s \longrightarrow 0} \sum_{j=0}^{t} \Delta s_j \Delta B_j = tB_t$$

and therefore:

$$\begin{aligned} tB_t = \lim_{\Delta s \longrightarrow 0} \sum_{j=0}^{t} \Delta s_j \Delta B_j &= \lim_{\Delta s \longrightarrow 0} \Big(\sum_{j=0}^{t} s_j \Delta B_j + \sum_{j=0}^{t} B_{j+1} \Delta s_j \Big) \\ &= \int_0^t s dB_s + \int_0^t B_s \, ds \end{aligned}$$

Therefore, rearranging terms:

$$\int_0^t s dB_s = tB_t - \int_0^t B_s \, ds$$

as expected. $\square$

Problem 2 (3.4)

Check whether the following processes are martingales w.r.t. $\{\mathcal{F}_t\}$:

(i) $X_t = B_t + 4t$

- $\mathbb{E}|X_t| = \mathbb{E}|B_t + 4t| \leq \mathbb{E}|B_t| + 4t < \infty \; \forall t < \infty$.
- $X_t \in \mathcal{F}_t$ because $4t$ is a constant and $B_t \in \mathcal{F}_t$.
- Let $s < t$. Then: $\mathbb{E}\{X_t|\mathcal{F}_s\} = \mathbb{E}\{B_t + 4t|\mathcal{F}_s\} = 4t + \mathbb{E}\{B_t|\mathcal{F}_s\} = 4t + B_s \neq 4s + B_s$. Therefore, X_t is **not** a martingale.

(ii) $X_t = B_t^2$

- $\mathbb{E}\{|B_t|^2\} = \mathbb{E}\{B_t^2\} = t < \infty \; \forall t < \infty$
- $B_t \in \mathcal{F}_t$ so therefore $B_t^2 \in \mathcal{F}_t$.
- Let $s < t$. Then: $\mathbb{E}\{X_t|\mathcal{F}_s\} = \mathbb{E}\{(B_t - B)s + B_s)^2|\mathcal{F}_s\} = \mathbb{E}\{(B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2|\mathcal{F}_s\} = B_s^2 + t - s > B_s^2$, so B_t^2 is **not** a martingale.

(iii) $X_t = t^2 B_t - 2\int_0^t sB_s ds$

- We w.t.s. $\mathbb{E}|X_t| = \mathbb{E}\left|t^2 B_t - 2\int_0^t sB_s\, ds\right| < \infty$:

$$\begin{aligned}\mathbb{E}\left|t^2 B_t - 2\int_0^t sB_s\, ds\right| &\leq \mathbb{E}\left|t^2 B_t + 2\int_0^t sB_s\, ds\right| \\ &\leq \mathbb{E}\left|t^2 B_t\right| + \mathbb{E}\left|2\int_0^t sB_s ds\right| \\ &\leq t^2\mathbb{E}|B_t| + 2\int_0^t s\mathbb{E}|B_s|\, ds < \infty\end{aligned}$$

  with the last inequality following because each term is finite.
- X_t is a linear function of measurable functions of B_s where $s \leq t$, so $X_t \in \mathcal{F}_t$ as well.
- We will show X_t is a martingale. Let $s < t$:

$$\begin{aligned}\mathbb{E}(X_t|\mathcal{F}_s) &= t^2 B_s - 2\int_0^t u\mathbb{E}(B_u|\mathcal{F}_s)du \\ &= t^2 B_s - 2\int_0^s uB_u du - 2\int_s^t u\mathbb{E}(B_u|\mathcal{F}_s)du \\ &= t^2 B_s - 2\int_0^s uB_u du - 2\int_s^t uB_s du \\ &= t^2 B_s - 2\int_0^s uB_u du - 2B_s \frac{u^2}{2}\Big|_s^t \\ &= t^2 B_s - 2\int_0^s uB_u du - B_s(t^2 - s^2) \\ &= s^2 B_s - 2\int_0^s uB_u du \\ &= X_s\end{aligned}$$

(iv) $X_t = B_1^{(t)} B_2^{(t)}$ where $\left(B_1^{(t)}, B_2^{(t)}\right)$ is a 2-dimensional Brownian Motion

- Clearly, $\mathbb{E}\left|X_t\right| = \mathbb{E}\left|B_t^{(1)} B_t^{(2)}\right| \leq \mathbb{E}\left|B_t^{(1)}\right| \mathbb{E}\left|B_t^{(2)}\right| < \infty$ because each expectation is finite.
- Clearly, $B_t^{(1)} \in \mathcal{F}_t$ and $B_t^{(2)} \in \mathcal{F}_t$ so $\left(B_t^{(1)} B_t^{(2)}\right) \in \mathcal{F}_t$ as well.
- Letting $s < t$:

$$
\begin{aligned}
\mathbb{E}\{X_t|\mathcal{F}_s\} &= \mathbb{E}(B_t^{(1)} B_t^{(2)}|\mathcal{F}_s) \\
&= \mathbb{E}\Big((B_t^{(1)} - B_s^{(1)} + B_s^{(1)})(B_t^{(2)} - B_s^{(2)} + B_s^{(2)})|\mathcal{F}_s\Big) \\
&= \mathbb{E}\Big\{(B_t^{(1)} - B_s^{(2)})(B_t^{(1)} - B_s^{(2)}) + B_s^{(2)}(B_t^{(1)} - B_s^{(1)}) + B_s^{(2)}(B_t^{(2)} - B_s^{(2)}) + B_s^{(1)} B_s^{(2)} \Big| \mathcal{F}_s\Big\} \\
&= B_s^{(1)} B_s^{(2)} + \mathbb{E}\{(B_t^{(1)} - B_s^{(2)})\}\mathbb{E}\{(B_t^{(2)} - B_s^{(2)})\} + 0 \\
&= (B_t^{(1)} - B_s^{(2)}) \\
&= X_s
\end{aligned}
$$

so X_t is an $\mathcal{F}_t$ martingale.

Problem 3 (3.9)

Suppose $f \in \mathcal{V}(0,T)$ and that $t \to f(t,\omega)$ is continuous for $a.a.\omega$. Then, we have shown that

$$\int_0^T f(t,\omega)dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_j f(t_j,\omega)\,\Delta B_j \quad \text{ in } L^2(P).$$

Similarly, we define the *Stratonovich Integral* of f by:

$$\int_0^T f(t,\omega) \circ dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_j f\left(t_j^*,\omega\right)\Delta B_j, \quad \text{ where } t_j^* = \frac{1}{2}(t_j + t_{j+1})$$

whenever the limit exists in $L^2(P)$. In general, these integrals are different. For example, compute

$$\int_0^T B_t \circ dB_t$$

and compare with Example 3.1.9.

Proof. We will show

$$\int_0^T B_t \circ dB_t = \frac{B_T^2}{2}$$

. Letting $t_j^* = \frac{t_{j+1}+t_j}{2}$

$$\int_0^T B_t \circ dB_t = \lim_{\Delta t \longrightarrow 0} \sum_{j=0}^T B_{t_j^*} \Delta B_j$$

Multiplying the term on the right-hand side by 2, note that:

$$\begin{aligned}
2\sum_{j=0}^T B_{t_j^*}\Delta B_j &= 2\sum_{j=0}^T B_{t_j^*}(B_{t_{j+1}} - B_{t_j^*} + B_{t_j^*} - B_{t_j}) \\
&= 2\sum_{j=0}^T B_{t_j^*}(B_{t_{j+1}} - B_{t_j^*}) + 2\sum_{j=0}^T B_{t_j^*}(B_{t_j^*} - B_{t_j}) \\
&= \sum_{j=0}^T \left(2B_{t_j^*}(B_{t_{j+1}} - B_{t_j^*}) + B_{t_j^*}^2\right) - \sum_{j=0}^T \left(2B_{t_j^*}(B_{t_j} - B_{t_{j*}}) + B_{t_j^*}^2\right) \\
&= \sum_{j=0}^T \left((B_{t_{j+1}} - B_{t_j^*} + B_{t_j^*})^2 - (B_{t_{j+1}} - B_{t_j^*})^2\right) - \sum_{j=0}^T \left((B_{t_j} - B_{t_{j^*}} + B_{t_j^*})^2 - (B_{t_j} - B_{t_{j*}})^2\right) \\
&= \sum_{j=0}^T B_{t_{j+1}}^2 - B_{t_j}^2 - (B_{t_{j+1}} - B_{t_j^*})^2 + (B_{t_j} - B_{t_j^*})^2
\end{aligned}$$

Note that by the time-inversion property of Brownian Motion:

$$B_{t_j} - B_{t_j^*} \stackrel{\mathcal{D}}{=} B_{t_j^*} - B_{t_j}$$

Rearranging terms, we have:

$$2\sum_{j=0}^{T} B_{t_j^*}\Delta B_j + (B_{t_{j+1}} - B_{t_j^*})^2 - (B_{t_j^*} - B_{t_j})^2 = \sum_{j=0}^{T} B_{t_{j+1}}^2 - B_{t_j}^2$$

where the right-hand side forms a telescoping series such that

$$\lim_{\Delta t \longrightarrow 0} \sum_{j=0}^{T} B_{t_{j+1}}^2 - B_{t_j}^2 = B_T^2$$

Therefore,

$$2\sum_{j=0}^{T} B_{t_j^*}\Delta B_j + (B_{t_{j+1}} - B_{t_j^*})^2 - (B_{t_j^*} - B_{t_j})^2 \xrightarrow{\Delta t \longrightarrow 0} B_T^2$$

Clearly,

$$2\sum_{j=0}^{T} B_{t_j^*}\Delta B_j \xrightarrow{\Delta t \longrightarrow 0} 2\int_0^T B_t \circ dB_t$$

Note that $\mathbb{E}(B_{t_{j+1}} - B_{t_j^*})^2 = t_{j+1} - t_j^*$ and $\mathbb{E}(B_{t_{j^*}} - B_{t_j})^2 = t_j^* - t_j$, and

$$\mathbb{E}\Big\{\sum_{j=0}^{T}(B_{t_{j+1}} - B_{t_j^*})^2\Big\} = \sum_{j=0}^{t} t_{j+1} - t_j^* = \frac{T}{2}$$

with the same holding true for the other half of the interval. We now show that

$$\sum_{j=0}^{T}(B_{t_{j+1}} - B_{t_j^*})^2 \xrightarrow{L^2} \frac{T}{2}.$$

Starting with the definition (and letting $\frac{\Delta t}{2} = t_{j+1} - t_j^* = t_j^* - t_j$):

$$\begin{aligned}
\mathbb{E}\Big\{\Big[\sum_{j=0}^{T}(B_{t_{j+1}} - B_{t_j^*})^2 - \frac{T}{2}\Big]^2\Big\} &= \mathbb{E}\Big\{\Big[\sum_{j=0}^{T}(B_{t_{j+1}} - B_{t_j^*})^2\Big]^2\Big\} - \frac{T^2}{2} \\
&= \mathrm{Var}\Big(\sum_{j=0}^{T}(B_{t_{j+1}} - B_{t_j^*})^2\Big) \\
&= \sum_{j=0}^{T}\mathrm{Var}\Big((B_{t_{j+1}} - B_{t_j^*})^2\Big) \\
&\overset{\mathcal{D}}{=} \sum_{j=0}^{T}\mathrm{Var}(B_{\frac{\Delta t}{2}}^2) \\
&= \sum_{j=0}^{T}\Big[\mathbb{E}(B_{\frac{\Delta t}{2}}^4) - \Big(\frac{\Delta t}{2}\Big)^2\Big] \\
\text{(because } \sqrt{t}B_1 \overset{\mathcal{D}}{=} B_t\text{)} &= \sum_{j=0}^{T}\Big[\Big(\Big(\frac{\Delta t}{2}\Big)^2(\mathbb{E}(B_1^4) - 1)\Big] \\
\text{(because } \mathbb{E}(B_1^4) - 1 = 2\text{)} &\le 2\max_j(t_{j+1} - t_j^*)\frac{T}{2} \xrightarrow{\max_j(t_{j+1}-t_j^*)\longrightarrow 0} 0
\end{aligned}$$

A nearly identical approach holds to show $\sum_{j=0}^{t}(B_{t_{j^*}} - B_{t_j})^2 \xrightarrow{L^2} \frac{T}{2}$. We have therefore shown that

$$2\sum_{j=0}^{T} B_{t_j^*}\Delta B_j + (B_{t_{j+1}} - B_{t_j^*})^2 - (B_{t_j^*} - B_{t_j})^2 \xrightarrow{\Delta t \longrightarrow 0} 2\int_0^T B_t \circ dB_t + \frac{T}{2} - \frac{T}{2}$$

with the L^2 convergence of the final sum guaranteed by square-integrability of each term and completeness of L^2. So, in the limit, $2\int_0^T B_t \circ dB_t = B_T^2$, or

$$\int_0^T B_t \circ dB_t = \frac{B_T^2}{2}$$

This implies that, in Stratonovich Calculus, $d(\frac{B_t^2}{2}) = B_t \circ dB_t$. $\square$

Problem 4 (3.18)

Let B_t be $1-$ dimensional Brownian motion and let $\sigma \in \mathbb{R}$ be constant. Prove directly from the definition that:

$$M_t := \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right); \quad t \geq 0$$

is an $\mathcal{F}_t$ -martingale.

Proof. (i) : L^1/Integrability condition:

$$\begin{aligned}
\mathbb{E}\left|M_t\right| &= \mathbb{E}\left|e^{\sigma B_t - \frac{1}{2}\sigma^2 t}\right| = \mathbb{E}(e^{\sigma B_t - \frac{1}{2}\sigma^2 t}) \\
&= \mathbb{E}(e^{\sigma B_t})e^{-\frac{1}{2}\sigma^2 t} = e^{-\frac{1}{2}\sigma^2 t}e^{\frac{1}{2}\sigma^2 t} = 1 < \infty
\end{aligned}$$

(ii) $M_t \in \mathcal{F}_t = \sigma(B_s; s \leq t)$ clearly, (i.e. M_t adapted)

(iii) Martingale Condition. Let $s < t$.

$$\begin{aligned}
\mathbb{E}(M_t|\mathcal{F}_s) &= \mathbb{E}\big(e^{\sigma B_t - \frac{1}{2}\sigma^2 t}\big|\mathcal{F}_s\big) \\
&= e^{-\frac{1}{2}\sigma^2 t}\mathbb{E}\big(e^{\sigma B_t}\big|\mathcal{F}_s\big) \\
&= e^{-\frac{1}{2}\sigma^2 t}e^{\sigma B_s}\mathbb{E}\big(e^{\sigma(B_t - B_s)}\big|\mathcal{F}_s\big) \\
&= e^{-\frac{1}{2}\sigma^2 t}e^{\sigma B_s}\mathbb{E}\big(e^{\sigma(B_t - B_s)}\big) \\
&= e^{-\frac{1}{2}\sigma^2 t}e^{\sigma B_s}e^{\sigma^2\frac{(t-s)}{2}} \\
&= e^{-\frac{1}{2}\sigma^2 2}e^{\sigma B_s} = M_s
\end{aligned}$$

Therefore, M_t is a martingale with respect to $\mathcal{F}_t$. $\square$

Problem 5 (4.1)

Use Itô's formula to write the following stochastic processes Y_t on the standard form $dY_t = u(t,\omega)dt + v(t,\omega)dB_t$ for suitable choices of $u \in \mathbb{R}^n, v \in \mathbb{R}^{n\times m}$ and dimensions n, m :

(i) $Y_t = B_t^2$, where B_t is 1 -dimensional

(ii) $Y_t = 2 + t + e^{B_t}$ (B_t is 1 dimensional)

(iii) $Y_t = B_1^2(t) + B_2^2(t)$ where (B_1, B_2) is 2- dimensional

(iv) $Y_t = (t_0 + t, B_t)$ (B_t is 1 dimensional)

(v) $Y_t = \big(B_1(t) + B_2(t) + B_3(t), B_2^2(t) - B_1(t)B_3(t)\big)$, where (B_1, B_2, B_3) is 3-dimensional.

Proof.

(i) $Y_t = B_t^2$, where B_t is 1 -dimensional

$$\begin{aligned} dY_t &= 0dt + 2B_t dB_t + \frac{1}{2}2\langle dB_t, dB_t\rangle \\ &= 2B_t dB_t + dt \end{aligned}$$

(ii) $Y_t = 2 + t + e^{B_t}$

$$\begin{aligned} dY_t &= dt + e^{B_t} dB_t + \frac{1}{2}e^{B_t}\langle dB_t, dB_t\rangle \\ &= (1 + \frac{1}{2}e^{B_t})dt + e^{B_t} dB_t \end{aligned}$$

(iii) $Y_t = B_1^2(t) + B_2^2(t)$

$$\begin{aligned} dY_t &= 0dt + 2\big(B_1(t)dB_1(t) + B_2(t)dB_2(t)\big) + \frac{1}{2}(dB_1(t)^2 + dB_2(t)^2) \\ &= 2\big(B_1(t)dB_1(t) + B_2(t)dB_2(t)\big) + dt \end{aligned}$$

(iv) $Y_t = (t_0 + t, B_t)$

$$dY_t = \begin{pmatrix}1\\0\end{pmatrix} dt + \begin{pmatrix}0\\1\end{pmatrix} dB_t + \begin{pmatrix}0\\0\end{pmatrix} \langle dB_t, dB_t\rangle = \begin{pmatrix}dt\\dB_t\end{pmatrix}$$

(v) $Y_t = \big(B_1(t) + B_2(t) + B_3(t), B_2^2(t) - B_1(t)B_3(t)\big)$.

$$\begin{aligned} dY_t &= \begin{pmatrix}0\\0\end{pmatrix} dt + \begin{pmatrix} dB_1(t) + dB_2(t) + dB_3(t) \\ -B_3(t)dB_1(t) + 2B_2(t)dB_2(t) - B_1(t)dB_3(t)\end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0\langle dB_1(t), dB_1(t)\rangle + 2\langle dB_2(t), dB_2(t)\rangle + 0\langle dB_3(t), dB_3(t)\rangle \end{pmatrix} \\ &= \begin{pmatrix}0\\2\end{pmatrix} dt + \begin{pmatrix} 1 & 1 & 1 \\ -B_3(t) & 2 & -B_1(t)\end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t)\end{pmatrix} \end{aligned}$$

□

Problem 6 (4.2)

Use Itô's formula to prove that

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

Proof. Let $Y_t = \frac{1}{3}B_t^3$. Then, applying Itô's formula, $dY_t = B_t^2 dB_t - B_t dt$. Rearranging terms and letting $B_0 = 0$,

$$B_t^2 dB_t = dY_t - B_t dt \quad \text{or}$$
$$\int_0^t B_s dB_s = \frac{1}{3}B_t^3 \int_0^t B_s ds.$$

$\square$

Problem 7 (4.5)

Let $B_t \in \mathbb{R}, B_0 = 0$. Define

$$\beta_k(t) = \mathbb{E}\left[B_t^k\right]; \quad k = 0, 1, 2, \dots; t \geq 0$$

(i) Use Itô's formula to prove that

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds; \quad k \geq 2$$

(ii) Deduce that

$$\mathbb{E}\left[B_t^4\right] = 3t^2$$

and find

$$\mathbb{E}\left[B_t^6\right]$$

(iii) Show that

$$\mathbb{E}\left[B(t)^{2k+1}\right] = 0$$

and

$$\mathbb{E}\left[B(t)^{2k}\right] = \frac{(2k)!t^k}{2^k k!}; \quad k = 1, 2, \dots \tag{1}$$

Proof.

(i) Let $Y_t = B_t^k$. Applying Itô's formula, $dY_t = kB_t^{k-1}dB_t + \frac{1}{2}k(k-1)B_t^{k-2}dt$. Therefore,

$$B_t^k = B_0 + k\int_0^t B_s^{k-1}dB_s + \frac{1}{2}k(k-1)\int_0^t B_s^{k-2}ds$$

Where $k\int_0^t B_s^{k-1}dB_s$ is a martingale by definition and $B_0 = 0$, so $\mathbb{E}\{k\int_0^t B_s^{k-1}dB_s\} = 0$, so Therefore,

$$\begin{aligned}
\beta_k(t) = \mathbb{E}(B_t^k) &= \mathbb{E}\Big\{\frac{1}{2}k(k-1)\int_0^t B_s^{k-2}ds\Big\} \\
\text{(Fubini's/Tonelli's Theorem)} &= \frac{1}{2}k(k-1)\int_0^t \mathbb{E}\{B_s^{k-2}\}ds \\
&= \frac{1}{2}k(k-1)\int_0^t \beta_k(s)ds
\end{aligned}$$

as expected

(ii) Again, letting $B_0 = 0$:

$$\begin{aligned}
\mathbb{E}(B_t^4) &= \frac{1}{2}(4)(3)\int_0^t \mathbb{E}(B_s^2)ds \\
&= 6\int_0^t sds = 6\frac{t^2}{2} = 3t^2
\end{aligned}$$

$$\begin{aligned}\mathbb{E}(B_t^6) &= \frac{1}{2}(6)(5)\int_0^t \mathbb{E}(B_s^4)ds \\ &= \frac{30}{2}\int_0^t 3s^2 ds = 45\frac{t^3}{3} = 15t^3\end{aligned}$$

(iii) From part (i),

$$\mathbb{E}\{B_t^{2k+1}\} = \frac{1}{2}(2k+1)(2k)\int_0^t \mathbb{E}(B_s^{2k-1})ds = 0$$

because $2k-1$ is odd, and all the finite-dimensional distributions of B_t are symmetric about 0, and thus all the odd moments are 0. For the second part, we use induction. Note that the formula in (1) holds for $k=2$. Assume it holds for the first k elements. We will show it also holds for $k+1$.

$$\begin{aligned}\mathbb{E}\{B_t^{2(k+1)}\} &= \frac{1}{2}(2k+2)(2k+1)\int_0^t \mathbb{E}(B_s^{2k})ds \\ &= \frac{1}{2}(2k+2)(2k+1)\int_0^t \frac{(2k)!s^k}{2^k k!}ds \\ &= \frac{1}{2}(2k+2)(2k+1)\frac{(2k)!}{2^k k!}\int_0^t s^k ds \\ &= \frac{(2k+2)!}{2^{k+1}k!}\frac{t^{k+1}}{k+1} \\ &= \frac{(2k+2)!t^{k+1}}{2^{k+1}(k+1)!}\end{aligned}$$

proving the inductive hypothesis.

$\square$

Problem 8 (4.10, Tanaka's Formula and Local Time)

What happens if we try to apply the Itô's formula to $g(B_t)$ where B_t is 1-dimensional and $g(x)=|x|$? In this case g is not C^2 at $x=0$, so we modify $g(x)$ near $x=0$ to $g_\epsilon(x)$ as follows:

$$g_\epsilon(x)=\begin{cases} |x| & \text{if } \ |x|\geq\epsilon \\ \frac{1}{2}\left(\epsilon+\frac{x^2}{\epsilon}\right) & \text{if } \ |x|<\epsilon \end{cases}$$

where $\epsilon>0$.

(i) Show that

$$g_\epsilon\left(B_t\right)=g_\epsilon\left(B_0\right)+\int_0 g_\epsilon'\left(B_s\right)dB_s+\frac{1}{2\epsilon}\cdot\mathscr{L}\left(\{s\in[0,t];B_s\in(-\epsilon,\epsilon)\}\right)$$

where $\mathscr{L}(F)$ denotes the Lebesgue measure of set F.

(ii) Prove that

$$\int_0^t g_\epsilon'\left(B_s\right)\cdot\mathcal{X}_{B_s\in(-\epsilon,\epsilon)}dB_s=\int_0^t\frac{B_s}{\epsilon}\cdot\mathcal{X}_{B_s\in(-\epsilon,\epsilon)}dB_s\to 0$$

in $L^2(P)$ as $\epsilon\to 0$.

(iii) By letting $\epsilon\to 0$ prove that

$$|B_t|=|B_0|+\int_0^t\operatorname{sgn}\left(B_s\right)dB_s+L_t(\omega)$$

where

$$L_t=\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\cdot\mathscr{L}\left(\{s\in[0,t];B_s\in(-\epsilon,\epsilon)\}\right)\quad\left(\text{limit in } L^2(P)\right)$$

and

$$\operatorname{sgn}(x)=\begin{cases} -1 & \text{for } \ x\leq 0 \\ 1 & \text{for } \ x>0 \end{cases}$$

L_t is called the local time for Brownian motion at 0 and (iii) is the Tanaka formula (for Brownian motion).

Proof. (i) We use the following approximation for $g(x)=|x|$:

$$g_\epsilon(x)=\begin{cases}|x| & |x|\geq\epsilon\\ \frac{1}{2}(\epsilon+\frac{x^2}{\epsilon}) & |x|<\epsilon\end{cases}$$

Taking the first derivative, we get:

$$g_\epsilon'(x)=\begin{cases}\operatorname{sgn}(x) & |x|\geq\epsilon\\ \frac{x}{\epsilon} & |x|<\epsilon\end{cases}$$

and the second:

$$g_\epsilon''(x)=\begin{cases}0 & |x|>\epsilon\\ \frac{1}{\epsilon} & |x|<\epsilon\end{cases}$$

with $g''_\epsilon(x)$ undefined at $x = \pm\epsilon$, which is a set that has Lebesgue measure 0. Applying Ito's formula in the integral form to $g_\epsilon(B_t)$, and using $\mathscr{L}(F)$ to denote the Lebesgue measure of set F, we get:

$$\begin{aligned}
g_\epsilon(B_t) &= g_\epsilon(0) + \int_0^t g_\epsilon^{'}(B_s)dB_s + \frac{1}{2}\int_0^t g_\epsilon^{''}(B_s)ds \\
&= g_\epsilon(0) + \int_0^t g_\epsilon^{'}(B_s)dB_s + \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{\{-\epsilon<B_s<\epsilon\}}ds \\
&= g_\epsilon(0) + \int_0^t g_\epsilon^{'}(B_s)dB_s + \frac{1}{2\epsilon}\mathscr{L}(\{s \in [0,t];\ -\epsilon < B_s < \epsilon\})
\end{aligned}$$

as expected.

(ii) As above, letting $\mathbb{1}_{\{F\}}$ be the indicator of a set F,

$$\int_0^t g_\epsilon^{'}(B_s)\mathbb{1}_{\{|Bs|<\epsilon\}}dB_s = \int_0^t \frac{B_s}{\epsilon}\mathbb{1}_{\{|Bs|<\epsilon\}}dB_s$$

Taking the Expectation of the Square and applying Itô's Isometry, we get:

$$\begin{aligned}
\mathbb{E}\Big\{\Big(\int_0^t g_\epsilon^{'}(B_s)\mathbb{1}_{\{|Bs|<\epsilon\}}dB_s\Big)^2\Big\} &= \mathbb{E}\Big\{\Big(\int_0^t \frac{B_s}{\epsilon}\mathbb{1}_{\{|Bs|<\epsilon\}}dB_s\Big)^2\Big\} \\
\text{(Itô's Isometry)} \quad &= \mathbb{E}\Big\{\int_0^t \big(\frac{B_s}{\epsilon}\mathbb{1}_{\{|B_s|<\epsilon\}}\big)^2 ds\Big\} \\
&= \mathbb{E}\Big\{\frac{1}{\epsilon^2}\int_0^t B_s^2\mathbb{1}_{\{|B_s|<\epsilon\}}\Big\} \\
\text{(Fubini's/Tonelli's Theorem)} \quad &= \frac{1}{\epsilon^2}\int_0^t \mathbb{E}(B_s^2\mathbb{1}_{\{|B_s|<\epsilon\}})ds \\
&\leq \frac{1}{\epsilon^2}\int_0^t \epsilon^2\mathbb{E}(\mathbb{1}_{\{|B_s|<\epsilon\}})ds \\
\text{(letting } B_0 = 0) \quad &= \int_0^t \mathcal{P}(|B_s| < \epsilon)ds \\
\big\{(\mathcal{P}(|B_s| < \epsilon) \xrightarrow{\epsilon\longrightarrow 0} 0\big\} &\longrightarrow 0
\end{aligned}$$

as expected.

(iii) First, note that $g_\epsilon(x) \xrightarrow{\epsilon\longrightarrow 0} |x|$ for all x. Breaking the expression for $g_\epsilon(B_t)$ into parts and taking $\epsilon \longrightarrow 0$, we have:

$$\begin{aligned}
g_\epsilon(B_t) &= g_\epsilon(B_0) + \int_0^t \operatorname{sgn}(B_s)\mathbb{1}_{\{|B_s|>\epsilon\}}ds + \frac{1}{\epsilon^2}\int_0^t \mathbb{E}(B_s^2\mathbb{1}_{\{|B_s|<\epsilon\}})ds + \frac{1}{2\epsilon}\mathscr{L}(\{s \in [0,t]; |B_s| < \epsilon\}) \\
&\xrightarrow{\epsilon\longrightarrow 0} g(B_0) + \int_0^t \operatorname{sgn}(B_s)ds + \lim_{\epsilon\longrightarrow 0}\frac{1}{2\epsilon}\mathscr{L}(\{s \in [0,t]; |B_s| < \epsilon\}) \\
&= |B_0| + \int_0^t \operatorname{sgn}(B_s)ds + L_t(\omega)
\end{aligned}$$

as expected, with $L_t(\omega)$ as defined in the problem statement.

$\square$

Problem 9 (4.11)

Use Itô's formula to prove that the following stochastic processes are $\{\mathcal{F}_t\}$-martingales:

(i) $X_t = e^{\frac{1}{2}t}\cos B_t \quad (B_t \in \mathbb{R})$

(ii) $X_t = e^{\frac{1}{2}t}\sin B_t \quad (B_t \in \mathbb{R})$

(iii) $X_t = (B_t + t)\exp\left(-B_t - \frac{1}{2}t\right) \quad (B_t \in \mathbb{R})$

Proof. For each of these examples, Øksendal Corollary 3.2.6, it suffices to show that in the differential representation, $dX_t = g(t,\omega)dB_t$ where $g(t,\omega) \in \mathcal{V}^{(n)}$, with $\mathcal{V}^{(n)}$ as defined in Øksendal Definition 3.1.4. (i.e. square-integrable, $\mathcal{F}_t^{(n)}$ adapted and Borel-measurable.) We can do this by applying Itô's Formula in each case.

(a) Let $X_t = e^{\frac{1}{2}t}\cos B_t$

$$\begin{aligned} dX_t &= d(e^{\frac{1}{2}t}\cos B_t) = e^{\frac{1}{2}t}d(\cos B_t) + \cos B_t d(e^{\frac{1}{2}t}) + d(e^{\frac{1}{2}t})d(\cos B_t) \\ &= e^{\frac{1}{2}t}\Big(-\sin B_t dB_t - \frac{\cos B_t}{2}dt\Big) + e^{\frac{1}{2}t}\frac{\cos B_t}{2}dt - \frac{e^{\frac{1}{2}t}}{2}dt\Big(-\sin B_t dB_t - \frac{\cos B_t}{2}dt\Big) \\ &= -e^{\frac{1}{2}t}\sin B_t dB_t \end{aligned}$$

where $-e^{\frac{1}{2}t}\sin B_t \in \mathcal{V}(0,T)$, $\forall T\ 0 < T < \infty$. Therefore, $X_t = e^{\frac{1}{2}t}\cos B_t$ is a martingale with respect to $\{\mathcal{F}_t\}$.

(b) Let $X_t = e^{\frac{1}{2}t}\sin B_t$

$$\begin{aligned} dX_t &= d(e^{\frac{1}{2}t}\sin B_t) = e^{\frac{1}{2}t}d(\sin B_t) + \sin B_t d(e^{\frac{1}{2}t}) + d(e^{\frac{1}{2}t})d(\sin B_t) \\ &= e^{\frac{1}{2}t}(\cos B_t dB_t - \frac{1}{2}\sin B_t dt) + \frac{\sin B_t}{2}e^{\frac{1}{2}t}dt - \frac{e^{\frac{1}{2}t}}{2}dt(\cos B_t dB_t - \frac{1}{2}\sin B_t dt) \\ &= e^{\frac{1}{2}t}\cos B_t dB_t \end{aligned}$$

where $e^{\frac{1}{2}t}\cos B_t \in \mathcal{V}(0,T)$, $\forall T\ 0 < T < \infty$. Therefore, $X_t = e^{\frac{1}{2}t}\sin B_t$ is a martingale with respect to $\{\mathcal{F}_t\}$.

(c) Let $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t)$. Note that there are two pieces, $(B_t + t)$ and $(e^{(-B_t - \frac{1}{2}t)})$. Appling Itô's formula to each piece, we get:

$$d(B_t + t) = dt + dB_t$$

and

$$\begin{aligned} d(e^{(-B_t - \frac{1}{2}t)}) &= -\frac{e^{(-B_t - \frac{1}{2}t)}2}{d}t - e^{(-B_t - \frac{1}{2}t)}dB_t + \frac{e^{(-B_t - \frac{1}{2}t)}2}{d}t \\ &= -e^{(-B_t - \frac{1}{2}t)}dB_t \end{aligned}$$

Therefore, plugging into the formula for the product of two processes, we get:

$$\begin{aligned} dX_t &= -(B_t + t)e^{(-B_t - \frac{1}{2}t)}dB_t + e^{(-B_t - \frac{1}{2}t)}(dt + dB_t) - e^{(-B_t - \frac{1}{2}t)}dB_t(dt + dB_t) \\ &= -(B_t + t)e^{(-B_t - \frac{1}{2}t)}dB_t + e^{(-B_t - \frac{1}{2}t)}(dt + dB_t) - e^{(-B_t - \frac{1}{2}t)}dt \\ &= e^{(-B_t - \frac{1}{2}t)}(1 - B_t - t)dB_t \end{aligned}$$

Clearly, $e^{(-B_t-\frac{1}{2}t)}(1-B_t-t) \in \mathcal{V}(0,T)$, $\forall T$ $0 < T < \infty$. Therefore $X_t = (B_t+t)e^{(-B_t-\frac{1}{2}t)}$ is a martingale with respect to $\{\mathcal{F}_t\}$.

$\square$