1. (25%) What Asymptotic solution, if any, does the master method give for each of the following recurrences?

(a) $T(n) = 16T(\frac{n}{2}) + 3n^4 \log^2(n) = \Theta(n^4 \log^2(n))$.

Proof. Let $a = 16$, $b = 2$, $f(n) = 3n^4 \log^2(n)$. Clearly,

$$\log_2(16) = 4 \text{ and}$$
$$f(n) = \Theta(n^4 \log_2^2(n)).$$

Therefore, by the Master Method Theorem,

$$T(n) = \Theta(n^4 \log_2^3(n)).$$

□

(b) $T(n) = T(\frac{n}{3}) + 4\log(n) = \Theta(\log^2(n))$.

Proof. Let $a = 1$, $b = 3$, and $f(n) = 4\log(n)$. Clearly,

$$\log_3(1) = 0.$$

And, we can say that:

$$f(n) = \Theta(n^0 \log^1(n)) = \Theta(\log(n)).$$

Therefore,

$$T(n) = \Theta(\log^2(n)).$$

□

(c) $T(n) = 4T(\frac{n}{4}) + \frac{n}{\log\log n}$ cannot be judged by the Master Method.

Proof. Let $a = 4$, $b = 4$, and $f(n) = \frac{n}{\log\log n}$. Clearly, $\log_b(a) = \log_4(4) = 1$. Also, comparing $g(n) = n$ to $f(n) = \frac{n}{\log\log n}n$ with the limit comparison test,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\frac{n}{\log\log n}}{n} = \lim_{n\to\infty} \frac{1}{\log\log n} = 0.$$

Therefore, we can say that

$$f(n) = \frac{n}{\log\log n} = \mathcal{O}(n).$$

However, for any $\epsilon \in \mathbb{R}^+$ (i.e. $\epsilon > 0$),

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\frac{n}{\log\log n}}{n^{1-\epsilon}} = \lim_{n\to\infty} \frac{n^\epsilon}{\log\log n} = \infty.$$

So, we can say that $f(n) = \Omega(n^{1-\epsilon})\ \forall\epsilon > 0$. For the same conditions on ϵ, we can say that $f(n) = \mathcal{O}(n^{1+\epsilon})$. Therefore we cannot extract any useful information about this recurrence from the Master Method. □

(d) $T(n) = 9T\left(\frac{n}{4}\right) + 2n^{\log_4 9} = \Theta\left(n^{\log_4(9)}\log(n)\right)$.

Proof. Let $a = 9$, $b = 4$, and $f(n) = 2n^{\log_4(9)}$. It is trivial to show that, $f(n) = \Theta\left(n^{\log_4(9)}\right) = \Theta\left(n^{\log_4(9)}\log^0(n)\right)$. Therefore, via the Master Method, we can say that $T(n) = \Theta\left(n^{\log_4(9)}\log(n)\right)$. $\square$

(e) $T(n) = 11T\left(\frac{n}{3}\right) + n^2 = \Theta\left(n^{\log_3(11)}\right)$.

Proof. Let $a = 11$, $b = 3$, and $f(n) = n^2$. Quick inspection tells us that $\log_3(11) > 2$, so we can say that $f(n) = \mathcal{O}\left(n^{\log_3(11)-\epsilon}\right)$ for a small $\epsilon > 0$. We can therefore say that $T(n) = \Theta\left(n^{\log_3(11)}\right)$ by the Master Method. $\square$

(f) $T(n) = 999T\left(\frac{n}{10}\right) + 2n^3 = \Theta(n^3)$.

Proof. Let $a = 999$, $b = 10$, and $f(n) = 2n^3$. Quick inspection tells us that $\log_1 0(999) < 3$, so $f(n) = 2n^3 = \Omega(n^{\log_{10}(999)+\epsilon})$ for a sufficent $\epsilon > 0$. Therefore, we can say that $T(n) = \Theta(n^3)$ by the Master Method. $\square$

(g) $T(n) = 7T\left(\frac{n}{2}\right) + n^3 \log n \log\log n = \Theta(n^3 \log n \log\log n)$.

Proof. Let $a = 7$, $b = 2$, and $f(n) = n^3 \log n \log\log n$. Quick inspection tells us that $\log_2(7) < 3$. We can therefore say that $f(n) = \Omega(n^{\log_2(7)+\epsilon})$ for a sufficient $\epsilon > 0$. Therefore, by the Master Method, $T(n) = \Theta(n^3 \log n \log\log n)$. $\square$