2. (25%) Consider the general recurrence

$$T(n) = \begin{cases} c_0 & \text{if } n = 1\\ aT(\frac{n}{b}) + f(n) & \text{otherwise} \end{cases}$$

In class, we showed that this recurrence has solution

$$T(n) = c_0 n^{\log_b a} + \sum_{k=0}^{\log_b n-1} a^k f(n/b^k).$$

Suppose now that we alter the recurrence so that it terminates not for n = 1 but for  $n = n_0$ , for some constant  $n_0 > 1$ . In other words, we replace the base case by " $c_0$  if  $n = n_0$ ." You may assume for simplicity that n and  $n_0$  are both powers of b.

(a) Thinking about the recursion tree for the revised recurrence, it isn't surprising that its solution has the form

$$T'(n) = c_0 a^x + \sum_{k=0}^{x-1} a^k f(n/b^k).$$

What expression should replace the x above? Justify your answer. You might find it helpful to sketch the recursion tree for the revised recurrence.

We will have that  $x = \log(\frac{n}{n_0})$ .

*Proof.* A table for the recurrence T(n) looks as follows:

Level	Nodes	Work Per Node
0	1	f(n)
1	a	$f(\frac{n}{h})$
2	$a^2$	$f(\frac{n}{h^2})$
k	$a^k$	$f(\frac{n}{h^k})$
$\log_b(\frac{n}{n_0})$ $\log_b(n)$	$a^{\log_b(\frac{n}{n_0})}$	$c_0$ (new base case)
$\log_b(n)$	$a^{\log_b(n)}$	_

Because our new base case is  $n = n_0$ , we only care about the levels above  $\log_b(\frac{n}{n_0})$ . Therefore, we will sum from  $0 \to \log_b(\frac{n}{n_0})$ , so the recurrence T'(n) becomes:

$$T'(n) = c_0 a^{\log_b(\frac{n}{n_0})} + \sum_{k=0}^{\log_b(\frac{n}{n_0})-1} a^k f(\frac{n}{b^k})$$

(b) Show that  $c_0 a^x$  above is still a constant times  $n^{\log_b a}$ .

*Proof.* We have  $n_0$  fixed, so  $a^{\log_b(n_0)}$  is fixed and not dependent on n. Therefore, using log rules, we have:

$$c_0 a^{\log_b(\frac{n}{n_0})} = c_0 a^{\log_b(n) - \log_b(n_0)}$$

$$= c_0 \left( a^{\log_b(n)} a^{-\log_b(n_0)} \right)$$

$$= c' \left( a^{\log_b(n)} \right)$$

$$= c' \left( n^{\log_b(a)} \right)$$

where  $c' = c_0 a^{-\log_b(n_0)}$ , which is a constant.

(c) The summation in (a) above can be rewritten as

$$\sum_{k=0}^{\log_b n - 1} a^k f(n/b^k) - \sum_{k=\log_b n - y}^{\log_b n - 1} a^k f(n/b^k)$$

for some constant y independent of n. What is this y?

$$y = \log_b(n_0).$$

*Proof.* This is evident from our use of logarithm rules.  $\log_b(\frac{n}{n_0}) = \log_b(n) - \log_b(n_0)$ , therefore we can rewrite the summation in (a) to be:

$$\sum_{k=0}^{\log_b(\frac{n}{n_0})-1} a^k f\big(\frac{n}{b^k}\big) = \sum_{k=0}^{\log_b n-1} a^k f(n/b^k) - \sum_{k=\log_b n-\log_b n_0}^{\log_b n-1} a^k f(n/b^k)$$

This is also evident, because we wish to remove all the terms between levels  $\log_b(\frac{n}{n_0})$  and  $\log_b(n)$ . That is, for all levels smaller than the base case  $n_0$ .

(d) Show that the right-hand summation in (c) can be simplified to  $c'n^{\log_b a}$  where c' is an expression independent of n.

*Proof.* We wish to show that:

$$\sum_{k=\log_b n - \log_b n_0}^{\log_b n - 1} a^k f(\frac{n}{b^k}) = c' n^{\log_b a}$$

for some c' not dependent on n. First, we need to adjust the bounds of our sum. Notice that each k will be of the form  $k = \log_b n - \log_b n_0 + i$  where i is an integer that ranges from  $0 \to \log_b n_0 - 1$ . Therefore, we can rewrite this sum as:

$$\sum_{k=\log_b n - \log_b n_0}^{\log_b n - 1} a^k f(\frac{n}{b^k}) = \sum_{i=0}^{\log_b n_0 - 1} a^{\log_b n - \log_b n_0 + i} f(\frac{n}{b^{\log_b n - \log_b n_0 + i}})$$

$$= \sum_{i=0}^{\log_b n_0 - 1} a^{\log_b (n)} a^{\log_b (n_0)} a^i f(\frac{n}{b^{\log_b n_0 - \log_b n_0 b^i}})$$

$$= a^{\log_b (n)} a^{-\log_b (n_0)} \sum_{i=0}^{\log_b n_0 - 1} a^i f(\frac{n}{n n_0^{-1} b^i})$$

$$= a^{\log_b (n)} a^{-\log_b (n_0)} \sum_{i=0}^{\log_b n_0 - 1} a^i f(\frac{n_0}{b^i})$$

$$= c' a^{\log_b (n)}$$

where

$$c' = a^{-\log_b(n_0)} \sum_{i=0}^{\log_b n_0 - 1} a^i f(\frac{n_0}{b^i})$$

which is independent of n.