4. (25%)

(a) Does $(n+1)^2 = \Omega(n \log n)$?
True.

Proof. Let $f(n) = (n+1)^2$ and $g(n) = n \log n$ Take

$$\begin{aligned}\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{(n+1)^2}{n \log n} \\ &= \lim_{n\to\infty} \frac{n^2+2n+1}{n \log n} \\ &= +\infty\end{aligned}$$

So, by the limit comparison test, $(n+1)^2 = \Omega(n \log n)$. $\square$

(b) Does $3^{9 \log n + \log \log n} = \Omega(n^3)$?
True.

Proof. Note that:

$$\begin{aligned}f(n) = 3^{9 \log n + \log \log n} &= 3^{9 \log n} 3^{\log \log n} \\ &= 3^{(\log n^9)} 3^{\log \log n} \\ &= (n^9)^{\log(3)} 3^{\log \log n} \\ &= n^{9 \log(3)} 3^{\log \log n}\end{aligned}$$

Also note both that $\log_2(\log_2(n)) \geq 1 \forall n \geq 4$, and $9(\log_2(3)) \approx 14$ (it's actually slightly greater). Therefore, $f(n)$ can be bounded below by $3n^{14}$ $\forall n \geq 4$. Therefore:

$$\lim_{n\to\infty} \frac{3^{9 \log n + \log \log n}}{n^3} \geq \lim_{n\to\infty} \frac{3n^{14}}{n^3} = +\infty$$

$\square$

(c) Does $n \log_5 n = \Theta(n \ln n)$?
True.

Proof. Note that:

$$n \log_5 5 = n \frac{\ln n}{\ln 5}$$

and that:

$$\lim_{n\to\infty} \frac{n \log_5 n}{n \ln n} = \lim_{n\to\infty} \frac{1}{\ln 5} = \frac{1}{\ln 5}$$

Because this limit is a constant, $n \log_5 n = \Theta(n \ln n)$ $\square$

(d) Does $(n-2)^2 = \Theta(n \log n)$?
False.

Proof. Note that:

$$\lim_{n\to\infty} \frac{(n-2)^2}{n \log n} = \lim_{n\to\infty} \frac{n^2 - 4n + 4}{n \log n} = +\infty$$
$$\neq c \ , \ c < \infty$$

Therefore, $(n-2)^2 \neq \Theta(n \log n)$. □

(e) Does $n^{\frac{61}{60}} = \mathcal{O}(n \log n)$?
False.

Proof.

$$\lim_{n\to\infty} \frac{n^{\frac{61}{60}}}{n \log n} = +\infty$$

Therefore, $n^{\frac{61}{60}}$ is not $\mathcal{O}(n \log n)$ but it is $\Omega(n \log n)$. □

(f) Let $f(n)$ and $g(n)$ be non-negative functions of n. If $f(n) = \mathcal{O}(g(n))$, does $f(n)+g(n) = \Theta(g(n))$?
True.

Proof. Because $f(n) = \mathcal{O}(g(n))$:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Therefore, we have:

$$\lim_{n\to\infty} \frac{f(n)+g(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)}{g(n)} + \frac{g(n)}{g(n)} = 1$$
$$= \lim_{n\to\infty} 0 + \lim_{n\to\infty} \frac{g(n)}{g(n)} = 1 < \infty.$$

Therefore, $f(n) + g(n) = \Theta(g(n))$. □

(g) Let $f(n)$ and $g(n)$ be non-negative functions of n. If $f(n) = \Theta(g(n))$, does $\frac{f(n)}{g(n)} = \Theta(1)$?
True.

Proof. By the provided relation, we know:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty.$$

Therefore, it is trivial to show that:

$$\lim_{n\to\infty} \frac{\left(\frac{f(n)}{g(n)}\right)}{1} = \lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty.$$

Therefore, $\frac{f(n)}{g(n)} = \Theta(1)$. □