4. (25%)

   (a) Does $(n+1)^2 = \Omega(n \log n)$?
   True.

   Proof. Let $f(n) = (n+1)^2$ and $g(n) = n \log n$ Take

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{(n+1)^2}{n \log n} \\
&= \lim_{n\to\infty} \frac{n^2 + 2n + 1}{n \log n} \\
&= +\infty
\end{aligned}$$

   So, by the limit comparison test, $(n+1)^2 = \Omega(n \log n)$. □

   (b) Does $3^{9\log n + \log\log n} = \Omega(n^3)$?
   True.

   Proof. Note that:

$$\begin{aligned}
f(n) = 3^{9\log n + \log\log n} &= 3^{9\log n} 3^{\log\log n} \\
&= 3^{(\log n)^9} 3^{\log\log n} \\
&= (3^{\log n})^9 3^{\log\log n} \\
&= (n^{\log 3})^9 3^{\log\log n} \\
&= n^{(\log 3)^9} 3^{\log\log n}
\end{aligned}$$

   Also note both that $\log_2(\log_2(n)) \geq 1 \forall n \geq 4$, and $(\log_2(3))^9 \approx 63$ (it's actually slightly greater). Therefore, $f(n)$ can be bounded below by $3n^{63}$ $\forall n \geq 4$. Therefore:

$$\lim_{n\to\infty} \frac{3^{9\log n + \log\log n}}{n^3} \geq \lim_{n\to\infty} \frac{3n^{63}}{n^3} = +\infty$$

   □

   (c) Does $n \log_5 n = \Theta(n \ln n)$?
   True.

   Proof. Note that:

$$n \log_5 5 = n \frac{\ln n}{\ln 5}$$

   and that:

$$\lim_{n\to\infty} \frac{n \log_5 n}{n \ln n} = \lim_{n\to\infty} \frac{1}{\ln 5} = \frac{1}{\ln 5}$$

   Because this limit is a constant, $n \log_5 n = \Theta(n \ln n)$ □

(d) Does $(n-2)^2 = \Theta(n \log n)$?
False.

Proof. Note that:

$$\lim_{n\to\infty} \frac{(n-2)^2}{n \log n} = \lim_{n\to\infty} \frac{n^2 - 4n + 4}{n \log n} = +\infty$$
$$\neq c\ ,\ c < \infty$$

Therefore, $(n-2)^2 \neq \Theta(n \log n)$. $\square$

(e) Does $n^{\frac{61}{60}} = \mathcal{O}(n \log n)$?

Proof. $\square$