2. (25%) Consider the general recurrence

$$T(n) = \begin{cases} c_0 & \text{if } n = 1 \\ aT(\frac{n}{b}) + f(n) & \text{otherwise} \end{cases}$$

In class, we showed that this recurrence has solution

$$T(n) = c_0 n^{\log_b a} + \sum_{k=0}^{\log_b n - 1} a^k f(n/b^k).$$

Suppose now that we alter the recurrence so that it terminates not for $n = 1$ but for $n = n_0$, for some constant $n_0 > 1$. In other words, we replace the base case by "$c_0$ if $n = n_0$." You may assume for simplicity that $n$ and $n_0$ are both powers of $b$.

(a) Thinking about the recursion tree for the revised recurrence, it isn't surprising that its solution has the form

$$T'(n) = c_0 a^x + \sum_{k=0}^{x-1} a^k f(n/b^k).$$

What expression should replace the $x$ above? Justify your answer. You might find it helpful to sketch the recursion tree for the revised recurrence.

We will have that $x = \log(\frac{n}{n_0})$.

*Proof.* A table for the recurrence $T(n)$ looks as follows:

| Level | Nodes | Work Per Node |
|---|---|---|
| 0 | 1 | $f(n)$ |
| 1 | $a$ | $f(\frac{n}{b})$ |
| 2 | $a^2$ | $f(\frac{n}{b^2})$ |
| k | $a^k$ | $f(\frac{n}{b^k})$ |
| $\log_b(\frac{n}{n_0})$ | $a^{\log_b(\frac{n}{n_0})}$ | $c_0$ (new base case) |
| $\log_b(n)$ | $a^{\log_b(n)}$ | – |

Because our new base case is $n = n_0$, we only care about the levels above $\log_b(\frac{n}{n_0})$. Therefore, we will sum from $0 \to \log_b(\frac{n}{n_0})$, so the recurrence $T'(n)$ becomes:

$$T'(n) = c_0 a^{\log_b(\frac{n}{n_0})} + \sum_{k=0}^{\log_b(\frac{n}{n_0})-1} a^k f(\frac{n}{b^k})$$

□

(b) Show that $c_0 a^x$ above is still a constant times $n^{\log_b a}$.

*Proof.* We have $n_0$ fixed, so $a^{\log_b(n_0)}$ is fixed and not dependent on $n$. Therefore, using log rules, we have:

$$\begin{aligned} c_0 a^{\log_b(\frac{n}{n_0})} &= c_0 a^{\log_b(n) - \log_b(n_0)} \\ &= c_0 \left(a^{\log_b(n)} a^{-\log_b(n_0)}\right) \\ &= c' \left(a^{\log_b(n)}\right) \\ &= c' \left(n^{\log_b(a)}\right) \end{aligned}$$

where $c' = c_0 a^{-\log_b(n_0)}$, which is a constant.

□

(c) The summation in (a) above can be rewritten as

$$\sum_{k=0}^{\log_b n-1} a^k f(n/b^k) - \sum_{k=\log_b n-y}^{\log_b n-1} a^k f(n/b^k)$$

for some constant $y$ independent of $n$. What is this $y$?

$y = \log_b(n_0)$.

*Proof.* This is evident from our use of logarithm rules. $\log_b(\frac{n}{n_0}) = \log_b(n) - \log_b(n_0)$, therefore we can rewrite the summation in **(a)** to be:

$$\sum_{k=0}^{\log_b(\frac{n}{n_0})-1} a^k f(\frac{n}{b^k}) = \sum_{k=0}^{\log_b n-1} a^k f(n/b^k) - \sum_{k=\log_b n-\log_b n_0}^{\log_b n-1} a^k f(n/b^k)$$

This is also evident, because we wish to remove all the terms between levels $\log_b(\frac{n}{n_0})$ and $\log_b(n)$. That is, for all levels smaller than the base case $n_0$. □

(d) Show that the right-hand summation in (c) can be simplified to $c'n^{\log_b a}$ where $c'$ is an expression independent of $n$.

*Proof.* We wish to show that:

$$\sum_{k=\log_b n-\log_b n_0}^{\log_b n-1} a^k f(\frac{n}{b^k}) = c'n^{\log_b a}$$

for some $c'$ not dependent on $n$. First, we need to adjust the bounds of our sum. Notice that each $k$ will be of the form $k = \log_b n - \log_b n_0 + i$ where $i$ is an integer that ranges from $0 \to \log_b n_0 - 1$. Therefore, we can rewrite this sum as:

$$\begin{aligned}
\sum_{k=\log_b n-\log_b n_0}^{\log_b n-1} a^k f(\frac{n}{b^k}) &= \sum_{i=0}^{\log_b n_0-1} a^{\log_b n-\log_b n_0+i} f(\frac{n}{b^{\log_b n-\log_b n_0+i}}) \\
&= \sum_{i=0}^{\log_b n_0-1} a^{\log_b(n)} a^{\log_b(n_0)} a^i f(\frac{n}{b^{\log_b n} b^{-\log_b n_0} b^i}) \\
&= a^{\log_b(n)} a^{-\log_b(n_0)} \sum_{i=0}^{\log_b n_0-1} a^i f(\frac{n}{nn_0^{-1} b^i}) \\
&= a^{\log_b(n)} a^{-\log_b(n_0)} \sum_{i=0}^{\log_b n_0-1} a^i f(\frac{n_0}{b^i}) \\
&= c' a^{\log_b(n)}
\end{aligned}$$

where

$$c' = a^{-\log_b(n_0)} \sum_{i=0}^{\log_b n_0-1} a^i f(\frac{n_0}{b^i})$$

which is independent of $n$. □