

1. (25%) What Asymptotic solution, if any, does the master method give for each of the following recurrences?

(a)  $T(n) = 16T(\frac{n}{2}) + 3n^4 \log^2(n) = \Theta(n^4 \log^2(n))$ .

*Proof.* Let  $a = 16$ ,  $b = 2$ ,  $f(n) = 3n^4 \log^2(n)$ . Clearly,

$$\log_2(16) = 4 \text{ and } f(n) = \Theta(n^4 \log_2^2(n)).$$

Therefore, by the Master Method Theorem,

$$T(n) = \Theta(n^4 \log_2^3(n)).$$

□

(b)  $T(n) = T(\frac{n}{3}) + 4 \log(n) = \Theta(\log^2(n))$ .

*Proof.* Let  $a = 1$ ,  $b = 3$ , and  $f(n) = 4 \log(n)$ . Clearly,

$$\log_3(1) = 0.$$

And, we can say that:

$$f(n) = \Theta(n^0 \log^1(n)) = \Theta(\log(n)).$$

Therefore,

$$T(n) = \Theta(\log^2(n)).$$

□

(c)  $T(n) = 4T(\frac{n}{4}) + \frac{n}{\log \log n}$  cannot be judged by the Master Method.

*Proof.* Let  $a = 4$ ,  $b = 4$ , and  $f(n) = \frac{n}{\log \log n}$ . Clearly,  $\log_b(a) = \log_4(4) = 1$ . Also, comparing  $g(n) = n$  to  $f(n) = \frac{n}{\log \log n}$  with the limit comparison test,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{\frac{n}{\log \log n}}{n} = \lim_{n \rightarrow \infty} \frac{1}{\log \log n} = 0.$$

Therefore, we can say that

$$f(n) = \frac{n}{\log \log n} = \mathcal{O}(n).$$

However, for any  $\epsilon \in \mathbb{R}^+$  (i.e.  $\epsilon > 0$ ),

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{\frac{n}{\log \log n}}{n^{1-\epsilon}} = \lim_{n \rightarrow \infty} \frac{n^\epsilon}{\log \log n} = \infty.$$

So, we can say that  $f(n) = \Omega(n^{1-\epsilon}) \forall \epsilon > 0$ . For the same conditions on  $\epsilon$ , we can say that  $f(n) = \mathcal{O}(n^{1+\epsilon})$ . Therefore we cannot extract any useful information about this recurrence from the Master Method. □

(d)  $T(n) = 9T(\frac{n}{4}) + 2n^{\log_4 9}$  cannot be solved via the Master Method.

*Proof.* Let  $a = 9$ ,  $b = 4$ , and  $f(n) = 2n^{\log_4(9)}$ . It is trivial to show that,  $f(n) = \Theta(n^{\log_4(9)})$ . Therefore,  $f(n) = \mathcal{O}(n^{\log_4(9)+\epsilon})$  and  $f(n) = \Omega(n^{\log_4(9)-\epsilon}) \forall \epsilon > 0$ . Therefore, we cannot extract any useful information from the Master Method for this problem.  $\square$

(e)  $T(n) = 11T\left(\frac{n}{3}\right) + n^2 = \Theta(n^{\log_3(11)})$ .

*Proof.* Let  $a = 11$ ,  $b = 3$ , and  $f(n) = n^2$ . Quick inspection tells us that  $\log_3(11) > 2$ , so we can say that  $f(n) = \mathcal{O}(n^{\log_3(11)-\epsilon})$  for a small  $\epsilon > 0$ . We can therefore say that  $T(n) = \Theta(n^{\log_3(11)})$  by the Master Method.  $\square$

(f)  $T(n) = 999T\left(\frac{n}{10}\right) + 2n^3 = \Theta(n^3)$ .

*Proof.* Let  $a = 999$ ,  $b = 10$ , and  $f(n) = 2n^3$ . Quick inspection tells us that  $\log_{10}(999) < 3$ , so  $f(n) = 2n^3 = \Omega(n^{\log_{10}(999)+\epsilon})$  for a sufficient  $\epsilon > 0$ . Therefore, we can say that  $T(n) = \Theta(n^3)$  by the Master Method.  $\square$

(g)  $T(n) = 7T\left(\frac{n}{2}\right) + n^3 \log n \log \log n = \Theta(n^3 \log n \log \log n)$ .

*Proof.* Let  $a = 7$ ,  $b = 2$ , and  $f(n) = n^3 \log n \log \log n$ . Quick inspection tells us that  $\log_2(7) < 3$ . We can therefore say that  $f(n) = \Omega(n^{\log_2(7)+\epsilon})$  for a sufficient  $\epsilon > 0$ . Therefore, by the Master Method,  $T(n) = \Theta(n^3 \log n \log \log n)$ .  $\square$