4. (25%)

(a) Does  $(n+1)^2 = \Omega(n \log n)$ ?

*Proof.* Let  $f(n) = (n+1)^2$  and  $g(n) = n \log n$  Take

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{(n+1)^2}{n \log n}$$
$$= \lim_{n \to \infty} \frac{n^2 + 2n + 1}{n \log n}$$
$$= +\infty$$

So, by the limit comparison test,  $(n+1)^2 = \Omega(n \log n)$ .

(b) Does  $3^{9 \log n + \log \log n} = \Omega(n^3)$ ? True.

*Proof.* Note that:

$$f(n) = 3^{9 \log n + \log \log n} = 3^{9 \log n} 3^{\log \log n}$$

$$= 3^{(\log n^9)} 3^{\log \log n}$$

$$= (n^9)^{\log(3)} 3^{\log \log n}$$

$$= n^{9 \log(3)} 3^{\log \log n}$$

Also note both that  $\log_2(\log_2(n)) \ge 1 \forall n \ge 4$ , and  $9(\log_2(3)) \approx 14$  (it's actually slightly greater). Therefore, f(n) can be bounded below by  $3n^{14} \ \forall n \ge 4$ . Therefore:

$$\lim_{n \to \infty} \frac{3^{9\log n + \log\log n}}{n^3} \geq \lim_{n \to \infty} \frac{3n^{14}}{n^3} = +\infty$$

(c) Does  $n \log_5 n = \Theta(n \ln n)$ ? True.

*Proof.* Note that:

$$n\log_5 5 = n\frac{\ln n}{\ln 5}$$

and that:

$$\lim_{n\to\infty}\frac{n\log_5 n}{n\ln n}=\lim_{n\to\infty}\frac{1}{\ln 5}=\frac{1}{\ln 5}$$

Because this limit is a constant,  $n \log_5 n = \Theta(n \ln n)$ 

(d) Does  $(n-2)^2 = \Theta(n \log n)$ ?

*Proof.* Note that:

$$\lim_{n \to \infty} \frac{(n-2)^2}{n \log n} = \lim_{n \to \infty} \frac{n^2 - 4n + 4}{n \log n} = +\infty$$

$$\neq c , c < \infty$$

Therefore,  $(n-2)^2 \neq \Theta(n \log n)$ .

(e) Does  $n^{\frac{61}{60}} = \mathcal{O}(n \log n)$ ? False.

Proof.

$$\lim_{n \to \infty} \frac{n^{\frac{61}{60}}}{n \log n} = +\infty$$

Therefore,  $n^{\frac{61}{60}}$  is not  $\mathcal{O}(n \log n)$  but it is  $\Omega(n \log n)$ .

(f) Let f(n) and g(n) be non-negative functions of n. If  $f(n) = \mathcal{O}(g(n))$ , does  $f(n) + g(n) = \Theta(g(n))$ ? True.

*Proof.* Because  $f(n) = \mathcal{O}(g(n))$ :

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

Therefore, we have:

$$\lim_{n \to \infty} \frac{f(n) + g(n)}{g(n)} = \lim_{n \to \infty} \frac{f(n)}{g(n)} + \frac{g(n)}{g(n)} = 1$$
$$= \lim_{n \to \infty} 0 + \lim_{n \to \infty} \frac{g(n)}{g(n)} = 1 < \infty.$$

Therefore,  $f(n) + g(n) = \Theta(g(n))$ .

(g) Let f(n) and g(n) be non-negative functions of n. If  $f(n) = \Theta(g(n))$ , does  $\frac{f(n)}{g(n)} = \Theta(1)$ ?

*Proof.* By the provided relation, we know:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=c<\infty.$$

Therefore, it is trivial to show that:

$$\lim_{n \to \infty} \frac{\left(\frac{f(n)}{g(n)}\right)}{1} = \lim_{n \to \infty} \frac{f(n)}{g(n)} = c < \infty.$$

Therefore,  $\frac{f(n)}{g(n)} = \Theta(1)$ .