Semi-explicit Solutions to Structured Quadratic Programming

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Allowing for some structure on a $p \times p$ matrix $\mathbf{Q}$, we derive semi-explicit solutions to constrained, quadratic optimization problems of the form[1]

$$\min_{x \in \mathbb{R}^p} x^\top \mathbf{Q} x \tag{1}$$
$$\text{subject to: } \begin{cases} \sum_k x_k = 1, \\ \text{each } x_k \geq 0. \end{cases}$$

These problems arise in a wide array of scientific and engineering applications. The constraint $x_i \geq 0$ is often a crucial ingredient and in its presence, the problem admits no (known) explicit solution in general. A numerical solver must be used. This is disadvantageous for two reasons we highlight.

(1) For p large, the computational efficiency may be unsatisfactory.

(2) Optimal points are not interpretable in the problem parameters.

Analytic expressions address both of these disadvantages. We derive semi-closed-form solutions to (1) under the assumption that $\mathbf{Q}$ arises from a factor model. That is, for a low-rank matrix $\mathbf{BVB}^\top = \sum_{k=1}^q \sigma_k^2 \beta_k \beta_k^\top$ (i.e. rank $q \ll p$) where $\beta_k \in \mathbb{R}^p$ and $\sigma_k \in \mathbb{R}$, we assume that $\mathbf{Q} - \mathbf{BVB}^\top$ is a diagonal matrix and prove that the minimizer of (1) is a function of $\mathbf{\Delta}$ and

$$\mathbf{B}\theta = \theta_1 \beta_1 + \cdots + \theta_q \beta_q; \qquad \theta \in \mathbb{R}^q. \tag{2}$$

The vector $\theta = (\theta_1, \ldots, \theta_q)$ is not known explicitly, but may be recovered efficiently by solving a certain fixed point problem (in dimension q). The fixed point characterization of θ also yields to interpretation of solutions in terms of each constituent factor β_k, an important feature for many applications.

Factor structure assumptions of the type above are pervasive in practice and give credence to statistical tools such as PCA. As an example we provide a case study on the Markowitz portfolio problem with the no-short sales constraint. Our results illustrate the computational advantages (item 1 above). We leave questions of portfolio composition (item 2 above) to future work.

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[1]Our results generalize to additional linear constraints of the form $x^\top a \geq b$ as well as a general lower bound constraints $x \geq \ell$ for $\ell \in \mathbb{R}^p$ may be added but excluded here for brevity.

1. Preliminaries. Consider a symmetric $p \times p$ matrix $\mathbf{Q}$ given by

$$\mathbf{Q} = \mathbf{B}\mathbf{V}\mathbf{B}^\top + \mathbf{\Delta} \tag{3}$$

for a $p \times q$ matrix $\mathbf{B}$ and diagonal matrices $\mathbf{V} = \mathbf{diag}(\sigma_1^2, \ldots, \sigma_q^2)$ and $\mathbf{\Delta} = \mathbf{diag}(\delta_1^2, \ldots, \delta_p^2)$ with $\delta_i^2, \sigma_k^2 \in (0, \infty)$. We regard q as much smaller than p.

To illustrate how $\mathbf{Q}$, structured as in (3), arises in applications, consider the covariance matrix of a linear q-factor model over p variables. That is, for a predictor $X = (X_1, \ldots, X_q)^\top$ we observe a $\mathbb{R}^p$-valued response Y defined by

$$Y = \mathbf{B}X + Z \tag{4}$$

that is subject to errors $Z = (Z_1, \ldots, Z_p)$. The variables X and Z are random while $\mathbf{B}$ is a constant factor structure generating the correlation in the model. If the $\{X_k, Z_i\}$ are all uncorrelated (a standard assumption), $\mathrm{Var}(Y) = \mathbf{Q}$ for the right side equal to (3) whenever every $\mathrm{Var}(X_k) = \sigma_k^2$ and every $\mathrm{Var}(Z_i) = \delta_i^2$.

2. Main Results. Our Main Theorem is:

Theorem 2.1. *Recall the minimization problem (1) . Without loss of generality, we let $\mathbf{B}^\top \mathbf{\Delta}^{-1} \mathrm{e} \geq 0$. Define $\varphi : \mathbb{R}^q \to \mathbb{R}^q$ by*

$$\varphi(\vartheta) = \mathbf{V}\mathbf{B}^\top \mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\vartheta)_+ \tag{5}$$

Let x be a solution of problem (1). Then

$$x = \frac{w}{\langle w, \mathrm{e}\rangle} \quad \textit{where} \quad w = \mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\theta)_+ \tag{6}$$

for $\theta \in \mathbb{R}^q$ the unique fixed point of φ (i.e., $\varphi(\theta) = \theta$).

Remark 2.2. *The condition such that $\mathbf{B}^\top \mathbf{\Delta}^{-1} \mathrm{e} \geq 0$ is without loss of generality, as we can flip the sign of any vector without altering the structure of the covariance matrix.*

Remark 2.3. *$\varphi(\cdot)$ is not a contraction mapping; in fact, fixed-point iterations of $\varphi(\cdot)$ generally do not converge.*

Define $\psi : \mathbb{R}^q \to \mathbb{R}^q$ as $\psi(\vartheta) = \mathbf{A}_\vartheta^{-1} b_\vartheta$ where

$$\begin{aligned} \mathbf{A}_\vartheta &= \mathbf{V}^{-1} + \mathbf{B}^\top \mathbf{\Delta}^{-1} \mathbf{diag}(\chi_\vartheta)\mathbf{B} \\ b_\vartheta &= \mathbf{B}^\top \mathbf{\Delta}^{-1} \chi_\theta \\ \chi_\vartheta &= (\mathrm{e} \geq \mathbf{B}\vartheta) \in \{0,1\}^p \end{aligned} \tag{7}$$

Lemma 2.4. *We have $\theta = \varphi(\theta)$ if and only if $\theta = \psi(\theta)$.*

Lemma 2.5. *The iterates $\vartheta^{k+1} = \psi(\vartheta^k)$ converge to the fixed point of ψ (i.e. $\theta = \psi(\theta)$) provided that $\vartheta^0 = 0 \in \mathbb{R}^q$.*

Theorem 2.6. *The iterates $\vartheta^{k+1} = \psi(\vartheta^k)$ converge to the fixed point of ψ (i.e. $\theta = \psi(\theta)$) provided that $\vartheta^0 = 0 \in \mathbb{R}^q$. This is also the fixed point $\varphi(\theta) = \theta$, which solves for the required θ in Theorem 2.1.*

A similar expression for the Long/Short portfolio exists and omits the positivity correction, but this leads to a vastly different fixed point. The portfolios correspond as follows:

Quantity	LS Min. Var.	LO Min. Var.
weights	$x = \frac{w}{\langle w, \mathrm{e}\rangle}$	$x = \frac{w}{\langle w, \mathrm{e}\rangle}$
w	$w = \mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\theta)$	$w = \mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\theta)_+$
θ	$\theta = \varphi(\theta)$	$\theta = \varphi(\theta)$
$\varphi(\vartheta)$	$\mathbf{V}\mathbf{B}^\top\mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\vartheta)$	$\mathbf{V}\mathbf{B}^\top\mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\vartheta)_+$
fixed point	$w = \Psi(w)$	$w = \Psi(w)$
$\Psi(v)$	$\mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\mathbf{V}\mathbf{B}^\top v)$	$\mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\mathbf{V}\mathbf{B}^\top v)_+$

3. Previous Work and the 1-Factor Case. A similar solution has existed for a few years (**?**), but only considers a single factor case. With a single factor, we can generalize the original problem slightly

$$\begin{aligned} &\min_{x\in\mathbb{R}^p} x^\top \mathbf{Q} x \\ &\text{subject to: } \begin{cases} \sum_k a_k x_k = 1, \\ \text{each } x_k \geq 0. \end{cases} \end{aligned} \tag{8}$$

where

$$Q = \beta\beta^\top + \Delta$$

and $\beta \in \mathbb{R}^p$ and Δ is a (strictly) positive-definite diagonal matrix. The solution contained a similar formalism for the elements of the factor vector, with the un-normalized weights w_i having the following form:

$$w_i = \frac{1}{\delta_i^2}\left(a - \theta\beta_i\right)_+ \tag{9}$$

where θ is found as the fixed point of:

$$\psi_{\text{CDT}}(z) = \frac{1/\sigma^2 + \sum_{\{i:\beta_i<z\}} \beta_i^2/\delta_i^2}{\sum_{\{i:\beta_i<z\}} \beta_i/\delta_i^2} \tag{10}$$

This result is almost correct; a fixed point may not exist of $sum_i\beta_i < 0$, and this is clearly a discontinuous map, so proving existence and uniqueness is still difficult. Furthermore, for some values of z, ψ_{CDT} may be $\pm\infty$. Our map, ψ, is essentially the reciprocal of ψ_{CDT}, generalized to multiple dimensions with a slightly different condition on the summation. Rather than checking if $\beta_i < z$, we check if $z\beta_i < 1$. Note that this alters the condition slightly, such that the signs of β_i and z affect which elements are being summed. Also, our assumption that $\sum_i \frac{\beta_i}{\delta_i^2} > 0$ guarantees the existence of a fixed point with our formulation. Our fixed-point finding map is therefore:

$$\psi(z) = \frac{\sum_{\{i:z\beta_i<1\}} \beta_i/\delta_i^2}{1/\sigma^2 + \sum_{\{i:z\beta_i<1\}} \beta_i^2/\delta_i^2} \tag{11}$$

This is simply the 1-dimensional version of Equation (7). This is still discontinuous, of course, but, as we have previously shown, the fixed points of this function correspond to the fixed points of

$$\phi(z) = \sigma^2 \left(\sum_{\{i: z\beta_i < 1\}} \frac{\beta_i}{\delta_i^2} (1 - z\beta_i) \right). \tag{12}$$

This is, of course, also the 1-dimensional version of Equation (5). This gives us the followinf set of lemmas and theorems:

Lemma 3.1. *The fixed-points of ϕ correspond with the fixed-points of ψ*

Proof. The implication both ways is straightforward algebraic manipulation under the assumption that $\sum_i \frac{\beta_i}{\delta_i^2} > 0$. □

Lemma 3.2. *The following are true about ϕ:*

1. *$\phi(z)$ is continuous*
2. *$\phi(0) > 0$*
3. *ϕ is differentiable a.e. and $\phi'(z) < 0$ for all $z > 0$*
4. *ϕ has a positive root*

Proof. These are all fairly straightforward to see in 1-dimension:

1. Continuity follows because the $\phi(\cdot)$ is piecewise-linear, and only changes slope when $z\beta_i = 1$, in which case the term being removed from the summation is identically 0.
2. $\phi(0) > 0$ follows from our assumption that $\sum_i \frac{\beta_i}{\delta_i^2} > 0$; if $z\beta_i < 1$ then

   $$\frac{\beta_i}{\delta_i^2} < (z\beta_i)\frac{\beta_i}{\delta_i^2}$$

   and at $z = 0$, $\phi(z)$ must include all elements of β as $0\beta_i < 1$.
3. Differentiability follows directly from piecewise linearity; the coefficient of z is also negative, meaning that the derivative is piecewise constant and negative where it exists. Furthermore, the derivative exists all points where $z\beta_i \neq 0$ for some i.
4. We have that $\phi(0) > 0$ and ϕ has a piecewise constant negative derivative a.e., so ϕ must cross 0 at some point of finite value.

□

Theorem 3.3. *ϕ has a unique fixed point, and therefore so does ψ.*

Proof. Applying the previous lemmas, we have that ϕ is continuous on a compact set, which guarantees existence of a fixed point via the Brouwer Fixed-Point Theorem.
Uniqueness follows from the fact that ϕ is monotonically decreasing on the entire compact set. $\square$

Define:

$$\psi(z) = \frac{\sum_{z\beta_i<1} \beta_i \delta_i^2}{1/\sigma^2 + \sum_{z\beta_i<1} (\beta_i/\delta_i)^2} \tag{13}$$

Assumption 3.4. *We assume $\psi(0) = 0$ and $\beta_1 > \beta_2 > \ldots > \beta_p$.*

Remark 3.5. *These assumptions are without loss of generality.*

Remark 3.6. *We prove the following theorem for the interval $\left[0, \frac{1}{\beta_1}\right)$. It generalizes inductively to positive intervals in $\mathbb{R}_+$.*

Theorem 3.7. *Beginning from $z_0 = 0$, the fixed point iterations of $z_{k+1} = \psi(z_k)$ converge to a fixed point, where $\psi(z_k)$ is defined above.*

Proof. First, we must define some notation. Consider the following:

$$A_k = \sum_{i=k+1}^{p} \frac{\beta_i}{\delta_i^2}$$
$$B_k = \frac{1}{\sigma^2} + \sum_{i=k+1}^{p} \frac{\beta_i^2}{\delta_i^2}$$

Note that because $\beta_1 > \beta_2 > \ldots > \beta_p$, $\frac{1}{\beta_1} < \frac{1}{\beta_2} < \ldots$ for all of the values $\beta_i > 0$. Clearly, we have that $\psi(0) = \frac{A_0}{B_0}$. Further, as long as $z < \frac{1}{\beta_1}$, we have $z\beta_1 < 1$, and so $\psi(z)$ is constant on the interval $[0, \frac{1}{\beta_1})$ (it will turn out, this is without loss of generality; $\psi(z)$ is constant between any interval $[\frac{1}{\beta_i}, \frac{1}{\beta_{i+1}})$). We will show that as z increases from 0, for $z \in [0, \frac{1}{\beta_1})$, the function $\psi(z)$ has two possibilities:

- $\psi(z) = z$ for some $z \in \left[0, \frac{1}{\beta_1}\right)$
- If $\psi(z) \neq z$ for all $z \in \left[0, \frac{1}{\beta_1}\right)$, $\psi(z)$ jumps upwards for the interval $\left[\frac{1}{\beta_1}, \frac{1}{\beta_2}\right)$; that is: the left limit $\psi\left(\frac{1}{\beta_1}^-\right) < \psi\left(\frac{1}{\beta_1}\right)$

Naively, it seems possible that $\psi\left(\frac{1}{\beta_1}^-\right) > \psi\left(\frac{1}{\beta_1}\right)$- that is, the function *decreases* at $\frac{1}{\beta_1}$. We will show this is not possible.
If it were possible (and $\psi(z) \neq z$ for $0 < z < \frac{1}{\beta_1}$), we would have:

$$\psi\left(\frac{1}{\beta_1}^-\right) = \frac{A_0}{B_0} > \frac{A_1}{B_1} = \psi\left(\frac{1}{\beta_1}\right). \tag{14}$$

Assume, this were true, that is assume

$$\frac{A_0}{B_0} > \frac{A_1}{B_1} \tag{15}$$

There are two cases:

1. $\frac{A_1}{B_1} > \frac{1}{\beta_1}$
2. $\frac{A_1}{B_1} < \frac{1}{\beta_1}$

In Case 1, we have:

$$\frac{A_0}{B_0} > \frac{A_1}{B_1} > \frac{1}{\beta_1} \tag{16}$$

where we note that:

$$\frac{A_0}{B_0} = \frac{\frac{\beta_1}{\delta_1^2} + \sum_{k=2}^{p} \frac{\beta_i}{\delta_i^2}}{\frac{1}{\sigma^2} + \frac{\beta_1^2}{\delta_1^2} + \sum_{k=2}^{p} \frac{\beta_i^2}{\delta_i^2}} = \frac{\delta_1^2 A_1 + \beta_1^2}{\delta_1^2 B_1 + \beta_1^2} \tag{17}$$

$$= \frac{A_1 + \frac{\beta_1}{\delta_1^2}}{B_1 + \frac{\beta_1^2}{\delta_1^2}} \tag{18}$$

It therefore follows that:

$$\frac{A_0}{B_0} = \frac{A_1 + \frac{\beta_1}{\delta_1^2}}{B_1 + \frac{\beta_1^2}{\delta_1^2}} > \frac{A_1}{B_1} \tag{19}$$

$$\implies \frac{\beta_1}{\delta_1^2} B_1 > \frac{\beta_1^2}{\delta_1^2} A_1 \tag{20}$$

It therefore follows that

$$\frac{A_1}{B_1} < \frac{1}{\beta_1} \tag{21}$$

which contradicts our assumption that $\frac{A_1}{B_1} > \frac{1}{\beta_1}$. We must now check Case 2. In Case 2, we have:

$$\frac{A_1}{B_1} < \frac{1}{\beta_1} \tag{22}$$

By similar computations to those with A_0 and B_0, we have:

$$\frac{A_1}{B_1} = \frac{\delta_2^2 A_2 + \beta_2}{\delta_2^2 B_2 + \beta_2^2} < \frac{1}{\beta_1} \tag{23}$$

Assumption 3.8. *we proceed under the assumption that* $\beta_1 > \beta_2 > 0$ *and* $A_1, A_2 > 0$.

If $\beta_2 \leq 0$, our ordering assumption means that $\beta_3, \ldots < 0$, and so $A_2 < 0$ and so $\frac{A_1}{B_1} < 0$. We therefore have that $\psi(z) < 0$ for all $z > \frac{1}{\beta_1}$, and so we never have a fixed point. Therefore, under the assumption that $\beta_2 > 0$, our previous assertion implies that

$$\beta_1 \left(\delta_2^2 A_2 + \beta_2\right) < \delta_2^2 B_2 + \beta_2^2 \tag{24}$$

Rearranging terms, we have

$$\delta_2^2 \left(A_2 \beta_1 - B_2\right) < \beta_2 \left(\beta_2 - \beta_1\right) \tag{25}$$

However, $\beta_2 < \beta_1$, so this implies that

$$A_2 \beta_1 - B_2 < 0 \tag{26}$$

$$\implies \frac{A_2}{B_2} < \frac{1}{\beta_1} \tag{27}$$

Now, assume that

$$\frac{A_2}{B_2} > \frac{A_1}{B_1} \tag{28}$$

we are only assuming this, positivity of β_1, β_2, and that $\frac{A_1}{B_1} < \frac{1}{\beta_1}$ (although it follows from our previous result that $\frac{A_1}{B_1} < \frac{1}{\beta_1}$). We have

$$\frac{A_2}{B_2} > \frac{\delta_2^2 A_2 + \beta_2}{\delta_2^2 B_2 + \beta_2^2} \tag{29}$$

and it follows that

$$A_2 \left(\delta_2^2 B_2 + \beta_2^2\right) > B_2 \left(\delta_2^2 A_2 + \beta_2\right) \tag{30}$$

$$\implies A_2 \beta_2^2 > B_2 \beta_2 \tag{31}$$

and so it follows that $(A_2\beta_2 - B_2) > 0$. However, from our earlier assumption, that $\beta_1 > \beta_2$, note that $A_2\beta_1 > A_2\beta_2$. Therefore, this contradicts that $A_2\beta_1 - B_2 < 0$. Therefore Case 2 cannot happen either.
It therefore follows that $\psi(z)$ either has a fixed point in $[0, \frac{1}{\beta_1})$ or increases when $z > \frac{1}{\beta_1}$. □

4. Sensitivity Analysis of Unnormalized Weights at Fixed Point. We wish to study the sensitivity of both the normalized weights x_i and the un-normalized weights w_i to the parameters σ^2, β_j and δ_i^2 at the fixed point, i.e. where

$$\theta = \psi(\theta) = \phi(\theta). \tag{32}$$

Note that $\psi(\cdot)$, $\phi(\cdot)$ are defined above. Note that at the fixed point,

$$\theta = \phi(\theta) \tag{33}$$

and

$$\frac{\partial}{\partial \alpha}\theta = \frac{\partial}{\partial \alpha}\phi(\theta) \tag{34}$$

where $\alpha \in \{\sigma^2, \beta_j, \delta_i^2\}$. We will take advantage of this fact for each of our computations.

To be more precise, define:

$$\theta = \theta(\alpha) \tag{35}$$

$$\frac{\partial}{\partial \alpha}\theta(\alpha) = \lim_{h \to 0} \frac{\theta(\alpha + h) - \theta(\alpha)}{h} \tag{36}$$

Note that at the fixed point,we must have $\phi(\theta) = \theta$ First we consider the derivative of x_i and w_i with respect to σ^2.

5. Algorithm. Algorithm 1 computes the solution in Theorem 2.6. It implements the fixed point iterations $\theta^{n+1} = \psi(\theta^n)$ to compute $\psi(\theta) = \theta$.

Algorithm 1 (FFP). Given ($\mathbf{B}$, $\mathbf{V}$, $\mathbf{\Delta}$), computes the optimizer of problem (1).

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1. Initialize $\theta_{\text{old}} \leftarrow (0, \dots, 0) \in \mathbb{R}^q$ and tolerance $\epsilon > 0$.

2. Assemble $\mathbf{A}_{\theta_{\text{old}}}$ and $b_{\theta_{\text{old}}}$ from $(\mathbf{B}, \mathbf{V}, \mathbf{\Delta})$ and θ_{old} as in (7).

3. Compute θ_{new} by solving $\mathbf{A}_{\theta_{\text{old}}} \theta_{\text{new}} = b_{\theta_{\text{old}}}$.

4. If $|\theta_{\text{new}} - \theta_{\text{old}}| > \epsilon$, update $\theta_{\text{old}} \leftarrow \theta_{\text{new}}$ and go to Step **2.**.

5. Compute $w = \mathbf{\Delta}^{-1}(\mathrm{e} - \mathbf{B}\theta_{\text{new}})_+$ and **return** $\frac{w}{\mathrm{e}^\top w}$.

We refer to Algorithm 1 as the factor fixed point (FFP) method. The considerable gain in efficiency of the FFP algorithm over traditional quadratic programming solvers is due to the fact that it requires only solutions to a $q \times q$ linear system. Traditional solvers require computations on the full problem size $p \times p$.

Under our assumptions on $\mathbf{Q}$, the FFP algorithm has a running time of $O(pq + q^2)$. The time consuming operations (those of order pq) involve assembling the linear system in (7). This complexity may potentially be reduced by solving (7) explicitly via Woodbury type identities and precomputing some of the variables. We do not pursue this here. We also leave open the question of convergence of the FFP iterations. We treat them as constant (relative to p).

A. Python Pseudocode.

`Python` implementation of the function ψ.

```
def psi(theta):

    chi = Dinv * (ones >= B @ theta)
    b = B.T @ chi
    A = Vinv + (B.T @ np.diag(chi)) @ B

    return solve(A,b)
```

`Python` implementation of the fixed point iterations.

```
def compute_fixed_point (t):

    s = t + 2 * tol

    while (norm(s - t) > tol * norm(s)):
        t = s
        s = psi(t)

    return(s)
```

`Python` computation of the long-only portfolio weights.

```
def lo_weights ():

    theta = psi (np.zeros(q))
    w = Dinv @ maximum(ones - B @ theta, 0)

    return w / sum(w)
```

References.