# PSTAT 222C Homework 1

Problem 1

We consider a corridor option for asset $X_t$ with payoff of \$1 if $X_t \in [L_1, L_2]$ and \$0 otherwise. $X_t$ is governed by the SDE

$$dX_t = rX_t dt + \sigma^2 X_t^\gamma dW_t \tag{1}$$

where $\gamma$ is the elasticity of the variance. Applying Ito's Lemma to the price of the option, $V(t,x)$, we get:

$$\begin{align}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}dX_t^2 \tag{2}\\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}(rX_t dt + \sigma^2 X_t^\gamma dW_t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(rX_t dt + \sigma^2 X_t^\gamma dW_t)^2 \tag{3}\\
&= \left(\frac{\partial V}{\partial t} + rX_t\frac{\partial V}{\partial x} + \frac{\sigma^2 X_t^{2\gamma}}{2}\frac{\partial^2 V}{\partial x^2}\right)dt + \sigma X_t^\gamma \frac{\partial V}{\partial x}dW_t \tag{4}
\end{align}$$

Under the assumption that this is the risk-neutral measure, we must have that the expected price of the option is equal to the present value of the risk free rate of the price of the option; that is:

$$\mathbb{E}[dV] = rVdt = \left(\frac{\partial V}{\partial t} + rX_t\frac{\partial V}{\partial x} + \frac{\sigma^2 X_t^{2\gamma}}{2}\frac{\partial^2 V}{\partial x^2}\right)dt \tag{5}$$

It therefore follows that the PDE for the CEV model can be written as:

$$\frac{\partial V}{\partial t} + rX_t\frac{\partial V}{\partial x} + \frac{\sigma^2 X_t^{2\gamma}}{2}\frac{\partial^2 V}{\partial x^2} - rV = 0 \tag{6}$$

Let the value of the discretization of the PDE at time $m\Delta t$ and $n\Delta x$ be $V(m\Delta t, n\Delta x) = V_n^m$. The explicit finite difference scheme is given by:

$$\frac{V_n^m - V_n^{m-1}}{\Delta t} + r(n\Delta x)\frac{V_{n+1}^{m-1} - V_{n-1}^{m-1}}{2\Delta x} + \frac{\sigma^2 (n\Delta x)^{2\gamma}}{2}\frac{V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}}{(\Delta x)^2} - rV_n^{m-1} = 0 \tag{7}$$

We can derive the implicit Finite Difference similarly

$$\frac{V_n^m - V_n^{m-1}}{\Delta t} + r(n\Delta x)\frac{V_{n+1}^m - V_{n-1}^m}{2\Delta x} + \frac{\sigma^2(n\Delta x)^{2\gamma}}{2}\frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{(\Delta x)^2} - rV_n^m = 0 \tag{8}$$

Solving for $V_n^{m-1}$, we find:

$$\begin{aligned}
V_n^{m-1} &= V_n^m + \Delta t\left(r(n\Delta x)\frac{V_{n+1}^m - V_{n-1}^m}{2\Delta x} + \frac{\sigma^2(n\Delta x)^{2\gamma}}{2}\frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{(\Delta x)^2} - rV_n^m\right) && (9)\\
&= \underbrace{\frac{\Delta t}{2}\left(-rn + \frac{\sigma^2(n\Delta x)^{2\gamma}}{(\Delta x)^2}\right)}_{\tilde{a}_n} V_{n-1}^m + \left(1 - \underbrace{r\Delta t + \frac{\sigma^2(n\Delta x)^{2\gamma}\Delta t}{(\Delta x)^2}}_{\tilde{b}_n}\right) V_n^m + \underbrace{\frac{\Delta t}{2}\left(rn + \frac{\sigma^2(n\Delta x)^{2\gamma}}{(\Delta x)^2}\right)}_{\tilde{c}_n} V_{n+1}^m && (10)\\
&= \tilde{a}_n V_{n-1}^m + (1 - \tilde{b}_n)V_n^m + \tilde{c}_n V_{n+1}^m && (11)
\end{aligned}$$

Imposing the exogenous boundary at $L_1$ and $L2$ such that for all $0 \le m \le T/\Delta t$, $V_{L_1}^m = V_{L_2}^m = 1$ and $V_{L_1-1}^m = V_{L_2+1}^m = 0$ we find that we can represent the discretized PDE as:

$$\begin{aligned}
AV^{m-1} + g^{m-1} &= BV^m && (12)\\
V^{m-1} &= A^{-1}\left(BV^m - g^{m-1}\right) && (13)
\end{aligned}$$

where $B = I$, the identity matrix.

To see how to apply the boundary conditions, it is helpful to visualize the matrix representation of this equation. We have:

$$\begin{pmatrix} \tilde{a}_1 & 1+\tilde{b}_1 & \tilde{c}_1 & 0 & \cdots & 0 \\ 0 & \tilde{a}_2 & 1+\tilde{b}_2 & \tilde{c}_2 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \tilde{a}_{N-1} & 1+\tilde{b}_{N-1} & \tilde{c}_{N-1} \end{pmatrix} \begin{pmatrix} V_0^{m-1} \\ V_1^{m-1} \\ \vdots \\ V_{N-1}^{m-1} \\ V_N^{m-1} \end{pmatrix} = \begin{pmatrix} V_1^m \\ V_1^m \\ \vdots \\ V_{N-1}^m \\ V_{N-1}^m \end{pmatrix} \tag{14}$$

and it becomes clear that $g^{m-1}$ is given by:

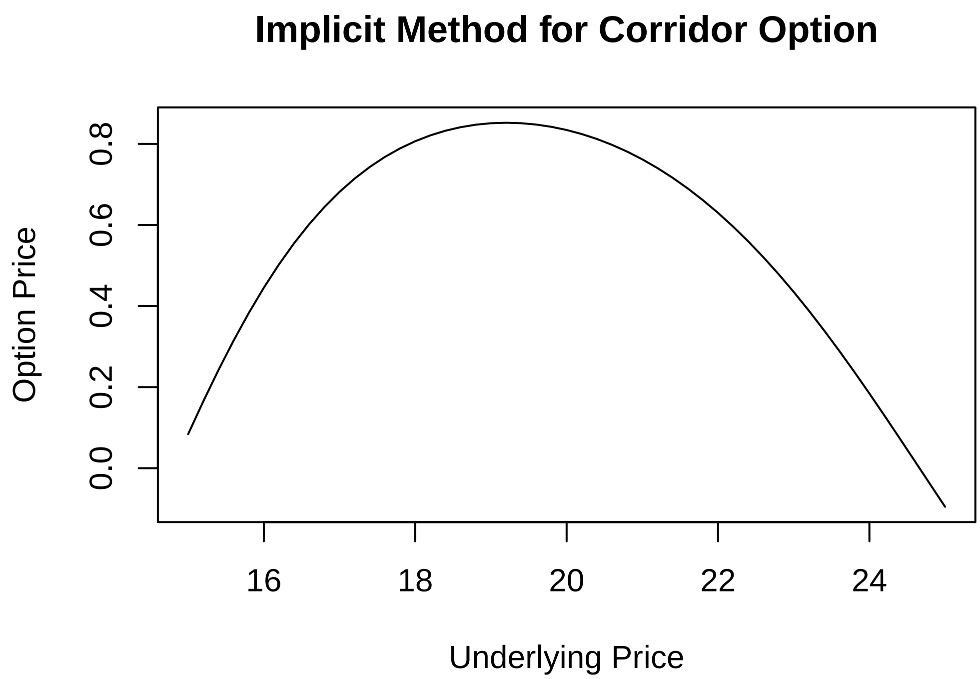
$$g^{m-1} = \begin{pmatrix} \tilde{a}_1 V_0^{m-1} \\ 0 \\ \vdots \\ 0 \\ \tilde{c}_{N-1} V_N^{m-1} \end{pmatrix} \tag{15}$$

## Implicit Finite Difference with Exogenous Boundary Conditions

```r
gam<-0.8
dt <- 0.1
T = 1/2
dx <- 0.2
L1 <- 15
L2 <- 25

implicitfd<- function(T,dt,dx,X_min,X_max,gam,opttype,K=0,other_asset=0,x0){
  r <- 0.05
  sigma <- 0.4
  time_steps <- as.integer(T/dt)
  space_steps <- as.integer((X_max-X_min)/dx)+1
  vetS <- X_min + dx*(0:(space_steps-1))

  V = matrix(0,space_steps,time_steps) # intitialize matrix

  vetI <- X_min/dx + 0:(space_steps-1)
  a_i <- dt/2*(sigma^2*(vetI*dx)^(2*gam)/(dx^2) - r*vetI)
  b_i <- -dt*(sigma^2*(vetI*dx)^(2*gam)/(dx^2) + r)
  c_i <- dt/2*(sigma^2*(vetI*dx)^(2*gam)/(dx^2) + r*vetI)

  Amatrix <- diag(1-b_i,space_steps)
  Amatrix[(row(Amatrix) - col(Amatrix)) == 1] <- -a_i[2:(space_steps)]
  Amatrix[(row(Amatrix) - col(Amatrix)) == -1] <- -c_i[1:(space_steps-1)]
  if(opttype=="cor"){
    V[,ncol(V)] <- 1 - ((vetS>L2 | vetS<L1)*1)
  }
  if(opttype=="call"){
    V[,ncol(V)] <- pmax(vetS-K,0)
    V[nrow(V),] <- X_max - K*exp(-r*T)
    }
  if(opttype=="compound"){
      V[,ncol(V)] <- pmax(other_asset-K,0)
      V[nrow(V),] <- other_asset[length(other_asset)] - 20*exp(-r*T) #hardcoded strike for other asset f
    }

  V[1,] = 0

  Bmatrix <- diag(1,space_steps)
  for (k in 1:time_steps-1){
    t = time_steps - k
    g <- c(rep(0,space_steps-1),c_i[length(c_i)]*(V[nrow(V),ncol(V)-1]))
    g[1] = a_i[1]*V[1]
    V[,t-1]<-solve(Amatrix, Bmatrix %*% V[,t] - g)
```

```
  }
  return(list(V=V,price=V[which(vetS==x0)]))
}
val<-implicitfd(T,dt,dx,L1,L2,gam,"cor",0,0,20)
```

```
val<-implicitfd(T,dt,dx,L1,L2,gam,"cor",0,0,20)
options(scipen=999)
plot(x = seq(15,25,dx), y= val$V[,1],type='l',main="Implicit Method for Corridor Option",
ylab="Option Price",xlab = "Underlying Price" )
```




### Crank-Nicholson Solver

The Crank-Nicholson Method is essentially an average of the Explicit and Implicit Finite Difference Methods. Writing this out, we have:

$$\frac{V_n^m - V_n^{m-1}}{\Delta t} = \frac{1}{2}\left(-r(n\Delta x)\frac{V_{n+1}^m - V_{n-1}^m}{2\Delta x} - \frac{\sigma^2(n\Delta x)^{2\gamma}}{2}\frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{(\Delta x)^2} + rV_n^m\right) \tag{16}$$

$$+\frac{1}{2}\left(-r(n\Delta x)\frac{V_{n+1}^{m-1} - V_{n-1}^{m-1}}{2\Delta x} - \frac{\sigma^2(n\Delta x)^{2\gamma}}{2}\frac{V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}}{(\Delta x)^2} + rV_n^{m-1}\right) \tag{17}$$

```
cnFd<-function(dt,dx,gam,opttype,T){
  r <- 0.05
  sigma <- 0.4
  L1 <- 15
  L2 <- 25
  X_min <- 15
  X_max <- 25

  time_steps <- as.integer(T/dt)
  space_steps <- as.integer((X_max-X_min)/dx)+1

  vetS <- X_min + dx*(0:(space_steps-1))


  vetI <- X_min/dx + 0:(space_steps-1)

  a_i <-  dt/4*(sigma^2*(vetI*dx)^(2*gam)/(dx^2)-r*vetI)
  b_i <- -dt/2*(sigma^2*(vetI*dx)^(2*gam)/(dx^2) + r)
  c_i <-  dt/4*(sigma^2*(vetI*dx)^(2*gam)/(dx^2) + r*vetI)

  # A = [-a below, 1-b on diag, -c above ]
  Amatrix <- diag(1-b_i,space_steps)
  Amatrix[(row(Amatrix) - col(Amatrix)) == 1] <- -a_i[2:(space_steps)] # row > col, i.e. below diag
  Amatrix[(row(Amatrix) - col(Amatrix)) == -1] <- -c_i[1:(space_steps-1)] #row < col, i.e. above diag

  # B = [a below, 1+b on diag, c above]
  Bmatrix<-diag(1+b_i,space_steps)
  Bmatrix[(row(Bmatrix)-col(Bmatrix)) == 1 ]<- a_i[2:space_steps]
  Bmatrix[(row(Bmatrix)-col(Bmatrix)) == -1 ]<- c_i[1:(space_steps-1)]

  if(opttype=="cor"){
    V <- 1 - ((vetS>L2 | vetS<L1)*1)
  }
  else if(opttype=="call"){

  }

  g <- c(rep(0,space_steps-1),c_i[length(c_i)]*V[length(V)])
```

```
  g[1] = a_i[1]*V[1]
  for(k in 1:time_steps){
    V <- solve(Amatrix, Bmatrix %*% V)
  }
  return(V)
}
```

```
gamma <- 0.8
dt <- 0.01
options(scipen = 999)
for (dX in c(0.1, 0.05, 0.02)) {
valX <- cnFd(dt, dX, gamma,"cor",T=1/2)
print(valX[(10/dX)/2+1])
}
```

```
[1] 0.7688451
[1] 0.7625108
[1] 0.7586165
```

## Compound Option

The Compound Option is priced like a Standard Call between $t = \frac{1}{4}$ and $T = \frac{1}{2}$, with strike 20. Let $S$ be the underlying asset, with discretization grid between 0 and $S_{max}$. Between these $\frac{1}{4}$ and $\frac{1}{2}$, we have the terminal condition:

$$V\left(S, \frac{1}{2}\right) = \max(S - 20, 0) \tag{18}$$

and boundary conditions for all $t \in [\frac{1}{4}, \frac{1}{2}]$:

$$V(0, t) = 0 \tag{19}$$
$$V(S_{max}, t) = S_{max} - 20e^{-rt} \tag{20}$$

Denote this call option $C_1$. The "mid-terminal'' condition we have for the compound option at $t = \frac{1}{4}$ is:

$$V\left(S, \frac{1}{2}\right) = \max\left(C_1\left(S, \frac{1}{4}\right) - 2, 0\right) \tag{21}$$

and we have the same boundary condition for $S = 0$, and the upper bound now becomes:

$$V\left(S_{max}, t\right) = C_1\left(S, \frac{1}{4}\right) - 2e^{-rt} \tag{22}$$

```
gam<-0.8
dt <- 0.1
```

```
T = 1/2
L1 <- 0
K_end = 20
L2 <- 2*K_end
K_compound<-2
L2_compound<-2 *K_compound

options(scipen=999)
prices = c()
dx_size = c(0.1,0.05,0.02)
for(i in 1:length(dx_size)){
  dx = dx_size[i]
  v_call_End <-implicitfd(1/4,dt,dx,L1,L2,gam,opttype="call",K_end,0,20)$V
  v_total <- implicitfd(1/4,dt,dx,L1,L2,gam,opttype="compound",K_compound,v_call_End[,1],20)$price
  prices<-c(prices,v_total)
}
```