PSTAT 222C Homework 1

Problem 1

We consider a corridor option for asset X_t with payoff of \$1 if $X_t \in [L_1, L_2]$ and \$0 otherwise. X_t is governed by the SDE

$$dX_t = rX_t dt + \sigma^2 X_t^{\gamma} dW_t \tag{1}$$

where γ is the elasticity of the variance. Applying Ito's Lemma to the price of the option, $V(t,x)$, we get:

$$\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}dX_t^2 && (2)\\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}(rX_t dt + \sigma^2 X_t^{\gamma} dW_t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(rX_t dt + \sigma^2 X_t^{\gamma} dW_t)^2 && (3)\\
&= \left(\frac{\partial V}{\partial t} + rX_t\frac{\partial V}{\partial x} + \frac{\sigma^2 X_t^{2\gamma}}{2}\frac{\partial^2 V}{\partial x^2}\right)dt + \sigma X_t^{\gamma}\frac{\partial V}{\partial x}dW_t && (4)
\end{aligned}$$

Under the assumption that this is the risk-neutral measure, we must have that the expected price of the option is equal to the present value of the risk free rate of the price of the option; that is:

$$\mathbb{E}[dV] = rVdt = \left(\frac{\partial V}{\partial t} + rX_t\frac{\partial V}{\partial x} + \frac{\sigma^2 X_t^{2\gamma}}{2}\frac{\partial^2 V}{\partial x^2}\right)dt \tag{5}$$

It therefore follows that the PDE for the CEV model can be written as:

$$\frac{\partial V}{\partial t} + rX_t\frac{\partial V}{\partial x} + \frac{\sigma^2 X_t^{2\gamma}}{2}\frac{\partial^2 V}{\partial x^2} - rV = 0 \tag{6}$$

Let the value of the discretization of the PDE at time $m\Delta t$ and $n\Delta x$ be $V(m\Delta t, n\Delta x) = V_n^m$. The explicit finite difference scheme is given by:

$$\frac{V_n^m - V_n^{m-1}}{\Delta t} + r(n\Delta x)\frac{V_{n+1}^m - V_{n-1}^m}{2\Delta x} + \frac{\sigma^2 (n\Delta x)^{2\gamma}}{2}\frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{(\Delta x)^2} - rV_n^m = 0 \tag{7}$$

We can derive the implicit Finite Difference similarly

$$\frac{V_n^m - V_n^{m-1}}{\Delta t} + r(n\Delta x)\frac{V_{n+1}^{m-1} - V_{n-1}^{m-1}}{2\Delta x} + \frac{\sigma^2(n\Delta x)^{2\gamma}}{2}\frac{V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}}{(\Delta x)^2} - rV_n^{m-1} = 0 \tag{8}$$

Solving for V_n^m, we find:

$$\begin{aligned}
V_n^m &= V_n^{m-1} + \Delta t\left(r(n\Delta x)\frac{V_{n+1}^{m-1} - V_{n-1}^{m-1}}{2\Delta x} + \frac{\sigma^2(n\Delta x)^{2\gamma}}{2}\frac{V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}}{(\Delta x)^2} - rV_n^{m-1}\right) && (9)\\
&= \underbrace{\frac{\Delta t}{2}\left(-rn + \frac{\sigma^2(n\Delta x)^{2\gamma}}{(\Delta x)^2}\right)}_{\tilde{a}_n} V_{n-1}^{m-1} + \underbrace{\left(1 - r\Delta t + \frac{\sigma^2(n\Delta x)^{2\gamma}\Delta t}{(\Delta x)^2}\right)}_{\tilde{b}_n} V_n^{m-1} + \underbrace{\frac{\Delta t}{2}\left(rn + \frac{\sigma^2(n\Delta x)^{2\gamma}}{(\Delta x)^2}\right)}_{\tilde{c}_n} V_{n+1}^{m-1} && (10)\\
&= \tilde{a}_n V_{n-1}^{m-1} + \tilde{b}_n V_n^{m-1} + \tilde{c}_n V_{n+1}^{m-1} && (11)
\end{aligned}$$

Imposing the exogenous boundary conditions on two assets N_- and N_+ such that $V_{N_-}^M = V_{N_+}^M = 0$ and $V_{N_-+1}^M = V_{N_+-1}^M = 1$ we find that we can represent the discretized PDE as:

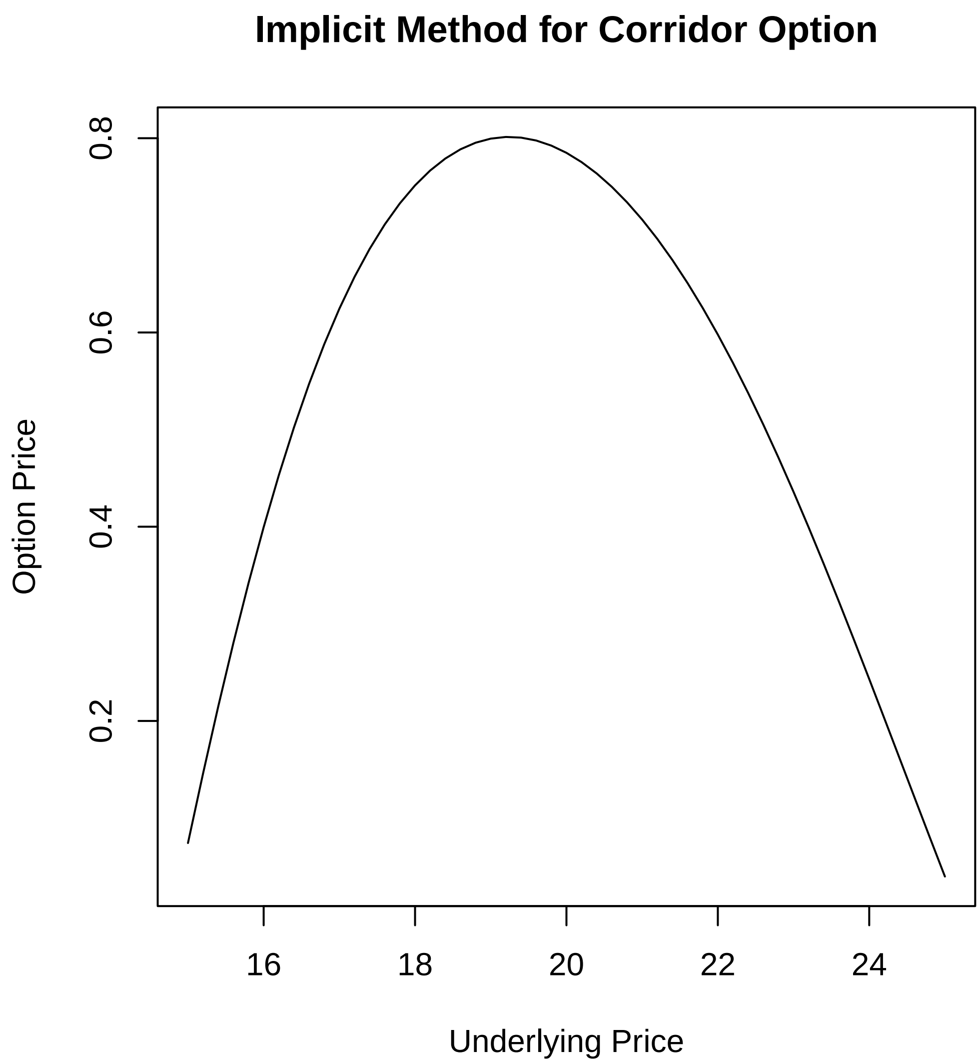
$$\begin{aligned}
AV^{m-1} &= BV^m + g && (12)\\
V^{m-1} &= A^{-1}\left(BV^m + g\right) && (13)
\end{aligned}$$

where $B = I$, the identity matrix.

```
implicitCorridor<- function(dt,dx,gam){
  r <- 0.05
  T <- 1/2
  sigma <- 0.4
  L1 <- 15
  L2 <- 25
  X_min <- 15
  X_max <- 25

  time_steps <- as.integer(T/dt)
  space_steps <- as.integer((X_max-X_min)/dx)+1

  space_grid <- X_min + dx*(0:(space_steps-1))

  V <- 1 - ((space_grid>L2 | space_grid<L1)*1)

  space_grid <- X_min/dx + 0:(space_steps-1)
```

```
  a_i <- dt/2*(sigma^2*(space_grid*dx)^(2*gam)/(dx^2) - r*space_grid)
  b_i <- -dt*(sigma^2*(space_grid*dx)^(2*gam)/(dx^2) + r)
  c_i <- dt/2*(sigma^2*(space_grid*dx)^(2*gam)/(dx^2) + r*space_grid)

  Amatrix <- diag(1-b_i,space_steps)
  Amatrix[(row(Amatrix) - col(Amatrix)) == 1] <- -1*a_i[2:(space_steps)]
  Amatrix[(row(Amatrix) - col(Amatrix)) == -1] <- -1*c_i[1:(space_steps-1)]

  Bmatrix <- diag(1,space_steps)
  for (k in 1:time_steps){
    V<-solve(Amatrix, Bmatrix %*% V)
  }
  return(V)
}
```

```
gam<-0.8
dt <- 0.1
dx <- 0.2
val<-implicitCorridor(dt,dx,gam)
options(scipen=999)
plot(x = seq(15,25,dx), y= val,type='l',main="Implicit Method for Corridor Option",
ylab="Option Price",xlab = "Underlying Price" )
```



Crank-Nicholson Solver

```
cnCorridor<-function(dt,dx,gam){
  r <- 0.05
  T <- 1/2
  sigma <- 0.4
  L1 <- 15
  L2 <- 25
  X_min <- 15
  X_max <- 25
```

```
  time_steps <- as.integer(T/dt)
  space_steps <- as.integer((X_max-X_min)/dx)+1

  space_grid <- X_min + dx*(0:(space_steps-1))

  V <- 1 - ((space_grid>L2 | space_grid<L1)*1)

  space_grid <- X_min/dx + 0:(space_steps-1)

}
```