Sheet 5

A2.1

$$\hat{F}_i \phi_i(\vec{r}) = \frac{k_F}{\pi}\left(1+\frac{1}{2}\left(\frac{k_F}{k_i}-\frac{k_i}{k_F}\right)\ln\left(\frac{k_F+k_i}{k_F-k_i}\right)\right)\phi_i$$

Ψ is a product of multiple single-wavefunctions ϕ_i:

$$\Psi = \frac{1}{\sqrt{N!}}\sum_{p,p'}(-1)^p \hat{p}^p\left[\varphi_a(\vec{r}_1)\dots\varphi_z(\vec{r}_N)\right]$$

$$\Rightarrow \int \Psi^* \Psi \, d^{3N}r = \frac{1}{N!}\sum_{p,p'}(-1)^{p+p'}\hat{p}^{p+p'}\left[\varphi_a\dots\varphi_z\right]\left[\varphi_a^*\dots\varphi_z\right]$$

$\Rightarrow$ only non-zero for $p=p'$ $\to$ $N!$ combinations

$$= \frac{1}{N!}\sum_{p,p'}(-1)^{p',p}\delta_{p,p'}\int\varphi_a\varphi_a^* d^3r_1 \dots \int\varphi_z\varphi_z^* d^3r_N$$

$$= \frac{1}{N!}\sum_{p=p'}\underbrace{(-1)^{2p}}_{1}\underbrace{\prod_i\int\varphi_i\varphi_i^* d^3r_N}_{1}$$

$$= \frac{1}{N!}\sum_{p=0}^{N!-1}\underbrace{(-1)^{2p}}_{1} = \frac{1}{N!}\cdot\underbrace{\left((-1)^2+(-1)^2+(-1)^4\dots\right)}_{N!} = 1$$

A2.2

$$E_x[n] = \langle\sum_i \hat{F}_i\rangle = \int d^{3N}\,\Psi^*\sum_i F_i\Psi$$

$$\Psi = \phi_1\phi_2\phi_3\dots\phi_N$$

$$\langle\sum_i\hat{F}_i\rangle = \int d^{3N}\,\Psi^*\sum_i F_i\Psi$$

$$= \sum_i\underbrace{\int d^{3N}\,\Psi^* F_i\Psi}_{\langle F_i\rangle}$$

$$\Rightarrow \langle \hat{F}_i \rangle = \int \Psi F_i \Psi \, d^{3N}r = \int \phi_a^* \phi_a \, d^3 r_1 \ldots \int \phi_i^* F_i \phi_i \, d^3 r_i \ldots \int \phi_z^* \phi_z \, d^3 r_N$$

$$= \frac{k_F}{\pi}\left(1 + \frac{1}{2}\left(\frac{k_F}{k_i} - \frac{k_i}{k_F}\right) \ln\left(\frac{k_F + k_i}{k_F - k_i}\right)\right)$$

$$\Rightarrow \sum_i \langle F_i \rangle = \frac{1}{(2\pi)^3} \int_0^{\pi} \sin\theta \int_0^{2\pi} \int_0^{k_F} \frac{k_F}{\pi}\left(1 + \frac{1}{2}\left(\frac{k_F}{k_i} - \frac{k_i}{k_F}\right) \ln\left(\frac{k_F + k_i}{k_F - k_i}\right)\right) k^2 \, d\theta \, d\varphi \, dk$$

$$= \frac{1}{2\pi^2} \int_0^{k_F} \frac{k_F}{\pi}\left(1 + \frac{1}{2}\left(\frac{k_F}{k} - \frac{k}{k_F}\right) \ln\left(\frac{k_F + k}{k_F - k}\right)\right) k^2 \, d\theta \, d\varphi \, dk$$

$$x = \frac{k}{k_F} \rightarrow dx = \frac{dk}{k_F} \qquad = \frac{k_F^4}{2\pi^3} \int_0^{k_F} \left(1 + \frac{1}{2}\left(\frac{1}{x} - x\right) \ln\left(\frac{k_F + x k_F}{k_F - x k_F}\right)\right) x^2 \, dx$$

$$= \frac{k_F^4}{2\pi^3} \int_0^1 \left(-\frac{1}{2}x^3 + x^2 + \frac{1}{2}x\right) \ln\left(\frac{1 + x}{1 - x}\right) dx$$

$$= \frac{k_F^4}{2\pi^3} C' \qquad \rightarrow \text{use } n = \frac{k_F^3}{3\pi^2}$$

$$\rightarrow E_x[n] = \frac{(3\pi^2 n)^{4/3}}{2\pi^3} = A\, n^{4/3}$$

since $E_x[n] = \int d^3 r \, \epsilon_x(r) \underbrace{n(r)}_{=n} \overset{!}{=} A\, n^{4/3}$

$$\Rightarrow \int d^3 r \, \epsilon_x(r) = A\, n^{1/3}$$

doesn't depend on r?

Then: $\epsilon_x(r) = A\, n^{1/3}$

where $A = C' \frac{(3\pi^2)^{4/3}}{2\pi^3}$

and C' is the Integral