Set Theory Proofs

Problem 1: Cardinality of Power Sets

Question: How does $|\mathcal{P}(A)|$ relate to $|A|$?

Small Examples

Let's examine some small sets:

- $A = \emptyset$: $\mathcal{P}(A) = \{\emptyset\}$, so $|\mathcal{P}(A)| = 1 = 2^0 = 2^{|A|}$
- $A = \{1\}$: $\mathcal{P}(A) = \{\emptyset, \{1\}\}$, so $|\mathcal{P}(A)| = 2 = 2^1 = 2^{|A|}$
- $A = \{1, 2\}$: $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, so $|\mathcal{P}(A)| = 4 = 2^2 = 2^{|A|}$
- $A = \{1, 2, 3\}$: $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, so $|\mathcal{P}(A)| = 8 = 2^3 = 2^{|A|}$

Conjecture: $|\mathcal{P}(A)| = 2^{|A|}$

Theorem: For any finite set A, $|\mathcal{P}(A)| = 2^{|A|}$.

Proof by induction on $|A|$:

Base case: When $|A| = 0$, we have $A = \emptyset$. Then $\mathcal{P}(A) = \{\emptyset\}$, so $|\mathcal{P}(A)| = 1 = 2^0 = 2^{|A|}$.

Inductive step: Assume the statement holds for all sets with n elements. Let A be a set with $n + 1$ elements.

We can write $A = B \cup \{x\}$

Then....its a bit tricky :-) work in progress....

Problem 2: Distributivity of Intersection over Union

Theorem: For all sets A, B, and C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$A \cap (B \cup C) = \{x \mid x \in A \text{ and } x \in B \cup C\}$$

$$= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\}$$

$$= \{x \mid x \in A \text{ and } x \in B \text{ or } x \in A \text{ and } x \in C\}$$

$$= \{x \mid x \in A \cap B \text{ or } x \in A \cap C\}$$

$$= (A \cap B) \cup (A \cap C)$$

Problem 3: Inclusion-Exclusion Principle

Theorem: For any finite sets A and B: $|A \cup B| = |A| + |B| - |A \cap B|$

Proof: We partition the union $A \cup B$ into three sets: 1. $A \setminus B = \{x \in A : x \notin B\}$ (elements only in A) 2. $B \setminus A = \{x \in B : x \notin A\}$ (elements only in B) 3. $A \cap B$ (elements in both A and B)

$$|A \cup B| = |A \setminus B| + |B \setminus A| + |A \cap B|$$

Now we express $|A \setminus B|$ and $|B \setminus A|$ in terms of the given quantities: - Since A is the union of $A \setminus B$ and $A \cap B$: $|A| = |A \setminus B| + |A \cap B|$ - Since B is the union of $B \setminus A$ and $A \cap B$: $|B| = |B \setminus A| + |A \cap B|$

Therefore: - $|A \setminus B| = |A| - |A \cap B|$ - $|B \setminus A| = |B| - |A \cap B|$

Substituting back:

$$|A \cup B| = (|A| - |A \cap B|) + (|B| - |A \cap B|) + |A \cap B|$$

$$= |A| + |B| - |A \cap B| - |A \cap B| + |A \cap B|$$

$$= |A| + |B| - |A \cap B|$$