

## Math Questions

### Question 1: Translate the following sentences into propositional logic

#### a) The flat is vacant.

Let  $V$  = "The flat is vacant"

**Translation:**  $V$

This is straightforward - a simple atomic proposition.

#### b) The flat can be let only if it is vacant and has been cleaned.

Let  $V$  = "The flat is vacant"

Let  $C$  = "The flat has been cleaned"

Let  $L$  = "The flat can be let"

The phrase "can be let only if" indicates a conditional relationship. "A only if B" means "if A then B", or  $A \rightarrow B$ .

**Translation:**  $L \rightarrow (V \wedge C)$

#### c) Don't drink and drive!

Let  $D$  = "You drink"

Let  $R$  = "You drive"

This imperative sentence is commanding someone not to do both actions simultaneously.

**Translation:**  $\neg(D \wedge R)$

### Question 2 [6]: Translate the following sentences into predicate logic

Your formalizations should be as detailed as possible.

#### a) Some teachers are strict and others are not.

This sentence asserts two things:

Some teachers are strict

Some teachers are not strict

There exists someone who is a teacher and is strict, and there exists someone who is a teacher and is not strict.

Let Domain: All people

Let  $T(x)$  = "x is a teacher"

Let  $S(x)$  = "x is strict"

**Translation:**  $\exists x(T(x) \wedge S(x)) \wedge \exists y(T(y) \wedge \neg S(y))$

**b) Some dogs' owners do not collect their dog's litter**

There exists a person x and a dog y and x owns y and x does not collect y's litter.

Let Domain: All people and dogs

Let  $P(x)$  = "x is a person"

Let  $D(x)$  = "x is a dog"

Let  $O(x,y)$  = "x owns y"

Let  $C(x,y)$  = "x collects the litter of y"

**Translation:**  $\exists x \exists y (P(x) \wedge D(y) \wedge O(x,y) \wedge \neg C(x,y))$

**c) All animals are equal.**

We interpret this as a universal statement about all pairs of animals.

For all x and y, if x is an animal and y is an animal, then x is equal to y.

Let Domain: All entities

Let  $A(x)$  = "x is an animal"

Let  $E(x,y)$  = "x is equal to y"

**Translation:**  $\forall x \forall y ((A(x) \wedge A(y)) \rightarrow E(x,y))$

**Question 3 [6]: Consider the following formula of propositional logic**

$$P = ((A \vee B) \wedge C \rightarrow (A \wedge B) \vee C)$$

**a) Assume that A is true, and B,C are false. Is P true or false? Briefly explain why.**

Given:  $A = T$ ,  $B = F$ ,  $C = F$

$$A \vee B = T \vee F = T$$

$$(A \vee B) \wedge C = T \wedge F = F$$

$$A \wedge B = T \wedge F = F$$

$$(A \wedge B) \vee C = F \vee F = F$$

$$P = ((A \vee B) \wedge C) \rightarrow ((A \wedge B) \vee C) = F \rightarrow F = T$$

**P is true.**

This is because a conditional statement ( $\rightarrow$ ) is true whenever the antecedent is false, regardless of the truth value of the consequent.

### **b) Which truth values for A,B,C will result in P being false?**

For P to be false, we need the antecedent  $((A \vee B) \wedge C)$  to be true and the consequent  $((A \wedge B) \vee C)$  to be false.

For the antecedent to be true:

$(A \vee B)$  must be true, so at least one of A or B must be true

C must be true

For the consequent to be false:

$(A \wedge B)$  must be false, so at least one of A or B must be false

C must be false

However, this creates a contradiction: C cannot be both true (for the antecedent) and false (for the consequent) simultaneously.

Therefore, **P is always true (it's a tautology)** - there are no truth value assignments that make P false.

## **Question 4 [8]: Predicate Logic Analysis**

Consider the sentence "There is a passenger on a coach such that, if that passenger reads, then all passengers on the coach read". In predicate logic, the formula Q formalises it, where  $Q = \exists x(Rx \rightarrow \forall yRy)$ . Answer these questions, giving some explanations.

**a) Suppose that the coach has just two passengers. When is Q true and when is it false?**

**b) Suppose that we know that there is at least one passenger on the coach, but we do not know how many, nor how many are reading. Can we know whether Q is true or false?**

### Part a) Coach with exactly two passengers

Let's call the two passengers a and b. The formula  $Q = \exists x(Rx \rightarrow \forall yRy)$  means "there exists a passenger x such that if x reads, then all passengers read."

For two passengers, this becomes:  $Q = (Ra \rightarrow \forall yRy) \vee (Rb \rightarrow \forall yRy)$

Since  $\forall yRy$  with two passengers means  $(Ra \wedge Rb)$ , we get:  $Q = (Ra \rightarrow (Ra \wedge Rb)) \vee (Rb \rightarrow (Ra \wedge Rb))$

**Case 1: Neither reads ( $Ra = F, Rb = F$ )** -  $Ra \rightarrow (Ra \wedge Rb) = F \rightarrow F = T$  -  $Rb \rightarrow (Ra \wedge Rb) = F \rightarrow F = T$  -  $Q = T \vee T = T$

**Case 2: Only a reads ( $Ra = T, Rb = F$ )** -  $Ra \rightarrow (Ra \wedge Rb) = T \rightarrow F = F$  -  $Rb \rightarrow (Ra \wedge Rb) = F \rightarrow F = T$  -  $Q = F \vee T = T$

**Case 3: Only b reads ( $Ra = F, Rb = T$ )** -  $Ra \rightarrow (Ra \wedge Rb) = F \rightarrow F = T$  -  $Rb \rightarrow (Ra \wedge Rb) = T \rightarrow F = F$  -  $Q = T \vee F = T$

**Case 4: Both read ( $Ra = T, Rb = T$ )** -  $Ra \rightarrow (Ra \wedge Rb) = T \rightarrow T = T$  -  $Rb \rightarrow (Ra \wedge Rb) = T \rightarrow T = T$  -  $Q = T \vee T = T$

**Q is always true** when there are exactly two passengers.

### Part b) At least one passenger, unknown number and reading status

We can determine that **Q is always true**, regardless of how many passengers there are or who is reading.

There are only two possibilities:

**Case 1: All passengers read** If  $\forall yRy$  is true, then for any passenger x, the conditional  $Rx \rightarrow \forall yRy$  is true (since the consequent is true). Therefore,  $\exists x(Rx \rightarrow \forall yRy)$  is true.

**Case 2: Not all passengers read** If  $\forall yRy$  is false, then there exists at least one passenger who doesn't read. Let's call such a passenger c, where  $Rc = F$ . Then the conditional  $Rc \rightarrow \forall yRy = F \rightarrow F = T$ . Since we found a passenger c for whom the conditional is true,  $\exists x(Rx \rightarrow \forall yRy)$  is true.

**Alternative explanation:** The formula Q is logically equivalent to  $(\exists x \neg Rx) \vee (\forall y Ry)$ , which reads "either someone doesn't read, or everyone reads." This is always true because in any situation, either everyone reads or someone doesn't read - there's no third possibility.

**Therefore, Q is a tautology** - it's true in all possible scenarios with any number of passengers and any reading configuration.

## Question 5 [12]: Binary Relations Analysis

Consider the binary relations provided below, and analyse them for these features: are they reflexive? Symmetric? Transitive? Are they equivalence relations? And if so, which are the equivalence classes? Clearly list which properties hold and which ones do not, in each relation. Give a short justification when you think that some property does not hold.

a)  $R_1 = \{(m, n) \mid m \geq n + 2\} \subseteq \mathbb{N} \times \mathbb{N}$

b)  $R_2 = \{(m, n) \mid m + n \text{ is odd}\} \subseteq \mathbb{Z} \times \mathbb{Z}$

c)  $S = \{0, 1, 2, 3, 4\}$ ,  $R_3 = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (3, 3), (4, 4), (3, 4), (4, 3)\} \subseteq S \times S$

d)  $S = \{0, 1, 2, 3, 4\}$ ,  $R_4 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\} \subseteq S \times S$

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### Analysis

I'll analyze each binary relation for reflexivity, symmetry, transitivity, and determine if they're equivalence relations.

**Part a)**  $R_1 = \{(m, n) \mid m \geq n + 2\} \subseteq \mathbb{N} \times \mathbb{N}$

**Reflexive?** NO

For reflexivity, we need  $(m, m) \in R_1$  for all  $m \in \mathbb{N}$ . This requires  $m \geq m + 2$ , which simplifies to  $0 \geq 2$ , which is false. So  $R_1$  is not reflexive.

**Symmetric?** NO

For symmetry, if  $(m, n) \in R_1$ , then  $(n, m) \in R_1$ . Consider  $(3, 1) \in R_1$  since  $3 \geq 1 + 2 = 3$ . But  $(1, 3) \notin R_1$  since  $1 \not\geq 3 + 2 = 5$ . So  $R_1$  is not symmetric.

**Transitive?** YES

If  $(m, n) \in R_1$  and  $(n, p) \in R_1$ , then  $m \geq n + 2$  and  $n \geq p + 2$ . From the second inequality:  $n \geq p + 2$ , so  $n + 2 \geq p + 4$ . From the first:  $m \geq n + 2 \geq p + 4 > p + 2$ . Therefore  $(m, p) \in R_1$ , so  $R_1$  is transitive.

**Equivalence relation?** NO (lacks reflexivity and symmetry)

**Part b)**  $R_2 = \{(m, n) \mid m + n \text{ is odd}\} \subseteq \mathbb{Z} \times \mathbb{Z}$

**Reflexive?** NO

For  $(m, m) \in R_2$ , we need  $m + m = 2m$  to be odd. But  $2m$  is always even for any integer  $m$ . So  $R_2$  is not reflexive.

**Symmetric?** YES

If  $(m, n) \in R_2$ , then  $m + n$  is odd. Since  $n + m = m + n$ , we have  $n + m$  is also odd. Therefore  $(n, m) \in R_2$ , so  $R_2$  is symmetric.

**Transitive?** NO

Consider  $(1, 2) \in R_2$  since  $1 + 2 = 3$  is odd. And  $(2, 3) \in R_2$  since  $2 + 3 = 5$  is odd. But  $(1, 3) \notin R_2$  since  $1 + 3 = 4$  is even. So  $R_2$  is not transitive.

**Equivalence relation?** NO (lacks reflexivity and transitivity)

**Part c)  $S = \{0, 1, 2, 3, 4\}$ ,  $R_3 = \{(0,0), (0,1), (1,0), (1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$**

**Reflexive?** YES

All pairs  $(0,0), (1,1), (2,2), (3,3), (4,4)$  are in  $R_3$ . So  $R_3$  is reflexive.

**Symmetric?** YES

For every  $(a,b) \in R_3$ , we also have  $(b,a) \in R_3$ : -  $(0,0) \leftrightarrow (0,0)$  -  $(0,1) \leftrightarrow (1,0)$  -  $(1,1) \leftrightarrow (1,1)$  -  $(2,2) \leftrightarrow (2,2)$  -  $(3,3) \leftrightarrow (3,3)$  -  $(4,4) \leftrightarrow (4,4)$  -  $(3,4) \leftrightarrow (4,3)$

**Transitive?** YES

Checking all possible chains: - From  $\{0,1\}$ :  $(0,0), (0,1), (1,0), (1,1)$  - all required pairs present - From  $\{2\}$ : only  $(2,2)$  - trivially transitive - From  $\{3,4\}$ :  $(3,3), (3,4), (4,3), (4,4)$  - all required pairs present

**Equivalence relation?** YES

**Equivalence classes:** -  $[0] = \{0, 1\}$  (elements related to 0) -  $[1] = \{0, 1\}$  (same as  $[0]$ ) -  $[2] = \{2\}$  -  $[3] = \{3, 4\}$  -  $[4] = \{3, 4\}$  (same as  $[3]$ )

The distinct equivalence classes are:  **$\{0,1\}, \{2\}, \{3,4\}$**

**Part d)  $S = \{0, 1, 2, 3, 4\}$ ,  $R_4 = \{(0,0), (1,1), (2,2), (3,3), (4,4)\}$**

**Reflexive?** YES

All pairs  $(i,i)$  for  $i \in S$  are in  $R_4$ .

**Symmetric?** YES

All pairs in  $R_4$  are of the form  $(i,i)$ , so  $(i,i) = (i,i)$ . Trivially symmetric.

**Transitive?** YES

For any  $(a,b), (b,c) \in R_4$ , we must have  $a = b$  and  $b = c$  (since all pairs are diagonal). Therefore  $a = c$ , so  $(a,c) \in R_4$ .

**Equivalence relation?** YES

**Equivalence classes:**

Since  $R_4$  is the identity relation, each element is only related to itself.

$$- [0] = \{0\} - [1] = \{1\} - [2] = \{2\} - [3] = \{3\} - [4] = \{4\}$$

The equivalence classes are:  $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}$

**Question 6 [8]: Function Analysis**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined as  $f(x) = (9x + 3)/(4x - 2)$

**a) Is  $f$  injective? Is it surjective? Justify your reasoning.**

**b) Find a formula that represents the graph of its inverse function**

**Analysis**

Looking at the function  $f(x) = (9x + 3)/(4x - 2)$ , I'll analyze its injectivity and surjectivity, then find its inverse.

**Part a) Injectivity and Surjectivity**

**Injectivity (One-to-one):** A function is injective if  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ .

$$\text{Assume } f(x_1) = f(x_2): (9x_1 + 3)/(4x_1 - 2) = (9x_2 + 3)/(4x_2 - 2)$$

$$\text{Cross-multiplying: } (9x_1 + 3)(4x_2 - 2) = (9x_2 + 3)(4x_1 - 2)$$

$$\text{Expanding: } 36x_1x_2 - 18x_1 + 12x_2 - 6 = 36x_1x_2 - 18x_2 + 12x_1 - 6$$

$$\text{Simplifying: } -18x_1 + 12x_2 = -18x_2 + 12x_1 \quad -18x_1 - 12x_1 = -18x_2 - 12x_2 \quad -30x_1 = -30x_2 \quad x_1 = x_2$$

Therefore,  **$f$  is injective.**

**Surjectivity (Onto):** A function is surjective if for every  $y \in \mathbb{R}$ , there exists an  $x \in \mathbb{R}$  such that  $f(x) = y$ .

$$\text{For any } y \in \mathbb{R}, \text{ we need to solve: } y = (9x + 3)/(4x - 2)$$

$$\text{Cross-multiplying: } y(4x - 2) = 9x + 3 \quad 4xy - 2y = 9x + 3 \quad 4xy - 9x = 2y + 3 \quad x(4y - 9) = 2y + 3$$

$$\text{If } 4y - 9 \neq 0 \text{ (i.e., } y \neq 9/4), \text{ then: } x = (2y + 3)/(4y - 9)$$

$$\text{However, when } y = 9/4, \text{ we get: } x(0) = 2(9/4) + 3 = 9/2 + 3 = 15/2$$

This gives  $0 = 15/2$ , which is impossible.

Therefore,  $y = 9/4$  cannot be achieved by  $f(x)$  for any real  $x$ .

**$f$  is not surjective** because  $9/4$  is not in the range of  $f$ .

### Part b) Inverse Function

Since  $f$  is injective but not surjective,  $f$  has an inverse function on its range (which is  $\mathbb{R} \setminus \{9/4\}$ ).

To find  $f^{-1}$ , I'll solve  $y = f(x)$  for  $x$ :  $y = (9x + 3)/(4x - 2)$   $y(4x - 2) = 9x + 3$   $4xy - 2y = 9x + 3$   $4xy - 9x = 2y + 3$   $x(4y - 9) = 2y + 3$   $x = (2y + 3)/(4y - 9)$

Therefore,  **$f^{-1}(x) = (2x + 3)/(4x - 9)$**  with domain  $\mathbb{R} \setminus \{9/4\}$ .

We can verify:  $f^{-1}(f(x)) = x$  and  $f(f^{-1}(x)) = x$  for appropriate values.

## Question 7 [16]: Calculus Functions Analysis

Answer the questions for the functions  $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$  defined below; show some working before the final answer.

**a) Find  $f'$ , for  $f(x) = \cos(2x^3 + 2x^2 + 3)$ .**

**b) Find  $g''$ , for  $g(x) = 4x^3 - 3x^2 + 2x$ , and calculate  $g''(3)$ .**

**c) Is the function  $h(x) = (4+2x)/(x^3+1)$  continuous? Is it derivable in all points in  $\mathbb{R}$ ? Justify your reasoning.**

**d) Consider again  $h(x) = (4+2x)/(x^3+1)$ , but defined as  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ . Is it continuous? Is this the same function as  $h_a(x) = (4+2x)/|x^3+1|$ ? Justify your reasoning.**

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## Solutions

I'll solve each part step by step, showing the working clearly.

**Part a) Find  $f'$ , for  $f(x) = \cos(2x^3 + 2x^2 + 3)$**

Using the chain rule: if  $f(x) = \cos(u)$  where  $u = 2x^3 + 2x^2 + 3$ , then  $f'(x) = -\sin(u) \cdot u'$

First, find  $u'$ :  $u = 2x^3 + 2x^2 + 3$   $u' = 6x^2 + 4x$

Therefore:  **$f'(x) = -\sin(2x^3 + 2x^2 + 3) \cdot (6x^2 + 4x)$**   **$f'(x) = -(6x^2 + 4x)\sin(2x^3 + 2x^2 + 3)$**



**Part b) Find  $g''$  for  $g(x) = 4x^3 - 3x^2 + 2x$ , and calculate  $g''(3)$**

First, find  $g'(x)$ :  $g(x) = 4x^3 - 3x^2 + 2x$   $g'(x) = 12x^2 - 6x + 2$

Then find  $g''(x)$ :  **$g''(x) = 24x - 6$**

Now calculate  $g''(3)$ :  **$g''(3) = 24(3) - 6 = 72 - 6 = 66$**

**Part c) Is  $h(x) = (4 + 2x)/(x^3 + 1)$  continuous? Is it derivable at all points in  $\mathbb{R}$ ?**

For continuity and differentiability, we need to check where the function might have problems.

The function  $h(x) = (4 + 2x)/(x^3 + 1)$  is a rational function (quotient of polynomials).

**Potential problems:** The denominator  $x^3 + 1 = 0$  Solving:  $x^3 = -1$ , so  $x = -1$

**Continuity:** - The numerator  $4 + 2x$  is continuous everywhere (polynomial) - The denominator  $x^3 + 1$  is continuous everywhere (polynomial) - A rational function is continuous everywhere except where the denominator equals zero

At  $x = -1$ : the denominator is 0 but the numerator is  $4 + 2(-1) = 2 \neq 0$ , so we have a vertical asymptote.

**Conclusion:**  $h(x)$  is continuous for all  $x \in \mathbb{R}$  except  $x = -1$ .

**Differentiability:** Since  $h(x)$  is continuous everywhere except  $x = -1$ , and rational functions are differentiable wherever they're continuous,  $h(x)$  is differentiable for all  $x \in \mathbb{R}$  except  $x = -1$ .

**Part d) Consider  $h(x) = (4 + 2x)/(x^3 + 1)$  defined as  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ . Is it continuous? Is this the same as  $h_a(x) = (4 + 2x)/|x^3 + 1|$ ?**

**With domain  $\mathbb{R}^+$  (positive real numbers):** Since the domain is now  $\mathbb{R}^+ = (0, +\infty)$ , we exclude  $x = -1$  from consideration.

For  $x > 0$ :  $x^3 > 0$ , so  $x^3 + 1 > 1 > 0$ . The denominator is never zero on this domain.

**Conclusion:**  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$  is continuous everywhere on its domain.

**Comparison with  $h_a(x) = (4 + 2x)/|x^3 + 1|$ :**

For  $x \in \mathbb{R}^+$ , we have  $x^3 + 1 > 0$ , so  $|x^3 + 1| = x^3 + 1$ .

Therefore:  $h_a(x) = (4 + 2x)/|x^3 + 1| = (4 + 2x)/(x^3 + 1) = h(x)$  for all  $x \in \mathbb{R}^+$ .

**Conclusion:** Yes, when restricted to the domain  $\mathbb{R}^+$ ,  $h(x)$  and  $h_a(x)$  are the same function.

However, if we considered  $h_a(x)$  on all of  $\mathbb{R}$ , they would differ at  $x = -1$ :  $h(-1)$  is undefined -  $h_a(-1) = (4 + 2(-1))/|(-1)^3 + 1| = 2/|0|$  is also undefined

But for  $x$  near  $-1$  ( $x \neq -1$ ),  $h_a(x) = (4 + 2x)/|x^3 + 1|$  would behave differently than  $h(x)$  since  $|x^3 + 1| \geq 0$  always, while  $x^3 + 1$  can be negative.

## Question 10 [10]: System of Linear Equations

Calculate the solution to the system of real linear equations, by showing your approach:

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -1 & x_1 + 3x_2 - 4x_3 &= 0 & 2x_1 + 3x_2 + x_3 &= 2 \end{aligned}$$


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### Solution using Direct Substitution (Simplest Method)

The easiest approach is to solve equation 1 for  $x_1$  and substitute.

#### Step 1: Solve equation 1 for $x_1$

From equation 1:  $x_1 - 2x_2 - 3x_3 = -1$

$$x_1 = -1 + 2x_2 + 3x_3$$

#### Step 2: Substitute into equations 2 and 3

**Substitute into equation 2:**  $x_1 + 3x_2 - 4x_3 = 0$   $(-1 + 2x_2 + 3x_3) + 3x_2 - 4x_3 = 0$   $-1 + 2x_2 + 3x_3 + 3x_2 - 4x_3 = 0$   $-1 + 5x_2 - x_3 = 0$   $5x_2 - x_3 = 1$  ... (4)

**Substitute into equation 3:**  $2x_1 + 3x_2 + x_3 = 2$   $2(-1 + 2x_2 + 3x_3) + 3x_2 + x_3 = 2$   $-2 + 4x_2 + 6x_3 + 3x_2 + x_3 = 2$   $-2 + 7x_2 + 7x_3 = 2$   $7x_2 + 7x_3 = 4$   $x_2 + x_3 = 4/7$  ... (5)

#### Step 3: Solve the $2 \times 2$ system

From equations (4) and (5):  $5x_2 - x_3 = 1$   $x_2 + x_3 = 4/7$

Add the equations:  $6x_2 = 1 + 4/7 = 7/7 + 4/7 = 11/7$

$$\mathbf{x_2 = 11/42}$$

#### Step 4: Find $\mathbf{x_3}$

From equation (5):  $\mathbf{x_3 = 4/7 - x_2 = 4/7 - 11/42}$

Convert to common denominator:  $\mathbf{4/7 = 24/42}$

$$\mathbf{x_3 = 24/42 - 11/42 = 13/42}$$

#### Step 5: Find $\mathbf{x_1}$

$$\begin{aligned} \mathbf{x_1} &= -1 + 2\mathbf{x_2} + 3\mathbf{x_3} \quad \mathbf{x_1} = -1 + 2(11/42) + 3(13/42) \quad \mathbf{x_1} = -1 + 22/42 + 39/42 \\ \mathbf{x_1} &= -1 + 61/42 \quad \mathbf{x_1} = -42/42 + 61/42 = \mathbf{19/42} \end{aligned}$$

#### Final Solution

$$\mathbf{x_1 = 19/42 \quad x_2 = 11/42 \quad x_3 = 13/42}$$

#### Verification

$$\mathbf{Equation \ 1:} \quad 19/42 - 2(11/42) - 3(13/42) = 19/42 - 22/42 - 39/42 = -42/42 = -1$$

$$\mathbf{Equation \ 2:} \quad 19/42 + 3(11/42) - 4(13/42) = 19/42 + 33/42 - 52/42 = 0$$

$$\mathbf{Equation \ 3:} \quad 2(19/42) + 3(11/42) + 13/42 = 38/42 + 33/42 + 13/42 = 84/42 = 2$$

This method is indeed the simplest - solving for  $\mathbf{x_1}$  from the first equation gives the cleanest substitution!

## Question 10 [10]: Gaussian Elimination

Calculate the solution to the system of real linear equations, by showing your approach:

$$\mathbf{x_1 - 2x_2 - 3x_3 = -1 \quad x_1 + 3x_2 - 4x_3 = 0 \quad 2x_1 + 3x_2 + x_3 = 2}$$


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#### Solution using Gaussian Elimination

I'll solve this system by converting the augmented matrix to row echelon form, then using back substitution.

**Step 1: Set up the augmented matrix**

$$\begin{bmatrix} 1 & -2 & -3 & | & -1 \\ 1 & 3 & -4 & | & 0 \\ 2 & 3 & 1 & | & 2 \end{bmatrix}$$

**Step 2: Eliminate the first column below the pivot**

$R_2 = R_2 - R_1$  (eliminate  $x_1$  from equation 2)

$$\begin{bmatrix} 1 & -2 & -3 & | & -1 \\ 0 & 5 & -1 & | & 1 \\ 2 & 3 & 1 & | & 2 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$  (eliminate  $x_1$  from equation 3)

$$\begin{bmatrix} 1 & -2 & -3 & | & -1 \\ 0 & 5 & -1 & | & 1 \\ 0 & 7 & 7 & | & 4 \end{bmatrix}$$

**Step 3: Eliminate the second column below the pivot**

$R_3 = R_3 - (7/5)R_2$  (eliminate  $x_2$  from equation 3)

First, let's calculate  $(7/5)R_2$ :  $(7/5) \times [0 \ 5 \ -1 \ | \ 1] = [0 \ 7 \ -7/5 \ | \ 7/5]$

$$R_3 = [0 \ 7 \ 7 \ | \ 4] - [0 \ 7 \ -7/5 \ | \ 7/5]$$

$$\begin{bmatrix} 1 & -2 & -3 & | & -1 \\ 0 & 5 & -1 & | & 1 \\ 0 & 0 & 42/5 & | & 13/5 \end{bmatrix}$$

**Step 4: Convert to exact fractions for clarity**

The last row gives us:  $(42/5)x_3 = 13/5$

Multiplying by  $5/42$ :  $x_3 = (13/5) \times (5/42) = 13/42$

Let's simplify:  $x_3 = 13/42$

**Step 5: Back substitution**

**Find  $x_3$ :** From the third equation:  $(42/5)x_3 = 13/5$   $x_3 = (13/5) \div (42/5) = 13/42$

Let's work with decimals for easier calculation:  $42x_3 = 13$ , so  $x_3 = 13/42$

Actually, let me recalculate this more carefully:

From R<sub>3</sub>:  $0x_1 + 0x_2 + (7 + 7/5)x_3 = 4 - 7/5 = (35/5 + 7/5)x_3 = (20/5 - 7/5) = (42/5)x_3 = 13/5$

Therefore:  $x_3 = (13/5) \times (5/42) = 13/42$

**Find  $x_2$ :** From the second equation:  $5x_2 - x_3 = 1$   $5x_2 - 13/42 = 1$   $5x_2 = 1 + 13/42 = 42/42 + 13/42 = 55/42$   $x_2 = 55/210 = 11/42$

**Find  $x_1$ :** From the first equation:  $x_1 - 2x_2 - 3x_3 = -1$   $x_1 - 2(11/42) - 3(13/42) = -1$   $x_1 - 22/42 - 39/42 = -1$   $x_1 - 61/42 = -1$   $x_1 = -1 + 61/42 = -42/42 + 61/42 = 19/42$

### Step 6: Final Solution

$$x_1 = 19/42 \quad x_2 = 11/42$$

$$x_3 = 13/42$$

### Verification

Let's check our solution in the original equations:

**Equation 1:**  $x_1 - 2x_2 - 3x_3 = 19/42 - 2(11/42) - 3(13/42) = 19/42 - 22/42 - 39/42 = -42/42 = -1$

**Equation 2:**  $x_1 + 3x_2 - 4x_3 = 19/42 + 3(11/42) - 4(13/42) = 19/42 + 33/42 - 52/42 = 0/42 = 0$

**Equation 3:**  $2x_1 + 3x_2 + x_3 = 2(19/42) + 3(11/42) + 13/42 = 38/42 + 33/42 + 13/42 = 84/42 = 2$

All equations check out, confirming our solution is correct.