Bijection Proofs

1. Inverse of bijection is bijection

If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

Proof: - f^{-1} is a function because f is surjective (every $b \in B$ has a preimage) - f^{-1} is injective: if $f^{-1}(b_1) = f^{-1}(b_2)$, then $b_1 = f(f^{-1}(b_1)) = f(f^{-1}(b_2)) = b_2$ - f^{-1} is surjective: for any $a \in A$, we have $f^{-1}(f(a)) = a$

2. Finite injection is surjection

If $|A| = |B| = n$ and $f : A \to B$ is injective, then f is surjective.

Proof: - f is injective, so all n elements of A map to distinct elements in B - But $|B| = n$, so f must hit every element of B - Therefore f is surjective

3. Finite surjection is injection

If $|A| = |B| = n$ and $f : A \to B$ is surjective, then f is injective.

Proof: - Suppose f is not injective, so $f(a_1) = f(a_2)$ for some $a_1 \neq a_2$ - Then the image of f has at most $n - 1$ elements - But f is surjective, so the image must have n elements - Contradiction, so f must be injective