Math Questions

Question 1: Translate the following sentences into propositional logic

a) The flat is vacant.

Let V = “The flat is vacant”

Translation: V

This is straightforward - a simple atomic proposition.

b) The flat can be let only if it is vacant and has been cleaned.

Let V = “The flat is vacant”

Let C = “The flat has been cleaned”

Let L = “The flat can be let”

The phrase “can be let only if” indicates a conditional relationship. “A only if B” means “if A then B”, or $A \rightarrow B$.

Translation: $L \rightarrow (V \wedge C)$

c) Don’t drink and drive!

Let D = “You drink”

Let R = “You drive”

This imperative sentence is commanding someone not to do both actions simultaneously.

Translation: $\neg(D \wedge R)$

Question 2 [6]: Translate the following sentences into predicate logic

Your formalizations should be as detailed as possible.

a) Some teachers are strict and others are not.

This sentence asserts two things:

Some teachers are strict

Some teachers are not strict

There exists someone who is a teacher and is strict, and there exists someone who is a teacher and is not strict.

Let Domain: All people

Let T(x) = “x is a teacher”

Let S(x) = “x is strict”

Translation: $\exists x(T(x) \land S(x)) \land \exists y(T(y) \land \neg S(y))$

b) Some dogs’ owners do not collect their dog’s litter

There exists a person x and a dog y and x owns y and x does not collect y’s litter.

Let Domain: All people and dogs

Let P(x) = “x is a person”

Let D(x) = “x is a dog”

Let O(x,y) = “x owns y”

Let C(x,y) = “x collects the litter of y”

Translation: $\exists x \exists y(P(x) \land D(y) \land O(x,y) \land \neg C(x,y))$

c) All animals are equal.

We interpret this as a universal statement about all pairs of animals.

For all x and y, if x is an animal and y is an animal, then x is equal to y.

Let Domain: All entities

Let A(x) = “x is an animal”

Let E(x,y) = “x is equal to y”

Translation: $\forall x \forall y((A(x) \land A(y)) \to E(x,y))$

Question 3 [6]: Consider the following formula of propositional logic

$P = ((A \lor B) \land C \to (A \land B) \lor C)$

a) Assume that A is true, and B,C are false. Is P true or false? Briefly explain why.

Given: A = T, B = F, C = F

$A \lor B = T \lor F = T$

$(A \lor B) \land C = T \land F = F$

$A \land B = T \land F = F$

$(A \land B) \lor C = F \lor F = F$

$P = ((A \lor B) \land C) \to ((A \land B) \lor C) = F \to F = T$

P is true.

This is because a conditional statement ($\to$) is true whenever the antecedent is false, regardless of the truth value of the consequent.

b) Which truth values for A,B,C will result in P being false?

For P to be false, we need the antecedent $((A \lor B) \land C)$ to be true and the consequent $((A \land B) \lor C)$ to be false.

For the antecedent to be true:

$(A \lor B)$ must be true, so at least one of A or B must be true

C must be true

For the consequent to be false:

$(A \land B)$ must be false, so at least one of A or B must be false

C must be false

However, this creates a contradiction: C cannot be both true (for the antecedent) and false (for the consequent) simultaneously.

Therefore, **P is always true (it's a tautology)** - there are no truth value assignments that make P false.

Question 4 [8]: Predicate Logic Analysis

Consider the sentence "There is a passenger on a coach such that, if that passenger reads, then all passengers on the coach read". In predicate logic, the formula Q formalises it, where $Q = \exists x(Rx \to \forall yRy)$. Answer these questions, giving some explanations.

a) Suppose that the coach has just two passengers. When is Q true and when is it false?

b) Suppose that we know that there is at least one passenger on the coach, but we do not know how many, nor how many are reading. Can we know whether Q is true or false?

Part a) Coach with exactly two passengers

Let's call the two passengers a and b. The formula $Q = \exists x(Rx \rightarrow \forall yRy)$ means "there exists a passenger x such that if x reads, then all passengers read."

For two passengers, this becomes: $Q = (Ra \rightarrow \forall yRy) \vee (Rb \rightarrow \forall yRy)$

Since $\forall yRy$ with two passengers means $(Ra \wedge Rb)$, we get: $Q = (Ra \rightarrow (Ra \wedge Rb)) \vee (Rb \rightarrow (Ra \wedge Rb))$

Case 1: Neither reads (Ra = F, Rb = F) - $Ra \rightarrow (Ra \wedge Rb) = F \rightarrow F = T$ - $Rb \rightarrow (Ra \wedge Rb) = F \rightarrow F = T$ - $Q = T \vee T =$ **T**

Case 2: Only a reads (Ra = T, Rb = F) - $Ra \rightarrow (Ra \wedge Rb) = T \rightarrow F = F$ - $Rb \rightarrow (Ra \wedge Rb) = F \rightarrow F = T$ - $Q = F \vee T =$ **T**

Case 3: Only b reads (Ra = F, Rb = T) - $Ra \rightarrow (Ra \wedge Rb) = F \rightarrow F = T$ - $Rb \rightarrow (Ra \wedge Rb) = T \rightarrow F = F$ - $Q = T \vee F =$ **T**

Case 4: Both read (Ra = T, Rb = T) - $Ra \rightarrow (Ra \wedge Rb) = T \rightarrow T = T$ - $Rb \rightarrow (Ra \wedge Rb) = T \rightarrow T = T$ - $Q = T \vee T =$ **T**

Q is always true when there are exactly two passengers.

Part b) At least one passenger, unknown number and reading status

We can determine that **Q is always true**, regardless of how many passengers there are or who is reading.

There are only two possibilities:

Case 1: All passengers read If $\forall yRy$ is true, then for any passenger x, the conditional $Rx \rightarrow \forall yRy$ is true (since the consequent is true). Therefore, $\exists x(Rx \rightarrow \forall yRy)$ is true.

Case 2: Not all passengers read If $\forall yRy$ is false, then there exists at least one passenger who doesn't read. Let's call such a passenger c, where Rc = F. Then the conditional $Rc \rightarrow \forall yRy = F \rightarrow F = T$. Since we found a passenger c for whom the conditional is true, $\exists x(Rx \rightarrow \forall yRy)$ is true.

Alternative explanation: The formula Q is logically equivalent to $(\exists x \neg Rx) \vee (\forall yRy)$, which reads "either someone doesn't read, or everyone reads." This is always true because in any situation, either everyone reads or someone doesn't read - there's no third possibility.

Therefore, Q is a tautology - it's true in all possible scenarios with any number of passengers and any reading configuration.

Question 5 [12]: Binary Relations Analysis

Consider the binary relations provided below, and analyse them for these features: are they reflexive? Symmetric? Transitive? Are they equivalence relations? And if so, which are the equivalence classes? Clearly list which properties hold and which ones do not, in each relation. Give a short justification when you think that some property does not hold.

a) $R_1 = \{(m, n) \mid m \geq n + 2\} \subseteq \mathbb{N} \times \mathbb{N}$

b) $R_2 = \{(m, n) \mid m + n \text{ is odd } \} \subseteq \mathbb{Z} \times \mathbb{Z}$

c) $S = \{0, 1, 2, 3, 4\}$, $R_3 = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (3, 3), (4, 4), (3, 4), (4, 3)\} \subseteq S \times S$

d) $S = \{0, 1, 2, 3, 4\}$, $R_4 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\} \subseteq S \times S$

Analysis

I'll analyze each binary relation for reflexivity, symmetry, transitivity, and determine if they're equivalence relations.

Part a) $R_1 = \{(m, n) \mid m \geq n + 2\} \subseteq \mathbb{N} \times \mathbb{N}$

Reflexive? NO
For reflexivity, we need $(m, m) \in R_1$ for all $m \in \mathbb{N}$. This requires $m \geq m + 2$, which simplifies to $0 \geq 2$, which is false. So R_1 is not reflexive.

Symmetric? NO
For symmetry, if $(m, n) \in R_1$, then $(n, m) \in R_1$. Consider $(3, 1) \in R_1$ since $3 \geq 1 + 2 = 3$. But $(1, 3) \notin R_1$ since $1 \ngeq 3 + 2 = 5$. So R_1 is not symmetric.

Transitive? YES
If $(m, n) \in R_1$ and $(n, p) \in R_1$, then $m \geq n + 2$ and $n \geq p + 2$. From the second inequality: $n \geq p + 2$, so $n + 2 \geq p + 4$. From the first: $m \geq n + 2 \geq p + 4 > p + 2$. Therefore $(m, p) \in R_1$, so R_1 is transitive.

Equivalence relation? NO (lacks reflexivity and symmetry)

Part b) $R_2 = \{(m, n) \mid m + n \text{ is odd}\} \subseteq \mathbb{Z} \times \mathbb{Z}$

Reflexive? NO
For $(m, m) \in R_2$, we need $m + m = 2m$ to be odd. But $2m$ is always even for any integer m. So R_2 is not reflexive.

Symmetric? YES
If $(m, n) \in R_2$, then $m + n$ is odd. Since $n + m = m + n$, we have $n + m$ is also odd. Therefore $(n, m) \in R_2$, so R_2 is symmetric.

Transitive? NO
Consider $(1, 2) \in R_2$ since $1 + 2 = 3$ is odd. And $(2, 3) \in R_2$ since $2 + 3 = 5$ is odd. But $(1, 3) \notin R_2$ since $1 + 3 = 4$ is even. So R_2 is not transitive.

Equivalence relation? NO (lacks reflexivity and transitivity)

Part c) $S = \{0, 1, 2, 3, 4\}$, $R_3 = \{(0,0), (0,1), (1,0), (1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$

Reflexive? YES
All pairs (0,0), (1,1), (2,2), (3,3), (4,4) are in R_3. So R_3 is reflexive.

Symmetric? YES
For every $(a,b) \in R_3$, we also have $(b,a) \in R_3$: - $(0,0) \leftrightarrow (0,0)$ - $(0,1) \leftrightarrow (1,0)$ - $(1,1) \leftrightarrow (1,1)$ - $(2,2) \leftrightarrow (2,2)$ - $(3,3) \leftrightarrow (3,3)$ - $(4,4) \leftrightarrow (4,4)$ - $(3,4) \leftrightarrow (4,3)$

Transitive? YES
Checking all possible chains: - From {0,1}: (0,0), (0,1), (1,0), (1,1) - all required pairs present - From {2}: only (2,2) - trivially transitive - From {3,4}: (3,3), (3,4), (4,3), (4,4) - all required pairs present

Equivalence relation? YES

Equivalence classes: - [0] = {0, 1} (elements related to 0) - [1] = {0, 1} (same as [0]) - [2] = {2} - [3] = {3, 4} - [4] = {3, 4} (same as [3])

The distinct equivalence classes are: **{0,1}, {2}, {3,4}**

Part d) $S = \{0, 1, 2, 3, 4\}$, $R_4 = \{(0,0), (1,1), (2,2), (3,3), (4,4)\}$

Reflexive? YES
All pairs (i,i) for $i \in S$ are in R_4.

Symmetric? YES
All pairs in R_4 are of the form (i,i), so $(i,i) = (i,i)$. Trivially symmetric.

Transitive? YES
For any $(a,b), (b,c) \in R_4$, we must have $a = b$ and $b = c$ (since all pairs are diagonal). Therefore $a = c$, so $(a,c) \in R_4$.

Equivalence relation? YES

Equivalence classes:
Since R_4 is the identity relation, each element is only related to itself.
- [0] = {0} - [1] = {1} - [2] = {2} - [3] = {3} - [4] = {4}

The equivalence classes are: **{0}, {1}, {2}, {3}, {4}**

Question 6 [8]: Function Analysis

Let f : ℝ → ℝ be the function defined as f(x) = (9x + 3)/(4x - 2)

a) Is f injective? Is it surjective? Justify your reasoning.

b) Find a formula that represents the graph of its inverse function

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Analysis

Looking at the function f(x) = (9x + 3)/(4x - 2), I'll analyze its injectivity and surjectivity, then find its inverse.

Part a) Injectivity and Surjectivity

Injectivity (One-to-one): A function is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Assume $f(x_1) = f(x_2)$: $(9x_1 + 3)/(4x_1 - 2) = (9x_2 + 3)/(4x_2 - 2)$

Cross-multiplying: $(9x_1 + 3)(4x_2 - 2) = (9x_2 + 3)(4x_1 - 2)$

Expanding: $36x_1x_2 - 18x_1 + 12x_2 - 6 = 36x_1x_2 - 18x_2 + 12x_1 - 6$

Simplifying: $-18x_1 + 12x_2 = -18x_2 + 12x_1$ $-18x_1 - 12x_1 = -18x_2 - 12x_2$ $-30x_1 = -30x_2$ $x_1 = x_2$

Therefore, **f is injective**.

Surjectivity (Onto): A function is surjective if for every y ∈ ℝ, there exists an x ∈ ℝ such that f(x) = y.

For any y ∈ ℝ, we need to solve: y = (9x + 3)/(4x - 2)

Cross-multiplying: y(4x - 2) = 9x + 3 4xy - 2y = 9x + 3 4xy - 9x = 2y + 3 x(4y - 9) = 2y + 3

If 4y - 9 ≠ 0 (i.e., y ≠ 9/4), then: x = (2y + 3)/(4y - 9)

However, when y = 9/4, we get: x(0) = 2(9/4) + 3 = 9/2 + 3 = 15/2

This gives 0 = 15/2, which is impossible.

Therefore, $y = 9/4$ cannot be achieved by $f(x)$ for any real x.

f is not surjective because 9/4 is not in the range of f.

Part b) Inverse Function

Since f is injective but not surjective, f has an inverse function on its range (which is $\mathbb{R} \ \{9/4\}$).

To find f^{-1}, I'll solve $y = f(x)$ for x: $y = (9x + 3)/(4x - 2)$ $y(4x - 2) = 9x + 3$ $4xy - 2y = 9x + 3$ $4xy - 9x = 2y + 3$ $x(4y - 9) = 2y + 3$ $x = (2y + 3)/(4y - 9)$

Therefore, $\mathbf{f^{-1}(x) = (2x + 3)/(4x - 9)}$ with domain $\mathbb{R} \ \{9/4\}$.

We can verify: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for appropriate values.

Question 7 [16]: Calculus Functions Analysis

Answer the questions for the functions $f, g, h : \mathbb{R} \to \mathbb{R}$ defined below; show some working before the final answer.

a) Find f', for $f(x) = \cos(2x^3 + 2x^2 + 3)$.

b) Find g'', for $g(x) = 4x^3 - 3x^2 + 2x$, and calculate $g''(3)$.

c) Is the function $h(x) = (4+2x)/(x^3+1)$ continuous? Is it derivable in all points in $\mathbb{R}$? Justify your reasoning.

d) Consider again $h(x) = (4+2x)/(x^3+1)$, but defined as $h : \mathbb{R}^+ \to \mathbb{R}$. Is it continuous? Is this the same function as $h_a(x) = (4+2x)/|x^3+1|$? Justify your reasoning.

Solutions

I'll solve each part step by step, showing the working clearly.

Part a) Find f', for $f(x) = \cos(2x^3 + 2x^2 + 3)$

Using the chain rule: if $f(x) = \cos(u)$ where $u = 2x^3 + 2x^2 + 3$, then $f'(x) = -\sin(u) \cdot u'$

First, find u': $u = 2x^3 + 2x^2 + 3$ $u' = 6x^2 + 4x$

Therefore: $\mathbf{f'(x) = -\sin(2x^3 + 2x^2 + 3) \cdot (6x^2 + 4x)}$ $\mathbf{f'(x) = -(6x^2 + 4x)\sin(2x^3 + 2x^2 + 3)}$

Part b) Find g'' for $g(x) = 4x^3 - 3x^2 + 2x$, and calculate $g''(3)$

First, find $g'(x)$: $g(x) = 4x^3 - 3x^2 + 2x$ $g'(x) = 12x^2 - 6x + 2$

Then find $g''(x)$: $\mathbf{g''(x) = 24x - 6}$

Now calculate $g''(3)$: $\mathbf{g''(3) = 24(3) - 6 = 72 - 6 = 66}$

Part c) Is $h(x) = (4 + 2x)/(x^3 + 1)$ continuous? Is it derivable at all points in $\mathbb{R}$?

For continuity and differentiability, we need to check where the function might have problems.

The function $h(x) = (4 + 2x)/(x^3 + 1)$ is a rational function (quotient of polynomials).

Potential problems: The denominator $x^3 + 1 = 0$ Solving: $x^3 = -1$, so $x = -1$

Continuity: - The numerator $4 + 2x$ is continuous everywhere (polynomial) - The denominator $x^3 + 1$ is continuous everywhere (polynomial) - A rational function is continuous everywhere except where the denominator equals zero

At $x = -1$: the denominator is 0 but the numerator is $4 + 2(-1) = 2 \neq 0$, so we have a vertical asymptote.

Conclusion: $h(x)$ is continuous for all $x \in \mathbb{R}$ except $x = -1$.

Differentiability: Since $h(x)$ is continuous everywhere except $x = -1$, and rational functions are differentiable wherever they're continuous, $h(x)$ is differentiable for all $x \in \mathbb{R}$ except $x = -1$.

Part d) Consider $h(x) = (4 + 2x)/(x^3 + 1)$ defined as $h : \mathbb{R}^+ \to \mathbb{R}$. Is it continuous? Is this the same as $h_a(x) = (4 + 2x)/|x^3 + 1|$?

With domain $\mathbb{R}^+$ (positive real numbers): Since the domain is now $\mathbb{R}^+ = (0, +\infty)$, we exclude $x = -1$ from consideration.

For $x > 0$: $x^3 > 0$, so $x^3 + 1 > 1 > 0$. The denominator is never zero on this domain.

Conclusion: $h : \mathbb{R}^+ \to \mathbb{R}$ is continuous everywhere on its domain.

Comparison with $h_a(x) = (4 + 2x)/|x^3 + 1|$:

For $x \in \mathbb{R}^+$, we have $x^3 + 1 > 0$, so $|x^3 + 1| = x^3 + 1$.

Therefore: $h_a(x) = (4 + 2x)/|x^3 + 1| = (4 + 2x)/(x^3 + 1) = h(x)$ for all $x \in \mathbb{R}^+$.

Conclusion: Yes, when restricted to the domain $\mathbb{R}^+$, h(x) and $h_a(x)$ are the same function.

However, if we considered $h_a(x)$ on all of $\mathbb{R}$, they would differ at x = -1: - h(-1) is undefined - $h_a(-1) = (4 + 2(-1))/|(-1)^3 + 1| = 2/|0|$ is also undefined

But for x near -1 ($x \neq -1$), $h_a(x) = (4 + 2x)/|x^3 + 1|$ would behave differently than h(x) since $|x^3 + 1| \geq 0$ always, while $x^3 + 1$ can be negative.

Question 8 [12]: Matrix Operations Analysis

We define the following matrices:

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 5 & 1 \\ -3 & -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -1 \\ 2 & 3 \\ -5 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} -3 & -2 & -4 \\ 3 & 1 & 0 \end{pmatrix}$$

Answer these questions, showing some working to justify your approach to the answer.

Question 1: Matrix Products D = ABC and E = CB

Does D = ABC exist?

For matrix multiplication to be valid, the number of columns in the first matrix must equal the number of rows in the second matrix.

Step 1: Check AB - A is 3×3, B is 3×2 - Since A has 3 columns and B has 3 rows, AB exists and will be 3×2

Step 2: Check (AB)C - AB is 3×2, C is 2×3 - Since AB has 2 columns and C has 2 rows, (AB)C exists and will be 3×3

Therefore, D = ABC exists and is a 3×3 matrix.

Does E = CB exist?

Check CB: - C is 2×3, B is 3×2 - Since C has 3 columns and B has 3 rows, CB exists and will be 2×2

Therefore, E = CB exists and is a 2×2 matrix.

Calculate d_{23} (element in row 2, column 3 of D)

First, calculate AB:

$$AB = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 5 & 1 \\ -3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 3 \\ -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(3) + 1(2) + (-2)(-5) & 2(-1) + 1(3) + (-2)(1) \\ 0(3) + 5(2) + 1(-5) & 0(-1) + 5(3) + 1(1) \\ (-3)(3) + (-1)(2) + 0(-5) & (-3)(-1) + (-1)(3) + 0(1) \end{pmatrix}$$

$$= \begin{pmatrix} 18 & -1 \\ 5 & 16 \\ -11 & 0 \end{pmatrix}$$

Now calculate D = (AB)C:

$$D = \begin{pmatrix} 18 & -1 \\ 5 & 16 \\ -11 & 0 \end{pmatrix} \begin{pmatrix} -3 & -2 & -4 \\ 3 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 18(-3) + (-1)(3) & 18(-2) + (-1)(1) & 18(-4) + (-1)(0) \\ 5(-3) + 16(3) & 5(-2) + 16(1) & 5(-4) + 16(0) \\ (-11)(-3) + 0(3) & (-11)(-2) + 0(1) & (-11)(-4) + 0(0) \end{pmatrix}$$

$$= \begin{pmatrix} -57 & -37 & -72 \\ 33 & 6 & -20 \\ 33 & 22 & 44 \end{pmatrix}$$

d_{23} = -20

Calculate e_{21} (element in row 2, column 1 of E)

Calculate E = CB:

$$E = \begin{pmatrix} -3 & -2 & -4 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 3 \\ -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-3)(3) + (-2)(2) + (-4)(-5) & (-3)(-1) + (-2)(3) + (-4)(1) \\ 3(3) + 1(2) + 0(-5) & 3(-1) + 1(3) + 0(1) \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -7 \\ 11 & 0 \end{pmatrix}$$

e_{21} = 11

Question 2: Matrix F = B + C^S

Does F = B + C^S exist?

For matrix addition, both matrices must have the same dimensions.

Check dimensions: - B is 3×2 - C is 2×3, so C^T is 3×2

Since both B and C^T are 3×2, **F = B + C^S exists.**

Calculate F

First, find C^T:

$$C^T = \begin{pmatrix} -3 & 3 \\ -2 & 1 \\ -4 & 0 \end{pmatrix}$$

Now calculate F = B + C^T:

$$F = \begin{pmatrix} 3 & -1 \\ 2 & 3 \\ -5 & 1 \end{pmatrix} + \begin{pmatrix} -3 & 3 \\ -2 & 1 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \\ -9 & 1 \end{pmatrix}$$

Question 3: Linear Independence and Transpose

What does it mean for columns of a square matrix M to be linearly independent?

The columns of a square matrix M are **linearly independent** if the only solution to the equation:

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$$

where $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ are the column vectors of M and $c_1, c_2, \dots, c_n$ are scalars, is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

Equivalently, the columns are linearly independent if and only if the matrix M is invertible (non-singular) or $\det(M) \neq 0$.

Will Mˢ also have linearly independent columns?

Yes, if M has linearly independent columns, then Mˢ will also have linearly independent columns.

Reasoning: - The columns of M are linearly independent $\Longleftrightarrow$ M is invertible $\Longleftrightarrow$ $\det(M) \neq 0$ - For any square matrix M: $\det(M^T) = \det(M)$ - Therefore: $\det(M) \neq 0 \Longleftrightarrow \det(M^T) \neq 0 \Longleftrightarrow M^T$ is invertible $\Longleftrightarrow$ columns of M^T are linearly independent

Note that the columns of M^T are the rows of M, so this also means: **the columns of M are linearly independent if and only if the rows of M are linearly independent.**

Effects on determinants

$$\det(M^T) = \det(M)$$

This is a fundamental property of determinants. Taking the transpose does not change the determinant value. Therefore, if M has linearly independent columns ($\det(M) \neq 0$), then M^T will have the same determinant value and also have linearly independent columns.

Question 9 [16]: Matrix Analysis

We define the matrix:

$$M = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$$

Question 1: Determinant, Inverse, Eigenvalues and Eigenvectors

Calculate the determinant

For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is $ad - bc$.

$$\det(M) = (-2)(3) - (4)(-1) = -6 + 4 = -2$$

Calculate the inverse matrix

For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant Δ, the inverse is:

$$M^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Since $\det(M) = -2 \neq 0$, the inverse exists:

$$M^{-1} = \frac{1}{-2} \begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

Calculate eigenvalues

The eigenvalues are found by solving $\det(M - \lambda I) = 0$:

$$M - \lambda I = \begin{pmatrix} -2 - \lambda & 4 \\ -1 & 3 - \lambda \end{pmatrix}$$

$$\begin{aligned} \det(M - \lambda I) &= (-2 - \lambda)(3 - \lambda) - (4)(-1) \\ &= (-2 - \lambda)(3 - \lambda) + 4 \\ &= -6 + 2\lambda - 3\lambda + \lambda^2 + 4 \\ &= \lambda^2 - \lambda - 2 \end{aligned}$$

Setting this equal to zero:

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

Eigenvalues: $\lambda_1 = 2$ **and** $\lambda_2 = -1$

Calculate eigenvectors

For $\lambda_1 = 2$: Solve $(M - 2I)\vec{v} = \vec{0}$:

$$\begin{pmatrix} -4 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives us: $-4x + 4y = 0$, so $x = y$.

Eigenvector for $\lambda_1 = 2$: $\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any scalar multiple)

For $\lambda_2 = -1$: Solve $(M + I)\vec{v} = \vec{0}$:

$$\begin{pmatrix} -1 & 4 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives us: $-x + 4y = 0$, so $x = 4y$.

Eigenvector for $\lambda_2 = -1$: $\vec{v_2} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ (or any scalar multiple)

Question 2: Geometric Effects on Point (0, -4)

Apply the transformation

When matrix M is applied to point $(0, -4)$:

$$M \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} (-2)(0) + (4)(-4) \\ (-1)(0) + (3)(-4) \end{pmatrix} = \begin{pmatrix} -16 \\ -12 \end{pmatrix}$$

The point $(0, -4)$ is transformed to $(-16, -12)$.

Geometric description

Analysis of the transformation:

1. **Scaling:** The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$
   - Along the eigenvector direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, vectors are scaled by factor 2
   - Along the eigenvector direction $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, vectors are scaled by factor -1 (reflection and scaling)

2. **Reflection:** The negative eigenvalue $\lambda_2 = -1$ indicates a reflection along the direction perpendicular to $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

3. **Overall effect:** The transformation combines scaling and reflection. The determinant is negative (-2), confirming orientation reversal.

4. **For the specific point $(0, -4)$:** The point moves from the negative y-axis to the third quadrant at $(-16, -12)$, demonstrating both the scaling and orientation-reversing effects.

Question 3: Information from the Determinant

The determinant of a matrix corresponding to a linear transformation provides several key pieces of geometric information:

1. Area/Volume Scaling Factor

The absolute value of the determinant $|\det(M)|$ tells us the factor by which areas (in 2D) or volumes (in higher dimensions) are scaled under the transformation.

For our matrix: $|\det(M)| = |-2| = 2$, so areas are doubled.

2. Orientation Preservation/Reversal

The sign of the determinant indicates whether orientation is preserved: - **Positive determinant:** Orientation is preserved (no reflection) - **Negative determinant:** Orientation is reversed (includes reflection)

For our matrix: $\det(M) = -2 < 0$, so orientation is reversed.

3. Invertibility

Non-zero determinant: The transformation is invertible (one-to-one and onto) **Zero determinant:** The transformation is not invertible (maps to lower dimension)

For our matrix: $\det(M) = -2 \neq 0$, so the transformation is invertible.

4. Linear Independence

Non-zero determinant: The columns (or rows) of the matrix are linearly independent **Zero determinant:** The columns (or rows) are linearly dependent

Summary for our transformation:

- Areas are scaled by factor 2
- Orientation is reversed (clockwise becomes counterclockwise)
- The transformation is invertible
- The transformation preserves the “shape” of objects but changes their size and orientation

Question 10 [10]: System of Linear Equations

Calculate the solution to the system of real linear equations, by showing your approach:

$x_1 - 2x_2 - 3x_3 = -1$ $x_1 + 3x_2 - 4x_3 = 0$ $2x_1 + 3x_2 + x_3 = 2$

Solution using Direct Substitution (Simplest Method)

The easiest approach is to solve equation 1 for x_1 and substitute.

Step 1: Solve equation 1 for x_1

From equation 1: $x_1 - 2x_2 - 3x_3 = -1$

$\mathbf{x_1 = -1 + 2x_2 + 3x_3}$

Step 2: Substitute into equations 2 and 3

Substitute into equation 2: $x_1 + 3x_2 - 4x_3 = 0$ $(-1 + 2x_2 + 3x_3) + 3x_2 - 4x_3 = 0$ $-1 + 2x_2 + 3x_3 + 3x_2 - 4x_3 = 0$ $-1 + 5x_2 - x_3 = 0$ $\mathbf{5x_2 - x_3 = 1}$ … (4)

Substitute into equation 3: $2x_1 + 3x_2 + x_3 = 2$ $2(-1 + 2x_2 + 3x_3) + 3x_2 + x_3 = 2$ $-2 + 4x_2 + 6x_3 + 3x_2 + x_3 = 2$ $-2 + 7x_2 + 7x_3 = 2$ $7x_2 + 7x_3 = 4$ $\mathbf{x_2 + x_3 = 4/7}$ … (5)

Step 3: Solve the 2×2 system

From equations (4) and (5): $5x_2 - x_3 = 1$ $x_2 + x_3 = 4/7$

Add the equations: $6x_2 = 1 + 4/7 = 7/7 + 4/7 = 11/7$

$\mathbf{x_2 = 11/42}$

Step 4: Find x_3

From equation (5): $x_3 = 4/7 - x_2 = 4/7 - 11/42$

Convert to common denominator: $4/7 = 24/42$

$x_3 = 24/42 - 11/42 =$ **$13/42$**

Step 5: Find x_1

$x_1 = -1 + 2x_2 + 3x_3$ $x_1 = -1 + 2(11/42) + 3(13/42)$ $x_1 = -1 + 22/42 + 39/42$ $x_1 = -1 + 61/42$ $x_1 = -42/42 + 61/42 =$ **$19/42$**

Final Solution

$x_1 = 19/42$ $x_2 = 11/42$ $x_3 = 13/42$

Verification

Equation 1: $19/42 - 2(11/42) - 3(13/42) = 19/42 - 22/42 - 39/42 = -42/42 = -1$

Equation 2: $19/42 + 3(11/42) - 4(13/42) = 19/42 + 33/42 - 52/42 = 0$

Equation 3: $2(19/42) + 3(11/42) + 13/42 = 38/42 + 33/42 + 13/42 = 84/42 = 2$

This method is indeed the simplest - solving for x_1 from the first equation gives the cleanest substitution!

Question 10 [10]: Gaussian Elimination

Calculate the solution to the system of real linear equations, by showing your approach:

$x_1 - 2x_2 - 3x_3 = -1$ $x_1 + 3x_2 - 4x_3 = 0$ $2x_1 + 3x_2 + x_3 = 2$

Solution using Gaussian Elimination

I'll solve this system by converting the augmented matrix to row echelon form, then using back substitution.

Step 1: Set up the augmented matrix

```
[  1  -2  -3 | -1 ]
[  1   3  -4 |  0 ]
[  2   3   1 |  2 ]
```

Step 2: Eliminate the first column below the pivot

$R_2 = R_2 - R_1$ (eliminate x_1 from equation 2)

```
[  1  -2  -3 | -1 ]
[  0   5  -1 |  1 ]
[  2   3   1 |  2 ]
```

$R_3 = R_3 - 2R_1$ (eliminate x_1 from equation 3)

```
[  1  -2  -3 | -1 ]
[  0   5  -1 |  1 ]
[  0   7   7 |  4 ]
```

Step 3: Eliminate the second column below the pivot

$R_3 = R_3 - (7/5)R_2$ (eliminate x_2 from equation 3)

First, let's calculate $(7/5)R_2$: $(7/5) \times [0\ 5\ -1 \mid 1] = [0\ 7\ -7/5 \mid 7/5]$

$R_3 = [0\ 7\ 7 \mid 4] - [0\ 7\ -7/5 \mid 7/5]$

```
[  1  -2  -3 | -1 ]
[  0   5  -1 |  1 ]
[  0   0  42/5 | 13/5 ]
```

Step 4: Convert to exact fractions for clarity

The last row gives us: $(42/5)x_3 = 13/5$

Multiplying by 5/42: $x_3 = (13/5) \times (5/42) = 13/42$

Let's simplify: $x_3 = 13/42$

Step 5: Back substitution

Find x_3: From the third equation: $(42/5)x_3 = 13/5$ $x_3 = (13/5) \div (42/5) = 13/42$

Let's work with decimals for easier calculation: $42x_3 = 13$, so $x_3 = 13/42$

Actually, let me recalculate this more carefully:

From R_3: $0x_1 + 0x_2 + (7 + 7/5)x_3 = 4 - 7/5 = (35/5 + 7/5)x_3 = (20/5 - 7/5) = (42/5)x_3 = 13/5$

Therefore: $x_3 = (13/5) \times (5/42) = 13/42$

Find x_2: From the second equation: $5x_2 - x_3 = 1$ $5x_2 - 13/42 = 1$ $5x_2 = 1 + 13/42 = 42/42 + 13/42 = 55/42$ $x_2 = 55/210 = 11/42$

Find x_1: From the first equation: $x_1 - 2x_2 - 3x_3 = -1$ $x_1 - 2(11/42) - 3(13/42) = -1$ $x_1 - 22/42 - 39/42 = -1$ $x_1 - 61/42 = -1$ $x_1 = -1 + 61/42 = -42/42 + 61/42 = 19/42$

Step 6: Final Solution

$x_1 = 19/42$ $x_2 = 11/42$
$x_3 = 13/42$

Verification

Let's check our solution in the original equations:

Equation 1: $x_1 - 2x_2 - 3x_3 = 19/42 - 2(11/42) - 3(13/42) = 19/42 - 22/42 - 39/42 = -42/42 = -1$

Equation 2: $x_1 + 3x_2 - 4x_3 = 19/42 + 3(11/42) - 4(13/42) = 19/42 + 33/42 - 52/42 = 0/42 = 0$

Equation 3: $2x_1 + 3x_2 + x_3 = 2(19/42) + 3(11/42) + 13/42 = 38/42 + 33/42 + 13/42 = 84/42 = 2$

All equations check out, confirming our solution is correct.