Mathematics for AI - Problem Sheet 2 Solutions

Question 1 [15%]

a) Three components of a probability space [6]

The three components that define a probability space are:[^1]

Sample Space (Ω): The set of all possible outcomes. For two dice: Ω = {(1,1), (1,2), (1,3), …, (6,6)} Total of 36 possible outcomes

Event Space (F): The collection of all possible events (subsets of the sample space) For example: "both dice show even numbers" = {(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)}

Probability Measure (P): A function that assigns probabilities to events For fair dice: P(each outcome) = 1/36 P must satisfy: 0 ≤ P(A) ≤ 1 for any event A, and P(Ω) = 1

b) Unfair dice [2]

If the dice are not fair, only the **probability measure** changes. The sample space and event space remain the same, but P(each outcome) is no longer 1/36. Some outcomes become more likely than others.

c) Summing dice values [3]

When we sum the dice values, the sample space changes to:

New sample space: {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

The event space now consists of subsets of these sums

Probabilities are no longer uniform: P(sum = 7) = 6/36 = 1/6, but P(sum = 2) = 1/36

d) Thousand dice [4]

With 1000 dice, the probability distribution of the sum would approach a **normal distribution** due to the Central Limit Theorem.[^2]

Expected value: E(sum) = 1000 × 3.5 = 3500

The distribution becomes bell-shaped and symmetric around 3500

Most values cluster near the mean, with very low probability of extreme values

This happens because we're adding many independent random variables

Question 2 [15%]

Setup: 4 red, 3 blue, 1 green ball (8 total). **With replacement** initially.

a) $P(BG)$ [3]

$P(BG) = P(\text{Blue first}) \times P(\text{Green second})$

$= (3/8) \times (1/8) = 3/64$

b) $P(G)$ [3]

$P(G) = P(\text{Green on first draw OR Green on second draw})$

$= P(G_1) + P(G_2) - P(G_1 \cap G_2)$

$= (1/8) + (1/8) - (1/8 \times 1/8)$

$= 1/8 + 1/8 - 1/64 = 8/64 + 8/64 - 1/64 = 15/64$

c) $P(R_2|B_1)$ [3]

Since we replace the ball, the second draw is independent of the first.

$P(R_2|B_1) = P(R_2) = 4/8 = 1/2$

d) Without replacement [6]

i) $P(BG)$ [2] $P(BG) = P(\text{Blue first}) \times P(\text{Green second} \mid \text{Blue first}) = (3/8) \times (1/7) = 3/56$

ii) $P(G)$ [2] $P(G) = P(G_1) + P(G_2|\text{not } G_1) = (1/8) + (7/8) \times (1/7) = 1/8 + 1/8 = 1/4$

iii) $P(R_2|B_1)$ [2] $P(R_2|B_1) = 4/7$ (since after removing a blue ball, 7 balls remain, 4 of which are red)

Question 3 [15%]

Given:

$P(\text{STEM}) = 0.3$, so $P(\text{non-STEM}) = 0.7$

$P(\text{Pass}|\text{STEM}) = 0.8$

$P(\text{Pass}|\text{non-STEM}) = 0.6$

Find: $P(\text{STEM}|\text{Pass})$

Using Bayes' theorem:

$P(\text{STEM}|\text{Pass}) = P(\text{Pass}|\text{STEM}) \times P(\text{STEM}) / P(\text{Pass})$

First, find $P(\text{Pass})$: $P(\text{Pass}) = P(\text{Pass}|\text{STEM}) \times P(\text{STEM}) + P(\text{Pass}|\text{non-STEM}) \times P(\text{non-STEM}) = 0.8 \times 0.3 + 0.6 \times 0.7 = 0.24 + 0.42 = 0.66$

Therefore: $P(\text{STEM}|\text{Pass}) = (0.8 \times 0.3) / 0.66 = 0.24 / 0.66 = 4/11 \approx 0.364$

Answer: 36.4%

Question 4 [15%]

Given: Average of 2 calls every 5 minutes, Poisson distribution
Find: Minimum agents needed so calls are on hold ≤ 1% of time

For a Poisson distribution with λ = 2:[^3]

We need P(X > n) ≤ 0.01, where n is the number of agents available and X is the number of calls arriving.

Poisson Probability Calculations

For a Poisson distribution with parameter λ = 2:

Individual probability formula:

$$P(X = k) = \frac{e^{-2} \cdot 2^k}{k!}$$

Where e^(-2) ≈ 0.1353

Individual probabilities:

P(X = 0) = (0.1353 × 2^0) / 0! = 0.1353

P(X = 1) = (0.1353 × 2^1) / 1! = 0.2706

P(X = 2) = (0.1353 × 2^2) / 2! = 0.2706

P(X = 3) = (0.1353 × 2^3) / 3! = 0.1804

P(X = 4) = (0.1353 × 2^4) / 4! = 0.0902

P(X = 5) = (0.1353 × 2^5) / 5! = 0.0361

P(X = 6) = (0.1353 × 2^6) / 6! = 0.0120

Cumulative probabilities:

P(X ≤ 4) = 0.1353 + 0.2706 + 0.2706 + 0.1804 + 0.0902 = 0.9471

P(X ≤ 5) = 0.9471 + 0.0361 = 0.9832

P(X ≤ 6) = 0.9832 + 0.0120 = 0.9952

Analysis

We need P(X > n) ≤ 0.01:

With 6 agents: P(X > 6) = 1 - P(X ≤ 6) = 1 - 0.9952 = 0.0048

With 5 agents: P(X > 5) = 1 - P(X ≤ 5) = 1 - 0.9832 = 0.0168

Since P(X > 6) = 0.0048 < 0.01, we need enough agents to handle up to 6 calls without anyone being placed on hold.

Answer: 7 agents are needed

Question 5 [10%]

descriptive statistics, covariance and sample and population sizes[^4]

Data: [11, 31], [8, 16], [5, 21], [9, 34], [12, 14], [13, 17]

a) Covariance calculation [8]

First, find the means:

$\bar{x} = \frac{11+8+5+9+12+13}{6} = \frac{58}{6} = 9.67$

$\bar{y} = \frac{31+16+21+34+14+17}{6} = \frac{133}{6} = 22.17$

Sample covariance formula: $s_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$
11	31	1.33	8.83	11.74
8	16	-1.67	-6.17	10.30
5	21	-4.67	-1.17	5.46
9	34	-0.67	11.83	-7.93
12	14	2.33	-8.17	-19.04
13	17	3.33	-5.17	-17.22

Sum = 11.74 + 10.30 + 5.46 - 7.93 - 19.04 - 17.22 = -16.69

$s_{xy} = \frac{-16.69}{5} = -3.34$

b) Population vs Sample [2]

If this was the whole population, we would use the population covariance formula with denominator n instead of n-1:

$\sigma_{xy} = \frac{-16.69}{6} = -2.78$

The difference is we divide by n rather than n-1 for population covariance.

Question 6 [15%]

Given: X ~ Uniform(0,1), Y = e^x

a) CDF of Y [6]

For Y = e^x where X ~ Uniform(0,1):

Range of Y: [e^0, e^1] = [1, e]

For y ∈ [1, e]: F_Y(y) = P(Y ≤ y) = P(e^x ≤ y) = P(X ≤ ln(y)) = ln(y)

(since X ~ Uniform(0,1), so F_X(x) = x for x ∈ [0,1])

Therefore:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 1 \\ \ln(y) & \text{if } 1 \leq y \leq e \\ 1 & \text{if } y > e \end{cases}$$

b) PDF of Y [4]

Using the transformation method for continuous random variables:

$$f_Y(y) = f_X(x(y)) \left| \frac{dx}{dy} \right|$$

Where:

$y = e^x \Rightarrow x = \ln(y)$

$\frac{dx}{dy} = \frac{d}{dy}[\ln(y)] = \frac{1}{y}$

$f_X(x) = 1$ for $x \in [0, 1]$

Therefore:

$$f_Y(y) = f_X(\ln(y)) \left| \frac{1}{y} \right| = 1 \cdot \frac{1}{y} = \frac{1}{y}$$

for $y \in [1, e]$, and 0 otherwise.

c) E(Y) [^5]

Using the definition of expected value for continuous random variables:

$$E(Y) = \int_1^e y \cdot \frac{1}{y} dy = \int_1^e 1 dy = [y]_1^e = e - 1$$

Answer: E(Y) = e - 1 ≈ 1.718

Question 7 [15%]

Using info learnt from Zedstatistics website [4]

Data: 1.647, 1.695, 1.703, 1.727, 1.784, 1.651, 1.764, 1.723

a) Hypotheses [4]

H_0: μ = 1.75 (filament meets industry standard)

H_1: μ ≠ 1.75 (filament does not meet industry standard)

b) P-value analysis [6]

Sample statistics:

n = 8

$\bar{x} = \frac{1.647+1.695+1.703+1.727+1.784+1.651+1.764+1.723}{8} = 1.712$

$s = 0.048$ (sample standard deviation)

Test statistic: $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{1.712-1.75}{0.048/\sqrt{8}} = \frac{-0.038}{0.017} = -2.24$

With df = 7 and $\alpha = 0.05$ (two-tailed), critical value $\approx \pm 2.365$

Since $|t| = 2.24 < 2.365$, we **fail to reject H_0**.

The p-value $\approx 0.06 > 0.05$, so the filament should be **approved**.

c) Type of test [2]

This is a **two-tailed test** because we're testing whether the diameter differs from 1.75mm (either too high or too low).

d) Type I and II Errors [2]

Type I Error: Rejecting good filament (saying it's defective when it's actually fine)

Consequence: Wasted materials and production costs

Type II Error: Accepting bad filament (saying it's fine when it's actually defective)

Consequence: Poor quality products reach customers, reputation damage

e) Reducing error risk [1]

Increase sample size - taking more measurements would reduce the standard error and make the test more reliable.

REFERENCES:

[^1] Bath University. (2024). Course Materials. Retrieved from

https://engage.bath.ac.uk/learn/mod/page/view.php?id=289625&forceview=1

[^2] Ross, S. M. (2017). *A First Course in Probability* (10th ed.). Pearson. Chapter 8: Limit Theorems.

[^3] Ross, S. M. (2017). *A First Course in Probability* (10th ed.). Pearson. Chapter 4.7: The Poisson Random Variable.

[^4] Zed Statistics. (2024). Covariance and Descriptive Statistics [Video series]. YouTube. https://www.youtube.com/@zedstatistics

[^5] Ross, S. M. (2017). *A First Course in Probability* (10th ed.). Pearson. Chapter 5 Continuous Random Variables

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