Technical Computing for the Earth Sciences, Lecture 6:

Solving some linearizable inverse problems for the Earth sciences

EARS 80.03

Linearizable inverse problems

You're already solving a nonlinear-but-linearizable inverse problem:

Project 2!

- We'll just review some of the concepts here

Linear inverse problems

Generalized linear inverse problem: you have

- some data y
- an equation of the form

  $$y = f(x,\ parameters...)\ [+\ g(\mathrm{x},\ parameters...) + ...]$$

  where f, g are nonlinear functions of x and one or more parameters for which we wish to invert

- Can't solve in one step any more, have to iterate!
- The derivative of a nonlinear function gives us one simple way to approximate a nonlinear function with a linear one (tangent line) -> *Newton's method*

Newton's method

18.2.1 Basic Gauss–Newton algorithm

The idea behind the Gauss–Newton algorithm is simple: We alternate between finding an affine approximation of the function f at the current iterate, and then solving the associated linear least squares problem to find the next iterate. This combines two of the most powerful ideas in applied mathematics: *Calculus* is used to form an affine approximation of a function near a given point, and *least squares* is used to compute an approximate solution of the resulting affine equations.

We now describe the algorithm in more detail. At each iteration k, we form the affine approximation $\hat{f}$ of f at the current iterate $x^{(k)}$, given by the Taylor approximation

$$\hat{f}(x; x^{(k)}) = f(x^{(k)}) + Df(x^{(k)})(x - x^{(k)}), \tag{18.5}$$

where the $m \times n$ matrix $Df(x^{(k)})$ is the Jacobian or derivative matrix of f (see §8.2.1 and §C.1). The affine function $\hat{f}(x; x^{(k)})$ is a very good approximation of $f(x)$ provided x is near $x^{(k)}$, *i.e.*, $\|x - x^{(k)}\|$ is small.

The next iterate $x^{(k+1)}$ is then taken to be the minimizer of $\|\hat{f}(x; x^{(k)})\|^2$, the norm squared of the affine approximation of f at $x^{(k)}$. Assuming that the derivative matrix $Df(x^{(k)})$ has linearly independent columns (which requires $m \geq n$), we have

$$x^{(k+1)} = x^{(k)} - \left(Df(x^{(k)})^T Df(x^{(k)})\right)^{-1} Df(x^{(k)})^T f(x^{(k)}). \tag{18.6}$$

This iteration gives the basic Gauss–Newton algorithm.

Algorithm 18.1 BASIC GAUSS–NEWTON ALGORITHM FOR NONLINEAR LEAST SQUARES

given a differentiable function $f : \mathbf{R}^n \to \mathbf{R}^m$, an initial point $x^{(1)}$.

For $k = 1, 2, \ldots, k^{\max}$

1. *Form affine approximation at current iterate using calculus.* Evaluate the Jacobian $Df(x^{(k)})$ and define

$$\hat{f}(x; x^{(k)}) = f(x^{(k)}) + Df(x^{(k)})(x - x^{(k)}).$$

2. *Update iterate using linear least squares.* Set $x^{(k+1)}$ as the minimizer of $\|\hat{f}(x; x^{(k)})\|^2$,

$$x^{(k+1)} = x^{(k)} - \left(Df(x^{(k)})^T Df(x^{(k)})\right)^{-1} Df(x^{(k)})^T f(x^{(k)}).$$

Newton's method

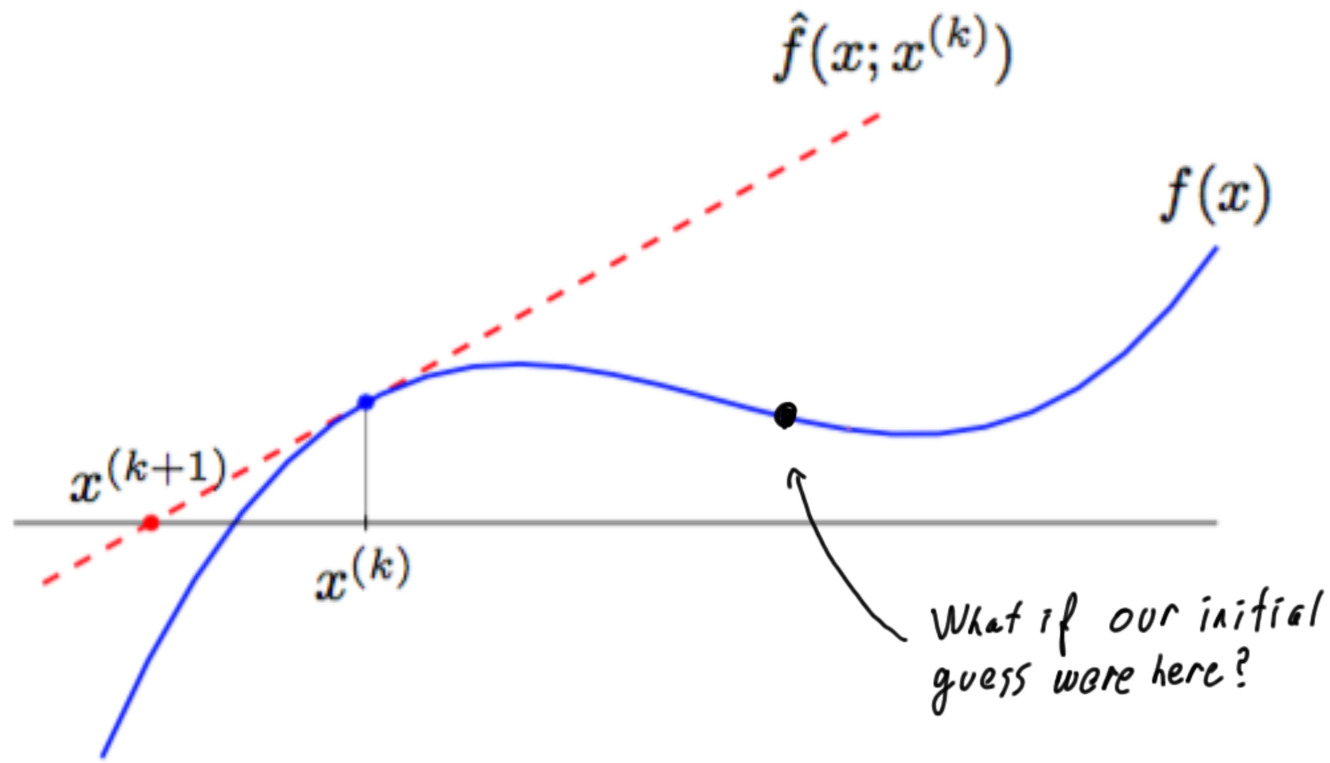
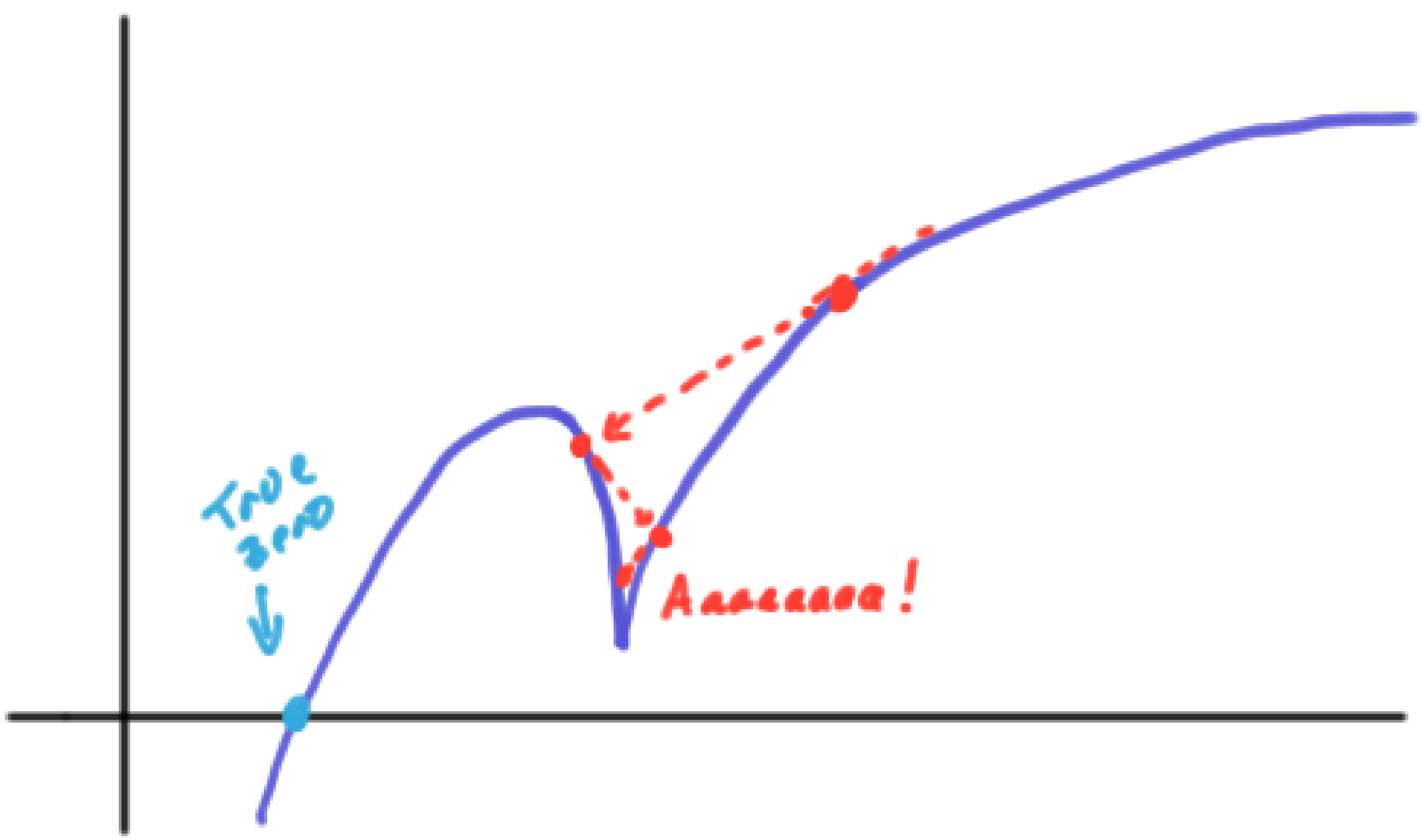


Figure 18.2 One iteration of the Newton algorithm for solving an equation $f(x) = 0$ in one variable.

Newton's method

Can get stuck:



Newton's method

Converges

Slightly worse initial guess - Diverges forever

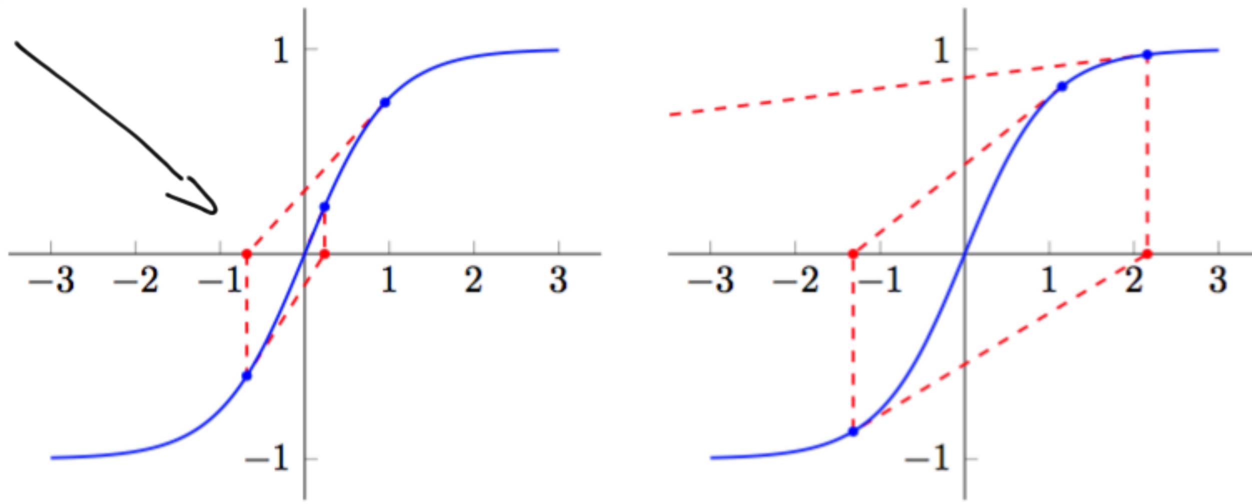


Figure 18.3 The first iterations in the Newton algorithm for solving $f(x) = 0$, for two starting points: $x^{(1)} = 0.95$ and $x^{(1)} = 1.15$.

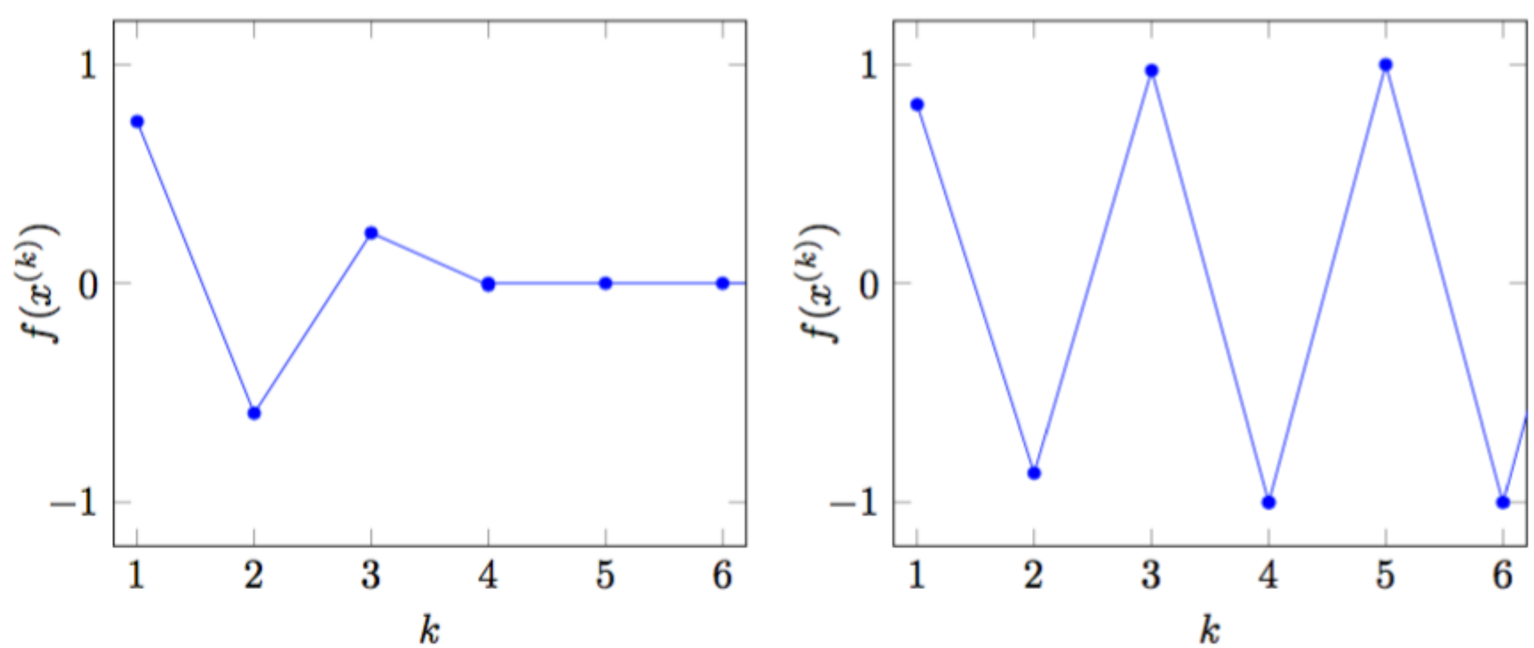


Figure 18.4 Value of $f(x^{(k)})$ versus iteration number k for Newton's method in the example of figure 18.3, started at $x^{(1)} = 0.95$ and $x^{(1)} = 1.15$.

Some more things to think about

- Thinking about Project 2, will there neccessarily be one point (x_0, y_0, z_0) where $\mathbf{t} - \hat{\mathbf{t}}$ will be zero at *all* of the seismic stations at the same time? **Only if there's no noise! Even then, our linear approximations generally won't get us there in one try. For real data with some noise, this will probably get harder the more data points and more dimensions we're working with.**

- What do we want to find instead? **The least-squares approximation!**

- So linear algebra saves the day again, because that's exactly what our left-inverse / "pseudoinverse" from Project 1 already gives us if there isn't an exact solution!

- Newton's method is the most approachable, but there are other more complicated methods for linearizing nonlinear inverse problems, e.g. *Levenberg–Marquardt* (discussed in B&V 18.3)