M50 Homework 2

Alex Craig

Exercise 1.

(Computing conditional averages): Consider a random variable $Y = (Y_1, Y_2)$ which takes values in the sample space:

$$S = \mathbb{N} \times \mathbb{N} = (i, j), i, j \in \mathbb{N}$$

That is, the sample space consists of all possible pairs of numbers (i, j). Now suppose we have some data:

$$(1, 2), (1, 2), (3, 1), (1, 4), (3, 3), (2, 2), (1, 5)$$

Give you best estimates of the following (either by hand, with Python, or a calculator)

$$E[Y_1], \quad E[Y_1 \mid Y_2 = 2], \quad E[Y_2 \mid Y_1 = 1], \quad E[Y_2 \mid Y_1 > 1]$$

Solution

$$E[Y_1] \approx \frac{1+1+3+1+3+2+1}{7} = \frac{12}{7}$$

$$E[Y_1 \mid Y_2 = 2] \approx \frac{1+1+2}{3} = \frac{4}{3}$$

$$E[Y_2 \mid Y_1 = 1] \approx \frac{2+2+4+5}{4} = \frac{13}{4}$$

$$E[Y_2 \mid Y_1 > 1] \approx \frac{1+3+2}{3} = 2$$

Exercise 2.

(Independence and conditional expectation): Let X and Y be two random variables with sample spaces S_X and S_Y.

Part A

Prove that if X and Y are independent $E[X \mid Y = y] = E[X]$ and $E[Y \mid X = x] = E[Y]$ for all $x \in S_X$ and $y \in S_Y$.

Solution

We know that conditional expectation is defined as:

$$E[X \mid Y = y] = \sum_{x \in S_X} x \cdot \mathbb{P}(X = x \mid Y = y)$$

Keep in mind that conditional probability is defined as:

$$\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

For independent random variables, we know that:

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$$

Therefore, if X and Y are independent we may rewrite the conditional probability as:

$$\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)}{\mathbb{P}(Y = y)} = \mathbb{P}(X = x)$$

Therefore, if X and Y are independent we may rewrite the conditional expectation as:

$$E[X \mid Y = y] = \sum_{x \in S_X} x \cdot \mathbb{P}(X = x \mid Y = y) = \sum_{x \in S_X} x \cdot \mathbb{P}(X = x) = E[X]$$

$$\Rightarrow E[X \mid Y = y] = E[X]$$

Part B

Prove the tower property of expectation that is stated in the class notes.

Solution

The notes state the tower property as:

$$E[E[X \mid Y]] = E[X]$$

Let's prove this in the discrete case:

$$E[E[X \mid Y]] = \sum_{y \in S_Y} E[X \mid Y = y] \cdot \mathbb{P}(Y = y)$$

We know that:

$$E[X \mid Y = y] = \sum_{x \in S_X} x \cdot \mathbb{P}(X = x \mid Y = y)$$

Therefore, we may rewrite the above as:

$$E[E[X \mid Y]] = \sum_{y \in S_Y} \sum_{x \in S_X} x \cdot \mathbb{P}(X = x \mid Y = y) \cdot \mathbb{P}(Y = y)$$

We know that:

$$\mathbb{P}(X = x \mid Y = y) \cdot \mathbb{P}(Y = y) = \mathbb{P}(X = x, Y = y)$$

Therefore, we may rewrite the above as:

$$E[E[X \mid Y]] = \sum_{y \in S_Y} \sum_{x \in S_X} x \cdot \mathbb{P}(X = x, Y = y)$$

When we sum over all y in S_Y, we marginalize out Y and are left with:

$$E[E[X \mid Y]] = \sum_{x \in S_X} x \cdot \mathbb{P}(X = x) = E[X]$$

Part C

Prove that if X and Y are independent, then $Var(X + Y) = Var(X) + Var(Y)$.

Hint: Use this formula for variance: $Var(X) = E[X^2] - E[X]^2$.

Solution

$$Var(X+Y) = E[(X+Y)^2] - E[X+Y]^2 = E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$$

Using the linearity of expectation, we may rewrite the above as:

$$= E[X^2] + 2E[XY] + E[Y^2] - (E[X]^2 + 2E[X]E[Y] + E[Y]^2)$$

If X and Y are independent, then $E[XY] = E[X]E[Y]$. Therefore, we may rewrite the above as:

$$= E[X^2] + 2E[X]E[Y] + E[Y^2] - (E[X]^2 + 2E[X]E[Y] + E[Y]^2)$$

$$= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 = Var(X) + Var(Y)$$

Exercise 3.

(Aspects of the binomial distribution): Suppose Y_1 and Y_2 are two independent binomial distributions:

$$Y_1 \sim Bin(n_1, p_1), \quad Y_2 \sim Bin(n_2, p_2)$$

with $n_1, n_2 \in \mathbb{N}$ and $p_1, p_2 \in (0, 1)$

Part A

If $p = p_1 = p_2$, what is the distribution of $Y_1 + Y_2$?

Solution

Given that $p = p_1 = p_2$, we may define the probability mass function of Y_k as:

$$\mathbb{P}(Y_k = y) = \binom{n_k}{y} p^y (1-p)^{n_k - y}, \quad y = 0, 1, \ldots, n_k, \quad k = 1, 2$$

Define $Y = Y_1 + Y_2$. We may then define the probability mass function of Y as:

$$\mathbb{P}(Y = y) = \mathbb{P}(Y_1 + Y_2 = y) = \sum_{i=0}^{y} \mathbb{P}(Y_1 = i, Y_2 = y - i)$$

Since Y_1 and Y_2 are independent, we may rewrite the above as:

$$\sum_{i=0}^{y} \mathbb{P}(Y_1 = i, Y_2 = y - i) = \sum_{i=0}^{y} \mathbb{P}(Y_1 = i) \cdot \mathbb{P}(Y_2 = y - i)$$

And now we may substitute in the probability mass function of Y_1 and Y_2:

$$\mathbb{P}(Y = y) = \sum_{i=0}^{y} \binom{n_1}{i} p^i (1-p)^{n_1 - i} \cdot \binom{n_2}{y-i} p^{y-i} (1-p)^{n_2 - y + i}$$

$$\mathbb{P}(Y = y) = \sum_{i=0}^{y} \binom{n_1}{i} \binom{n_2}{y-i} p^y (1-p)^{n_1 + n_2 - y}$$

$$\mathbb{P}(Y = y) = p^y (1-p)^{n_1 + n_2 - y} \sum_{i=0}^{y} \binom{n_1}{i} \binom{n_2}{y-i}$$

And, by Vandermonde's Identity:

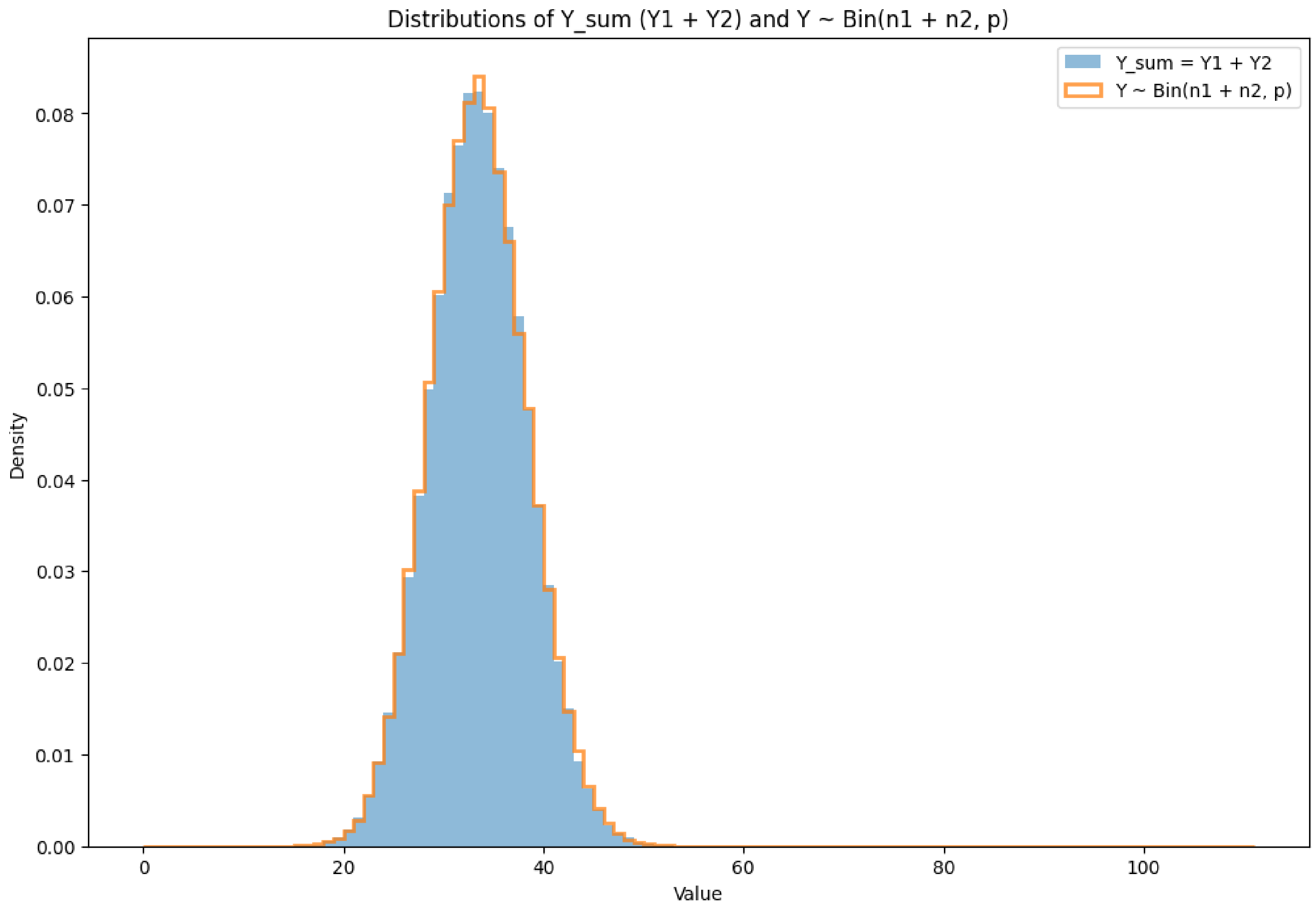
$$\mathbb{P}(Y = y) = p^y(1-p)^{n_1+n_2-y}\binom{n_1+n_2}{y}$$

From this probability mass function, we may conclude that $Y \sim Bin(n_1 + n_2, p)$.

Part B

Confirm you answer to part (a) with simulations with $n_1 = 100$, $n_2 = 10$ and $p = 0.3$.

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
n1 = 100
n2 = 10
p = 0.3

# Number of simulations
num_simulations = 100000

# Simulating Y1 and Y2
Y1 = np.random.binomial(n1, p, num_simulations)
Y2 = np.random.binomial(n2, p, num_simulations)

# Y = Y1 + Y2
Y_sum = Y1 + Y2

# Simulating Y
Y = np.random.binomial(n1 + n2, p, num_simulations)

# Plotting histograms
plt.figure(figsize=(12, 8))

plt.hist(
    Y_sum, bins=range(n1 + n2 + 2), alpha=0.5, label="Y_sum = Y1 + Y2", density=True
)
plt.hist(
    Y,
    bins=range(n1 + n2 + 2),
    alpha=0.75,
    label="Y ~ Bin(n1 + n2, p)",
    density=True,
    histtype="step",
    linewidth=2,
)

plt.xlabel("Value")
plt.ylabel("Density")
plt.title("Distributions of Y_sum (Y1 + Y2) and Y ~ Bin(n1 + n2, p)")
plt.legend()

plt.show()
```



Part C

Now suppose $p_1 \neq p_2$. Let

$$Y_3 \sim Bin(n_1 + n_2, \frac{n_1}{n_1 + n_2} p_1 + \frac{n_2}{n_1 + n_2} p_2)$$

Here is an **erroneous** argument for why Y_3 might have the same distribution as $Y_1 + Y_2$ (it doesn't!):

$Y_1 + Y_2$ is the sum of n Bernoulli random variables. Denote these as $X_1, X_2, ..., X_n$ where $n = n_1 + n_2$. Assume they are in order, so that the first n_1 terms are the Bernoulli random variables corresponding to the first binomial distribution (the one with success probability p_1). Note that with this notation we are not specifying whether X_i comes from the first or second Bernoulli sequence. If we randomly select one of these, X_i, then the chance it is equal to 1 is:

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 1 | i \leq n_1)\mathbb{P}(i \leq n_1) + \mathbb{P}(X_i = 1 | i > n_1)(i > n_1)$$

Now observe that:

$$\mathbb{P}(i \leq n_1) = \frac{n_1}{(n_1 + n_2)}, \quad \mathbb{P}(i > n_1) = \frac{n_2}{(n_1 + n_2)}$$
$$\mathbb{P}(X_i = 1 \mid i \leq n_1) = p_1, \quad \mathbb{P}(X_i = 1 \mid i > n_1) = p_2$$

Plugging these into the formula for $\mathbb{P}(X_i = 1)$ gives the probability of success in the definition of Y_3.

Part D

Explain why this argument above is flawed. **Hint:** Is the variable X_i and X_j independent for all $i \neq j$? If not, why is independence important?

Solution

The primary flaw in the argument is its treatment of the combined sequence of Bernoulli variables from Y_1 and Y_2 as a set of independent Bernoulli trials. However, this is misleading as within each binomial distribution, Y_1 and Y_2, the Bernoulli trials are indeed independent. But when combining trials from both Y_1 and Y_2 into one sequence, the trials are no longer independent in the context of the entire sequence. Specifically, a trial X_i originating from Y_1 has a success probability of p_1, while a trial X_j from Y_2 has a different success probability of p_2. If we randomly select a trial from the combined sequence, knowing which binomial distribution it comes from immediately informs us about its success probability, illustrating the dependence.

For a concrete example, let us assume that Y_1 has a very high probability of success, and Y_2 has a very low probability of success. If we draw a random X_i from the combined sequence and observe that it is a success, we can be fairly certain that it originated from Y_1 and not Y_2. Assuming $n_1 \neq 1$ (if $n_1 = 1$ then we probably would have just drawn the only trial from Y_1), this means that probability of success for the next trial X_{i+1} is higher than another random trial X_j from the combined sequence.

In essence, the argument mistakenly assumes that the trials from the two different binomial distributions are interchangeable and independent when considered in a unified sequence, which is not the case.

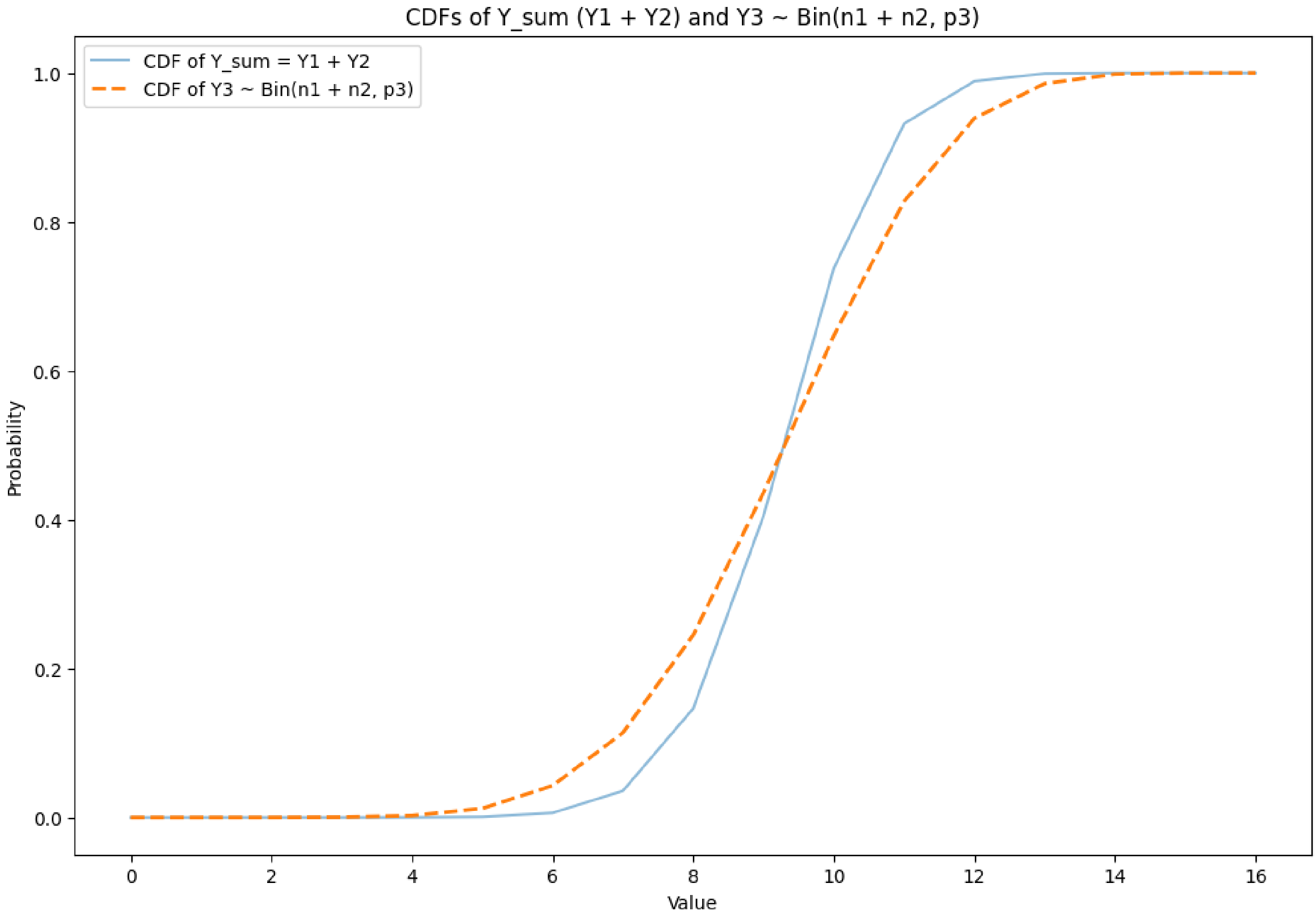
Part E

Confirm that the argument above is incorrect using simulations, that is, confirm via simulations of an example that Y_3 does not have the same probability distribution as $Y1 + Y2$. You can do this many ways, for example, by plotting $\mathbb{P}(Y_3 > k)$ as a function of k and comparing to $\mathbb{P}(Y_1 + Y_2 > k)$.

Solution

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
n1 = 10
n2 = 5
p1 = 0.9
p2 = .15

# Number of simulations
num_simulations = 100000

# Simulating Y1 and Y2
Y1 = np.random.binomial(n1, p1, num_simulations)
Y2 = np.random.binomial(n2, p2, num_simulations)

# Y = Y1 + Y2
Y_sum = Y1 + Y2

# Weighted average probability for Y3
p3 = (n1 * p1 + n2 * p2) / (n1 + n2)

# Simulating Y3
Y3 = np.random.binomial(n1 + n2, p3, num_simulations)

# Calculate CDFs
values = range(n1 + n2 + 2)
cdf_Y_sum = [np.mean(Y_sum <= v) for v in values]
cdf_Y3 = [np.mean(Y3 <= v) for v in values]

# Plotting CDFs
plt.figure(figsize=(12, 8))

plt.plot(values, cdf_Y_sum, label="CDF of Y_sum = Y1 + Y2", alpha=0.5)
plt.plot(values, cdf_Y3, label="CDF of Y3 ~ Bin(n1 + n2, p3)", linestyle='--', linewidth=2)
```

```python
plt.xlabel("Value")
plt.ylabel("Probability")
plt.title("CDFs of Y_sum (Y1 + Y2) and Y3 ~ Bin(n1 + n2, p3)")
plt.legend()

plt.show()
```



Exercise 4.

(Election modeling): Suppose again that we are interested in predicting the outcome of an election with two candidates and N voters. Based on or polling data, people's preferences are equally split between the two candidates ($q = \frac{1}{2}$). However, there is one particular person – person 1 – who is particularly influential. If person 1 votes for candidate one, then everyone else votes for candidate one, while if person 1 votes for candidate two, everyone sticks with their original preference.

The total vote for candidate two can be written as:

$$Y = \sum_{i=1}^{N} y_i \times y_1$$

where

$$y_i \sim Bernoulli(\frac{1}{2}), \quad i = 1, ..., N$$

Each y_i is 0 if they vote for candidate one and 1 if they vote for candidate two. Y_1 represents the vote of the very influential person. The code below simulates this model.

```python
def sample_Y (N, n_samples):
    Y = np.zeros(n_samples)

    for i in range (n_samples):
        y_sample = np.random.choice([0,1], N)
        Y[i] = np.sum(y_sample) * y_sample[0]

    return Y
```

Part A

Let ϕ denote the fraction of votes for candidates one. How do you think the CV (coefficient of variation) of ϕ depends on N as N becomes large? Test your hypothesis with simulations.

Solution

Remember that the coefficient of variation is defined as:

$$CV = \frac{\sigma}{\mu}$$

Where σ is the standard deviation and μ is the mean.

ϕ is the fraction of votes for candidates one. We may define ϕ as:

$$\phi = \frac{N - Y}{N}$$

We know that Y is following a distribution where it had a $\frac{1}{2}$ chance of being 0 and a $\frac{1}{2}$ chance of being a binomial random variable with $N-1$ trials and a probability of success of $\frac{1}{2}$ (because the votes are independent bernoulli events) plus the additional vote from person one.

Let's then find the mean Y, denoted μ_Y.

$$\mu_Y = E[Y] = E[Y \mid y_1 = 0] \cdot \mathbb{P}(y_1 = 0) + E[Y \mid y_1 = 1] \cdot \mathbb{P}(y_1 = 1)$$

Remember that $Y \mid y_1 = 0$ is always 0, so $E[Y \mid y_1 = 0] = 0$.

$$\mu_Y = 0 \cdot \frac{1}{2} + E[Y \mid y_1 = 1] \cdot \frac{1}{2}$$

Remember that $Y \mid y_1 = 1$ is a binomial random variable with $N-1$ trials and a probability of success of $\frac{1}{2}$, plus an additional vote from person one. Therefore

$$E[Y \mid y_1 = 1] = 1 + \frac{N-1}{2} = \frac{N+1}{2}$$

Therefore:

$$\mu_Y = 0 \cdot \frac{1}{2} + \frac{N+1}{2} \cdot \frac{1}{2} = \frac{N+1}{4}$$

We may now find the standard deviation of Y, denoted σ_Y:

$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{E[Y^2] - E[Y]^2}$$

We already know that $E[Y] = \frac{N+1}{4}$, so $E[Y]^2 = \frac{(N+1)^2}{16}$. We may now find $E[Y^2]$:

$$E[Y^2] = E[Y^2 \mid y_1 = 0] \cdot \mathbb{P}(y_1 = 0) + E[Y^2 \mid y_1 = 1] \cdot \mathbb{P}(y_1 = 1)$$

$E[Y^2 \mid y_1 = 0] = 0$ because $Y \mid y_1 = 0$ is always 0.

Remember $Y \mid y_1 = 1$ is a binomial random variable with $N-1$ trials and a probability of success of $\frac{1}{2}$ plus the additional first vote. We already found that such a random variable has a mean of $\frac{N+1}{2}$. Because the additional first vote doesn't add any variation as it is guaranteed, $Y \mid y_1 = 1$ will only have variance from the binomial random variable and thus it has a variance of $\frac{N-1}{4}$.

Therefore:

$$E[Y^2 \mid y_1 = 1] = Var(Y \mid y_1 = 1) + E[Y \mid y_1 = 1]^2 = \frac{N-1}{4} + \frac{(N+1)^2}{4} = \frac{N^2+3N}{4}$$

We may now find $E[Y^2]$:

$$E[Y^2] = E[Y^2 \mid y_1 = 0] \cdot \mathbb{P}(y_1 = 0) + E[Y^2 \mid y_1 = 1] \cdot \mathbb{P}(y_1 = 1) = 0 \cdot \frac{1}{2} + \frac{N^2+3N}{4} \cdot \frac{1}{2} = \frac{N^2+3N}{8}$$

We may now find σ_Y:

$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{E[Y^2] - E[Y]^2} = \sqrt{\frac{N^2+3N}{8} - \frac{(N-1)^2}{16}}$$

$$\sigma_Y = \sqrt{\frac{2N^2+6N-(N^2-2N+1)}{16}} = \sqrt{\frac{N^2+4N-1}{16}} = \frac{\sqrt{N^2+4N-1}}{4}$$

Now let us find CV_ϕ, by first finding μ_ϕ and σ_ϕ:

$$\mu_\phi = E[\phi] = E[\frac{N-Y}{N}] = \frac{N-E[Y]}{N} = \frac{N-\frac{N+1}{4}}{N}$$

$$\mu_\phi = 1 - \frac{N+1}{4N} = \frac{4N-N-1}{4N} = \frac{3N-1}{4N}$$

σ_ϕ is a bit more complicated. Remember that $Var(aX+b) = a^2 Var(X)$.

$$\sigma_\phi = \sqrt{Var(\phi)} = \sqrt{Var(\frac{N-Y}{N})}$$

In this case N is fixed and Y is our random variable, so $a = -\frac{1}{N}$ and $b = \frac{N}{N} = 1$.

$$\sigma_\phi = \sqrt{(-\frac{1}{N^2})^2 Var(Y)} = \frac{1}{N}\sqrt{Var(Y)} = \frac{1}{N}\sigma_Y$$

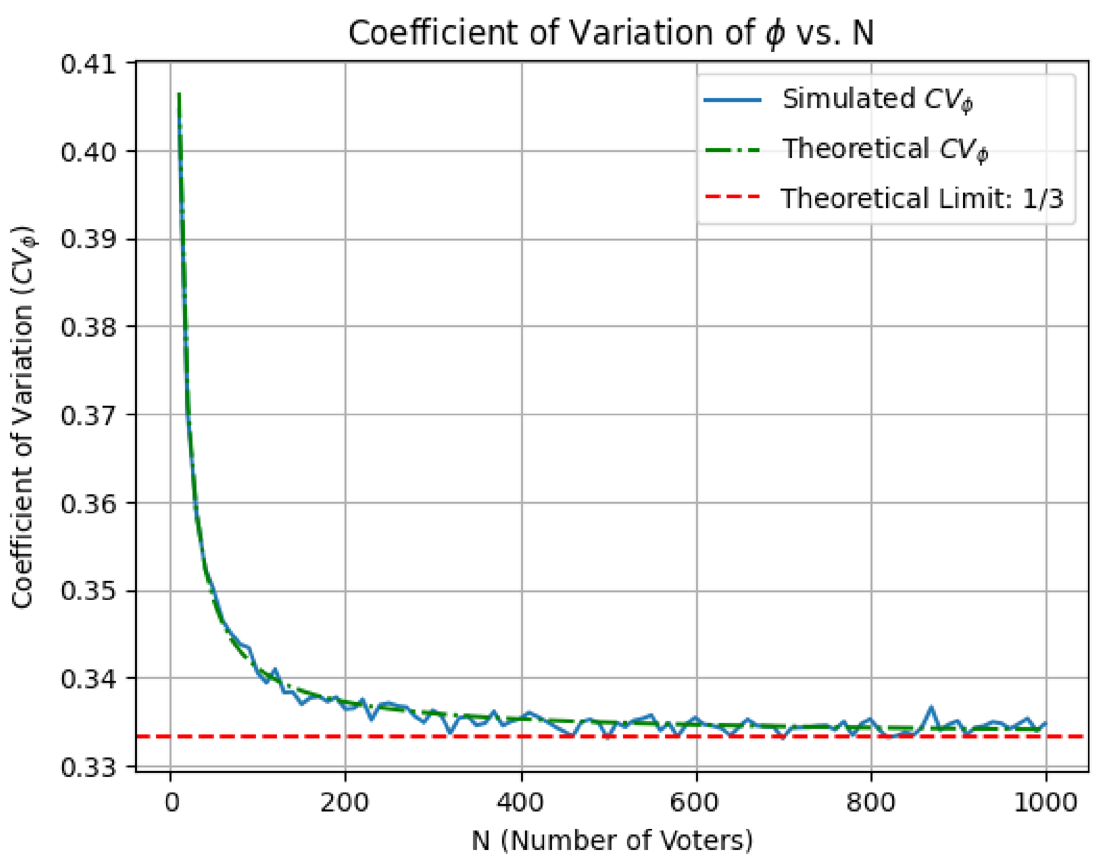
$$\sigma_\phi = \frac{1}{N}\frac{\sqrt{N^2+4N-1}}{4} = \frac{\sqrt{N^2+4N-1}}{4N}$$

We may now find CV_ϕ:

$$CV_\phi = \frac{\sigma_\phi}{\mu_\phi} = \frac{\frac{\sqrt{N^2+4N-1}}{4N}}{\frac{3N-1}{4N}} = \frac{\sqrt{N^2+4N-1}}{3N-1}$$

We see that as N becomes large, CV_ϕ approaches $\frac{1}{3}$. Let's confirm this with simulations.

```python
def coefficient_of_variation_phi(samples, N):
    phi_values = (N - samples) / N
    mean_phi = np.mean(phi_values)
    std_phi = np.std(phi_values)
    return std_phi / mean_phi

def test_hypothesis():
    N_values = np.arange(10, 1001, 10)
    CV_values = []

    n_samples = 30000  # Number of samples for each N

    for N in N_values:
        samples = sample_Y(N, n_samples)
        CV_phi = coefficient_of_variation_phi(samples, N)
        CV_values.append(CV_phi)

    # Theoretical CV_phi values for the N_values
    theoretical_values = np.sqrt(N_values**2 + 4*N_values - 1) / (3*N_values - 1)

    # Plotting both simulated CV, theoretical line, and theoretical values
    plt.plot(N_values, CV_values, label='Simulated $CV_{\phi}$')
    plt.plot(N_values, theoretical_values, label='Theoretical $CV_{\phi}$', linestyle='-.', color='g')
    plt.axhline(y=1/3, color='r', linestyle='--', label='Theoretical Limit: 1/3')
    plt.xlabel('N (Number of Voters)')
    plt.ylabel('Coefficient of Variation ($CV_{\phi}$)')
    plt.title('Coefficient of Variation of $\phi$ vs. N')
    plt.legend()
    plt.grid(True)
    plt.show()

test_hypothesis()
```



Part B

What are $E[\phi]$ and $E[\phi \mid y_1 = 0]$? Confirm your answers with simulations of the model.

Solution

We have already calculated $E[\phi]$ above.

$$E[\phi] = \mu_\phi = \frac{3N - 1}{4N}$$

Now let's find $E[\phi \mid y_1 = 0]$:

$$E[\phi \mid y_1 = 0] = E[\frac{N - Y}{N} \mid y_1 = 0] = \frac{N - E[Y \mid y_1 = 0]}{N}$$

We have already calculated $E[Y \mid y_1 = 0] = 0$, above, so:

$$E[\phi \mid y_1 = 0] = \frac{N - 0}{N} = 1$$

```python
def compute_phi(samples, N):
    return (N - samples) / N

def simulate_expected_phi():
    N_values = np.arange(10, 1001, 10)
    expected_phi_values = []
    conditional_phi_values = []

    n_samples = 100000  # Number of samples for each N

    for N in N_values:
        samples = sample_Y(N, n_samples)

        # E[phi]
        phi_values = compute_phi(samples, N)
        expected_phi = np.mean(phi_values)
        expected_phi_values.append(expected_phi)

        # E[phi | y_1 = 0]
        conditional_samples = samples[samples == 0]
        conditional_phi = compute_phi(conditional_samples, N)
        conditional_phi_mean = np.mean(conditional_phi) if len(conditional_phi) > 0 else 0
        conditional_phi_values.append(conditional_phi_mean)

    plt.figure(figsize=(12, 5))

    plt.subplot(1, 2, 1)
    plt.plot(N_values, expected_phi_values, label="Simulation")
    plt.plot(N_values, [(3*N - 1)/(4*N) for N in N_values], 'r--', label="Theory")
    plt.xlabel('N (Number of Voters)')
    plt.ylabel('$E[\phi]$')
    plt.legend()
    plt.grid(True)

    plt.subplot(1, 2, 2)
    plt.plot(N_values, conditional_phi_values, label="Simulation")
    plt.plot(N_values, [1 for _ in N_values], 'r--', label="Theory")
    plt.xlabel('N (Number of Voters)')
    plt.ylabel('$E[\phi \mid y_1 = 0]$')
    plt.legend()
```

```
    plt.grid(True)

    plt.tight_layout()
    plt.show()

simulate_expected_phi()
```