M50 Homework 3

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Exercise 1.

(Bias and consistency): Let

$$X \sim \text{Bernoulli}(q)$$

and $X_1, ..., X_N$ denote N samples of X. For each of the following estimators of q, (i) write down the standard error and (ii) state whether they are un-biased and/or consistent. In each case, you can write down an exact formula for the standard error, so you do NOT need to use the CLT.

Part A

$$\hat{q}_0 = \frac{1}{N}\sum_{i=1}^{N} X_i$$

Solution

We know that $SE(\hat{q}_0) = \sqrt{Var(\hat{q}_0)}$.

$$Var(\hat{q}_0) = Var(\frac{1}{N}\sum_{i=1}^{N} X_i) = \frac{1}{N^2}Var(\sum_{i=1}^{N} X_i)$$

Because each X_i is independent, we can write:

$$Var(\hat{q}_0) = \frac{1}{N^2}\sum_{i=1}^{N} Var(X_i) = \frac{1}{N^2}\sum_{i=1}^{N} q(1-q) = \frac{1}{N^2} \times N \times q(1-q) = \frac{q(1-q)}{N}$$

$$\Rightarrow SE(\hat{q}_0) = \sqrt{\frac{q(1-q)}{N}}$$

Let's check if $\hat{q}_0$ is unbiased:

$$E(\hat{q}_0) = E(\frac{1}{N}\sum_{i=1}^{N} X_i) = \frac{1}{N}\sum_{i=1}^{N} E(X_i) = \frac{1}{N} \times N \times q = q$$

$$\Rightarrow \hat{q}_0 \text{ is unbiased}$$

Let's check if $\hat{q}_0$ is consistent:

$$\lim_{N\to\infty} Var(\hat{q}_0) = \lim_{N\to\infty} \frac{q(1-q)}{N} = 0$$

$$\text{Variance approaches zero around } E[\hat{q}_0] = q \Rightarrow \hat{q}_0 \text{ is consistent}$$

Part B

$$\hat{q}_1 = \frac{X}{N} + \frac{1}{\sqrt{N}}$$

Solution

We know that $SE(\hat{q}_1) = \sqrt{Var(\hat{q}_1)}$.

$$Var(\hat{q}_0) = Var(\frac{X}{N} + \frac{1}{\sqrt{N}})$$

$\frac{1}{\sqrt{N}}$ is constant, so $Var(\frac{X}{N} + \frac{1}{\sqrt{N}}) = Var(\frac{X}{N})$.

$$\Rightarrow Var(\hat{q}_1) = Var(\frac{X}{N}) = \frac{1}{N^2}Var(X) = \frac{1}{N^2} \times q(1-q) = \frac{q(1-q)}{N^2}$$

$$\Rightarrow SE(\hat{q}_1) = \sqrt{\frac{q(1-q)}{N^2}} = \frac{\sqrt{q(1-q)}}{N}$$

Let's check if $\hat{q}_1$ is unbiased:

$$E(\hat{q}_1) = E(\frac{X}{N} + \frac{1}{\sqrt{N}}) = \frac{1}{N}E(X) + \frac{1}{\sqrt{N}} = \frac{1}{N}q + \frac{1}{\sqrt{N}} \neq q$$

$$\Rightarrow \hat{q}_1 \text{ is biased}$$

Part C

$$\hat{q}_2 = \frac{X}{\lfloor N/2 \rfloor} + \sum_{i=1}^{\lfloor N/2 \rfloor} X_i$$

Note: $\lfloor x \rfloor$ denotes the floor function, which rounds x down to the nearest integer.

Solution

Exercise 2.

(Estimator of mean in exponential model): Let

$$T \sim exp(\lambda)$$

Recall that $E[T] = \frac{1}{\lambda}$. We can estimate $E[T]$ via the sample average of measurements $T_1, ..., T_n$:

The suggests that a natural way to estimate λ is by:

$$\hat{\lambda} = \frac{1}{\bar{T}} = \frac{n}{\sum_{i=1}^{n} T_i}$$

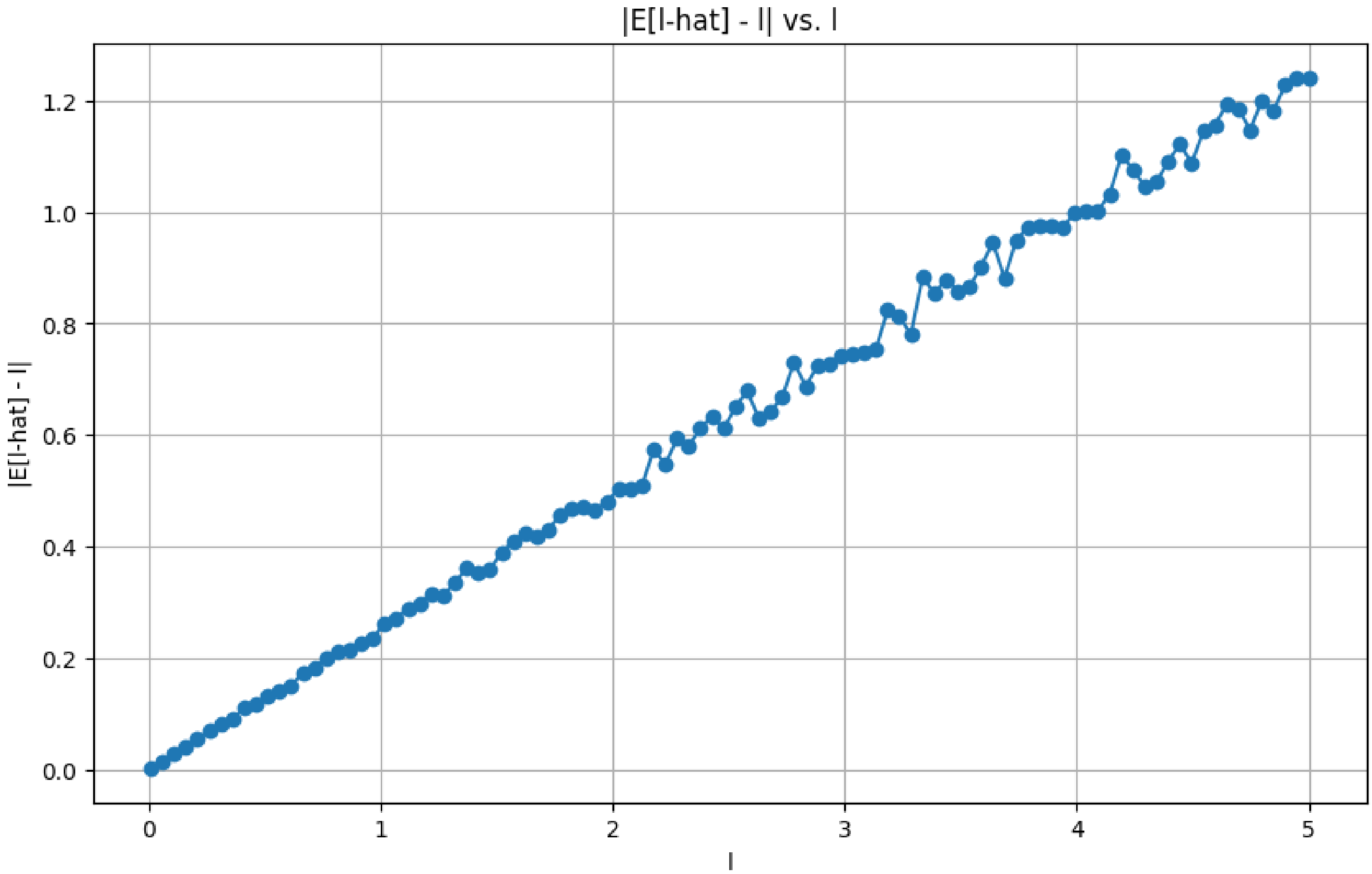
Part A

The goal of the first part of this problem is to show, using simulations, that this is in-fact a biased estimator of λ, although the bias decreases with n. To achieve this, you should do the following:

- Make a list of 100 values of λ in any range.
- For each value of λ:
  - Simulate 10000 replicates of an experiment, where each replicate includes $n = 5$ values of T.
  - For each of these replicates, compute $\hat{\lambda}$ as defined above.
  - Then estimate the average $E[\hat{\lambda}]$ and save this value is a list.
- Make a plot of λ vs. $|E[\hat{\lambda}] - \lambda|$.

Solution

```
import numpy as np
import matplotlib.pyplot as plt

# Number of lambda values
num_lambdas = 100

# Number of replicates per lambda
num_replicates = 10000

# Number of T values per replicate
n = 5

# Create a list of 100 lambda values
lambdas = np.linspace(0.01, 5, num_lambdas)

# Initialize a list to store the average estimated lambdas
average_estimated_lambdas = []

for l in lambdas:
    # Simulate 10000 replicates of n values of T
    T_values = np.random.exponential(scale=1/l, size=(num_replicates, n))

    # Compute lambda hat for each replicate
    lambda_hats = n / np.sum(T_values, axis=1)

    # Compute the average of lambda hat and store in the list
    average_estimated_lambdas.append(np.mean(lambda_hats))

# Calculate absolute differences
differences = np.abs(np.array(average_estimated_lambdas) - lambdas)

# Plotting
plt.figure(figsize=(10,6))
plt.plot(lambdas, differences, '-o')
plt.title('|E[l-hat] - l| vs. l')
plt.xlabel('l')
plt.ylabel('|E[l-hat] - l|')
plt.grid(True)
plt.show()
```


|E[l-hat] - l| vs. l
|E[l-hat] - l|
l
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
1
2
3
4
5