MATH 86, WINTER 24, PROBLEM SET 2

This problem set is due on Thursday February 08 at the start of lecture.

(1) Wilmott, Chapter 3, Problem 8
(2) Wilmott, Chapter 7, Problems 1, 8, 9
(3) Suppose W_t and Z_t are two independent Brownian motions. This means that each satisfies the 4 defining properties of Brownian motion, and the increments $Z_t - Z_s$ and $W_t - W_s$ are independent random variables for all $0 \le s < t$.

For a constant ρ with $-1 \le \rho \le 1$, consider the random process

$$X_t = \rho W_t + \sqrt{1-\rho^2} Z_t$$

Is X_t a Brownian motion? Explain your answer.

(4) If W_t is the Wiener process, show that the limit

$$\lim_{\Delta t \to 0} \frac{(\Delta W)^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{(W_{t+\Delta t} - W_t)^2}{\Delta t}$$

does not exist.

(5) Use Ito's Lemma to derive a formula for $\int_0^T W^3 dW$.

(6) For a stochastic price process $S_t : [0,T] \to \mathbb{R}$, define the approximate quadratic variation

$$Q_m = \sum_{j=1}^{m} (S_{t_j} - S_{t_{j-1}})^2 \qquad t_j = \frac{jT}{m} \quad j = 0, 1, 2, \ldots, m$$

partitioning $[0,T]$ into intervals of equal size $\Delta t = \frac{T}{m}$.

(a) Assume S_t is an Ito process with $dS = \alpha S dt + \sigma S dW$. Derive formulas for $\mathbb{E}(Q_m)$ and $\mathrm{Var}(Q_m)$.
(b) Explain why the distribution of Q_m is approximately normal.
(c) Use your code from *Problem Set 1, Problem (4)* that generates a random stock price path for one year (with $\alpha = 10\%, \sigma = 30\%$). Modify your code to use $N = 2000$ time steps, instead of $N = 252$.
Then add a function that calculates Q_{50} and Q_{250} for the price data generated by the previous function (effectively using $\Delta t = 5$ day and $\Delta t = 1$ days respectively, assuming there are 250 trading days in the year).
Generate 1000 random stock price paths, and calculate Q_{50} and Q_{250} for each path. Plot the two histograms for the distributions of the values of Q_{50} and Q_{250} for the 1000 sample paths.
(d) What are the average and standard deviation for the empirical distributions of Q_{50} and Q_{250} respectively? Do the empirical results confirm your calculation in item (a)?

(e) In practice, is Q_{50} or Q_{250} a better estimator of quadratic variation? Why?

(7) (a) Modify your code from *Problem Set 1, Problem (5)*, so that in one loop through the binomial tree it calculates the price of a European put option as well as of an American put option.

(b) In a single graph, plot the price of a European and American put option with $S_0 = \$100, T = 1, \sigma = 45\%, r = 5\%$ as a function of strike K (take strikes at regular intervals \$5 apart).