MATH 86, WINTER 24, PROBLEM SET 3

This pset is due on Tuesday February 27 at the end of the day.

(1) Wilmot, Chapter 8, problems 1, 2, 9, 14

(2) The Breeden-Litzenberger formula for the pdf $\varphi(x)$ of the future price S_T under the risk-neutral measure $\mathbb{Q}$ is

$$\varphi(x) = \frac{\partial^2 C}{\partial K^2}$$

Where $C = C(K,T)$ is the market price of a call option as a function of strike K and expiration T.

(a) For a stock of your choice, choose the nearest available expiration date that is at least 4 weeks in the future, and download the call prices for all strikes at that expiration.
Use the package *scipy.interpolate.CubicSpline* to interpolate the list of call prices using a "natural" cubic spline (use *bc_type*). Use the mid prices (average of bid and ask). Plot the prices and the interpolating spline in a single graph.

(b) Use *scipy.interpolate.CubicSpline.derivative* to create a plot of the second derivative $\frac{\partial^2 C}{\partial K^2}$ of the cubic spline. This graph is a representation of the risk-neutral density for the stock price S_T that is implicit in the market prices of the call options.

(c) Create a second plot with a logarithmic x-axis. Does the distribution seem log-normal, or is it skewed (and if so, in which direction)?

(3) (a) Implement a function that calculates implied volatility using Newton's method (for example, you can follow Wilmot's sample code). Plot the implied volatilities of the call options you used in the previous problem.

As interest rate, use the rate of a US Treasury bond with expiration close to the expiration of your call options (e.g. 1 month).

Note. Newton's method may or may not converge, depending on your initial guess for the volatility. You may need to experiment to get your code to converge for all prices in your range.

(b) Plot the volatility smile for your call options.

(c) Do you see a connection between features of the risk-neutral price distribution you found in the previous problem and the shape of the smile?

HOW TO CALCULATE GAMMA EXPOSURE

The goal of this exercise is to:

- Create a simple spot gamma chart for your favorite stock index.
- Compute its total gamma exposure.
- Construct a gamma profile chart
- Calculate a Zero Gamma level (aka “Flip Gamma”).

OPTIONS DATA

CBOE provides an excellent options data tool, which includes gamma, among other things.

cboe.com/delayed_quotes

Search for "SPX" -> click on Options tab -> set Options Range == All -> set Expiration == All -> click View Chain.

To download the entire options chain, scroll all the way down, and click on Download CSV.

We now have an entire snapshot of the SPX options data in Excel.

We want to find out two things:

- How much gamma the dealers are sitting on across the index levels, and
- What's their current total gamma exposure.

For these calculations, ideally, we need to know how many of each particular option the dealers hold.

Unfortunately, they're a secretive bunch and keep their cards close to their chest. Hence, we need to make an assumption about that.

A crude approximation is that the dealers are long the calls and short the puts, which is true to some expend on an index level.

For single-stocks, it's more debatable, as the street can be short the calls due to some speculative buying frenzy. (Examples: TSLA or GME)

It is possible to build up a more accurate dealer positioning picture from trade reporting data. **We will assume that calls carry positive gamma and puts carry negative gamma.**

GAMMA CONTRIBUTION FORMULA

Let's now calculate the total gamma contribution from each option.

The formula is:

Option's Gamma * Contract Size * Open Interest * Spot Price * (-1 if puts)

This gives the total option's gamma per ONE POINT move in the index.

To convert into percent, we must multiply by how many points 1% is.

Hence, we multiply by 1% * Spot Price, giving the final formula:

Option's Gamma * Contract Size * Open Interest * Spot Price ^ 2 * 0.01

Summing gamma contributions across the options gives us the total gamma exposure.

Doing this in Excel for 1 Feb 2022 gives me -$19Bn.

In plain English:

Option dealers need to SELL $19Bn worth of $SPX index for each 1% move DOWN, and BUY $19Bn for each 1% move UP.

Frequently the gamma exposure is an overstatement, and the actual delta-hedging flows are typically smaller than this calculation would imply.

This is because investors:
• Also sell puts for yield (structured products)
• Buy calls for leverage
• Trade spreads and combos

Let's now look at how gamma exposure is distributed across strikes and expiries.

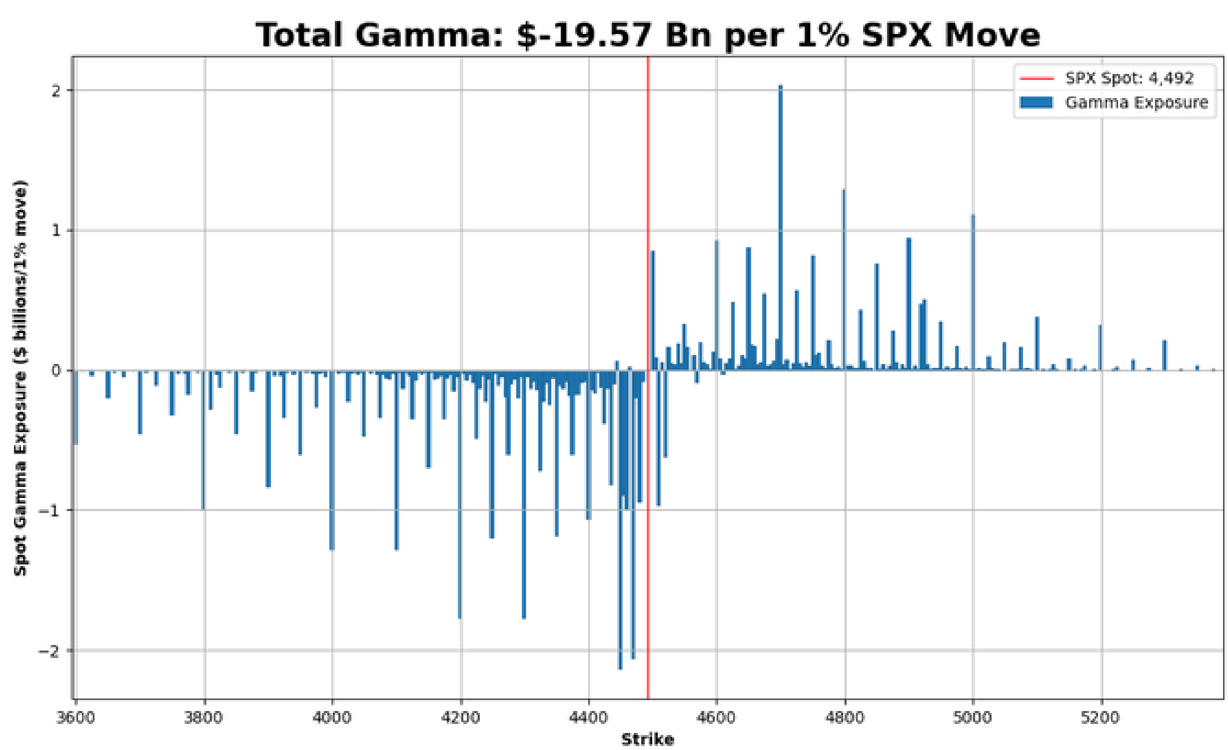
In Excel, this is done by selecting the area with the headings and the data and click on Insert -> Pivot Table.

Pivot Table allows us to easily aggregate the gamma values for a range of strikes and expiries.

Under PivotTable Fields, drag-and-drop Strike into the Rows and drag Total Gamma into Values.

This will give you a total gamma exposure per strike!

If you now click on the Pivot Table Analyze tab -> Pivot Chart, you'll get an excellent visualization.



This chart shows how much gamma is outstanding at each strike from the point of view of the current spot.

This is why it's called spot gamma exposure.

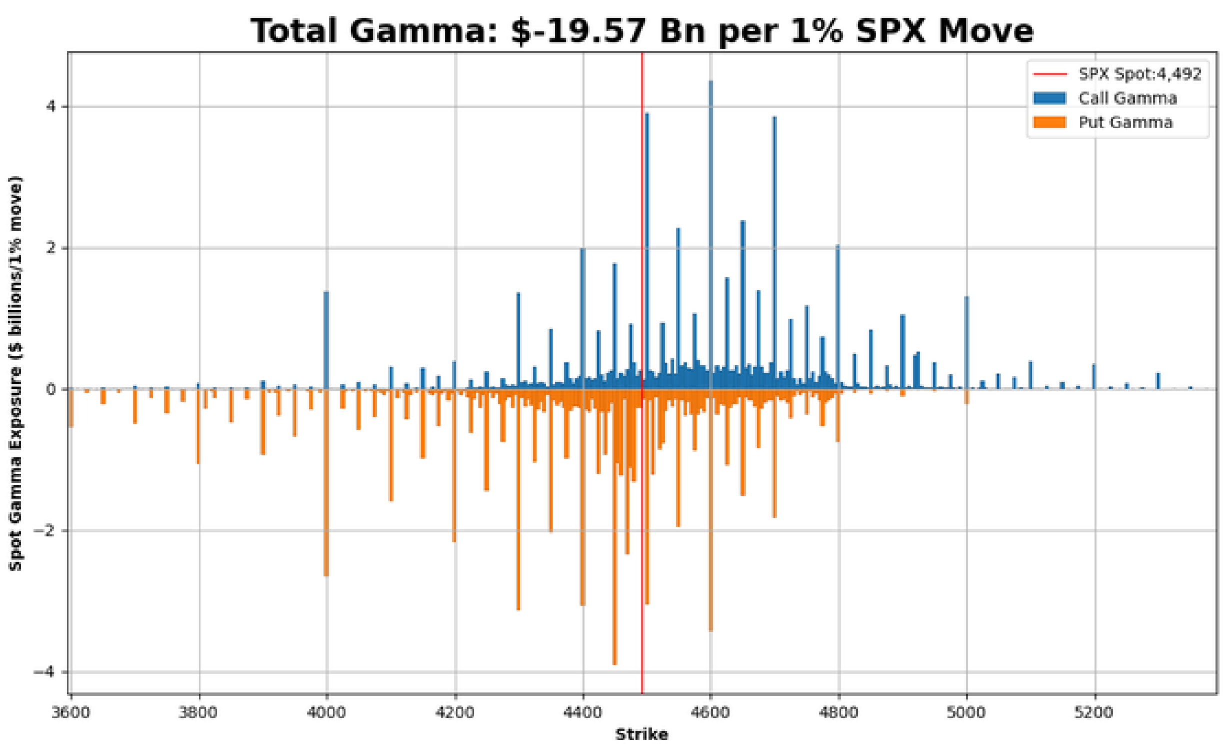
Each bar represents how much delta needs to be bought or sold if the market moves by 1% from where it is now.

Want to see the largest gamma expiries?

To do so, remove the Strikes from Rows and add Expiration Date instead.

Want to see gamma for calls and puts separately?

Replace Total Gamma with Put Gamma and Call Gamma and put the Strike back in place of Expiration Date.



So far, we calculated the gamma exposure for the current spot level only.

But what if we wanted to know an approximate gamma exposure if the market drops by 10%?

Or how about if it rallies by 10%?

Would the dealers be short or long gamma, and at what point would it flip?

This is a slightly more complex question since we must compute the gamma exposure not just for the current spot but across the index levels.

As in, we need to do that Excel calculation for different spot values.

Well, that's great news - we already know how to do that!

The challenge is that, when we change the spot, options gammas will change.

This means we can no longer use the same gammas so thoughtfully provided by CBOE.

Those have to be recalculated.

For every option.

For every spot price

And while it's possible to do it in Excel, it's a bit of an overkill...

It's time to dig into your toolbox and pull out some Python.

So how do we go about that?

Example: Consider a 4,000 put option, expiring on 18 Feb.

Using a Black-Scholes formula, we can calculate its gamma since we have most of the ingredients we need.

The CBOE data provides strikes, expiries and, most importantly, IV levels for each option. We will use those.

We also need to source the interest rate data (ideally swap rates) and a projected dividend yield.

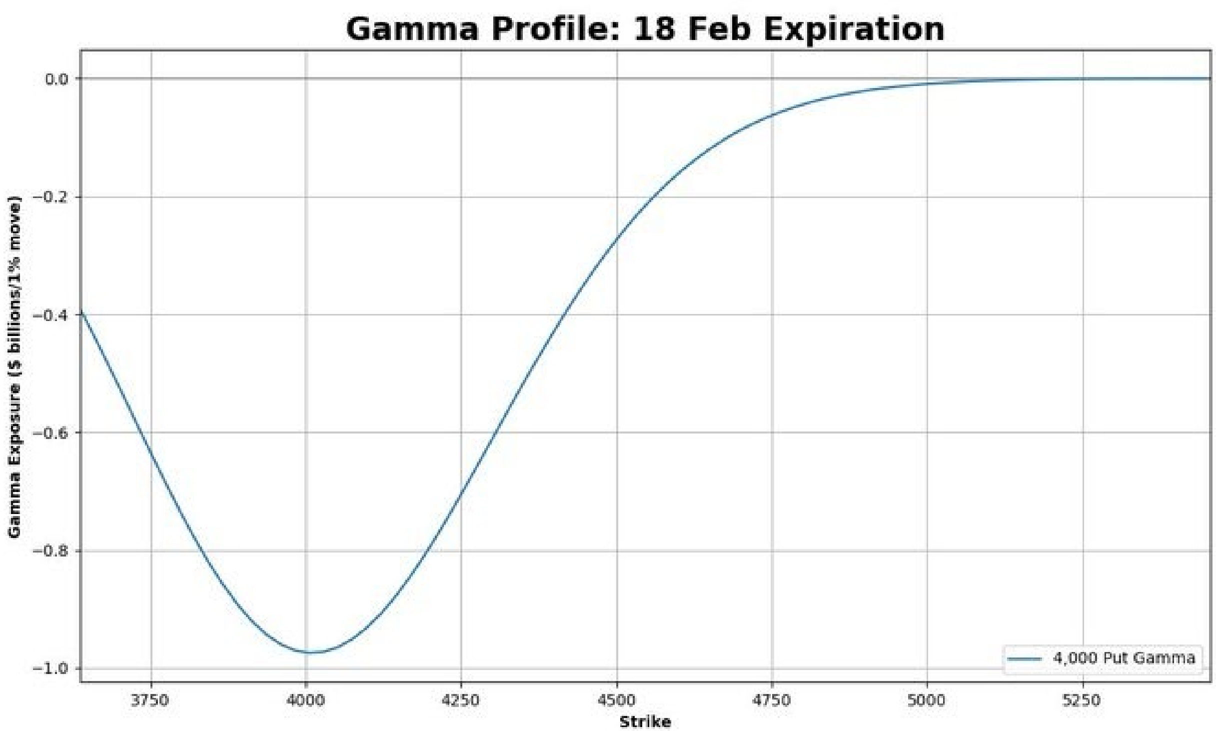
In this example, we'll set rates and dividends to zero (or to 1.5-2% to be closer to the current levels for $SPX).

Ultimately, these choices don't really impact the final figures that much.

Now, back to our 4,000 put. For this option, we do the following:

• Download its data into Python.
• Apply the Black-Scholes and calculate unit gamma for a range of spot levels.
• Using unit gamma, calculate the total gamma exposure at each spot price.

This will give us the following chart:

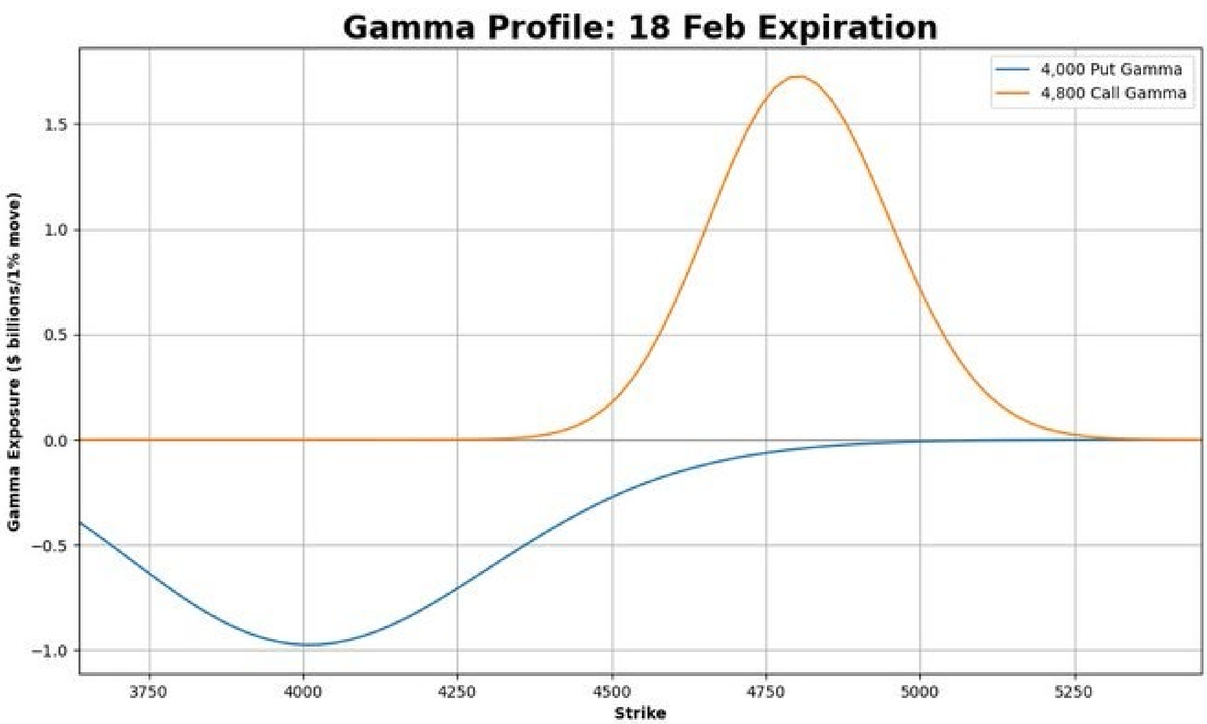


It shows the gamma exposure at each spot level resulting from this single option.

As expected, it's lowest at the strike, and its gamma impact diminishes as we move closer to the current spot.

Ok, how about we add a few more in the mix?

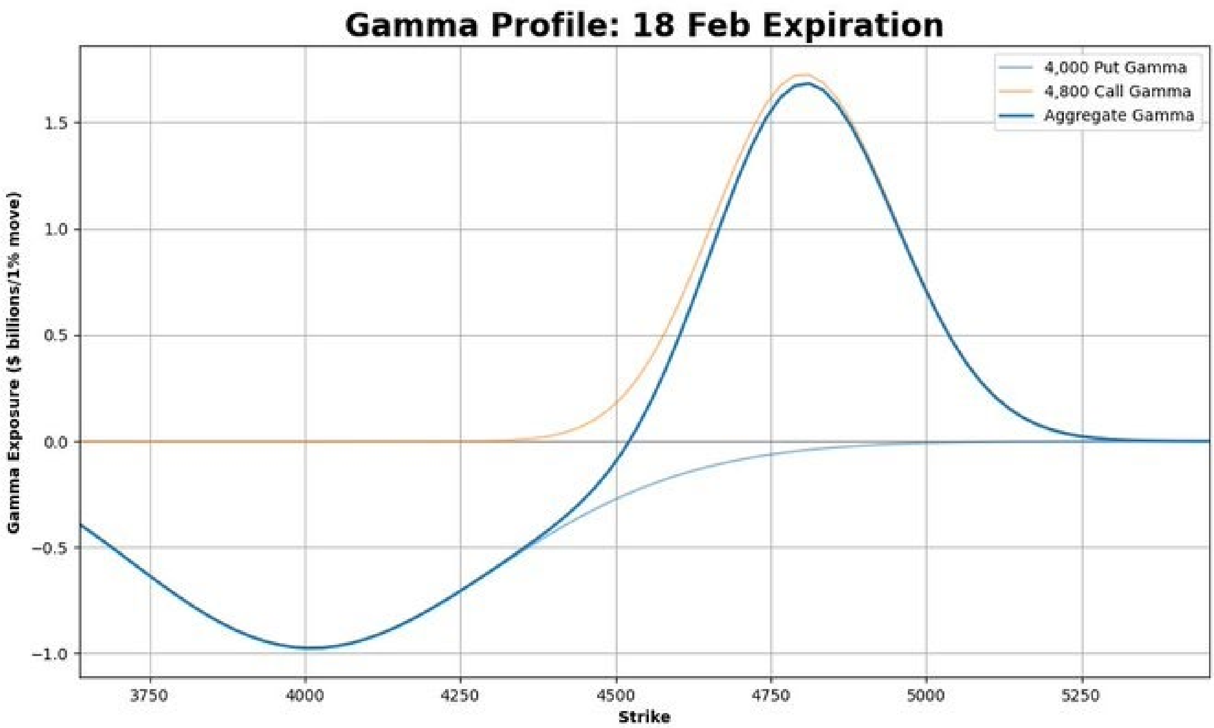
Consider the 4,800 call:



Now we have the gamma exposure for two options.

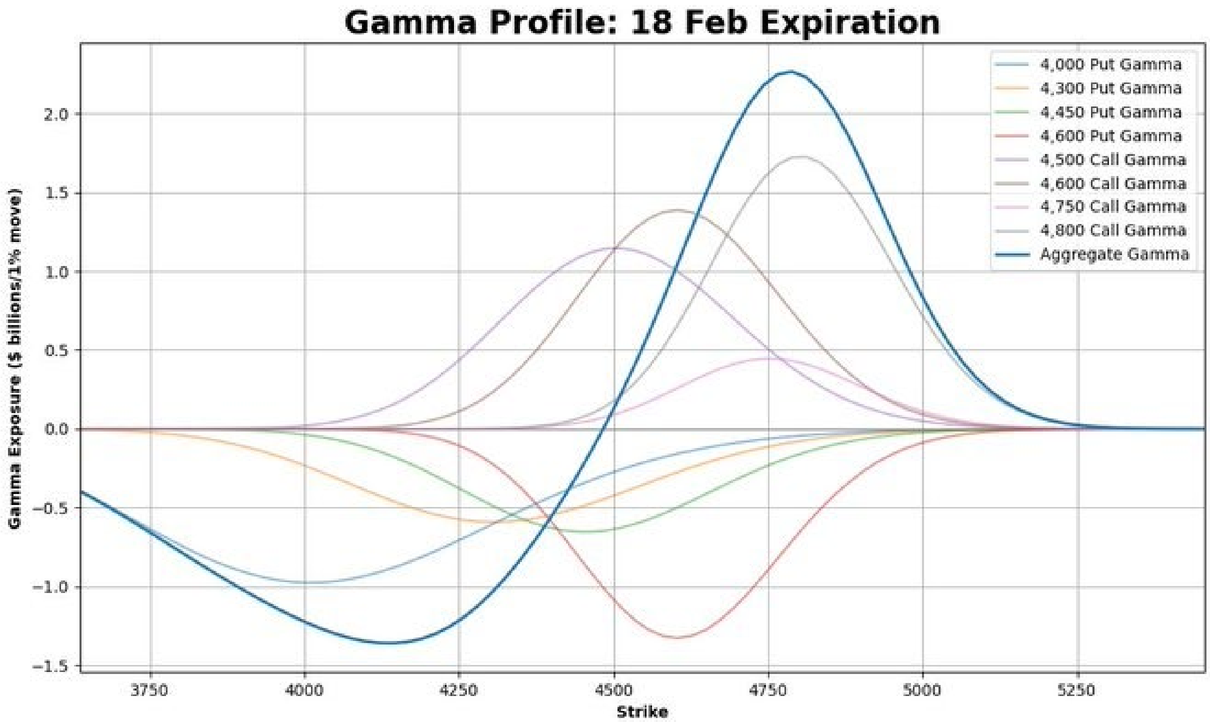
As you can see, they have their own sphere of influence, sometimes offsetting each other.

If we add them, we'll net out their impacts and get the overall gamma exposure

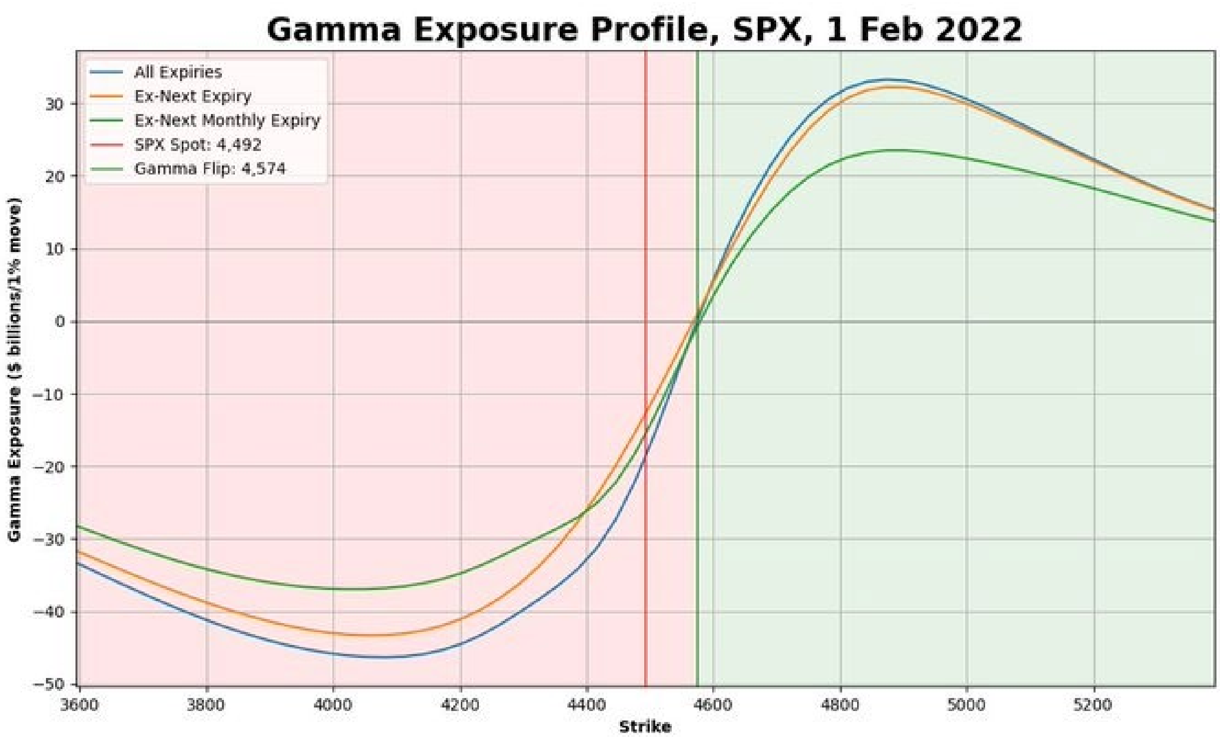


Let's now throw in 4,300 put and 4,600 call

As we keep adding more options, we get a more complete picture of the gamma exposure for this expiration.



To get the overall gamma exposure profile, we'll need to do this exercise for all options across all expiries. And if we do that, we'll get the following chart!



You might have seen this kind of chart before.

It shows an approximate amount of dealer hedging flows across index levels, assuming the current state of the world.

Short gamma to the left. Long gamma to the right.

Two levels are notable on this chart.

The first one is where the Total Gamma Exposure blue line crosses the current spot red line.

This point represents the current spot gamma exposure and occurs at around -$19Bn.

Yes, the same -$19Bn that we calculated earlier in Excel!

The second level is where the Total Gamma Exposure blue line crosses zero.

On the chart, it occurs at around 4,575 SPX level.

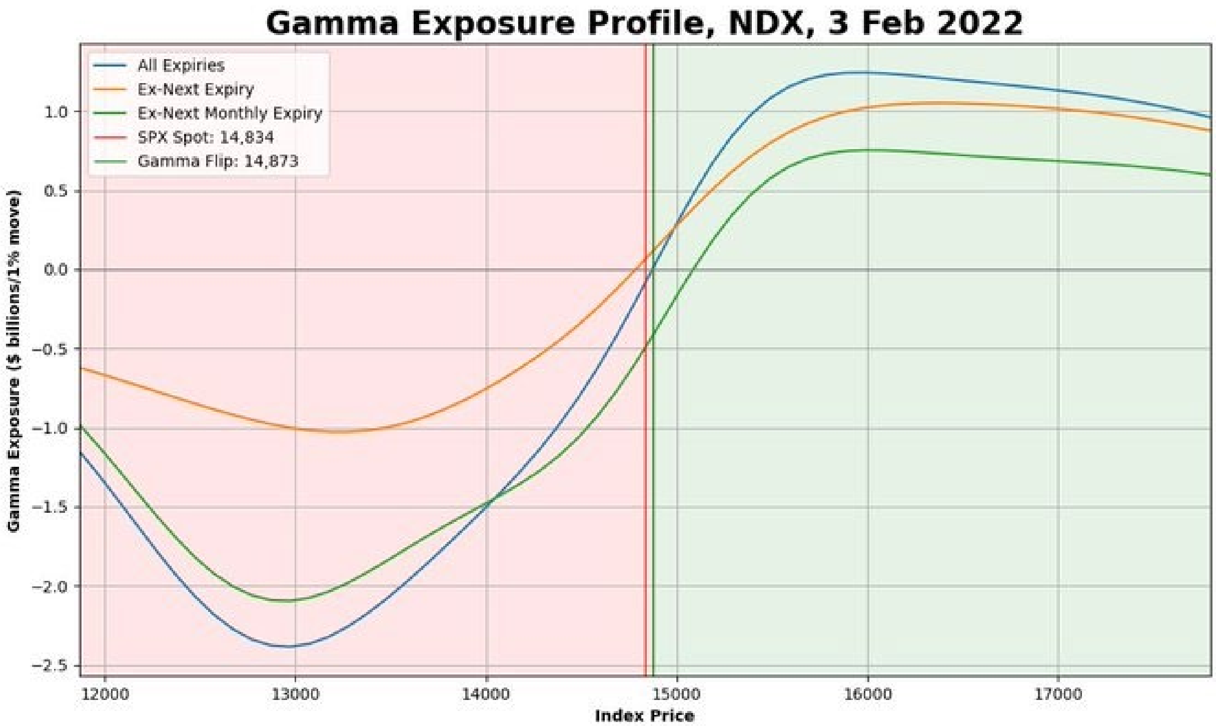
This is the Zero Gamma point, where the dealer gamma exposure flips from positive to negative.

Above this level, dealer hedging flows are stabilizing and add liquidity to the market.

Below this level, dealer hedging flows are destabilizing and add to the volatility instead.

At this level, flows are zero, as delta doesn't need rebalancing due to spot moves.

Here is another example for the Nasdaq index ($NDX).



And here is an example for the Russell 2000 Index.