MATH 86, WINTER 24, PROBLEM SET 1

This problem set is due on Thursday, January 25 at the start of lecture.

(1) From Wilmott, Chapter 1, problems 1, 2, 3, 5, 6
(2) From Wilmott, Chapter 2, problems 1bcd, 2, 3, 4
(3) In lecture, we showed how to calibrate a binomial tree using parameters

$$u = e^{\sigma\sqrt{\Delta t}} \qquad d = e^{-\sigma\sqrt{\Delta t}} \qquad p = \frac{1}{2} + \frac{\mu\sqrt{\Delta t}}{2\sigma}$$

where σ is volatility, and μ is the drift of $\log S(t)$ (cf. Van Erp, lecture 1). Wilmott uses instead:

$$u = 1 + \sigma\sqrt{\Delta t} \qquad d = 1 - \sigma\sqrt{\Delta t} \qquad p = \frac{1}{2} + \frac{\alpha\sqrt{\Delta t}}{2\sigma}$$

where α is the mean rate-of-return, i.e. $\mathbb{E}(S(T)) = S_0 e^{\alpha T}$.

(a) Show that, with Wilmott's parameters, we get

$$\mathbb{E}(S_1) \approx S_0 e^{\alpha\Delta t}$$

(b) Show that if we do not assume $ud = 1$, then we must require

$$\mu\Delta t = p\log u + (1-p)\log d \qquad \sigma\sqrt{\Delta t} = (\log u - \log d)\sqrt{p - p^2}$$

(c) Show that, with Wilmott's parameters, we have approximately

$$\mu\Delta t \approx p\log u + (1-p)\log d \qquad \sigma\sqrt{\Delta t} \approx (\log u - \log d)\sqrt{p - p^2}$$

Hint. Use a Taylor expansion to approximate $\log u$ and $\log d$.

(4) Suppose a stock has annualized mean rate-of-return 10% and volatility 30%.
In Python, implement the binomial tree model to simulate one year of stock price movements. Use $N = 252$ time-steps, i.e. one time-step is $\Delta t = 1$ day. Run a simulation, and create a plot of the daily prices $S_0, S_1, S_2, \dots, S_N$ showing three random paths.

(5) (a) Write a Python function that calculates the price of a call option using the binomial tree model. The function should take as input parameters:
- spot price S_0
- strike price K
- time to expiration T (in years)
- number of time steps N
- volatility σ (annualized)
- risk free rate r (annualized)

(b) Use your function to calculate the price of an at-the-money call option with $S_0 = 100, K = 100, T = 1, \sigma = 30\%, r = 5\%$. Do this for all values of N in the range $50, 100, 150, ..., 2500$ (with increments of 50). Plot the resulting price as a function of the number of time steps N.

(c) Find a Black-Scholes calculator online. What is the price of the call option of item (b) according to the Black-Scholes calculator? Does your code confirm that the price calculated by the binomial tree model converges to the Black-Scholes price as $N \to \infty$?

(d) For which value of N is the price calculated by your code within \$0.001 of the Black-Scholes price?

(e) Plot the price of an at-the-money call option with $S_0 = 100, T = 1, \sigma = 30\%, r = 5\%$ as a function of strike K. The x-axis shows K. The y-axis shows the price of the option. (Use your code to find all the option prices for $K = 20, 25, 30, \ldots, 200$.)
Combine in one graph a plot for the option price if volatility is 30%, and a plot if volatility is 60%.

(f) Based on your graph, what is the effect of increased volatility on the option price?