M86 Homework 1

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Exercise 1

Wilmott Ch. 1 Questions

Question 1.1 A company makes a three-for-one stock split. What effect does this have on the share price?

Answer The share price will decrease by a factor of three. So if the share price was \$90 before the split, it will be \$30 after the split. The total value of the shares will remain the same.

Question 1.2 A company whose stock price is currently S pays out a dividend DS, where $0 \leq D \leq 1$. What is the price of the stock just after the dividend date?

Answer If the stock price is \$100 and the dividend is 0.01, the price of the stock after the dividend date will be $S - DS = \$100 - \$100 \times 0.01 = \$99$.

Question 1.3 The dollar sterling exchange rate (colloquially known as 'cable') is 1.83, £1 = \$1.83. The sterling euro exchange rate is 1.41, £1 = €1.41. The dollar euro exchange rate is 0.77, \$1 = €0.77. Is there an arbitrage, and if so, how does it work?

Answer Let us test if there is an arbitrage opportunity. Let's start with \$1.

$\$1 \rightarrow £\frac{1}{1.83} \rightarrow €\frac{1.41}{1.83} \rightarrow \$\frac{1.41}{1.83 \times 0.77} = \$\frac{1.41}{1.4091} = \$1.0006387$

So we have made a profit of \$0.0006387. Very small, but still an arbitrage opportunity. The exchange fees would probably eat up the profit, but if we had a large amount of money, we might make a profit.

Another way to represent the conversion is:

$\$1 \times \frac{£1}{\$1.83} \times \frac{€1.41}{£1} \times \frac{\$1}{€0.77} = \$1.0006387$

Question 1.5 A spot exchange rate is currently 2.350. The one-month forward is 2.362. What is the one-month interest rate assuming there is no arbitrage?

Answer The one-month interest rate is $\frac{2.362-2.350}{2.350} = 0.005106383$ or 0.5106383%.

Another way to calculate is the following:

$$2.362 = 2.350 \times e^r \Rightarrow r = \ln(\frac{2.362}{2.350}) = 0.0050933896191$$

This gives us approximately the same answer. The difference is due to one being a continuous rate and the other being a discrete rate.

Question 1.6 A particular forward contract costs nothing to enter into at time t and obliges the holder to buy the asset for an amount F at expiry T. The asset pays a dividend DS at time t_d, where $0 \leq D \leq 1$ and $t \leq t_d \leq T$. Use an arbitrage argument to find the forward price $F(t)$.

Hint: Consider the point of view of the writer of the contract when the dividend is reinvested immediately in the asset.

Answer The value of the dividend including interest rate can be represented as $DS(t_d) \times e^{r(T-t_d)}$. For no-arbitrage, the party entering the forward must be compensated for this opportunity cost, so the forward price will be the interest-adjusted price of the underlying stock minus the interest-adjusted price of the dividend, or $F(t) = S(t) \times e^{r(T-t)} - DS(t_d) \times e^{r(T-t_d)}$.

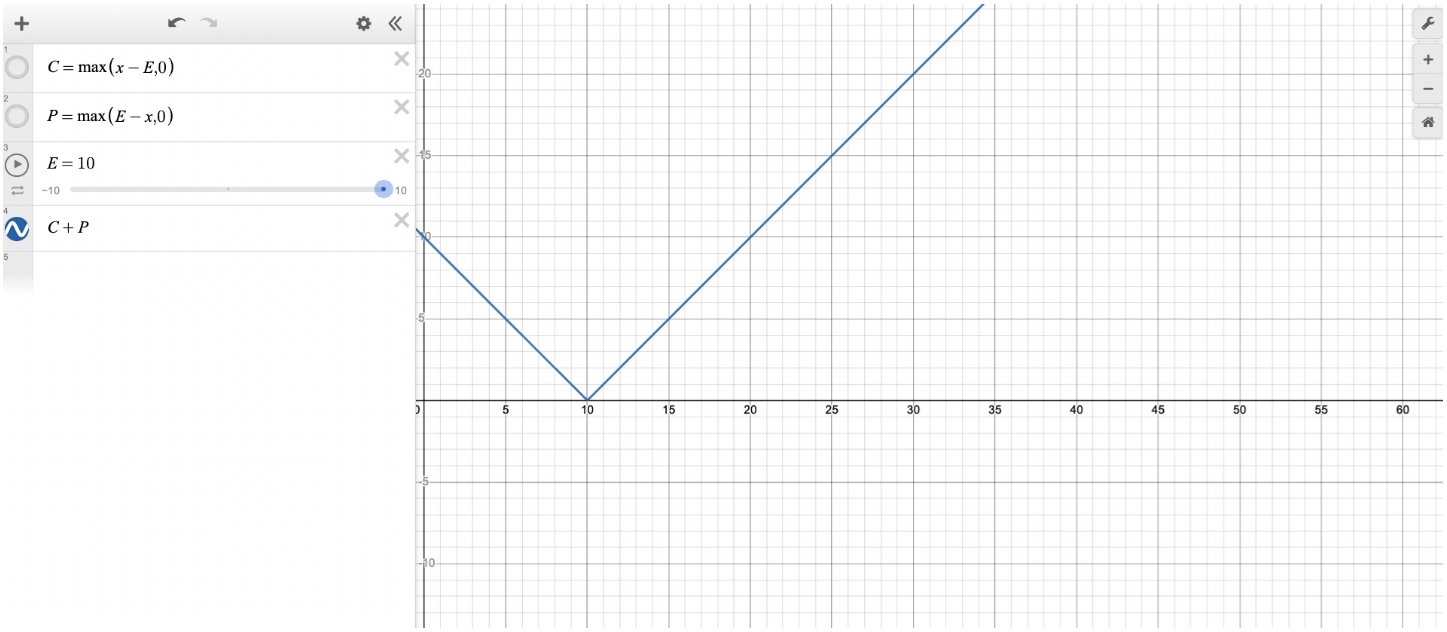
Exercise 2

Wilmott Ch. 2 Questions

Question 2.1 Find the value of the following portfolios of options at expiry, as a function of the share price:

2.1b Long one call and one put, both with exercise price E.

Answer This is a straddle. The value of the call is $C = \max(S - E, 0)$ and the value of the put is $P = \max(E - S, 0)$. The value of the portfolio is $C + P = \max(S - E, 0) + \max(E - S, 0) = \max(S - E, E - S, 0) = |S - E|$.



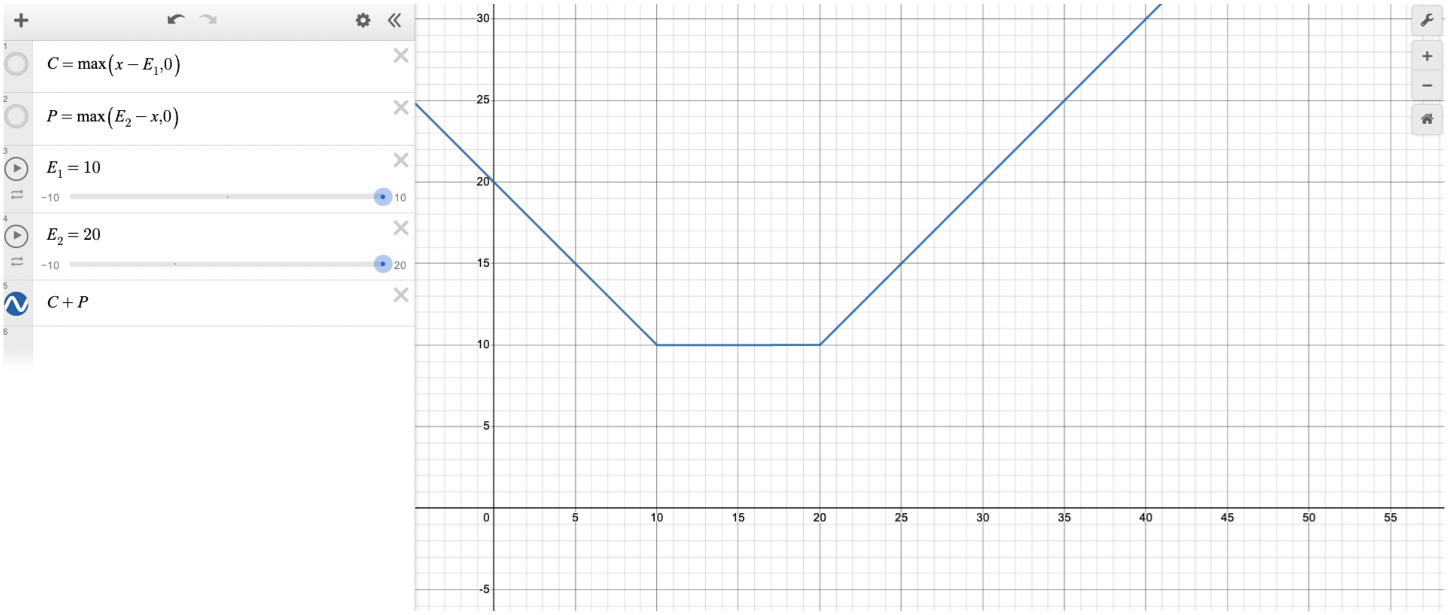
2.1c Long one call, exercise price E_1, short one call, exercise price E_2, where $E_1 < E_2$.

Answer This is a bull spread. The value of the portfolio is $C_1 - C_2 = \max(S - E_1, 0) - \max(S - E_2, 0)$.

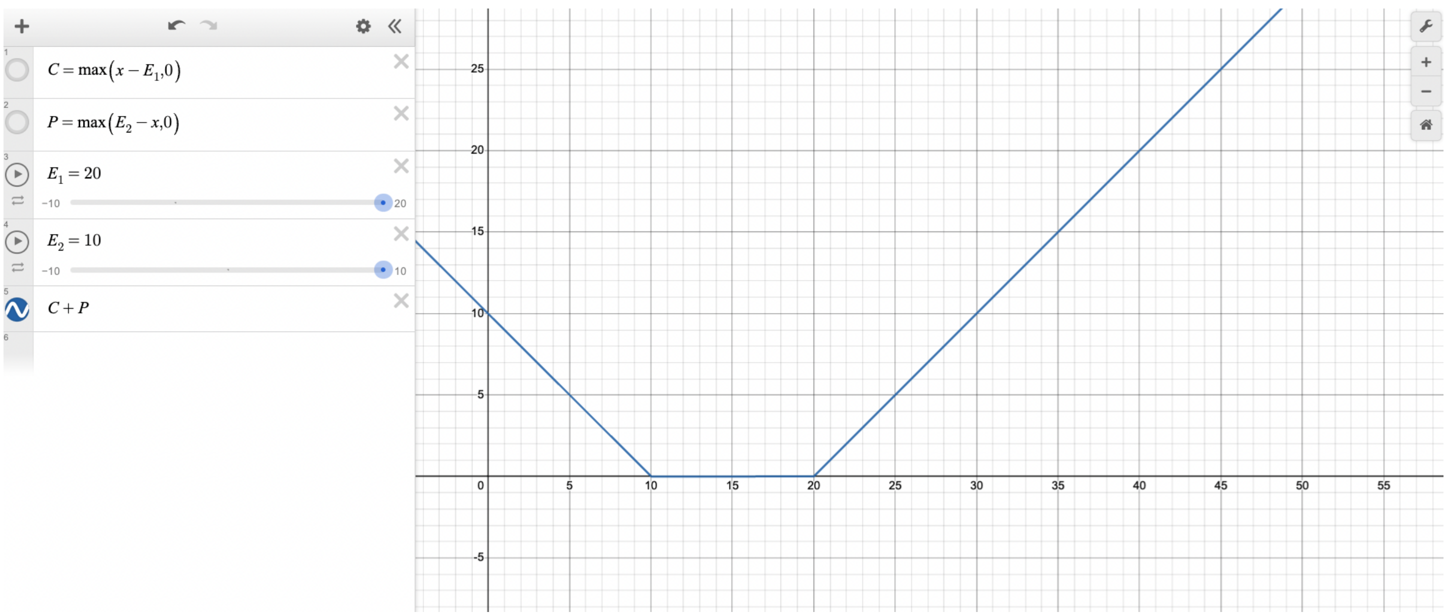
2.1d Long one call at exercise price E_1, long one put at exercise price E_2 . There are three cases to consider.

Answer In all three cases, the value of the portfolio is $C + P = \max(S - E_1, 0) + \max(E_2 - S, 0)$.

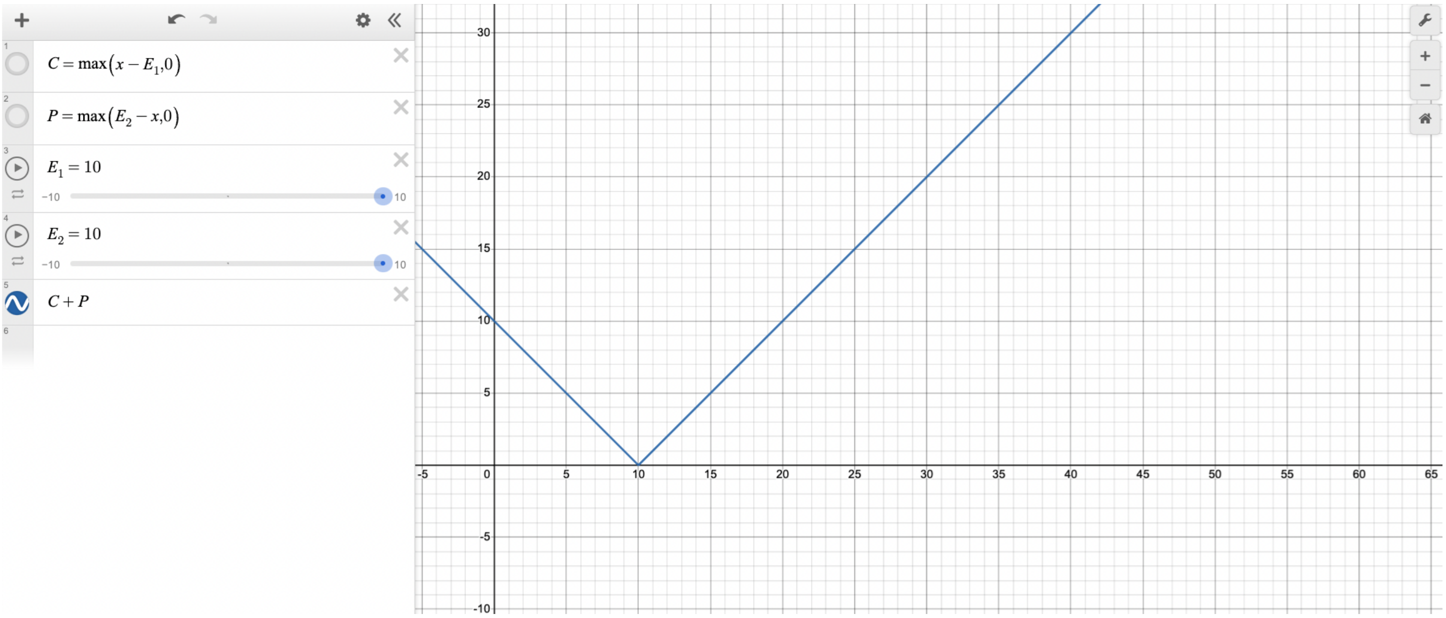
Case 1: $E_1 < E_2$



Case 2: $E_1 > E_2$

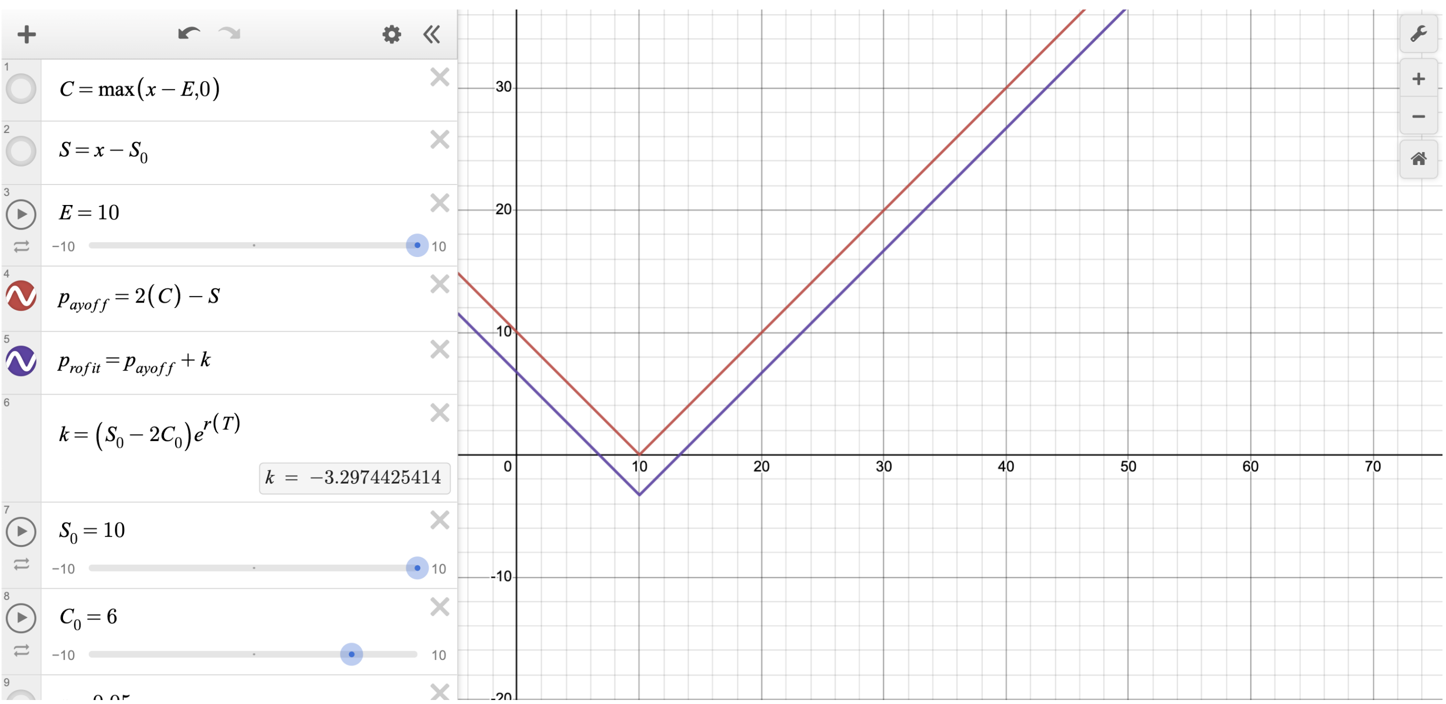


Case 3: $E_1 = E_2$



Question 2.2 What is the difference between a payoff diagram and a profit diagram? Illustrate with a portfolio of short one share, long two calls with exercise price E.

Answer The payoff diagram is simply the payoff of being short one share and long two calls. The profit diagram takes into account the interest-adjusted value of the cash from selling the share at $t = 0$ and the value of buying two calls at $t = 0$. In the illustration, we used strike price $E = 10$, initial stock from $S_0 = 10$, call price $C_0 = 6$, interest rate $r = 0.05$, and time to maturity $T = 10$. The red line is the payoff and the purple line is the profit.



Question 2.3 A share currently trades at \$60. A European call with exercise price \$58 and expiry in three months trades at \$3. The three month default-free discount rate is 5%. A put is offered on the market, with exercise price \$58 and expiry in three months, for \$1.50. Do any arbitrage opportunities now exist? If there is a possible arbitrage, then construct a portfolio that will take advantage of it. (This is an application of put-call parity.)

Answer Under the **put-call parity**, if we long a share, long a put and short a call, we can simulate the performance of cash plus interest that has an expected value of the strike price. So $S(t) + P(t) - C(t) = PV_t(E)$, where E is the strike price, S is the stock, P is the put, C is the call, and PV_t is the discounted value of the strike price at time t.

On the left-hand-side, $S(t) + P(t) - C(t) = 60 + 1.5 - 3 = \$58.5$. On the right-hand-side, given $r = 0.05$, we have $PV_t(E) = PV_t(58) = 58 \cdot \frac{1}{e^{0.05}} \approx \$55.17$. Since $\$58.5 > \$55.17$, an arbitrage opportunity exists because the LHS is overpriced.

We can carry out a **"reverse conversion"** and take advantage of the arbitrage if we short a share and a put, and long a call.

Question 2.4 A three-month, 80 strike, European call option is worth \$11.91. The 90 call is \$4.52 and the 100 call is \$1.03. How much is the butterfly spread?

Answer We can construct a butterfly spread if we buy an 80 and a 100 call, and write two 90 calls. So the price will be $11.91 + 1.03 - 2 \cdot 4.52 = \$3.9$

Exercise 3

Question 3 from your document is as follows:

In lecture, we showed how to calibrate a binomial tree using parameters:

$$\begin{aligned} u &= e^{\sqrt{\sigma \Delta t}} \\ d &= e^{-\sqrt{\sigma \Delta t}} \\ p &= \frac{1}{2} + \frac{\mu \sqrt{\Delta t}}{2\sigma} \end{aligned}$$

where σ is volatility and μ is the drift of $\log S(t)$ (cf. Van Erp lecture 1). Wilmott uses instead:

$$\begin{aligned} u &= 1 + \sqrt{\sigma \Delta t} \\ d &= 1 - \sqrt{\sigma \Delta t} \\ p &= \frac{1}{2} + \frac{\alpha \sqrt{\Delta t}}{2\sigma} \end{aligned}$$

where α is the mean rate-of-return i.e. \$E(S(T)) = S_0 e^{\alpha T} \$.

(a) Show that with Wilmott's parameters we get \$ E(S_1) \approx S_0 e^{\alpha \Delta t} \$

(b) Show that if we do not assume ud = 1 then we must require \$ \mu \Delta t = p \log u + (1 - p) \log d \$ and \$ \sigma \Delta t = (\log u - \log d) \sqrt{p - p^2}\$

(c) Show that with Wilmott's parameters we have approximately \$ \mu \Delta t \approx p \log u + (1 - p) \log d \$ and \$ \sigma \Delta t \approx (\log u - \log d) \sqrt{p - p^2} \$

Hint: Use a Taylor expansion to approximate log u and log d.

(a)

$E(S_1)$ is the expected value of the stock price at time $t = \Delta t$. We can calculate this as follows:

$$\begin{aligned} E(S_1) &= S_0 \cdot p \cdot u + S_0 \cdot (1-p) \cdot d \\ &= S_0 ((\frac{1}{2} + \frac{\alpha \sqrt{\Delta t}}{2\sigma})(1 + \sigma \sqrt{\Delta t}) + (\frac{1}{2} - \frac{\alpha \sqrt{\Delta t}}{2\sigma})(1 - \sigma \sqrt{\Delta t})) \end{aligned}$$

Let $a = \frac{1}{2}$, $b = \frac{\alpha \sqrt{\Delta t}}{2\sigma}$, $c = 1$, and $d = \sigma \sqrt{\Delta t}$.

We then have a polynomial of the form $(a+b)(c+d)+(a-b)(c-d)$. Expanding this out, we get $ac+ad+bc+bd+ac-ad-bc+bd = 2ac + 2bd = 2 \cdot \frac{1}{2} \cdot 1 + 2 \cdot \frac{\alpha \sqrt{\Delta t}}{2\sigma} \cdot \sigma \sqrt{\Delta t} = 1 + \alpha \Delta t$. This is approximately $e^{\alpha \Delta t}$.

So we have $E(S_1) = S_0(1 + \alpha \Delta t) \approx S_0 e^{\alpha \Delta t}$.

(b)

Because μ is the drift of $\log S(t)$, by definition $\mu\Delta t$ is the expected change in $\log S(t)$ over the time period Δt. We can calculate this as follows:

$$E(\log S(\Delta t)) = \mu\Delta t = p\log u + (1-p)\log d$$

We took log of u and d because we are dealing with $\log S(t)$ and we know that $E(S(\Delta t)) = p \cdot u + (1-p)\cdot d$

We can also calculate the variance of $\log S(t)$ over the time period Δt as follows:

$$\begin{gathered}
\mathrm{Var}(\log S(\Delta t)) = \sigma^2\Delta t = E((\log S(\Delta t))^2) - (E(\log S(\Delta t)))^2 \\
[p(\log(u))^2 + (1-p)(\log(d))^2] - [p\cdot\log(u) + (1-p)\cdot\log(d)]^2 \\
[p(\log(u))^2 + (1-p)(\log(d))^2] - [p^2\cdot\log(u)^2 + (1-p)^2\cdot\log(d)^2 + 2p(1-p)\cdot\log(u)\cdot\log(d)] \\
p(1-p)(\log(u^2)) + (1-p)(p)(\log(d^2)) - 2p(1-p)\log(u)\log(d) \\
p(1-p)[\log(u)^2 + \log(d)^2 - 2\log(u)\log(d)] \\
p(1-p)(\log(u) - \log(d))^2
\end{gathered}$$

Thus, it follows that the standard deviation of the return to one time-step is:

$ \text{SD}(\log(S(\Delta t))) = \sigma \sqrt{\Delta t} = \sqrt{p-p^2}(\log(u) -\log(d)) $.

(c)