M86 Homework 2

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Exercise 1

Wilmott, Chapter 3, Problem 8

A share price is currently \$180. At the end of one year, it will be either \$203 or \$152. The risk-free interest rate is 3% p.a. with continuous compounding. Consider an American put on this underlying. Find the exercise price for which holding the option for the year is equivalent to exercising immediately. This is the break-even exercise price. What effect would a decrease in the interest rate have on this break-even price?

Solution

Let's denote:

- $S_0 = \$180$ as the current share price,
- $S_u = \$203$ as the share price at the end of the year if it goes up,
- $S_d = \$152$ as the share price at the end of the year if it goes down,
- $r = 3\%$ as the continuous compounding annual risk-free interest rate,
- K as the exercise price of the American put option.

The value of exercising the option immediately is $V_0 = K - S_0$. We know $K \geq S_0$, or else exercising immediately yields a negative return, while holding the put for a year yields a minimum return of 0 if the put is not exercised, and no equivalence can be formed.

The value of the option at the end of the year in each scenario is:

- If the price goes up: $V_u = \max(K - S_u, 0)$,
- If the price goes down: $V_d = \max(K - S_d, 0)$.

The present/discounted value of the option exercised at the end of the year ($T = 1$) at rate r, for both upward and downward movements, is given by:

$$PV_u = V_u \cdot e^{-r} = \max(K - S_u, 0) \cdot e^{-r}$$
$$PV_d = V_d \cdot e^{-r} = \max(K - S_d, 0) \cdot e^{-r}$$

We first find the risk-neutral probabilities for the share price going up, denoted as q_u and going down, denoted as q_d.

The up-factor, denoted as u, and the down-factor, denoted as d, are given by:

$$u = \frac{S_u}{S_0} = \frac{203}{180} = 1.127778, \quad d = \frac{S_d}{S_0} = \frac{152}{180} = 0.844444$$

The risk-neutral probabilities are given by:

$$q_u = \frac{e^r - d}{u - d} = \frac{e^{0.03} - 0.844444}{1.127778 - 0.844444} = 0.6565$$

$$q_d = 1 - q_u = 0.3435$$

The method given in Wilmott, Ch.3, p.72, for finding the risk-neutral probabilities is as follows:

$$e^{-r} \cdot (q_u \cdot S_u + (1 - q_u) \cdot S_d) = S_0$$

$$e^{-0.03} \cdot (q_u \cdot 203 + (1 - q_u) \cdot 152) = 180$$

$$q_u = 0.6565, \quad q_d = (1 - q_u) = 0.3435$$

To find the break-even exercise price K for which holding the option for the year is equivalent to exercising immediately, we set the immediate exercise value equal to the present value of exercising at the end of the year.

Thus, the equation to solve for K is:

$$K - S_0 = e^{-r}(q_u \cdot \max(K - S_u, 0) + q_d \cdot \max(K - S_d, 0))$$

$$K - 180 = e^{-0.03}(0.6565 \cdot \max(K - 203, 0) + 0.3435 \cdot \max(K - 152, 0))$$

Using the solver on WolframAlpha, we get the **break-even strike price,** $K = \$194$.

Exercise 2

Wilmott, Chapter 7, Problems 1, 8, 9

Question 7.1 Consider an option with value $V(S,t)$, which has payoff at time T. Reduce the Black–Scholes equation, with final and boundary conditions, to the diffusion equation, using the following transformations:

(a) $S = Ee^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V(S,t) = Ev(x,\tau)$

(b) $v = e^{\alpha x + \beta t}u(x,\tau)$

For some α and β. What is the transformed payoff? What are the new initial and boundary conditions? Illustrate with a vanilla European call option.

Solution We start with the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

where σ is the volatility, r is the risk-free interest rate, and $V(S,t)$ is the value of the option at time t given the stock price S.

Let us apply the substitution to each term in the Black-Scholes equation using chain rule.

For the first term $\frac{\partial V}{\partial t}$:

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = \frac{\partial Ev}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = -\frac{E\sigma^2}{2} \cdot \frac{\partial v}{\partial \tau}$$

For the second term $rS\frac{\partial V}{\partial S}$:

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial S} = \frac{\partial Ev}{\partial x} \cdot \frac{\partial x}{\partial S}$$

We know that $S = Ee^x$, so $\frac{\partial S}{\partial x} = Ee^x$, and $\frac{\partial x}{\partial S} = \frac{1}{Ee^x}$. Thus,

$$\frac{\partial V}{\partial S} = \frac{\partial Ev}{\partial x} \cdot \frac{1}{Ee^x}$$

Substituting this into the entire term:

$$rS\frac{\partial V}{\partial S} = rEe^x \cdot \frac{\partial Ev}{\partial x} \cdot \frac{1}{Ee^x} = Er \cdot \frac{\partial v}{\partial x}$$

For the third term $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$:

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 V}{\partial x^2} \cdot \left(\frac{\partial x}{\partial S}\right)^2 = \frac{\partial^2 Ev}{\partial x^2} \cdot \left(\frac{1}{Ee^x}\right)^2$$

Substituting this into the entire term:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = \frac{1}{2}\sigma^2 E^2 e^{2x} \cdot \frac{\partial^2 Ev}{\partial x^2} \cdot \frac{1}{E^2 e^{2x}} = \frac{E\sigma^2}{2} \cdot \frac{\partial^2 v}{\partial x^2}$$

For the fourth term rV:

$$rV = rEv$$

Substituting all of these into the Black-Scholes equation, we get:

$$-\frac{E\sigma^2}{2} \cdot \frac{\partial v}{\partial \tau} + Er \cdot \frac{\partial v}{\partial x} + \frac{E\sigma^2}{2} \cdot \frac{\partial^2 v}{\partial x^2} - rEv = 0$$

Let's divide through by $\frac{E}{2}$:

$$\frac{E}{2}\left(-\sigma^2 \cdot \frac{\partial v}{\partial \tau} + 2r \cdot \frac{\partial v}{\partial x} + \sigma^2 \cdot \frac{\partial^2 v}{\partial x^2} - 2r \cdot v\right) = 0$$

Because $\frac{E}{2}$ is constant, multiplying by it does not contribute to the condition of the equation. Therefore, we have:

$$-\sigma^2 \cdot \frac{\partial v}{\partial \tau} + 2r \cdot \frac{\partial v}{\partial x} + \sigma^2 \cdot \frac{\partial^2 v}{\partial x^2} - 2r \cdot v = 0$$

Question 7.8 Show that if

$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$, on $-\infty < x < \infty$, $\tau > 0$,

with

$u(x,0) = u_0(x) > 0$,

then $u(x,\tau) > 0$ for all τ.

Use this result to show that an option with positive payoff will always have a positive value.

Solution: The solution to the heat equation is given by:

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\tau}} u_0(y)dy$$

Since $u_0(x) > 0$, the integral is a positive number, and the exponential term is always positive. Also, because $\tau > 0$, the fraction is always positive. Thus, the value of $u(x,\tau)$ is always positive.

Question 7.9 If $f(x,\tau) \geq 0$ in the initial value problem

$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + f(x,\tau)$, on $-\infty < x < \infty$, $\tau > 0$,

with

$u(x,0) = 0$, and $u \to 0$ as $|x| \to \infty$,

then $u(x,\tau) \geq 0$. Hence show that if C_1 and C_2 are European calls with volatilities σ_1 and σ_2 respectively, but are otherwise identical, then $C_1 > C_2$ if $\sigma_1 > \sigma_2$.

Use put-call parity to show that the same is true for European puts.

Solution (NEED MORE ON HOW VALUE RELATES TO VOLATILITY): The solution to the non-homogenous heat equation is given by:

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\tau}} u_0(y)dy + \int_0^{\tau} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(\tau-s)}} f(y,s)dyds$$

In our case, $u_0(x) = 0$, so the first integral is 0. Since $f(x,\tau) \geq 0$, the second integral is also non-negative. Thus, we have

$$u(x,\tau) = \int_0^{\tau} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(\tau-s)}} f(y,s)dyds \geq 0$$

Now, we can use put-call parity to show that if C_1 and C_2 are European calls with volatilities σ_1 and σ_2 respectively, but are otherwise identical, then $C_1 > C_2$ if $\sigma_1 > \sigma_2$.

Exercise 3

Suppose W_t and Z_t are two independent Brownian motions. This means that each satisfies the 4 defining properties of Brownian motion and the increments $Z_t - Z_s$ and $W_t - W_s$ are independent random variables for all $0 \leq s < t$. For a constant ρ with $-1 \leq \rho \leq 1$, consider the random process:

$$X_t = \rho W_t + \sqrt{1-\rho^2} Z_t.$$

Is X_t a Brownian motion? Explain your answer.

Solution The four conditions of Brownian motion are:

1. $W_0 = 0$
2. The distribution of the increment $W_{t2} - W_{t1}, \quad 0 \leq t_1 \leq t_2$ is normal with mean 0 and variance $t_2 - t_1$
3. For non-overlapping intervals $t_1 < t_2 \leq t_3 < t_4$, the increments $W_{t2} - W_{t1}$ and $W_{t4} - W_{t3}$ are independent random variables
4. The function $t \to W_t$ is almost surely continuous

Let's go through each of these conditions for the process $X_t = \rho W_t + \sqrt{1-\rho^2} Z_t$:

1. $X_0 = \rho W_0 + \sqrt{1-\rho^2} Z_0 = \rho \cdot 0 + \sqrt{1-\rho^2} \cdot 0 = 0$. This condition is satisfied.
2. The distribution of the increment $X_{t2} - X_{t1}, \quad 0 \leq t_1 \leq t_2$ is normal with mean 0 and variance $t_2 - t_1$:

$$X_{t2} - X_{t1} = \rho(W_{t_2} - W_{t_1}) + \sqrt{1-\rho^2}(Z_{t_2} - Z_{t_1})$$

Let's take the expectation and variance of this increment:

$$E[X_{t2} - X_{t1}] = \rho E[W_{t_2} - W_{t_1}] + \sqrt{1-\rho^2} E[Z_{t_2} - Z_{t_1}] = \rho \cdot 0 + \sqrt{1-\rho^2} \cdot 0 = 0$$

$$Var[X_{t2} - X_{t1}] = \rho^2 Var[W_{t_2} - W_{t_1}] + (1-\rho^2) Var[Z_{t_2} - Z_{t_1}] = \rho^2(t_2 - t_1) + (1-\rho^2)(t_2 - t_1) = t_2 - t_1$$

1. For non-overlapping intervals $t_1 < t_2 \leq t_3 < t_4$, the increments $X_{t2} - X_{t1}$ and $X_{t4} - X_{t3}$ are independent random variables:

$$X_{t2} - X_{t1} = \rho(W_{t_2} - W_{t_1}) + \sqrt{1-\rho^2}(Z_{t_2} - Z_{t_1}), \quad X_{t4} - X_{t3} = \rho(W_{t_4} - W_{t_3}) + \sqrt{1-\rho^2}(Z_{t_4} - Z_{t_3})$$

Given that W_t and Z_t describe Brownian motion, the increments $W_{t2} - W_{t1}$ and $W_{t4} - W_{t3}$ are independent random variables, and the increments $Z_{t2} - Z_{t1}$ and $Z_{t4} - Z_{t3}$ are independent random variables. Since W_t and Z_t are independent, the increments $W_{t2} - W_{t1}$ and $Z_{t2} - Z_{t1}$ are independent random variables. Therefore, the increments $X_{t2} - X_{t1}$ and $X_{t4} - X_{t3}$ are independent random variables.

1. The function $t \to X_t$ is almost surely continuous:

The function $t \to X_t$ is a linear combination of two Brownian motions, W_t and Z_t, and is therefore continuous.

Exercise 4

If W_t is the Wiener process, show that the limit

$$\lim_{\Delta t \to 0} \frac{(\Delta W)^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{(W_{t+\Delta t} - W_t)^2}{\Delta t}$$

does not exist.

Solution:

TBD

Exercise 5

Use Ito's Lemma to derive a formula for $\int_0^T W^3 dW$.

Solution:

TBD