STATISTICAL COMPUTATIONAL METHODS

Simulation of Random Variables and Monte Carlo Methods

1. Discrete Methods for $X\begin{pmatrix} x_1 & x_2 & \dots \\ p_1 & p_2 & \dots \end{pmatrix}$, $\sum p_i = 1$.

Algorithm 1.

1. Divide interval $[0,1]$ into subintervals

$$\begin{aligned} A_1 &= [0, p_1) \\ A_2 &= [p_1, p_1 + p_2) \\ A_3 &= [p_1 + p_2, p_1 + p_2 + p_3) \\ &\dots \end{aligned}$$

   Then length$(A_i) = p_i$, for a finite or countably infinite number of values x_i.

2. Get $U \in Unif(0,1)$.
3. If $U \in A_i$, then let $X = x_i$.

2. Inverse Transform Method for a random variable X with cdf $F : \mathbb{R} \to \mathbb{R}$.

Algorithm 2.

1. Get $U \in Unif(0,1)$.
2. – if X is continuous, then let $X = F^{-1}(U)$.

   – if X is discrete, then let $X = \min\{x \mid F(x) \geq U\}$.

3. Rejection Method for a continuous random variable X with pdf $f : \mathbb{R} \to \mathbb{R}$.

Algorithm 3.

1. Find a *bounding box* $[a,b] \times [0,c]$, i.e. numbers $a, b \in \mathbb{R}, c \in \mathbb{R}_+$, such that $f(x) \in [0,c]$, for $x \in [a,b]$.
2. Get $U, V \in Unif(0,1)$.
3. Let $X = a + (b-a)U$ and $Y = cV$. Then $X \in Unif(a,b)$, $Y \in Unif(0,c)$ and $(X,Y) \in Unif\Big([a,b] \times [0,c]\Big)$.
4. If $Y > f(X)$, reject the point and return to step 2. If $Y \leq f(X)$, then X is a random variable with pdf f.

4. Special Methods

– for a $Poiss(\lambda), \lambda > 0$, random variable $X\begin{pmatrix} k \\ \dfrac{\lambda^k}{k!}e^{-\lambda} \end{pmatrix}_{k \in \mathbb{N}}$.

Algorithm 4.

1. Get $U_1, U_2, \dots \in Unif(0,1)$.
2. Let $X = \max\{k \mid U_1 \cdot U_2 \cdot \ldots \cdot U_k \geq e^{-\lambda}\}$.

– for a $Norm(\mu, \sigma), \mu \in \mathbb{R}, \sigma > 0$, random variable, pdf $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$.

Box-Muller Transform

Algorithm 5.

1. Get $U, V \in Unif(0, 1)$.
2. Let
$$\begin{cases} Z_1 &= \sqrt{-2\ln(U)}\cos(2\pi V), \\ Z_2 &= \sqrt{-2\ln(U)}\sin(2\pi V). \end{cases}$$
Then Z_1, Z_2 are independent $Norm(0, 1)$ random variables.
3. Let $X = \sigma Z + \mu$. Then $X \in Norm(\mu, \sigma)$.

5. Accuracy of an MC Study of size N

– estimating probabilities: $p = P(X \in A)$ is estimated by $\overline{p} = \dfrac{\text{number of } X_1, \ldots, X_N \in A}{N}$;

– to ensure that $P(|\overline{p} - p| > \varepsilon) \leq \alpha$, take $N \geq \dfrac{1}{4}\left(\dfrac{z_{\alpha/2}}{\varepsilon}\right)^2$;

– estimating means: $\mu = E(X)$ is estimated by $\overline{X} = \dfrac{X_1 + \ldots + X_N}{N}$;

– estimating variances: $\sigma^2 = V(X)$ is estimated by $s^2 = \dfrac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \overline{X}\right)^2$.