STATISTICAL COMPUTATIONAL METHODS

Seminar Nr. 5, Counting Processes

1. On the average, 2 airplanes per minute land at a certain international airport. Assume the number of landings is modeled by a Binomial counting process.
a) what frame length should be used to guarantee that the probability of a landing does not exceed 0.1?
b) using the chosen frames, compute the probability of no landings during the next 2 minutes;
c) using the chosen frames, compute the probability of more than 120 landed airplanes during the next hour.

Solution:

a) We have $\lambda = 2/\min$. So, if we want $p \le 0.1$, then

$$\Delta \;=\; \frac{p}{\lambda} \le \frac{0.1}{2}\min \;=\; 0.05\min = 3\,\mathrm{sec}.$$

b) Let $\Delta = 3$ sec. In $t = 2$ min. $= 120$ sec., there are $n = \dfrac{t}{\Delta} = \dfrac{120}{3} = 40$ frames. The number of landings $X(40)$ during 40 frames has Binomial distribution with $n = 40$ and $p = 0.1$. We want

$$P(X(40) = 0) \;=\; \mathit{binopdf}(0, 40, 0.1) \;=\; 0.0148.$$

c) Similarly, in 1 hour $= 3600$ sec., there are $n = \dfrac{3600}{3} = 1200$ frames. Thus, the number of landings $X(1200)$ during the next hour has Binomial distribution with $n = 1200$ and $p = 0.1$. We have

$$P(X(1200) > 120) \;=\; 1 - P(X(1200) \le 120) \;=\; 1 - \mathit{binocdf}(120, 1200, 0.1) \;=\; 0.4757.$$

2. Messages arrive at a communications center according to a Binomial counting process with 30 frames per minute. The average arrival rate is 40 messages per hour. How many messages can be expected to arrive between 10 a.m. and 10:30 a.m.? What is the standard deviation of that number of messages?

Solution:

The arrival rate is $\lambda = 40/\text{hr} = \frac{2}{3}/$ min. and the frame length is $\Delta = \frac{1}{30}$ min. Then, the probability of a new message arriving during any given frame is

$$p = \lambda\Delta = \frac{2}{3} \cdot \frac{1}{30} = \frac{1}{45}.$$

Between 10 and 10:30 a.m., there are $t = 30$ minutes, so there are $n = \frac{t}{\Delta} = \frac{30}{1/30} = 900$ frames, hence, the number of new messages $X(900)$ arriving during this time has Binomial distribution with $n = 900$ and p = $1/45$, so

$$\begin{aligned} E(X) &= np = 20 \text{ messages}, \\ \sigma(X) &= \sqrt{np(1-p)} = 4.4222 \text{ messages}. \end{aligned}$$

3. An internet service provider offers special discounts to every third connecting customer. Its customers connect to the internet according to a Poisson process with the rate of 5 customers per minute. Compute

a) the probability that no offer is made during the first 2 minutes;

b) the probability that no customers connect for 20 seconds;

c) expectation and standard deviation of the time of first offer.

Solution:

a) We have $\lambda = 5\,/\,\text{min}$. In $t = 2$ minutes, the number of connections X has Poisson distribution with parameter $\lambda t = 5 \cdot 2 = 10$. No offer is made if there are fewer than three connections. So, we want

$$P(\text{no offer}) = P(X < 3) = P(X \le 2) = poisscdf(2, 10) = 0.0028.$$

b) The time between connections (interarrival time) T has Exponential(λ) distribution. No customers connect for 20 sec. $= 1/3$ minutes, if the interarrival time exceeds that. So, we want

$$P(T > 1/3) = 1 - P(T \le 1/3) = 1 - expcdf(1/3, 1/5) = 0.1889.$$

Or we can express λ in seconds, $\lambda = 5/60 = 1/12\,/\text{sec}$. Then we compute

$$P(T > 20) = 1 - P(T \le 20) = 1 - expcdf(20, 12) = 0.1889.$$

c) The time T_3 of the third connection (arrival) (and therefore, the first offer) is the sum of 3 independent $Exp(5)$ times, so it has Gamma distribution with parameters $\alpha = 3$ and $\lambda = 1/5$. Then

$$\begin{aligned} E(T_3) &= \alpha\lambda = 3/5 = 0.6 \text{ min}, \\ \sigma(T_3) &= \sqrt{V(T_3)} = \sqrt{\alpha\lambda^2} = 0.3464 \text{ min}. \end{aligned}$$

4. On the average, Mr. X drinks and drives once in 4 years. He knows that

- every time he drinks and drives, he is caught by the police;
- according to the law of his state, the third time he is caught drinking and driving, he loses his driver's license;
- a Poisson counting process models such "rare events" as drinking and driving.

What is the probability that Mr. X will keep his driver's license for at least 10 years?

Solution:

The arrival rate of drinking and driving is $\lambda = 1/4$/year. Let X be the number of times Mr. X is caught drinking and driving during 10 years. Then X has Poisson distribution with parameter $\lambda t = (1/4)(10) = 2.5$. Keeping the driver's license is equivalent to being caught *less* than three times in 10 years. Then,

$$\begin{aligned} P(\text{Mr. X keeps his driver's license}) &= P(X < 3) = P(X \le 2) \\ &= poisscdf(2, 2.5) = 0.5438. \end{aligned}$$

5. Simulation and illustration of Binomial and Poisson counting processes.

a) Given sample path size N_B and probability of arrival p, simulate a Binomial counting process $X(t)$.

Application: For a frame size of 1 second, simulate the number of airplane landings from Problem 1., for 1 minute.

b) Given frequency λ and a time frame $[0, T_{max}]$, simulate a Poisson counting process $X(t)$.

Application: Simulate the number of internet connections from Problem 3., for a period of half an hour.

Solution:

a) We simulate a sequence of Bernoulli trials, where we count the number of successes. Recall that a "success" with probability p is simulated as $U < p$, where U is Standard Uniform. We use Algorithm 3.6, Lecture 6.

Algorithm

1. Given: N_B = sample path length
2. Generate $U \in U(0,1)$, let $Y = (U < p)$, let $X(1) = Y$.
3. At each time t, let $Y = (U < p)$, let $X(t) = X(t-1) + Y$.
4. Return to step 3 until length N_B is achieved.

Let us do just that.

```
% Simulation of Binomial counting process.
% and illustration of the generated discrete-time process
% with Delta frame size.
clear all
NB = input('length of sample path = ');
p = input('prob. of success (arrival) = ');
X = zeros(1, NB); % allocate memory for X
X(1) = (rand < p); % first Bernoulli trial; X nr. of successes
for t = 2 : NB
    X(t) = X(t - 1) + (rand < p); % count the nr. of successes
end
X
```

This returns a sequence of the number of arrivals (messages) each second that looks like this:

$$0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2$$

$$2\ 2\ 2\ 2\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 6$$

These are the numbers of landings during one minute.

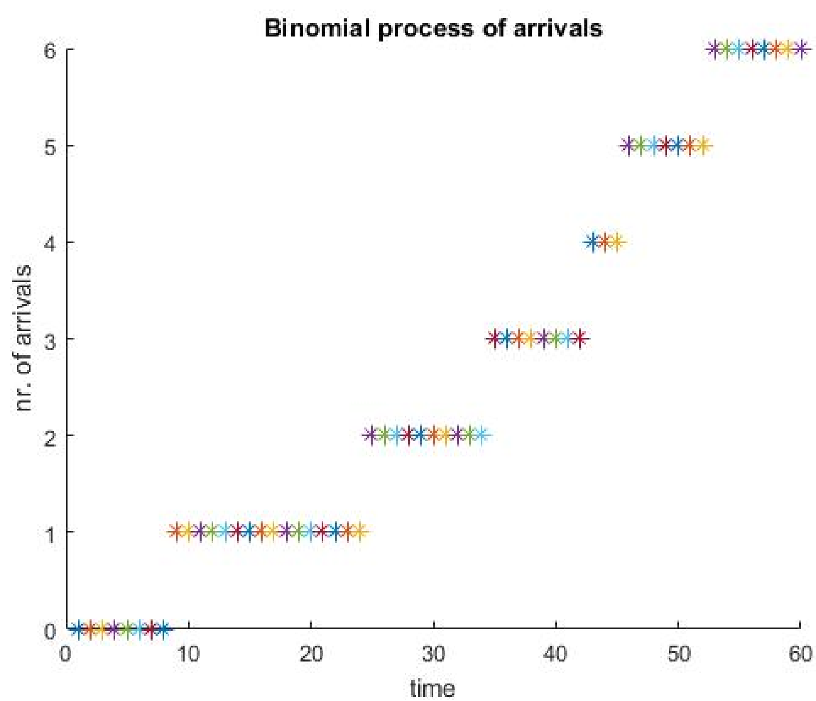
Let us also see it graphically. To see it in real time, we will change the frame size Δ from 3 seconds to 1 second, while keeping the same probability of arrival $p = 0.1$. That means now the arrival rate is

$$\lambda = \frac{p}{\Delta} = \frac{1}{10}/\text{ sec.} = 6/\text{ min.}$$

```
clf
% illustration
Del = input('frame size (in seconds) = ');
```

```
axis([ 0 N 0 max(X)]); % allocate the box for the entire
                       % simulated segment
hold on
title('Binomial process of arrivals')
xlabel('time');
ylabel('nr. of arrivals')
for t = 1 : NB
    plot(t, X(t), '*', 'MarkerSize', 8) % plot each point with '*'
    pause(Del) % to see it in real time
end
hold off
```

It looks like this.



b) For a Poisson counting process, we generate the moments of arrival, which are $Exp(\lambda)$ variables. We use Algorithm 3.3 from Lecture 6.

```
% Simulate a [0, Tmax] segment of a Poisson counting process.
clear all
lambda = input('frequency lambda = '); % given frequency
Tmax = input('time frame (in minutes) Tmax = '); % given
                         % time period
arr_times = -1/lambda * log(rand); % array containing
                                    % arrival times
last_arrival = arr_times; % each interarriv. time is Exp(lambda)
while last_arrival <= Tmax
    last_arrival = last_arrival - 1/lambda * log(rand);
    arr_times = [arr_times, last_arrival];
end;
arr_times = arr_times(1 : end - 1) % nr. of arrivals during
                                    % time Tmax
% last last_arrival should not be included
step = 0.01; % small step size, simulate continuity
t = 0 : step : Tmax; % time variable
Nsteps = length(t);
X = zeros(1, Nsteps); % Poisson process X(t)
for s = 1 : Nsteps;
    X(s) = sum(arr_times <= t(s));
end; % X(s) is the number of arrivals by the time t(s)
X
```

This returns the arriving times and a sequence for the number of messages arriving every 0.01 of a minute (or every 0.6 of a second) for 30 minutes. Now, for a cool graph, do this:

```
% illustration
axis([0 max(t) 0 max(X)]); hold on
title('Poisson process of arrivals');
xlabel('time'); ylabel('number of arrivals');
% plot(t, X, 'r')
comet(t, X)
```