## 2.3 Steady-State Distribution; Regular Markov Chains

It is sometimes necessary to be able to make long-term forecasts, meaning we want $\lim_{h\to\infty} P_h$, so we need to compute $\lim_{h\to\infty} p_{ij}^{(h)}$.

**Definition 2.13.** *Let $X$ be a Markov chain. The vector $\pi = [\pi_1, \dots, \pi_n]$, consisting of the limiting probabilities*

$$\pi_k = \lim_{h\to\infty} P_h(k), k = 1, \dots, n, \tag{2.13}$$

*if it exists, is called a **steady-state distribution** of $X$.*

In order to find it, let us notice that

$$P_h P \;=\; (P_0 P^h) P \;=\; P_0 P^{h+1} \;=\; P_{h+1}.$$

Taking the limit as $h \to \infty$ on both sides, we get

$$\pi P \;=\; \pi. \tag{2.14}$$

System (2.14) is an $n \times n$ singular linear system (multiplication by a constant on each side leads to infinitely many solutions). However, since $\pi$ must also be a *stochastic* matrix, the sum of its components must equal 1. We state the following result, without proof.

**Proposition 2.14.** *The steady-state distribution of a homogeneous Markov chain $X$, $\pi = [\pi_1, \dots, \pi_n]$, if it exists, is unique and is the solution of the $(n+1) \times n$ linear system*

$$\begin{cases} \pi P & = \; \pi \\ \displaystyle\sum_{k=1}^{n} \pi_k & = \; 1. \end{cases} \tag{2.15}$$

**Example 2.15.** Let us find the steady-state distribution of the Markov chain in Example 2.9 (Lecture 5). What is the weather forecast in Rainbow City for Christmas Day next year?

**Solution.** Recall that in Example 2.9 we had a homogeneous Markov chain with two states, (1-sunny, 2-rainy), the initial situation (on Monday) was $80\%$ chance of rain, i.e.

$$P_0 = [0.2 \;\; 0.8]$$

and the transition probability matrix was

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}.$$

We write system (2.14). We have

$$[\pi_1 \;\; \pi_2]P = [\pi_1 \;\; \pi_2] \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.7\pi_1 + 0.4\pi_2 \\ 0.3\pi_1 + 0.6\pi_2 \end{bmatrix}.$$

Then system (2.15) is

$$\begin{cases} 0.7\pi_1 + 0.4\pi_2 & = & \pi_1 \\ 0.3\pi_1 + 0.6\pi_2 & = & \pi_2 \\ \pi_1 + \pi_2 & = & 1, \end{cases}$$

which then becomes

$$\begin{cases} -0.3\pi_1 + 0.4\pi_2 & = & 0 \\ \pi_1 + \pi_2 & = & 1, \end{cases}$$

with solution

$$[\pi_1 \;\; \pi_2] = [4/7 \;\; 3/7].$$

Interpretation: in the long-run, in the future, $4/7 \approx 57\%$ of days are sunny and $3/7 \approx 43\%$ of days are rainy. Recall that the forecast for Wednesday was $53.8\%/46.2\%$ and for Friday, $56.84\%/43.16\%$, which is already getting close to the steady-state distribution.

Since Christmas Day next year is *many* steps from now, we use the steady-state distribution. So *that* would be the forecast for Christmas Day next year, too!

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**Remark 2.16.**

1. When we need to make predictions after a large number of steps, instead of the lengthy computation of $P_h$, it may be easier to try to find the steady-state distribution, $\pi$, directly.
2. If a steady-state distribution exists, then $P^{(h)} = P^h$ also has a limiting matrix, given by

$$\mathbf{\Pi} = \lim_{h\to\infty} P^{(h)} = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \dots & \pi_n \\ \vdots & \vdots & \dots & \vdots \\ \pi_1 & \pi_2 & \dots & \pi_n \end{bmatrix}.$$

3. Notice that $\pi$ and $\mathbf{\Pi}$ *do not* depend on the initial state $X_0$. Actually, in the long run, the probabil-

ities of transitioning from any state to a given state are the same, $p_{ik} = p_{jk},\ \forall i, j, k = \overline{1, n}$ (all the rows of $\mathbf{\Pi}$ coincide). Then, it is just a matter of "reaching" a certain state (from anywhere), rather than "transitioning" to it (from another state). That should, indeed, depend only on the pattern of changes, i.e. only on the transition probability matrix.

Now, a natural question arises: does a steady-state distribution always exist? The answer is **no**! Here is a simple example:

**Example 2.17.** In a game of chess, a knight (română "calul") can only move to a field of different color (white-black or black-white) at any time. Then the transition probability matrix of the color of its field is

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For this matrix, a simple computation yields

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

so $P^{2k} = I$ and $P^{2k+1} = P, \forall k \in \mathbb{N}$ these are the only possible values. Thus, $\lim\limits_{h\to\infty} P^h$ *does not exist*.
This is a **periodic** Markov chain, $X_t = X_{t+2}$. Periodic Markov chains do not have a steady-state distribution.

So, when *does* a steady-state distribution exist? This is an ongoing research problem. We mention (without proof) one case, which is really easy to check, when such a distribution does exist.

**Definition 2.18.** *A Markov chain is called* ***regular*** *if there exists $h \geq 0$, such that*

$$p_{ij}^{(h)} > 0, \tag{2.16}$$

*for all $i, j = 1, \ldots, n$.*

This is saying that at some step $h$, $P^{(h)}$ has only non-zero entries. meaning that $h$-step transitions from any state to any state are possible.

**Proposition 2.19.** *Any regular Markov chain has a steady-state distribution.*

**Remark 2.20.**

1. Regularity of Markov chains does not mean that *all* $p_{ij}^{(h)}$ should be positive, for all $h$. The transition probability matrix $P$, or some of its powers, may have some $0$ entries, but there must exist some power $h$, for which $P^{(h)}$ has all non-zero entries.

2. If there exists a state $i$ with $p_{ii} = 1$, then that Markov chain cannot be regular. There is no exit (no transition possible) from state $i$. Such a state is called an **absorbing state**. For example, state $4$ in Figure 1(a) is absorbing, therefore, the Markov chain is irregular. There may be several absorbing states or an entire *absorbing zone*, from which the remaining states can never be reached. For example, states $3, 4$ and $5$ in Figure 1(b) form an absorbing zone, some kind of a "Bermuda triangle". When this process finds itself in the set $\{3, 4, 5\}$, there is no route from there to the set $\{1, 2\}$. As a result, e.g. probability $p_{31}^{(h)}$ is $0$ for all $h$. Notice that both Markov chains *do* have steady-state distributions. The first process will eventually reach state $4$ and will stay there for good. Therefore, the limiting distribution of $X_h$ is $\pi = \lim_{h\to\infty} P_h = [0\ \ 0\ \ 0\ \ 1]$. The second Markov chain will eventually leave states $1$ and $2$ for good, thus its limiting (steady-state) distribution has the form $\pi = [0\ \ 0\ \ \pi_3\ \ \pi_4\ \ \pi_5]$.

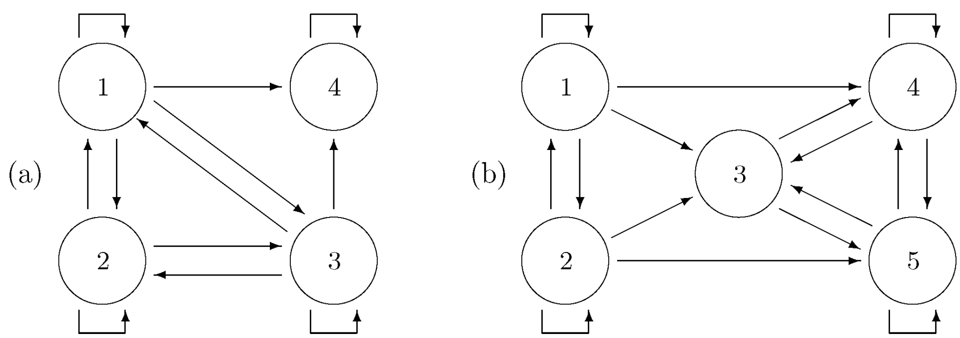


Fig. 1: Absorbing states and absorbing zones

# 3 Counting Processes

A special case of stochastic processes are the ones where one needs to count the occurrences of some types of events over time. These are described by *counting processes*.

**Definition 3.1.** *A* ***counting process*** $X(t), t \geq 0$, *is a stochastic process that represents the number of items counted by the time* $t$.

Counting processes deal with the number of occurrences of something over time, such as customers arriving at a supermarket, deleted errors, transmitted messages, number of job arrivals to a queue, holding times (in renewal processes), etc. In general, we refer to the occurrence of each event that is being counted as an "arrival". Since their sample paths (values) are always non-decreasing, non-negative integers $\{0, 1, \ldots\}$, **all** counting processes are **discrete-state** stochastic processes. They can be discrete-time or continuous-time. Next, we consider the most widely used examples, Binomial (discrete-time) and Poisson (continuous-time) counting processes.

## 3.1 Binomial Counting Process

Consider a sequence of Bernoulli trials with probability of success $p$ and count the number of "successes".

**Definition 3.2.** *A **Binomial counting process** $X(n)$ is the number of successes in $n$ Bernoulli trials,* $n = 0, 1, \ldots$ .

**Remark 3.3.**
1. Obviously, a Binomial process $X(n)$ is a discrete-state, discrete-time stochastic process.
2. The pdf of $X(n)$ is Binomial $B(n, p)$ at any time $n$. Recall that $E(X(n)) = np$.
3. The number of trials between two consecutive successes, $Y$, is the number of trials needed to get the next (first) success, so it has a $SGeo(p)$ pdf. Recall that $E(Y) = \dfrac{1}{p}, V(Y) = \dfrac{q}{p^2}$.

It is important to make the distinction between real time and the "time" variable $n$ ("time" as in a stochastic process). Variable $n$ is not measured in time units, it measures the number of trials. Suppose that Bernoulli trials occur at equal time intervals, say every $\Delta$ seconds (or other time measurement units). That means that $n$ trials occur during time $t = n\Delta$. The value of the process at time $t$ has Binomial pdf with parameters $n = \dfrac{t}{\Delta}$ and $p$. Then the expected number of successes *during $t$ seconds* is

$$E(X(n)) = E\left(X\left(\frac{t}{\Delta}\right)\right) = np = \frac{t}{\Delta}p,$$

so the expected number of successes *per second* is

$$\lambda = \frac{p}{\Delta}.$$

**Definition 3.4.**
*The quantity* $\lambda = \dfrac{p}{\Delta}$ *is called the* ***arrival rate****, i.e. the average number of successes per one unit of time.*
*The quantity* $\Delta$ *is called a* ***frame****, i.e the time interval of each Bernoulli trial.*
*The* ***interarrival time*** *is the time between successes.*

We can now rephrase: $p$ is the probability of arrival (success) during one frame (trial), $n = \dfrac{t}{\Delta}$ is the number of frames during time $t$, $X\left(\dfrac{t}{\Delta}\right)$ is the number of arrivals by time $t$.

The concepts of *arrival rate* and *interarrival time* deal with modeling arrival of jobs in discrete-time queuing systems by Binomial counting processes. The key assumption in such models is that no more than 1 arrival can occur during each $\Delta-$second frame (otherwise, a smaller $\Delta$ should be considered), so each frame is a Bernoulli trial.

The interarrival period, $Y$, measured in number of frames, has a $SGeo(p)$ pdf (as mentioned earlier). Since each frame takes $\Delta$ seconds, the interarrival time is $T = \Delta Y$, a rescaled $SGeo(p)$ variable, whose expected value and variance are given by

$$\begin{aligned} E(T) &= \Delta E(Y) = \Delta \frac{1}{p} = \frac{1}{\lambda}, \\ V(T) &= \Delta^2 V(Y) = \Delta^2 \frac{q}{p^2} = \frac{q}{\lambda^2}. \end{aligned} \tag{3.1}$$

**Example 3.5.** Messages arrive at a communications center at the rate of 6 messages per minute. Assume arrivals of messages are modeled by a Binomial counting process.
a) What frame size should be used to guarantee that the probability of a message arriving during each frame is $0.1$?
b) Using the chosen frames, find the probability of no messages arriving during the next 1 minute.
c) Compute the probability of more than 35 messages arriving during the next 6 minutes.
d) Find the probability of more than 350 messages arriving during the next hour.
e) What is the average interarrival time and its standard deviation?
f) Compute the probability that the next message does not arrive during the next 20 seconds.

**Solution.**
a) We have $\lambda = 6$ / min. and $p = 0.1$. Thus,

$$\Delta = \frac{p}{\lambda} = \frac{1}{60} \text{ min.} = 1 \text{ sec.}$$

b) So $\Delta = 1$ sec. In $t = 1$ minute $= 60$ seconds, there are $n = \dfrac{t}{\Delta} = 60$ frames. The number of messages arriving during 60 frames, $X(60)$, has a Binomial distribution with parameters $n = 60$ and $p = 0.1$. So the desired probability is

$$\begin{aligned} P\big(X(60) = 0\big) &= \mathrm{pdf}_{X(60)}(0) \\ &= \mathrm{binopdf}(0, 60, 0.1) \\ &= 0.0018. \end{aligned}$$

c) Similarly, in $t = 6$ minutes $= 360$ seconds, there are $n = \dfrac{t}{\Delta} = 360$ frames. So, the number of messages arriving during the next 6 minutes, $X(360)$, has Binomial distribution with parameters $n = 360$ and $p = 0.1$. Then the probability of more than 35 messages arriving during the next 6 minutes is

$$\begin{aligned} P\big(X(360) > 35\big) &= 1 - P\big(X(360) \leq 35\big) \\ &= 1 - \mathrm{cdf}_{X(360)}(35) \\ &= 1 - \mathrm{binocdf}(35, 360, 0.1) \\ &= 0.5257. \end{aligned}$$

d) Again, in $t = 1$ hour $= 3600$ seconds, there are $n = \dfrac{t}{\Delta} = 3600$ frames. Thus, the number of messages arriving during one hour, $X(3600)$, has Binomial distribution with parameters $n = 3600$ and $p = 0.1$. Then the probability of more than 350 messages arriving during the next hour is

$$\begin{aligned} P\big(X(3600) > 350\big) &= 1 - P\big(X(3600) \leq 350\big) \\ &= 1 - \mathrm{cdf}_{X(3600)}(350) \\ &= 1 - \mathrm{binocdf}(350, 3600, 0.1) \\ &= 0.6993. \end{aligned}$$

<u>**Notice**</u> that "more than 35 messages in 6 minutes" is **not** the same as "more than 350 messages in 60 minutes"!! These are *random* variables ...

e) By (3.1), we have

$$\begin{aligned} E(T) &= \frac{1}{\lambda} = \frac{1}{6}\text{ minutes} = 10\text{ seconds}, \\ Std(T) &= \sqrt{V(T)} = \sqrt{\frac{1-p}{\lambda^2}} = \sqrt{0.0250}\text{ minutes} \approx 9.5\text{ seconds}. \end{aligned}$$

f) Recall that the interarrival time $T = \Delta Y$, where $Y$ has a $SGeo(p)$ distribution and, hence, $Y - 1$ has a $Geo(p)$ pdf. The next message does not arrive during the next 20 seconds, if $T > 20$. So,

$$\begin{aligned} P(T > 20) &= P(\Delta Y > 20) = P(Y > 20) \\ &= 1 - P(Y \le 20) = 1 - P(Y - 1 \le 19) \\ &= 1 - \mathrm{cdf}_{Y-1}(19) = 1 - \mathrm{geocdf}(19, 0.1) \\ &= 0.1216. \end{aligned}$$

Alternatively, this is also the probability of 0 arrivals during the next $t = 20$ seconds, i.e. during $n = \dfrac{t}{\Delta} = 20$ frames. The number of messages arriving during the next 20 seconds, $X(20)$, has a Binomial distribution with parameters $n = 20$ and $p = 0.1$. Thus, the probability that no messages arrive during the next 20 seconds is

$$P\big(X(20) = 0\big) = \mathrm{pdf}_{X(20)}(0) = \mathrm{binopdf}(0, 20, 0.1) = 0.1216.$$

■

**Markov property of Binomial counting processes**

It is clear that the number of successes in $n$ trials depends *only* on the number of successes in $n - 1$ trials (not on previous values $n - 2, n - 3, \ldots$), so a Binomial process has the Markov property. Thus, it is a **Markov chain**.

Let us find the transition probability matrix. At each trial (i.e. during each frame), the number of successes $X(n)$ either increases by 1 (in case of success), or stays the same (in case of failure). Then,

$$p_{ij} = \begin{cases} p, & j = i + 1 \\ q = 1 - p, & j = i \\ 0, & \text{otherwise} \end{cases}. \tag{3.2}$$

Obviously, transition probabilities are constant over time and independent of past values of $X(n)$. Hence, $X(n)$ is a **homogeneous Markov chain** with transition probability matrix given by

$$P = \begin{bmatrix} 1-p & p & 0 & \dots & 0 & \dots \\ 0 & 1-p & p & \dots & 0 & \dots \\ 0 & 0 & 1-p & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix} \tag{3.3}$$

and transition diagram depicted in Figure 2.

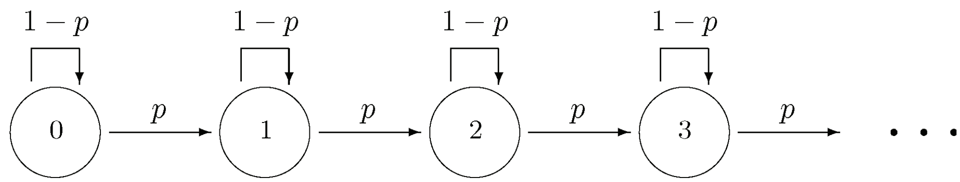


Fig. 2: Transition diagram for a Binomial counting process

Notice that it is an *irregular* Markov chain. Since $X(n)$ is non-decreasing, e.g. $p_{10}^{(h)} = 0$, for all $h \geq 0$ (once we have a success, the number of successes will *never* go back to $0$). Hence, a Binomial counting process *does not* have a steady-state distribution.

Another interesting fact: the $h$-step transition probabilities simply form a Binomial distribution. Indeed, $p_{ij}^{(h)}$ is the probability of going from $i$ to $j$ successes in $h$ transitions, i.e.,

$$\begin{aligned} p_{ij}^{(h)} &= P\big((j-i) \text{ successes in } h \text{ trials}\big) \\ &= \begin{cases} C_h^{j-i} p^{j-i} q^{h-j+i}, & 0 \leq j-i \leq h \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

**Simulation of Binomial counting processes**

This is straightforward, a sequence of Bernoulli trials, where we count the number of successes.

**Algorithm 3.6.**

1. Given: $N_B =$ sample path length
2. Generate $U \in U(0,1)$, let $Y = (U < p)$, let $X(1) = Y$.
3. At each time $t$, let $Y = (U < p)$, let $X(t) = X(t-1) + Y$.
4. Return to step $3$ until length $N_B$ is achieved.