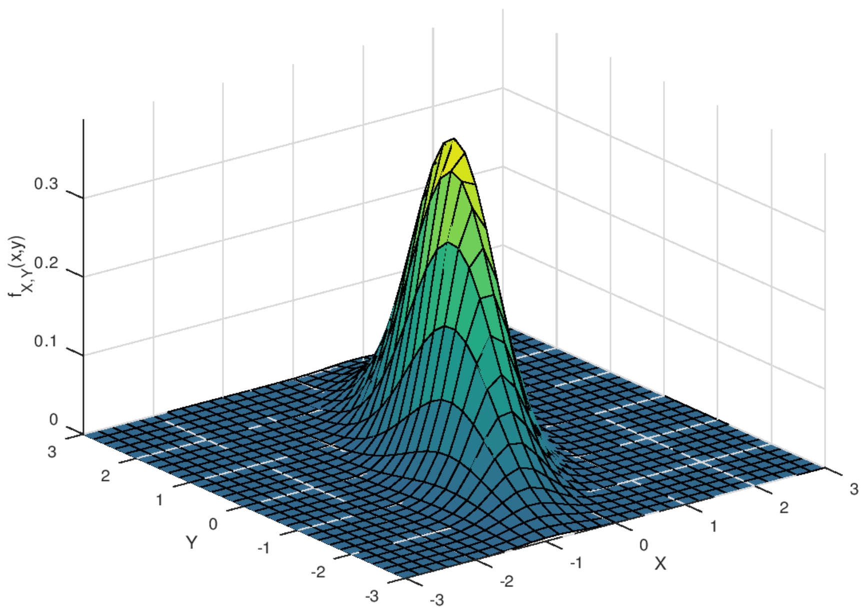
Dependence Structures in Quantitative Finance



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MA489 Report

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1. **Introduction**

In the world of finance understanding the association between random variables is crucial. The ability to identify and accurately model dependencies can provide a significant advantage in an extremely competitive marketplace.

This paper will uncover copula theory and examine the types of dependencies that exist. There will be a brief probability review, concerning some fundamentals that the reader must be familiar with in order to fully understand and appreciate the copula. Once the copula and the dependence measures that it relies upon have been examined, a specific focus will be given to exotic rainbow option pricing, i.e. derivatives with payoffs that are dependent on more than one asset. However, this is just one application. Copulas provide a flexible method in dependence modelling and are used extensively in insurance, risk management and portfolio optimization.

2. **Probability Review**

Before introducing the copula method of dependence modelling we must first review some basic probability concepts. This paper will focus on the bivariate case going forward, however copulas in general may be extended to n dimensions. We will also restrict ourselves to continuous random variables.

Definition 2.1. *Bivariate Cumulative Distribution Function*
The joint distribution, F of random variables X and Y is given by

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

where the R.H.S represents the probability that the random variable X takes on a value less than or equal to x and that Y takes on a value less than or equal y. Every multivariate CDF is;

- *Monotonically non-decreasing for each of its variables*
- *Right continuous in each of its variables*

- $0 \leq F_{X<Y}(x, y) \leq 1$
- $\lim_{x,y\to\infty} F_{X,Y}(x, y) = 1$ *and* $\lim_{x,y\to-\infty} F_{X,Y}(x, y) = 0$

Definition 2.2. *Bivariate Probability Density Function*
The joint density, f of random variables X and Y is given by

$$f_{X,Y}(x, y) = P(X, Y \in A) = \iint_A f_{X,Y}(x, y)dxdy$$

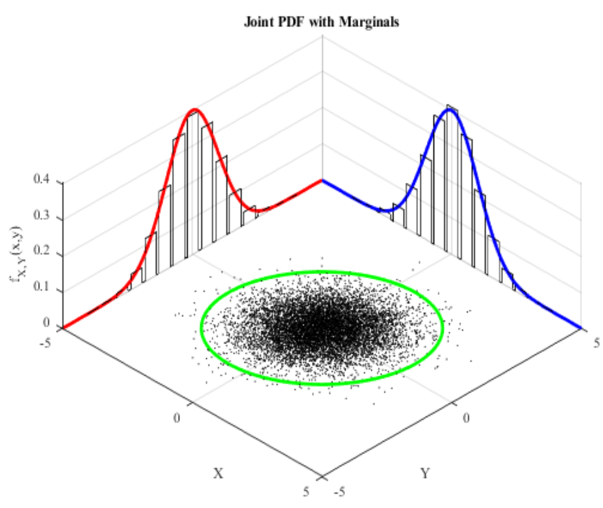
where the double integral evaluates the probability that random variables X and Y are within region A of the sample space. The PDF can be found by differentiating the CDF w.r.t its random variables, i.e.

$$f_{X,Y}(x, y) = \partial_X \partial_Y F(x, y)$$

Definition 2.3. *Marginal Density Function*
The marginal density of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

which is simply concerned only with the probability of X, and no longer considers Y. We may evaluate the marginal of X by fixing the value of Y.



An understanding of the brief probability review conducted will allow for a seamless transition into dependence measures.

3. **Dependence Measures**

There are four well-known desired properties of dependence (Embrechts);

(1) $\delta(X,Y) = \delta(Y,X)$, symmetry condition
(2) -1 $\leq \delta(X,Y) \leq 1$, normalization condition
(3) $\delta(X,Y) = 1$, then (X,Y) are comonotonic
$\delta(X,Y) = -1$, then (X,Y) are countermonotonic
(4) $\delta(T(X),Y) = \begin{cases} \delta(T(X),Y) & \text{if T is increasing} \\ -\delta(T(X),Y) & \text{if T is decreasing} \end{cases}$

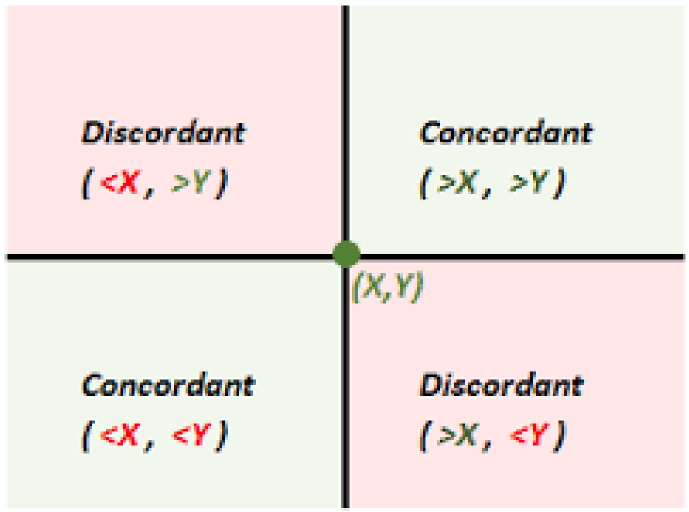
for the dependence measure δ

There are three different dependence measures we will look at; Pearson linear correlation, Spearman's Rank and Kendall's Tau. The latter two are *rank correlation* measures. Before we dig into the abilities, pro's and con's of each measure above we will discuss the concept of rank correlation and concordance.

3.1. **Rank Correlation & Concordance.**

Rank correlation is a statistic that measures an ordinal association - the relationship between rankings of two variables, where a "ranking" is the assignment of the labels "first", "second", "third", and so on to different observations of a particular variable. A rank correlation coefficient measures the degree of similarity between two rankings, and can be used to assess the significance of the relation between them. It's main advantage, over the Pearson correlation, is its ability to model nonlinear dependence and its independence from the marginal distributions which is crucially important when dealing with copulas.

The Kendall's Tau rank correlation uses concordance, which is the 'agreement' between two random variables. In other words, variables are ranked on their degree of concordance where concordant pairs associate high (low) values of RV X with high (low) values of RV Y and discordant pairs associate high (low) values of X with low (high) values of Y. The plot below gives a good representation.



3.2. **Kendall's Tau.** Kendall's Tau measures the difference between probability of concordance and of discordance between random variables.An estimation of Kendall's Tau using concordance is;

$$\tau_K = \frac{(C - D)}{n(n-1)/2},$$

where C is the number of concordant pairs, D is the number of discordant pairs and n is the number observations. Furthermore, each of the copulas we will look at have a unique relationship between Kendall's Tau and the dependence parameter - this will be discussed further in the paper.

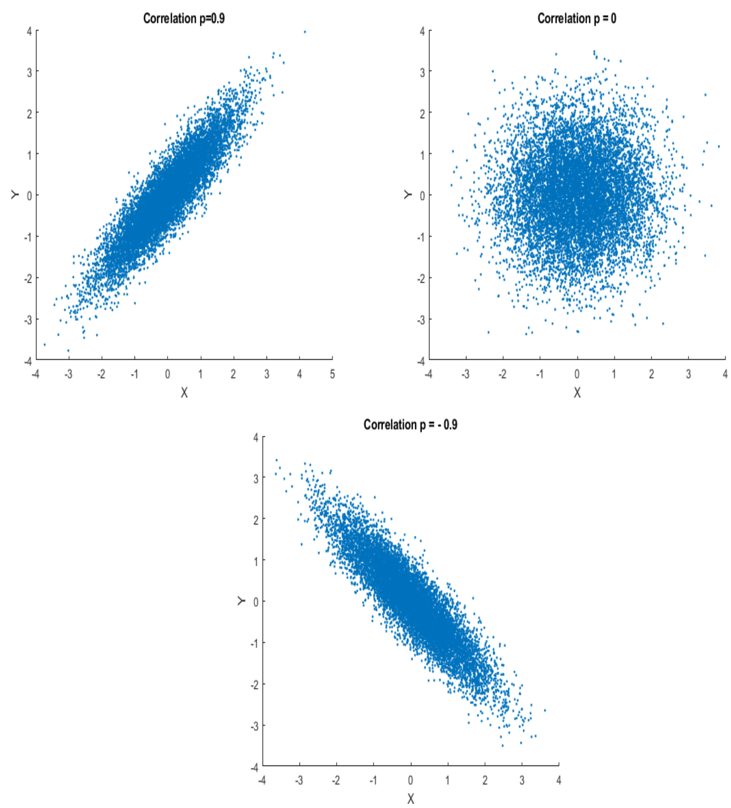
3.3. **Spearman's Rank.** The Spearman's rank is also a rank correlation. It is a non-parametric correlation measure that unlike Pearson correlation, has the ability the capture non-linear dependence. The variables are simply ranked highest to lowest in value, then the difference in the pairs is considered, d, an estimation is;

$$\rho_S = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}.$$

3.4. **Pearson's Linear Correlation.** Last but not least, the most well-know dependence measure, the Pearson linear correlation captures linear association between random variables. However, unlike Kendall's and Spearman's it may not be used for non-linear dependencies. If we consider X $\sim N(0,1)$ and $Y = X^2$, the correlation between X and Y is zero even though they are dependent. Another downfall of Pearson correlation is dependence on the marginal distributions. Clearly from its formula,

$$\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y},$$

there is a direct dependence on the marginals of both X and Y.



4. **Basic Concepts of Copulas**

One method of modelling the dependence between random variables is the concept of copulas. The theory was first introduced in 1959 by Abe Sklar. The way copulas work, is actually described within its name, copula means *a link* or *a tie* in Latin. Copulas are parametrically specified joint distributions generated from given marginals. They have the ability to model all dependence structures and essentially allow us to separate the marginal behaviours and the dependence structure from the joint distribution. This modelling flexibility is the main advantage of copulas and is why it has gained so much traction in quantitative finance.

Definition 4.1. *Copula Function*
A copula is the distribution function of a random vector in $\mathbb{R}^2$ *with Uniform(0,1) marginals, U and V. It is defined as any function* $C : [0,1]^2 \to [0,1]$ *where the distribution and density functions are defined as,*

$$C(u,v) = P(U \leq u, V \leq v),$$
$$c(u,v) = \frac{\partial^2}{\partial u \partial v} C(u,v).$$

It's vital to note that, the copula is *unique* for continuous uniform U and V. The copula function enables us to separate any continuous bivariate CDF into a component which describes the dependence structure and components which describe marginal behaviour only. This is formally written as Sklar's theorem.

Theorem 4.1. *Sklar's Theorem*
Every bivariate cumulative distribution function of random variables X and Y can be expressed in terms of its marginals, $F_X(x) = P(X \leq x)$, $F_Y(y) = P(Y \leq y)$ *and a copula, C. It follows that,*

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y))$$

The density function can be derived by differentiating $F_{X,Y}$ w.r.t X and Y,

$$f_{X,Y}(x,y) = c(F_X(x), F_Y(y)) f_X(x) f_Y(y).$$

where $c(F_X(x), F_Y(y)) = c(u,v)$ is the copula density.

Notice that a crucial step for Sklar's theorem is utilizing the probability integral transformation (P.I.T) to transform the random vector (X,Y) component-wise to standard uniform-(0,1) random variables (U,V). The P.I.T is as follows for $X \to U$. Let $U = F_X(X)$;

$$\begin{aligned} \mathrm{F}_U(u) &= \mathrm{P}(\mathrm{U} \leq u) \\ &= \mathrm{P}(\mathrm{F}_X(X) \leq u) \\ &= \mathrm{P}(\mathrm{X} \leq F_X^{-1}(u)) \\ &= \mathrm{F}_X(F_X^{-1}(u)) \\ &= u \end{aligned} \tag{1}$$

Thus U is uniform and V is proved similarly. In summary, by using the P.I.T along with Sklar's theorem, the copula method can be utilized for any continuous random variables (X,Y) with known distributions.

Beyond its flexibility for modelling dependence, another attractive aspect of copulas is the invariance property. This means that the copula is invariant under monotone transformation, i.e. a strictly increasing (logarithmic) or decreasing transformation can be applied to the marginals without affecting the dependence structure.

There are two main families of copulas, the Archimedean approach and the compounding approach. The latter represent a class of explicit copulas that have become extremely popular due to the ease of their implementation and construction. The Archimedean representation are also defined by one dependence parameter, enabling us to simplify a complex multivariate copula to a univariate function. The remainder of this research paper will be focused on Archimedean and Gaussian copulas.

4.1. **Bounds on Copulas.** Another interesting aspect of the copula function is its bounds. The late Maurice Frechet showed there exist upper and lower bounds for a copula function,

$$max(0, u + v - 1) \leq C(u, v) \leq min(u, v)$$

Which can be more generally thought of as being bounded above by perfect dependence and below perfectly negative dependence. In terms of Pearson linear correlation and Kendall's Tau, we are bounded below by the dependence measures $\rho_p =$-1, $\tau_K =$-1 and above by the measures $\rho_p = 1$, $\tau_K = 1$, since ρ,τ ϵ (-1,1).

This is an idea we will revisit during our Rainbow option pricing application.

5. **Gaussian and Archimedean Copulas**

The discussion about rank correlations now presents a dependence measure that is independent from the marginals. If we revert back to the joint PDF, using Sklar's Theorem, we notice that the copula density, $c(u, v)$, is completely isolated from the marginals, f_x and f_y, hence we require a dependence measure independent from the marginals.

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)\mathbf{c(u,v)}$$

There are several different copulas within the Archimedean family, and each has specific properties that enable it to model certain types of dependence. Each copula has a unique relationship between Kendall's Tau and it's dependence controlling parameter, derived from;

$$\tau(X, Y) = 4 \iint C(u, v)c(u, v)dudv - 1 \tag{2}$$

where C is the CDF and c is the copula PDF. After the appropriate integration, you arrive at a simple relationship between the dependence parameter α and the dependence measure τ.

The following table gives the copula function (CDF) and dependence relationship for the Clayton, Gumbel, Frank and Gaussian copulas.

Family	Bivariate Copula $C(u, v)$	τ
Clayton	$(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$	$\frac{\alpha}{1+\alpha}$, 1 α
Gumbel	$exp-[(-ln(u)^{\alpha}) + (-ln(v)^{\alpha})]^{1/\alpha}$	1 - $\alpha^{-1}, 1 \leq \alpha$
Frank	$\frac{1}{\alpha}ln(1 + \frac{(e^{\alpha u}-1)(e^{\alpha v}-1)}{e^{\alpha}-1})$	1 - $\frac{4}{\alpha}(D_1(-\alpha) - 1)$
Gaussian	$\Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); \rho)$	No closed form

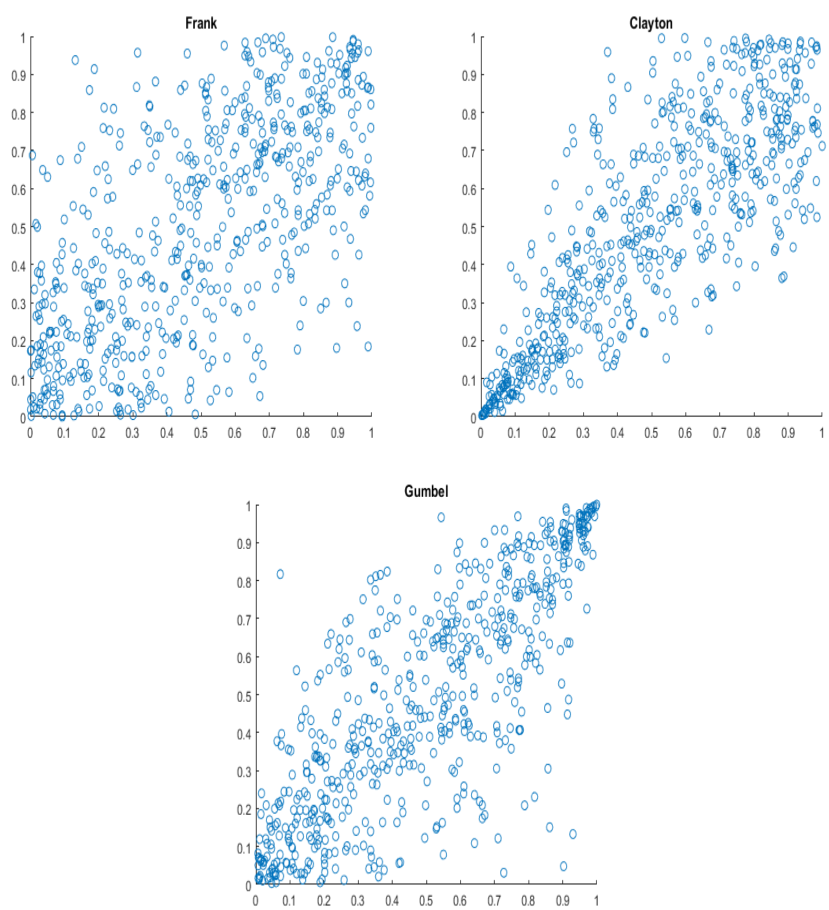
In order to make reasonable and legitimate comparison's between the above copula functions, a level of dependence must be fixed between them, i.e. fix τ_K. This provides a basis on which we can accurately analyze the differences between copula functions.

5.1. **Clayton Copula.** The fact that θ is bounded between one and ∞ implies that the Clayton copula cannot be used to model negative dependence. The Clayton copula allows for the occurrence of extreme downside events and has been successfully applied in many areas of financial risk and pricing. Analyzing the third column, you can see that as $\alpha \to \infty$, $\tau_K \to 1$, and as $\alpha \to 1$, $\tau_K \to 1/3$.

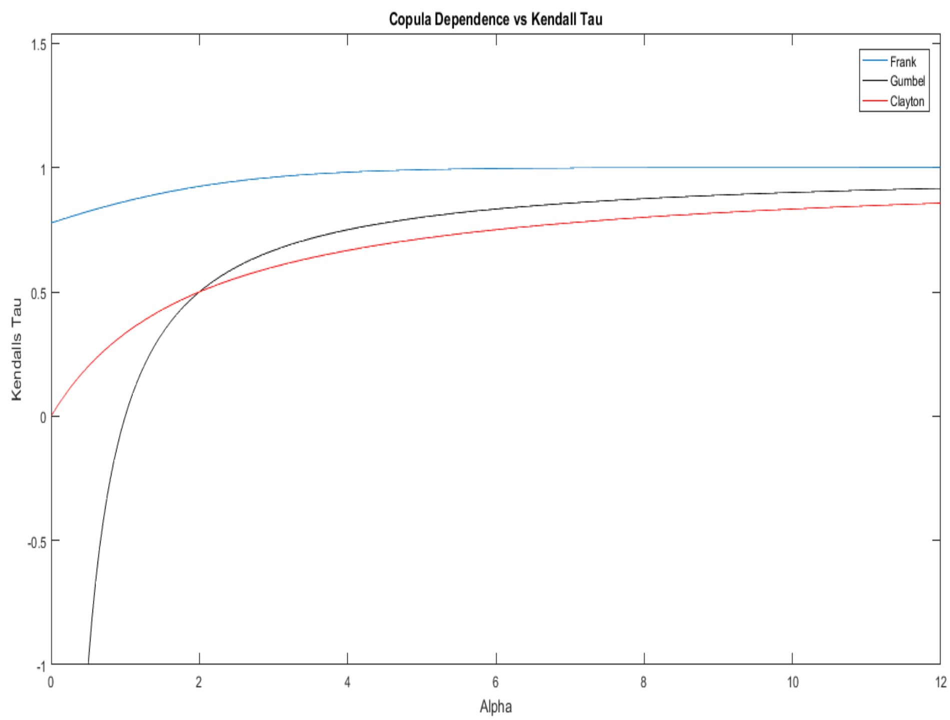
5.2. **Gumbel Copula.** The Gumbel is used primarily to model asymmetric dependence between random variables. It has the ability to model strong upper tail and weak lower tail dependence. Meaning if values are highly correlated at large values but less correlated at smaller values, the Gumbel could be an appropriate choice. Analyzing the third column, you can see that as $\alpha \to \infty$, $\tau_K \to 1$, and as $\alpha \to 1$, $\tau_K \to 0$.

5.3. **Frank Copula.** Unlike, the Clayton or Gumbel, the Frank has the ability to model negative dependence. However for the purposes of this report, in order to effectively compare each copula we will focus on positive dependence. Additionally, the Frank lacks tail dependence on either end of the scatter seen below, indicating it's inability to model 'unlikely' events. As such, the Frank would often be inappropriate choice when working in financial risk management. Similarly to both Clayton and Gumbel, as $\alpha \to \infty$, $\tau_K \to 1$.

5.4. **Gaussian Copula.** where Φ is the standard normal CDF, Φ_2 is the joint normal CDF and ρ is Pearson linear correlation. The Gaussian is the most commonly used as it is easy to implement in a wide variety of problems. However, like the Frank copula it fails to model dependence within the tails. I was also unable to capture the relationship between Kendall's Tau and it's dependence parameter using Equation (2) above. As such it was difficult to compare the Gaussian with Clayton, Gumbel and Frank since there was no ability to fix the dependence level, τ_K and analyze differences. Instead, real data was taken from Yahoo Finance regarding two stocks and the Pearson linear correlation was used as the Gaussian dependence parameter proxy.



For each of these scatterplot's, a level of dependence was fixed between RV's X and Y, $\tau_K = 0.6$. It is easy to see the differences between the copula functions, particularly in the tail behaviours. Another intriguing plot shows the relationships between the dependence controlling parameter, α and the measure Kendall's Tau. We will observe only the positive dependence.



It's clear in the figure above, that as $\alpha \to \infty$, $\tau_K \to 1$. This connects with the Frechet proof on bounds, since each copula converges to perfect dependence, i.e. $\tau_K = 1$, implying each copula is bounded above by perfect dependence between its RVs.

5.5. **Copula Selection.** The focus of this paper is to gain a better understanding of the copula theory and its applications to finance. Hence, proper copula selection and copula fitting techniques are not the main focus or concern of this report. However, in industry, proper copula selection is of the utmost importance as it can either perfect or cripple your model.

Suppose a risk analyst at a large financial institution was responsible for modelling loss-given-default metrics. From the ever so brief review above, you would know immediately that the Gaussian or Frank would be ill-suited, as they lack the necessary strong positive tail dependence required. This was a major issue when copulas came into widespread use on Wall Street as Gaussian copulas had been inappropriately applied to the pricing of CDO's (Collateralized Debt Obligation) and other structured credit products. This led to the copula and it's 'mainstream' advocate, Dr. David X. Li from the University of Waterloo, to be severely scrutinized and partly blamed for the 2008-09 global economic crisis.

This vast and somewhat unfortunate past surrounding copulas in finance reminds us that copula selection is crucial and must be done accurately based on the data and variables specific to your model/situation.

6. **Applications of Copulas**

Copula theory in finance originated in the mid 1990s, and became very popular in areas like portfolio optimization, risk management and pricing. As previously stated, it's misuse in the mid 2000s gained lots of attention and scrutiny. Although, if used correctly, by analysts who truly understand the concepts, the copula is an invaluable tool in the world of quantitative finance due to its flexibility in dependence modelling.

6.1. **Common Uses in Finance & Insurance.** The insurance and finance industries have a constant need to accurately model dependence structures between random variables for pricing and risk management purposes.

A simple and straightforward example in insurance is a Joint Life Survivor Insurance product. This contract pays a death benefit, B, to the recipient upon the death of the second spouse (both spouses have passed). In an extremely simple pricing model, we have the dependence structure contained within the RV T which represents the death time (payout time).

$$V = \mathbb{E}[Be^{-rT}],$$

where T is $max(T_1, T_2)$ and T_1, T_2 are the death times of each spouse respectively.

Another common practice is pricing CDO products and working with portfolio optimization. The copula is often used due to its flexibility but also because many copula functions can be extended to the n-th dimension (n number of assets). These applications can be extremely complex and will not be covered in this report, but are very interesting and I hope to study their use in the financial system more during my post-undergrad studies.

Finally, the application we will be looking most closely at is derivative pricing. But before we get into more complicated exotic pricing we will define a vanilla option.

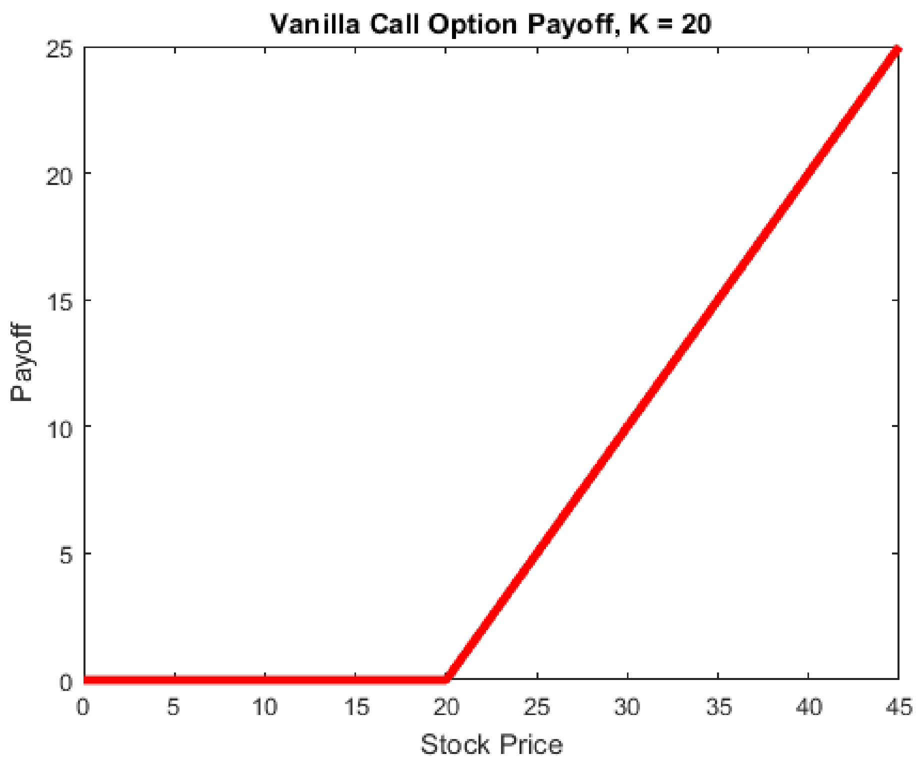
6.2. **Option Pricing.** A vanilla option is the most basic option you can have. The holder of said option has the right but *not* the obligation to buy or sell an underlying asset at a predetermined price and maturity date. Specifically, a European vanilla call option gives that holder the right to purchase the asset at some strike price, $\$K$ at some maturity date in the future. It's payoff function would be,

$$\Lambda(S_T) = max(0, S_T - K),$$

where S_T is the asset price at time T, K is the strike. It's important to notice that if we are sitting at time t, t $\leq$ T, then S_T is a RV of the asset price. Hence our valuation/pricing formula must take it's discounted expected value,

$$
\begin{aligned}
V_t &= e^{-r(T-t)}\,\mathbb{E}[\Lambda(S_T)|S_t = S] \\
&= e^{-r(T-t)}\,\mathbb{E}_{t\,,S}[max(0, S_T - K)]
\end{aligned}
$$

where we discount the expected value of the payoff conditioned on the stock's value at time t. A simple plot shows the payoff structure of the vanilla call option with strike, $K = 20$,



Now that we understand the fundamentals of option pricing, we can extend this case to consider options whose payoffs depend on more than one underlying asset. These are exotic derivatives called Rainbow Options. For this presentation we will hold our application to two-dimensions but it is important to note that this method could be extended well beyond 2 assets.

We start by defining a Call-on-Max rainbow option. We use this specific option because we can take our knowledge of the vanilla call and simply add one more asset to the payoff function. Hence, we have the new Call-on-Max payoff,

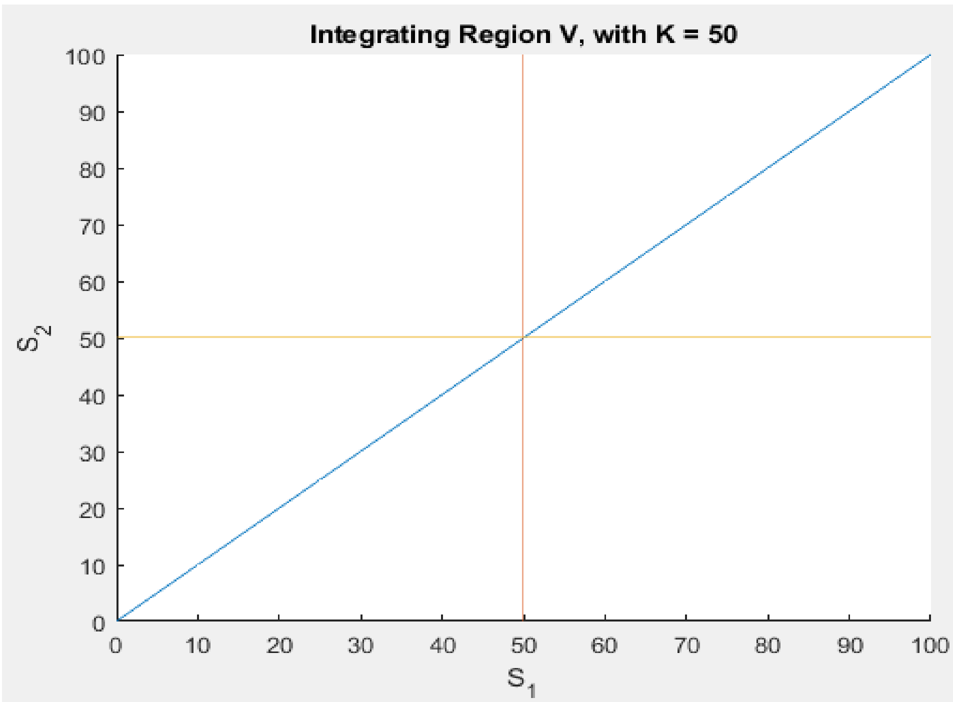
$$\Lambda_{max}(S_T^1, S_T^2) = max(max(S_T^1, S_T^2) - K, 0)$$

where S_T^1 and S_T^2 are time T RVs which represent log-normally distributed stock prices, and K is the given strike price. Just as with any other derivative, including the vanilla call above, the price is simply the risk-neutral discounted value of the expected payoff.

$$\begin{aligned} V_t &= e^{-r(T-t)}\,\mathbb{E}[\Lambda_{max}(S_T^1, S_T^2)|S_t^1 = S^1, S_t^2 = S^2] \\ &= e^{-r(T-t)}\,\mathbb{E}_{t\ ,S1,S2}[max(max(S_T^1, S_T^2) - K, 0)], \end{aligned}$$

where we are conditioning on the values of S^1 and S^2 at time t and discounting $T - t$ by the discount rate r. It's important to note that the dependence structure we're interested in is between S_1 and S_2 contained within the payoff function. The relationship between stock 1 and stock 2 will have a direct and unequivocal impact on the price, or "fair value" of this option. We will consider one case where the stocks are positively correlated and one where they are negatively correlated. If the stocks are negatively correlated, then as S^1 increases, S^2 decreases and vice versa while if they are positively correlated they will trend in the same direction. Since the payoff of a Call-on-Max considers the maximum value between S^1 and S^2, we only need one of the stocks to increase by time T. Thus, as this would happen more frequently with negative correlated stocks - the price of a Max-on-Call will be higher for negatively correlated stocks. This will be seen in our later analysis.

6.3. **Analytical Solution.** To go about solving this Max-on-Call option price, we will use two methods. The analytical approach which provides the exact solution by discounting the integration the joint distribution, $f_{S1,S2}(s^1, s^2)$ and the payoff function. As such, we need to consider our region of integration before we begin to tackle the integrand. Our integration region, Q will allow us to simplify our payoff function and thus our integrand. The region is broken into quadrants pictured below where either both S^1 and S^2 are greater than K or just one of them is, (essentially eliminating 0 to 50 on both the S^1 and S^2-axis,



Additionally, if we recall Sklar's theorem, we are able to dissect the joint distribution, $f_{S1,S2}(s^1, s^2)$ into the marginals, $f_{Si}(s^i)$ for i = 1,2 and a copula density $c(U, V)$, i.e.

$$f_{S1,S2}(s^1, s^2) = f_{S1}(s^1) f_{S2}(s^2) c(u, v)$$

where $U = F(S_1(s_1)$ and $V = F(S_2(s_2)$.

Hence, if we utilize our region and the copula joint distribution above we arrive at the pricing function,

$$V_t \ = e^{-r(T-t)}\, \mathbb{E}_{t\ S1S2}[\Lambda_{max}(S_T^1, S_T^2)]$$
$$= e^{-r(T-t)} \iint_Q max(max(S_1, S_2) - K, 0) f_{S1,S2}(s_1, s_2)\, dS_1\, dS_2$$
$$= e^{-rT} \iint_Q max(max(S_1, S_2) - K, 0) f_{S1}(s^1) f_{S2}(s^2) c(u, v)\, dS_1\, dS_2$$

If we consider the first quadrant in the top-left corner, where $S_1 \leq K$ and $S_2 \geq K$, we price it as follows,

$$V_t = e^{-r(T-t)} \int_K^\infty \int_0^K (S_2 - K) f_{S1}(s^1) f_{S2}(s^2) c(u,v)\, dS_1\, dS_2$$

If we were to conduct the same process for each of the other three regions, then sum them all together, we would arrive at the exact price for the Max-on-Call option.

6.4. **Monte Carlo Simulation.** While the analytical solution should be used as often as possible, it can sometimes lead to extremely complex - even impossible solutions. As such, another common pricing technique is by Monte Carlo simulation. Monte Carlo methods are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. Essentially, we are able to generate RVs from the respective copula functions then sub them into the stock price processes, and average our results.

To start, our goal is to construct an algorithm to generate dependent RVs X, Y with the known copula distribution. To do this we will utilize a method of inverse transform sampling. Our procedure is as follows,

(1) Generate U, V independent uniform-(0,1) RVs.
(2) Set X = $F^{-1}_{S1}(U)$, where $F^{-1}_{S1}(.)$ is the inverse CDF of S_1
(3) Given X, use V and the conditional distribution of Y given X, i.e. $F_{S2}(Y|X)$, to solve for Y
(4) Output X and Y.

The above process generates a dependent set of points X and Y from our desired copula with a fixed level of τ_K. We can use these points to simulate correlated stock prices by making use of the strong solution of a geometric brownian motion stock price process,

$$S_T^1 = S_t^1 e^{(r-0.5\sigma_1^2)(T-t)+\sigma_1\sqrt{T-t}X}$$

$$S_T^2 = S_t^2 e^{(r-0.5\sigma_2^2)(T-t)+\sigma_2\sqrt{T-t}Y}$$

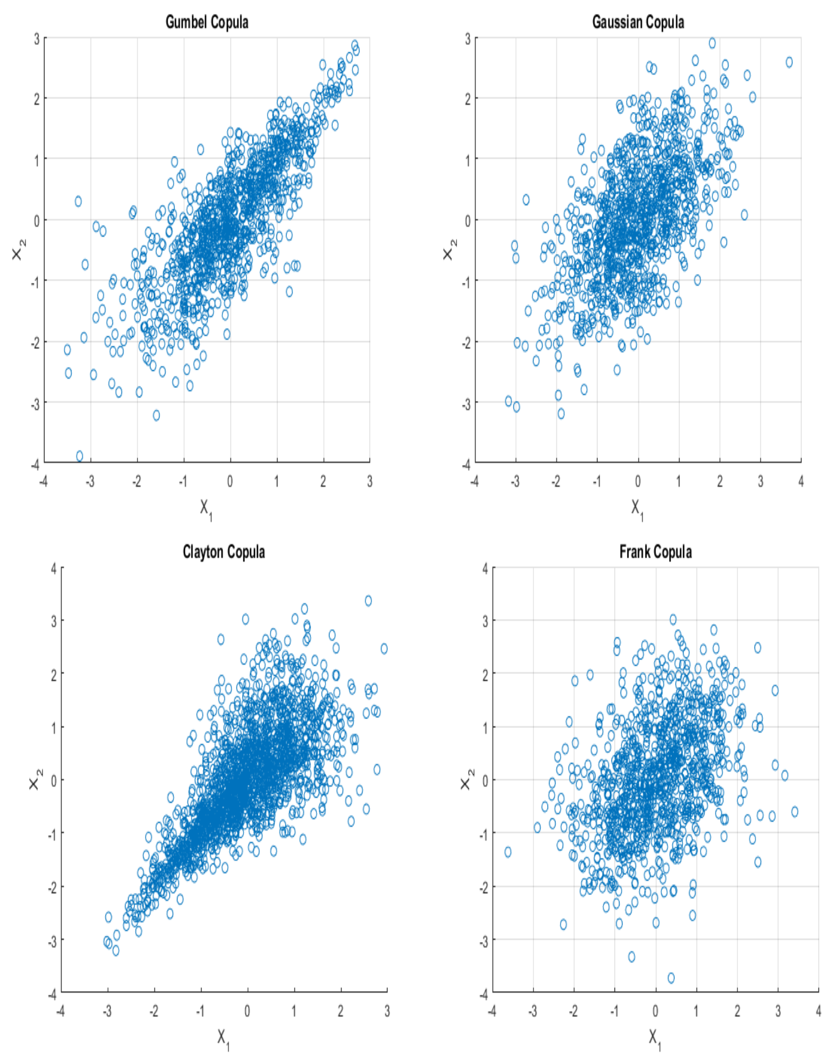
where X and Y are the correlated RVs generated from the above algorithm. By subbing these dependent RVs into S_T^1 and S_T^2 we obtain correlated stock prices at time T. By Monte Carlo, we can obtain a Max-on-Call estimator by averaging a large number of simulations,

$$\tilde{V} = \frac{e^{-r(T-t)}}{n_{sim}} \sum_{j=1}^{n_{sim}} (max(S_T^{1,j}, S_T^{2,j}) - K)^+$$

where $S_T^{i,j} = S_t^i e^{(r-0.5\sigma_i^2)(T-t)+\sigma_i\sqrt{T-t}X_j}$ for $j = 1, ..., n_{sim}$ and $i = 1, 2$. For a large number of simulations, this Monte Carlo estimator will utilize randomness to efficiently price the Max-on-Call option.

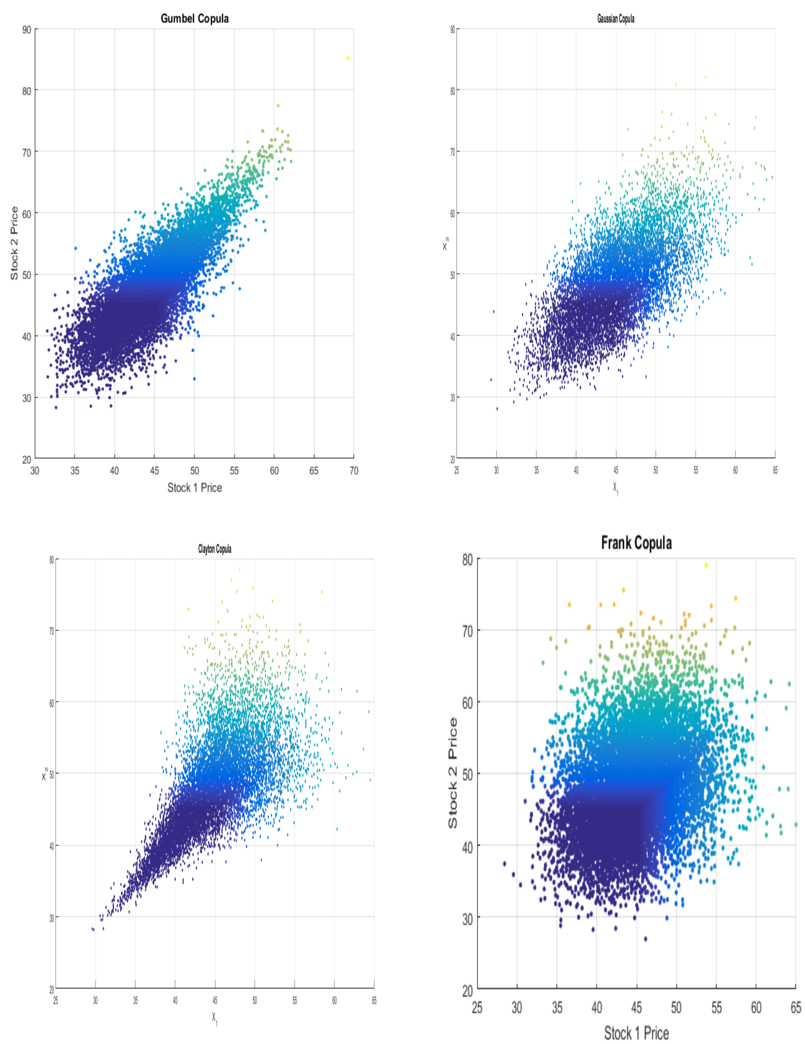
6.5. **Max-on-Call Simulation.** In order to effectively compare the Gaussian with Clayton, Gumbel and Frank, two stocks were selected at random from Yahoo Finance and Kendall's Tau and Pearson linear correlation were estimated. This provided a basis on which comparisons could be made.

From the Bivariate copula table in *Section 5*, τ_K was used to solve for the dependence parameter, α. Following which we applied the Monte Carlo algorithm to obtain dependent RVs X and Y from each copula. With a $\tau_K = 0.56$ and $\rho = 0.61$ we generated the following scatters,



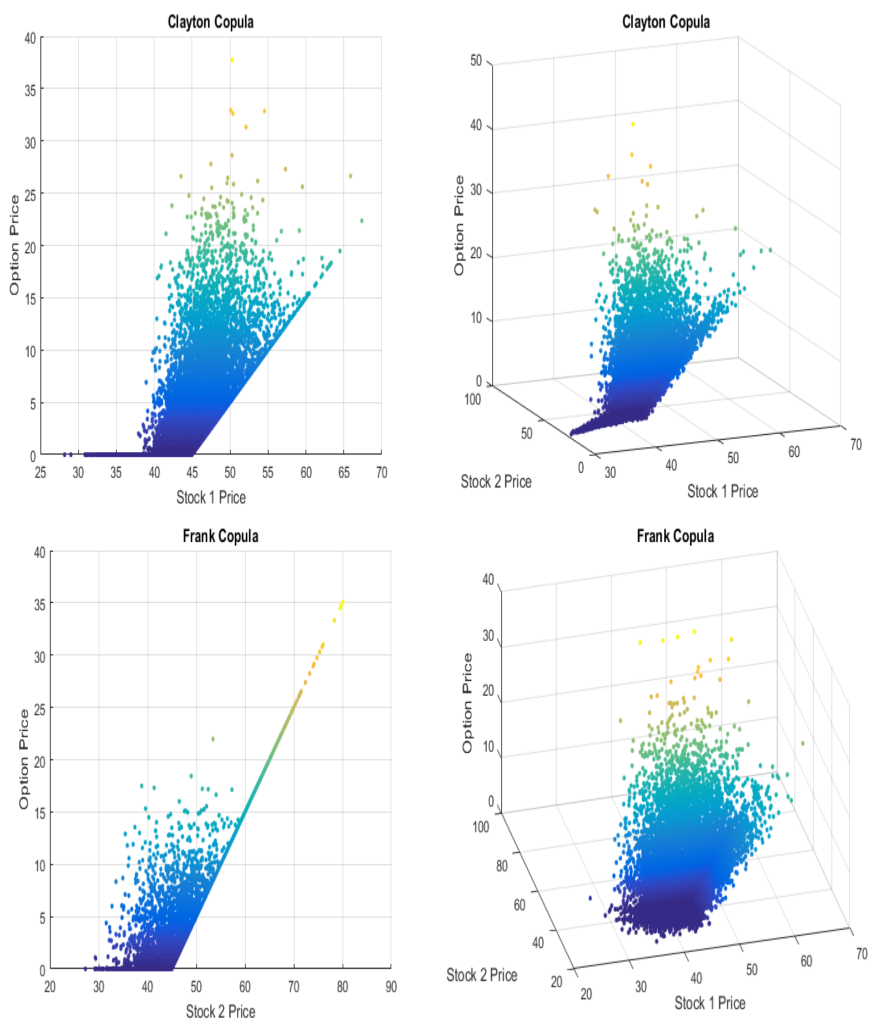
As you can see there are distinct differences between the RVs generated. The Gaussian and Frank hold an elliptical shape with minimal tail dependence while the Clayton and Gumbel each display strong positive and negative tail dependence. By mapping these RVs to stock price RVs, S_T^1 and S_T^2, using the Monte Carlo estimator we can achieve similar scatterplot's of stock prices. The market data used was;

Initial stock prices and strike : $S_1 = 42$, $S_2 = 45$, $K = 45$
Volatilities: $\sigma_1 = 0.075$, $\sigma_2 = 0.1$
Risk-free rate and time-to-maturity (T-t): $r = 0.03$ and $T - t = 1$
Dependence: $\tau_K = 0.56$ and $\rho = 0.61$



These scatter's not only provide an XY-perspective in which we see the n_{sim} stock price simulations but Z-axis adds the heat map which indicates the payoff of the option - where dark blue represents the option expiring worthless. As such, by our integrating region and the nature of the payoff function, it makes sense that higher values of S_1 and S_2 lead to larger option payoffs.

The 3-dimensional nature of this scatter also gives us unique images where we can appreciate the Monte Carlo simulation better where perceiving the scatter from the XZ-perspective creates a plot similar to that of a vanilla call.

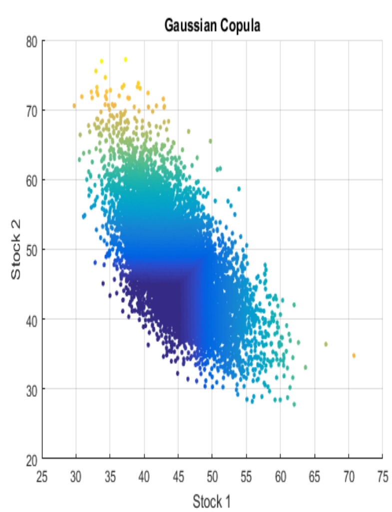


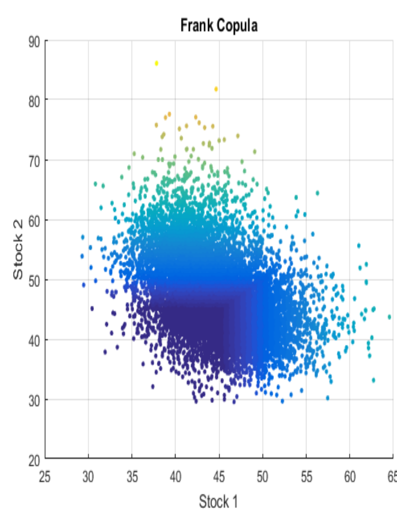
The Monte Carlo estimator prices the Max-on-Call numerically and resulted in the following prices for each copula given 10000 simulations the market data listed above;

Copula	Option Price ($)
Clayton	4.49
Gumbel	4.29
Frank	4.74
Gaussian	4.68

Clearly, as expected from the heat maps - the Gaussian and Frank copulas result in the highest option payoffs since the probability of one of the stock prices being greater than K is greater than with Clayton or Gumbel. As previously touched upon when we first introduced the Max-on-Call, the option price would be greater for negative dependence between stocks 1 and 2. However for the purpose of comparing all four copulas we fixed a positive dependence.

By *Section 5*, both the Gaussian (using Pearson correlation) and Frank have the ability to model negative dependence. As such, let's now take $\tau_K = -0.56$ and $\rho = -0.61$ and conduct similar analysis.



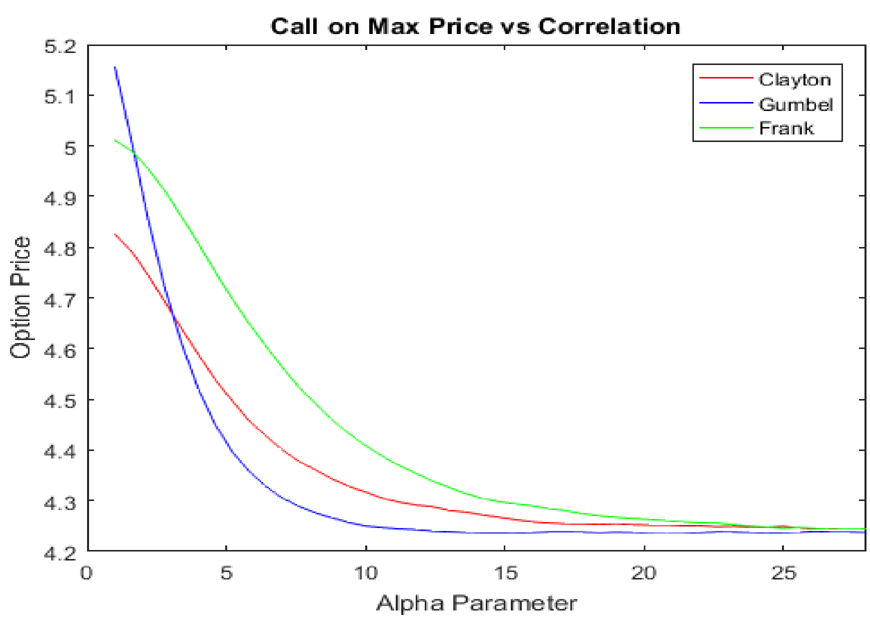


With a quick comparison to the Gaussian and Frank with positive dependence, its clear that the option pays of more frequently due to the increased points of a lighter blue shade. This fact is confirmed in the Monte Carlo estimate of the option price with negative dependence;

Copula	Option Price ($)
Frank	5.83
Gaussian	5.69

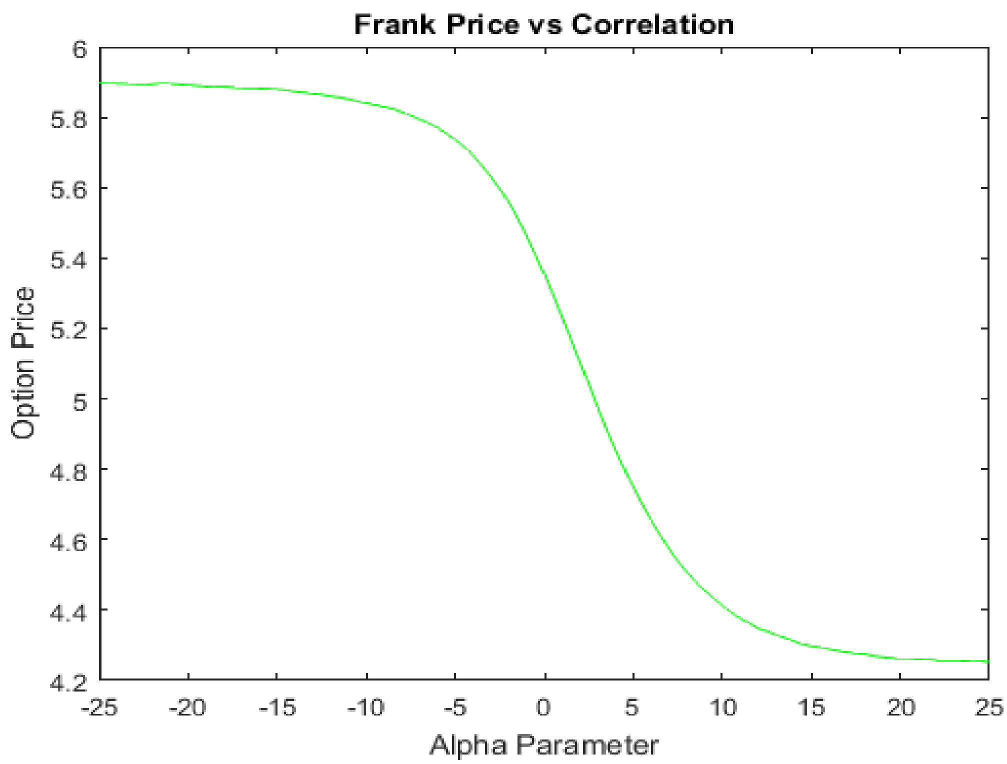
6.6. **Option Price vs Dependence.** Another interesting analysis to consider was plotting the option price as a function of Kendall's Tau through the dependence controlling parameter. Recall that as $\alpha \to \infty$, $\tau_K \to 1$ which is perfect dependence.

Each copula has arrived at unique option prices solely due to the fact that the dependence structures are different, since the marginals have remained constant. It's interesting to notice that if we price the Max-on-Call's as a function of α (which represents τ_K), the Clayton, Gumbel and Frank converge to approx. $ 4.23 as the stocks become 'perfectly dependent'.



This implies that the option price is being bounded below due to the fact that the copula functions are bounded above by perfect dependence (Frechet), as was discussed earlier in the report.

Similarly, if we plot the option priced by Frank copula on its full range of dependence, i.e. τ_K (-1,1), we uncover a lower bound on the copula that produces an upper bound on the Max-on-Call option price. This reflects the option price as $\alpha \to -\infty$, $\tau_K \to -1$, or the price under 'perfect negative dependence' between the stocks.



As noted earlier in the paper, this plot shows that the price of a Max-on-Call increases as correlation decreases.

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