Exercisesheet No.3

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Ex.1

a) In the card scenario an atomic event is the drawing of one specific card. As the deck consists of 52 cards we have 52 atomic events.

b) The probability of drawing one specific card is 1 divided by the amount of cards, thus:

$$\frac{1}{52}$$

c) As there are as many black cards as there are red cards, 26 out of the total of 52 cards are black. Hence the probability is:

$$\frac{26}{52} = \frac{1}{2}$$

Ex.2

a) $\binom{52}{5} = 2598960$

b) $\frac{1}{\binom{52}{5}} = \frac{1}{2598960}$

c)

i) There are four Royal Straigth Flushes (Heart, Spades, Clubs, Diamonds). Thus, the answer is: $4 * \frac{1}{2598960}$

ii) There are 13 possibilities of a Four of a kind. Since one card does not matter, every remaining card is fine. $\frac{13*(52-4)}{2598960}$

Ex.3

a)
i) $P(\neg a) = P(\Omega \setminus a) = P(\Omega) - P(a) = 1 - P(a)$

ii) $P(a \wedge \neg b) = P(\Omega \setminus a) \cap P(\Omega \setminus \neg b)$
$= P(a) \cap P(\neg b) = P(a) \cap (P(\Omega) - P(b)) = (P(\Omega) \cap P(a)) \setminus (P(a) \cap P(b))$
$= P(a) \setminus (P(a) \cap P(b)) = P(a) - P(a \wedge b)$

b) If a and b are disjoint then: $P(a \cup b) = P(a) + P(b) = \frac{1}{3} + \frac{5}{6} > 1$
This cannot be the case since $P(\Omega) = 1$. Thus, a and be are not disjoint.

Ex.4

a)
i) $P(a \wedge \neg b) = P(a) - P(a \wedge b)$ (see Ex. 3a)
Since a and b are disjoint we can write:
$= P(a) - P(a) \cdot P(b) = P(a) \cdot (1 - P(b)) = P(a) \cdot P(\neg b)$

ii) $P(\neg a \wedge \neg b) = P(\neg b \wedge \neg a) = P(\neg b) - P(a \wedge \neg b)$
As we already proved Ex.4 a)i) we know that $P(a)$ and $P(\neg b)$ are independent and we can substitute them in the formula:
$= P(\neg b) - P(a) \cdot P(\neg b) = P(\neg b) \cdot (1 - P(a)) = P(\neg b) \cdot P(\neg a)$

b)

i) At first Alice has the probability of $\frac{1}{2}$ to win. When Alice does not win, it is Rob's turn. When he loses (also with P = $\frac{1}{2}$), Alice has again the possiblity to win with $\frac{1}{2}$ resulting in $\frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2}$. The third step is that, Alice does not win in the first round, Rob does not win in his first try, Alice does not win on her second try, Rob does not win on his second try and Alice wins on her third try. This results in $\frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2}$. This goes on forever. This can be described with the following sum: $\sum_{i=0}^{\infty}(\frac{1}{2^{2i+1}})$. This sum converges to the probability P(AliceWins) $= \frac{2}{3}$ - shown as follows by making use of the geometric series: $\sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} = \sum_{i=0}^{\infty} \frac{1}{2^{2i}*2} = \sum_{i=0}^{\infty} \frac{1}{2}\frac{1}{4^i} = \sum_{i=0}^{\infty} \frac{1}{2}(\frac{1}{4})^i = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{4}{6} = \frac{2}{3}$

ii) P(head) =: p; P(AliceWins) $= p + (1-p)*(1-p)*p + (1-p)*(1-p)*(1-p)*(1-p)*p + \ldots$ We can express this by the following formula: $\sum_{i=0}^{\infty} p*(1-p)^{2i} = \frac{p}{1-(1-p)^2} = \frac{p}{1-(1-2p+p^2)} = \frac{p}{2p-p^2} = \frac{1}{2-p}$ The first equation is by the geometric series as $(1-p)^{2i}$ is smaller than 1 for any $p \in (0,1]$. The second equation is by the binomial theorem.

iii) Since it holds that $P(FirstWins) = \frac{1}{2-p}$, where $p \in (0,1]$ it follows that $P(FirstWins) \in (\frac{1}{2}, 1]$ and by the rules of the game $P(SecondWins) = 1 - P(FirstWins) \in [0, \frac{1}{2})$. To put it in other words, the chance to win if we flip first is greater than the chance to win if we flip second for every $p \in (0,1]$. Thus we would flip first.

Ex.5

a) P(toothache)=$0.108 + 0.012 + 0.016 + 0.064 = \frac{1}{5}$
b) P(catch)=$0.108 + 0.016 + 0.072 + 0.144 = 0.34$
c) P(cavity|catch)=$0.108 + 0.072 = 0.18$
d) P(toothache $\vee$ catch) $= 0.108+0.016+0.012+0.064+0.072+0.144 = 0.416$
P(cavity|toothache $\vee$ catch)=$(0.108 + 0.012 + 0.072)/0.416 = 0.0.4615$

Ex.6

Given:
v: (virus) present
p: prognosis
A: $P_A(p|v) = 0.95; P_A(p|\neg v) = 0.1$
B: $P_B(p|v) = 0.9; P_B(p|\neg v) = 0.05$
$P(v) = 0.01$

We can use Bayes Theorem to transform the conditional probability:

$$P_A(p|v) = \frac{P_A(v|p) * P_A(p)}{P(v)}$$

We need to calculate $P_A(r)$. We can do this by enumeration:

$$P_A(p) = P_A(p|v) \cdot P(v) + P_A(p|\neg v) \cdot P(\neg v) = 0.1085$$

Thus we can transform the formula above to determine $P(v|p)$:

$$P_A(v|p) = \frac{P_A(p|v) \cdot P(v)}{P_A(p)} = \frac{0.95 \cdot 0.01}{0.1085} = 0.08755760369$$

The same approach can be done with procedure B:

$$P_B(p) = P_B(p|v) \cdot P(v) + P_B(p|\neg v) \cdot P(\neg v) = 0.0585$$

$$P_B(v|p) = \frac{P_B(p|v) \cdot P(v)}{P_B(p)} = \frac{0.9 \cdot 0.01}{0.0585} = 0.1538461538$$

Test B is more indicative as the probability of having the virus given a positive result is nearly 2 times the probability of test A.

Ex.7

First we setup the basic formula:

$$P(B|j,m) = \alpha P(B) \cdot \sum_e P(e) \cdot \sum_a P(a|B,e) \cdot P(j|a) \cdot P(m|a)$$

Afterwards the probabilities are replaces with factors:

$$P(B|j,m) = \alpha f1(B) \times \sum_e f2(E) \times \sum_a f3(A,B,E) \times f4(A) \times f5(A)$$

We can sum out f3, f4 and f5 as they all share A. First we combine f4 and f5 (and call it f6):

$$f4(A) = \begin{pmatrix} P(j|a) \\ P(j|\neg a) \end{pmatrix} = \begin{pmatrix} 0.9 \\ 0.05 \end{pmatrix}$$

$$f5(A) = \begin{pmatrix} P(m|a) \\ P(m|\neg a) \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.01 \end{pmatrix}$$

$$f6(A) = \begin{pmatrix} P(j|a) * P(m|a) \\ P(j|\neg a) * P(m|\neg a) \end{pmatrix} = \begin{pmatrix} 0.9 \cdot 0.7 \\ 0.05 \cdot 0.01 \end{pmatrix} = \begin{pmatrix} 0.63 \\ 0.0005 \end{pmatrix}$$

Then we can sum out A in f3 and combine it with f6. We call this factor f7:

$$f7(B,E) = f3(a,B,E) * 0.63 + f3(\neg a,B,E) * 0.0005$$

$$f3(a,B,E) = \begin{pmatrix} P(a|b,e) & P(a|b,\neg e) \\ P(a|\neg b,e) & P(a|\neg b,\neg e) \end{pmatrix} \cdot 0.63 = \begin{pmatrix} 0.95 & 0.94 \\ 0.29 & 0.001 \end{pmatrix} \cdot 0.63 = \begin{pmatrix} 0,5985 & 0,5922 \\ 0,1827 & 0.00063 \end{pmatrix}$$

$$f3(\neg a,B,E) = \begin{pmatrix} P(\neg a|b,e) & P(\neg a|b,\neg e) \\ P(\neg a|\neg b,e) & P(\neg a|\neg b,\neg e) \end{pmatrix} \cdot 0.0005 = \begin{pmatrix} 0.05 & 0.06 \\ 0.71 & 0.999 \end{pmatrix} \cdot 0.0005$$

$$= \begin{pmatrix} 0.000025 & 0.00003 \\ 0.000355 & 0.0004995 \end{pmatrix}$$

$$f7(B,E) = \begin{pmatrix} 0.5985 & 0.5922 \\ 0.1827 & 0.00063 \end{pmatrix} + \begin{pmatrix} 0.000025 & 0.00003 \\ 0.000355 & 0.0004995 \end{pmatrix} = \begin{pmatrix} 0.598525 & 0.592226 \\ 0.183055 & 0.0011295 \end{pmatrix}$$

In the next step we can combine f2 and f7 by summing out E. This factor will be called f8:

$$f8(B) = f7(e,B) + f7(\neg e,B)$$

$$f7(e,B) = \begin{pmatrix} 0.598525 \\ 0.183055 \end{pmatrix} \cdot 0.002 = \begin{pmatrix} 0.00119705 \\ 0.00036611 \end{pmatrix}$$

$$f7(\neg e,B) = \begin{pmatrix} 0.592226 \\ 0.0011295 \end{pmatrix} \cdot 0.998 = \begin{pmatrix} 0.591041548 \\ 0.001127241 \end{pmatrix}$$

$$f8(B) = \begin{pmatrix} 0.00119705 \\ 0.00036611 \end{pmatrix} + \begin{pmatrix} 0.591041548 \\ 0.001127241 \end{pmatrix} = \begin{pmatrix} 0.592238598 \\ 0.001493351 \end{pmatrix}$$

Finally we can combine f1 and f2:

$$f9(B) = \begin{pmatrix} 0.592238598 \cdot P(b) \\ 0.001493351 \cdot P(\neg b) \end{pmatrix} = \begin{pmatrix} 0.000592238598 \\ 0.001491857649 \end{pmatrix}$$

Now we only have to calculate α by dividing 1 by the sum of the probabilities of f9(B):

$$\alpha = 1 \div (0.000592238598 + 0.001491857649) = 479.82428904$$

By normalizing with α we get the final result:

$$f(B) = \alpha \cdot f9(B) = \begin{pmatrix} 0.2841704642 \\ 0.7158295358 \end{pmatrix}$$