Aug 25th

Functions.

Def A <u>relation</u> or <u>mapping</u> from a set A to a set B is a subset of $A \times B$, usually denoted by R

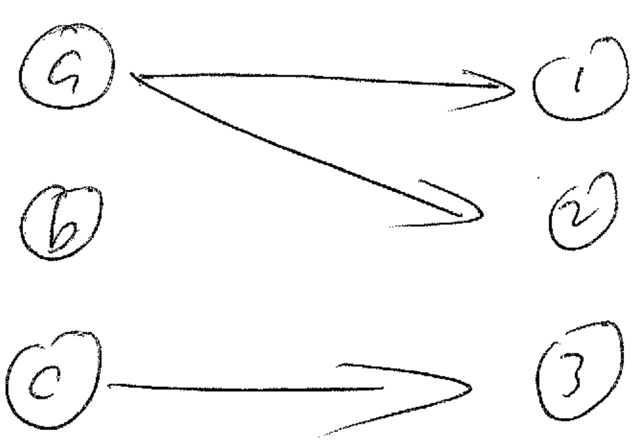
Recall $\underline{A \times B} = \{\underline{(a,b)} \mid a \in A, b \in B\}$

All possible relations from A to B is $2^{A \times B}$

Examples:

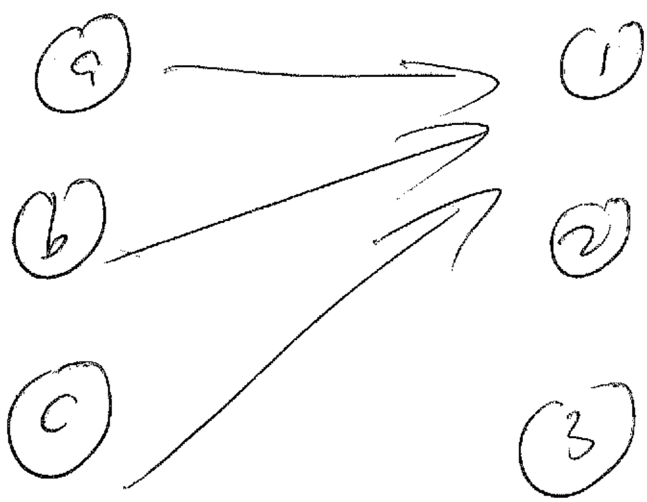
$A = \{a, b, c\}$ $\qquad B = \{1, 2, 3\}$

$R = \{ (a,1), (a,2), (c,3) \}$



A function f from A to B is a special relation such that for each element $a \in A$, there is one and only one element $b \in B$ ~~that~~ is related to a by f.

Ex.



R from A to A is called a relation on A

$R \subseteq A \times A$

$A = \{a, b, c\}$

$R = \{(a, b), (a, c), (b, c), (a, a)\}$

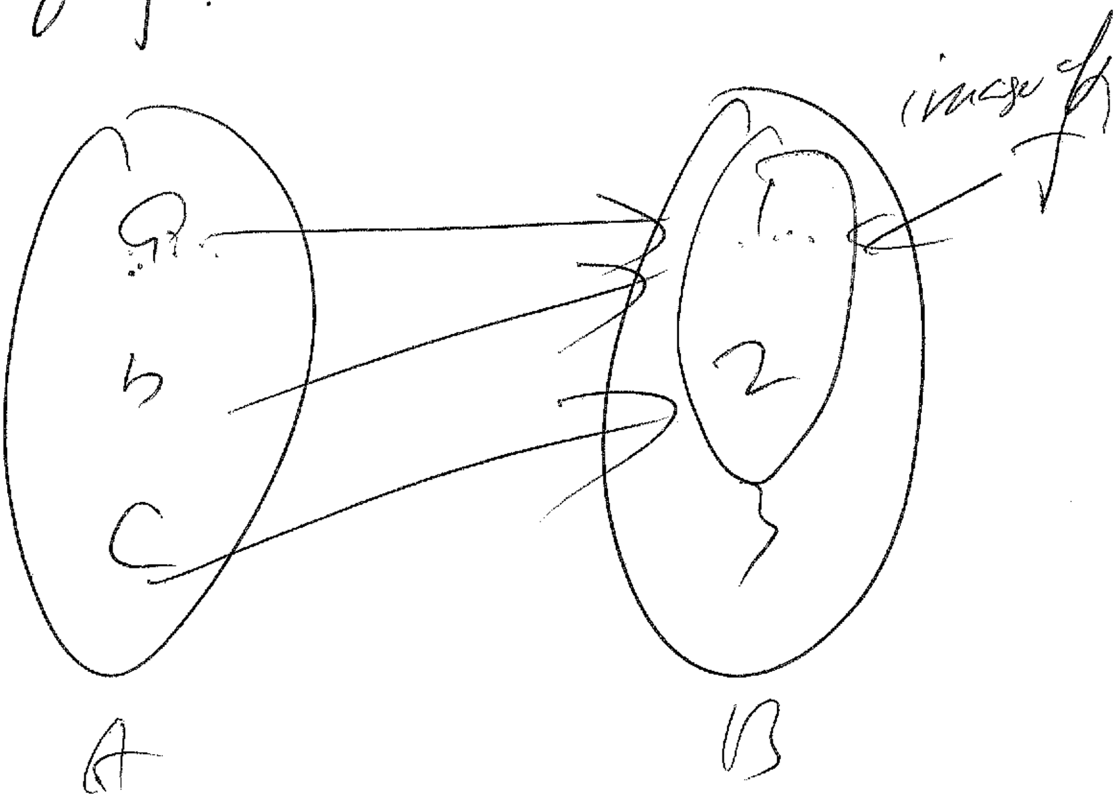
Let $f: A \to B$

the domain   range

When $(a,b) \in f$, we write $f(a)=b$

argument   image

The collection of all the images of f is called the image of f.



The ceiling function $\lceil x \rceil$ is a function from $\mathbb{R}$ to $\mathbb{Z}$ such that for $x \in \mathbb{R}$

$\lceil x \rceil$ returns the smallest integer greater than or equal to x

$$\lceil 3.5 \rceil = 4$$

$$\lceil 3 \rceil = 3$$

Similarly the floor function $\lfloor x \rfloor$ returns the largest integer $\leq x$.

The Characteristic Function f_A of a set A is a function, from U to $\{0,1\}$ ~~and~~ such that

for $a \in A$, $f_A(a) = 1$

for $a \notin A$, $f_A(a) = 0$

The identity function I_A is a function from A to A such that $I_A(a) = a$ for $a \in A$

Special Functions

Let f be a function from A to B.

1. f is one-to-one (injection) if for $a_1, a_2 \in A$, $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$

E.g. $\lceil x \rceil$ is not one to one becaus

$$\lceil 3.5 \rceil = \lceil 3.6 \rceil = 4$$

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2. f is onto (surjection) if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$

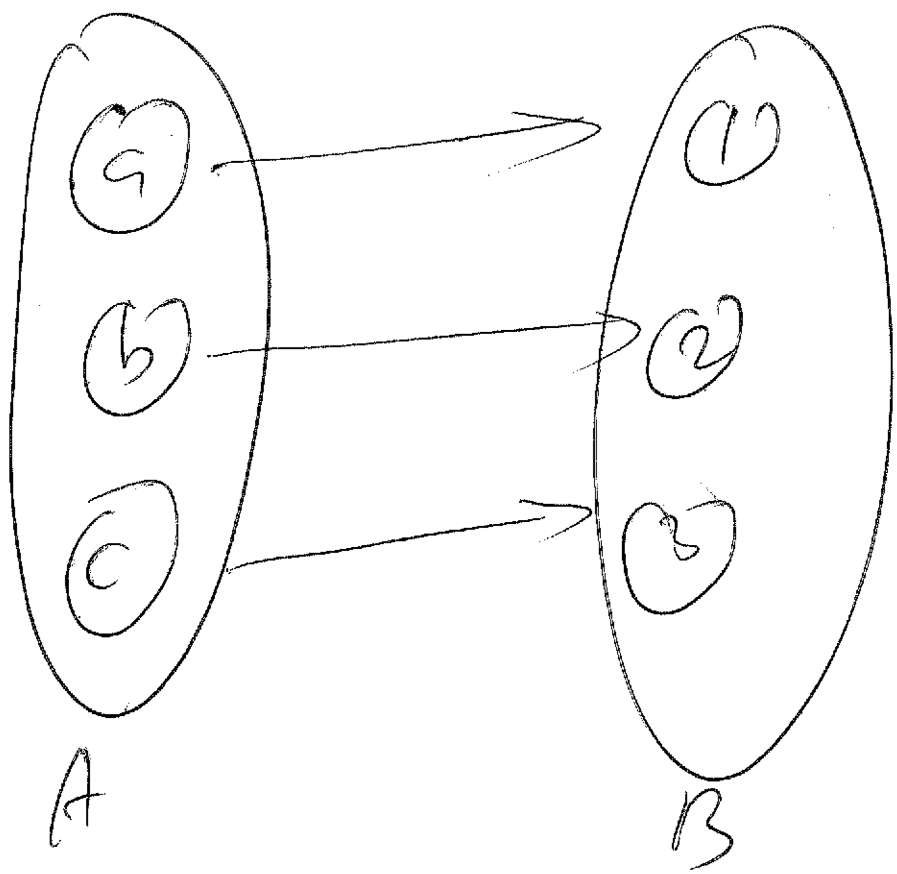
$\lceil x \rceil$ is onto (Domain $\mathbb{R}$, Range $\mathbb{Z}$)

for $z \in \mathbb{Z}$ $\lceil z - 0.1 \rceil = z$

(bijection)

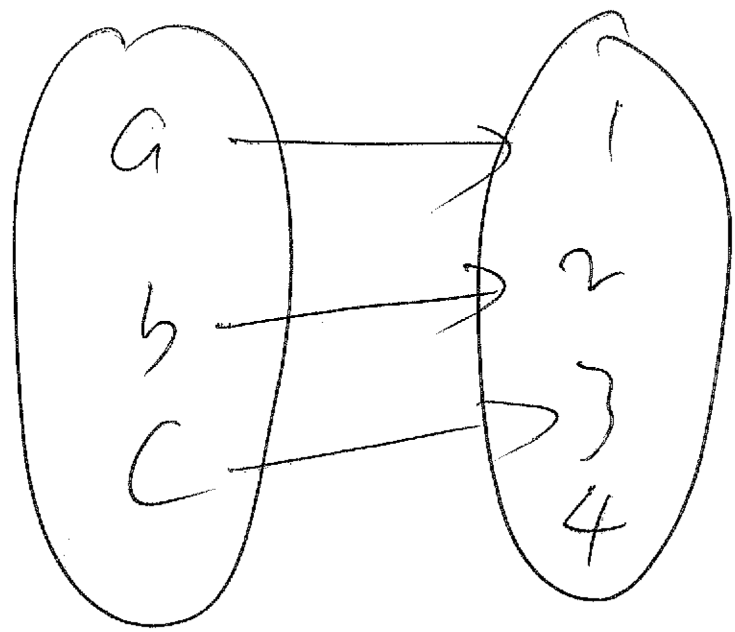
f is one-to-one correspondence if f is both one-to-one and onto

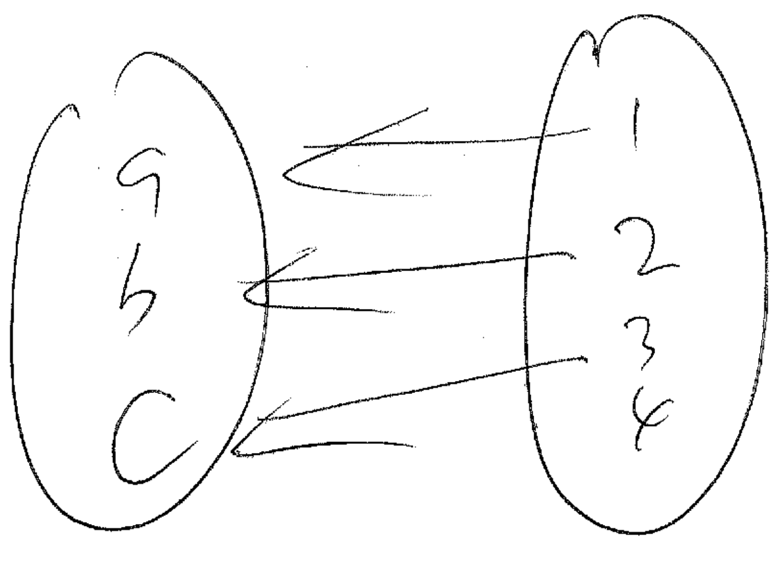
(injection) (surjection)



Let f be a one-to-one correspondence from A to B.
The inverse of f denoted by f^{-1} is a function from B to A such that

for every $b \in B$, $f^{-1}(b) = a$ if $f(a) = b$



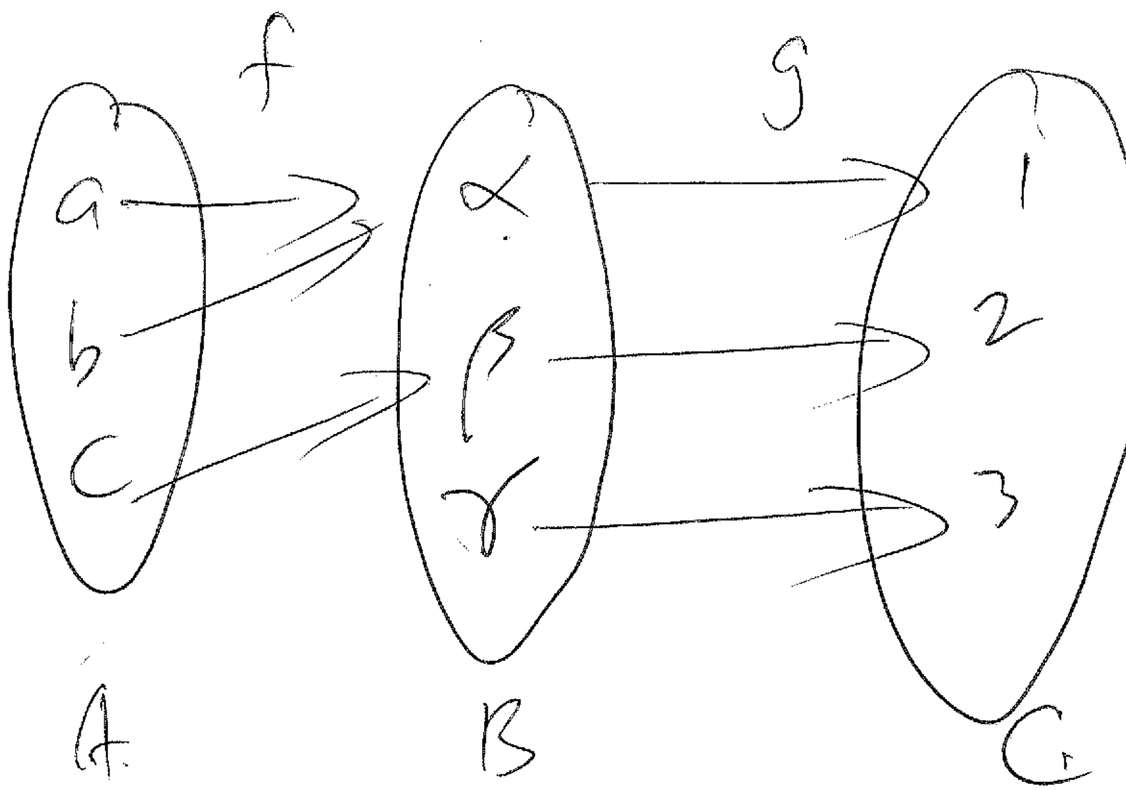


e^{x^2+2x+1}

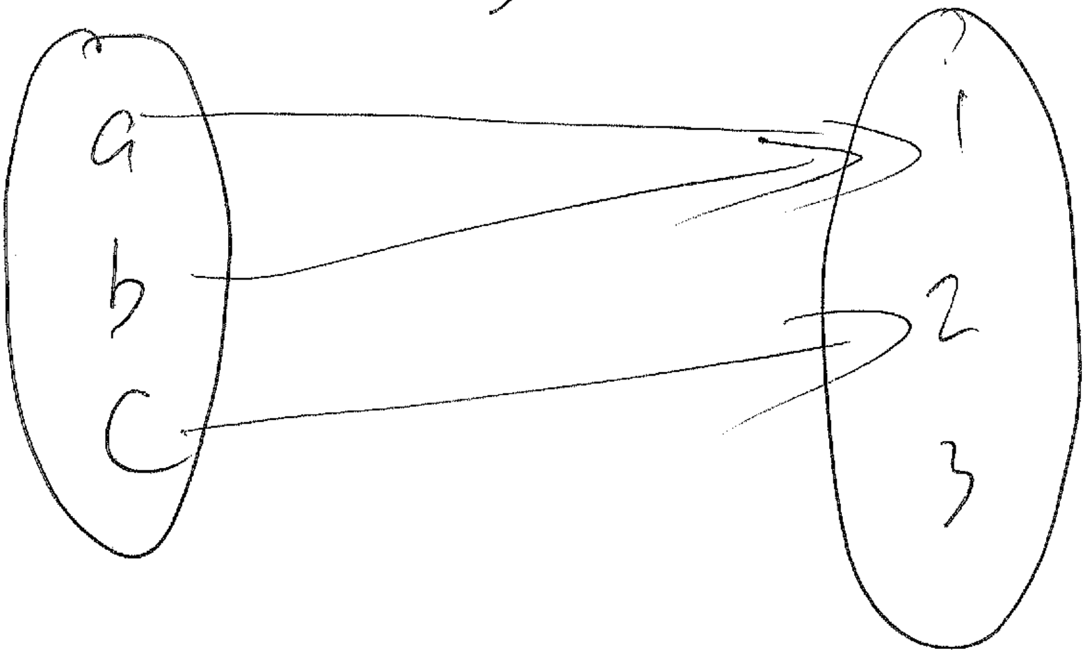
Def. Let $f: A \to B$, $g: B \to C$ are functions

the composite function of f and g is from $A \to C$ denoted by $g \circ f$ such that for $a \in A$,

$$(g \circ f)(a) = g(f(a))$$



$g \circ f$



$(g \circ f)(a) = g(f(a)) = g(\alpha)$

Cardinality of Finite Sets

Let A be a finite set, the cardinality of A denoted by $|A|$, is the number of elements in A.

~~$A = \{1, 2, 3\}$ $|A$~~

$A = \{a, b, c, d\}$ $\qquad |A| = 4$

$$\begin{array}{ccccc} A = & \{a, & b, & c, & d\} \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & 1, & 2, & 3, & 4 \end{array}$$

Observation: counting is building a one to one correspondence from A to $\{1, 2, \cdots, |A|\}$

Let A, B be two sets can you compare their cardinalities $|A|$ and $|B|$ without counting?

Observation;

If there is a one-to-one correspondence from A to B then $|A| = |B|$

If there is a one-to-one function from A to B

$|A| \leq |B|$

If there is an onto function from A to B

then $|A| \geq |B|$

~~ex~~ Let $f : A \to B$ be a one-to-one correspondence

~~B~~ Can you show that $|A| = |B|$?

$$\begin{array}{ccc} \{1, 2, \cdots, |A|\} & \to & A \\ & & \downarrow \\ \{1, 2, \cdots, |B|\} & \to & B \end{array}$$

Cardinality of Infinite Sets

① We will retain the word "cardinality" as a "measure" of how many elements an infinite set has

② ~~We will~~ Instead of counting, we will use functions to measure the cardinality of infinite sets.

Def Let A, B be two (possibly) infinite sets

We say that A and B have the same cardinality if there is a one-to-one correspondence from A to B, and write $|A| = |B|$

If one can prove that there is no one-to-one correspondence from A to B, then we say that ~~(~~A and B) have different cardinalities and write $|A| \neq |B|$

If there is a one-to-one function from A to B, then $|A| \leq |B|$

If there is an onto function from A to B, then $|A| \geq |B|$

We say $|A| > |B|$ if $|A| \geq |B|$ and $|A| \neq |B|$

Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$

$$|\mathbb{N}| = |\mathbb{Z}|$$

To show this, we have to build a one-to-one correspondance from $\mathbb{Z}$ to $\mathbb{N}$.

$x \in \mathbb{Z}$

$$f(x) = \begin{cases} 1 & x = 0 \\ 2x & x \in \mathbb{Z}^+ \\ -2x+1 & x \in \mathbb{Z}^- \end{cases}$$

$\mathbb{Z}$: $0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \ldots$

$\mathbb{N}$ $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$