Homework 1
Due: Sept 6 (Tuesday)

1. Let $A = \{a, b, c\}$ and $B = \{b, c, d, e, f\}$. Find:

   (a) $A \cup B$.
   (b) $A \cap B$.
   (c) $A - B$.
   (d) $B - A$.
   (e) $A \oplus B$.
   (e) 2^A.

2. Determine whether these statements are true or false.

   (a) $\phi \in \{\phi\}$.
   (b) $\{\phi\} \in \{\phi\}$.
   (c) $\phi \subset \{\phi\}$.
   (d) $\{\phi\} \subset \{\phi\}$.
   (e) $\{\phi\} \subset \{\phi, \{\phi\}\}$.
   (f) $\{\{\phi\}\} \subset \{\phi, \{\phi\}\}$.

3. Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

4. Let A and B be two **finite** sets. Argue that if $2^A = 2^B$, then $A = B$.

5. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

6. Are $A \times B \times C$ and $(A \times B) \times C$ the same? Explain why?

7. Show if A and B are sets, then $A \cup (A \cap B) = A$

8. Find the sets A and B if $A - B = \{a, b, c, d\}$, $B - A = \{e, f\}$, and $A \cap B = \{g, h, i, j\}$.

9. Let A and B be sets. Show that $A \cup (B - A) = A \cup B$ and $A \cap (B - A) = \phi$.

10. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

11. Show that if A and B are sets, then $A - B = A \cap \overline{B}$ and $A = (A - B) \cup (A \cap B)$.

12. Let A, B, and C be sets. Does $(A - B) - C = (A - C) - (B - C)$? Explain why?

13. Let A, B, and C be sets. Can you conclude that $A = B$ if

    (a) $A \cup C = B \cup C$?
    (b) $A \cap C = B \cap C$?
    (c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

    Explain why?

14. Let A and B be sets. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

15. Let A, B, and C be sets. Determine if $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

16. Find the domain and image of these functions. Note that in each case, to find the image, determine the set of elements assigned values by the function.

    (a) The function f that assigns to each positive integer its last decimal digit. For example, $f(123) = 3$.

    (b) The function f that assigns the next largest integer to a non-positive integer. For example, $f(-10) = -9$.

    (c) The function f that assigns to a positive integer the number of distinct decimal digits. For example, $f(11244) = 3$.

    (d) The function f that assigns to each pair of real numbers the first number of the pair. For example, $f(1.1, -0.5) = 1.1$.

17. For each of the functions below, determine if it is an injection, surjection, and bijection. Explaine why?

    (a) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = -3x + 4$.

    (b) $f : \mathcal{Z} \to \mathcal{Z}$, $f(x) = -3x + 4$.

    (c) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = x^2$.

    (d) $f : \mathcal{R}^+ \to \mathcal{R}^+$, $f(x) = x^2$.

    (e) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = x^3 - x$.

18. Recall a function f is invertible if it is a one-to-one correspondence, and the inverse of f denoted by f^{-1} is defined as $f^{-1}(b) = a$ if and only if $f(a) = b$. Is the function $f : \mathcal{R} \to \mathcal{R}$, $f(x) = |x|$ invertible? What about the function $g : \mathcal{R}^+ \to \mathcal{R}^+$, $g(x) = |x|$, is g invertible?

19. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are both functions from $\mathcal{R}$ to $\mathcal{R}$.

20. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Explain why?

21. What is the value of $\lceil x \rceil - \lfloor x \rfloor$?

22. Show that if A and B are two sets each with n elements, where n is a positive integer, then there is a one-to-one correspondence between A and B.

23. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, construct a one-to-one correspondence between the set of positive integers and that set.

    (a) The integers greater than 10.

    (b) The non-positive integers.

    (c) The integers with absolute value less than $1,000,000,000$.

    (d) The real numbers between 1 and 2.

    (e) The set $\mathcal{Z}^+ \times \mathcal{Z}^+$.

24. Give an example of two uncountable sets A and B such that $A \cap B$ is:

   (a) Finite.

   (b) Countably infinite.

   (c) Uncountable.

25. Prove that $|\mathcal{N}| = |\mathcal{Z}^+ \times \mathcal{Z}^+|$ by developing a one-to-one correspondence f from $\mathcal{N}$ to $\mathcal{Z}^+ \times \mathcal{Z}^+$.