Set Theory

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Acknowledgement
This notes is compiled based on the following textbooks and variety of Internet resources, such as wikipedia for example.

- Discrete Mathematics with Applications by Susanna S. Epp.
- Discrete Mathematics and Its Applications by Kenneth Rosen.
- Discrete Mathematics by John A. Dossey, Albert D. Otto, Lawrence E. Spence, and Charles Vanden Eynden.
- Concrete Mathematics: A Foundation for Computer Science by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik.
- Discrete Mathematics by Kenneth A. Ross and Charles R. Wright.
- Discrete and Combinatorial Mathematics: An Applied Introduction by Ralph P. Grimaldi.

1 What is a Set?

Definition 1 *A set is a* **well-defined** **unordered** *collection of objects, where "well-defined" means that given an object, one can determine if the object is in the set or not, and it is possible to distinguish one object from another in a set.*

Commonly, we use upper case letters A, B, C, ... to represent a set, and use lower case letters a, b, c, ... to represent the objects in a set. Given a set, the objects belonging to the set are also referred to as its **elements** or **members**. If an element a belongs to a set A, we write $a \in A$. We write $a \notin A$ if a is not a member of A.

There are two ways commonly used to represent a set, the *roster notation* and the *set builder notation*.

In roster notation, we list all the elements of a set one by one separated by commas and enclose them with braces.

Example 1 *The set of months in a year is the set: {January, Feburary, March, April, May, June, July, August, Septemper, October, November, December}.*

Example 2 *The set of programming languages normally taught in today's computer science curriculum is the set: {Java, C++, Python, Scheme}.*

Note that since a set is an unordered collection of objects, order is not important when listing elements from a set. The set above is the same as {*C++, Python, Scheme, Java*} .

When the set has infinite number of elements, it is impossile to list all the elements of a set. In this situation, using the roster notation to define a set will require the user to come up with a general formula or rule of how elements are included in a set. For example, consider the set of natural numbers $A = \{1, 2, ...\}$. As one can see, using the roster notation to represent infinite sets may cause confusions. For the set A above, it could also represent the set of the powers of 2, i.e., $\{1, 2, 4, 8, ...\}$. To avoid such ambiguities, the set builder notation is introduced, in which we will specify the property that the elements in a set have in common.

Example 3 $\mathcal{N}$={ x | x *is a natural number.*}

Example 4 $\mathcal{Z}$={ x | x *is an integer.*}

Example 5 $\mathcal{Q}$={ x | x *is a rational number.*}

Example 6 $\mathcal{Q}$={ $x = \frac{p}{q}$ | $p, q \in \mathcal{Z}$ *and* $q \neq 0$.}

Example 7 $\mathcal{R}$={ x | x *is a real number.*}

Example 8 $\mathcal{R}$={ x | x *is the distance from a point on the real line to the defined origin.*}

2 Relations between Sets

Definition 2 *Two sets A and B are said to be equal if they contain the same collection of objects, and we write $A = B$. We say A and B are not equal if they don't contain the same collection of objects, and we write $A \neq B$.*

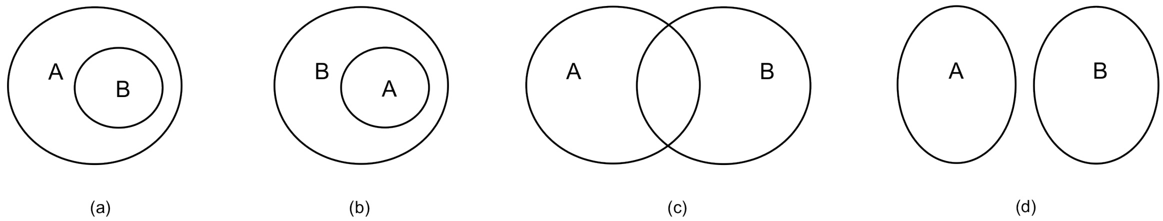

Figure 1: Illustrating the four scenarios of $A \neq B$. (a) $B \subset A$. (b) $A \subset B$. (c) $A \not\subset B$ and $B \not\subset A$. (d) A and B are disjoint, i.e., $A \cap B = \phi$

Definition 3 *A set A is a subset of another set B if every element of A belongs to B, and we write $A \subseteq B$. We say A is not a subset of B and write $A \not\subseteq B$ if there exists an element in A that doesn't belong to B.*

Definition 4 *If $A \subseteq B$ and $A \neq B$, then we say A is a proper subset of B and write $A \subset B$.*

Note that, when we write $A \subseteq B$, then A is either the same set as B or a proper subset of B.

Similar to the subset and proper subset, we can also define **superset** and **proper superset**. When A is a subset of B, we say that B is a superset of A, and write $B \supseteq A$. When A is a proper subset of B, we say that B is a proper superset of A, and write $B \supset A$.

The best way to visualize the relation between different sets is the *Venn Diagram*. In a Venn Diagram, the sets are denoted by planar objects such as rectangles, disks, etc. Figure 1 illustrates the Venn Diagrams of the four scenarios when two sets are not equal.

In most situations it is difficult to prove two sets are equal directly. In these situations, we can use the following equivalent definition.

Definition 5 *Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$.*

With the introduction of subset, this is a good place to introduce the concept of *universal set*, *empty set*, and *power set*.

Definition 6 *We will assume in the context of each discussion, there is a universal set U that contains all the elements that are meaningful to the discussion.*

Observe that any set mentioned in the discussion is a subset of U.

Definition 7 *The set that contains no elements is called the empty set and is denoted by ϕ or $\{\ \}$.*

Note that the empty set is a subset of any set.

Definition 8 *For any given set A, the power set of A, denoted by 2^A or $P(A)$ is the set of all subsets of A.*

Observe that both the empty set ϕ and the set A itself belong to the power set 2^A.

Example 9 *Let $A = \{a, b, c\}$, then $2^A = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$.*

A tricky questions is what is the power set of an empty set? The answer is $\{\phi\}$. Note that the power set of an empty set is not an empty set. It contains one member, which is an empty set.

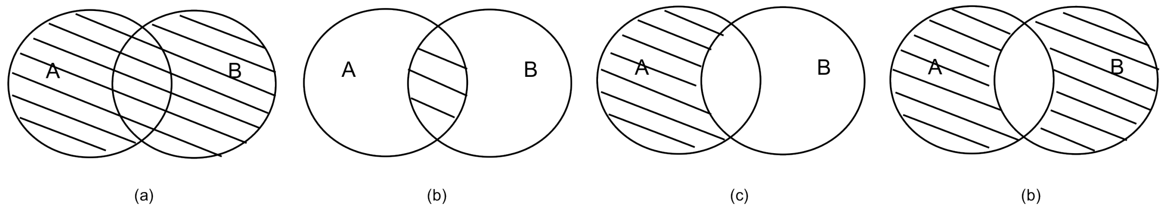

Figure 2: Illustrating the operations between sets, where the result is highlighted. (a) $A \cup B$. (b) $A \cap B$. (c) $A - B$, and (d) $A \oplus B$.

3 Set Operations

Definition 9 *The union of two sets A and B is denoted by $A \cup B$ and is $\{x \mid x \in A$ or $x \in B\}$.*

Definition 10 *The intersection of two sets A and B is denoted by $A \cap B$ and is $\{x \mid x \in A$ and $x \in B\}$.*

Definition 11 *Given two sets A and B. The complement of B with respect to A is denoted by $A - B$ and is $\{x \mid x \in A$ and $x \notin B\}$.*

Recall that the universal set U is the set that contains all the elements in the context of a discussion. For any set A, the complement of A with respect U is the set $U - A$, which contains everything that is not in A, and is denoted by $\overline{A}$ or A^c, and simply referred to as the complement of A. Since $A - B$ contains all elements that are in A but not in B, it contains all elements that are in both A and the complement of B, thus $A - B = A \cap \overline{B}$.

Definition 12 *The symmetric difference of two sets A and B is denoted by $A \oplus B$ and is $(A - B) \cup (B - A)$.*

Figure 2 illustrate these common set operations.

It turns out that we can also multiply two sets. This leads to the definiton of *Cartesian Product*. To introduce the concept of Cartesian Product, we need the concept of ordered pair.

Definition 13 *An ordered pair is a pair of elements (a, b), where a is the first element and b is the second element.*

Note that unless $a = b$, $(a, b) \neq (b, a)$.

Definition 14 *The Cartesian Product between two sets A and B is denoted by $A \times B$ and consists of all the ordered pairs, where the first element is from A and the second element is from b, i.e., $A \times B = \{(a, b) \mid a \in A, b \in B\}$.*

Example 10 *Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$, then $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.*

We can generalize Cartesian Product for n given sets. This requires the definiton of *n-tuple*.

Definition 15 *An ordered $n-$tuple $(a_1, a_2, ..., a_n)$ is the ordered collection that has a_1 as the first element, a_2 as the second element, $\ldots$, and a_n as the $n-$th element.*

Definition 16 *The Cartesian Product between n sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \cdots \times A_n$ consists of all the ordered $n-$tuples $(a_1, a_2, \ldots, a_n)$ whose first element $a_1 \in A_1$, whose second element $a_2 \in A_2$, $\ldots$, and whose $n-$th element $a_n \in A_n$.*

Some of the more difficult problems that arise in set theory is when one combines these various set operations. We will take a look at a few examples here.

Example 11 *For three given sets A, B and C, does $A - C = B - C$ imply $A = B$?*

Solution: Note that the "subtraction" in set theory is different from the algebraic subtraction, and one can't cancel C on both sides of the equation. To see that the above claim is false consider $A = \{a, b\}$, $B = \{a, b, c\}$, and $C = \{b, c\}$. Then $A - B = B - C = \{a\}$, however, $A \neq B$.

This technique of using an example to disprove a claim is called the use of counter examples. $\square$

Example 12 *Prove or disprove if $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$.*

Solution: Observe that $A = (A \cup C - C) \cup (A \cap C)$ and $B = (B \cup C - C) \cup (B \cap C)$. The rigthhand terms are equal, thus $A = B$.

$\square$

Example 13 *Prove or disprove that if $A - B = B - A$, then $A = B$.*

Solution: Observe that $A = (A \cap B) \cup (A - B)$ and $B = (A \cap B) \cup (B - A)$. Thus if $A - B = B - A$, $A = B$. $\square$

Example 14 **Distributive Law:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: Use Venn Diagrams. $\square$

Example 15 **DeMorgan's Law:** $\overline{A \cap B} = \overline{A} \cup \overline{B}$, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Solution: Use Venn Diagrams. $\square$

4 Functions

Definition 17 *A relation or mapping R from a set A to another set B is a subset of $A \times B$.*

Definition 18 *A function f from a set A to another set B is a special relation or mapping that maps each element $a \in A$ to one and only one element $b \in B$.*

Definition 19 *Let f be a function from the set A to the set B. We call the set A the demain of f and the set B the range of f. We write $f(a) = b$ if f maps a to b, and refer a as the argument, and b as the image. The collection of all the images under f is a subet of B and is called the image of f.*

Example 16 *$f(x) = x^2$ is a function from the set of real numbers $\mathcal{R}$ to $\mathcal{R}$.*

Example 17 *$f(x) = \sqrt{x}$ is not a function from $\mathcal{R}$ to $\mathcal{R}$ for two reasons: (1) -1 is not mapped to any real number by f, and (2) $+1$ is mapped to two numbers $+1$ and -1.*

Example 18 *$f(x) = \sqrt{x}$ is a function from the set of all positive real numbers $\mathcal{R}^+$ to the $\mathcal{R}^+$, where $\mathcal{R}^+$ is the set of all positive real numbers.*

Example 19 *The ceiling function $\lceil x \rceil$ is a function, which given a number x returns the smallest interger greater than or equal to x is a function from $\mathcal{R}$ to $\mathcal{R}$. For examples, $\lceil 3.5 \rceil = 4$, $\lceil 3 \rceil = 3$.*

Example 20 *The floor function $\lfloor x \rfloor$ is a function, which given a number x returns the largest interger less than or equal to x is a function from $\mathcal{R}$ to $\mathcal{R}$. For examples, $\lfloor 3.5 \rfloor = 3$, $\lfloor 3 \rfloor = 3$.*

Example 21 *The characteristic function defined by a subset A of a universal set U is a function f_A from U to $\{0,1\}$, where $f_A(u) = 1$ if $u \in A$ and 0 otherwise.*

Example 22 *The identity function defined by a set A is a function 1_A from A to itself, where $1_A(a) = a$ for all $a \in A$.*

There are some special functions that are particularly important.

Definition 20 *Let f be a function from the set A to the set B.*
We say that f is one-to-one or injection if for $a_1, a_2 \in A$ and $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$.
We say that f is onto or surjection if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$.
We say that f is one-to-one correspondence or bijection if f is both one-to-one and onto.

Example 23 *$f(x) = e^x$ from $\mathcal{R}$ to $\mathcal{R}$ is one-to-one, but not onto. $f(x) = x^2$ is from $\mathcal{R}$ to $\mathcal{R}^+ \cup \{0\}$ is onto, but not one-to-one. $f(x) = x^3$ from $\mathcal{R}$ to $\mathcal{R}$ is one-to-one, onto, and one-to-one correspondence.*

Definition 21 *Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is a function f^{-1} from B to A that maps every $b \in B$ to an element $a \in A$ with $f(a) = b$.*

Note that since the definition of a function requires that every element from the domain to be mapped to a unique element in the range, the inverse function is only well-defined for one-to-one correspondence.

Definition 22 *Let f be a function from A to B, and g a function from B to C. The composite function of f and g, denoted by $g \circ f$ is a function from A to C, such that for any $a \in A$, $(g \circ f)(a) = g(f(a))$.*

5 Cardinality of Finite Set

One of the important concept in set theory is how many elements are there in a set.

Definition 23 *For a finite set A, the cardinality of A denoted by $|A|$ is the number of elements in A.*

Thus, to figure out the cardinatliy of a finite set, we will need to count the number of elements in the set. So, what is the essence of counting then? Surprisingly, counting is actually a one-to-one correspondence.

5.1 Relation between Counting and One-to-one Correspondence

Consider two little kids Charlie and Lucy who each have a large collection of toys. Neither of them knows how to count yet. How do they determine who has more toys?

Here is how they settle the question. First they find a big basket. Each time, Charlie brings out one of his toys not yet in the bastket, while Lucy brings out one of her toys not yet in the basekt. They pair up the toys and put the pair in the basket. They repeat the process until one of them has no toys. Whoever still has toys left has more toys. Figure 3 illustrates the idea.

What if Charlie and Lucy have the same number of toys? Then by the time the game stops, neither of them will have any toys left. Observe that at this point, the game essentially created a one-to-one

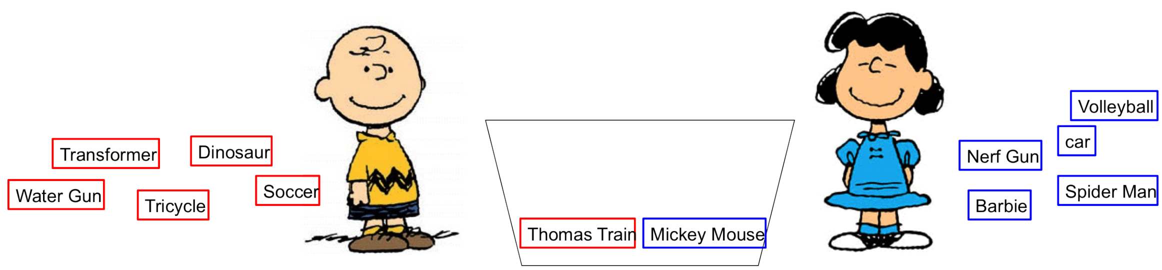

Figure 3: Comparing the number of elements from two sets without counting.

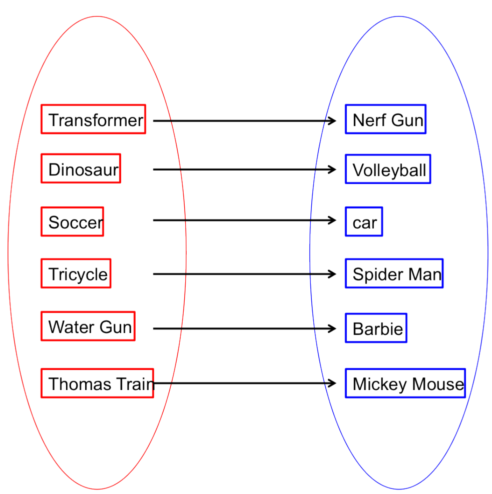

Figure 4: A one-to-one correspondence between two **finite** sets implies they have the same number of elements.

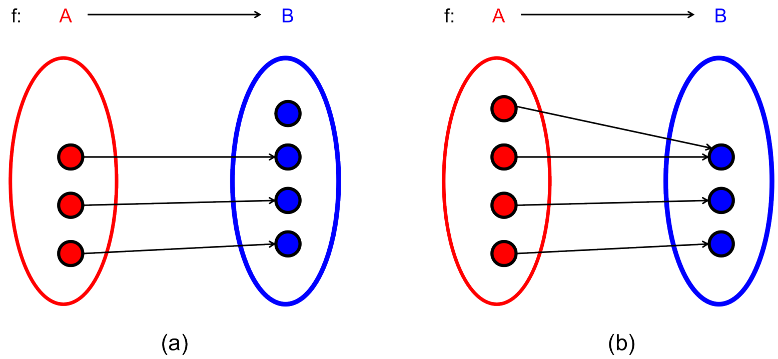

Figure 5: Illustrating Observation 1.

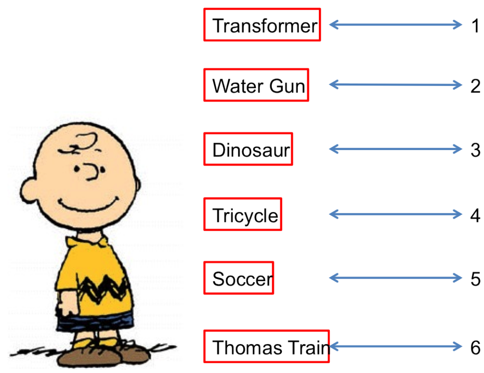

Figure 6: The process of counting forms a one-to-one correspondence.

correspondence from Charlie's collection of toys to Lucy's collection of toys, where each pair of toys form the argument and image pair. See Figure 4 for illustrations.

What if Charlie has less toys than Lucy? Then by the time the game stops, Charlie will have no toys left. Observe that at this point, the game essentially created a one-to-one function from Charlie's collection of toys to Lucy's collection of toy.

Observation 1 *Let A and B be two finite sets. A one-to-one correspondence or bijection from A to B implies that $|A| = |B|$. A one-to-one function or injection from A to B implies that $|A| \leq |B|$. A onto function or surhection from A to B implies that $|A| \geq |B|$.*

Observation 1 is illustrated in Figures 4 and 5.

Observation 2 *Let A be a finite set. The process of counting the number of elements in A forms a one-to-one correspondence from A to the numbers $\{1, 2, ..., |A|\}$.*

To see Observation 2, notice that when we count the elements from a finite set A, i.e., 1, 2, ..., we are assigning the number 1 to an element in the set, the number 2 to another element in the set, ... Since every time, we pick out an element that has not been counted before, this assignment from the collection of elements to numbers is obviously one-to-one. Therefore, by the time we have counted all the elements, we have established a one-to-one correspondence from A to the numbers $\{1, 2, ..., |A|\}$. This is also illustrated in Figure 6.

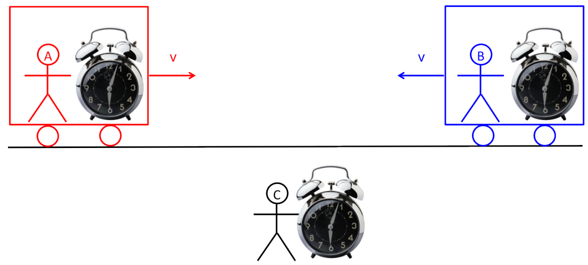

Figure 7: The Galilean transform.

6 Cardinality of Infinite Set

So how many elements are there in an infinite set? Should we just say the number of elements is ∞? Can one infinite set has more elements than another infinite set? What does it mean when we say two inifite sets have the same number of elements? Since infinity is not a number, maybe it is even technically incorrect to use the phrase "the number of elements of an inifite set". To answer these questions, we will have to introduce some new ways to thinking.

Generally speaking, the understand the "number of elements" of an infinite set, we will need to accept the following:

- We will retain the word "cardinaliiy" as a measure of the "number of elements" for an infinite set.
- To compare the cardinality of two infinite sets, we will embrace the notion of one-to-one correspondence. In other words, two infinite sets have the same cardinatliy if there is a one-to-one correspondence between them.

To ease the transition to cardinality of infinite sets, we shall use the example of special relatively as an analogy here.

6.1 Special Relativity

Consider the following example. Suppose that a person A is in a train moving at a speed of v to the right, a person B is in a train moving at speed v to the left, while a person C is standing still. See Figure 7 for illustrations. In addition, each person is carrying a clock. Our daily intuitition tells us that there the time that each person reads from his clock must be the same, i.e., there is a universal time independent of the speed of the clock. As a result of this universal time, one can conclude that in the eyes of the person A, person B is moving at a (relative) speed of $2v$ toward him. This is the so-called Galilean transform.

Now assume that both A and B are moving at the speed of light c ($c = 299,792,458$ metres per second in vacuum and $225,056,264$ metres per second in water) toward each other. Then according to Galilean transform, A will see that B is moving at $2c$ toward him. Beginning in the 19th centery, many experiments were performed and indicated that the two-way speed of light remains the same in every direction. In other words, even though both A and B are both moving at the speed of light toward each other, in the eyes of A, B is still moving at the speed of light, rather than twice the speed of light toward him!

Since the Galilean transform is the direct consequence of the universal time, therefore, the only way to explain the aparent contradiction between experimental results and Galilean transform is to

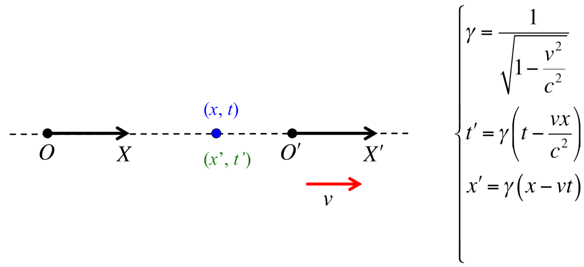

Figure 8: The Lorentz transform between two reference frames with a relative speed of v.

give up the idea of the universal time, and accept that the speed of light is the limit of motions, which results in the Theory of Special Relativity. To accomodate that the speed of light is limit of motions, one has to use the Lorentz Transform instead of the Galilean Transform (see Figure 8 for illustrations):

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t' = \gamma(t - \frac{vx}{c^2})$$

$$x' = \gamma(x - vt)$$

Observe that when v is small compared to c, i.e., $\frac{v}{c} \approx 0$, the Lorentz Transform becomes the Galiliean Transform:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1$$

$$t' = \gamma(t - \frac{vx}{c^2}) = t$$

$$x' = \gamma(x - vt) = x - vt$$

One interesting consequence of special relativity is the so-called "time dilation". Observe that from $t' = \gamma(t - \frac{vx}{c^2})$, thus $dt' = \gamma dt$. Since $\gamma \geq 1$, $dt' \geq dt$. In other words, the clock runs slower in a moving frame.

In summary, the theory of special relativity is the direct consequence of rejection of universal time and the acceptance that speed of light is absolute. To understand the cardinality of infinite sets, we will have to do the same by accepting that the only way to compare the cardinality of two sets is through one-to-one correspondence and giving up the intuition that the "whole cannot be the same size as the part".

6.2 Definition of Cardinality of Infinite Sets

We can generalize Observation 1 to infinite sets with the following definitions.

Definition 24 *Let A and B be two sets possibly with infinite number of elements. Then we say that A and B have the same cardinality if one can establish a one-to-one correspondence between A and B. Following the notations for finite sets, we still use $|A|$ and $|B|$ to represent their cardinalities and write $|A| = |B|$ if they have the same cardinality.*

Definition 25 *Let A and B be two sets possibly with infinite number of elements. Then we say that $|A| \neq |B|$ if one can prove that there is no one-to-one correspondence between A and B.*

Definition 26 *Let A and B be two sets possibly with infinite number of elements. Then we say that the cardinality of A is less than or equal to that of B if one can establish a one-to-one function from A to B, and write $|A| \leq |B|$.*

Definition 27 *Let A and B be two sets possibly with infinite number of elements. Then we say that the cardinality of A is greater than or equal to that of B if one can establish an onto function from A to B, and write $|A| \geq |B|$.*

Definition 28 *Let A and B be two sets possibly with infinite number of elements. Then we say that $|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$. Similarly, we say that $|A| > |B|$ if $|A| \geq |B|$ but $|A| \neq |B|$.*

Theorem 1 *If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.*

Theorem 2 (Cantor-Bernstein-Schroeder Theorem) *For two sets, if $|A| \leq |B|$ and $|A| \geq |B|$, the $|A| = |B|$.*

The proof for the Cantor-Bernstein-Schroeder Theorem is rather involved and will be skipped in our lectures. However, it is worthwhile to point out that since $|A|$ and $|B|$ are infinite, the only way to prove the theorem is to show that there is a one-to-one correspondence between A and B.

Note that with the above definitions, we can compare the cardinalities of infinite sets, and rank them based on their cardinalities. Is there an infinite set with the least number of elements? Turns out that it is the set of natural numbers.

Definition 29 *Let $\mathcal{N} = \{1, 2, 3, \ldots,\}$ be the set of natural numbers. The cardinailty of $\mathcal{N}$ is denoted by $\aleph_0$.*

Let $\mathcal{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots,\}$ denote the set of integers. What is the cardinality of $\mathcal{Z}$?

Observe that since $\mathcal{N} \subset \mathcal{Z}$, it is clear that $|\mathcal{N}| \leq |\mathcal{Z}|$. The question is whether we can replace $\leq$ with $<$. The answer may seem obvoius to a lot of readers because negative numbers belong to $\mathcal{Z}$ but not $\mathcal{N}$. However, surprisingly, $|\mathcal{N}| = |\mathcal{Z}|$, i.e., $|\mathcal{Z}| = \aleph_0$.

Theorem 3 *$|\mathcal{Z}| = \mathcal{N}$.*

Proof: From Definition 24, we know to show that $|\mathcal{Z}| = \mathcal{N}$, all we need to do is to create a one-to-one correspondence between the integers and the natural numbers. Consider the following function $f : \mathcal{Z} \to \mathcal{N}$, which is defined as follows:

$$f(x) = \begin{cases} 2x, \text{ if } x > 0. \\ 1, \text{ if } x = 0. \\ -2x + 1, \text{ if } x < 0. \end{cases}$$

Figure 9 illustrate the construction of f, where 0 is mapped to 1, the positive integers are mapped to the even natural numbers, and the negative integers are mapped to the odd natural numbers larger than 1. It is very easy to show that the function f thus defined is a one-to-one correspondence. □

Theorem 3 is a very surprising result. How can there be the same number of integers as the same number of natural numbers? The theorem and proof reveals something more profound about infinite

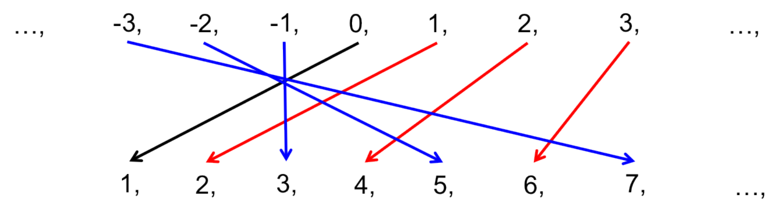

Figure 9: Illustrating the proof of Theorem 3.

sets. Our intuition developed with finite sets breaks down with infinite sets. This means that we have to reject the intuition that the "whole cannot be the same size as the part". This is mainly due to the fact that counting is not defined for infinite set. As we have already discussed previously, for finite sets, counting essentially yields a one-to-one correspondence from the finite set to an initial segment of the natural integers. Because counting is well-defined for finite sets, we form the intuition that removing an element from a finite set always makes the set "smaller", because if we apply counting, the one-to-one correspondence formed will be from the finite set to a "smaller" segment of the natural numbers. All of these intuition however is not true for infinite sets because we can't count the number of elements in the set. Georg Cantor, the founder of set theory was one of the first mathemeticians to realize the problem and define the cardinality of the set based on one-to-one correspondence as shown in Definition 24. Once one accept the fact that one-to-one correspondence is the method to compare the cardinaliy between sets, a lot of fascinating results follow.

Theorem 4 *Let A and B be two possibly infinite sets, and $A \subseteq B$. Then $|A| \leq |B|$.*

Definition 30 *Any set A whose cardinality is less than or equal to $\aleph_0$ is said to be a countable set.*

Theorem 5 *For any countable set A, one can list all the elements of A in the form $a_1, a_2, a_3, \ldots,$ just like the natural numbers.*

Proof: Observe that if A is finite, the correctness of the theorem is obvious. Now assume that A is infinite. Since A is countable, there is a one-to-one correspondence between A and the natural numbers $\mathcal{N}$. Let $f : \mathcal{N} \to A$ be one such one-to-one correspondence. Let a_j be the element of A that is mapped by f from the natural number j, i.e., $f(j) = a_j$. The list $a_1, a_2, a_3, \ldots,$ is obviously a list of all the elements in A. $\square$

Theorem 6 *Let $\mathcal{Z}_{\geq 0}$ be the set of nonnegative integers, then $|\mathcal{Z}_{\geq 0} \times \mathcal{Z}_{\geq 0}| = |\mathcal{N}|$.*

Proof:

$\mathcal{Z}_{\geq 0} + \times \mathcal{Z}_{\geq 0} = \{(p, q) | \, p, q \in \mathcal{Z}_{\geq 0}\}$. We need to develop a systematic way to list all the ordered pairs (p, q). Consider the following scheme, where we are listing (p, q) based on how big $p + q$ is.

$$\begin{array}{ccccccc}
 & & (0,0) & & & & p+q=0 \\
 & & (0,1) \quad (1,0) & & & & p+q=1 \\
 & & (0,2) \quad (1,1) \quad (2,1) & & & & p+q=2 \\
 & & (0,3) \quad (1,2) \quad (2,1) \quad (3,1) & & & & p+q=3 \\
 & & (0,4) \quad (1,3) \quad (2,2) \quad (3,1) \quad (4,0) & & & & p+q=4 \\
 & & (0,5) \quad (1,4) \quad (2,3) \quad (3,2) \quad (4,1) \quad (5,0) & & & & p+q=5 \\
 & & \cdots & & & & \cdots
\end{array}$$

Now observe that if we list all the elements of $\mathcal{Z}_{\geq 0} \times \mathcal{Z}_{\geq 0}$ one by one from the above triangular arrangement, the ones with the smaller $p + q$ are listed first, while the ones with the larger $p + q$

are listed later; the ordered pairs with the same $p+q$ value are listed from left to right, i.e., in the increasing order of p; we end up with $(0,0)$ being the 1^{st} element, $(0,1)$ being the 2^{nd} element, $(1,0)$ being the 3^{rd} element, $(0,2)$ being the 4^{th} element, ...

Thus $|\mathcal{Z}_{\geq 0} \times \mathcal{Z}_{\geq 0}| = |\mathcal{N}|$.

$\square$

Theorem 7 *Let $\mathcal{Q}_{\geq 0}$ be the set of non-negative rational numbers, then $|\mathcal{Q}_{\geq 0}| = |\mathcal{N}|$. In otherwords, rational numbers are countable.*

Proof: We will use Theorem 5 and show that we can list all the rational numbers. Recall that the rational numbers are defined as $\mathcal{Q}_{\geq 0} = \left\{ \frac{p}{q} \middle| p, q \in \mathcal{Z}_{\geq 0}, q \neq 0 \right\}$. We need to develop a systematic way to list all the rational numbers. Consider the following scheme, where we are listing a rational number $\frac{p}{q}$ based on how big $p+q$ is. The rational numbers with a smaller $p+q$ will be listed first, while the ones with a larger $p+q$ will be listed afterwards. This is illustratedin the table below, where the numbers being crossed out are the ones that has already been listed. This we can list all the rational numbers and the rational numbers are countable.

$$
\begin{array}{cl}
\frac{0}{1} & p+q=1 \\
\cancel{\frac{0}{2}} \quad \frac{1}{1} & p+q=2 \\
\cancel{\frac{0}{3}} \quad \frac{1}{2} \quad \frac{2}{1} & p+q=3 \\
\cancel{\frac{0}{4}} \quad \frac{1}{3} \quad \cancel{\frac{2}{2}} \quad \frac{3}{1} & p+q=4 \\
\cancel{\frac{0}{5}} \quad \frac{1}{4} \quad \frac{2}{3} \quad \frac{3}{2} \quad \frac{4}{1} & p+q=5 \\
\cancel{\frac{0}{6}} \quad \frac{1}{5} \quad \cancel{\frac{2}{4}} \quad \cancel{\frac{3}{3}} \quad \cancel{\frac{4}{2}} \quad \frac{5}{1} & p+q=6 \\
\cdots & \cdots
\end{array}
$$

$\square$

Theorem 8 *Let $\mathcal{R}$ be the set of real numbers, then $|\mathcal{R}| > |\mathcal{N}|$.*

Proof: Since $\mathcal{N} \subset \mathcal{R}$, we know that $|\mathcal{R}| \geq |\mathcal{N}|$. The difficult part of the proof is to show that $|\mathcal{R}| \neq |\mathcal{N}|$. According to Definition 25, we must show that there is no one-to-one corespondence between $\mathcal{R}$ and $\mathcal{N}$.

For this will use proof by contradiction. We will assume that the conclusion of the theorem is wrong and show that the assumption leads to a contradiction. This implies that the assumption must be wrong, for otherwise it should not lead to a contradiction.

Assume that $\mathcal{R}$ is countable. Then any subset of $\mathcal{R}$ should also be countable. In parciular, we consider the real numbers between 0 and 1. Note that every real number has a unique decimal expansion when the tail end consists of entirely of 9's are excluded. Let such decimal expansion of the real numbers be used. For examples, the number 0.2 is represented as $0.200\cdots$, the number $\frac{1}{3}$ is represented as $0.333\cdots$, the number π is represented as $3.14159265358979323846264338327950\cdots$. Since we assume these real numbers are countable, we must be able to list all the numbers one by one, just like natural numbers. Let the numbers be listed as:

$$
\begin{array}{l}
0.d_{1,1}d_{1,2}d_{1,3}d_{1,4}d_{1,5}\cdots \\
0.d_{2,1}d_{2,2}d_{2,3}d_{2,4}d_{2,5}\cdots \\
0.d_{3,1}d_{3,2}d_{3,3}d_{3,4}d_{3,5}\cdots \\
0.d_{4,1}d_{4,2}d_{4,3}d_{4,4}d_{4,5}\cdots \\
0.d_{5,1}d_{5,2}d_{5,3}d_{5,4}d_{5,5}\cdots \\
\cdots
\end{array}
$$

Consider the following number: $0.d_1d_2d_3d_4d_5\cdots$, where for $j = 1, 2, \ldots,$

$$d_j = \begin{cases} 5, \text{ if } d_{j,j} \neq 5. \\ 4, \text{ if } d_{j,j} = 5. \end{cases}$$

Clearly the number $0.d_1d_2d_3d_4d_5\cdots$ is real number different from all the numbers in the list. Thus we have found a real number not on the list. This contradicts to that all the real numbers between 0 and 1 is on the list. Thus the assumption is wrong, and R is not countable. $\square$

Definition 31 *The cardinality of $\mathcal{R}$ is denoted by $\aleph_1$.*

Question: An interesting question is that if one can apply the above proof for rational numbers and claim that rational numbers are not countable. The answer is negative. Can you explain why?

The answer lies in the fact that not all decimal numbers are rational numbers. Thus the number created may not be rational.

We now move on to present some important results regarding the cardinalities of infinite sets.

Theorem 9 *For any set S, $|S| < |2^S|$.*

Proof: We will use proof by contradiction. Assume that the theorem is wrong. Then $|S| \geq |2^S|$. From Definition 28, there must exists an onto function f from S to its power set 2^S. We will show that this will lead to a contradiciton.

Let $X = \{x \mid x \in S, x \notin f(x)\}$. In other words, X is the subset of S that contains all the element x of S that are mapped to a subset $f(x)$ of S not containing x. Since f is an onto function, there exists an element $x^* \in S$, such that $f(x^*) = X$. Since X is a subset of S, either x^* belongs to X or doesn't belong to X. We will show that both cases lead to contradictions.

Suppose $x^* \in X$, then from the definition of X, we know that $x^* \notin X$, contradiction to the assumptiong that $x^* \in X$. Therefore, we assume that $x^* \notin X$. Thus x^* is mapped to a setset X not containing it. Therefore, it must be the case that $x^* \in X$, a contradiciton. Thus both cases lead to contraductions and can't be valid. $\square$

Theorem 10 *For any infinite set S, $|S| \geq \aleph_0$.*

Theorem 11 $|\mathcal{R}| = |2^{\mathcal{N}}|$.

7 Russel's Paradox

Consider the set S defined as the collection of objects that doesn't belong to itself. In other words $S = \{x | x \notin x\}$.

Recall that according to Definition 1, a set must be well-defined, which means for any object one should be able to determine if the object is in a set or not. The question here is if S belongs to itself.

Observe that if $S \in \mathcal{S}$, where we use $\mathcal{S}$ to indicate that we are referring to it as a set. Then according to the definition of $\mathcal{S}$, we have $S \notin S$. A contradiction. If $S \notin \mathcal{S}$, then S doesn't meet the definition of $\mathcal{S}$, and we have $S \in S$, a contradiction too! Thus we can't determine if S belongs to itself or not.

How does this happen?

The problem of the paradox stems from the naive defiinition of set. To avoid this Russell's basic idea we must avoid the commitment to the set of all sets (in other words, a set can be a member of itself) and thus organizing thing in hierarchy. The lowest level consist of all individuals. The next level will consist of sets of individuals. The next lowest level will consist of collection of sets of individuals, and so on.