Sept 1, 2016

Sequences : a list of objects arranged in a <u>definite order</u> : a 1st element, a ~~sec~~ 2^{nd}, a 3^{rd}, ...

1, 2, 3, 4, 5, ...

2, 4, 6, 8, ...

1, 1, 1, 1, 1, ...

0, 1, 10, 11, 100, ...

Why Sequences?

Problem: insertion sort.

Given n distinct integers, sort the integers.

Solve the problem by induction

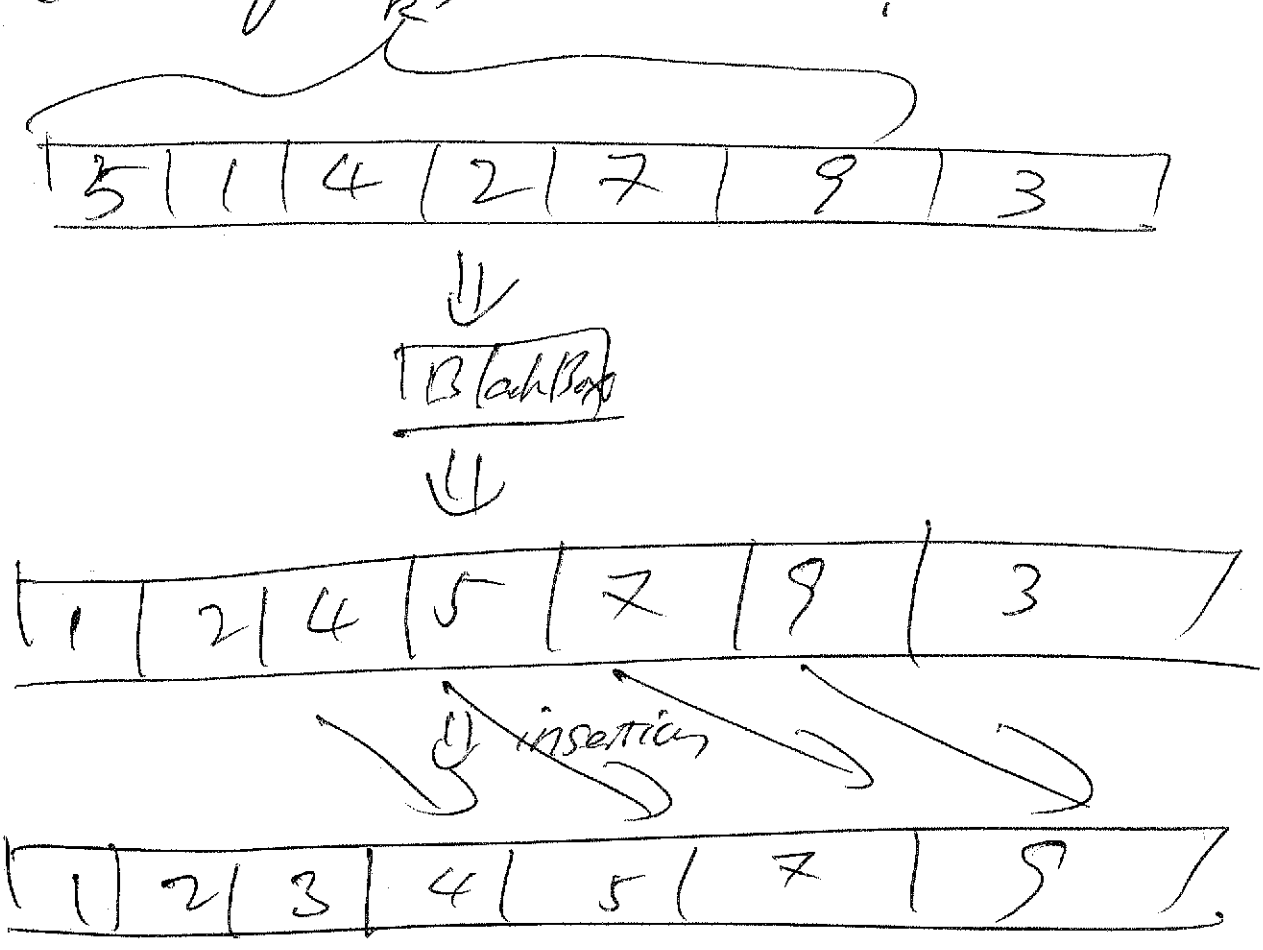
complexity: the number of integers given

Basis: We sort 1 number ✓.

Induction Hypothesis.

Assume we can sort ~~any~~ k numbers.

What if we get one more?



Running Time Analysis based on the RAM model

random access machine

(1) A single CPU, executing instructions sequentially

(2) Unlimited main memory and no memory hierarchy.

(3) Each memory access takes unit time (step)

(4) Each standard data type occupies a single memory cell.

(5) Each simple instruction takes unit time (step)

$+, -, \times, /,$

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input size n: the number of memory cells the input occupies

running time: worst case number of CPU steps the algorithm takes: $T(n)$

Pseudo code.

input: an array $A[1..n]$ of n distinct integers

output: A sorted

// Basis $A[1]$ is already sorted by itself.

for $i = 2$ to n

// $A[1..i-1]$ are already sorted.

for $j = i-1$ to 1

if $A[j] > A[j+1]$

swap.

else

break.

How well does our alg perform?

$T(n)$: insertion sort.

* input size n the size of A.

$i =$ 2 3 4 5 . — n

steps

$2 + 4 + 6 + 8$

$$= \sum_{i=2}^{n} 2(i-1)$$

A sequence of numbers is said to be arithmetic if the difference between adjacent numbers are constant, i.e., $a_j - a_{j-1} = d$. ← difference

a_1 a_2 a_3 a_4 a_5, ...

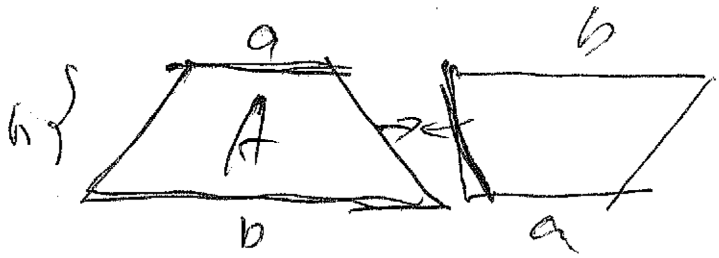
2, 4, 6, 8, 10, 12, . . .

2 2 2 2 2

Sum of Arithmetic Sequence.

$$S_n = 1 + 2 + 3 + 4 + \cdots + n$$

This arithmetic with constant difference $d=1$.



$$A = \frac{(a+b)\,h}{2}$$

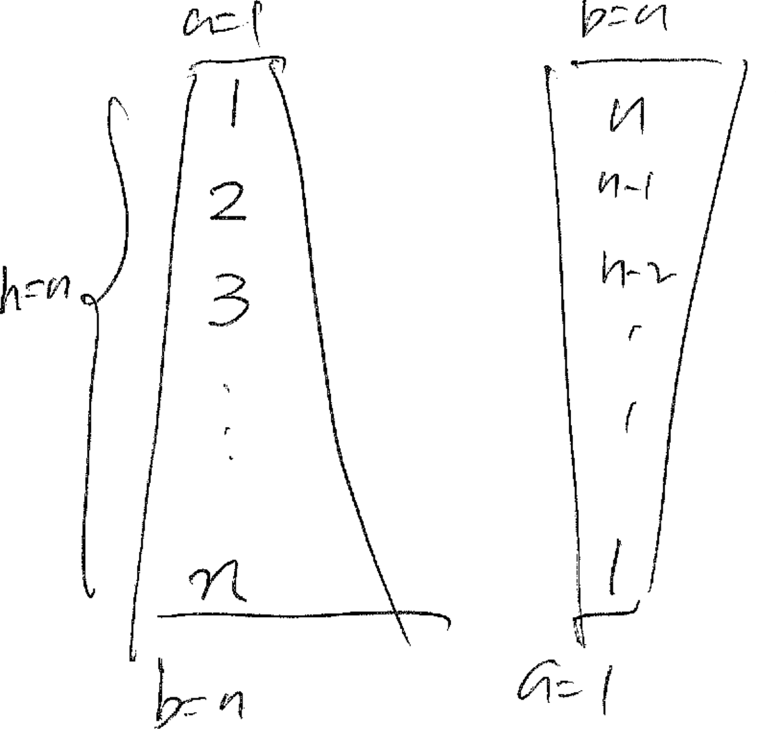
$$S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

$$+\ S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

$$2S_n = \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n}$$

$$= n(n+1)$$

$$S_n = \frac{(1+n)\,n}{2}$$



$$\frac{(1+n)n}{2}$$

$$S' = 2+4+6+8+ \quad + 2(n-1)$$

$$S' = 2(n-1)+2(n-2)+2(n-3)+2(n-4)+ \quad + 2$$

$$2S' = \underbrace{2n+2n+2n+ \quad +2n}_{n-1}$$

$$= \frac{2n(n-1)}{2} = n(n-1) = n^2-n$$

Let $a_1, a_2, \cdots, a_n$ be an arithmetic sequence with constant difference d, then the sum $S_n = \frac{(a_1+a_n)n}{2}$

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a_1	a_1
a_2	a_1+d
a_3	a_1+2d
a_4	a_1+3d
a_5	a_1+4d
a_6	a_1+5d
a_7	a_1+6d
$\vdots$	$\vdots$
a_n	$a_1+(n-1)d$

⑧

$$S = a_1 + a_2 + a_3 + \cdots + a_n$$

$$S = a_n + a_{n-1} + a_{n-2} + \cdots + a_1$$

$$S = a_1 + (a_1+d) + (a_1+2d) + \cdots + (a_1+(n-1)d)$$

$$+\ S = a_1+(n-1)d + (a_1+((n-1)-1)d) + (a_1+((n-2)-1)d) + \cdots + a_1$$

$$2S = \underbrace{[2a_1+(n-1)d] + [2a_1+(n-1)d] + \cdots + [2a_1+(n-1)d]}_{n}$$

$$= (2a_1+(n-1)d)\,n$$

$$S = \frac{(2a_1+(n-1)d)n}{2} = \frac{(a_1 + \overbrace{a_1+(n-1)d}^{a_n})\,n}{2}$$

$$= \frac{(a_1+a_n)\,n}{2}$$

Arithmetic — constant difference

Geometric — constant ratio

$1, 1, 1, 1, \cdots$

$1, 2, 3, 4, \cdots$

$2^0, 2^1, 2^2, 2^3, \cdots$

$2^0, 2^{-1}, 2^{-2}, 2^{-3}, \cdots$

$(\frac{1}{2})^0, (\frac{1}{2})^1, (\frac{1}{2})^2, (\frac{1}{2})^3, \cdots$

$$a^m = \underbrace{a \cdots a}_{m}$$

$$a^m \cdot a^n = \underbrace{a \cdots a}_{m} \underbrace{a \cdots a}_{n} = \underbrace{a \cdots a}_{m+n} = a^{m+n}$$

$$(a^m)^n = \underbrace{\underbrace{a \cdots a}_{m} \underbrace{a \cdots a}_{m} \cdots \underbrace{a \cdots a}_{m}}_{n} = a^{mn}$$

$$S = 2^0 + 2^1 + 2^2 + \cdots + 2^n$$

$$2S = 2^1(2^0 + 2^1 + 2^2 + \cdots + 2^n)$$

$$2S = \cancel{2^1} + \cancel{2^2} + \cancel{2^3} + \cdots + \cancel{2^n} + 2^{n+1}$$

$$- \; S = 2^0 + \cancel{2^1} + \cancel{2^2} + \cdots + \cancel{2^n}$$

$$S = 2^{n+1} - 2^0 = 2^{n+1} - 1$$

$$S = 3^0 + 3^1 + 3^2 + \cdots + 3^n \quad ①$$

$$3S = 3^1 + 3^2 + \cdots + 3^n + 3^{n+1} \quad ②$$

② − ①

$$(3-1)S = 3^{n+1} - 3^0$$

$$S = \frac{3^{n+1} - 3^0}{3-1} = \frac{r a_n - a_1}{r-1}$$

(12)

$$S = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n \qquad (1)$$

Geometric $r = \frac{1}{2}$

$$S = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^0}{\frac{1}{2} - 1}$$

$$\frac{1}{2}S = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n+1} \qquad (2)$$

(2) − (1)

$$\left(\frac{1}{2} - 1\right)S = \left(\frac{1}{2}\right)^{n+1} - \left(\frac{1}{2}\right)^0$$

$$S = \frac{\left(\frac{1}{2}\right)^{n+1} - \left(\frac{1}{2}\right)^0}{\frac{1}{2} - 1}$$

Geometric with ratio r.

$$
\begin{array}{ll}
a_1 & a_1 \\
a_2 & a_1 r \\
a_3 & a_1 r^2 \\
a_4 & a_1 r^3 \\
\vdots & \\
a_n & a_1 r^{n-1}
\end{array}
$$

$S = a_1 + a_2 + a_3 + \qquad + a_n$

$S = a_1 + a_1 r + a_1 r^2 + \qquad + a_1 r^{n-1}$ ①

$rS = \qquad a_1 r + a_1 r^2 + \qquad + a_1 r^{n-1} + a_1 r^n$ ②

② − ①

$(r-1)S = a_1 r^n - a_1$

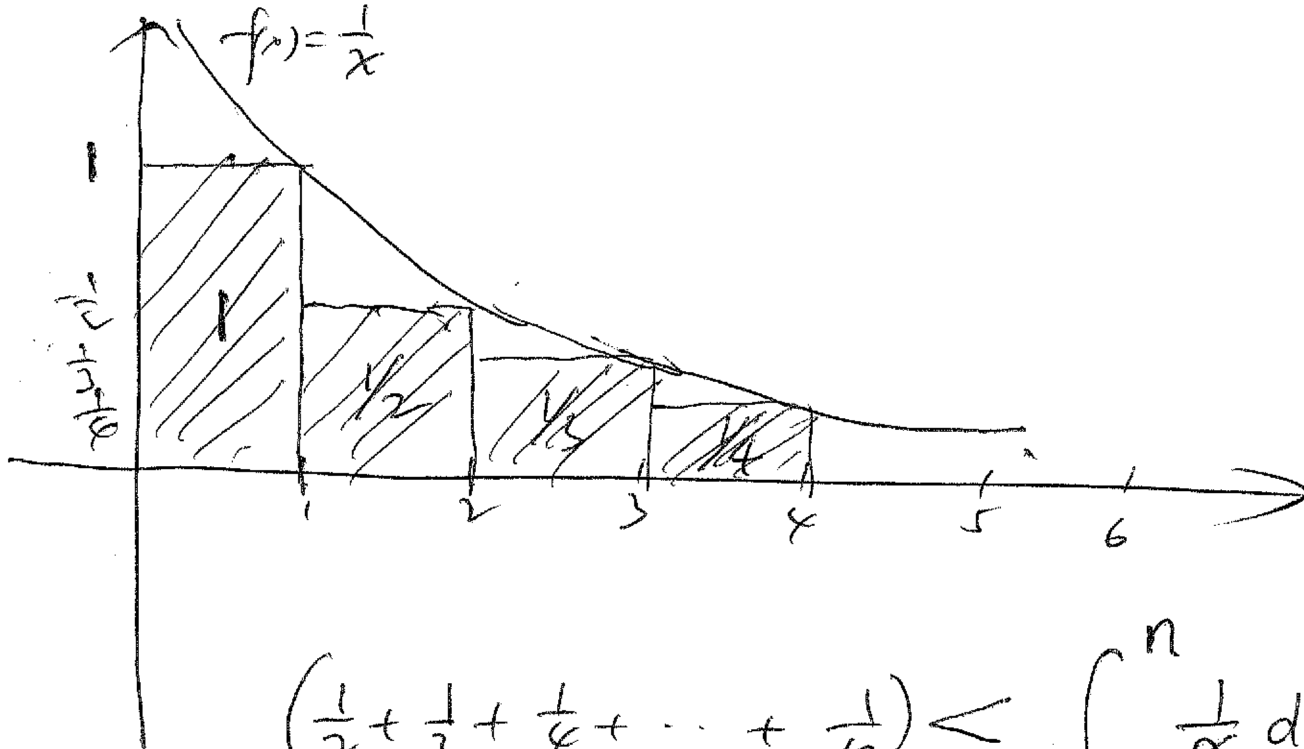
$$S = \frac{a_1 r^n - a_1}{r-1} = \frac{r a_n - a_1}{r-1}$$

Harmonic Sequence

1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\cdots$ $\frac{1}{n}$

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

$f(x) = \frac{1}{x}$



$$\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) < \int_1^n \frac{1}{x}\,dx$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}\,dx$$

$$S < 1 + \ln x \Big|_1^n = 1 + \ln n$$