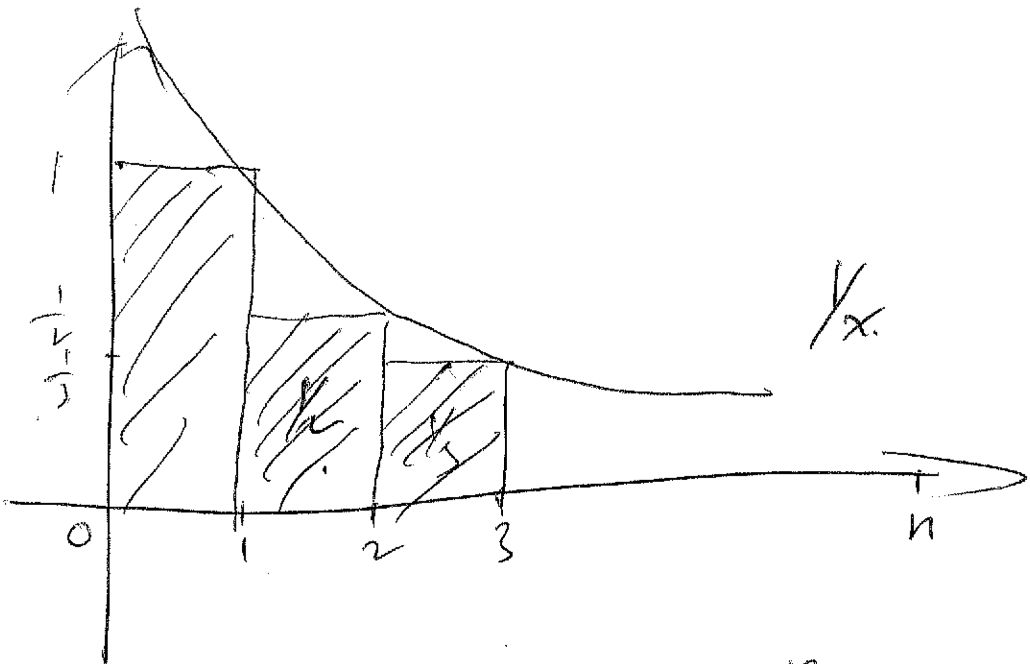
Sept 6, 2016

Harmonic Sequence

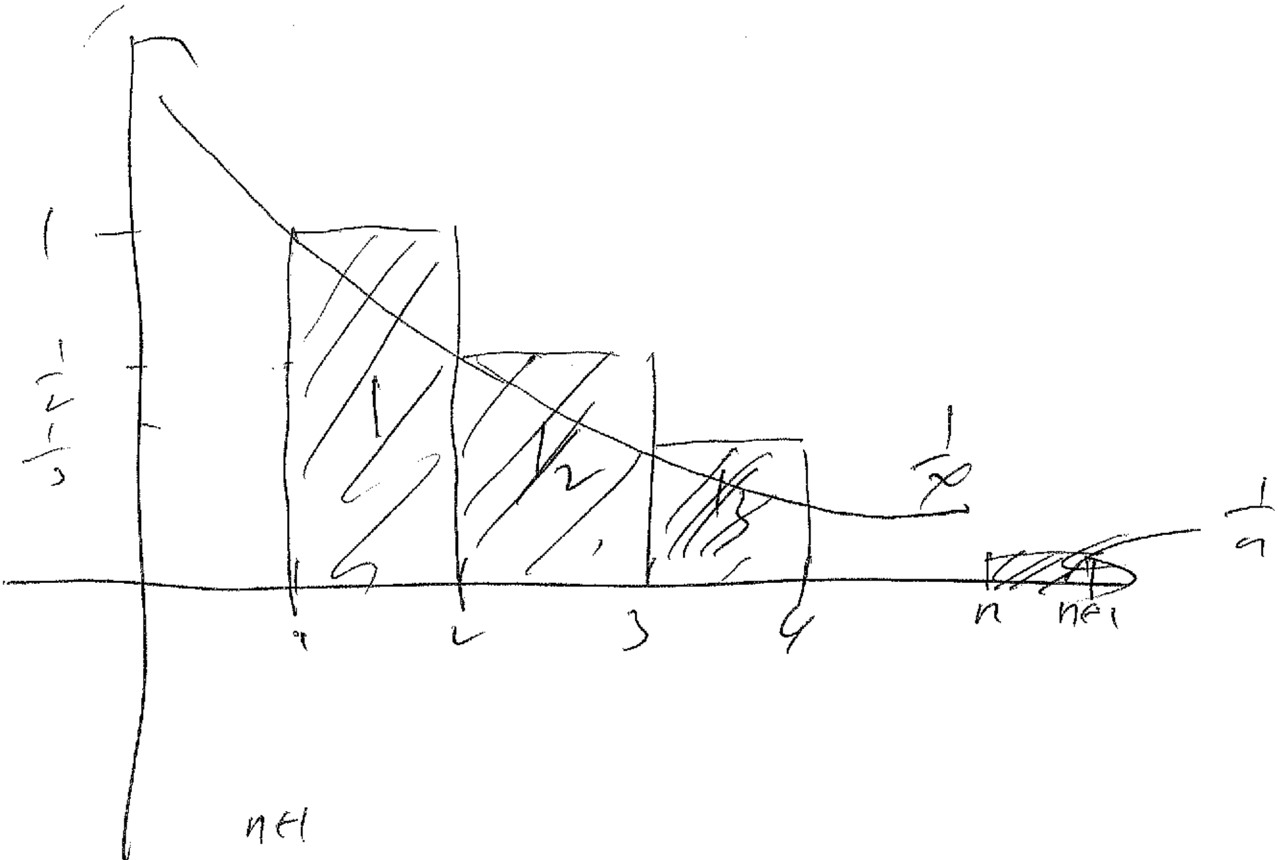
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$



$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} \leq \int_1^n \frac{1}{x}\,dx + 1$$

$$= \ln n + 1$$

②



$$\int_1^{n+1} \frac{1}{x}\,dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$\ln(n+1) \leq \sum_{j=1}^{n} \frac{1}{j} \leq \ln n + 1$$

Arithmetic : constant difference

$$a_1, a_2, \cdots, a_n$$

$$a_j - a_{j-1} = d$$

$$S = \sum_{j=1}^{n} a_j = \frac{(a_1 + a_n)\, n}{2}$$

Geometric : constant ratio

$$a_1, a_2, \cdots, a_n$$

$$\frac{a_j}{a_{j-1}} = r$$

$$S = \sum_{j=1}^{n} a_j = \frac{r a_n - a_1}{r - 1}$$

Harmonic Sequence :

$$1, \frac{1}{2}, \frac{1}{3}, \cdots$$

$$\sum_{j=1}^{n} \frac{1}{j} \approx \ln n$$

$$1 + 2^1 + 2^2 + \cdots + 2^n = \frac{2 \cdot 2^n - 1}{2 - 1} = 2^{n+1} - 1$$

$$2^{n+1} = 2^1 \cdot 2^n$$

1 ✗

2 ✗

2^2

2^3

$\vdots$

2^n ✗

2^{n+1} ✓

$2^{n+1} - 1$

$\Big\}$ \$1

Recurrence relations.

Consider a sequence

$$a_1, a_2, a_3, \ldots \quad a_n, \ldots$$

We can view a sequence as a function of natural numbers.

$$\begin{array}{ccccc} 1 & 2 & 3 & & n \\ \downarrow & \downarrow & \downarrow & \cdots\cdots & \downarrow f(n) \quad - \\ a_1 & a_2 & a_3 & & a_n \end{array}$$

$a(1)$

domain $\mathbb{N}$ or $\mathbb{Z}_{\geq 0}$

A recurrence relation is an integral function whose definition involves the function itself at a smaller argument and an exit condition.

Example

$$\begin{cases} f(n) = f(n-1) + 1 \\ f(1) = 1 \end{cases}$$

$f(1)$, $f(2)$, $f(3)$, ...

1 2 3

$$\begin{cases} n! = (n-1)! \cdot n \\ 0! = 1 \end{cases}$$

Example

$$a_n = r \cdot a_{n-1}$$

$$a_{\cancel{0}}$$

$$\begin{cases} a_n = 2a_{n-1} \\ a_1 = 2^1 \end{cases}$$

Insertion sort viewed recursively

insertion sort

input $A[1..n]$

output: A sorted

if $n=1$, return A

else.

insertion sort $A[1..n-1]$

update with $A[n]$

$$T(n) = \cancel{T(n-1)}$$

$$T(n-1) + 2(n-1)$$

$$T(1) = 1$$

$$\begin{cases} T(n) = T(n-1) + 2(n-1) \\ T(1) = 1 \end{cases}$$

$$\begin{cases} T(x) = T(x-1) + 2(x-1) \\ T(1) = 1 \end{cases}$$

$T(1)$	$T(2)$	$T(3)$	$T(4)$	$T(5)$
1	3	7	13	21

$T(2) = T(2-1) + 2(2-1) = T(1) + 2$

$T(3) = \cancel{2T(2) + 2(3-1)}$

$= T(2) + 2(3-1) = 3 + 4 = 7$

$T(4) = T(3) + 2(4-1) = 7 + 6$

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$$T(n) = T(n-1) + 2(n-1)$$

$$= [T((n-1)-1) + 2((n-1)-1)] + 2(n-1)$$

$$= T(n-2) + 2(n-2) + 2(n-1)$$

$$= [T(n-3) + 2(n-3)] + 2(n-2) + 2(n-1)$$

$$= T(n-3) + 2(n-3) + 2(n-2) + 2(n-1)$$

$$= T(n-4) + 2(n-4) + 2(n-3) + \cdots + 2(n-1)$$

$$= \cdots =$$

$$= T(n-(n-1)) + 2 + 4 + \cdots + 2(n-1)$$

$$= T(1) + 2 + 4 + \cdots + 2(n-1)$$

$$= 1 + (2 + 4 + 6 + \cdots + 2(n-1))$$

$$= 1 + \frac{(2 + 2(n-1))(n-1)}{2} =$$

$$= 1 + \frac{n(n-1)}{} = n^2 - n + 1$$

$n(n-1)$

Claim for the recurrence relation

$$\begin{cases} T(n) = T(n-1) + 2(n-1) \\ T(1) = 1 \end{cases}$$

the solution is $T(n) = n^2 - n + 1$ for $n = 1, 2, 3, \ldots$

Verification

$T(1) = \frac{1}{2} - \frac{1}{2} + 1 = 1$

$T(2) = 2^2 - 2 + 1 = 3$

$T(3) = 3^2 - 3 + 1 = 7$

$T(4) = 4^2 - 4 + 1 = 13$

Induction Theorem.

Given a statement P with respect to natural numbers n. To prove that $P(n)$ is correct for all natural number n, it suffices to do the following.

Basis: check if P is correct for $n=1$

I.S.: Assume that ~~P(k)~~ $P(k)$ is correct.

and show ~~that~~ P is also correct for $k+1$

by using the assumption $P(k)$ is correct

Induction Hypothesis

Back to the recursion

Basis ✓

I.S.

Assume that $T(k) = k^2 - k + 1$

Need to show $T(k+1) = (k+1)^2 - (k+1) + 1$

by ~~u~~ using the assuption.

$$\boxed{\begin{aligned} &= k^2 + 2k + 1 - k - 1 + 1 \\ &= k^2 + k + 1 \end{aligned}}$$

From the def of the recurrence relation, we know

$$T(k+1) = T(k) + 2k$$

$$= (k^2 - k + 1) + 2k$$

$$= k^2 + k + 1 = (k+1)^2 - (k+1) + 1$$

The induction step went through, hence the claim is correct.

Some advanced techniques for solving recurrence relations

$$\begin{cases} F_n = F_{n-1} + F_{n-2} \\ F_1 = 1 \\ F_0 = 1 \end{cases}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right)$$

Characteristic Equation Method

Linear homogeneous recurrence relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Example

$$\begin{cases} a_n = 2a_{n-1} \\ a_1 \end{cases} \quad \checkmark \qquad a_n = a_1 2^{n-1}$$

$$a_n = a_{n-1} + 1 \quad \times$$