University of New Mexico
Department of Computer Science
Exam III
CS261: Mathematical Foundations of Computer Science

Last Name:______________________________
First Name:______________________________
Email:_______________________________

Instructions:

1. Write your name and email address legibly in the space provided above.
2. Write your name legibly at the upper right hand corner on each page.
3. There are 4 problems in the exam.
4. This is a close-book exam. You must not discuss the questions with anyone except the professor in charge.
5. You are allowed to use a one page double-sided handwritten “cheating sheet” that you have brought to the exam and a “dumb” calculator. Nothing else permitted.
6. Write your answers legibly.
7. Don’t spend too much time on any single problem. All questions are weighted equally. If you get stuck, move on to something else and get back later.
8. Good luck, enjoy the exam, and have great summer break!

1. (Relation) Answer the following questions with respect to relatio:

(a) Let $A = \{1,2,3,4,5,6,7,8,9\}$. $R = \{(a,b) \lfloor\, a \equiv b \bmod 4\}$.

(a.1) Write out the Boolean matrix representing R.

(a.2) Draw the digraph of the relation R.

(a.3) Is R an equivalence relation? Explain why? If yes, write out its equivalence classes.

(a.4) Is R a partial order? Explain why? If yes, is R a linear order?

(b) Let $G(V, E)$ be a directed graph as shown in Figure 1. Consider the relation S defined on V, where two vertices $u, v \in V$ satisfy the relation S, i.e., $(u, v) \in S$ if there is a directed path from u to v.

(b.1) Is S reflexive? Explain why?

(b.2) Is S anti-symmetric? Explain why?

(b.3) Is S transitive? Explain why?

(b.4) Is S a partial order? If yes, is S a lattice? Explain why or why not?

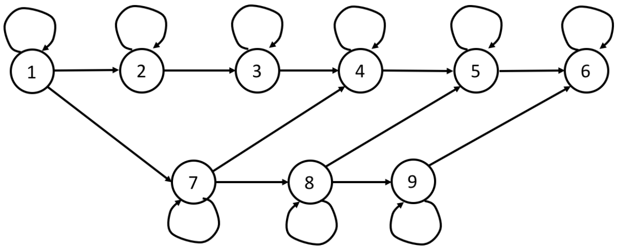

Figure 1

Solution:

(a) This is an equivalence relation, but NOT a partial order.

(b) The relation is transitive, because if one can go from i to j and from j to k, then there is a path from i to k.

The relation is a partial order and a lattice.

2. (Five Room Puzzle) Consider the room layout as shown in Figure 2. Is it possible to go through every door exactly once? Explain why or why not.

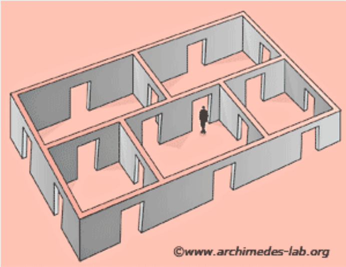

Figure 2

Solution:

Introduce a vertex for each room, introduce a vertex for the outside.

Connect two vertices together if there is a door connecting them.

For the graph thus constructed, there is no Euler circuit of Euler path.

Note, one can’t construct a graph treating each door as a vertex. The problem will become Hamiltonian, which is not computational tractable.

3. (Linear Diophantine Equations) For this problem, we are interested in linear Diophantine Equations of the form: $ax + by = c$, where a, b, c are integers, and x and y are integer unknowns.

(a) Consider the linear Diophantine Equation $13x + 11y = 1$. Can you find integers x and y satisfying the equation? Explain why?

(b) Consider the linear Diophantine Equation $13x + 11y = 5$. Can you find integers x and y satisfying the equation? Explain why?

(c) Consider the linear Diophantine Equation $4x + 6y = 3$. Can you find integers x and y satisfying the equation? Explain why?

Hint: Bezout's Identity.

Solution:

(a) From Bezout's identity, there exists s and t, such that $13x + 11y = 1$.

(b) By multiplying 5 to both sides of the equation in (a), one obtains the solution to (b).

(c) As discussed in class, this is not solvable. The left-hand side is an even number, while the right-hand side is odd.

4. (Probability) Answer the following questions:

(a) Let $A = \{1, 2, 3, \ldots, 100\}$, i.e., the set of integers from 1 to 100.

(a.1) Suppose two distinct integers X and Y are chosen uniformly at random from A, meaning any subset of two integers are equally likely to be chosen. What is the probability that the two integers are 1 and 100?

(a.2) Suppose two distinct integers X and Y are chosen uniformly at random from A. Suppose someone told you that X is > 50.5. Given this, what is the probability that Y is < 50.5?

(b) Suppose a prisoner is sentenced to death, but offered a chance to live by playing a simple game. The prisoner was given 50 black marbles, 50 white marbles and 2 empty bowls. The prisoner can divide the 100 marbles into the 2 bowls any way he likes provided all marbles are used. After this is done, the prisoner will be blindfolded and given one of the two bowls with equal probability. The prisoner will then remove ONE marble. If the marble is WHITE the prisoner will live, otherwise the prisoner will die. How should the prisoner divide the marbles up to maximize his probability of survival?

Solution:

(a) $\frac{1}{\binom{100}{2}}$ or $2 \cdot \frac{1}{100P2}$

Let E be the event that $X > 50.5$, and F be the event $Y < 50.5$.

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{|E \cap F|}{|E|} = \frac{50 \times 50}{50 \times 99} = \frac{50}{99}$$

(b) Put a single white marble in one bowl and the rest in the other. The probability of survival is $0.5 + \frac{49}{99}$.