Sept 8th

Characteristic Equation

$$\begin{cases} F_n = F_{n-1} + F_{n-2} \\ F_1 = 1 \\ F_0 = 1 \end{cases}$$

Assume $F_n = r^n$ for some $r \neq 0$

$F_0 = r^0 \quad F_1 = r \cancel{\phi} = 1 \neq 0$

$F_n = F_{n-1} + F_{n-2}$

$r^n = r^{n-1} + r^{n-2}$

Factor out r^{n-2} we get $r^2 = r + 1$

$r^2 - r - 1 = 0 \qquad r = \dfrac{1 \pm \sqrt{5}}{2}$

$r_1 = \dfrac{1+\sqrt{5}}{2} \qquad r_2 = \dfrac{1-\sqrt{5}}{2}$

Observe that if r satisfies the recurrence relation

$$r^n = r^{n-1} + r^{n-2},$$

then $c r^n$ also satisfies the recursion for some constant $c \neq 0$

$$\underbrace{c r^n}_{F_n} = \underbrace{c r^{n-1}}_{F_{n-1}} + \underbrace{c r^{n-2}}_{F_{n-2}}$$

The solutions of the recursion is of the format

$$F_n = c_1 r_1^n + c_2 r_2^n$$

$$= c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$F_1 = 1$. $\quad \begin{cases} c_1 \dfrac{1+\sqrt{5}}{2} + c_2 \dfrac{1-\sqrt{5}}{2} = 1 \\ c_1 + c_2 = 1 \end{cases}$

$F_0 = 1$

$$\begin{cases} c_1 = \dfrac{1+\sqrt{5}}{2\sqrt{5}} \\ c_2 = -\dfrac{1-\sqrt{5}}{2\sqrt{5}} \end{cases}$$

$$C_1 + C_2 = \frac{1+\sqrt{5}}{2\sqrt{5}} + \left(-\frac{1-\sqrt{5}}{2\sqrt{5}}\right) = \frac{1+\sqrt{5} - 1+\sqrt{5}}{2\sqrt{5}} = 1$$

$$C_1 \frac{1+\sqrt{5}}{2} + C_2 \frac{1-\sqrt{5}}{2}$$

$$= \frac{1+\sqrt{5}}{2\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2} + -\frac{1-\sqrt{5}}{2\sqrt{5}} \frac{1-\sqrt{5}}{2}$$

$$= \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{4\sqrt{5}}$$

$$= \frac{(\cancel{1}+2\sqrt{5}+\cancel{5}) - (\cancel{1}-2\sqrt{5}+\cancel{5})}{4\sqrt{5}}$$

$$= 1$$

$$\begin{cases} a_n = 3a_{n-1} - 2a_{n-2} \\ a_1 = 1 \\ a_0 = 0 \end{cases}$$

Solution,

Use characteristic equation method

Assume $a_n = r^n$ for $r \neq 0$

$$r^n = 3r^{n-1} - 2r^{n-2}$$

Factor r^{n-2}, we get $r^2 = 3r - 2$

$r^2 - 3r + 2 = 0$ ⟵ characteristic equation

$(r-1)(r-2) = 0$

$r_1 = 1 \quad r_2 = 2$

The solution is of the form

$$a_n = \underline{c_1} 1^n + \underline{c_2} 2^n$$

$c_1 + 2c_2 = 1$

$c_1 + c_2 = 0$

$$\begin{cases} c_1 = -1 \\ c_2 = +1 \end{cases}$$

$$a_n = a_{n-1} + 2$$

$$a_n - a_{n-1} = 2$$

$$\frac{a_n - a_{n-1}}{n-(n-1)} = 2$$

$$\frac{da}{dx} = 2$$

$$\begin{cases} a_n = 2a_{n-1} - a_{n-2} \\ a_1 = 4 \\ a_0 = 2 \end{cases}$$

Solution. characteristic equation:

$$r^2 = 2r - 1$$

$$r^2 - 2r + 1 = 0 \qquad (r-1)^2 = 0$$

$$r_{1,2} = 1$$

Assume the solution is of the form $\cancel{c_1 1^n + c_2 1^n}$

Assume the solution is of the form $c_1 + n c_2$

Zxample

$$\begin{cases} a_n = 2a_{n-1} + 1 \\ a_0 = 1 \end{cases}$$

Observation :

$$a_n = 2a_{n-1} + 1 \qquad ①$$

$$a_{n-1} = 2a_{n-2} + 1 \qquad ②$$

①−② $\qquad a_n - a_{n-1} = 2a_{n-1} - 2a_{n-2}$

$$\begin{cases} a_n = 3a_{n-1} - 2a_{n-2} \\ a_1 = 2a_0 + 1 = 3 \\ a_0 = 1 \end{cases}$$

Generating Functions.

$$\begin{cases} a_n = 2a_{n-1} + 1 \\ a_0 = 0 \end{cases}$$

The generating function ~~is~~ for the sequence a_n is defined as

$$G(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots +$$

$$G(x) = a_0 + \quad a_1 x + \quad a_2 x^2 + \quad a_3 x^3 +$$

$$= a_0 + (2a_0+1)x + (2a_1+1)x^2 + (2a_2+1)x^3 +$$

$$= a_0 +$$
$$2a_0 x \quad + \quad 2a_1 x^2 \quad + \quad 2a_2 x^3 + \cdots$$
$$x \quad + \quad x^2 \quad + \quad x^3 \quad + \cdots$$

$$= a_0 +$$
$$+ 2x(\underline{a_0 + a_1 x + a_2 x^2 + \cdots})$$
$$+ x + x^2 + x^3 + \cdots$$

$$= \cancel{a_0} +$$
$$+ 2x G$$
$$+ x + x^2 + x^3 + \cdots$$

$$G = 2x G + (x + x^2 + x^3 + \cdots)$$

$$G = 2x G + \frac{x}{1-x}$$

$$(1-2x)G = \frac{x}{1-x}$$

$$G = \frac{1}{1-2x} \cdot \frac{x}{1-x}$$

To get the a_n, we do the Taylor expansion of G.

Taylor Expansion

$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$+ \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$(x^a)' = ax^{a-1}$$

$$f'(x) = -1\,(1-x)^{-2}\,(-1) = (1-x)^{-2}$$

$$f''(x) = (-2)\,(1-x)^{-3}(-1) = 2!\,(1-x)^{-3}$$

$$f'''(x) = (f''(x))' = (-3)\left(2!\,(1-x)^{-4}\right)(-1) = 3!\,(1-x)^{-4}$$

$$\vdots$$

$$f^{(n)}(x) = n!\,(1-x)^{-n-1}$$

$$f(x) \approx 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$x + x^2 + x^3 + x^6 + \cdots = x\left(1 + x + x^2 + \cdots\right) = \frac{x}{1-x}$$

(11)

$$\frac{1}{1-2x}$$

Let $y = 2x$

$$\frac{1}{1-2x} = \frac{1}{1-y} = 1+y+y^2+y^3+\cdots$$

$$= 1+(2x)+(2x)^2+(2x)^3+\cdots$$

$$= 1+2x+2^2x^2+2^3x^3+\cdots$$

$(1x + 2x^2 + 3x^3 + 4x^6)(4x + 3x^2 + 2x^3 + 1x^6)$

What's the coefficient for x^6

$2x^2 \cdot 1x^6 + 3x^3 2x^3 + 4x^6 \cdot 3x^2$

$= (2 + 6 + 12)x^6 = 20x^6$

Back to $G(x)$

$$G(x) = \frac{1}{1-2x} \cdot \frac{x}{1-x}$$

$$= (1 + 2x + 2^2x^2 + 2^3x^3 + \cdots) \cdot (x + x^2 + x^3 + \cdots)$$

$$= \underset{a_0}{0}x^0 + \underset{a_1}{?}x^1 + \underset{a_2}{?}x^2 + \cdots + \underset{a_n}{?}x^n + \cdots$$

$$a_n = 1 \cdot 1 + 2 \cdot 1 + \cdots + 2^{n-1} \cdot 1$$

$$= 2^n - 1$$

$$\begin{cases} a_n = 2a_{n-1} + 1 \\ a_0 = 0 \end{cases}$$

From Generating Function, we found $a_n = 2^n - 1$

Prove that this is correct for all n.

Proof by Induction.

Basis $n=0$

$$a_0 = 2^0 - 1 = 0 \checkmark$$

I.S.

Assume for $n=k$, $a_k = 2^k - 1$

Need to show $a_{k+1} = 2^{k+1} - 1$ using the induction hypothesis

$$a_{k+1} = 2a_k + 1 = 2(2^k - 1) + 1$$

$$= 2 \cdot 2^k - 2 + 1$$

$$= 2^{k+1} - 1$$

$$
\begin{cases}
F_n = F_{n-1} + F_{n-2} \\
F_1 = 1 \\
F_0 = 1
\end{cases}
$$

Solution

$$G(x) = F_0 x^0 + F_1 x^1 + F_2 x^2 + F_3 x^3 + F_4 x^4 \ldots$$

$$= F_0 + F_1 x + (F_1 + F_0) x^2 + (F_2 + F_1) x^3 + (F_3 + F_2) x^4$$

$$+ \cdots$$

$$= F_0 + F_1 x$$
$$+ (F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots)$$
$$+ (F_0 x^2 + F_1 x^3 + F_2 x^4 + \cdots)$$

$$= F_0 + F_1 x$$
$$+ x^2 (F_1 + F_2 x + F_3 x^2 + \cdots)$$
$$+ x^2 \underline{(F_0 + F_1 x + F_2 x^2 + \cdots)}$$

$G(x) = F_0 + F_1 x +$

$x^2 (F_1 + F_2 x + F_3 x^2 \cdots)$

$+ x^2 G$

$\Rightarrow$ $\cancel{F_0 + x^2 G}$

$\cancel{+ x (F_1 + F}$

$= F_0 + F_1 x + x^2 G$

$+ x (F_1 x + F_2 x^2 + F_3 x^3 + \cdots)$

$= F_0 + F_1 x + x^2 G$

$+ x (G - F_0)$

$G = F_0 + F_1 x + x^2 G + x (G - F_0)$