Aug 30, 2016

Cardinality of Infinite Sets

a measure of how many elements in an infinite set

Let A, B be two infinite sets

If there is a one-to-one correspondance from A to B, then $|A| = |B|$

If there is ~~an~~ a one-to-one function from A to B, then $|A| \leq |B|$

If there is an onto function from A to B ~~then~~ ~~|A|~~ $|A| \geq |B|$

Recall $R \subseteq A \times B$

Functions

one-to-one functions $f: A \to B$

For $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$

onto :

$|\mathbb{N}| = |\mathbb{Z}|$

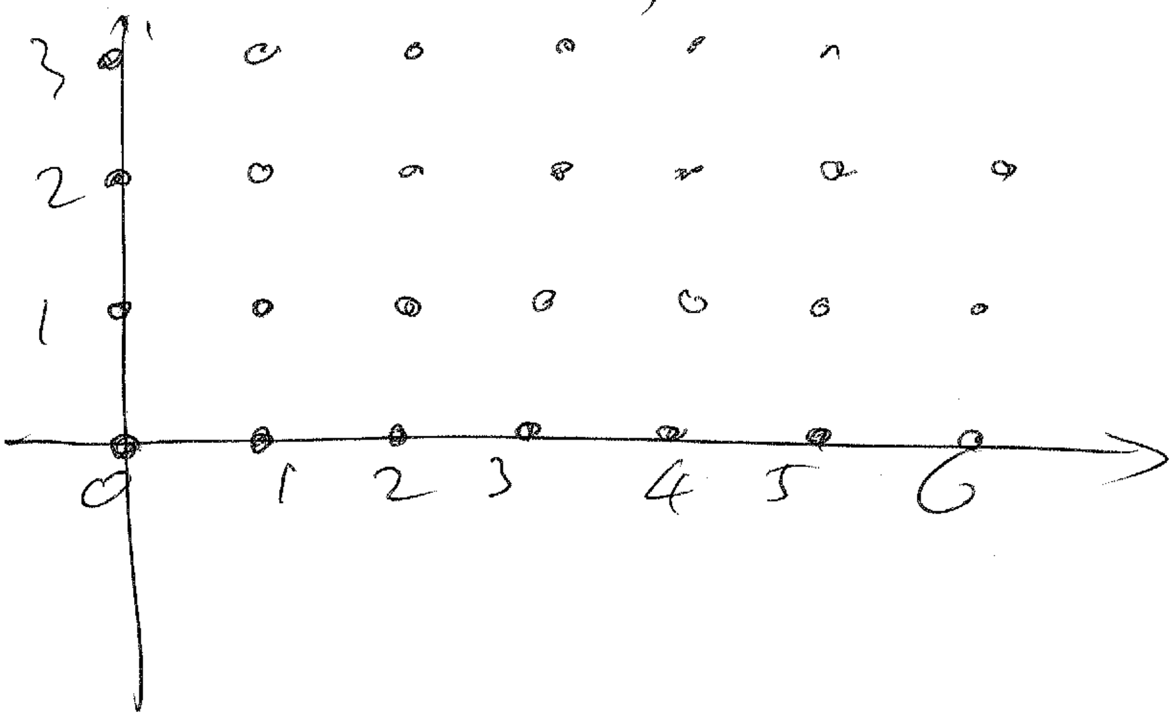
~~Prove~~ Prove that $\cancel{\mathbb{Z}^+ \times \mathbb{Z}}$

$$\left| \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \right| = |\mathbb{N}|$$

where $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \cdots\}$

$\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} = \{(0,0), (0,1), (1,0), \cdots\}$

$\mathbb{N} = \{1, 2, \cdots\}$



(0,0) 1

(0,1) (1,0) 2 3

(0,2) (1,1) (2,0) 4 5 6

(0,3) (1,2) (2,1) (3,0) 7 8 9 10

. . .

Let $\mathbb{N}=\{1,2,3,\dots\}$ be the set of natural numbers

The cardinality of $\mathbb{N}$ is denoted by $\aleph_0$

Any set whose cardinality is $\leq \aleph_0$ is said to be countable

E.g. $S=\{1,2,3,\dots,100\}$

$$f: S \to \mathbb{N}$$

$$f(s) = s$$

$S=\{1,3,5,7,9,\dots\}$ $f: S \to \mathbb{N}$ $|S| \leq |\mathbb{N}|$

$\mathbb{Z}$

$\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad \cdots$

$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots$

$$f(s) = \frac{s+1}{2}$$

for $n \in \mathbb{N}$. $\quad f(2n-1) = \frac{(2n-1)+1}{2} = n$

Let A be a <u>countably infinite</u> set, and $|A| = \aleph_0$

Then there is a one-to-one correspondence f from the elements of A to the natural numbers

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6, \ldots$

$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \ldots$

Consider the following proof $|\mathbb{Z}| = |\mathbb{N}|$

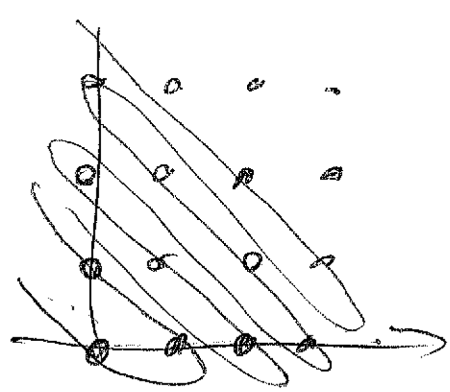
Recall $\quad 0 \quad +1 \quad -1 \quad +2 \quad -2 \quad +3 \ldots$

$\quad\quad\quad\; 1 \quad\; 2 \quad\;\; 3 \quad\;\; 4 \quad\;\; 5 \quad\;\; 6 \ldots$

Mapping

~~$0, 1, 2, \ldots, -1, -2, \ldots$~~

$\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$



~~$(0,0),\ (0,1),\ (0,2),\ (0,3),\ \ldots$~~

~~$(1,0),\ (1,1),\ (1,2),\ (1,3),\ \ldots$~~

~~$\ldots$~~

Prove that $|\mathbb{Q}^+| = |\mathbb{N}|$

$$\mathbb{Q}^+ = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z}^+ \right\}$$

$$\sqrt{2} \notin \mathbb{Q}^+$$

$$\frac{2}{3} \in \mathbb{Q}^+$$

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① Consider $f: \mathbb{Q}^+ \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

$$f\left(\frac{p}{q}\right) = (p, q)$$

f is one-to-one. $|\mathbb{Q}^+| \leq |\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}| = |\mathbb{N}|$

On the other hand $\mathbb{N} \subseteq \mathbb{Q}^+$, hence

$$|\mathbb{N}| \leq |\mathbb{Q}^+|$$

Thus $|\mathbb{Q}^+| = |\mathbb{N}|$

$$\frac{1}{1} \qquad 2$$

$$\frac{1}{2} \quad \frac{2}{1} \qquad 3$$

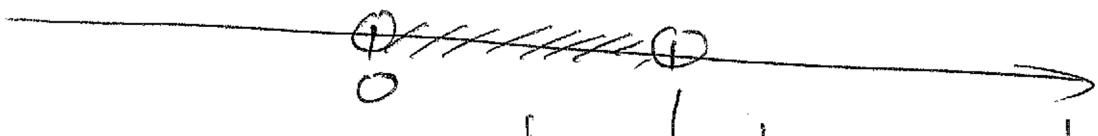
$$\frac{1}{3} \quad \cancel{\frac{2}{2}} \quad \frac{3}{1} \qquad 4$$

$$\frac{1}{4} \quad \frac{2}{3} \quad \frac{3}{2} \quad \frac{4}{1} \qquad 5$$

Thus Rational Numbers are contable

$\mathbb{R}$ are not countable.

$(0, 1)$ are not countable.



We need to show $|(0,1)| \neq |\mathbb{N}|$

there is no one-to-one correspondence between ~~$|(0,1)|$~~ $(0,1)$ and $\mathbb{N}$.

Proof by Contradiction

Assume the claim is wrong
and show that the assumption will lead to a contradiction.

Using proof by contradiction.

Assume there is a one-to-one correspondence from $(0,1)$ to $\mathbb{N}$.

$$
\begin{array}{ll}
0.d_{11}\,d_{12}\,d_{13}\,d_{14}\,d_{15}\cdots & 1 \\
0.d_{21}\,d_{22}\,d_{23}\,d_{24}\,d_{25}\cdots & 2 \\
0.d_{31}\,d_{32}\,d_{33}\,d_{34}\,d_{35}\cdots & 3 \\
0.d_{41}\,d_{42}\,d_{43}\,d_{44}\,d_{45}\cdots & 4 \\
\cdots &
\end{array}
$$

$$0.d_1\,d_2\,d_3\,d_4\,d_5\cdots$$

$$d_j = \begin{cases} 4 & \text{if } d_{jj} = 5 \\ 5 & \text{otherwise} \end{cases} \qquad j = 1, 2, \cdots$$

Observe that the number $0.d_1d_2d_3\cdots$ is not on the line a Contradiction!

Thus we know that

$$\left.\begin{array}{l} |\mathbb{R}| \neq |\mathbb{N}| \\ |\mathbb{N}| \leq |\mathbb{R}| \end{array}\right\} \qquad |\mathbb{N}| < |\mathbb{R}| = \aleph_1$$

Theorom. For any set S, $|S| < |2^S|$

$$2^{\mathbb{R}} \qquad\qquad |\mathbb{R}| = |2^{\mathbb{N}}|$$

$$2^{2^{\mathbb{R}}}$$

Assume the claim is wrong., $|S| \geq |2^S|$

Then there is an onto function $f: S \to 2^S$

for $s \in S$, $f(s) \subseteq S$

Consider $X = \{ x \mid x \in S, x \notin f(x) \}$

Since $X \subseteq S$, f is an onto function

there exists $x^* \in S$, such $f(x^*) = X$

There two possibilities

if $x^* \in X$, so x^* is mapped to a set X

~~that contains~~ x^*, $x^* \notin X$

$x^* \notin X$, x^* is mapped to a set X that

doesn't contain x^*, so $x^* \in X$