UNM SCHOOL of ENGINEERING
Department of Computer Science

Floating Point

CS 341: Intro. to Computer Architecture & Organization

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(Slides courtesy of
Prof. Dorian Arnold)

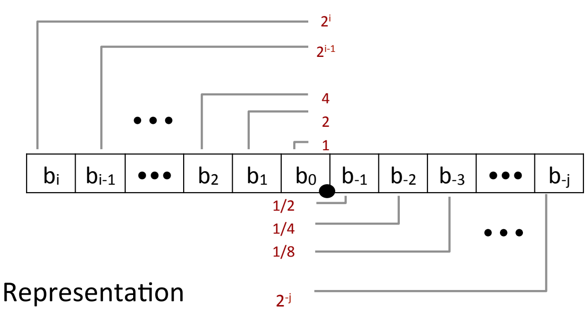
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \times 2^k$

Fractional Binary Numbers: Examples

- Value Representation
  5 3/4 101.11_2
  2 7/8 10.111_2
  63/64 0.111111_2

- Observations
- Divide by 2 by shifting right
- Multiply by 2 by shifting left
- Numbers of form $0.111111\ldots_2$ are just below 1.0
  - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
  - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation
  - Can only exactly represent numbers of the form $x2^k$
  - Other rational numbers have repeating bit representations

- Value Representation
  - 1/3 $0.0101010101[01]\ldots_2$
  - 1/5 $0.001100110011[0011]\ldots_2$
  - 1/10 $0.0001100110011[0011]\ldots_2$

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IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- Numerical Form:

  $$(-1)^s\ M\ \ 2^E$$

  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0).
  - Exponent E weights value by power of two
- Encoding
  - MSB `s` is sign bit s
  - `exp` field encodes E (but is not equal to E)
  - `frac` field encodes M (but is not equal to M)

s	exp	frac

Precisions

- Single precision: 32 bits

s	exp	frac
1	8-bits	23-bits

- Double precision: 64 bits

s	exp	frac
1	11-bits	52-bits

- Extended precision: 80 bits (Intel only)

s	exp	frac
1	15-bits	63 or 64-bits

Normalized Values

- Condition: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as biased value: E = Exp – Bias
  - Exp: unsigned value `exp`
  - Bias = 2^{k-1} - 1, where k is number of exponent bits
    - Single precision: Bias: 127 (Exp: 1…254, E: -126…127)
    - Double precision: Bias: 1023 (Exp: 1…2046, E: -1022…1023)
- Significand coded with implied leading 1: M = `1.xxx…x`$_2$
  - `xxx…x`: bits of `frac`
  - Minimum when `000…0` (M = 1.0)
  - Maximum when `111…1` (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

- Value: `Float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$
- Significand
  - *M* = `1.1101101101101`$_2$
  - **`frac`** = `11011011011010000000000`$_2$
- Exponent
  - *E* = 13
  - *Bias* = 127
  - *Exp* = 140 = `10001100`$_2$
- Result:

s	exp	frac
0	10001100	11011011011010000000000

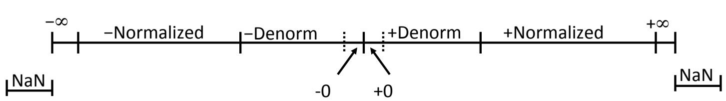
Denormalized Values

- Condition: exp = 000…0
- Exponent value: E = 1 – Bias (instead of E = 0 – Bias)
- Significand coded with implied leading 0: M = 0.xxx…x_2
  - xxx…x: bits of frac
- Cases
  - exp = 000…0, frac = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - exp = 000…0, frac ≠ 000…0
    - Numbers very close to 0.0
    - Lose precision as get smaller
    - Equispaced

Special Values

- Condition: exp = 111…1
- Case: exp = 111…1, frac = 000…0
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞
- Case: exp = 111…1, frac ≠ 000…0
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

Visualization: Floating Point Encodings



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Tiny Floating Point Example

s	exp	frac
1	4-bits	3-bits

- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`
- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

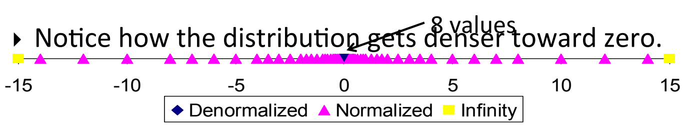
	s exp frac	E	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8 * 1/64 = 1/512	closest to zero
	0 0000 010	-6	2/8 * 1/64 = 2/512	
	…			
	0 0000 110	-6	6/8 * 1/64 = 6/512	
	0 0000 111	-6	7/8 * 1/64 = 7/512	largest denorm
Normalized numbers	0 0001 000	-6	8/8 * 1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8 * 1/64 = 9/512	
	…			
	0 0110 110	-1	14/8 * 1/2 = 14/16	
	0 0110 111	-1	15/8 * 1/2 = 15/16	closest to 1 below
	0 0111 000	0	8/8 * 1 = 1	
	0 0111 001	0	9/8 * 1 = 9/8	closest to 1 above
	0 0111 010	0	10/8 * 1 = 10/8	
	…			
	0 1110 110	7	14/8 * 128 = 224	
	0 1110 111	7	15/8 * 128 = 240	largest norm
	0 1111 000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

s	exp	frac
1	3-bits	2-bits

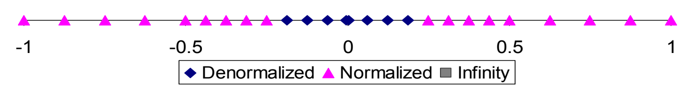
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

s	exp	frac
1	3-bits	2-bits



Interesting Numbers

{single,double}

Description	*exp*	*frac*	*Numeric Value*
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
◦ Single ≈ 1.4×10^{-45}			
◦ Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
◦ Single ≈ 1.18×10^{-38}			
◦ Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
◦ Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
◦ Single ≈ 3.4×10^{38}			
◦ Double ≈ 1.8×10^{308}			

Special Properties of Encoding

- FP Zero Same as Integer Zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Floating Point Operations: Basic Idea

- `x +f y = Round(x + y)`
- `x ×f y = Round(x × y)`
- Basic idea
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`

Rounding

- Rounding Modes (illustrate with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1	$1	$1	$2	–$1
Round down (−∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

- What are the advantages of the modes?

Closer Look at Round-To-Even

- Default Rounding Mode
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated
- Applying to Other Decimal Places / Bit Positions
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- Binary Fractional Numbers
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100\ldots_2$
- Examples
  - Round to nearest 1/4 (2 bits right of binary point)

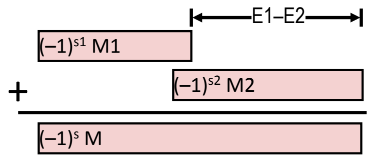
Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \times\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s: s1 ^ s2
  - Significand M: M1 x M2
  - Exponent E: E1 + E2
- Fixing
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision
- Implementation
  - Biggest chore is multiplying significands

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ +\ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume E1 > E2
- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1
- Fixing
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision



Mathematical Properties of FP Add

- Compare to those of Abelian Group
  - Closed under addition? Yes
    - But may generate infinity or NaN
  - Commutative? Yes
  - Associative? No
    - Overflow and inexactness of rounding
  - 0 is additive identity? Yes
  - Every element has additive inverse Almost
    - Except for infinities & NaNs
- Monotonicity
  - a ≥ b ⇒ a+c ≥ b+c? Almost
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- Compare to Commutative Ring
  - Closed under multiplication?
    - But may generate infinity or NaN
  - Multiplication Commutative?
  - Multiplication is Associative?
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity?
  - Multiplication distributes over addition?
    - Possibility of overflow, inexactness of rounding
- Monotonicity
  - a ≥ b & c ≥ 0 ⇒ a * c ≥ b *c?
    - Except for infinities & NaNs

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Floating Point in C

- C Guarantees Two Levels
  - `float` single precision
  - `double` double precision
- Conversions/Casting
  - Casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - `int` → `double`
    - Exact conversion, as long as `int` has ≤ 53 bit word size
  - `int` → `float`
    - Will round according to rounding mode

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither `d` nor `f` is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form M x 2^E
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

More Slides

Creating Floating Point Number

s	exp	frac
1	4-bits	3-bits

- Steps
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding
- Case Study
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

128	10000000
15	00001101
33	00010001
35	00010011
138	10001010
63	00111111

Normalize

s	exp	frac
1	4-bits	3-bits

- Requirement
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	Binary	Fraction	Exponent
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- Round up conditions
  - Round = 1, Sticky = 1 ➙ > 0.5
  - Guard = 1, Round = 1, Sticky = 0 ➙ Round to even

Value	Fraction	GRS	Incr?	Rounded
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- Issue
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Result
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64