March 31

Reminder Exam II Next Thursday (Apr 7)

Min-cut Algorithm

While there are > 2 vertices,

randomly choose an edge and contract.

return all edges between the 2 remaining vertices

What's the probability that ~~the~~ a mincut survives to the very end?

$$\geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3}$$

$$= \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}$$

Suppose we run the algorithm $\binom{n}{2}$ times,

~~and~~ each run will produce a cut, we will return the smallest cut out of these $\binom{n}{2}$ runs.

What's the probability that this returned cut is a min cut?

What's the probability that the returned cut is not a mincut?

$$\left(1-\frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}} \approx \left(e^{-\frac{1}{\binom{n}{2}}}\right)^{\binom{n}{2}} = e^{-1}$$

Recall the Taylor expansion of

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

When x is small, $e^x \approx 1+x$, $e^{-x} \approx 1+(-x)$

$$= 1-x$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

What's the probability of not finding a mincut when we run the algorithm $\binom{n}{2} \ln n$ time?

$$\left(\left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}}\right)^{\ln n}$$

$$= \left(e^{-1}\right)^{\ln n} = \frac{1}{n}.$$

For exam,

(1) understand the difference between Las Vegas and Monte Carlo Algorithms

(2) Basic tail analysis technique

Markov Inequality $\Pr(X \geq \delta) \leq \frac{E[X]}{\delta}$

Chebyshev Inequality

Randomized Linear Programming

$$\max \quad c_1 x_1 + c_2 x_2$$

$$\text{s.t.} \quad a_{11} x_1 + a_{12} x_2 \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 \leq b_2$$

$$\vdots$$

$$a_{n1} x_1 + a_{n2} x_2 \leq b_n$$

$$\max \quad x_1 + x_2 \quad \leftarrow \text{objective function}$$

$$\text{s.t.} \quad \left.\begin{aligned} 2x_1 + x_2 &\leq 4 \\ -x_1 &\leq 0 \\ -x_2 &\leq 0 \end{aligned}\right\} \text{linear constraints}$$

$$\begin{cases} x_1 = 0 \\ x_2 = 4 \end{cases}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad$ max $c^T x$

$C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$= (c_1, c_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= c_1 x_1 + c_2 x_2$$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \\ a_{n1} & a_{n2} \end{bmatrix}$ $\quad$ $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

then the constraint becomes $\quad Ax \le b$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \\ a_{n1} & a_{n2} \end{bmatrix}_{n\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} \le \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 \end{bmatrix} \le \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The condensed form

$$\max \ c^T x \qquad \min \ -c^T x$$

$$\text{s.t.} \ Ax \le b \qquad \text{s.t.} \ -Ax \ge -b$$

$$\max \ x_1 + x_2 \qquad \max \ (1,1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{s.t.} \ 2x_1 + x_2 \le 4 \qquad \text{s.t.} \ \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$3x_1 - 2x_2 \le 5$$

$$\Downarrow$$

$$\max \ x_1 + x_2 + 0 \cdot S_1 + 0 \cdot S_2$$

$$\text{s.t.} \ (2x_1 + x_2) + S_1 = 4$$

$$(3x_1 - 2x_2) + S_2 = 5$$

$$S_1 \ge 0$$

$$S_2 \ge 0$$

$$\max \; c^T x$$
$$\text{s.t.} \quad Ax \leq b$$

$$\min \; c^T x$$
$$\text{s.t.} \quad Ax \geq b$$

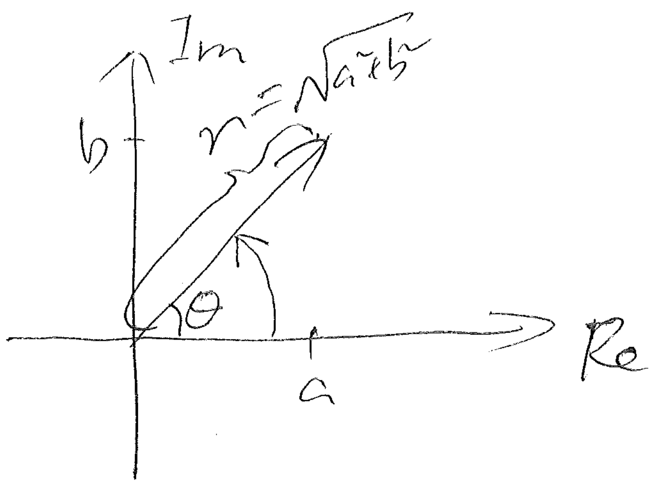
$$\max \; c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

$\cos(\alpha+\beta)$ $\qquad$ $\cos(\alpha-\beta)$

$\sin(\alpha+\beta)$ $\qquad$ $\sin(\alpha-\beta)$

Complex numbers $\quad \sqrt{-1} = i$ or j.

$$\underbrace{a}_{\text{real part}} + i\underbrace{b}_{\text{imaginary part}}$$



$a = r\cos\theta$

$b = r\sin\theta$

$$a + ib = \cancel{a}\ r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}$$

$$(a+ib)(c+id)$$

$$=(ac-bd)+i(ad+bc)$$

$$=(ac-bd)+i\left((a+b)(c+d)-ac-bd\right)$$

$$(a+ib)+(c+id)$$

$$=(a+c)+i(b+d)$$

$$\cos\alpha + i\sin\alpha = e^{i\alpha}.$$

$$\cos\beta + i\sin\beta = e^{i\beta}.$$

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = e^{i\alpha} \cdot e^{i\beta}$$

$$\begin{matrix}(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + \\ i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\end{matrix} = e^{i(\alpha+\beta)}$$

$$\begin{matrix}(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + \\ i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\end{matrix} = \begin{matrix}\cos(\alpha+\beta) + \\ i\sin(\alpha+\beta)\end{matrix}$$

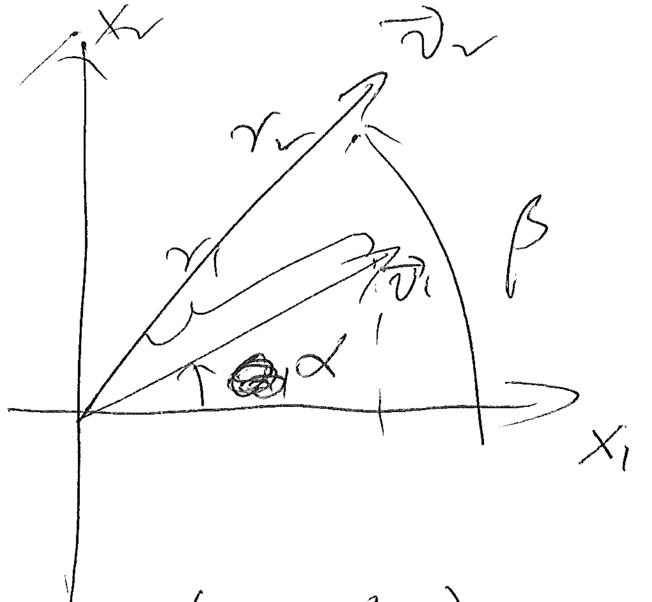
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + (-\beta)) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

$$\cos(\alpha-\beta) = \underline{\cos\alpha\cos\beta} + \underline{\sin\alpha\sin\beta}$$

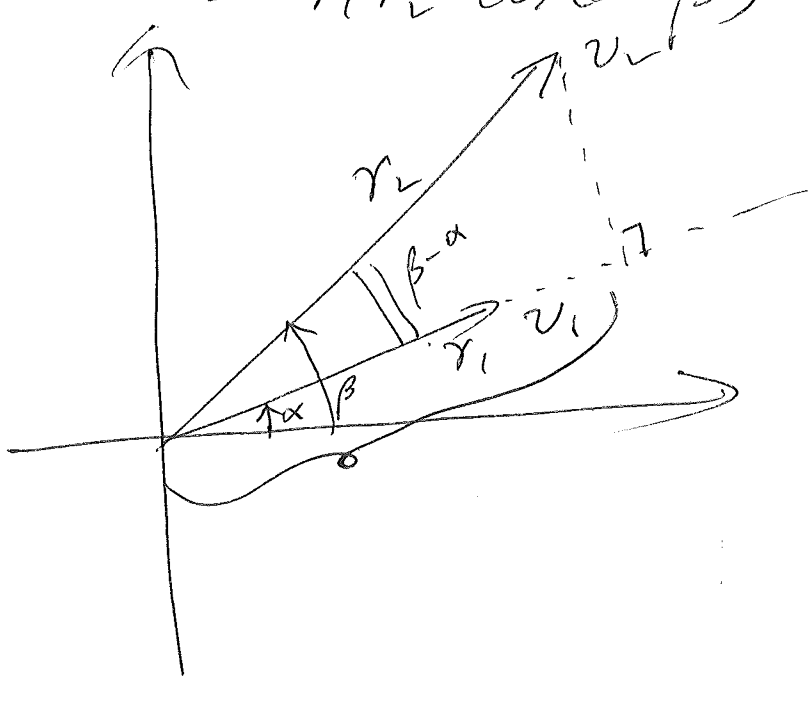
inner product



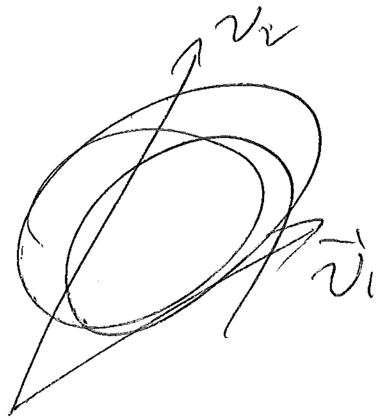
$$\vec{v}_1 = \begin{pmatrix} r_1 \cos\alpha \\ r_1 \sin\alpha \end{pmatrix} \qquad \vec{v}_2 \begin{pmatrix} r_2 \cos\beta \\ r_2 \sin\beta \end{pmatrix}$$

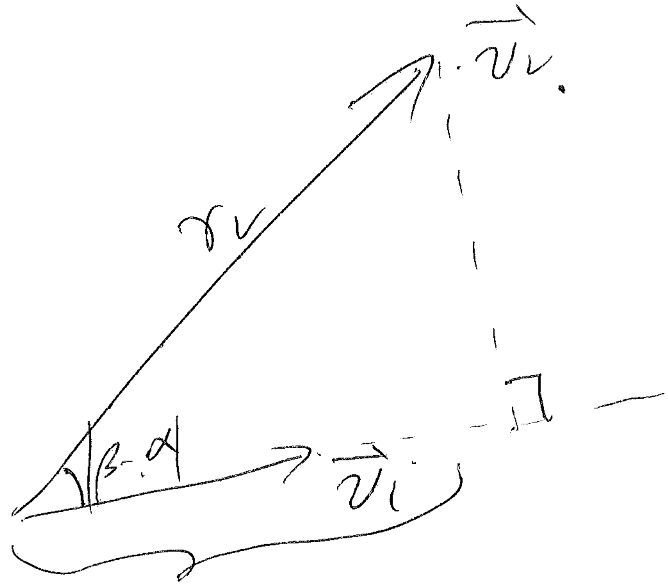
$$\langle \vec{v}_1, \vec{v}_2 \rangle = (\cos\alpha\cos\beta + \sin\alpha\sin\beta)\, r_1 r_2$$

$$= r_1 r_2 \cos(\alpha - \beta) = r_1 (r_2 \cos(\beta - \alpha))$$



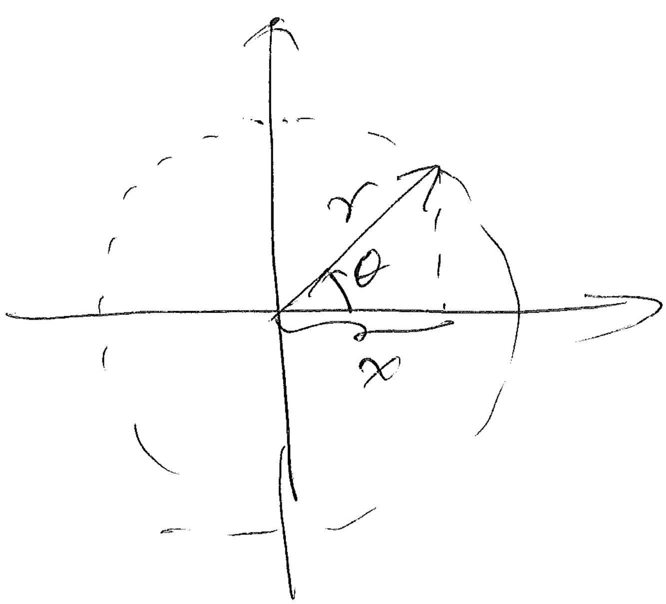
$\vec{v}_1, \vec{v}_2$ $\quad |\vec{v}_1| = 1$, $r_1 = 1$





$$\langle v_1, v_2\rangle = r_1 r_2 \cos(\beta-\alpha) = r_2 \cos(\beta-\alpha)$$

$$\langle v_1, v_2\rangle = |v_1|\cdot|v_2|\cos\theta$$

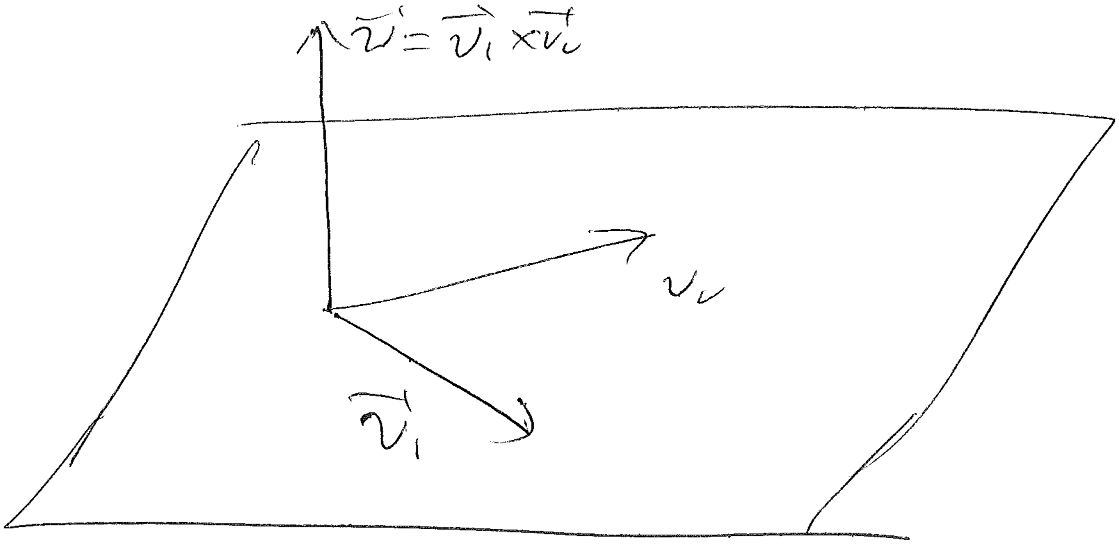


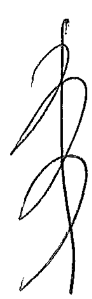
$$\frac{x}{r} = \cos\theta$$

when $\theta = \frac{\pi}{2}$ $\quad \langle v_1, v_2\rangle = 0$

Cross produre.

$$\vec{v}_1 \times \vec{v}_2 = \vec{w}$$





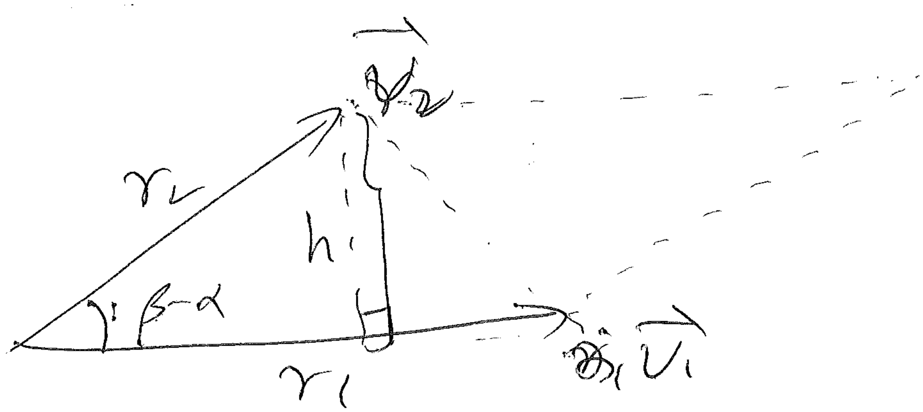
Let $\vec{v}_1 = \begin{pmatrix} r_1 \cos\alpha \\ r_2 \sin\alpha \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} r_2 \cos\beta \\ r_2 \sin\beta \end{pmatrix}$

$$|\vec{v}_1 \times \vec{v}_2| = \left| \begin{pmatrix} r_1 \cos\alpha & r_2 \cos\beta \\ r_2 \sin\alpha & r_2 \sin\beta \end{pmatrix} \right|$$

$$= \underline{r_1} \cos\alpha\, \underline{r_2} \sin\beta - \underline{r_1} \sin\alpha\, r_2 \cos\beta$$

$$= r_1 r_2 (\cos\alpha \sin\beta - \sin\alpha \cos\beta)$$

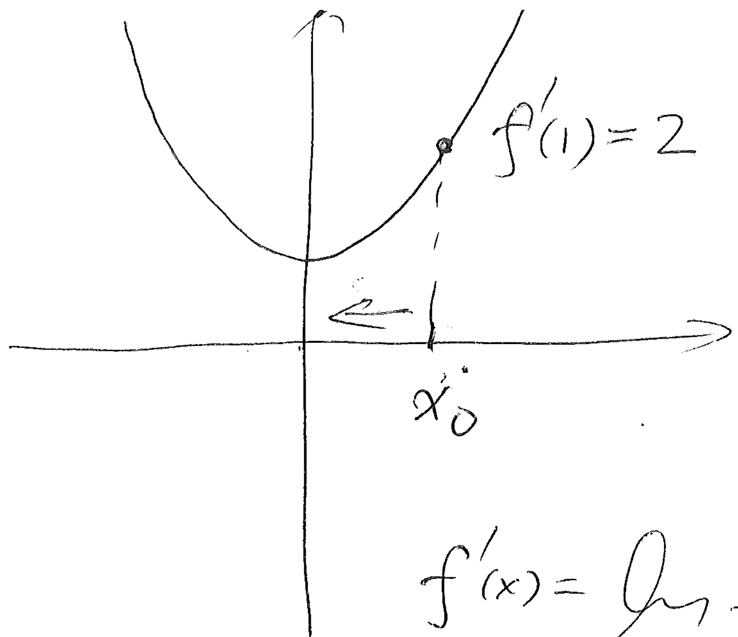
$$= r_1 r_2 \sin(\beta - \alpha) = -r_1 r_2 \sin(\alpha - \beta)$$



Gradient Search

$f(x) = x^2 + 1$ $\qquad x_0 = 1$

min $f(x)$



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

assuming Δx is small

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) \cdot \Delta x = f(x+\Delta x) - f(x)$$

$$f(x+\Delta x) = f(x) + f'(x)\Delta x$$

$$f(x_0+\Delta x) = f(x_0) + f'(x_0)\Delta x$$

We need $f(x_0+\Delta x) < f(x_0)$

$f'(x_0)\Delta x < 0$ $\qquad$ Δx and $f'(x_0)$ have different signs

$x \in \mathbb{R}^n$

$f(x)$ $\qquad$ $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$

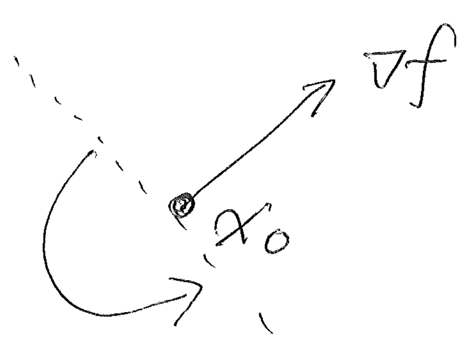
$$f(x+\Delta x) = f(x) + \langle \nabla f, \Delta x \rangle$$

$\min f(x)$

we want $f(x+\Delta x) < f(x)$

$$\langle \nabla f, \Delta x \rangle < 0$$

$$|\nabla f| \cdot |\Delta x| \cos\theta < 0$$



Descent direcing