Homework 4
Due: 3:30pm Tuesday April 12

1. Ramesh can get to work in three different ways: by bicycle, by car, or by bus. Because of commuter traffic, there is a 50% chance that he will be late when he drives his car. When he takes the bus, which uses a special lane reserved for buses, there is a 20% chance that he will be late. The probability that he is late when he rides his bicycle is only 5%. Ramesh arrives late one day. His boss wants to estimate the probability that he drove his car to work that day.

   (a) Suppose the boss assumes that there is a 1/3 chance that Ramesh takes each of the three ways he can get to work. What estimate for the probability that Ramesh drove his car does the boss obtain from Bayes' theorem under this assumption?

   (b) Suppose the boss knows that Ramesh drives 30% of the time, takes the bus only 10% of the time, and takes his bicycle 60% of the time. What estimate for the probability that Ramesh drove his car does the boss obtain from Bayes' theorem using this information?

2. Calculate the expectation and variance of the Binomial $B(n, p)$, geometric, and Poisson distributions.

3. A ship arrives at a port and the 40 sailors on board go ashore for revelry. Later that night, the 40 sailors return to the ship, all drunk. As a result, each randomly chooses a cabin to sleep in. How many sailors are expected to sleep in their own cabins?

   Hints:

   (1) Assume each sailor has his own cabin.

   (2) You may assume that the sailors are coming back sequentially, i.e., one by one and randomly chooses an unoccupied cabin to sleep if that makes it easier.

4. A cereal company is doing a promotion based on the movie TRANSFORMER. Each box of cereal contains one of the n different transformers. Once a customer obtains one of every type of transformer, he or she can send them in and collect a prize. Assume that the transformer in each box is put in independently and uniformly at random from the n possibilities and that customers can't collaborate with each other to collect the transformers together.

   (a) How many boxes of cereal must a customer buy in order to expect collecting every type of transformer?

   (b) If the profit for each box of cereal (including the transformer) is \$1, and the company would like to give away an iPhone 6 (costs about \$700), what n should the company choose such that it will not lose money?

   Hint:

   (1) Let X be the random variable of the number of boxes bought until at least every type of transformer is obtained. Then we need to calculate E[X].

   (2) Let X_i be the number of boxes bought to collect the $i-$th new transformer while you already have exactly $i-1$ different transformers, then clearly $X = \sum_{i=1}^{n} X_i$.

5. In class, we have seen the Markov Inequality and the Chebyshev Inequality. Please use these two inequalities to estimate the probability of obtaining more than 70 heads when tossing a fair coin 100 times.

6. Let X_1, X_2 be 2 independement random variables. Prove that $Var(X_1 + X_2) = Var(X_1) + Var(X_2)$. (Note that this can also be generalized to n independent random variables.)

7. Suppose there are two candidates and n voters. A common algoroithm is to ask k random voters and take the average response. Assume that exactly one-half of the voters favor each of the candidates. What is the probability that the results of the survey (with k voters) are in the range of 45 to 55 percent?

8. In class, we have seen that the worst case running time of insertion sort is $O(n^2)$. Suppose that we randomly perturb the input sequence just like in randomized quicksort, what is the expected running time of randomized insertion sort now?

9. For a given a graph $G = (V, E)$, a vertex cover of G is a subset of vertices $C \subseteq V$ such that each edge of G has at least one endpoint in C. The goal of the vertex cover problem is to find the optimal vertex cover C^* with the minimum number of vertices.

   Consider the following randomized algorithm for vertex cover.

Step 1: Start with $C = \phi$.

Step 2: Pick an edge e uniformly at random from the edges that are not covered by C (i.e., if e has endpoints u and v, then $\{u, v\} \cap C = \phi$. and add a random endpoint of e to C.

Step 3: If C is a vertex cover, terminate and output C; else go to Step 2.

   Answer the following questions:

   (a) Consider the very first iteration of the algorithm. What is the probability that a vertex from the smallest vertex cover C^* is added to C? (Hint: for each edge $e \in E$, at least one endpoint of e must be in C^*.)

   (b) Consider the second iteration of the algorithm. What is the probability that a vertex from the smallest vertex cover C^* is added to C? (Hint: you should discuss the two scenarios of whether a vertex from C^* is added to C in the first iteration or not.)

   (c) Let k be the number of vertices in the smallest vertex cover C^*. Show that the expectation of the number of vertices of C is $2k$.