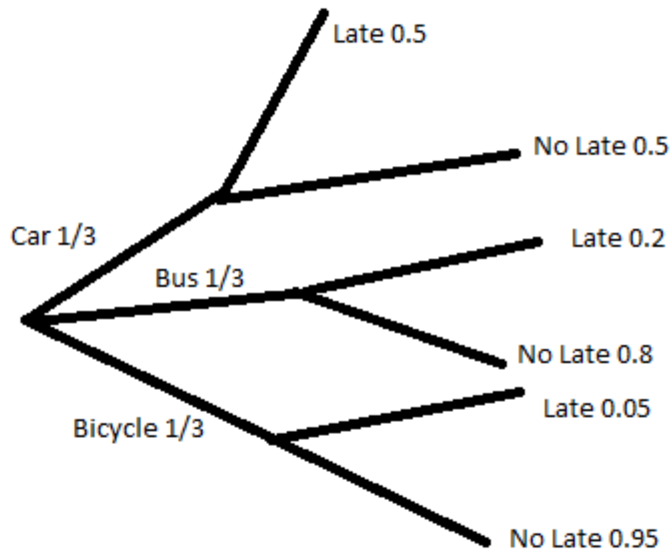
Homework 4 Solution

Solution 1:



P(Late|Car) = 0.5, P(Late|Bus) = 0.2, P(Bicycle) = 0.05

a. P(Car) = 1/3, P(Bus) = 1/3 , P(Bicycle) = 1/3

P(Car| Late) = (P(Late|Car)P(Car))/ {P(Late|Car)P(Car) + P(Late|Bus)P(Bus) + P(Late|Bicycle)P(Bicycle) }

= (1/3 * 0.5) / {(1/3 * 0.5) + (1/3 * 0.2) + (1/3 * 0.05)}
= 0.16667 / (0.16667 + 0.0667+ 0.016667)
= 0.66668

b. P(Car) = 0.3, P(Bus) = 0.2, P(Bicycle) = 0.6
P(Car| Late) = (0.3*0.5) /{ (0.3*0.5) + (0.2*0.1) +(0.05*0.6) }

= 0.15 / (0.15+.02+.03)

= 0.75

Solution 2:

Expectation and variance of the Binomial B(n, p):

Let X_i be the i^{th} Bernoulli trail, let p the probability of success and q = 1-p
Then,$E[X_i]$ = p $Var[X_i]$ = pq

$E[B(n,p)] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = np$
$Var[B(n,p)] = Var[\sum_{i=1}^{n} X_i]$

Since X_i are independent
$Var[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} X_i\ Var(X_i) = npq$

Expectation and variance of the Binomial Geometric distribution:

Let X be the random variable representing the number of trials until the first success occur.

Let , on k Bernoulli trail success occurs
$P(X=k) = (1-p)^{k-1}p$

$E[X] = (1-p)^0 p + (1-p)^1 p\ 2 + (1-p)^2 p\ .3 + \ldots..$
let $S_n = \sum_{j=1}^{n} (1-p)^{j-1} p$

Then we have $E[X] = \lim_{n\to inf} S_n$

Now, $S_n = (1-p)^0 p + (1-p)^1 p\ 2 + (1-p)^2 p\ .3 + \ldots.. + (1-p)^{n-1} p\ .n$
$(1-p)\ S_n = (1-p)^1 p + (1-p)^2 p\ 2 + (1-p)^3 p\ .3 + \ldots.. + (1-p)^n p\ .n$

$S_n - S_n(1-p) = (1 - p)^0 p + (1 - p)^1 p + (1 - p)^3 p + \ldots + (1 - p)^{n-1} p - (1 - p)^n p \cdot n$
Solving this,

$pS_n = 1 - (1 - p)^n \cdot (n + 1)$

$Sn = 1/p - (1 - p)^n \cdot (n + 1)/p$
$E[X] = \lim_{n\to inf} (1/p - (1 - p)^n \cdot (n + 1)/p)$

Variance:

$Var(X) = E[X^2] – (E[X])^2$

$E[X] = 1/p$
$E[X^2] = 1^2 (1-p)^0 p + (1-p)^1 p\ 2^2 + (1-p)^2 p\ .3^2 + \ldots..$
$(1-p)E[X^2] = 1^2 (1-p)^1 p + (1-p)^2 p\ 2^2 + (1-p)^3 p\ .\ 3^2 + \ldots.$

$pE[X^2] = 1^2(1-p)p + 3(1-p)p + 5(1-p)^2 p + 7(1-p)^3 p + \ldots.$

$= \sum_{k=1}^{inf}(2k-1)(1-p)^{k-1}p$
$pE[X^2] = 2/p – 1$

$E[X^2] = 2/p^2 – 1/p^2$

$Var[X] = E[X^2] – (E[X])^2$

$= (2/p^2 – 1/p^2) - (i/p^2)$
$=(1-p)/p^2$

Poisson Distribution:

$E[X] = \sum_{k=0}^{inf}(\lambda^k e^{-\lambda} k/k!)$
$= \sum_{k=0}^{inf}(\lambda^k e^{-\lambda}/(k-1)!)$
$=\lambda\sum_{k=0}^{inf}(\lambda^{k-1} e^{-\lambda}/(k-1)!)$
$= \lambda\sum_{k=1}^{inf}(\lambda^k e^{-\lambda}/(k)!) = \lambda$

Variance:

$E[X^2] = \sum_{k>0} \lambda^k e^{-\lambda} k^2 /k!$
$= \lambda e^{-\lambda} \sum_{k>0} \lambda^{k-1} k /(k-1)!$
$= \lambda e^{-\lambda} (\sum_{k>0} \lambda^{k-1} (k-1) /(k-1)! + \sum_{k>0} \lambda^{k-1}/(k-1)!)$
$= \lambda e^{-\lambda} (\lambda\sum_{k>1} \lambda^{k-2}/(k-2)! + \sum_{k>0} \lambda^{k-1}/(k-1)!)$
$= \lambda e^{-\lambda} (\lambda\sum_{i\geq 0} \lambda^i /i! + \sum_{j\geq 0} \lambda^j/j!)$
$= \lambda e^{-\lambda} (\lambda e^{\lambda} + e^{\lambda})$
$= \lambda(\lambda+1)$

$Var(X) = E[X^2] – (E[X])^2 = \lambda$

Solution 3:

Using the hint

$E[\sum_{i=1}^{40} X_i] = \sum_{i=1}^{40} E[X_i]$

For $X_1, X_2, X_3 \ldots, X_{40}$

$P(X_1 = 1) = 1/40$ and $E[X_1] = 1/40$

1st Sailor didn't pick 2nd sailors room = 39/40

2nd Sailor picked his own room = 1/39

$P(X_2 = 1) = 39/40*1/39$ and $E[X_2] = 1/40$

$P(X_3=1) = 39/40* 38/39*1/38 = 1/40$, $E[X_3] = 1/40$

………………………………………………………

$P(X_4=1) = 1/40$ $E[X_4] = 1/40$

Thus $\sum_{i=1}^{40} E[X_i]=1$

Solution 4:

$P(X_1=1) = 1$ $E[X_1] = 1$

$P(X_2=1) = (n-1)/n$ $E[X_2] = n/(n-1)$

$P(X_3=1) = (n-2)/n$ $E[X_3] = n/(n-2)$

………………………………………………………………

$P(X_n=1) = (n-(n-1))/n$ $E[X_n] = n$

Thus

$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} n/(n-i+1)$

$= n*(\sum_{i=1}^{n} 1/(n-i+1)) = n*(\sum_{i=1}^{n} 1/j) = n\ln n$

Thus one need to open nlnn boxes to collect all n transformers.

Solution 5:

Since this is a fair coin, the odds of success is ½. We are looking at the random variable B(100,1/2) => 100 Bernoulli trails

$E[B] = 50$, $Var(B) = 25$, $\sigma(B) = 5$

Using Markov Inequality:

$P(B \geq 70) \leq E[B]/70 = 50/70 = 0.71$

Using Chebyshev:

$P(|B-50| \geq 20) \leq ½\ (25/20^2) = 0.03125$

Solution 6:

$$
\begin{aligned}
Var(X_1+X_2) &= E((X_1+X_2) - E[X_1+X_2])^2 \\
&= E((X_1+X_2) - E[X_1] - E[X_2])^2 \\
&= E((X_1 - E[X_1]) + (X_2 - E[X_2]))^2 \\
&= E((X_1 - E[X_1])^2 - 2 (X_1 - E[X_1]) (X_2 - E[X_2]) + (X_2 - E[X_2])^2) \\
&= E[(X_1 - E[X_1])^2] - 2E[X_1 - E[X_1]]\ E[X_2 - E[X_2]] + E[(X_2 - E[X_2])^2] \\
&= Var(X_1) - 0 + Var(X_2) \\
&= Var(X_1) + Var(X_2)
\end{aligned}
$$

Solution 7:

Let the two candidates be A and B.

Since 50% of the voters favor each candidate, if we pick a vote uniformly at random, the odds is 0.5 for picking a voter favoring each candidate.

There are two scenarios:

If $n \gg k$, this is just like flipping a fair coin k times, and the resulting distribution is a binomial distribution:

$Pr(X=j)=\binom{k}{j}\left(\frac{1}{2}\right)^{j}\left(1-\frac{1}{2}\right)^{k-j}=\binom{k}{j}\left(\frac{1}{2}\right)^{k}$**, where *X* is the r.v., indicating the number of voters out of these *k* voters that favors candidate A.**

Thus, the probability that 45-55% of the voters favoring A is:

$\sum_{j=0.45k}^{0.55k} Pr(X=j)$.

If $n \sim k$, i.e., roughly in the same order of magnitude of *k*, then let *X* be the number of voters of the chosen *k* voters favoring A:

Since any group of *k* voters are equally likely to be chosen, this probably can be calculated using product rule:

$$Pr(X=j)=\frac{\binom{n/2}{j}\binom{n/2}{k-j}}{\binom{n}{k}}$$

Thus, the probability that 45-55% of the voters favoring A is:

$\sum_{j=0.45k}^{0.55k} Pr(X=j)$.

Solution 8:

Similar to the analysis of randomized quick sort. Let X_{ij} be the indicator variable of whenever X_i and X_j are compared ($X_i < X_j$).

However unlike quicksort, in selection sort

$P(X_{ij}) = ½$, in other words the two are compared when X_j is infront of X_i

Hence, $E[\sum_{i=1}^{n-1} \sum_{i=j+1}^{n} X_{ij}] = \sum_{i=1}^{n-1} \sum_{i=j+1}^{n} \mathrm{E}[X_{ij}] =$ n(n-1)/4

Solution 9:

a) Each edge of G has at least one endpoint in C. So, in the first iteration the probability that a vertex from the smallest vertex cover C* is added to C is $\geq \frac{1}{2}$
b) This is still $\geq \frac{1}{2}$.
   Note that whether the vertex chosen in the first iteration belong to C* or not, the same argument still holds
c) 2k. This is like flipping a fair coin, and asking for the number of flips to expected to get k heads. Since 2 flips are expected to get one head, one would need 2k flips.