Homework 1
Due: Jan 28th (Thursday), 2016

1. (20pt) Using the $c-n_0$ definitions of the asymptotic notations to answer the following questions:

   (a) Let $f(n) = 2n$, is $f(n) = O(n)$? Why or why not?

   (b) Let $f(n) = 2n$, is $f(n) = \Omega(n)$? Why or why not?

   (c) Let $f(n) = 2n$, is $f(n) = \Theta(n)$? Why or why not?

   (d) Let $f(n) = 2n$, is $f(n) = o(n)$? Why or why not?

   (e) Let $g(n) = n$, is $g(n) = O(n^2)$? Why or why not?

   (f) Let $g(n) = n$, is $g(n) = o(n)$? Why or why not?

2. (20pt) Sort the following functions based on their asympotic growth rate with brief explanations. $f(n) = 10^{-10}$, $g(n) = 10^{10}$, n^2, $(\log_2 n)^2$, 2^n, 3^n, $n \log_2 n$, $\log_2 n$, $\log_3 n$, $2^{\log_2 n}$, $(\sqrt{2})^{\log_2 n}$, $(\log_2 n)^{\log_2 n}$, $n^{\log_2 n}$, $\sqrt{n}$, $\log_2 \sqrt{n}$, $\log_2 (\log_2 n)$, $n^n(1 + (-1)^n)$.

3. (10pt) Consider a special type of tree data structure called the **Short Tree**, which is defined as follows.

   - Each leaf node of a *Short Tree* is associated with a distinct key, i.e., no two nodes of the tree have the same key.
   - Each non-leaf node of a *Short Tree* is associated with a pair of keys, v_{min} and v_{max}, indicating the smallest key v_{min} and the largest key v_{max} in its sub-tree.
   - Let v be an aibitrary node of a *Short Tree*. Let m be the total number of leaf nodes in the sub-tree rooted at v. For ease of explanation, let the keys be $\{k_0 < k_1 < k_2 < ... < k_{m-1}\}$. Then v has $\Theta(\sqrt{m})$ children. Let the children be indexed $0, 1, ..., \sqrt{m} - 1$, then the $j-$th child is responsible for storing the keys within the range $\{k_{j\sqrt{m}}, ..., k_{(j+1)\sqrt{m}-1}\}$, and has a *min* of $k_{j\sqrt{m}}$ and a *max* of $k_{(j+1)\sqrt{m}-1}$. Note that the above definiion is applied recursively to the children of v.

   Answer the following questions:

   (a) Let the keys of a particular *Short Tree* be $0, 1, 2, ..., 255$. What is the height of the tree?

   (b) What is the assmptotic height of a *Short Tree* with a total of n keys $\{0, 1, ..., n - 1\}$?

4. (10pt) In computational geometry, an arrangement of lines is the partition of the plane formed by a collection of lines. Observe that the lines partition the plane into disjoint regions. Calculate the maximum number of disjoint regions in an arragement created by n lines.

5. (10pt) Use the recursion tree method to determine a good upper bound on the recurrence

$$T(n) = 3T\left(\lceil \frac{n}{2} \rceil\right) + n$$

6. (10pt) Use the guess and substitution method to prove that the T(n) in the following recurrence relation is $O(n)$.

$$\begin{cases} T(n) = T(\frac{n}{2}) + n & \text{for } n \geq 2 \\ T(1) = 1 \end{cases}$$