March 29

Exam II April 7th (Thursday)

Randomized Quick Sort. ← Las Vegas Alg

Mincut Algorithm ← Monte Carlo Alg

Randomized Quick Sort.

~~Pick~~ Randomly Pick a pivot x

Partition $S_1 \leq x < S_2$, $S_1 \cup S_2 = S$

recursively quick sort S_1 and S_2

Randomized Quick Sort

Generate a random permutation of S

Use $S[i]$ as the pivot

Partition $S_1 \leq S[i] < S_2$

Recursively sort S_1 and S_2

Let A be a set of n elements

an r-permutation is an ordered sequence of r distinct elements from A

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

an n-permutation is an ordered sequence of all elements in A and every element appeared once and only once

$${}_nP_n = n!$$

$$= \frac{n!}{(n-n)!} = n!$$

Let the input be a ~~ra~~ permutation of

$$x_1 < x_2 < \cdots < x_n$$

Since we will randomly permutate the input, we shall assume the input to quick sort phase is a random permutation σ of $x_1, x_2 \cdots x_n$

The running time of quick sort is the number of comparisons

Observation: every pair of numbers x_i and x_j $(i \neq j)$ will be compared at most once during the quick sort phase.

Introduce r.v. X_{ij} to describe whether x_i and x_j are compared or not

$$\begin{cases} X_{ij} = 1 & x_i \text{ and } x_j \text{ are compared} \\ X_{ij} = 0 & \text{otherwise} \end{cases}$$

Total Number of Comparisons

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

the expected running time is

$$E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

Thus it all comes down to the calculation of

$$P(X_{ij}=1)$$

$$E[X_{ij}] = \cancel{P(X_{ij}=0) \times 0} + P(X_{ij}=1) \times 1$$

$$= P(X_{ij}=1)$$

1, 3, 4, 7, 8, 9, 13, 21

The input to the quick sort phase is a random permutation of ~~those~~ 8 elements

$\boxed{4}$, 3, 7, 21, 13, 9, 8, 1

3 1 $\boxed{4}$ 7, 21, 13, 8, 9

$\boxed{3}$ $\boxed{7}$

1

$\boxed{21}$, 13, 8, 9

13, 8, 9

$P(X_{ij}=1) = ?$

what's the probability that x_i and x_j are compared during the execution.

Observations

(1) in order to compare x_i and x_j, one of them has to be the pivot during the execution

(2) The two numbers must be in the same subset during the recursion

$\underbrace{x_1 < \cdots x_{i-1}} < x_i < \underbrace{x_{i+1}, \cdots\cdots, x_{j-1}} < x_j < x_{j+1} < x_n$

in order for x_i and x_j to be compared, those two numbers must be in front of

($x_{i+1}, \cdots, x_{j-1}$ in the random permutation

what's the odds?

$$\frac{2}{j-i+1}$$

$$P(X_{ij}=1) = \frac{2}{j-i+1}$$

$$E\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E[X_{ij}]$$

$$= \sum_{i=1}^{n-1}\left(\sum_{j=i+1}^{n} \frac{2}{j-i+1}\right)$$

$$= 2\sum_{i=1}^{n-1}\left(\sum_{j=i+1}^{n} \frac{1}{j-i+1}\right)$$

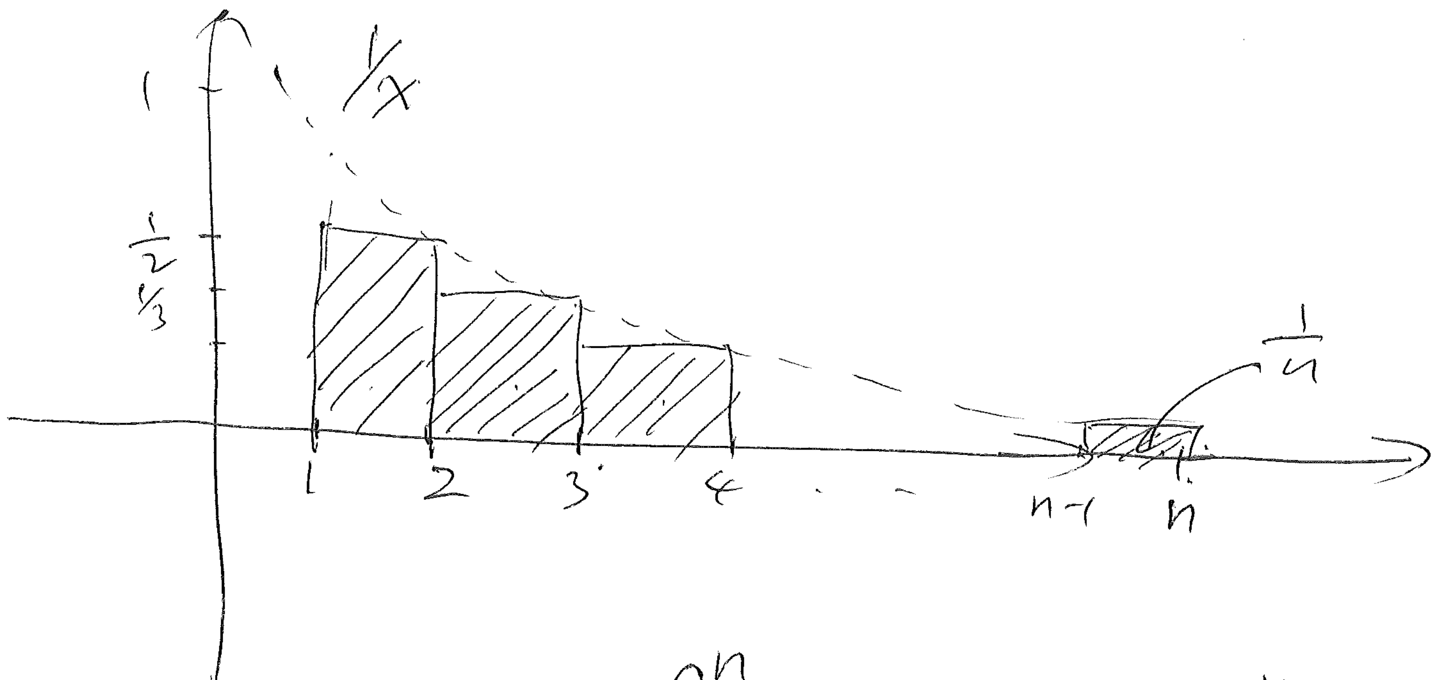
$$= 2\sum_{i=1}^{n-1}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-i+1}\right)$$

$$\leq 2\sum_{i=1}^{n-1}\left(\frac{1}{2}+\cdots+\frac{1}{n-i+1}+\left(\frac{1}{n-i+2}+\cdots+\frac{1}{n}\right)\right)$$

$$= 2\sum_{i=1}^{n-1}\left(\sum_{k=2}^{n}\frac{1}{k}\right)$$

$$\leq 2\sum_{i=1}^{n-1} \ln n < 2n\ln n$$

$$\sum_{k=2}^{n} \frac{1}{k} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$



$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{1}{x}\,dx = \ln x \Big|_1^n = \ln n$$

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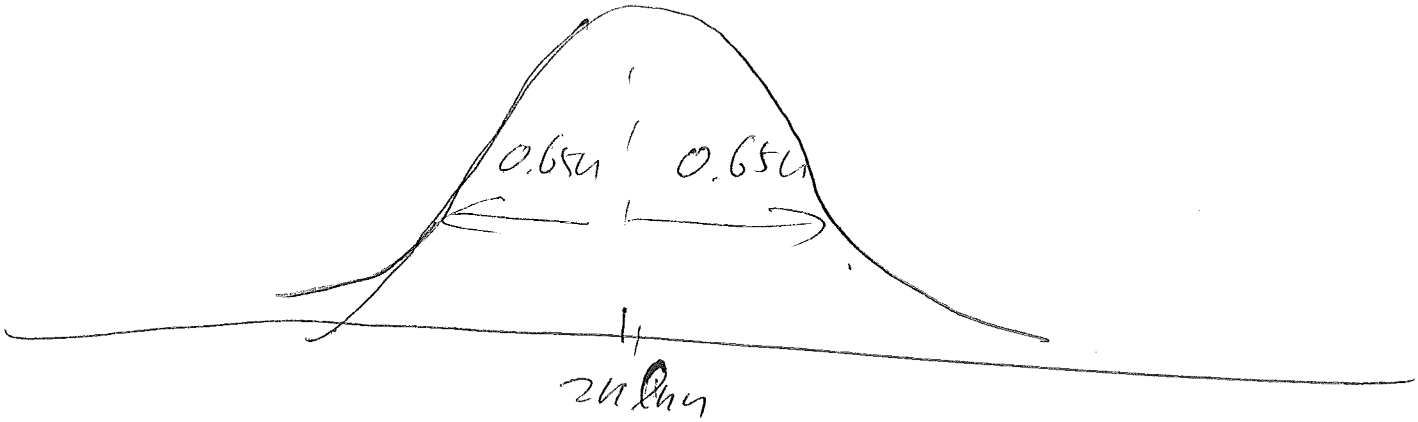
The expected running time of randomized quick sort is bounded by $\boxed{2n \ln n}$.

The standard deviation $\sigma = 0.65n$.

Variance σ^2



From Markov

$$\Pr(X \geq 20 n \ln n) \leq \frac{2n \cancel{\ln n}}{20 n \cancel{\ln n}} = 0.1$$

From Chebyshev

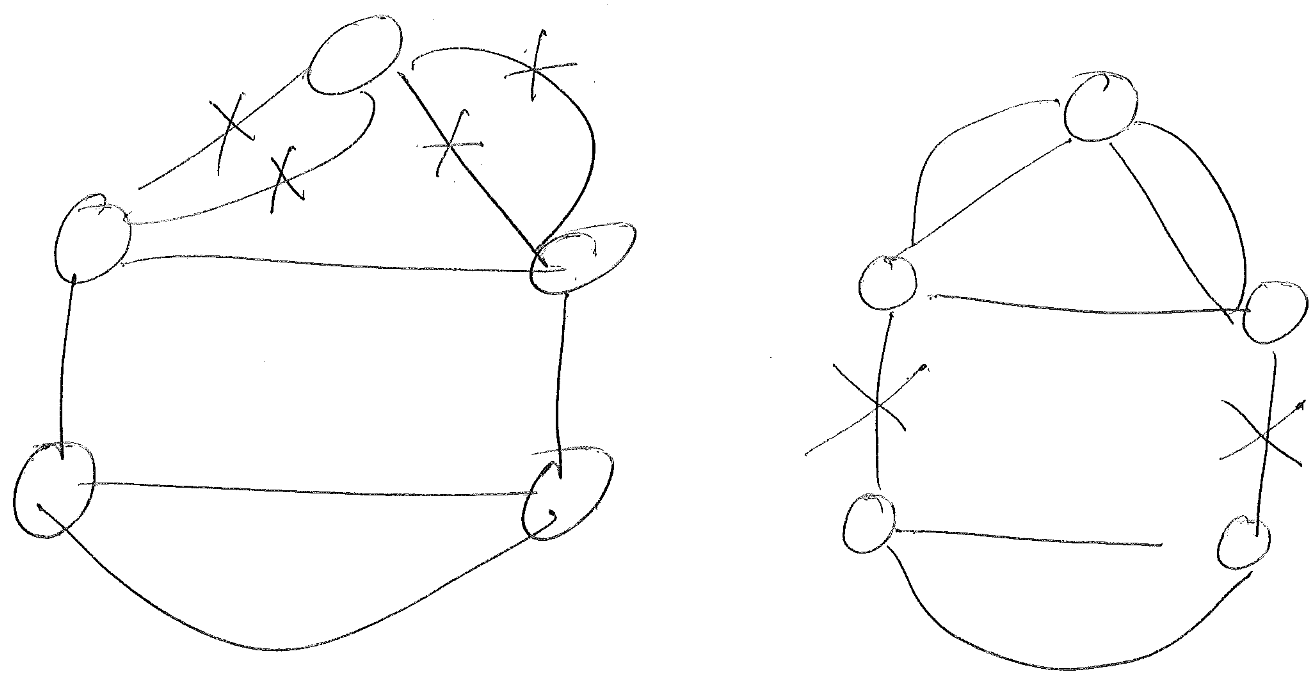
$$\Pr(|X - 2n\ln n| \geq n \ln n)$$

$$\leq \frac{(0.65n)^2}{(n \ln n)^2} = \frac{0.425}{(\ln n)^2}$$

Karger's Mincut Algorithm

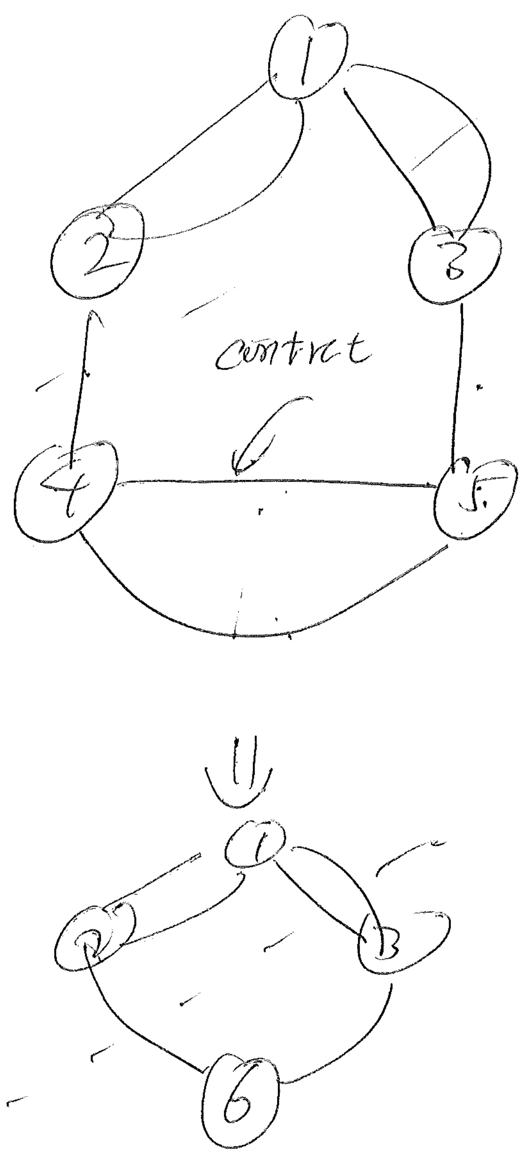
Let G be a connected undirected multigraph with no self loops. A cut of G is a set of edges whose removal results in G disconnected.

A mincut is a cut of minimum number of edges



Mincut Problem: Find a mincut

Contraction (Edge Contraction)



By contracting an edge e connecting $\underline{u}$ and $\underline{v}$ we mean:

(1) remove all edges between u and v.

(2) merge u, v to be a new vertex w.

(3) connecting all edges adjacent to u and v to w

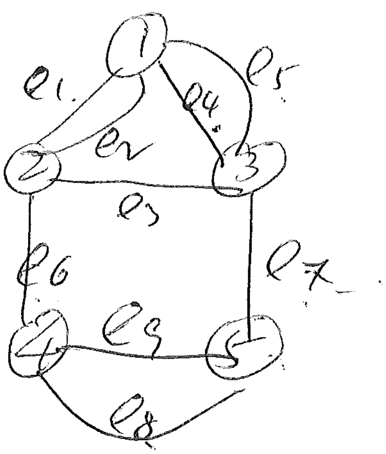
Let G be a graph, G' be a graph after contracting some edge in G.

Then ~~th~~ any cut in G' is still a cut in G

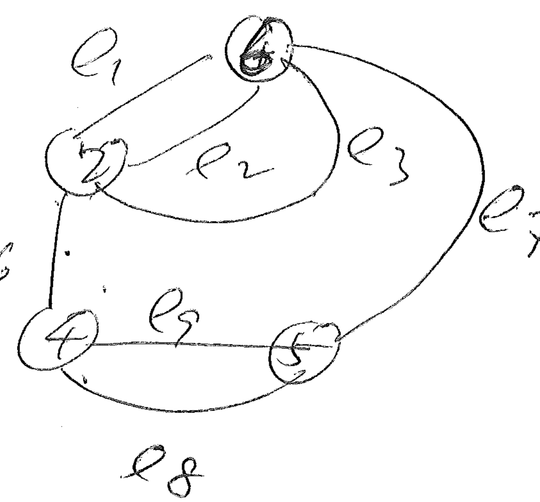
Randomized MinCut Algorithm

While G has more than 2 vertices
  Pick an edge e uniformly at random
  contract e

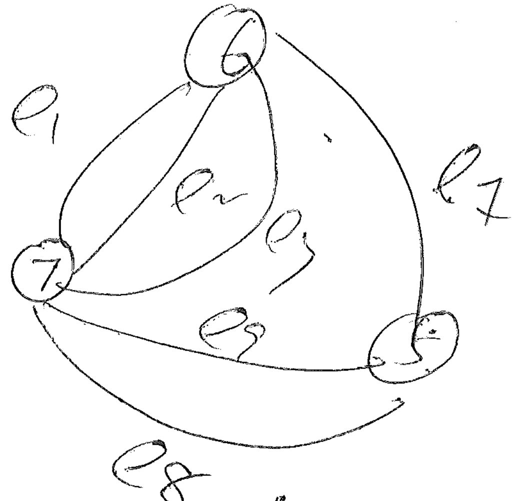
Return the remaining edges.

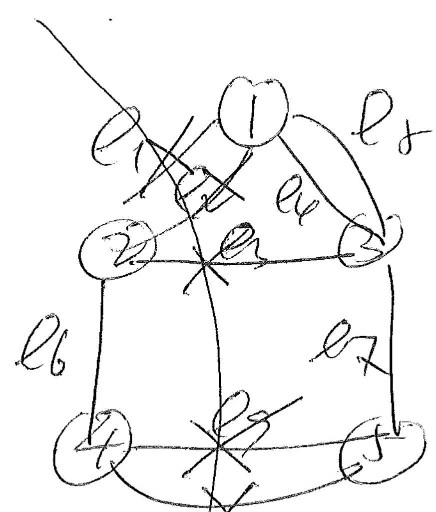


Contract e_4 $\Longrightarrow$

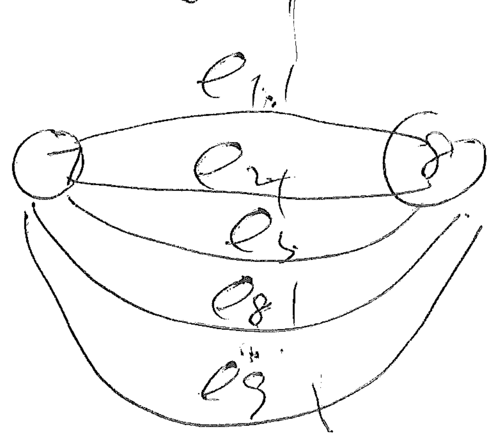


contract e_6 $\Longrightarrow$





contract e_7 $\Longrightarrow$



What's the probability that the algorithm will succeed in generating a min-cut?

Let $|V| = n$ $\quad |E| = m$ $\quad$ the size of the mincut $= k$.

Consider the very first iteration, ~~what's~~ what's the odds that the mincut survives the first contraction?

$$1 - \frac{k}{m}$$

The relation of m and n, k

Since the mincut has size k, every vertex will have a degree $\geq k$.

$$m \geq \frac{nk}{2} \Rightarrow 1 \geq \frac{n}{2} \frac{k}{m}$$

$$\Rightarrow \frac{2}{n} \geq \frac{k}{m} \Rightarrow -\frac{2}{n} \leq -\frac{k}{m}$$

$$\Rightarrow 1 - \frac{2}{n} \leq 1 - \frac{k}{m} \Rightarrow 1 - \frac{k}{m} \geq \frac{n-2}{n}$$

The probability that the mincut survives the 1st contraction is $\geq \frac{n-2}{n}$

Note this only ~~on~~ depends on n

Assuming the min-cut survives the 1st contraction what's the probability it survives the 2nd?

$$\geq \frac{(n-1)-2}{(n-1)} = \frac{n-3}{n-1}$$

$\vdots$

Surviving the last contraction

$$\geq \frac{3-2}{3} = \frac{1}{3}$$

The probability of surviving all contractions is:

$$\geq \frac{\cancel{n-2}}{n} \cdot \frac{\cancel{n-3}}{n-1} \cdot \frac{\cancel{n-4}}{\cancel{n-2}} \cdot \frac{\cancel{n-5}}{\cancel{n-3}} \cdots \frac{\cancel{3}}{\cancel{5}} \cdot \frac{2}{\cancel{4}} \cdot \frac{1}{\cancel{3}}$$

$$= \frac{2}{n(n-1)} = \frac{1}{\frac{n(n-1)}{2}} = \frac{1}{\binom{n}{2}}$$