Jan 28, 2016

Guess and Sustitution:

$$T(n) = T(\frac{n}{5}) + T(\frac{7n}{10}) + n$$

(Deterministic Selection Algorithm)

Recall from recursion tree

$$T(n) \leq \sum_{j=0}^{\infty} (\frac{9}{10})^j n \simeq 10n$$

$$S = \sum_{j=0}^{k} (\frac{9}{10})^j$$

For $j=0$ to k

sum = sum + $(\frac{9}{10})^j$

$$= \underbrace{(\frac{9}{10})^0 + (\frac{9}{10})^1}_{\frac{9}{10}} + \underbrace{(\frac{9}{10})^2 + \cdots}_{\frac{9}{10}}$$

$$S = (\frac{9}{10})^0 + (\frac{9}{10})^1 + (\frac{9}{10})^2 + \cdots + (\frac{9}{10})^k$$

$$-\ \frac{9}{10}S = (\frac{9}{10})^1 + (\frac{9}{10})^2 + \cdots + (\frac{9}{10})^k + (\frac{9}{10})^{k+1}$$

$$(1-\frac{9}{10})S = (\frac{9}{10})^0 + 0 + 0 + 0 + 0 - (\frac{9}{10})^{k+1}$$

$$\left(1-\frac{9}{10}\right)S = \left(\frac{9}{10}\right)^0 - \left(\frac{9}{10}\right)\cdot\left(\frac{9}{10}\right)^k$$

$$S = \frac{\left(\frac{9}{10}\right)^0 - \left(\frac{9}{10}\right)\left(\frac{9}{10}\right)^k}{1-\frac{9}{10}}$$

$$= \frac{a_0 - r a_k}{1-r}$$

$$\sum_{j=0}^{\infty} n\left(\frac{9}{10}\right)^j = n \sum_{j=0}^{\infty}\left(\frac{9}{10}\right)^j$$

$$= n \lim_{k\to\infty} \sum_{j=0}^{k}\left(\frac{9}{10}\right)^j$$

$$= n \lim_{k\to\infty} \frac{\left(\frac{9}{10}\right)^0 - \frac{9}{10}\left(\frac{9}{10}\right)^k}{1-\frac{9}{10}}$$

$$= n \lim_{k\to\infty} \frac{1-\frac{9}{10}\left(\frac{9}{10}\right)^k}{\frac{1}{10}}$$

$$= 10n$$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n$$

Guess $T(n) = O(n)$

There exists positive constants c and n_0 such that $T(n) \le cn$ for all $n \ge n_0$

Pf. using guess and substitution.

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n$$

$$\le c \cdot \frac{n}{5} + c \cdot \frac{7n}{10} + n \overset{?}{\le} cn$$

$$= \frac{9}{10} c \cdot n + n \overset{?}{\le} c \cdot n$$

$$\frac{9}{10} c + n \le cn$$

$$n \le cn - \frac{9}{10} cn$$

$$\cancel{n} \le \frac{1}{10} c\cancel{n}$$

$$c \ge 10$$

Design of Algorithm using Induction and ~~Recursion~~ Recursion

Mathematical Induction

To prove a claim $P(n)$ with respect to natural number $n = 1, 2, \ldots$ is correct for all n, it suffices to do following:

(1) $P(1)$ ✓

(2) I.H. Assume that $P(k)$ is correct for all k

(3) I.S. Need to show that $P(k+1)$ is correct using I.H.

I.S.

$$\forall n\, P(n) \iff \underbrace{P(1)}_{\text{Basis}} \wedge \left(\forall k\, (\underbrace{P(k)}_{\text{I.H.}} \rightarrow P(k+1))\right)$$

Earliest citation of induction occurs in a dialogue between Parmenides and Socrates

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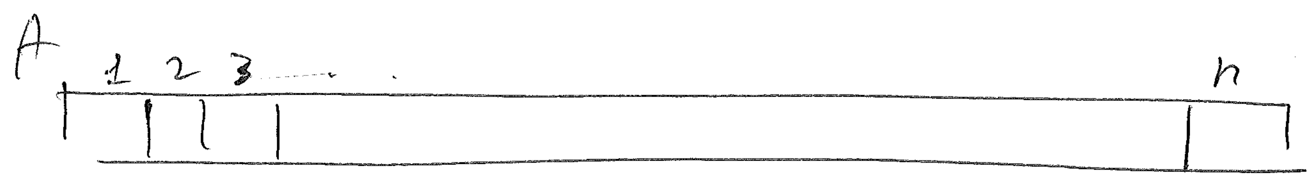
Insertion Sort.

Give n distinct numbers in an array A.
Sort the numbers.

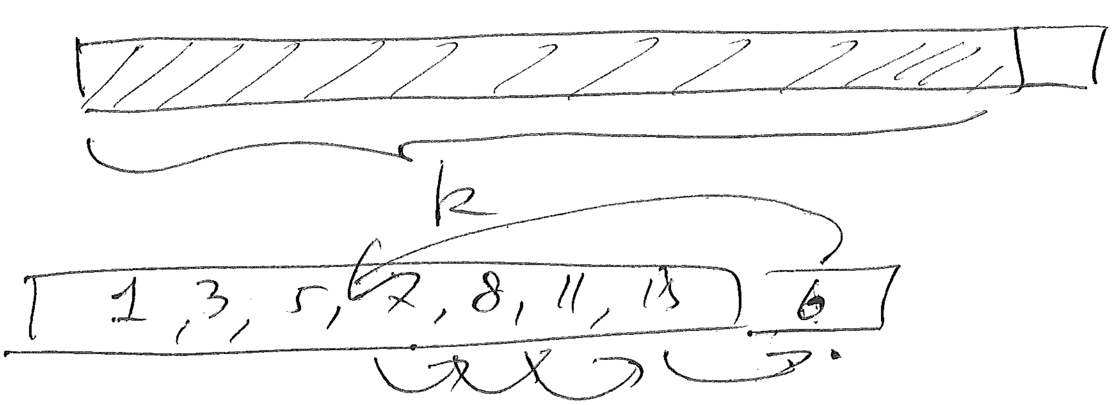


Using induction to design insertion sort.

Basis: $n=1$. ✓

I.H: Assume we know how to sort ~~n~~ k numbers.

I.S. Can we sort $k+1$ numbers?



input. $A[1..n]$, n distinct numbers

output. A sorted in increasing order

For $k=2$ to n

insert $A[k]$ into $A[1..k]$
so that $A[1..k]$ is sorted in increasing order

Running time: $\sum_{k=2}^{n} k = 2+3+4+\cdots+n$

$$= \frac{(2+n)(n-1)}{2}$$

$$= \Theta(n^2)$$

Merge Sort.

$$T(n) = 2T(\tfrac{n}{2}) + n \Rightarrow T(n) = O(n \log n)$$

Radix Sort

Given an array A of n non-negative integers

each integer has at most k digits

Sort ~~the~~ A. in increasing order

Basis $k = 1$

Given n integer numbers all from 0 to 9

Sort them

Bucket Sort.

Build 10 "Buckets", each Bucket is a queue.

Buckets are indexed $0, 1, 2, \cdots, 9$

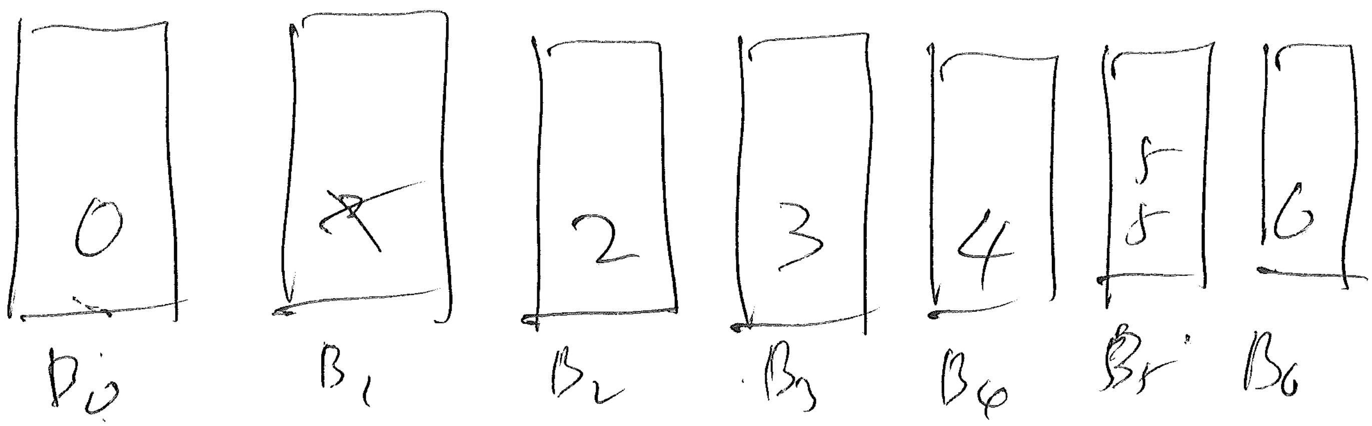
For $j = 1$ to n

  Put $A[j]$ in Bucket $B_{A[j]}$

For $j = 0$ to 9

  output all numbers in B_j

$A = \{0, 3, 4, 2, 6, 5, 5$



0, 2, 3, 4, 5, 5, 6

Running time $\Theta(n)$

I.H. Assume we know how to sort n k-digit numbers.

I.S. Can we sort $(k+1)$-digit numbers?

$$d_k \mid \boxed{d_{k-1} \cdots d_2 d_1 d_0}$$

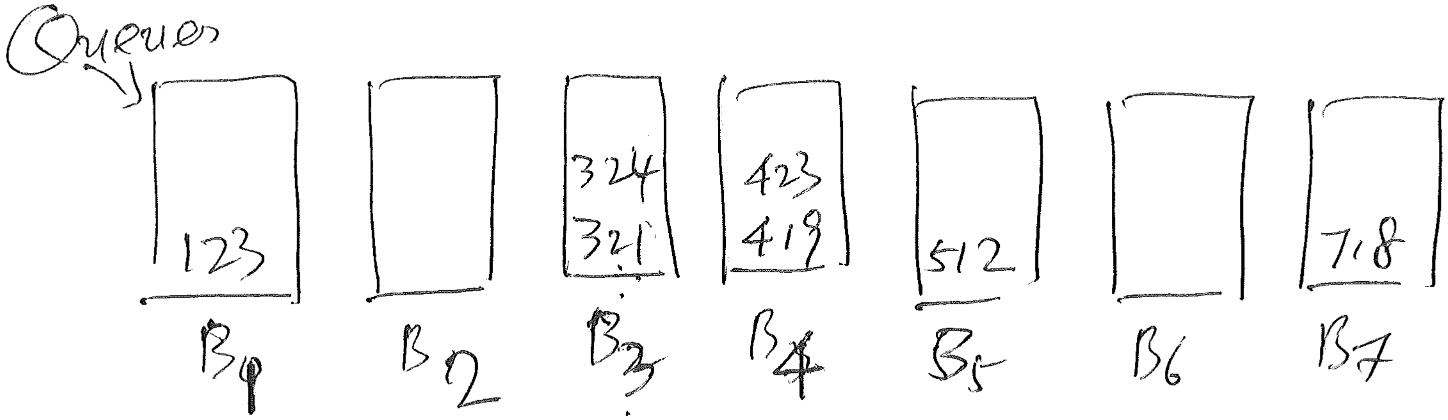
idea. sort A based on the k least significant digits

Then sort using the most significant digits

$A = \{321, 123, 419, 718, 423,$
$324, 512\}$ $k=3$

Step 1 output:

$A = \{512, 718, 419, 321, 123, 423, 324\}$



123, 321, 324, 419, 423, 512, 718

Radix Sort

For j = 0 to k-1   from least significant to the most significant

Sort ~~the numb~~ A based on the j^{th} digit.

$$\sum_{j=0}^{k-1} n = kn$$

if k is small $\sim n$

Pf:

Basis $k=1$ Algorithm ✓ ✓

I.H. Assume the algorithm sorts correctly for k-digit numbers.

I.S. Need to show the algorithm sorts correctly for $(k+1)$-digit numbers

Consider two number x and y belong to A.

Clearly if $x=y$ they will be sorted correctly.

So assume $x \neq y$

Let $x = a_k a_{k-1} \cdots a_0$

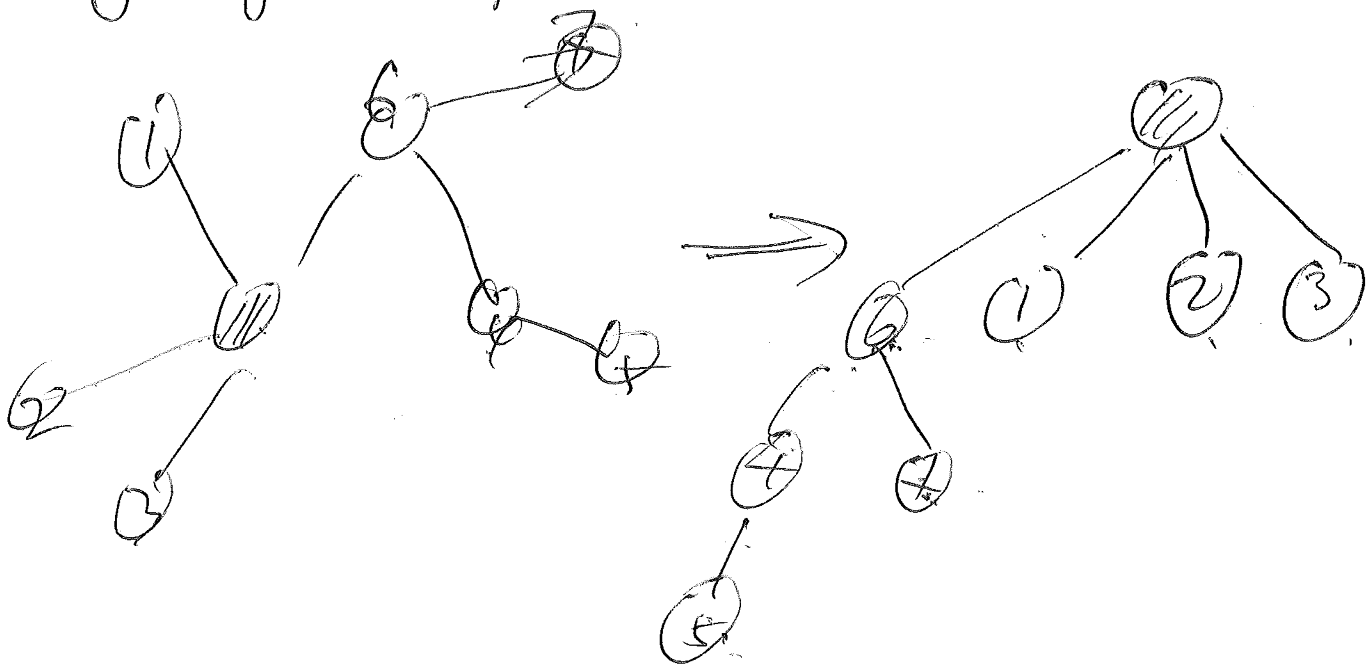
$y = b_k b_{k-1} \cdots b_0$

Case 1 $a_k = b_k$ ✓

case 2 $a_k > b_k$ ✓

Some other Sorting Algorithms

Sorting using a heap and a BST



The number of edges on the simple path from a node v to r

is the level of v

The maximum level of a node in a tree

is the height of ~~the~~ tree