Feb 11, 2010

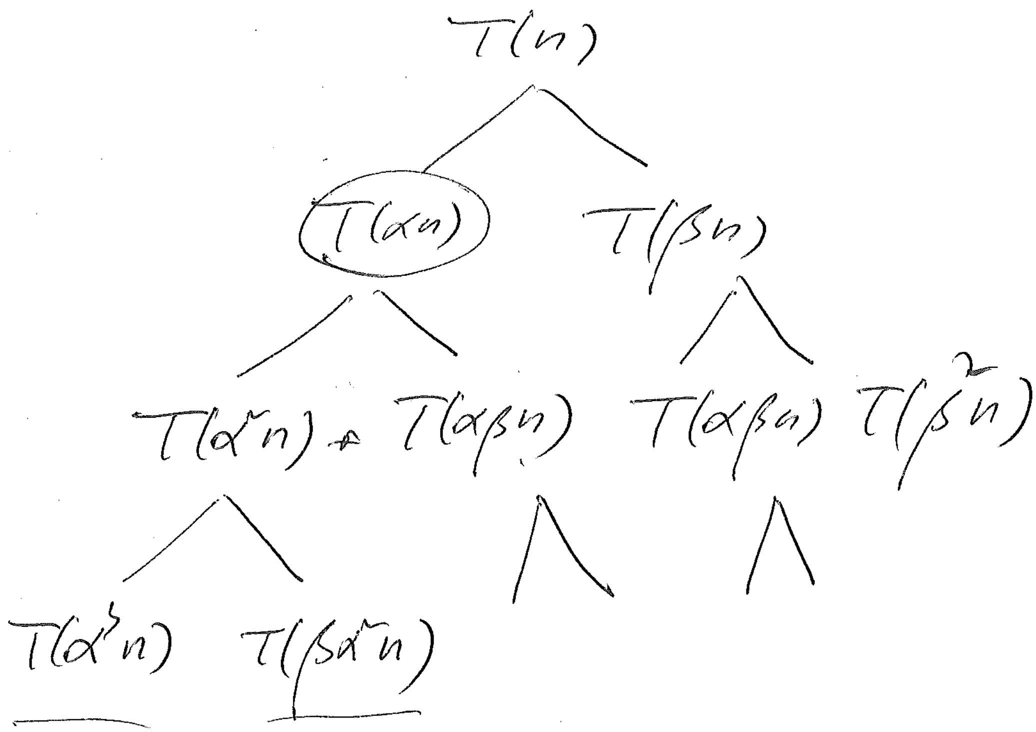
$$T(n) = T(\alpha n) + T(\beta n) + n$$

(1) $\alpha + \beta = 1$ $\quad T(n) = T(\frac{n}{7}) + T(\frac{6n}{7}) + n$

(2) $\alpha + \beta < 1$ $\quad T(n) = T(\frac{n}{5}) + T(\frac{3n}{5}) + n$

Case 1.

$T(n) = T(\alpha n) + T(\beta n) + n$ $\qquad \alpha + \beta = 1$



n

$\alpha n + \beta n = n$

$\alpha^2 n + \alpha\beta n$

$\alpha\beta n + \beta^2 n$

$= (\alpha + \beta)^2 n = n$

height $= \log_{1/\alpha} n$

Total running time $\quad n \log n$

Example : Merge Sort :

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n \Rightarrow T(n) = O(n\log n)$$

Case (2)

$$T(n) = T(\alpha n) + T(\beta n) + n \qquad \alpha + \beta < 1$$

Let $\alpha + \beta = r < 1$

$T(n)$ $\qquad\qquad$ n

$T(\alpha n)$ $\quad$ $T(\beta n)$ $\qquad\qquad$ $(\alpha+\beta)n = rn$

$T(\alpha^2 n)$ $\quad$ $T(\beta\alpha n)$ $\quad$ $T(\alpha\beta n)$ $\quad$ $T(\beta^2 n)$ $\qquad\qquad$ $(\alpha+\beta)^2 n = r^2 n$

Total cost : $n + rn + r^2 n + \cdots r^{\log n} n$

$$\leq n + rn + r^2 n + \cdots + \cdots$$

$$= n \sum_{j=0}^{\infty} r^j = n \frac{1 - \cancel{r^{\infty}}}{1 - r} = n\frac{1}{1-r}$$

~~T(α~~

$$\left.\begin{array}{l} T(n) = T(\alpha n) + T(\beta n) + n \\ \alpha + \beta < 1 \end{array}\right\} \Rightarrow T(n) = \Theta(n)$$

$$\approx \frac{1}{1-(\alpha+\beta)} n$$

Example:

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n$$

$$\alpha = \frac{1}{5} \qquad \beta = \frac{7}{10}$$

$$\alpha + \beta = \frac{9}{10}$$

$$T(n) = \frac{1}{1-\left(\frac{1}{5}+\frac{7}{10}\right)} n = 10n$$

$$T(n) = T(\alpha n) + T(\beta n) + n$$

if $\alpha + \beta = 1 \Rightarrow T(n)\ \Theta(n \log n)$

Divide and Conquer

if $\alpha + \beta < 1 \Rightarrow T(n) = \Theta(n)$

Decrease and Conquer

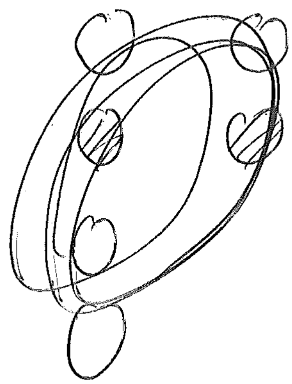
Selection Algorithm of Different ~~portion~~ partition numbers.

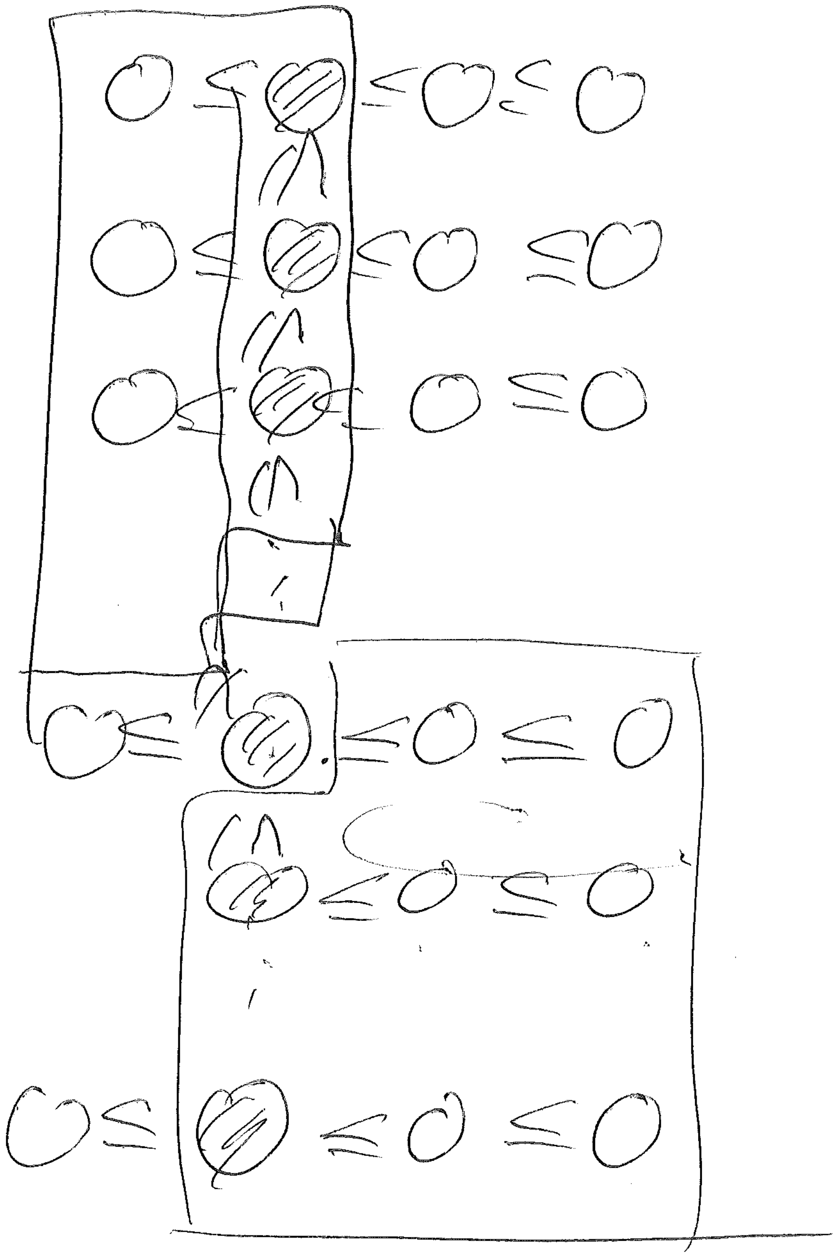
(1) Group of 4.

$$T(n) = n + T\left(\frac{n}{4}\right) + T\left(\frac{3n}{4}\right)$$

$$S_1 \le x < S_2$$

$\Rightarrow T(n) = \Theta(n \lg n)$





$$|S_1| \ge \frac{n}{4} \cdot \frac{1}{2} \cdot 2 = \frac{n}{4}$$

$$|S_2| \ge \frac{n}{4} \cdot \frac{1}{2} \cdot 3 = \frac{3n}{8}$$

$$|S_1| + |S_2| = n$$

$$|S_2| \le \frac{3n}{4}$$

$$|S_1| \le \frac{5n}{8}$$

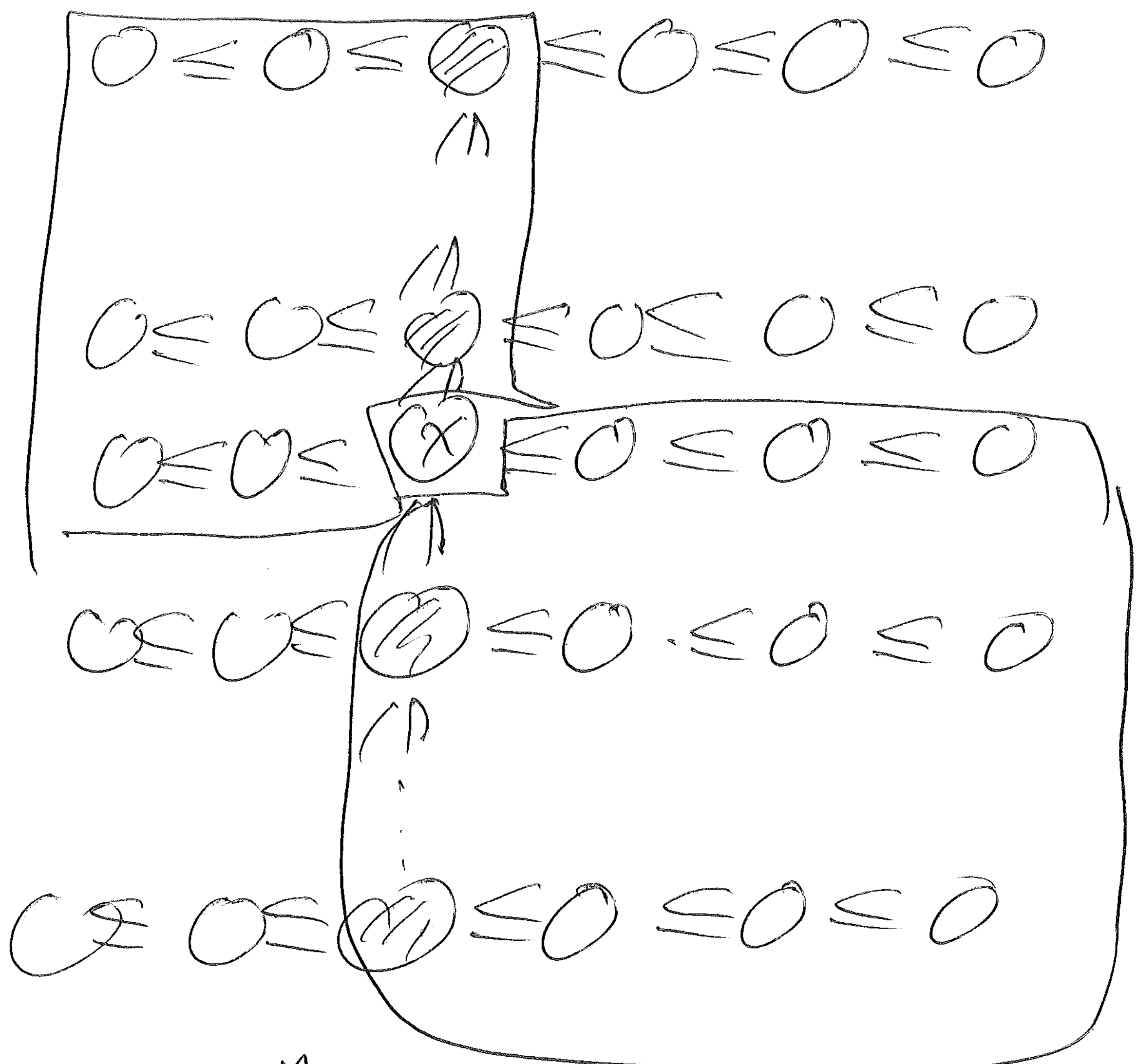
Partition into groups of 6:

$\alpha + \beta = \frac{11}{12}$

$$T(n) = n + T\left(\frac{n}{6}\right) + T\left(\frac{3n}{4}\right)$$

$\alpha + \beta \le \frac{11}{12}$

$$S_1 \le x < S_2$$



$$|S_1| \ge \frac{n}{6} \cdot \frac{1}{2} \cdot 3 = \frac{3n}{12}$$

$$|S_2| \ge \frac{n}{6} \cdot \frac{1}{2} \cdot 4 = \frac{n}{3}$$

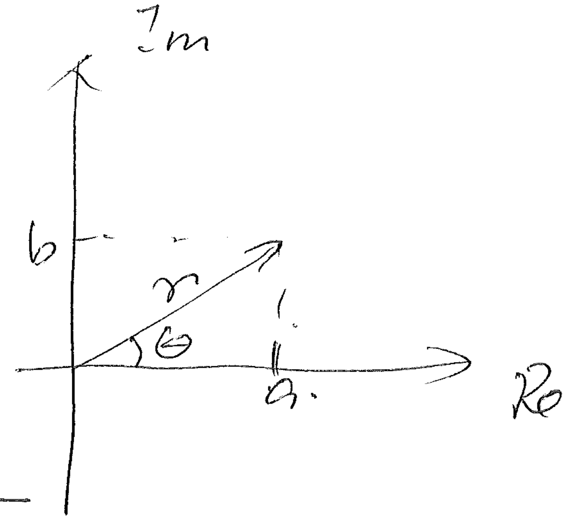
$$|S_1| + |S_2| = n$$

$$|S_2| \le \frac{9n}{12} = \frac{3n}{4} \qquad |S_1| = \frac{2n}{3}$$

Gauss:

$a+ib$

$i=\sqrt{-1}$



$|a+ib|=\sqrt{a^2+b^2}$

$a+ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$(a+ib)(c+id)$ $\qquad i^2=-1$

$= ac + iad + ibc + \underline{i^2bd}$

$= (\underline{ac} - \underline{bd}) + i(\underline{ad} + \underline{bc})$

We need 4 multiplications!

$(\underline{ac} - \underline{bd}) + i\ ((a+b)(c+d) - ac - bd)$

$\cancel{ac} + ad + bc + \cancel{bd}$
$- \cancel{ac} - \cancel{bd}$

Thus we can do this in 3 multiplications!

~~1+2~~

~~1 5~~

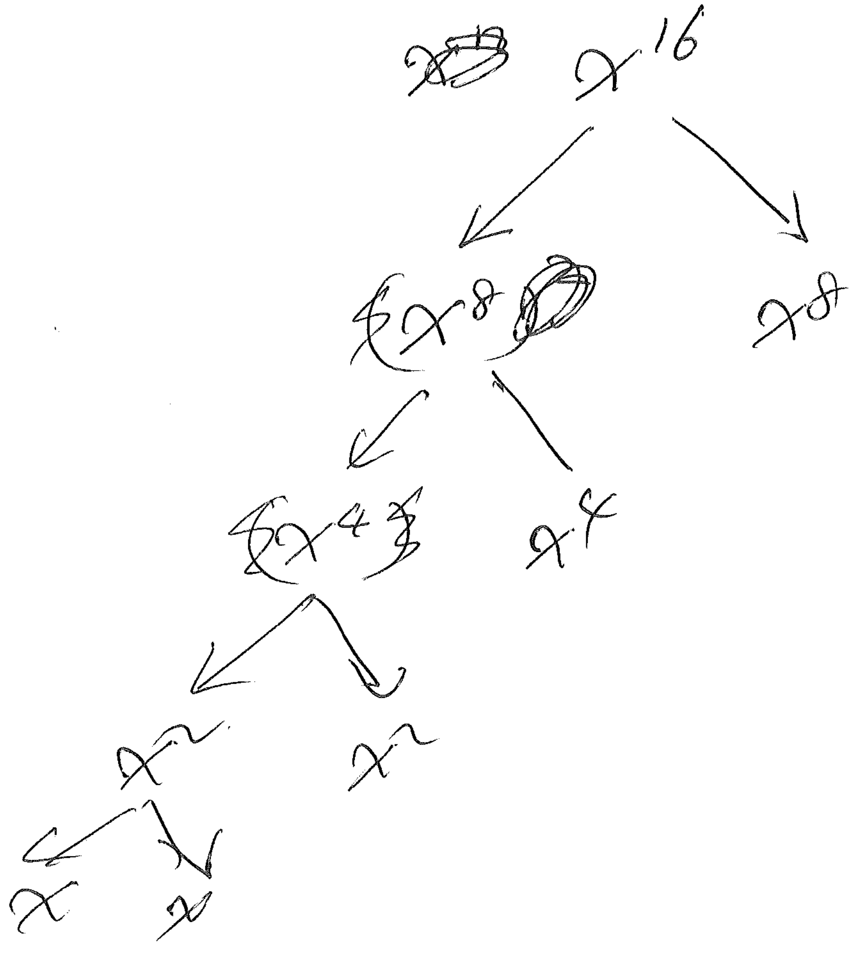
Polynomial Multiplication

$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

n^{th}-degree polynomial

~~(1) how to evaluate a polynomial?~~ ?

(1) how to calculate x^n? $\log n$ multiplica



4 multiplica

$\log_2 16$

(2) how to evaluate

$$P(x) = \cancel{a_0 x} \; a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n ?$$

Naive approach: $O(n \lg n)$ multiplications

$$P(x) = \underline{a_0 + x(a_1 + x(a_2 + x(a_3 + \cdots}\;))\cdots)$$

E.g.:

$$P(x) = a_0 + \cancel{x a_1}\; a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

$$= a_0 + x(a_1 + x(a_2 + x(a_3 + x a_4))))$$

We can ~~do the that~~ evaluate $P(x)$ using $\Theta(n)$ multiplications

(3) Let $P(x)$ and $Q(x)$ be two polynomials of degree n, how to calculate $\underline{P(x) \cdot Q(x)}$

(3)

$$(a_0+a_1x+a_2x^2+a_3x^3)(b_0+b_1x+b_2x^2+b_3x^3)$$

$$= c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6$$

$$c_3 = a_0b_3+a_1b_2+a_2b_1+a_3b_0$$

$$P(x)\cdot Q(x)$$

$$= (a_0+a_1x+\cdots+a_nx^n)$$

$$(b_0+b_1x+\cdots+b_nx^n)$$

$$= c_0+c_1x+\cdots+c_kx^k+\cdots+c_{2n}x^{2n}$$

$$c_k = a_0b_k+a_1b_{k-1}+\cdots+a_kb_0$$

$$c_{n+3} = a_3b_n+a_4b_{n-1}+\cdots+a_nb_3$$

Total number of multiplications:

$$\sum_{j=0}^{n}(j+1) + \sum_{j=n+1}^{2n}(2n-j+1) = \Theta(n^2)$$

The Divide and Conquer Algorithm

$$(a_0 + a_1 x)(b_0 + b_1 x)$$

$$= a_0 b_0 + (\underline{a_0 b_1 + a_1 b_0})x + a_1 b_1 x^2$$

$$= a_0 b_0 + ((a_0 + a_1)(b_0 + b_1) - a_0 b_0 - a_1 b_1)x$$

$$+ a_1 b_1 x^2$$

$$P(x) = a_0 + a_1 x + \cdot \quad + a_n x^n$$

$$= (a_0 + a_1 x + \cdots + a_{\frac{n}{2}} x^{n/2})$$

$$+ (a_{\frac{n}{2}+1} x^{\frac{n}{2}+1} + \cdots + a_n x^n)$$

$$= (a_0 + a_1 x + \cdot + a_{\frac{n}{2}} x^{n/2})$$

$$+ x^{\frac{n}{2}} (a_{\frac{n}{2}+1} x + \cdot \quad + a_n x^{\frac{n}{2}})$$

$$= P_0(x) + x^{n/2} P_1(x)$$

Note $P_0(x)$, $P_1(x)$ are both $\frac{n}{2}$-degree polynomials

Similarly $Q(x) = Q_0(x) + x^{\frac{n}{2}} Q_1(x)$

$Q_0(x)$ and $Q_1(x)$ are both $\frac{n}{2}$-degree.

$P(x)\ Q(x)$

$= (P_0(x) + x^{n/2} P_1(x))(Q_0(x) + x^{n/2} Q_1(x))$

$= P_0(x) Q_0(x) + x^{\frac{n}{2}} P_0 Q_1 + x^{n/2} P_1 Q_0 + x^n P_1 Q_1$

$= \underline{\underline{P_0 Q_0}} + x^{n/2} (\underline{\underline{P_0 Q_1}} + \underline{\underline{P_1 Q_0}}) + x^n \underline{\underline{P_1 Q_1}}$

$= P_0 Q_0 + x^{n/2} ((P_0 + P_1)(Q_0 + Q_1) - P_0 Q_0 - P_1 Q_1)$

$+ x^n P_1 Q_1$

$T(n)$: time for multiply two n-degree polynomials

$T(\frac{n}{2})$: time for multiplying two $\frac{n}{2}$-degree polynomials

$$T(n) = 3T\left(\frac{n}{2}\right) + n \Rightarrow n^{\log_2 3}$$

$$\approx n^{1.585} \quad vs. \; n^2$$

$$\lim_{n\to\infty} \frac{n^{1.585}}{n^2} = \lim_{n\to\infty} n^{-0.415} = \lim_{n\to\infty} \frac{1}{n^{0.415}} = 0$$

Let $n = 2^k$

$$T(2^k) = 3T(2^{k-1}) + 2^k$$

$$= 3[3T(2^{k-2}) + 2^{k-1}] + 2^k$$

$$= 3^2 T(2^{k-2}) + 3 \cdot 2^{k-1} + 2^k$$

$$= 3^2[3T(2^{k-3}) + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k$$

$$= 3^3 T(2^{k-3}) + 3^2 \cdot 2^{k-2} + 3 \cdot 2^{k-1} + 2^k$$

$$= \cdots =$$

$$= 3^k \cancel{T(2^{k-k})} + 3^{k-1}2^1 + 3^{k-2}2^2 + \cdots + 3\,2^{k-1} + 2^k$$

$= 3^k + 3^{k-1}2^1 + 3^{k-2}2^2 + \cdots + 3 \cdot 2^{k-1} + 2^k$

Geometric with ratio $\frac{2}{3}$

$$= \frac{3^k - \frac{2}{3}2^k}{1 - \frac{2}{3}} \qquad 2^n = o(3^n)$$

$= O(3^k)$ Recall $k = \log_2 n$

$3^k = 3^{\log_2 n} = \left(2^{\log_2 3}\right)^{\log_2 n}$

$= \left(2^{\log_2 n}\right)^{\log_2 3} = n^{\log_2 3}$