Jan 21, 2016

Given an array $A[1..n]$ of n distinct numbers from the set $\{1, 2, \cdots, n, n+1\}$, find the missing integer.

Alg 1. return $\frac{(n+1)(n+2)}{2} - \sum_{j=1}^{n} A[j]$ T: $(2n+2)$ S: n

2 return $\frac{(n+1)!}{\prod_{j=1}^{n} A[j]}$ T: $3n+4$ S: n

3 Create a Boolean array $B[1..n+1]$ T: $4n+2$
mark all the numbers in A. S: $2n$
return the index j where $B[j] == F$

Big O notations. $3.7 n \log n + 4.778 \log n$

(1) Focus on the leading growth term of the running time with constant coefficient stripped off

(2). For space analysis, we must be very careful when using the Big O notation

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$$\begin{array}{ll} x & 1101 \\ y & 0101 \\ \hline x \oplus y & 1000 \end{array}$$

Properties:

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(1) $x \oplus 0 = x$

(2) $x \oplus x = 0$

(3) $x \oplus y = y \oplus x$

(4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

Observation:

$$x \oplus y \oplus x = (x \oplus x) \oplus y = y$$

$$y \oplus x \oplus y = x$$

int, x, y, tmp

tmp = x

x = y

y = tmp

$x = x \oplus y$

$y = x \oplus y = (x \oplus y) \oplus y = x$

$x = x \oplus y = (x \oplus y) \oplus x = y$

Alg 4: return $\left(\bigoplus_{j=1}^{n+1} j\right) \oplus \left(\bigoplus_{j=1}^{n} A[j]\right)$.

Ex. Let $A = \{1, 2, 4\}$ from $\{1, 2, 3, 4\}$

$(\cancel{1} \oplus \cancel{2} \oplus 3 \oplus \cancel{4}) \oplus (\cancel{1} \oplus \cancel{2} \oplus \cancel{4})$.

$= 3$

Time: $\cancel{4}n+1 \longrightarrow O(n)$

Space: n.

Def: (c_0, n_0)

Let $f(n)$ and $g(n)$ be non-negative increasing functions

We say $f(n) = O(g(n))$ if

there exists positive constants c and n_0 such that

for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$

asympotic $\leq$

$f(n) = 3n^2 \qquad g(n) = n^2$

Let $c = 4$, $n_0 = 1$, then

for $n \geq 1$, $\quad 3n^2 \leq 4 \cdot n^2$

therefore $\quad 3n^2 = O(n^2)$

$c = 10^6$

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what is $g(n) = n^3$.

For $c = 4$ and $n_0 = 1$ for all $n \geq 1$,

$3n^2 \leq 4 \cdot n^3$, therefore $3n^2 = O(n^4)$

Def. We say $f = \Omega(g)$ if there exists c and n_0 such that for $n \geq n_0$, $\quad f \geq c \cdot g$

asymptotic "$\geq$"

Def We say $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$

Def (small o notation)

We say $f(n) = o(g(n))$ if for any positive c, there exires n_0 such that

for $n \geq n_0$, $f(n) < c \cdot g(n)$

Example $f(n) = n^2$ $g(n) = n^3$.

Let c be arbitray positive real number

$n^2 < c \cdot n^3$ $n_0 > \frac{1}{c}$

What if $c = 10^{-6}$

Can we find an n_0 such that

For $n \geq n_0$, $n^2 < 10^{-6} n^3$

$f = \underline{10^6}$ $g = 1$ is $f = O(1)$?

Yes $c = 10^8$ $n_0 = 1$ then $10^6 < 10^8 \cdot 1$

Small ω notation:

$f = \omega(g)$ if for any $c > 0$, there exists $n_0 > 0$ such that for all $n \ge n_0$, $f > c \cdot g$

Vs. limit.

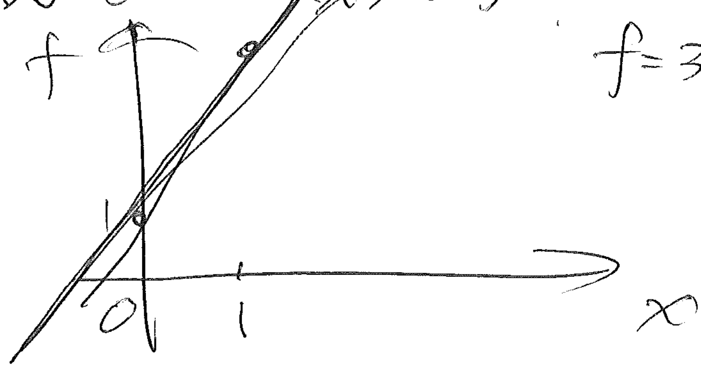
$\lim_{n\to\infty} \frac{f(n)}{g(n)} = a$ $\quad 0 < a < +\infty$, then $f = \Theta(g)$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ $\quad f = o(g)$ $\quad f \in O(g)$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$ $\quad f = \omega(g)$ $\quad f \in \Omega(g)$

L'hospital Rule: Let $\lim_{x\to\infty} f(x) = \infty$ $\quad \lim_{x\to\infty} g(x) = \infty$

$$\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{f'}{g'}$$



$f = 3x + 1$

$f' = 3$

Example 1 $\quad f = \sqrt{x} \qquad g = \ln x$

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{x^{1/2}}{\ln x} = \lim_{x\to\infty} \frac{(x^{1/2})'}{(\ln x)'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2} x^{-1/2}}{1/x} = \lim_{x\to\infty} \frac{x^{1/2}}{2} = \infty$$

$$\sqrt{x} = \omega(\ln x)$$

Example 2 $\displaystyle \lim_{x\to\infty} \frac{e^x}{x^2}$

Sol: $$\lim_{x\to\infty} \frac{(e^x)'}{(x^2)'} = \lim_{x\to\infty} \frac{e^x}{2x}$$

$$= \lim_{x\to\infty} \frac{(e^x)'}{(2x)'} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$e^x = \omega(x^2)$$

polylogs " < " polynomial " < " exponential

Exponetiation and Exponential Functions

Def Let a be a real number
n be a positive integer, then

$$a^n = \underbrace{a \cdot a \cdots a}_{n}$$

$$a^m \cdot a^n = (\underbrace{a \cdots a}_{m})(\underbrace{a \cdots a}_{n})$$

$$= \underbrace{a \cdots a}_{m+n}$$

$$= a^{m+n}$$

$$(a^m)^n = \underbrace{a^m \cdots a^m}_{n}$$

$$= \underbrace{(\underbrace{a \cdots a}_{m}) \cdots (\underbrace{a \cdots a}_{m})}_{n}$$

$$= \underbrace{a \cdots a}_{mn} = a^{mn}$$

~~@~~ Define $a^0 = 1$ for $a \neq 0$

$0^0 = 1$ ✓

$$(a^{\frac{1}{n}})^n = a^1 = a$$

$$\|$$

$$\underbrace{a^{1/n} \cdots a^{1/n}}_{n}$$

a^{-n}

$a^{-n} \cdot a^n = a^{(-n)+n}$

$= a^0 = 1$

$a^{-n} = \frac{1}{a^n}$

$a^{1/n}$ is a solution to ~~the~~ equation $x^n = a$

Let $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$ $\quad e = 2.718$

The exponential function: e^x

$e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \frac{\overbrace{(2)\cdots(2)}^{n}}{\underbrace{(3)\cdots(3)}_{n}} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n \to 0$$

$$\lim_{x\to\infty} \frac{(2^x)'}{(3^x)'} \overset{?}{=}$$

$$2^n = o(3^n)$$

Logarithmic Functions:

Let a and x be positive real numbers, $a \neq 1$

Then if $a^y = x$, we say $\log_a x = y$

Properties:

(1) $a^{\log_a x} = a^y = x$

(2) $\log_a xy = \log_a x + \log_a y$

Pf $a^m = x \quad a^n = y$

$xy = a^m \cdot a^n = a^{m+n}$

$\log_a xy = m+n = \log_a x + \log_a y$

$$\log_a(x^p) = p\log_a x$$

$$\log_a(x^p) = \log_a(\overbrace{x \cdot x \cdot x \cdots x}^{p})$$

$$= \underbrace{\log_a x + \log_a x + \cdots + \log_a x}_{p}$$

$$= p\log_a x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x} = \lim_{x\to\infty} \frac{\frac{\cancel{\ln x}}{\ln 2}}{\frac{\cancel{\ln x}}{\ln 3}}$$

$$= \lim_{x\to\infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2}.$$

Thus $\log_2 x = \Theta(\log_3 x)$

2^x

$(e^x)' = e^x$

$2 = e^{\ln 2}$

$2^x = (e^{\ln 2})^x = e^{x \ln 2}$

$= (e^x)^{\ln 2}$