April 26, 2016

Reminder: Exam 3 on Thursday May 5

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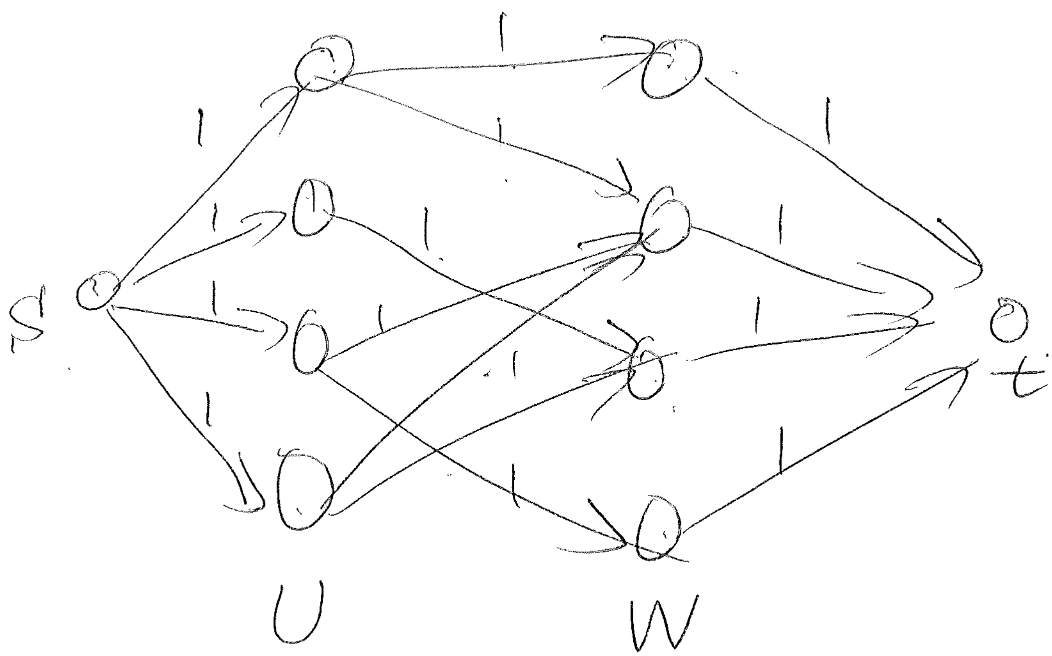
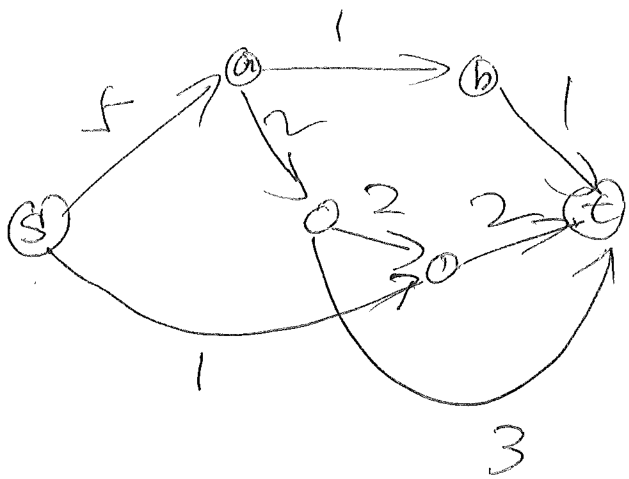
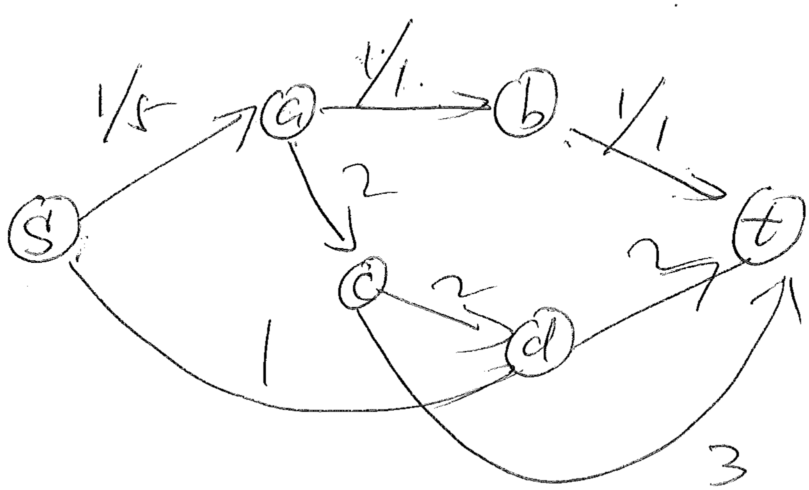




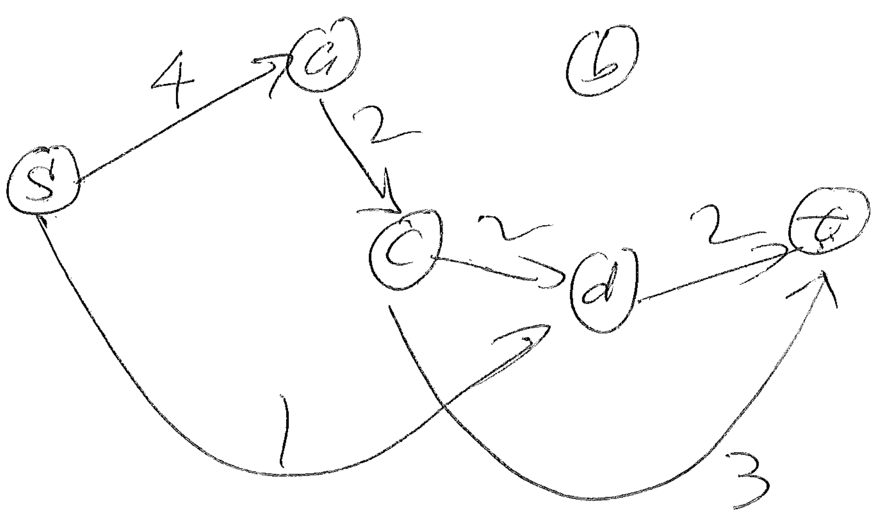
Greedy Strategy:
We will try to send flow from s to t as long as we can without violating the capacities

Observations,
We can use DFS to find paths from s to t

$s \xrightarrow{5} a \xrightarrow{1} b \xrightarrow{1} t$

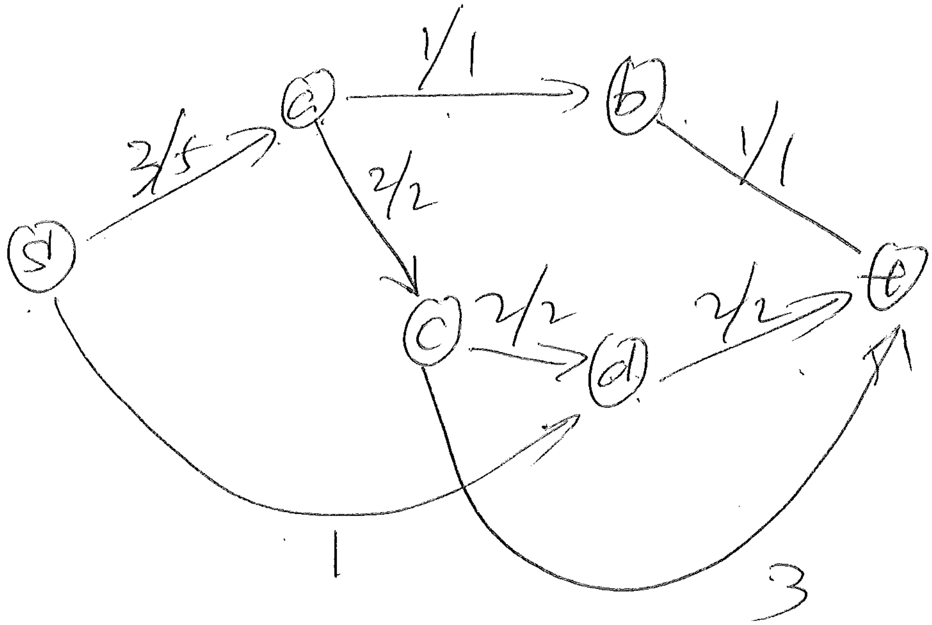


original network

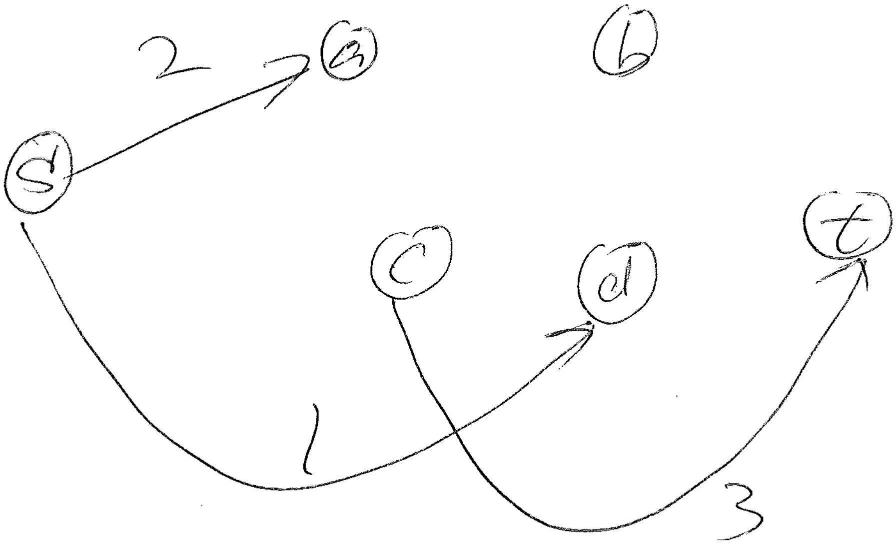


residue graph

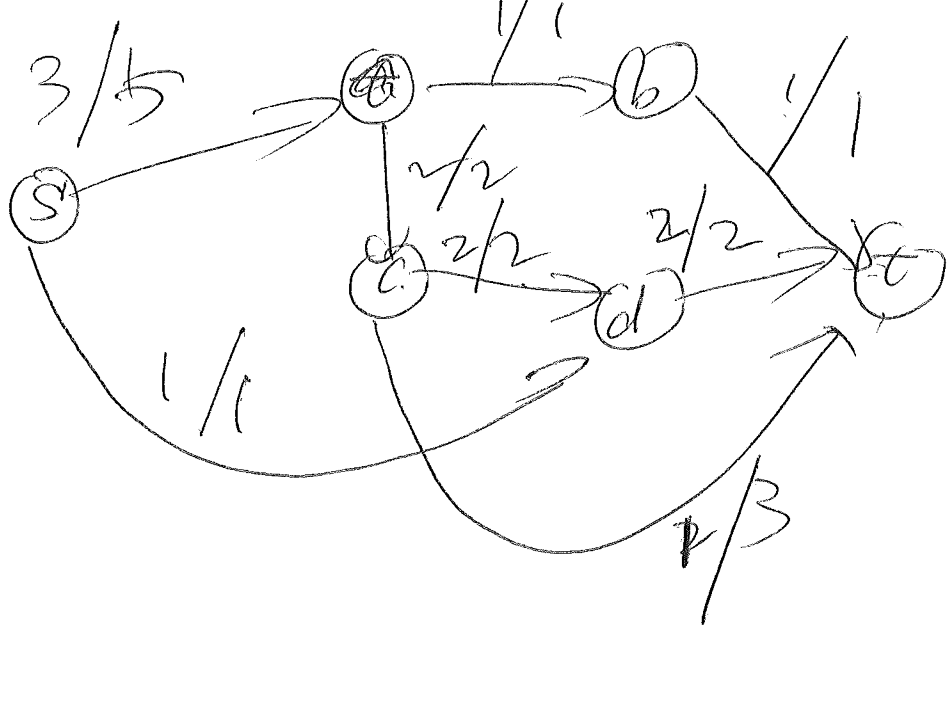
③

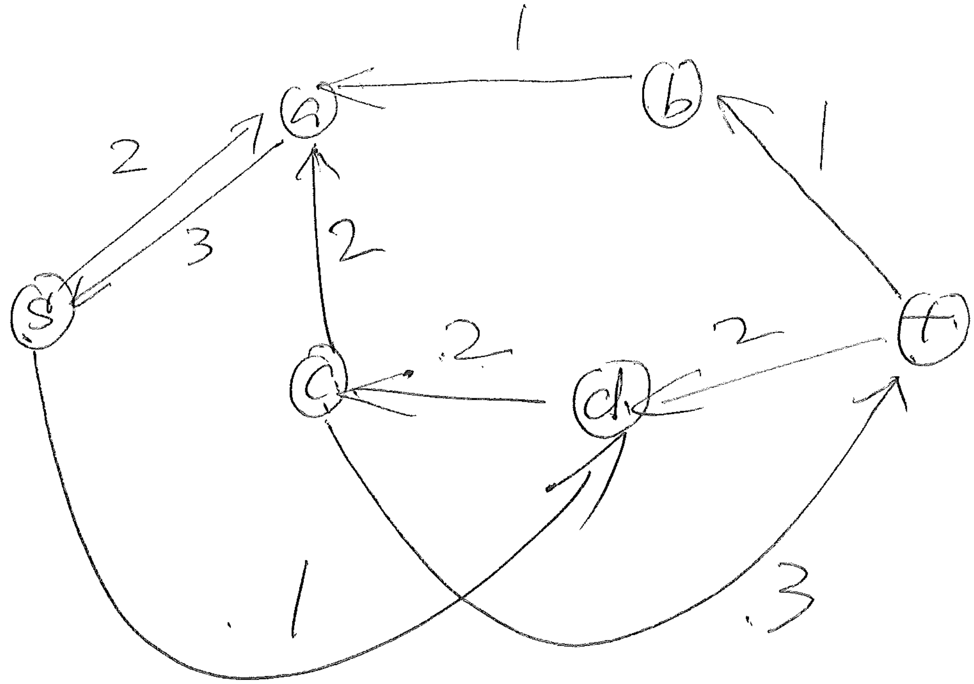



original network

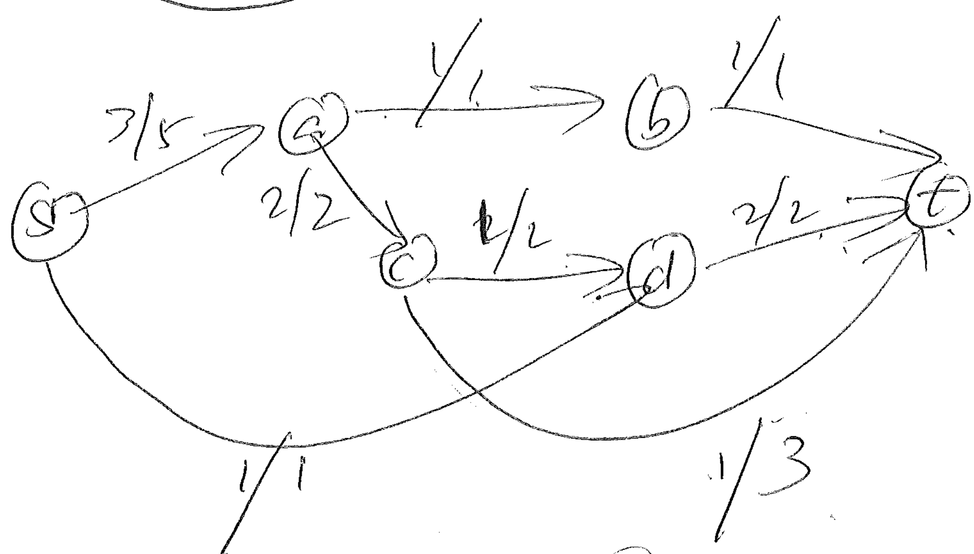



residue

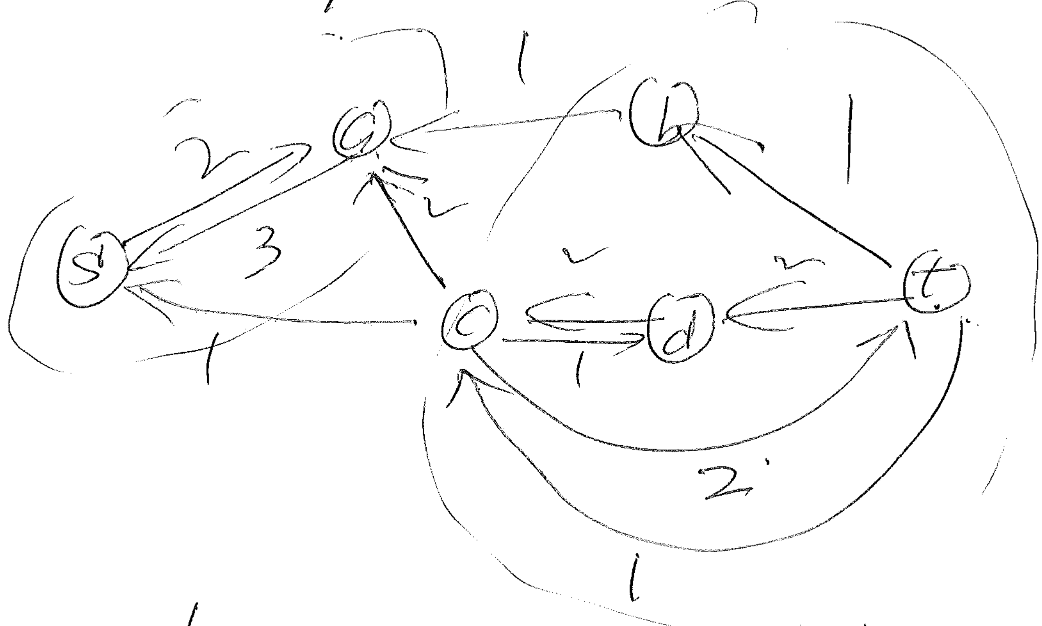





residue graph




original




residue

The maximum flow is 4 since there are
no more paths from S to t

# Augmenting Path

1. Create residue graph
2. Find if there is directed path from $s$ to $t$ and update $f$

repeat (1) (2) until there are no paths

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## MCF

Given a network (digraph) $G(V,E)$, $s,t \in V$

each edge $e \in E$ is associated with a capacity $u_e \geq 0$

and a cost $c_e \geq 0$ per unit capacity.

The goal is to send $b$ units of flow from

$s$ to $t$ minimizing the cost

Assuming that $b$ is feasible, i.e. $\leq$ max flow.

How to solve it?

# Max Flow – Mincut Algorithm

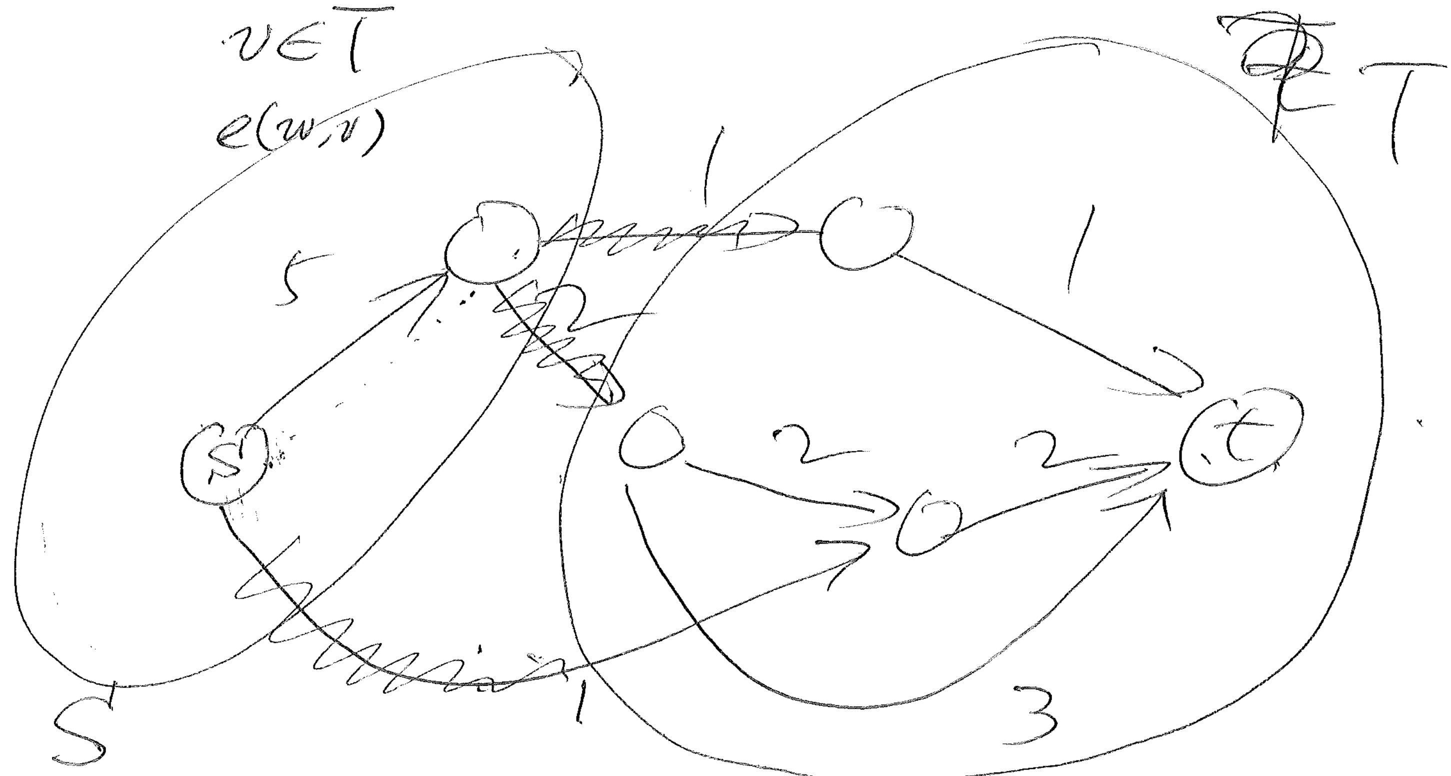
Let $G(V, E)$ be a network where each edge $e \in E$ is associated with a non-negative capacity $u_e$

An s-t cut (source-sink cut) $C(S, T)$ is a partition of $V$ into two disjoint subsets $S$, $T$. such that $s \in S$, $t \in T$.

cut

The capacity of the s,t-cut $C(S, T)$ is

$$|C| = \sum_{\substack{w \in S,\\ v \in T\\ e(w,v)}} u(e)$$




$|C| = 4$

Theorem ① The maximum flow is $\leq$ the capacity of any s-t cut.

② The augmenting path algorithm saturates a cut.

$s \in S \quad t \in T$

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Proof. of ① Let $C(S; T)$ be any cut.

Let $f$ be an arbitrary flow function.

$|f|$ To simplify discussions, we shall assume $G(V, E)$ is a <u>complete</u> directed graph

$$|f| = \sum_{e(s,v)} f(\underline{s,v}) - \sum_{e(v,s)} f(\underline{v,s})$$

$$= \cancel{\left(\sum_{e(s,v)} f(s,v) - \sum_{e(v,s)} f(v,s)\right)} +$$

$$\cancel{\sum_{w \in S-s} \left(\sum_{e(v,w)} f(v,w) - \sum_{e(w,v)} f(w,v)\right)}.$$

$$= \sum_{e(s,v)} f(s,v) - \sum_{e(v,s)} f(v,s)$$

$$+ \sum_{w \in S'-s} \left( \sum_{e(w,v)} f(w,v) - \sum_{e(v,w)} f(v,w) \right)$$

$$= \sum_{w \in S} \left( \sum_{e(w,v)} f(w,v) - \sum_{e(v,w)} f(v,w) \right)$$

$$= \sum_{w \in S} \sum_{e(w,v)} f(w,v) - \sum_{w \in S'} \sum_{e(v,w)} f(v,w)$$

$$= \sum_{w \in S} \left( \sum_{v \in S} f(w,v) + \sum_{v \in T} f(w,v) \right)$$

$$\oplus - \sum_{w \in S} \left( \sum_{v \in S} f(v,w) + \sum_{v \in T} f(v,w) \right)$$

(8)

$$= \cancel{\sum_{w\in S}\sum_{v\in S} f(w,v)} + \underbrace{\sum_{w\in S}\sum_{v\in T} f(w,v)}$$

$$- \cancel{\sum_{w\in S}\sum_{v\in S} f(v,w)} - \sum_{w\in S}\sum_{v\in T} f(v,w)$$

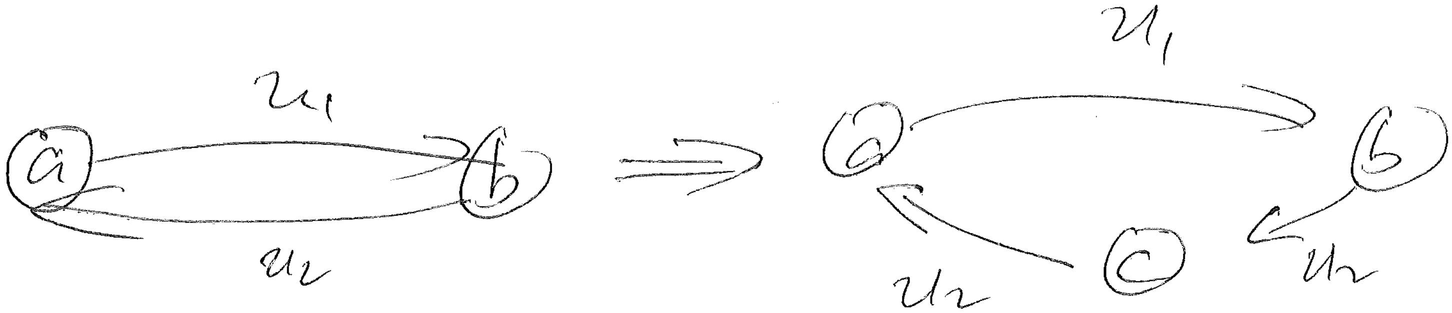
$$= |C| - \underline{\sum_{w\in S}\sum_{v\in T} f(v,w)} \le |C|$$

Proof 2

Let $f$ be the flow funcion generaced by ~~the~~ the augmenting pach algorichm.

~~Since there is no m~~

To simplify discussion, we will assume there is only one ~~are~~ arc between any pair of vertices




~~Sn~~ Let $G(V, E)$ be the network.

Let $G'$ be its residue graph.

Since the algorichm terminaced, there is no more pach from $S$ to $t$ in $G'$, hence $s$ and $t$ are disconneced. Let $S$ contain all the vertices reachable from $s$, and $T = V - S$

We will show that ~~|~~ $|C(S,T)| = |f|$

Consider ~~u∈~~ $v \in S'$ and $w \in T$

$$|C(S,T)| = \sum_{v \in S, w \in T} u(v,w) = \sum_{v \in S, w \in T} f(v,w)$$

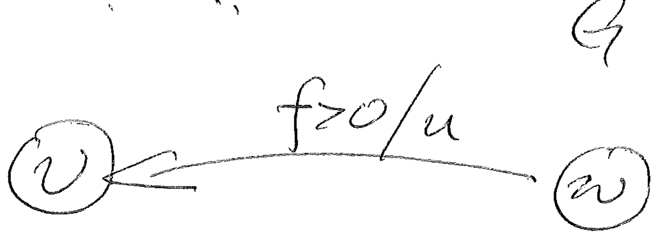
There are two possibilities

(1) $(v,w) \in G$, then $(v,w) \notin G'$

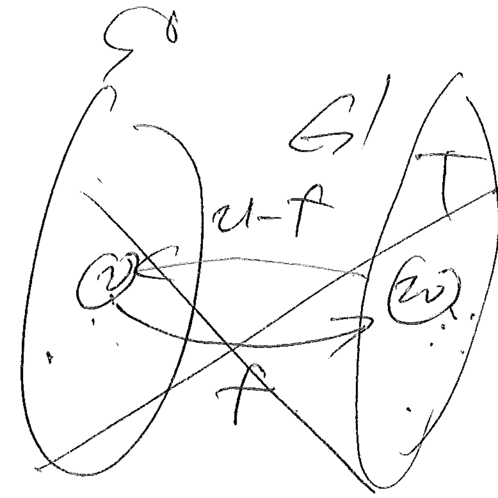
$$f(v,w) = u(v,w)$$

(2) $(v,w) \notin G$, $(w,v) \in G$

then $f(w,v) = 0$




$\Rightarrow$




Contradiction and $f=0$

① ~~Note~~ What if the capacity is integer?

② What if the capacity is rational numbers?

$42 \times \frac{3}{7}$ $\quad 42 \times \frac{4}{14}$ $\quad 42 \times \frac{5}{12}$

③ what if the capacity is irrational?

The algorithm may not terminate,