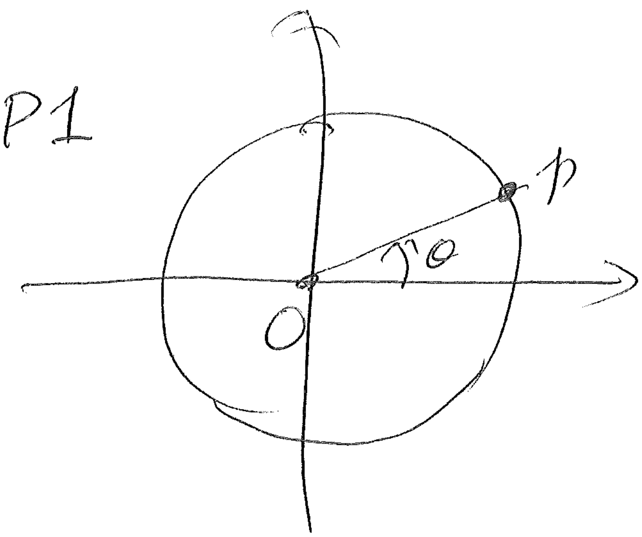
March 1

Grades posted are raw marks

Your Grade
—
50

Average: 30 ~ 60%

P1



n points on the unit circle

Sort counterclockwise

P2

Given a string / sentence

algorithm is fun ⟹ fun is algorithm

optimize time and space

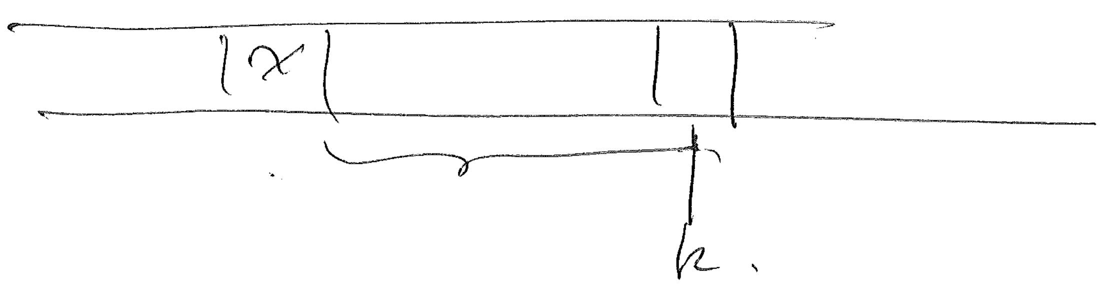
P3 Design a data structure and optimize

insertion

deletion

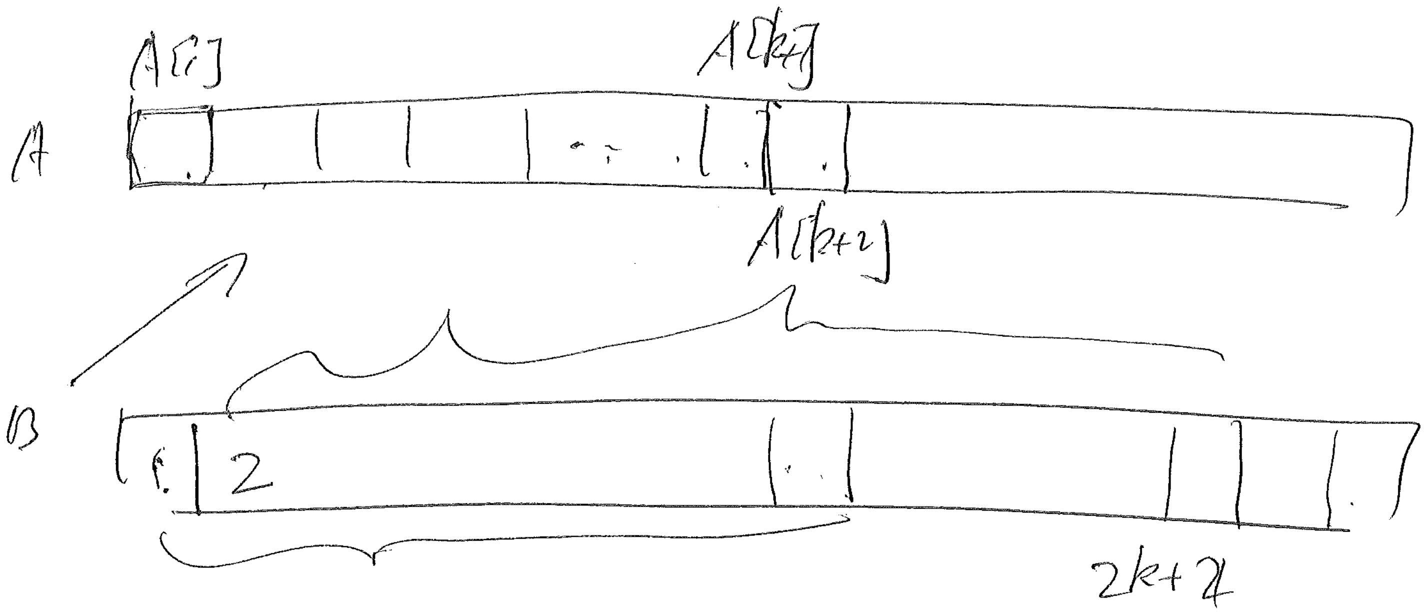
search (x, k)

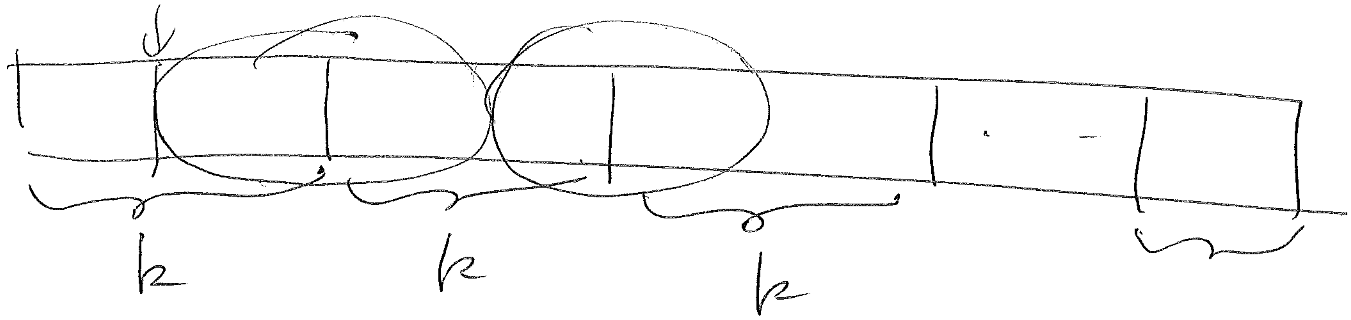
BST



maintain number of nodes in each subtree

P4 sorting an almost sorted array





P5

$$S = \{k_1, \dots, k_n\}$$

$$S_1 \cup S_2 = S, \quad S_1 \cap S_2 = \phi$$

$$\sum_{k_i \in S_1} k_i = \sum_{k_j \in S_2} k_j = \frac{1}{2} \sum_{k \in S} k$$

$$S = \{1, 2, 3, 4, 5, 6, 9\}$$

$$S_1 = \{2, 4, 9\} \quad S_2 = \{1, 3, 5, 6\}$$

$$\sum_{k_i \in S_1} = 15 \qquad \sum_{k_j \in S_2} = 15$$

Bellman-Ford Algorithm

For calculating single source shortest paths in a general graph with possible negative edge costs/distance.

Function

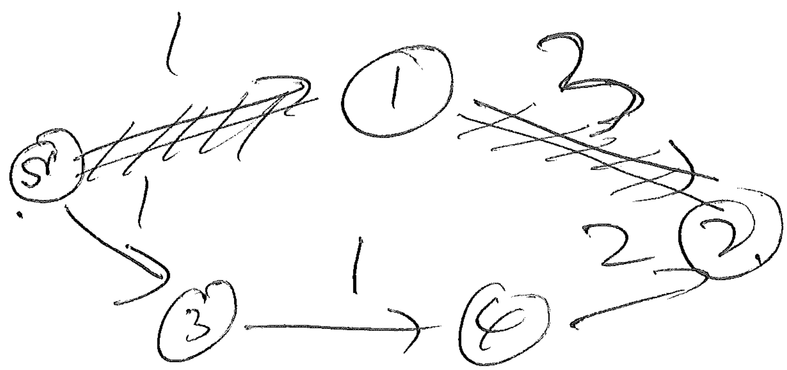
1. if the graph has a negative cycle reachable from the source, the algorithm will terminate and report there is a negative cycle.

2. if there is no negative cycles, then the algorithm correctly calculates the single source shortest paths.

induction on the # of edges on the shortest path

in k^{th} iteration, the algorithm finds ~~the~~

shortest path using $\leq k$ edges



Algorithm

$SP[k][n] \leftarrow$ shortest path from $\overset{0}{\overset{\shortparallel}{s}}$ to n using $\leq k$ edges

Basis ~~$SP[0][0] = 0$~~

~~For $k = 1$ to $n-1$~~

Let $s = 1$

$SP[k][n] \leftarrow$ shortest path from s to n using $\leq k$ edges

Basis $SP[0][1] = 0$

$SP[0][j] = \infty$ for $j \neq 1$

$SP[1][1] = 0$

$SP[1][j] = c(1,j)$ for $j \neq 1$

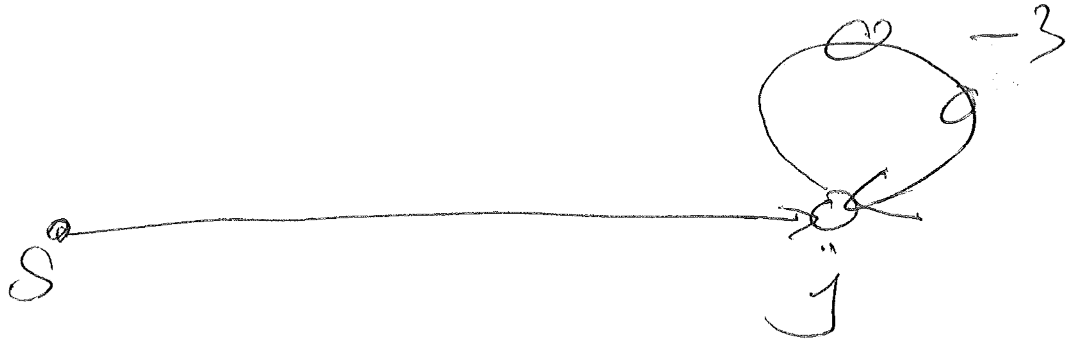
For $k = 2$ to $n-1$

$$SP[k][j] = \min \left\{ \begin{array}{l} SP[k-1][1] + c(1,j) \\ SP[k-1][2] + c[2,j], \\ \vdots \\ SP[k-1][n] + c[n,j] \end{array} \right\}$$

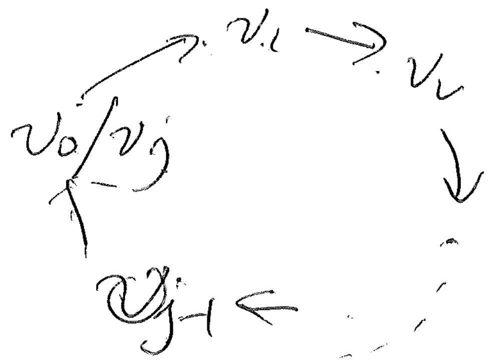
Calculate $SP[n][j]$ for all j

if $\exists j$, $SP[n][j] < SP[n-1][j]$

then report there is a negative cycle



Pf. Let $v_0, v_1, v_2, \dots v_j = v_0$ be a negative cycle



$$\sum_{i=0}^{j-1} c(v_i, v_{i+1}) < 0$$

Assume that this cycle is reachable from the source s

We will prove that if we run Bellman-Ford one more round, then ~~SP~~ $\exists$ there exists some vertex v, such $SP[n][v] < SP[n-1][v]$

Proof By Contradiction

Assume Not

Then with the negative cycles $SP[n][v] \geq SP[n-1][v]$ forall v..

$$\neg\left(\exists v \quad SP[n][v] < SP[n-1][v]\right)$$

$$\forall v \quad SP[n][v] \geq SP[n-1][v]$$

In particular

$$SP[n-1][v_0] + c(v_0, v_1) \geq SP[n][v_1] \geq SP[n-1][v_1]$$

$$SP[n-1][v_1] + c(v_1, v_2) \geq SP[n][v_2] \geq SP[n-1][v_2]$$

$$SP[n-1][v_2] + c(v_2, v_3) \geq SP[n][v_3] \geq SP[n-1][v_3]$$

$$\vdots$$

$$SP[n-1][v_{j-1}] + c(v_{j-1}, v_j) \geq SP[n][v_j] \geq SP[n-1][v_j]$$

$$\sum_{i=0}^{j-1} C(v_i, v_{i+1}) \geq 0$$

$$\cancel{\sum_{i=0}^{j-1} SP[n-1][v_i]} + \sum_{i=0}^{j-1} C(v_i, v_{i+1}) \geq \cancel{\sum_{i=0}^{j-1} SP(n-1)[v_i]}$$

$$\sum_{i=0}^{j-1} C(v_i, v_{i+1}) \geq 0$$

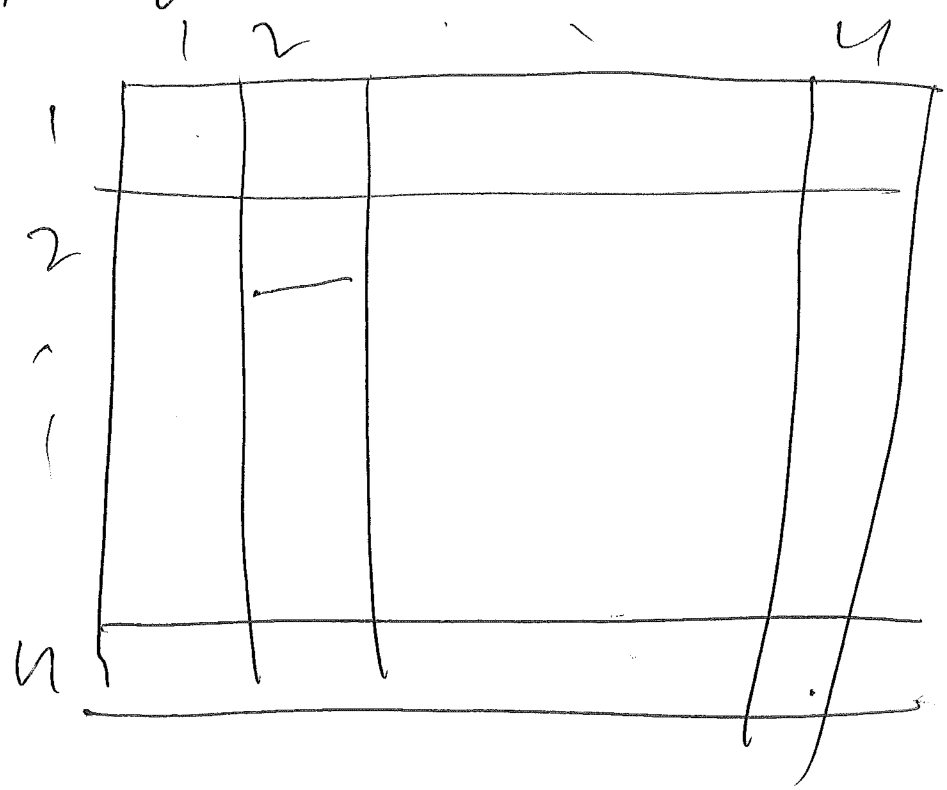
However $v_0, v_1, \ldots, v_{j-1}, v_j = v_0$ is a negative cycle

$$\sum_{i=0}^{j-1} C(v_i, v_{i+1}) < 0$$

A contradiction!

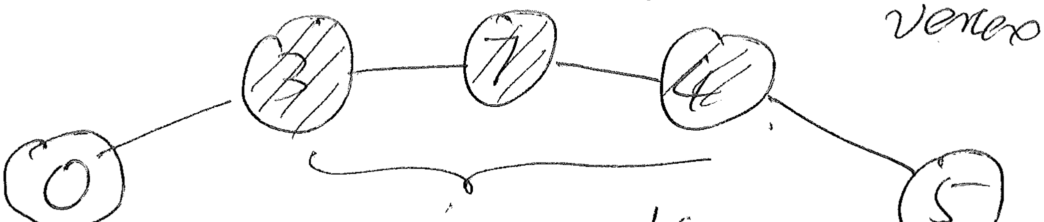
Floyd-Warshall Algorithm for all pair shortest paths

it calculates the shortest path between every pair of vertices



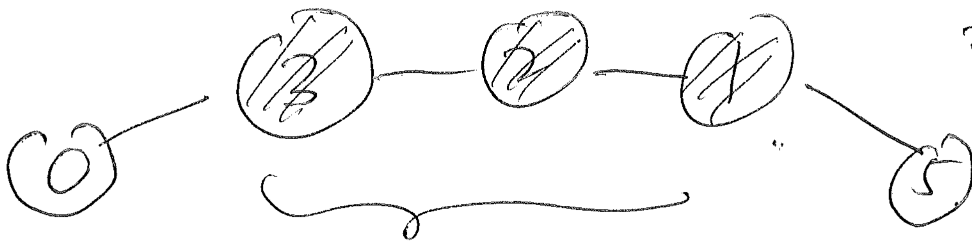
induction strategy : induction on the vertex labels

max label of the intermediate vertex = 7



intermediate vertices

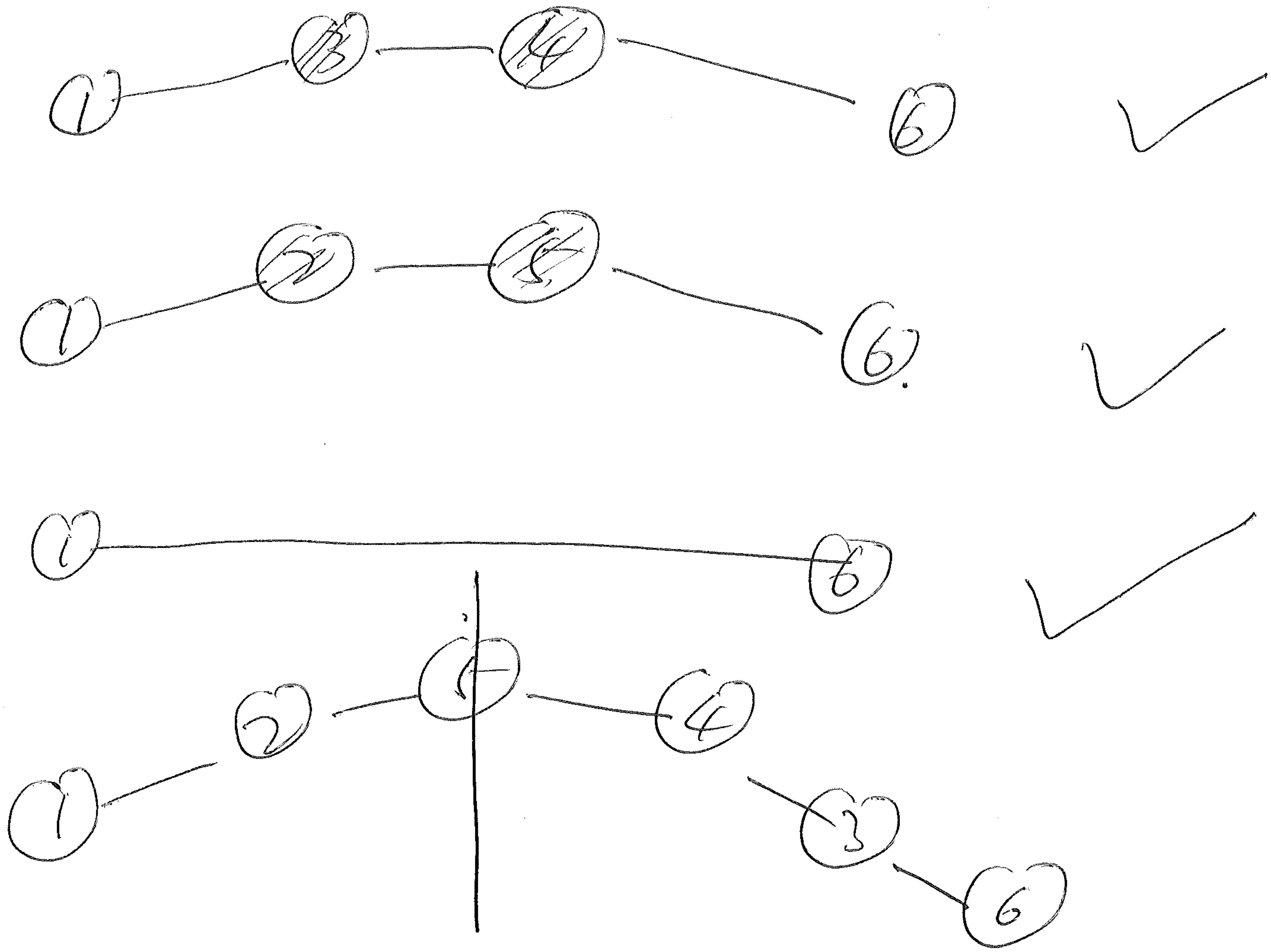
maximum label = 3



intermediate vertices

A k-shortest path from v to w is a shortest path whose intermediate vertices all have a label $\leq k$

5 - path



4 path

4 path