PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, May 13, 2015

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Assume that $f\colon [0,1] \to [0,\infty)$ is a Lebesgue measurable function such that $f(x) > 0$ for a.e x. Show that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every Lebesgue measurable set E with Lebesgue measure $\lambda(E) \geq \varepsilon$ we have $\int_E f d\lambda \geq \delta$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space with a finite measure μ and let f be a measurable function such that $\int_X |f|^q \, d\mu < \infty$ for some $0 < q < \infty$. Show that
$$\lim_{p \to 0^+} \int_X |f|^p \; d\mu = \mu(\{x \in X : \; f(x) \neq 0\}).$$

3. Let f and g be integrable functions on $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, respectively, and let $F(x,y) = f(x)g(y)$. Show that F is measurable and integrable on $X \times Y$ and that
$$\int F \, d(\mu \times \nu) = \int f \, d\mu \int g \, d\nu.$$

4. Let $f\colon [0,\infty) \to \mathbb{R}$ be Lebesgue integrable. Show that if f is uniformly continuous, then $\lim_{x\to\infty} f(x) = 0$.