Analysis Qualifying Examination

Part A - August 19, 2010

Solve at least two of the following problems.

Write clearly and justify your answers.

1. Let $f : I\!R \to I\!R$ be an absolutely continuous function. Prove that f maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.

2. Consider a sequence of functions $f_n : I\!R \mapsto I\!R$.

(i) If $\lim_{n\to\infty} \|f_n - f\|_{\mathbf{L}^1(I\!R)} = 0$, prove that f_n converges in measure to f.

(ii) Give an example of a sequence of functions $f_n \in \mathbf{L}^1(I\!R)$ which converges to a function f in measure, but not in $\mathbf{L}^1$.

3. Construct a sequence of measurable functions $f_n : [0,1] \mapsto [0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \;=\; 0$$

but, for each $y \in [0,1]$, the sequence $f_n(y)$ has no limit.

4. Let $(f_n)_{n\geq 1}$ be a sequence of absolutely continuous functions such that

$$f_n(0) = \cos n\,, \qquad \int_0^1 |f_n'(t)|^3\, dt \;\leq\; 1 \qquad \text{for every } \; n \geq 1\,.$$

(i) Prove that each f_n is Hölder continuous, namely

$$|f_n(b) - f_n(a)| \;\leq\; C\,|b-a|^{2/3} \qquad \text{for all} \quad 0 \leq a < b \leq 1\,,$$

for some constant C independent of n.

(ii) Using (i), prove that there exists a subsequence that converges to a continuous function f, uniformly on $[0,1]$.