PhD QUALIFYING EXAMINATION IN ANALYSIS
Paper A, August 2011

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let K be a subset of $\mathbb{R}^n$. Assume that every continuous real-valued function on K is bounded. Prove that K is compact.

2. Suppose that $(X, \mathfrak{M}, \mu)$ is a finite measure space and that $\{E_k\}_{k=1,2,\ldots}$ is a sequence of measurable subsets of X such that $\mu(E_k) \geq \frac{1}{2}$ for each k. Let E be the subset of X consisting of points x that belong to E_k for infinitely many values of k. Show that E is measurable and that $\mu(E) \geq \frac{1}{2}$.

3. A function $f\colon [a,b] \to \mathbb{R}$ is called *Lipschitz* if there is a constant $C > 0$ such that
$$|f(x) - f(y)| \leq C|x - y|$$
for all $x, y \in [a,b]$. Show that $f\colon [a,b] \to \mathbb{R}$ is Lipschitz if and only if it is absolutely continuous and there is a constant $K > 0$ such that $|f'(x)| \leq K$ for all $x \in [a,b]$ for which the derivative is defined.

4. Find the limit of the sequence of integrals
$$I_n := \int_0^{n^2} \frac{x^n}{1 + x^{n+2}} \sin\left(\frac{\pi x}{n}\right) dx$$
as $n \to \infty$. Justify each step in your calculation.