PhD QUALIFYING EXAMINATION IN ANALYSIS
Paper A, May 2011

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Suppose that (X, d) is a compact metric space and that $f\colon X \to X$ is a continuous function satisfying $d(x, y) \leq d(f(x), f(y))$ for all $x, y \in X$. Show that $f(X) = X$.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ of finite measure and let (f_k) be a sequence of measurable functions $E \to \mathbb{R}$. Suppose that for every $x \in E$ there exists a constant M_x such that $|f_k(x)| \leq M_x$ for all $k \geq 1$. Prove that, given $\epsilon > 0$, there exist a closed set $F \subseteq E$ and a constant M such that

$$\lambda(E \setminus F) < \epsilon \quad \text{and} \quad |f_k(x)| \leq M \text{ for all } x \in F \text{ and all } k \geq 1.$$

(Here, λ denotes Lebesgue measure.)

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function. Suppose that

$$\lim_{t \to 0^+} \int_{\mathbb{R}} \frac{|f(x+t) - f(x)|}{t}\, dx = 0.$$

Show that f is a constant function.

4. For $f, g \in L^1(\mathbb{R}^n, \mathbb{R})$ consider the convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy, \quad x \in \mathbb{R}^n.$$

Prove that if g is bounded, then $f * g$ is continuous on $\mathbb{R}^n$.