Analysis Qualifying Examination
Part A, Fall 2008

Instructions. To pass the exam you must correctly solve two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity. Write your solutions carefully!

Problem 1. Prove that if h is a one-to-one map from a metric space X onto a metric space Y, then h is a homeomorphism if and only if $h(\overline{A}) = \overline{h(A)}$ for each $A \subset X$.

Problem 2. Let f be a real-valued Lebesgue measurable function on $\mathbb{R}$ and g a continuous real-valued function on $\mathbb{R}$. Are the functions $f \circ g$ and $g \circ f$ necessarily Lebesgue measurable?

Problem 3. Prove that

$$\int \left| |f|^p - |g|^p \right| d\mu \leq \int |f - g|^p \, d\mu$$

for any $f, g \in L_p([0, 1], \mu)$ and $0 < p < 1$.

Problem 4. Let f be a non-negative function defined on a measurable subset E of the unit interval $[0, 1]$. Show that f is measurable if the region

$$\{(x, y) : x \in E,\ f(x) \geq y\}$$

is a measurable subset of $\mathbb{R}^2$.