# PhD QUALIFYING EXAMINATION IN ANALYSIS
## Paper A, May 7th, 2012

**Instructions:** To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let $E$ be a noncompact subset of $\mathbb{R}^n$. Show that there is a bounded continuous real-valued function on $E$ that does not assume its maximum on $E$.

2. Let $(X, \mu)$ be a measure space. Show that if $f \in L^p(X, \mu)$ with $1 \leq p < \infty$, then
$$\lim_{t \to \infty} t^p \mu(\{x \in X : |f(x)| > t\}) = 0.$$

3. Let $(X, \mu)$ be a measure space of finite measure and let $f$ be a nonnegative integrable function on $X$ such that for every integer $n = 1, 2, \ldots,$
$$\int_X f(x)^n \, dx = \int_X f(x) \, dx.$$
Show that $f$ must be almost everywhere equal to the characteristic function $\chi_E$ of some measurable set $E$.

4. Show that the graph of a continuous function $f \colon [0, 1] \to \mathbb{R}$ has measure zero with respect to the two-dimensional Lebesgue measure.