PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, May 8th, 2013

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let μ and ν be finite measures on a measure space $(X, \mathcal{A})$. Show that there is a nonnegative measurable function f on X such that for all $E \in \mathcal{A}$,
$$\int_E (1-f)\, d\mu = \int_E f\, d\nu.$$

2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence of real-valued measurable functions on X. Suppose that there is a constant M such that $|f_n(x)| \leq M$ for all n and $x \in X$. Suppose also that the sequence $f_n(x)$ converges almost everywhere to a function f. Show that f is measurable and for each function $g \in L^2(X, \mu)$
$$\int f g d\mu = \lim_{n\to\infty} \int f_n g d\mu.$$

3. Construct an example of a continuous function $f\colon [0,1] \to [0,1]$, which has the following property: there exists a Lebesgue measurable set $E \subset [0,1]$ such that for every $n > 0$ the set
$$F_n = \left\{ y \in \left[\frac{1}{n+1}, \frac{1}{n}\right] : y = f(x) \text{ for some } x \in E \right\}$$
is not Lebesgue measurable.

4. Let f be a non-negative function defined on a Lebesgue measurable subset $E \subseteq [0,1]$. Show that f is Lebesgue measurable if the region
$$\{(x,y) : x \in E,\ f(x) \geq y\}$$
is a Lebesgue measurable subset of $\mathbb{R}^2$.