# PhD QUALIFYING EXAMINATION IN ANALYSIS
## Part A, May 9, 2016

**Instructions:** To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let $\lambda$ be the Lebesgue measure on $[0,1]$ and $\mathcal{A}$ be the $\sigma$-algebra of Lebesgue measurable sets in $[0,1]$. Suppose that a set $N \subset [0,1]$ is not in $\mathcal{A}$. Prove that $\lambda$ can be extended to a measure on a $\sigma$-algebra $\mathcal{B}$ that contains $\mathcal{A}$ and $N$.

2. Let $\mu$ be a finite Borel measure on a metric space $(X,d)$. Prove that for each $x \in X$ and for each $\varepsilon > 0$ there exists a number $0 < r < \varepsilon$ such that

$$\lim_{\rho \to r} \mu(B_x(\rho)) = \mu(B_x(r)), \tag{1}$$

   where $B_x(r)$ denotes the open ball in $X$ of radius $r$ centered at $x$.

3. Let $f$ be a function which is in $L^p = L^p((0,1),\lambda)$ for each $p \geq 1$, where $\lambda$ is the Lebesgue measure on the interval $(0,1)$. Give an example of such a function which is not in $L^\infty = L^\infty((0,1),\lambda)$. Prove that if there is a constant $C$ so that $\|f\|_p \leq C$ for all $p \geq 1$, then $f \in L^\infty$.

4. Let $f$ be a function in $L^1 = L^1(\mathbb{R},\lambda)$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}$. Let $E$ be a measurable set with $\lambda(E) < \infty$. Prove that the following function is continuous on $\mathbb{R}$

$$g(x) = \int_E f(x-t)\,d\lambda(t).$$

   You may use without proof that for each $\varepsilon > 0$ there exists a continuous function $\phi$ on $\mathbb{R}$ with compact support such that $\|f - \phi\|_1 < \varepsilon$.