PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, August 19, 2014

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Note: In the following, λ denotes the Lebesgue measure on $\mathbb{R}^m$.

1. Assume that E is Borel subset of $\mathbb{R}^2$. Show that for every $y \in \mathbb{R}$, the slice $E^y = \{x \in \mathbb{R} \,|\, (x, y) \in E\}$ is a Borel subset of $\mathbb{R}$.

2. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous and increasing function. Show that f is absolutely continuous if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\lambda^*(f(E)) < \varepsilon$$
for Lebesgue measurable subsets $E \subset [a, b]$ satisfying $\lambda(E) < \delta$. (Here λ^* is the outer Lebesgue measure.)

3. Assume that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of Lebesgue measurable functions such that $\int_{\mathbb{R}} |f_n| \leq 1/n^2$. Show that $f_n \to 0$ λ-a.e.

4. Let $0 < \alpha < d$ and $K(x) = \dfrac{1}{|x|^\alpha}$ for $x \in \mathbb{R}^d$. For nonnegative Lebesgue integrable function $f : \mathbb{R}^d \to \mathbb{R}$, define
$$g(x) = \int_{\mathbb{R}^d} f(x - y)K(y)\; d\lambda(y).$$
Show that $g(x)$ is finite for λ-a.e. $x \in \mathbb{R}^d$.