PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, August 21, 2012

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let $\{f_n\}$ be a non-increasing sequence of non-negative integrable functions on a measure space $(X, \mathcal{A}, \mu)$. Assume that

$$\lim_{n\to\infty} \int f_n \, d\mu = 0.$$

Show that $\lim_{n\to\infty} f_n(x) = 0$ a.e. Does this statement remain true if the assumption that "the sequence $\{f_n\}$ is non-increasing" is dropped?

2. Let $C([0,1])$ be the metric space of all continuous functions on the unit interval $[0,1]$ equipped with the usual supremum metric. Fix $k > 0$ and let E be the set of all polynomial functions on $[0,1]$ of degree $\leq k$. Prove that E is a closed and nowhere dense subset of $C([0,1])$.

3. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that $V(f) \leq 3$, where $V(f)$ is the total variation of the function f.

4. Prove that if f is a Lipschitz function on an interval $[a, b]$ then f is absolutely continuous and $f' \in L^\infty([a, b])$.