PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, December 12, 2015

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Note: In the following, λ denotes the Lebesgue measure on $\mathbb{R}^m$.

1. Denote by λ^* the Lebesgue outer measure on $\mathbb{R}^m$. Assume that $E_1, E_2 \subset \mathbb{R}^m$ satisfy $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2$ is Lebesgue measurable with $\lambda(E_1 \cup E_2) < \infty$, and
$$\lambda(E_1 \cup E_2) = \lambda^*(E_1) + \lambda^*(E_2).$$
Prove that E_1, E_2 are Lebesgue measurable.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(f_n) \subset L^1(X, \mathcal{A}, \mu)$, and $f \in L^1(X, \mathcal{A}, \mu)$. Assume that
$$\lim_{n\to\infty} \|f_n - f\|_1 = 0.$$
Prove that there exists a subsequence (f_{n_k}) such that $f_{n_k} \to f$ μ-a.e.

3. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $1 < p < \infty$. Assume that the sequence $(f_k) \subset L^p(X, \mathcal{A}, \mu)$ satisfies
$$\|f_k\|_p \le C \quad \text{and} \quad \lim_{k\to\infty} f_k(x) = 0 \quad \text{for all } x \in X.$$
Prove that $f_k \to 0$ in $L^1(X, \mathcal{A}, \mu)$.

4. Let $f\colon \mathbb{R} \to (0, \infty)$ be a Borel measurable function and let E be a Borel measurable subset of $\mathbb{R}$ such that $\lambda(E) > 0$. Define
$$F(t) = \int_E f(t + x)\, d\lambda(x), \quad t \in \mathbb{R}.$$
Prove that F is Borel measurable. Prove further that if $F \in L^1(\mathbb{R})$, then $f \in L^1(\mathbb{R})$ and $\lambda(E) < \infty$.