Analysis Qualifying Examination
Section A, Spring 2009

Instructions. To pass the exam you must correctly solve two of the following four problems. ***Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate all your effort on two solutions.*** Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly!

1. Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[0, 1]$ that converges to a function f pointwise. Assume that

$$\sup_n \sup_{x\in[0,1]} |f'(x)| < \infty.$$

Show that the sequence $\{f_n\}$ converges *uniformly* to f. Show also that this may not be true if the displayed assumption is dropped.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and assume that $\mu(X) < \infty$. Let g be a nonnegative measurable function on X. For each measurable set $E \in \mathcal{A}$ let $\nu(E) = \int_E g(x)\, d\mu(x)$. Show that ν is a measure on X, and that if f is a nonnegative measurable function on X, then

$$\int_X f(x)\, d\nu(x) = \int_X f(x)g(x)\,d\mu(x).$$

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of functions in $L^p(\mu)$, where $1 \le p < \infty$. Assume the sequence converges in $L^p(\mu)$ to a function f. Show that if $\{g_n\}$ is a sequence of measurable functions, and if there is a constant M such that $|g_n(x)| \le M$ for all n and all x, and if $\lim_{n\to\infty} g_n(x) = g(x)$ for almost every $x \in X$, then the sequence $\{g_n f_n\}$ converges to gf in $L^p(\mu)$.

4. Let $S = [0, 1] \times [0, 1]$ and define $f\colon S \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Is f integrable with respect to area in S?