Analysis A - Lebesgue Measure Theory

Sample Exam Problems

1. Let $S_1, S_2, \dots$ be closed sets such that $\cup_j S_j = \mathbb{R}$. Prove that at least one of the sets S_j has nonempty interior.

2. Let A be any countable subset of the real numbers. Construct a monotonically increasing function whose set of points of discontinuity is precisely the set A.

3. Let E and F be disjoint closed subsets of $\mathbb{R}^n$. Prove that there is a continuous function f in $\mathbb{R}^n$ such that $\{x : f(x) = 0\} = E$ and $\{x : f(x) = 1\} = F$.

4. Let $\{p_j(x)\}$ be a sequence of polynomial functions on $I\!R$, each of degree not exceeding a number $k \geq 1$. Assume that this sequence converges pointwise to a limit function f. Prove that f is a polynomial of degree not exceeding k.

5. Consider the functions

$$f_n(x) = \sin nx, \quad n = 1, 2, 3, \dots, \quad -\pi \leq x \leq \pi$$

as points of L^2. Prove that the set of these points is closed and bounded, but is not compact.

6. Let X be an uncountable set and $\mathcal{A}$ the collection of all sets $E \subset X$ such that either E or its compliment E^c is at most countable, and define $\mu(E) = 1$ in the second. Prove that $\mathcal{A}$ is a σ-algebra in X and that μ is a measure on $\mathcal{A}$. Describe the corresponding measurable functions and their integrals.

7. Suppose $f_n : X \to [0, \infty]$ is a sequence of measurable functions for $n = 1, 2, 3, \dots$ satisfying $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$ and $f_1 \in L^1(\mu)$. Prove that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

8. If $0 < \varepsilon < 1$, construct an open set $E \subset [0, 1]$, which is dense in $[0, 1]$ and such that $m(E) = \varepsilon$ where m is the Lebesgue measure.

9. Construct a sequence of continuous function f_n on $[0,1]$ such that $0 \leq f_n(x) \leq 1$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0,$$

but, for every $x \in [0,1]$, the sequence of pointwise values $\{f_n(x)\}$ does not converge.

10. Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^x e^{-xt}\, dt, \qquad x > 0$$

to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

11. Using a fixed point argument, show that the equation

$$u(x) \; = \; x + \frac{1}{2}\int_0^1 \sin(x+y)\, u(y)\, dy \qquad\qquad \forall x \in [0,1]$$

has a unique solution $u \in \mathcal{C}([0,1])$.

12. Let $(f_n)_{n\geq 1}$ be a sequence of functions in $\mathbf{L}^p(I\!R)$, $1 < p < \infty$, such that $f_n(x) \to f(x)$ for a.e. $x \in I\!R$. Assume that $\|f_n\|_{\mathbf{L}^p} \leq M$, for some constant M and all $n \geq 1$.

Prove that

$$\int fg\, dx \; = \; \lim_{n\to\infty} \int f_n\, g\, dx$$

for every $g \in \mathbf{L}^q(I\!R)$, with $\frac{1}{p} + \frac{1}{q} = 1$.

13. Let $f : I\!R \mapsto I\!R$ be an absolutely continuous function, with bounded variation. Consider the sequence of divided differences $g_n(x) \; = \; n\left[f\left(x+\frac{1}{n}\right) - f\left(x - \frac{1}{n}\right)\right]$. Prove that there exists a function $g \in \mathbf{L}^1(I\!R^n)$ such that $g_n(x) \to g(x)$ for a.e. $x \in I\!R$, and moreover $\|g_n - g\|_{\mathbf{L}^1} \to 0$.
Identify the function g.

14. Consider a sequence of functions $f_n \in \mathbf{L}^1(I\!R)$ with $\|f_n\|_{\mathbf{L}^1} \leq C$ for every $n \geq 1$. Define

$$f(x) \; \doteq \; \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Lebesgue measurable and $\|f\|_{\mathbf{L}^1} \leq C$.

15. If $g(x) = f(x+c)$, show that the Fourier transforms of f and g are related as $\hat{g}(y) = \hat{f}(y)e^{icy}$.

16. If the Fourier transform of f is identically zero, show that $f = 0$ almost everywhere.

17. Construct a continuous and monotonic function f on $[0,1]$ for which $f(0) = 0$, $f(1) = 1$ and $f'(x) = 0$ for almost every $x \in [0,1]$.