PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, December 10, 2016

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Notation: In the problems below, λ denotes the Lebesgue measure.

1. Let f be a non-negative function in $L^1 = L^1([0,1], \lambda)$. Prove that for each $\varepsilon > 0$ there exists a finite linear combination of characteristic functions of intervals, $\phi = \sum c_i \chi_{[a_i, b_i)}$, such that $\|f - \phi\|_1 < \varepsilon$.

   You may use the standard facts from the construction of the Lebesgue measure and integral as well as convergence theorems, but you may not use without a proof any density results in L^1 for other classes of functions, such as density of continuous functions.

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative functions in $L^1 = L^1(\mathbb{R}, \lambda)$ such that $\|f_n\|_1 \leq 1$ for each n. Consider the function f defined as $f(x) = \lim_{n\to\infty} f_n(x)$ if the limit exists and $f(x) = 0$ otherwise. Prove that f is Lebesgue measurable and $\|f\|_1 \leq 1$.

3. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a positive, Lebesgue measurable function. Define the set

$$A := \{(x,y) \in \mathbb{R}^2 \mid 0 < y < f(x)\}.$$

   Show that A is Lebesgue measurable and that

$$\int_{\mathbb{R}} f(x)\, dx = m_2(A),$$

   where m_2 denotes the two-dimensional Lebesgue measure.

4. Prove that the function

$$f(x) = \begin{cases} \sqrt{x} \sin\left(\frac{\pi}{2\sqrt{x}}\right), & 0 < x \leq 1, \\ 0, & x = 0, \end{cases}$$

   is not of bounded variation on $[0,1]$. Show that f is not Lipschitz continuous.