PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, August 22, 2013

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let X be a metric space and $\{f_n\}$ a sequence of continuous functions from X to a metric space Y. Assume that f_n converge to a function f uniformly on each compact subset $K \subset X$. Show that f is continuous.

2. Let $g(x,y) = x^{-3/2} \cos\left(\frac{\pi y}{2x}\right)$.

   (a) Prove that
   $$\int_0^1 \int_0^x |g(x,y)| dy dx < \infty.$$

   (b) Evaluate the integral
   $$\int_0^1 \int_y^1 g(x,y)\, dx dy.$$

   Justify your reasoning carefully.

3. Let $\{f_n\}$ be a sequence of integrable functions on $[0,1]$ (i.e., function in $L^1([0,1],m)$, where m is the Lebesgue measure on $[0,1]$). Suppose that f_n converges almost everywhere to a function $f \in L^1([0,1],m)$. Prove that $\|f_n - f\|_1 \to 0$ if and only if $\|f_n\|_1 \to \|f\|_1$.

4. Let μ and ν be finite positive measures on a measurable space $(X, \mathcal{A})$, which are equivalent (i.e., μ is absolutely continuous with respect to ν and ν is absolutely continuous with respect to μ). Let $\lambda = \mu + \nu$. Show that the Radon-Nikodym derivative $d\nu/d\lambda$ satisfies
   $$0 < \frac{d\nu}{d\lambda} < 1$$
   almost everywhere.