PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, May 14, 2014

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Note: In the following, λ denotes the Lebesgue measure on $\mathbb{R}^m$.

1. Let X be a compact metric space and let μ be a finite measure on the σ-algebra of Borel measurable subsets of X. Assume that $\mu(\{x\}) = 0$ for every $x \in X$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\mu(B) < \varepsilon$$
for every Borel subset B of X satisfying diam $(B) < \delta$.

2. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Show that the following conditions are equivalent:
   (a) For every $N \subset [a,b]$ satisfying $\lambda(N) = 0$, $\lambda(f(N)) = 0$.
   (b) For Lebesgue measurable subset $E \subset [a,b]$, $f(E)$ is Lebesgue measurable.

3. Let (f_n) be a sequence of Lebesgue integrable functions $f_n\colon \mathbb{R}^k \to [-\infty, \infty]$ such that $\|f_n\|_1 \leq 1$ for all $n \geq 1$ and $f_n \to f$ λ-a.e. in $\mathbb{R}^k$. Show that $f \in L^1(\mathbb{R}^k)$, $\|f\|_1 \leq 1$, and
$$\|f\|_1 = \lim_{n\to\infty} \big(\|f_n\|_1 - \|f_n - f\|_1\big).$$

4. Assume that f is a Lebesgue integrable function on $[0,a]$ and
$$g(x) = \int_x^a \frac{f(t)}{t}\, dt \quad \text{for } 0 < x \leq a.$$
Show that g is Lebesgue integrable on $[0,a]$ and
$$\int_0^a g(x)\, dx = \int_0^a f(t)\, dt.$$