August 2009 Exam
Analysis Qualifying Course A – Real Analysis

Solve at least two of the following problems.
Write clearly and justify your answers.

1) Let (X, μ) be a measure space and let $\{f_n\}$ be a sequence of measurable functions $X \to \mathbb{R}$. Say what it means for $\{f_n\}$ to *converge in measure* to a measurable function f. If $g\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous and f_n converges to f in measure, show that the compositions $g \circ f_n$ converge to $g \circ f$ in measure.

2) Consider $X = [0,1]$ equipped with its Borel σ-algebra $\mathcal{B}_X$. Let m be the Lebesgue measure and μ the counting measure on X. Show that $m \ll \mu$, but there does not exist any μ-integrable function f such that $dm = f\,d\mu$.

3) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz continuous* if $\exists M > 0$ constant such that

$$|f(x) - f(y)| \leq M\,|x - y|, \qquad \forall x,\, y \in \mathbb{R}.$$

Show that f is Lipschitz continuous if and only if f is absolutely continuous and $|f'(x)| \leq M$ a.e. x.

4) Let $u, w \in L^2(\mathbb{R})$. Show that the convolution

$$u * w(x) := \int_{\mathbb{R}} u(x - y)w(y)\,dy,$$

is a continuous function on $\mathbb{R}$.