Analysis Qualifying Examination

Part A - May 12, 2010

Solve at least two of the following problems.

Write clearly and justify your answers.

1. Using a fixed point argument, show that the equation

$$u(x) \;=\; x + \frac{1}{2}\int_0^1 \sin(x+y)\, u(y)\, dy \qquad\qquad \text{for all} \quad x \in [0,1]$$

has a unique solution $u \in C([0,1])$.

2. Let $(f_n)_{n\geq 1}$ be a sequence of functions in $\mathbf{L}^p(I\!R)$, $1 < p < \infty$, such that $f_n(x) \to f(x)$ for a.e. $x \in I\!R$. Assume that $\|f_n\|_{\mathbf{L}^p} \leq M$, for some constant M and all $n \geq 1$.

Prove that

$$\int fg\, dx \;=\; \lim_{n\to\infty} \int f_n\, g\, dx$$

for every $g \in \mathbf{L}^q(I\!R)$, with $\frac{1}{p} + \frac{1}{q} = 1$.

3. Let $f : I\!R \mapsto I\!R$ be an absolutely continuous function. Consider the sequence of divided differences $g_n(x) \;=\; n\left[f\left(x+\frac{1}{n}\right) - f\left(x-\frac{1}{n}\right)\right]$. Prove that there exists a function $g \in \mathbf{L}^1(I\!R^n)$ such that $g_n(x) \to g(x)$ for a.e. $x \in I\!R$, and moreover $\|g_n - g\|_{\mathbf{L}^1} \to 0$.
Identify the function g.

4. Consider a sequence of functions $f_n \in \mathbf{L}^1(I\!R)$ with $\|f_n\|_{\mathbf{L}^1} \leq C$ for every $n \geq 1$. Define

$$f(x) \;\doteq\; \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Lebesgue measurable and $\|f\|_{\mathbf{L}^1} \leq C$.