COMPUTING EIGENVALUES OF THE LAPLACIAN ON ROUGH DOMAINS

FRANK RÖSLER AND ALEXEI STEPANENKO

Abstract. We prove that a simple universal algorithm for the computation of the eigenvalues of the Dirichlet Laplacian on a domain in $\mathbb{R}^2$ converges, provided the domain satisfies a collection of mild, topological hypotheses and had boundary with upper box-counting dimension < 2. Conversely, we show that there does not exist a universal algorithm of the same type which converges for an arbitrary bounded domain. Along the way, we prove a Mosco convergence result for rough domains and an explicit, uniform Poincaré-type inequality.

1. Introduction

The eigenvalues of the Dirichlet Laplacian on a domain, which we shall refer to more simply as the eigenvalues that domain, play a fundamental role in many areas of physics, engineering and mathematics. It is known that it is possible to numerically compute the eigenvalues of the Dirichlet Laplacian on a domain, provided the boundary of that domain is sufficiently regular [9, 10, 27]. In this paper, we investigate the question:

How bad can boundary regularity get before numerical computation of eigenvalues is no longer possible?

The question is formulated in a rigorous framework using the theory of *Solvability Complexity Indices (SCI)* [17, 8, 25], particularly the notion of an *arithmetic algorithm*. Our results address this question from two angles.

On one hand, we explicitly construct a simple univeral algorithm and prove it's convergence for the eigenvalue problem on any domain in $\mathbb{R}^2$ which satisfies a collection of mild, purely topological hypotheses and whose boundary has upper box-counting dimension < 2. The set of domains satisfying these hypotheses include many domains exhibiting fractal boundaries, such as the Koch snowflake, and domains with cusp singularities. The novelty of the suggested algorithm is that the only input it requires is the information of whether any given point lies in the domain. The reader is referred to [2, 6, 14] for examples of numerical algorithms for specific rough domain and to [11, 19, 1] for examples of algorithms for rough domains for which one has a-priori access to a sequence of pre-fractal approximations for the boundary. See also [15, 22, 21].

Converse to the existence of this algorithm, we show that there does not exist a universal algorithm of the same type capable of computing the eigenvalues of an arbitrary bounded domain, indicating that sufficiently pathological boundary regularity eventually leads to failure of computation.

The key to the proof of convergence for the simple universal algorithm (cf. Theorem 1.3) is a proof of Mosco convergence for a certain type of approximation of an open set, which we call *pixelation* (cf. Theorem 1.7). A highlight of this Mosco convergence result is the absence of the standard stability assumption [5] for the H_0^1 Sobolev space of the domain. Underlying the proof of this result is an explicit, uniform Poincaré-type inequality which is proved using a novel and elementary method. These technical results are stated and discussed in Sections 1.3 and 1.4.

1.1. Computational problems and arithmetic algorithms.

Let us first describe the necessary elements of SCI theory needed to formulate our problem.

Definition 1.1 (Computational problem). A *computational problem* is a quadruple $(\Omega, \Lambda, \Xi, \mathcal{M})$, where

(A) Ω is a set, called the *primary set*,
(B) Λ is a set of complex-valued functions on Ω, called the *evaluation set*,
(C) $\mathcal{M}$ is a metric space,
(D) $\Xi : \Omega \to \mathcal{M}$ is a map, called the *problem function*.

Intuitively, elements of the primary set Ω are the objects giving rise to the computational problems, the evaluation set Λ represents the information available to an algorithm, the metric space $\mathcal{M}$ is the output of an algorithm and the problem function Ξ represents the true solutions of the computational problems.

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Now we are in position to define the notion of an arithmetic algorithm. The definition here differs slightly from the one in [8, Definition 6.3] as it is convenient for us to explicitly indicate the evaluation set Λ.

Definition 1.2 (Arithmetic algorithm)**.** Let $(\Omega, \Lambda, \mathcal{M}, \Xi)$ be a computational problem. An *arithmetic algorithm with input* Λ is a map $\Gamma : \Omega \to \mathcal{M}$ such that for each $T \in \Omega$ there exists a finite subset $\Lambda_\Gamma(T) \subset \Lambda$ such that

(i) the action of Γ on T depends only on $\{f(T)\}_{f\in\Lambda_\Gamma(T)}$,
(ii) (consistency) for every $S \in \Omega$ with $f(T) = f(S)$ for all $f \in \Lambda_\Gamma(T)$ one has $\Lambda_\Gamma(S) = \Lambda_\Gamma(T)$,
(iii) the action of Γ on T consists of performing only finitely many arithmetic operations on $\{f(T)\}_{f\in\Lambda_\Gamma(T)}$.

Arithmetic algorithms give a notion of computability - a computational problem $(\Omega, \Lambda, \mathcal{M}, \Xi)$ is considered computable if

$$\exists \text{ arithmetic algorithms } \Gamma_n : \Omega \to \mathcal{M},\ n \in \mathbb{N},\ \text{ with input } \Lambda \text{ s.t. } \lim_{n\to\infty} d_{\mathcal{M}}(\Gamma_n(T), \Xi(T)) = 0\, \forall\, T \in \Omega$$

where $\mathrm{d}_{\mathcal{M}}$ denote the metric for the metric space $\mathcal{M}$. The above condition is equivalent to $\mathrm{SCI}(\Omega, \Lambda, \mathcal{M}, \Xi) = 1$ (cf. [8, Definition 6.9]).

1.2. **Computational eigenvalue problem for the Laplacian.** For simplicity, we only consider domains in $\mathbb{R}^2$. The sole input for the arithmetic algorithms that we consider is the information of whether a finite set of points are or are not in the domain. An arithmetic algorithm produces its output by performing only a finite number of arithmetic operations on this input. The computational eigenvalue problem we consider is described by the following elements:

(A) The primary set Ω is a subset of the set of domains
$$\Omega_0 := \big\{\mathcal{O} \subset \mathbb{R}^2 : \mathcal{O} \text{ open, bounded and connected}\big\}.$$
(B) The evaluation set is
$$\Lambda_0 := \big\{\mathcal{O} \mapsto \chi_{\mathcal{O}}(x) : x \in \mathbb{R}^2\big\}$$
where χ is the characteristic function.
(C) The metric space is $\mathcal{M} := (\mathrm{cl}(\mathbb{C}), \mathrm{d}_{\mathrm{AW}})$, where $\mathrm{cl}(\mathbb{C})$ denotes the set of closed subsets of $\mathbb{C}$ and d_{AW} denotes the Attouch-Wets metric. Recall that the Attouch-Wets metric is defined by
$$\mathrm{d}_{\mathrm{AW}}(A, B) := \sum_{j=1}^{\infty} 2^{-j} \min\left\{1, \sup_{|x|\le j} |\mathrm{dist}(x, A) - \mathrm{dist}(x, B)|\right\}$$
for any subsets $A, B \subseteq \mathbb{C}$ (cf. [7, Ch. 3]). Note that for bounded sets $A, B \subset \mathbb{C}$, d_{AW} is equivalent to the Haussdorff distance d_{H}.
(D) The problem function $\Xi_\sigma : \Omega \to \mathcal{M}$ is defined by $\Xi_\sigma(\mathcal{O}) := \sigma(\mathcal{O})$, where $\sigma(\mathcal{O})$ denotes the spectrum of the Laplacian $-\Delta$ on $L^2(\mathcal{O})$ endowed with Dirichlet boundary conditions on $\partial\mathcal{O}$.

Note that the Dirichlet Laplacian on any bounded open set is self-adjoint with compact resolvent so for any $\mathcal{O} \in \Omega_0$, $\sigma(\mathcal{O})$ is equal to the set of eigenvalues of $\mathcal{O}$ - a discrete subset of $\mathbb{R}$ (see e.g. [13, Th. VI.1.4]). Note also that the above description of the computational eigenvalue problem is a slight idealisation in the sense that a real world computer can only perform floating point operations (as opposed to real arithmetic operations) and it is not necessarily always the case that one has complete access to the characteristic function of a domain (consider the interior of the Mandelbrot set for instance). Allowing these idealisations allows us to present the main ideas of our analysis more clearly.

Algorithm for eigenvalues. Our main result is an explicit construction of a sequence of arithmetic algorithms, describing a simple numerical method for the computation of eigenvalues of the Dirichlet Laplacian on a large class of bounded domains. The next theorem follows immediately from Proposition 4.4.

Theorem 1.3. *Let*
$$\Omega_1 := \big\{\mathcal{O} \in \Omega_0 : \mathcal{O} = \mathrm{int}(\overline{\mathcal{O}}),\ \mathrm{vol}(\partial\mathcal{O}) = 0,\ \partial\mathcal{O} \text{ path-connected and } \mathrm{int}(\mathcal{O}^c) \text{ connected}\big\}.$$
There exists a sequence of arithmetic algorithms $\Gamma_n : \Omega_1 \to \mathrm{cl}(\mathbb{C})$ *with input* Λ_0 *such that*
$$\mathrm{d}_{\mathrm{AW}}(\Gamma_n(\mathcal{O}), \sigma(\mathcal{O})) \to 0 \quad \text{as} \quad n \to \infty \qquad \text{for all} \qquad \mathcal{O} \in \Omega_1.$$

The condition that $\mathcal{O} = \mathrm{int}(\overline{\mathcal{O}})$ is commonly referred to as the condition that $\mathcal{O}$ is (topologically) *regular* (cf. Definition 3.3). A simple sufficient conditon for the hypothesis $\mathrm{vol}(\partial\mathcal{O}) = 0$ is $\dim_b(\partial\mathcal{O}) < 2$ where $\dim_b$ denotes the upper box counting dimension.

The basic idea behind the construction of the algorithm is to combine pixelation approximations of the domain with computable error bounds for the finite-element method. The computable error bounds for the finite element method that we employ are those of Liu and Oishi [20]. The pixelation approximations are defined as follows.

Definition 1.4. For any open set $\mathcal{O} \subseteq \mathbb{R}^d$, *pixelated domains* for $\mathcal{O}$ are the open sets $\mathcal{O}_n \subseteq \mathbb{R}^d$, $n \in \mathbb{N}$, defined by

$$\mathcal{O}_n := \operatorname{int}\Big(\bigcup_{j \in L_n} (j + [-\tfrac{1}{2n}, \tfrac{1}{2n}]^d) \Big),$$

where

$$L_n := \big\{ j \in \mathbb{Z}_n^d : j \in \mathcal{O} \big\} \quad \text{and} \quad \mathbb{Z}_n^2 := (n^{-1}\mathbb{Z})^2.$$

The statement that the eigenvalues of the pixelated domains $\mathcal{O}_n$ converge to $\mathcal{O}$ as $n \to \infty$ provided $\mathcal{O}$ satisfies the hypotheses of Theorem 1.3 is an immediate corollary our main technical result on the Mosco convergence of pixelated domains (cf. Theorem 1.7 below).

The algorithm of Theorem 1.3 can be summarised as:

(1) Approximate $\mathcal{O}$ by a corresponding pixelated domain $\mathcal{O}_n$.
(2) Approximate the eigenvalues of $\mathcal{O}_n$ to an error $1/n$ in the Attouch-Wets metric, using computable error bounds for the finite element method on a uniform triangulation of $\mathcal{O}_n$ and the Jacobi method combined a-posteriori error bounds for the associated matrix pencils.

Counter-example. Converse to Theorem 1.3, we prove that there is no hope of constructing an arithmetic algorithm which is capable of computing the eigenvalues an arbitrary bounded domain. The next proposition follows immediatly from Proposition 4.5.

Proposition 1.5. *There does not exist a sequence of arithmetic algorithms* $\Gamma_n : \Omega_0 \to \mathrm{cl}(\mathbb{C})$ *with input* Λ_0 *which satisfy*

$$\mathrm{d_{AW}}\left(\Gamma_n(\mathcal{O}), \sigma(\mathcal{O})\right) \to 0 \quad \textit{as} \quad n \to \infty \qquad \textit{for all} \qquad \mathcal{O} \in \Omega_0.$$

The counter-example comprising the proof is based on the geometric observation that for any $\epsilon > 0$ and any countable set of points $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^2$ (even a set dense in $\mathbb{R}^2$) there exists a connected, open set $U \subset \mathbb{R}^2$ with $x_n \in U$ for all n and $\mathrm{vol}(U) < \epsilon$.

1.3. **Mosco Convergence.** The key notion that is employed in the proof of Theorem 1.3 is Mosco convergence. Recall that if $U \subseteq \mathbb{R}^d$ is an open set for some $d \in \mathbb{N}$, then $H_0^1(U)$ is defined as the closure of the space $C_c^\infty(U)$ of compactly supported smooth functions with respect to the H^1-norm $\|\cdot\|_{H^1} := (\|\cdot\|_{L^2}^2 + \|\nabla \cdot\|_{L^2}^2)^{1/2}$ and $H^1(U)$ is defined as the closure of $C^\infty(U)$ with respect to the H^1-norm. Mosco convergence is defined as follows.

Definition 1.6. The sequence of open sets $\mathcal{O}_n \subseteq \mathbb{R}^d$, $n \in \mathbb{N}$, converges to an open set $\mathcal{O} \subseteq \mathbb{R}^d$ in the *Mosco sense*, denoted $\mathcal{O}_n \xrightarrow{\mathrm{M}} \mathcal{O}$ as $n \to \infty$, if:

(1) The weak limit points of every sequence $u_n \in H_0^1(\mathcal{O}_n)$, $n \in \mathbb{N}$, lies in $H_0^1(\mathcal{O})$.
(2) For every $u \in H_0^1(\mathcal{O})$ there exists $u_n \in H_0^1(\mathcal{O}_n)$ such that $u_n \to u$ as $n \to \infty$ in $H^1(\mathbb{R}^d)$.

Mosco convergence can be thought of as a notion of convergence for the Sobolev spaces $H_0^1(\mathcal{O}_n)$ to $H_0^1(\mathcal{O})$. If the domains $\mathcal{O}_n$ and $\mathcal{O}$ are bounded, then Mosco convergence $\mathcal{O}_n \xrightarrow{\mathrm{M}} \mathcal{O}$ ensures that the eigenvalues of $\mathcal{O}_n$ converge to the eigenvalues of $\mathcal{O}$ as $n \to \infty$ (more precisely, Mosco convergence for bounded domains ensures spectral exactness). Our main technical result establishes Mosco convergence for pixelated domains under a collection of mild hypotheses.

Theorem 1.7. *Suppose that* $\mathcal{O} \subseteq \mathbb{R}^2$ *is a connected, bounded, regular open set such that* $\mathrm{vol}(\partial\mathcal{O}) = 0$, $\partial\mathcal{O}$ *is path connected and* $\mathrm{int}(\mathcal{O}^c)$ *is connected. Then the pixelated domains* $(\mathcal{O}_n)_{n\in\mathbb{N}}$ *for* $\mathcal{O}$ *converge to* $\mathcal{O}$ *in the Mosco sense as* $n \to \infty$.

Crucially, in Theorem 1.7, we do not explicitly assume that the limit domain $\mathcal{O}$ is *stable*. A domain $\mathcal{O}$ is said to be stable if

$$H_0^1(\mathcal{O}) = H_0^1(\overline{\mathcal{O}}) := \big\{ u \in H^1(\mathbb{R}^d) : u = 0 \text{ a.e. on } \mathcal{O}^c \big\}.$$

A sufficient condition for a bounded domain to be stable is that the boundary is locally the image of a continuous map [5, Proposition 2.2]. For many domains satisfying our hypotheses, particularly those with fractal boundaries like the Koch snowflake, this does *not* hold. Stability allows for the application of many powerful results (see, for instance, [24][12, Section 5][3, Chapter 11]). It is not yet clear whether there do in fact exist non-stable domains satisfying the hypotheses of Theorem 1.7 or whether these hypotheses ensure, in a non-trivial way, that stability holds.

Nested approximations of a domain converge in the Mosco sense even if the domain is not stable: if $\mathcal{O}_n \subseteq \mathcal{O}_{n+1} \subseteq \mathcal{O}$ for all n and $\mathcal{O} = \cup_{n=1}^\infty \mathcal{O}_n$, then $\mathcal{O}_n$ tends to $\mathcal{O}$ as $n \to \infty$ in the Mosco sense [12, Proposition 5.4.1]. We emphasise the pixelation approximations for domains satisfying our hypotheses are not nested in general. In fact, since fractal boundaries can have structure at arbitrarily small scales, it is not possible for an arithmetic algorithm with input Λ_0 to construct nested approximations for an arbitrary domain satisfying our hypotheses.

In Proposition 3.2, we show that an *arbitrary* sequence of open sets $\mathcal{O}_n \subseteq \mathbb{R}^d$ (with compact boundaries) converges in the Mosco sense to an open set $\mathcal{O} \subseteq \mathbb{R}^d$ (also with compact boundary) in the Mosco sense provided that

$$d_H(\mathcal{O}_n, \mathcal{O}) + d_H(\partial\mathcal{O}_n, \partial\mathcal{O}) \to 0 \quad \text{as} \quad n \to \infty, \tag{1}$$

where d_H denotes Haussdorff distance, and that one can verify a certain Poincaré-type inequality on $\mathcal{O}$ and on $\mathcal{O}_n$. In particular, the necessary Poincaré-type inequality on $\mathcal{O}_n$ must be uniform in n.

The pixelation approximations for a regular open set with compact boundary are shown to converge in the sense of (1) in Proposition 3.5. The uniform Poincaré-type inequality in established using our general explicit Poincaré-type inequality (cf. Theorem 1.8 below) and a result on the large n geometry of the boundary of pixelated domains (cf. Proposition 3.10).

1.4. **An explicit Poincaré-type Inequality.** The application of our general result Proposition 3.2 to the proof of Mosco convergence for pixelated domains requires the development of the following Poincaré-type inequality for collar neighbourhoods of the boundary of domain. As far as we are aware, this is the first Poincaré-type inequality of its form to be reported.

Theorem 1.8. *Let $\mathcal{O} \subseteq \mathbb{R}^2$ be any open set and let $Q(\partial\mathcal{O})$ denote the infimum of the set of diameters of the path-connected components of $\partial\mathcal{O}$. Let $\partial^r\mathcal{O} := \{x \in \mathcal{O} : \mathrm{dist}(x, \partial\mathcal{O}) < r\}$ for any $r > 0$. If $Q(\partial\mathcal{O}) > 0$ and $r > 0$ satisfies $4\sqrt{2}r < Q(\partial\mathcal{O})$, then*

$$\|u\|_{L^2(\partial^r\mathcal{O})} \leq 5r\|\nabla u\|_{L^2(\partial^{(1+\sqrt{2})r}\mathcal{O})} \tag{2}$$

for all $u \in H_0^1(\mathcal{O})$.

The key properties of (2) we require are that the constant on the right hand side is independent of $\mathcal{O}$ and decays like $O(r)$ as $r \to 0$.

The proof of Theorem 1.8 is inspired by the simple proof of the Poincaré inequality in the classic text of Adams and Fournier [4, Theorem 6.30]. The idea is to express the value of the function u at any point x in the domain as a line integral from the boundary to that point x. In our proof, we explicitly construct such paths, aided by the introduction of various geometric structures.

1.5. **Organisation of the paper.** In Section 2, we prove the Poincaré-type inequality Theorem 1.8. In Section 3, we prove the result on Mosco convergence of pixelated domains Theorem 1.7. In the final Section 4, we prove the convergence of the main algorithm Theorem 1.3 and construct the counter-example Proposition 1.5.

2. An explicit Poincaré-type inequality

Definition 2.1. The *minimum component diameter* $Q(\partial A)$ for the boundary of a set $A \subset \mathbb{R}^d$ is defined by

$$Q(\partial A) = \inf\{\mathrm{diam}(\Gamma) : \Gamma \subseteq \partial A \text{ path-connected component of } \partial A\}$$

where the *diameter* of a bounded set $A \subset \mathbb{R}^d$ is defined by

$$\mathrm{diam}(A) = \sup_{x\in A}\sup_{y\in A}|x - y|.$$

Definition 2.2. The *r-collar neighbourhood* $\partial^r A$ for an open set $A \subset \mathbb{R}^d$ is defined by

$$\partial^r A = \{x \in A : \mathrm{dist}(x, \partial A) < r\}.$$

In the remainder of the section, and fix $r > 0$ and let $\mathcal{O} \subset \mathbb{R}^2$ be an arbitrary open set such that the connected components of $\partial\mathcal{O}$ are path-connected.

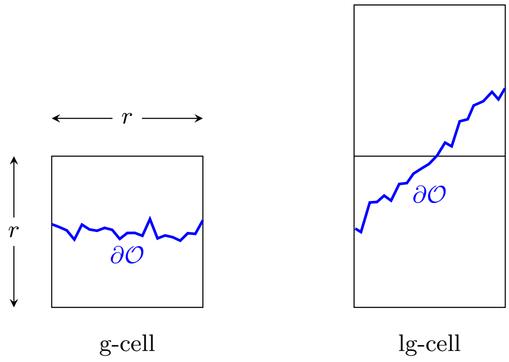
Definition 2.3. (a) A *cell* is a closed box $j + [0, r]^2$ for some $j \in (r\mathbb{Z})^2$.
(b) An *edge* of a cell is one of the 4 connected, closed straight line segments whose union comprises the boundary of the cell.

Definition 2.4. A *g-cell* is a cell c_0 such that two parallel edges e_1 and e_2 of c_0 are connected by a segment of $\partial\mathcal{O}$, that is,

$$\exists \Gamma \subseteq \partial\mathcal{O} \cap c_0 : \Gamma \text{ path-connected}, \Gamma \cap e_1 \neq \emptyset \text{ and } \Gamma \cap e_2 \neq \emptyset.$$

e_1 and e_2 are called the *connected edges* of the g-cell c_0. The other two edges of c_0 are called the *normal edges*.

Definition 2.5. (a) A *long-cell* is a set of two cells $\{c_1, c_2\}$ such that c_1 and c_2 share a common edge.
(b) An *edge* of a long-cell $\{c_1, c_2\}$ is one of the 4 connected, closed straight line segments whose union comprises the boundary of the cell $c_1 \cup c_2$.
(c) A *short-edge* of the long-cell is an edge of the long cell which is also the edge of a cell.



(d) A *long-edge* of a long-cell is an edge of the long-cell which is not a short-edge.

Definition 2.6. An *lg-cell* is a long cell $\{c_1, c_2\}$ for which there exist distinct long-edges e_1 and e_2 are connected by a segment of $\partial\mathcal{O}$, that is,

$$\exists \Gamma \subseteq \partial\mathcal{O} \cap (c_1 \cup c_2) : \Gamma \text{ path-connected}, \Gamma \cap e_1 \neq \emptyset \text{ and } \Gamma \cap e_2 \neq \emptyset.$$

The *normal edges* of a lg-cell refers to its short edges.

Definition 2.7. A *cell path* from a cell c_0 to a g-cell c_n (or to a lg-cell $\{c_n, c_{n+1}\}$) is a sequence of cells $(c_1, ..., c_{n-1})$ such that

(1) For each $j \in \{1, ..., n-1\}$, c_j shares a common edge with c_{j-1}.
(2) There exists an edge of c_{n-1} which is also a normal edge of the g-cell c_n (or of the lg-cell $\{c_n, c_{n+1}\}$ respectively).

We allow the possibility that $n = 0$, corresponding to the case that c_0 is itself a g-cell (or in a long-cell).

Definition 2.8. (a) The *1-cell neighbourhood* $D_1[c_0]$ of a cell c_0 is the union of all cells sharing an edge or a corner with c_0, that is,

$$D_1[c_0] = \bigcup \{c : c \text{ is a cell and } c \cap c_0 \neq \emptyset\}.$$

(b) The *2-cell neighbourhood* $D_2[c_0]$ of a cell c_0 is the union of all cells sharing an edge or a corner with $D_1[c_0]$, that is,

$$D_2[c_0] = \bigcup \{c : c \text{ is a cell and } c \cap \tilde{c} \neq \emptyset \text{ for some } \tilde{c} \in D_1[c_0]\}.$$

Definition 2.9. (a) A *covering cell* is a cell which intersects $\partial^r\mathcal{O}$.
(b) A *filled cell* is a cell which intersects $\partial\mathcal{O}$.
(c) A *non-filled cell* is a cell which is a covering cell but is not a filled cell.

Lemma 2.10. *For any bounded set $A \subset \mathbb{R}^d$, we have*

$$\operatorname{diam}(A) \leq 2 \inf_{x \in A} \sup_{y \in A} |x - y|.$$

Proof. Let x_1, $x_2 \in \overline{A}$ be such that

$$\inf_{x \in A} \sup_{y \in A} |x - y| = \sup_{y \in A} |x_1 - y| \quad \text{and} \quad \sup_{x \in A} \sup_{y \in A} |x - y| = \sup_{y \in A} |x_2 - y|.$$

The lemma follows from the observation that

$$\sup_{y \in A} |x_2 - y| \leq |x_1 - x_2| + \sup_{y \in A} |x_1 - y| \leq 2 \sup_{y \in A} |x_1 - y|.$$

$\square$

Lemma 2.11. *If $Q(\partial\mathcal{O}) > 4\sqrt{2}r$, then for any filled cell c_0, there exists a g-cell or an lg-cell in $D_1[c_0]$.*

Proof. Let c_0 be a filled cell. There exists a path-connected component $\Gamma \subseteq \partial\mathcal{O}$ such that $\Gamma \cap c_0 \neq \emptyset$. Let $x \in \Gamma \cap c_0$. Using Lemma 2.10 and the hypothesis on $Q(\partial\mathcal{O})$,

$$2\sqrt{2}r < \frac{1}{2} Q(\partial\mathcal{O}) \leq \frac{1}{2} \operatorname{diam}(\Gamma) \leq \sup_{y \in \Gamma} |x - y|$$

so there exists $y \in \Gamma$ with $|x - y| > 2\sqrt{2}r$. Hence, $y \in \Gamma$ lies outside of $\partial D_1[c_0]$. Since Γ is path-connected, there exists a continuous path from x to y. Restricting this path, we deduce there exist a continuous path

$\gamma : [0,1] \to D_1[c_0]$ such that

$$\gamma(t) \in \begin{cases} \partial D_1[c_0] & \text{if } t = 0 \\ \mathrm{int} D_1[c_0] \backslash c_0 & \text{if } t \in (0,1) \\ \partial c_0 & \text{if } t = 1 \end{cases} \quad \text{and} \quad \forall t \in [0,1] : \gamma(t) \in \Gamma.$$

Let us fix some notions that will allow us to prove the lemma. An edge e is a *zeroth edge* if $\gamma(0) \in e$ and $e \subset \partial D_1[c_0]$. Let

$$t_1 := \inf\{t > 0 : \exists \text{ edge } e \text{ such that } \gamma(t) \in e\} \in [0,1].$$

A *first edge* is defined as an edge e such that $\gamma(t_1) \in e$ and e is not a zeroth edge. If $t_1 < 1$, then let

$$t_2 := \inf\{t > 0 : \exists \text{ edge } e \text{ such that } \gamma(t) \in e \text{ and } e \text{ is not a first edge}\} \in (t_1, 1]$$

and a *second edge* is defined as an edge e such that $\gamma(t_2) \in e$ and e is not a first edge. If $t_1 = 1$, then t_2 and the second edges are not defined. Finally, if $t_1 < 1$, then there exists a unique first edge e_1 and we define

$$\tilde{t}_1 := \sup\{t \leq t_2 : \gamma(t) \in e_1\}.$$

Let us now proceed onto the main part of the proof, in which we repeatedly use the continuity of the path γ.

(A) Suppose that $t_1 \in (0,1)$. Then, since $\gamma(t)$ belongs to at most one edge for $t \in (0,1)$, there exists a unique first edge e_1 and a unique cell c_1 containing $\gamma([0,t_1])$. c_1 must contain e_1 - let c_2 be the other cell containing e_1.

- (A1) If there exists a zeroth edge contained in c_1 which is parallel to e_1, then c_1 is a g-cell.
- (A2) Suppose there exists a second edge e which is contained in a cell $c \in \{c_1, c_2\}$ and is which parallel to e_1. Then $e \neq e_1$ by the definition of a second edge so $\tilde{t}_1 < t_2$. $\gamma([\tilde{t}_1, t_2])$ is contained in c and connects the distinct parallel edges e and e_1 of c, therefore, c is a g-cell.
- (A3) In the only other case, there exists a zeroth edge e_0 contained in the cell c_1 and a second edge e_2 contained in a cell $c \in \{c_1, c_2\}$ such that both e_0 and e_2 are perpendicular to the edge e_1. It follows that e_0 and e_2 are distinct, parallel edges. Furthermore, the edges e_0 and e_2 are contained in distinct long edges of the long-cell $\{c_1, c_2\}$, hence the long edges of $\{c_1, c_2\}$ are connected by $\gamma([0,t_2])$. Since $\gamma([0,t_2])$ is contained in $c_1 \cup c_2$, $\{c_1, c_2\}$ is an lg-cell.

We conclude that if $t_1 \in (0,1)$, then there is a g-cell or an lg-cell in $D_1[c_0]$.

(B) Suppose that $t_1 = 1$. Then, $\gamma([0,1])$ is contained entirely in one cell.

- (B1) Suppose $\gamma(1)$ is in the interior of an edge e belonging to c_0. Then the cell $c \neq c_0$ containing e also contains a zeroth edge parallel to e, as well as $\gamma([0,1])$ in its entirety. In this case, c is a g-cell.
- (B2) In the only other case, $\gamma(1)$ is not in the interior an edge, then $\gamma(1)$ is a corner of c_0. Then,' there are four first edges, each of which is parallel and sharing a cell with one of the possible four zeroth edges. Consequently, in this case the cell containing $\gamma([0,1])$ in its entirety is a g-cell.

We conclude that if $t_1 = 1$, then there is a g-cell in $D_1[c_0]$.

(C) Suppose that $t_1 = 0$. Then, there exist a unique first edge e_1 which must satisfy $\gamma([0,\epsilon)) \subset e_1$ for all small enough $\epsilon > 0$.

- (C1) Suppose that $\tilde{t}_1 = t_2$. Let c_1 and c_2 be the cells sharing the edge e_1. In this case $\gamma(0)$ and $\gamma(t_2)$ belong to opposite extremal points of the edge e_1 hence belong to distinct long-edges of the long cell $\{c_1, c_2\}$. Furthermore, $\gamma([0,t_2]) \subset c_1 \cup c_2$ hence $\{c_1, c_2\}$ forms an lg-cell.

Suppose, on the other hand, that $\tilde{t}_1 < t_2$. Then there exists a unique cell c_1 containing $\gamma([\tilde{t}_1, t_2])$. c_1 must contain the edge e_1 - let c_2 denote the other cell containing the edge e_2. Note that c_1 must contain at least one second edge and, by the definition of $\tilde{t}_1$, e_1 is not a second edge.

- (C2) If there exists a second edge e contained in c_1 which is parallel to e_1, then c_1 is a g-cell.
- (C3) In the only other case, there exists a second edge contained in c_1 which is perpendicular to e_1. In this case, $\gamma(0)$ and $\gamma(t_2)$ are contained in distinct long-edges of the long-cell $\{c_1, c_2\}$, hence $\{c_1, c_2\}$ forms a lg-cell.

We conclude that if $t_1 = 0$, then there is a g-cell or a lg-cell in $D_1[c_0]$.

We have covered every possible case, proving the lemma. □

Corollary 2.12. *For any covering cell c_0, there exists a g-cell c_n (or an lg-cell $\{c_n, c_{n+1}\}$) and a cell-path $(c_1, ..., c_{n-1})$ from c_0 to c_n (or to $\{c_n, c_{n+1}\}$ respectively), such that*

$$c_n \subset D_2[c_0] \quad \textit{or} \quad c_n \cup c_{n+1} \subset D_2[c_0]$$

and

$$\forall j \in \{1, ..., n-1\} : c_j \subset D_2[c_0].$$

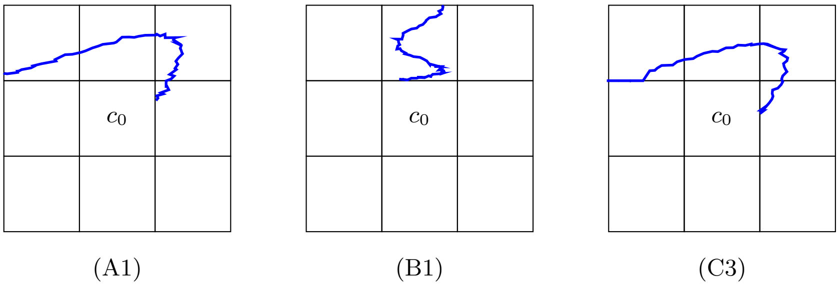


FIGURE 1. Example sketches for some of the cases (A), (B), (C) in the proof of Lemma 2.11

Lemma 2.13. *Let c_0 be a cell and let $(c_1, ..., c_{n-1})$ be a cell path from c_0 to a g-cell c_n or an lg-cell $\{c_n, c_{n+1}\}$. Then, for any $u \in H^1_0(\mathcal{O})$,*

$$\|u\|^2_{L^2(c_0)} \le r^2 \sum_{j=0}^{n+1} \|\nabla u\|^2_{L^2(c_j)}. \tag{3}$$

Here, c_{n+1} is considered to be the empty set in the case of a cell path to a g-cell.

Proof. Assume without loss of generality that $c_0 = [0, r]^2$. In the case of a cell path to an lg-cell, assume that c_n shares an edge with c_{n-1}. For each $j \in \{0, ..., n-1\}$, let e_j denote the edge shared by c_j and c_{j+1} Assume without loss of generality that $e_0 = [0, r] \times \{0\}$.

Let us parameterise each of the edges e_j by $(e_j(s))_{s\in[0,r]}$. Assume that $s \mapsto e_j(s)$ is a unit speed segment of a straight-line. It suffices to specify the point $e_j(0)$ or the point $e_j(r)$ in order to define the entire parameterisation $(e_j(s))_{s\in[0,r]}$.

(1) Define $e_0(s)$ by $e_0(0) = (0, 0)$, so that $e_0(s) = (s, 0)$.
(2) For $j \in \{1, ..., n-1\}$, if e_{j-1} is parallel to e_j, then we call c_j a *straight tile*. In this case, define $(e_j(s))_{s\in[0,r]}$ by the condition that $e_j(0)$ is connected by an edge of c_j to $e_{j-1}(0)$, so that $e_j(r)$ is connected by an edge of c_j to $e_{j-1}(r)$.
(3) For $j \in \{1, ..., n-1\}$, if e_{j-1} is perpendicular to e_j, then we call c_j an *corner tile*. If e_j and e_{j-1} share the point $e_{j-1}(0)$, then c_j is said to the *positively oriented*. In this case, define e_j by the condition that $e_j(0) = e_{j-1}(0)$. On the other hand, if e_j and e_{j-1} share the point $e_{j-1}(r)$, then c_j is said to the *negatively oriented*. In this case, define e_j by the condition that $e_j(r) = e_{j-1}(r)$.[1]

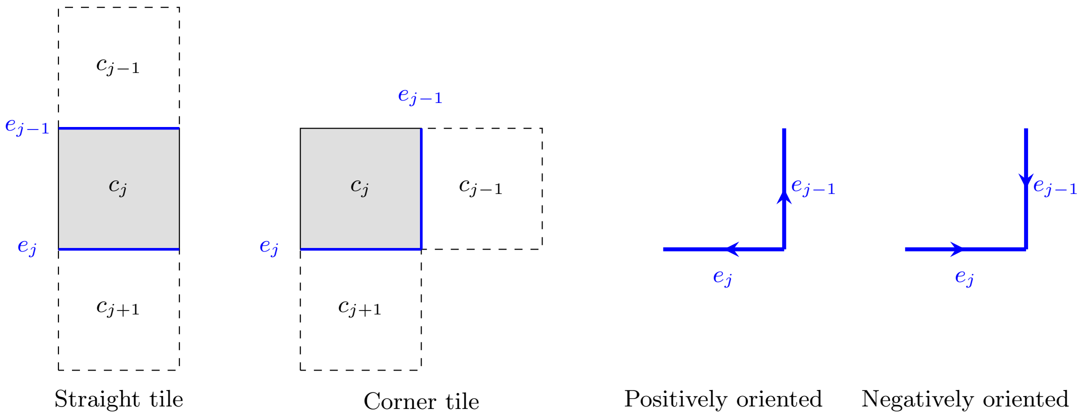


FIGURE 2. Sketch of the different types of tiles and orientation.

Next, we construct a family of isometries $(\iota_j : \mathbb{R}^2 \to \mathbb{R}^2)_{j\in\{1,...,n\}}$ each of which maps $[0, r]^2$ to the cell c_j. The purpose of this is to simplify the later construction of certain paths within each cell. The reader is reminded that any composition of translations, rotations and reflections in the plane is an isometry - this fact is what guarantees the existence of isometries satisfying the below conditions.

(1) For $j \in \{1, ..., n-1\}$, if c_j is a straight tile, then choose ι_j such that $\iota_j(s, 0) = e_j(s)$ and $\iota_j(s, r) = e_{j-1}(s)$

[1] The name positively oriented corner tile comes from the fact that if such a cell is mapped isometrically so that e_j lies along $[0, r]$, with $e_j(0)$ at the origin, the edges e_j and e_{j-1} form an L shape. For a similarly mapped negatively oriented corner tile, e_j and e_{j-1} form a backwards L shape.

(2) For $j \in \{1, ..., n-1\}$, if c_j is a positively oriented corner tile, then choose ι_j so that $\iota_j(r-s, 0) = e_j(s)$ and $\iota(r, s) = e_{j-1}(s)$.
(3) For $j \in \{1, ..., n-1\}$, if c_j is a negatively oriented corner tile, then choose ι_j so that $\iota_j(s, 0) = e_j(s)$ and $\iota_j(r, r-s) = e_{j-1}(s)$.
(4) Choose ι_n so that $\iota_n(s, r) = e_{n-1}(s)$ and $\iota_n([0, r]^2) = c_n$. In the case of a cell-path to an lg-cell, this implies that $\iota_n([0, r] \times [-r, 0]) = c_{n+1}$.

By Definitions 2.4 and 2.6 for a g-cell and an lg-cell respectively, there exists a function $w : [0, r] \to [-r, r]$ such that $u \circ \iota_n(x, w(x)) = 0$. Note that in the case of a cell-path to a g-cell, w only takes values in $[0, r]$. By the density of $C_0^\infty(\mathcal{O})$ in $H_0^1(\mathcal{O})$, it suffices to show that (3) holds for all $u \in C_0^\infty(\mathcal{O})$. Hence, let $u \in C_0^\infty(\mathcal{O})$.

Firstly, for any $s \in [0, r]$,

$$u(e_{n-1}(s)) = u \circ \iota_n(s, r) = \int_{w(s)}^{r} \frac{\partial}{\partial t} u \circ \iota_n(s, t) \mathrm{d}t =: I_g(s).$$

Let $j \in \{1, ..., n-1\}$. If c_j is a straight tile, then for any $s \in [0, r]$,

$$u(e_{j-1}(s)) - u(e_j(s)) = u \circ \iota_j(s, r) - u \circ \iota_j(s, 0) = \int_0^r \frac{\partial}{\partial t} u \circ \iota_j(s, t) \mathrm{d}t =: I_j(s).$$

If c_j, is an corner tile, then let

$$\tilde{I}_j(s) := \int_0^s \frac{\partial}{\partial t} u \circ \iota_j(r-s, t) \mathrm{d}t + \int_{r-s}^{r} \frac{\mathrm{d}}{\mathrm{d}t} u \circ \iota_j(t, s) \mathrm{d}t.$$

If c_j is a positively oriented corner tile, then for any $s \in [0, r]$,

$$u(e_{j-1}(s)) - u(e_j(s)) = u \circ \iota_j(r, s) - u \circ \iota_j(r-s, 0) = \tilde{I}_j(s) =: I_j(s).$$

If c_j is a negatively oriented corner tile, then for any $s \in [0, r]$,

$$u(e_{j-1}(s)) - u(e_j(s)) = u \circ \iota_j(r, r-s) - u \circ \iota_j(s, 0) = \tilde{I}_j(r-s) =: I_j(s).$$

We can now express the value of u at any point in $c_0 = [0, r]^2$ as sum of line integrals. For any $x, y \in [0, r]$,

$$u(x, y) = u(e_0(x)) + \int_0^y \frac{\partial}{\partial t} u(x, t) \mathrm{d}t = I_g(x) + \sum_{j=1}^{n-1} I_j(x) + \int_0^y \frac{\partial}{\partial t} u(x, t) \mathrm{d}t$$

hence

$$\begin{aligned} \|u\|^2_{L^2(c_0)} &= \int_0^r \int_0^r |u(x, y)|^2 \mathrm{d}x \mathrm{d}y \\ &\leq r \left[\int_0^r |I_g(x)|^2 \mathrm{d}x + \sum_{j=1}^{n-1} \int_0^r |I_j(x)|^2 \mathrm{d}x + \int_0^r \left(\int_0^r \left| \frac{\partial}{\partial t} u(x, t) \right| \mathrm{d}t \right)^2 \mathrm{d}x \right]. \end{aligned} \tag{4}$$

Focusing on the final term in the square brackets of (4) and applying Cauchy-Schwarz,

$$\int_0^r \left(\int_0^r \left| \frac{\partial}{\partial t} u(x, t) \right| \mathrm{d}t \right)^2 \mathrm{d}x \leq r \int_0^r \int_0^r \left| \frac{\partial}{\partial t} u(x, t) \right|^2 \mathrm{d}x \mathrm{d}t \leq r \|\nabla u\|^2_{L^2(c_0)}. \tag{5}$$

To estimate the remaining terms, we need to use the fact that

$$\left| \frac{\partial}{\partial t} (u \circ \iota_j)(x, t) \right| = \left| \nabla u(\iota_j(x, t)) \cdot \frac{\mathrm{d}\iota_j}{\mathrm{d}t}(x, t) \right| \leq |\nabla u(\iota_j(x, t))| \left| \frac{\mathrm{d}\iota_j}{\mathrm{d}t}(x, t) \right| \leq |\nabla u(\iota_j(x, t))|,$$

where the final inequality holds since ι_j is an isometry, and similarly,

$$\left| \frac{\partial}{\partial t} (u \circ \iota_j)(t, y) \right| \leq |\nabla u(\iota_j(t, y))|.$$

Focusing on the middle terms in the square brackets of (4), let $j \in \{1, ..., n\}$. If c_j is a straight tile, then

$$\int_0^r |I_j(x)|^2 dx \leq \int_0^r \left(\int_0^r |\nabla u(\iota_j(x, t))| \mathrm{d}t \right)^2 \mathrm{d}x \leq r \int_0^r \int_0^r |\nabla u(\iota_j(x, t))|^2 \mathrm{d}x \mathrm{d}t = r \|\nabla u\|^2_{L^2(c_j)}. \tag{6}$$

If c_j is an corner tile, then

$$\begin{aligned}\int_0^r |\tilde{I}_j(x)|^2 \mathrm{d}x &\leq \int_0^r \left(\int_0^x |\nabla u(\iota_j(r-x,t))|\mathrm{d}t\right)^2 \mathrm{d}x + \int_0^r \left(\int_{r-x}^r |\nabla u(\iota_j(t,x))|\mathrm{d}t\right)^2 \mathrm{d}x \\ &= \int_0^r \left(\int_0^{r-x} |\nabla u(\iota_j(x,t))|\mathrm{d}t\right)^2 \mathrm{d}x + \int_0^r \left(\int_{r-x}^r |\nabla u(\iota_j(t,x))|\mathrm{d}t\right)^2 \mathrm{d}x \\ &\leq r\int_0^r \int_0^r |\nabla u(\iota_j(t,x))|^2 \mathrm{d}t\mathrm{d}x = r\|\nabla u\|^2_{L^2(c_j)},\end{aligned}$$

hence if c_j is a positively oriented corner tile, then

$$\int_0^r |I_j(x)|^2 \mathrm{d}x = \int_0^r |\tilde{I}_j(x)|^2 \mathrm{d}x \leq r\|\nabla u\|^2_{L^2(c_j)} \tag{7}$$

and if c_j is a negatively oriented corner tile, then

$$\int_0^r |I_j(x)|^2 \mathrm{d}x = \int_0^r |\tilde{I}_j(r-x)|^2 \mathrm{d}x = \int_0^r |\tilde{I}_j(x)|^2 \mathrm{d}x \leq r\|\nabla u\|^2_{L^2(c_j)}. \tag{8}$$

Finally, letting $h=0$ in the case of cell-path to a g-cell and $h=-r$ in the case of a cell-path to an lg-cell, we have

$$\int_0^r |I_g(x)|^2 \mathrm{d}x \leq \int_0^r \left(\int_{w(x)}^r |\nabla u(\iota_j(x,t))|\mathrm{d}t\right)^2 \mathrm{d}x \tag{9}$$

$$\leq \int_0^r \left(\int_0^r |\nabla u(\iota_j(x,t))|\mathrm{d}t\right)^2 \mathrm{d}x + \int_0^r \left(\int_h^0 |\nabla u(\iota_j(x,t))|\mathrm{d}t\right)^2 \mathrm{d}x \tag{10}$$

$$\leq r\Big(\|\nabla u\|^2_{L^2(c_n)} + \|\nabla u\|^2_{L^2(c_{n+1})}\Big). \tag{11}$$

where c_{n+1} is considered to the the empty set in the case of a cell-path to a g-cell. The proof is completed by substituting estimates (5)-(9) into (4). □

Theorem 2.14. *If $\mathcal{O} \subset \mathbb{R}^2$ is an open set such that $Q(\partial\mathcal{O}) > 0$ and $r > 0$ is such that $4\sqrt{2}r < Q(\partial\mathcal{O})$, then for any $u \in H_0^1(\mathcal{O})$,*

$$\|u\|_{L^2(\partial^r\mathcal{O})} \leq 5r\|\nabla u\|_{L^2(\partial^{(1+\sqrt{2})r}\mathcal{O})}.$$

Proof. Let $\{c_j\}$ be the set of covering cells, so that $\partial^r\mathcal{O} \subseteq \bigcup_j c_j$. For each c_j, fix an associated g-cell $\mathrm{as}[c_j]_{n_j}$ or an associated lg-cell $\{\mathrm{as}[c_j]_{n_j}, \mathrm{as}[c_j]_{n_j+1}\}$ and an associated cell-path $(c_1, ..., c_{n_j-1})$ from c_j to $\mathrm{as}[c_j]_{n_j}$ or to $\{\mathrm{as}[c_j]_{n_j}, \mathrm{as}[c_j]_{n_j+1}\}$. By Corollary 2.12, this can be done such that

$$\forall j : \forall k \in \{1, ..., n_j+1\} : \mathrm{as}[c_j]_k \subset D_2[c_j]. \tag{12}$$

By Lemma 2.13,

$$\|u\|^2_{L^2(\partial^r\mathcal{O})} \leq \sum_j \|u\|^2_{L^2(c_j)} \leq r^2 \sum_j \sum_{k=1}^{n_k+1} \Big(\|\nabla u\|^2_{L^2(c_j)} + \|\nabla u\|^2_{L^2(\mathrm{as}[c_j]_k)}\Big). \tag{13}$$

Since there are 25 cells in a 2-cell neighbourhood, each cell can be an associate to at most 24 other cells so (12) and (13) imply that

$$\|u\|^2_{L^2(\partial^r\mathcal{O})} \leq 25r^2 \sum_j \|\nabla u\|^2_{L^2(c_j)}.$$

The proof is completed by noting that $\mathrm{int}(\cup_j c_j) \subseteq \partial^{(1+\sqrt{2})r}\mathcal{O}$ □

3. Mosco convergence for pixelated domains

Throughout the section let $d \geq 1$ denote any positive integer, d is the dimension.

Lemma 3.1. *For any $\mathcal{O} \subset \mathbb{R}^d$ and $\mathcal{O}_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, we have*

$$\sup_{x \in \mathcal{O}^c \cap \mathcal{O}_n} \mathrm{dist}(x, \partial\mathcal{O}_n) \leq d_H(\mathcal{O}, \mathcal{O}_n) + d_H(\partial\mathcal{O}, \partial\mathcal{O}_n)$$

and

$$\sup_{x \in \mathcal{O}_n^c \cap \mathcal{O}} \mathrm{dist}(x, \partial\mathcal{O}) \leq d_H(\mathcal{O}, \mathcal{O}_n) + d_H(\partial\mathcal{O}, \partial\mathcal{O}_n).$$

Proof. The first statement holds because

$$\begin{aligned}\sup_{x\in\mathcal{O}^c\cap\mathcal{O}_n} \operatorname{dist}(x,\partial\mathcal{O}_n) &\le \sup_{x\in\mathcal{O}^c\cap\mathcal{O}_n} \operatorname{dist}(x,\partial\mathcal{O}) + \mathrm{d}_H(\partial\mathcal{O},\partial\mathcal{O}_n)\\ &\le \sup_{x\in\mathcal{O}^c\cap\mathcal{O}_n} \operatorname{dist}(x,\mathcal{O}) + \mathrm{d}_H(\partial\mathcal{O},\partial\mathcal{O}_n)\\ &\le \mathrm{d}_H(\mathcal{O},\mathcal{O}_n) + \mathrm{d}_H(\partial\mathcal{O},\partial\mathcal{O}_n).\end{aligned}$$

The proof of the second statement is similar. □

Proposition 3.2. *Let $\mathcal{O} \subseteq \mathbb{R}^d$ and $\mathcal{O}_n \subseteq \mathbb{R}^d$, $n \in \mathbb{N}$ be open sets with compact boundaries such that the following holds:*

(1) $l(n) := d_H(\mathcal{O},\mathcal{O}_n) + d_H(\partial\mathcal{O},\partial\mathcal{O}_n) \to 0$ *as* $n\to\infty$.
(2) *There exists*
 (i) $(f(n))_{n\in\mathbb{N}}$ *such that* $2l(n) \le f(n)$ *for all* $n\in\mathbb{N}$ *and* $f(n)\to 0$ *as* $n\to\infty$,
 (ii) *constants* $C,\alpha>0$ *independent of* n, u *and* r
 such that, if either $V=\mathcal{O}$ *or* $V=\mathcal{O}_n$ *for some large enough* $n\in\mathbb{N}$, *then for all* $u\in H_0^1(V)$ *we have*

$$\|u\|_{L^2(\partial^{f(n)}V)} \le Cf(n)\|\nabla u\|_{L^2(\partial^{\alpha f(n)}V)}. \tag{14}$$

(3) $\operatorname{vol}(\partial\mathcal{O}) = 0$.

Then, $\mathcal{O}_n$ *converges to* $\mathcal{O}$ *in the Mosco sense as* $n\to\infty$.

Proof. Throughout the proof, let L^2 denote $L^2(\mathbb{R}^d)$ and let H^1 denote $H^1(\mathbb{R}^d)$. All limits will be as $n\to\infty$.

Define function $\tilde\chi:\mathbb{R}_+\to[0,1]$ by

$$\tilde\chi(t) := \begin{cases} t & \text{if } t\in[0,1)\\ 1 & \text{if } t\in[1,\infty).\end{cases} \tag{15}$$

$\tilde\chi$ is weakly differentiable with $\|\tilde\chi'\|_{L^\infty}=1$. $\tilde\chi$ will be used in the construction of a cut-off function in both step 1 and step 2 below. We shall also require the following two facts. Firstly, for any $A\subset\mathbb{R}^d$ with smooth boundary, the function $x\mapsto\operatorname{dist}(x,A)$ is piecewise smooth hence weakly differentiable and, since

$$|\operatorname{dist}(x,A)-\operatorname{dist}(y,A)| \le |x-y| \qquad (x,y\in\mathbb{R}^d),$$

the L^∞ norm of $x\mapsto\nabla\operatorname{dist}(x,A)$ is bounded by 1. Secondly, it holds that $\operatorname{vol}(\partial^r\mathcal{O})\to 0$ as $r\to 0$. To see the latter fact, note that one has $\partial\mathcal{O}=\bigcap_{r>0}\partial^r\mathcal{O}$ so by continuity of measures from above and hypothesis (3), it follows that $0=\operatorname{vol}(\partial\mathcal{O})=\lim_{r\to0}\operatorname{vol}(\partial^r\mathcal{O})$.

Step 1 (Mosco convergence condition (1)). Let $u_n\in H_0^1(\mathcal{O}_n)$, $n\in\mathbb{N}$, and suppose that $u_n\rightharpoonup u$ in H^1 for some $u\in H^1$. We aim to show that $u\in H_0^1(\mathcal{O})$.

Let $P:H^1\to H_0^1(\mathcal{O})$ be the orthogonal projection. If $(w_n)\subset H_0^1(\mathcal{O})$ and $w_n\rightharpoonup u$ in H^1 then

$$\langle u,(1-P)\phi\rangle_{H^1} = \lim_{n\to\infty}\langle w_n,(1-P)\phi\rangle_{H^1} = 0 \qquad (\phi\in H^1)$$

so $u\in H_0^1(\mathcal{O})$. Hence it suffices to show that there exists $w_n\in H_0^1(\mathcal{O})$ such that $w_n\rightharpoonup u$ in H^1.

For each $n\in\mathbb{N}$, let $A_n\subset\mathbb{R}^d$ be an open neighbourhood of $\mathcal{O}^c$ with smooth boundary such that $A_n\cap\mathcal{O}\subseteq\partial^{f(n)/4}\mathcal{O}$. Define a cut-off function $\chi_n:\mathbb{R}^d\to[0,1]$ by

$$\chi_n(x) = \tilde\chi(4f(n)^{-1}\operatorname{dist}(x,A_n)) \qquad (x\in\mathbb{R}^d). \tag{16}$$

Then, $\chi_n=0$ on an open neighbourhood of $\mathcal{O}^c$ and $\chi_n(x)=1$ for any $x\in\mathcal{O}_n$ outside the set

$$\mathcal{U}_n := \{x\in\mathcal{O}_n : \operatorname{dist}(x,\mathcal{O}^c)\le f(n)/2\}.$$

χ_n is weakly differentiable and, by an application of the chain rule,

$$\|\nabla\chi_n\|_{L^\infty} \le 4f(n)^{-1}. \tag{17}$$

By Lemma 3.1,

$$\sup_{x\in\mathcal{O}^c\cap\mathcal{O}_n}\operatorname{dist}(x,\partial\mathcal{O}_n) \le l(n) \le \frac{f(n)}{2}$$

so $\mathcal{U}_n\subseteq\partial^{f(n)}\mathcal{O}_n$. Furthermore, for any $x\in\mathcal{U}_n$,

$$\operatorname{dist}(x,\partial\mathcal{O}) \le l(n)+\operatorname{dist}(x,\partial\mathcal{O}_n) \le l(n)+f(n)$$

so $\operatorname{vol}(\mathcal{U}_n)\to0$ as $n\to\infty$.

Let $w_n:=\chi_n u_n$. Then, since $\chi_n=0$ on an open neighbourhood of $\mathcal{O}^c$, we have $w_n\in H_0^1(\mathcal{O})$ and it suffices to show that $w_n=\chi_n u_n\rightharpoonup u$ in H^1. Let $\phi\in H^1$ be an arbitrary test function. Firstly, we have,

$$|\langle\chi_n u_n-u,\phi\rangle_{H^1}| \le \underbrace{|\langle\chi_n u_n-u,\phi\rangle_{L^2}|}_{(A1)} + \underbrace{|\langle\nabla(\chi_n u_n)-\nabla u,\nabla\phi\rangle_{L^2}|}_{(A2)}.$$

Focusing on the term (A1),

$$\begin{aligned}|\langle \chi_n u_n - u, \phi\rangle_{L^2}| &\leq |\langle \chi_n u_n - u_n, \phi\rangle_{L^2}| + |\langle u_n - u, \phi\rangle_{L^2}| \\ &\leq 2\|u_n\|_{L^2(\mathcal{U}_n)}\|\phi\|_{L^2(\mathcal{U}_n)} + |\langle u_n - u, \phi\rangle_{L^2}| \to 0\end{aligned}$$

where the second inequality holds since $\chi_n u_n = u_n$ outside $\mathcal{U}_n$ and the limit holds by the weak convergence of (u_n) as well as the fact that $\mathrm{vol}(\mathcal{U}_n) \to 0$. Focusing on the term (A2),

$$|\langle \nabla(\chi_n u_n) - \nabla u, \nabla\phi\rangle_{L^2}| \leq \underbrace{|\langle \nabla(\chi_n u_n) - \nabla u_n, \nabla\phi\rangle_{L^2}|}_{(B1)} + \underbrace{|\langle \nabla u_n - \nabla u, \nabla\phi\rangle_{L^2}|}_{(B2)}.$$

The term (B2) tends to zero by the weak convergence of (u_n). Focusing on the term (B1),

$$\begin{aligned}|\langle \nabla(\chi_n u_n) - \nabla u_n, \nabla\phi\rangle_{L^2}| &\leq |\langle \chi_n \nabla u_n - \nabla u_n, \nabla\phi\rangle_{L^2}| + |\langle \nabla(\chi_n) u_n, \nabla\phi\rangle_{L^2}| \\ &\leq \underbrace{2\|\nabla u_n\|_{L^2(\mathcal{U}_n)}\|\nabla\phi\|_{L^2(\mathcal{U}_n)}}_{(C1)} + \underbrace{\|\nabla\chi_n\|_{L^\infty}\|u_n\|_{L^2(\mathcal{U}_n)}\|\nabla\phi\|_{L^2(\mathcal{U}_n)}}_{(C2)}\end{aligned}$$

where in the second inequality we used the fact that $\chi_n \nabla u_n = \nabla u_n$ outside $\mathcal{U}_n$ and $\mathrm{supp}(\nabla(\chi_n)u_n) \subseteq \mathcal{U}_n$. The term (C1) tends to zero since (u_n) is bounded in H^1 and $\mathrm{vol}(\mathcal{U}_n) \to 0$. Focusing on the term (C2), notice first that, by the assumed Poincaré-type inequality (14),

$$\|u_n\|_{L^2(\mathcal{U}_n)} \leq \|u_n\|_{L^2(\partial^{f(n)}\mathcal{O}_n)} \leq Cf(n)\|\nabla u_n\|_{L^2(\partial\mathcal{O}_n^{\alpha f(n)})},$$

and so, by (17),

$$\begin{aligned}\|\nabla\chi_n\|_{L^\infty}\|u_n\|_{L^2(\mathcal{U}_n)}\|\nabla\phi\|_{L^2(\mathcal{U}_n)} &\leq 4f(n)^{-1}\|u_n\|_{L^2(\mathcal{U}_n)}\|\nabla\phi\|_{L^2(\mathcal{U}_n)} \\ &\leq 4C\|\nabla u_n\|_{L^2(\partial^{\alpha f(n)}\mathcal{O}_n)}\|\nabla\phi\|_{L^2(\mathcal{U}_n)} \to 0.\end{aligned}$$

It follows that the term (A2) tends to zero, that $w_n \rightharpoonup u$ in H^1 and hence that $u \in H_0^1(\mathcal{O})$.
Step 2 (Mosco convergence condition (2)). Let $u \in H_0^1(\mathcal{O})$ - we aim to show that there exists $u_n \in H_0^1(\mathcal{O}_n)$ such that $u_n \to u$ in H^1. Note that in this part of the proof, we shall redefine A_n, χ_n and $\mathcal{U}_n$.

For each $n \in \mathbb{N}$, let $A_n \subset \mathbb{R}^d$ be an open neighbourhood of $\mathcal{O}_n^c$ with smooth boundary such that $A_n \cap \mathcal{O}_n \subseteq \partial^{f(n)/4}\mathcal{O}_n$. Define a cut-off function $\chi_n : \mathbb{R}^d \to [0,1]$ by

$$\chi_n(x) = \tilde{\chi}(4f(n)^{-1}\mathrm{dist}(x, A_n)) \qquad (x \in \mathbb{R}^d). \tag{18}$$

Then, $\chi_n = 0$ on an open neighbourhood of $\mathcal{O}_n^c$ and $\chi_n(x) = 1$ for any $x \in \mathcal{O}$ outside the set

$$\mathcal{U}_n := \{x \in \mathcal{O} : \mathrm{dist}(x, \mathcal{O}_n^c) \leq f(n)/2\}.$$

χ_n is weakly differentiable and, by an application of the chain rule,

$$\|\nabla\chi_n\|_{L^\infty} \leq 4f(n)^{-1}. \tag{19}$$

By Lemma 3.1,

$$\sup_{x \in \mathcal{O}_n^c \cap \mathcal{O}} \mathrm{dist}(x, \partial\mathcal{O}) \leq l(n) \leq \frac{f(n)}{2}$$

so $\mathcal{U}_n \subseteq \partial^{f(n)}\mathcal{O}$ and so $\mathrm{vol}(\mathcal{U}_n) \to 0$.

Let $u_n := \chi_n u$. Then clearly $u_n \in H_0^1(\mathcal{O}_n)$, because $u \in H^1(\mathbb{R}^d)$ and $\chi_n \in W_0^{1,\infty}(\mathcal{O}_n)$. Firstly,

$$\|u - u_n\|_{H^1} \leq \underbrace{\|\chi_n u - u\|_{L^2}}_{(D1)} + \underbrace{\|\nabla(\chi_n u) - \nabla u\|_{L^2}}_{(D2)}.$$

Focusing on the term (D1) and using the fact that $\chi_n u = u$ outside $\mathcal{U}_n$,

$$\|\chi_n u - u\|_{L^2} = \|\chi_n u - u\|_{L^2(\mathcal{U}_n)} \leq 2\|u\|_{L^2(\mathcal{U}_n)} \to 0.$$

Focusing on the term (D2), we have

$$\|\nabla(\chi_n u) - \nabla u\|_{L^2} \leq \underbrace{\|\chi_n \nabla u - \nabla u\|_{L^2}}_{(E1)} + \underbrace{\|\nabla(\chi_n) u\|_{L^2}}_{(E2)}.$$

The term (E1) tends to zero by the same reasoning that was applied to (D1). Focusing on the term (E2), notice first that, by the assumed Poincaré-type inequality (14),

$$\|u\|_{L^2(\mathcal{U}_n)} \leq \|u\|_{L^2(\partial^{f(n)}\mathcal{O})} \leq Cf(n)\|\nabla u\|_{L^2(\partial^{\alpha f(n)}\mathcal{O})},$$

and so, by (19),

$$\|\nabla(\chi_n)u\|_{L^2} = \|\nabla(\chi_n)u\|_{L^2(\mathcal{U}_n)} \leq 4f(n)^{-1}\|u\|_{L^2(\mathcal{U}_n)} \leq 4C\|\nabla u\|_{L^2(\partial^{\alpha f(n)}\mathcal{O})} \to 0.$$

It follows that the term (D2) tends to zero hence $u_n \to u$ strongly in H^1 as required. □

Definition 3.3. An open set $\mathcal{O}$ is said to be *regular* if $\mathcal{O} = \mathrm{int}(\overline{\mathcal{O}})$.

Note that an open set $\mathcal{O}$ is regular if and only if $\mathcal{O}^c$ is the closure of an open set.

Lemma 3.4. *If $A \subset \mathbb{R}^d$ and $A_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, are bounded open sets such that $A_n \subseteq A_{n+1} \subseteq A$ for all $n \in \mathbb{N}$ and $A = \cup_{n=1}^{\infty} A_n$, then*

(a) $d_H(A_n, A) + d_H(\partial A_n, \partial A) \to 0$ *as* $n \to \infty$,

(b) $\mathrm{vol}(A_n) \to \mathrm{vol}(A)$ *as* $n \to \infty$.

Proof. Let $B_\epsilon(x) \subset \mathbb{R}^d$ denote an open ball of radius $\epsilon > 0$ about $x \in \mathbb{R}^d$.

First we look at the term

$$\mathrm{d}_H(A_n, A) = \max\{\underbrace{\sup_{x\in A} \mathrm{dist}(x, A_n)}_{(A1)}, \underbrace{\sup_{x\in A_n} \mathrm{dist}(x, A)}_{(A2)}\}.$$

(A2) vanishes for all $n \in \mathbb{N}$ since $A_n \subseteq A$. Focusing on (A1), let $\epsilon > 0$ and $x \in \overline{A}$. Fix $y \in A$ such that $|x - y| < \epsilon$. There exists $N(\epsilon, x) \in \mathbb{N}$ such that $y \in A_n$ for all $n \geq N(\epsilon, x)$ and so $\mathrm{dist}(x, A_n) < \epsilon$ for all $n \geq N(\epsilon, x)$. Since $\overline{A}$ is compact, $N(\epsilon) = \sup_{x\in A} N(\epsilon, x) < \infty$. It follows that

$$\forall n \geq N(\epsilon) : \sup_{x\in \overline{A}} \mathrm{dist}(x, A_n) < \epsilon$$

proving that $(A1) \to 0$ and hence $\mathrm{d}_H(A, A_n) \to 0$ as $n \to \infty$.

Next we look at the term

$$\mathrm{d}_H(\partial A_n, \partial A) = \max\{\underbrace{\sup_{x\in \partial A} \mathrm{dist}(x, \partial A_n)}_{(B1)}, \underbrace{\sup_{x\in \partial A_n} \mathrm{dist}(x, \partial A)}_{(B2)}\}.$$

Focusing on (B1), let $\epsilon > 0$ and $x \in \partial A$. Let $N(\epsilon, x) \in \mathbb{N}$ be large enough so that $B_\epsilon(x) \cap A_n \neq \emptyset$ for all $n \geq N(\epsilon, x)$. The ball $B_\epsilon(x)$ also intersects $A^c \subseteq A_n^c$ for all n so in fact $B_\epsilon(x)$ intersects ∂A_n for all $n \geq N(\epsilon, x)$. This proves that $\mathrm{dist}(x, \partial A_n) < \epsilon$ for all $n \geq N(\epsilon, x)$. Letting $N(\epsilon) := \sup_{x\in\partial A} N(\epsilon, x)$, we have

$$\forall n \geq N(\epsilon) : \sup_{x\in \partial A} \mathrm{dist}(x, \partial A_n) < \epsilon$$

which proves that $(B1) \to 0$ as $n \to \infty$.

Focusing on (B2), suppose for contradiction that there exists a subsequence (∂A_{n_k}) such that

$$\sup_{x\in \partial A_{n_k}} \mathrm{dist}(x, \partial A) \geq C$$

for some $C > 0$ independent of k. Then there exists $x_{n_k} \in \partial A_{n_k}$ such that $\mathrm{dist}(x_{n_k}, \partial A) \geq C$. Let $\epsilon \in (0, C)$. $B_\epsilon(x_{n_k})$ intersects $A_{n_k}^c$ for all k so there exists a sequence $y_{n_k} \in B_\epsilon(x_{n_k})$ such that $y_{n_k} \in A_{n_k}^c \cap A$ for all k. (y_{n_k}) satisfies $\mathrm{dist}(y_{n_k}, \partial A) \geq C - \epsilon > 0$ for all k. Let $y \in A_{n_k}^c \cap \overline{A}$ be an accumulation point of (y_{n_k}). Then y must satisfy $\mathrm{dist}(y, \partial A) \geq C - \epsilon$ so in fact $y \in \mathrm{int}(A)$. But the fact that $A = \cup_{n=1}^{\infty} A_n$ implies that $y \in A_n$ for large enough n, which is the desired contradiction proving that (B2)$\to 0$ as $n \to \infty$. It follows that $\mathrm{d}_H(\partial A_n, \partial A) \to 0$ as $n \to \infty$, completing the proof of (a).

(b) holds by continuity of measures from below. □

Proposition 3.5. *If $\mathcal{O} \subseteq \mathbb{R}^d$ is a regular open set with compact boundary and $\mathcal{O}_n$, $n \in \mathbb{N}$, are the pixelated domains for $\mathcal{O}$, then*

$$l(n) := d_H(\mathcal{O}, \mathcal{O}_n) + d_H(\partial\mathcal{O}, \partial\mathcal{O}_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. All limits in the proof are as $n \to \infty$.

The collection of open sets

$$\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_n \quad \text{where} \quad \mathcal{B}_n := \{j + (-\tfrac{1}{n}, \tfrac{1}{n})^d : j \in \mathbb{Z}_n^d\}.$$

Note that the elements of $\mathcal{B}_n$ are open boxes of side-length $2/n$ and hence overlap. Furthermore, $\mathcal{B}$ forms a basis for the standard topology on $\mathbb{R}^d$. Let A_n denote the union of all elements of $\mathcal{B}_n$ which are subsets $\mathcal{O}$ and let B_n denote the union of all elements of $\mathcal{B}_n$ which are subsets of $\mathrm{int}(\mathcal{O}^c)$. Then, for all $n \in \mathbb{N}$, we have $A_n \subseteq A_{n+1} \subseteq \mathcal{O}$ and $B_n \subseteq B_{n+1} \subseteq \mathrm{int}(\mathcal{O}^c)$. Furthermore, since $\mathcal{B}$ is a basis,

$$\mathcal{O} = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \mathrm{int}(\mathcal{O}^c) = \bigcup_{n=1}^{\infty} B_n. \tag{20}$$

Assume without loss of generality that $\mathcal{O}$ is bounded. We can do this because if $\mathcal{O}$ is *not* bounded, then, since $\partial\mathcal{O}$ is assumed to be compact, there exists $X > 0$ such that for all $n \in \mathbb{N}$,

$$\mathcal{O}\backslash B_X(0) = \mathcal{O}_n\backslash B_X(0) = A_n\backslash B_X(0) = B_n^c\backslash B_X(0) = \mathbb{R}^d\backslash B_X(0)$$

.

Firstly, we claim that

$$\tilde{A}_n \subseteq \mathcal{O}_n \subseteq B_n^c \tag{21}$$

where

$$\tilde{A}_n := \{x \in A_n : \mathrm{dist}_\infty(x, \partial A_n) > \tfrac{1}{2n}\}. \tag{22}$$

To see the first inclusion in (21), note that for any grid point $j \in A_n \cap \mathbb{Z}_n^d$, the corresponding cell $j + [-1/2n, 1/2n]^d$ is a subset of $\overline{\mathcal{O}}_n$ by the definition of pixelated domains and so

$$\tilde{A}_n = \mathrm{int}\left(\bigcup_{j \in A_n \cap \mathbb{Z}_n^d} (j + [-\tfrac{1}{2n}, \tfrac{1}{2n}]^d) \right) \subseteq \mathcal{O}_n.$$

To see the second inclusion in (21), note that any grid-point in $\overline{B}_n$ lies in $\mathcal{O}^c$ so

$$B_n \subseteq \bigcup_{j \in \overline{B}_n} (j + [-\tfrac{1}{2n}, \tfrac{1}{2n}]^d) \subseteq O_n^c.$$

Secondly, we have

$$\mathrm{vol}(B_n^c \backslash A_n) \to 0. \tag{23}$$

This holds because, using Lemma 3.4 (b),

$$\mathrm{vol}(B_X(0) \cap A_n) \to \mathrm{vol}(B_X(0) \cap \mathcal{O}), \quad \mathrm{vol}(B_X(0) \cap B_n) \to \mathrm{vol}(B_X(0) \cap \mathcal{O}^c)$$

and

$$\mathrm{vol}(B_n^c \backslash A_n) = \mathrm{vol}(B_X(0)) - \mathrm{vol}(B_X(0) \cap A_n) - \mathrm{vol}(B_X(0) \cap B_n).$$

Next we claim that

$$\sup_{x \in \partial O_n} \mathrm{dist}(x, \partial A_n \cup \partial B_n) \to 0. \tag{24}$$

This can be seen by considering an expanding ball around any point in $B_n^c \backslash A_n$. More precisely, for any $x \in B_n \backslash A_n$, define the quantity

$$r(x) := \inf\{r > 0 : B_r(x) \cap (\partial A_n \cup \partial B_n) \neq \emptyset\}.$$

For all $x \in B_n^c \backslash A_n$ we have

$$\mathrm{dist}(x, \partial A_n \cap \partial B_n) = r(x) \quad \text{and} \quad B_{r(x)}(x) \subseteq B_n^c \backslash A_n.$$

The set $B_n^c \backslash A_n$ is compact so we can define

$$r_n := \sup_{x \in B_n^c \backslash A_n} r(x) < \infty$$

and there exists $x_n \in B_n^c \backslash A_n$ such that $r_n = r(x_n)$. The fact that $B_{r_n}(x_n) \subseteq B_n^c \backslash A_n$ combined with the limit (23) yields

$$\mathrm{vol}(B_{r_n}(x_n)) \leq \mathrm{vol}(B_n^c \backslash A_n) \to 0$$

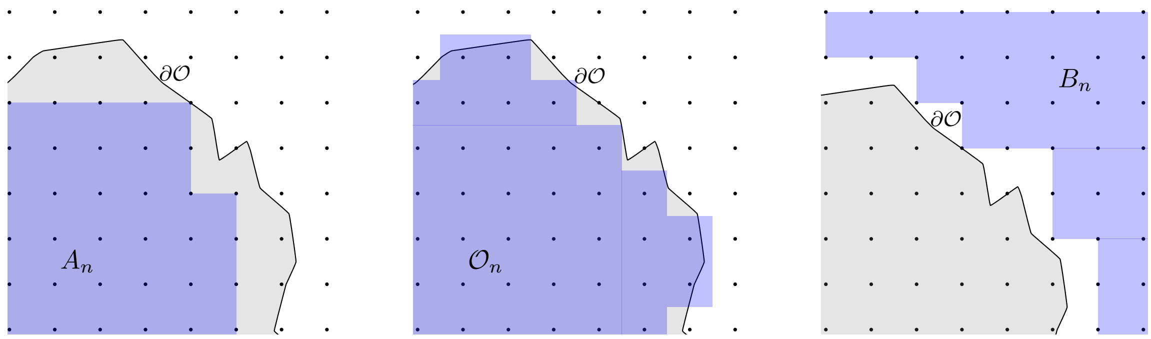


FIGURE 3. Sketch of the domains A_n (left), $\mathcal{O}_n$ (centre) and B_n (right).

which implies that $r_n \to 0$. (24) is obtained by applying the inclusions (21) to get

$$\begin{aligned}\sup_{x\in\partial\mathcal{O}_n} \operatorname{dist}(x,\partial A_n\cup\partial B_n) &\le \sup_{x\in B_n^c\setminus\tilde{A}_n} \operatorname{dist}(x,\partial A_n\cup\partial B_n)\\ &\le \max\left\{\frac{1}{\sqrt{2}n}, \sup_{x\in B_n^c\setminus A_n} \operatorname{dist}(x,\partial A_n\cup\partial B_n)\right\}\\ &= \max\left\{\frac{1}{\sqrt{2}n}, r_n\right\} \to 0,\end{aligned}$$

where the second inequality holds because any point in $A_n\setminus\tilde{A}_n$ is at most a distance $1/\sqrt{2}n$ from ∂A_n.

By the properties of A_n and B_n^c, Lemma 3.4 (a) can be applied to (A_n) and (B_n^c) to get,

$$\mathrm{d}_H(\mathcal{O},A_n)+\mathrm{d}_H(\partial\mathcal{O},\partial A_n)\to 0 \tag{25}$$

and

$$\mathrm{d}_H(\mathcal{O},B_n^c)+\mathrm{d}_H(\partial\mathcal{O},\partial B_n)\to 0. \tag{26}$$

Crucially, the hypothesis that $\mathcal{O}$ is regular was used in (27), to ensure that $\partial\mathrm{int}(\mathcal{O}^c)=\partial\mathcal{O}$. It follows from (25) and (26) that

$$\mathrm{d}_H(\partial\mathcal{O},\partial A_n\cup\partial B_n)\to 0 \tag{27}$$

and

$$\mathrm{d}_H(A_n,B_n^c)+\mathrm{d}_H(\partial A_n,\partial B_n)\to 0. \tag{28}$$

Furthermore, we claim that

$$\sup_{x\in\partial A_n\cup\partial B_n} \operatorname{dist}(x,\partial\mathcal{O}_n)\to 0. \tag{29}$$

This can be seen by considering length minimising lines between ∂A_n and ∂B_n. More precisely, let $x\in\partial B_n$ and let $y=y(x)\in\partial\tilde{A}_n$ such that $|x-y|=\operatorname{dist}(x,\partial\tilde{A}_n)$. The straight line connecting x and y must intersect $\partial\mathcal{O}_n$ since one end is in $\mathcal{O}_n$ and the other is in $\mathcal{O}_n^c$. This fact implies that

$$\begin{aligned}\sup_{x\in\partial B_n} \operatorname{dist}(x,\partial\mathcal{O}_n) &\le \sup_{x\in\partial B_n} \operatorname{dist}(x,\partial\tilde{A}_n)\\ &\le \sup_{x\in\partial B_n} \operatorname{dist}(x,\partial A_n)+\mathrm{d}_H(\partial A_n,\partial\tilde{A}_n)\to 0\end{aligned}$$

where limit holds by (28) and the definition of $\tilde{A}_n$. It can be similarly seen that

$$\sup_{x\in\partial A_n} \operatorname{dist}(x,\partial\mathcal{O}_n)\to 0.$$

giving us (29)

The limits (24) and (29) prove that $\mathrm{d}_H(\partial\mathcal{O}_n,\partial A_n\cup\partial B_n)\to 0$ which, combined with (27), gives $\mathrm{d}_H(\partial\mathcal{O},\partial\mathcal{O}_n)\to 0$. Finally, using the inclusions (21), the limit (27) and the definition of $\tilde{A}_n$, we have

$$\mathrm{d}_H(B_n^c,\mathcal{O}_n)=\sup_{x\in B_n^c}\operatorname{dist}(x,\mathcal{O}_n)\le\sup_{x\in B_n^c}\operatorname{dist}(x,\tilde{A}_n)\to 0$$

which, in combination with (26), shows that $\mathrm{d}_H(\mathcal{O},\mathcal{O}_n)\to 0$, completing the proof.

□

In what follows, note that the complement of a bounded set in $\mathbb{R}^d$ has a unique unbounded connected component.

Definition 3.6. The *outer boundary component* $\partial^{\mathrm{out}}A$ of a bounded, connected set $A\subset\mathbb{R}^d$ is defined as the boundary $\partial\Gamma$ of the unique unbounded component Γ of $\overline{A^c}$.

Definition 3.7. A *path* γ in $A\subseteq\mathbb{R}^d$ from $x_0\in A$ to $x_1\in A$ refers to a set $\gamma^*([0,1])$, where $\gamma^*:[0,1]\to A$ is a continuous map such that $\gamma^*(0)=x_0$ and $\gamma^*(1)=x_1$.

Definition 3.8. For any set $A\subseteq\mathbb{R}^d$ and $r>0$, we define the set $\mathrm{dil}_r(A)$ by

$$\mathrm{dil}_r(A):=\{x\in\mathbb{R}^d:\operatorname{dist}(x,A)<r\}.$$

Lemma 3.9. *If $A\subset\mathbb{R}^d$ is a bounded, connected, regular open set such that $\mathrm{int}(A^c)$ is connected, then*

$$\sup_{x\in\partial A}\operatorname{dist}(x,\partial^{\mathrm{out}}\,\mathrm{dil}_\epsilon(A))\to 0 \quad \textit{as} \quad \epsilon\to 0.$$

Proof. Let $x \in \partial A$. By regularity, A^c is the closure of $\mathrm{int}(A^c)$ so there exists a sequence $(x_n) \subset \mathrm{int}(A^c)$ with $x_n \to x$. We claim that for each n, there exists $\epsilon_n > 0$ such that x_n lies in the unbounded component of $\mathrm{int}(\mathrm{dil}_\epsilon(A)^c)$ for all $\epsilon \in (0, \epsilon_n]$. To see this, first note that, since $\mathrm{int}(A^c)$ is connected and A is bounded, there exists an unbounded, connected open set V such that $\overline{V} \subset \mathrm{int}(A^c)$ and $x_n \in V$. The claim follows from the fact that V is a subset of $\mathrm{int}(\mathrm{dil}_\epsilon(A)^c)$ for small enough ϵ.

Without loss of generality, assume that $\epsilon_{n+1} < \epsilon_n$ for all n. For each $\epsilon \in (\epsilon_{n-1}, \epsilon_n]$, x lies in $\mathrm{dil}_\epsilon(A)$ and x_n lies in the unbounded component of $\mathrm{int}(\mathrm{dil}_\epsilon(A)^c)$, so,

$$\delta_x(\epsilon) := \mathrm{dist}(x, \partial^{\mathrm{out}} \mathrm{dil}_\epsilon(A)) \leq |x - x_n|.$$

Since ϵ_n tends monotonically to zero as $n \to \infty$, we have $\delta_x(\epsilon) \to 0$ as $\epsilon \to 0$. $\delta_x(\epsilon)$ is equal to the distance from x to the unbounded component of $\mathrm{dil}_\epsilon(A)^c$. Since the latter set is nested for decreasing $\epsilon > 0$, $\delta_x(\epsilon)$ in fact tends to zero monotonically as $\epsilon \to 0$. Finally, ∂A is compact and $\delta_x(\epsilon)$ is continuous in x so an application of Dini's theorem yields

$$\delta(\epsilon) := \sup_{x \in \partial A} \delta_x(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0,$$

as required. □

Proposition 3.10. *Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded, regular, connected open set such that $\mathrm{int}(\mathcal{O}^c)$ is connected. Suppose that $\mathcal{O}_n$, $n \in \mathbb{N}$, are the pixelated domains for $\mathcal{O}$. Then there exists a sequence $(\epsilon(n))_{n\in\mathbb{N}}$ with $\epsilon(n) \to 0$ as $n \to \infty$ and a connected components γ_n of $\partial\mathcal{O}_n$ for each sufficiently large n, such that*

$$\mathrm{diam}(\gamma_n) \geq \mathrm{diam}(\mathcal{O}) - 2\epsilon(n) \tag{30}$$

and

$$\sup_{x \in \partial\mathcal{O}_n} \mathrm{dist}(x, \gamma_n) \leq 2\epsilon(n). \tag{31}$$

Proof. Let $l(n)$ be defined as in Proposition 3.2 and Proposition 3.5.

For each $k \in \{1, ..., M\}$, let $\tilde{F}_n := \mathrm{int}(\mathcal{O}) \backslash \partial^{l(n)}\mathcal{O})$. Then, by the definition of $l(n)$, $\tilde{F}_n$ does not intersect $\partial\mathcal{O}_n$ hence we must have $\tilde{F}_n \subseteq \mathcal{O}_n$. Fix $x_0 \in \mathcal{O}$. For all n large enough so that $x_0 \in \tilde{F}_n$, define F_n as the set of points x for which there exists a path γ in $\tilde{F}_n$ from x_0 to x. It follows immediately F_n is open, bounded, path connected and

$$\forall n : F_n \subseteq F_{n+1} \subseteq \mathcal{O}_{n+1} \cap \mathcal{O}.$$

Furthermore, since $\mathcal{O}$ is path connected and any path in $\mathcal{O}$ lies in $\tilde{F}_n$ for large enough n, we have that

$$\mathcal{O} = \bigcup_{n=1}^{\infty} F_n.$$

By Lemma 3.4,

$$\epsilon_1(n) := \mathrm{d}_H(\mathcal{O}, F_n) + \mathrm{d}_H(\partial\mathcal{O}, \partial F_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{32}$$

Let $\mathcal{O}_{n,0}$ denote the connected component of $\mathcal{O}_n$ such that $F_n \subseteq \mathcal{O}_{n,0}$. We shall define

$$\gamma_n := \partial^{\mathrm{out}}\mathcal{O}_{n,0}$$

for all n large enough so that F_n is defined. Since F_n is contained in $\mathcal{O}_n$ and by the definition of $\epsilon_1(n)$, we have

$$\mathrm{diam}(\partial^{\mathrm{out}}\mathcal{O}_{n,0}) \geq \mathrm{diam}(F_n) \geq \mathrm{diam}(\mathcal{O}) - 2\epsilon_1(n). \tag{33}$$

Let

$$\epsilon_2(n) := \sup_{x \in \partial\mathcal{O}} \mathrm{dist}(x, \partial^{\mathrm{out}} \mathrm{dil}_{l(n)}(\mathcal{O})). \tag{34}$$

By Lemma 3.9, we have $\epsilon_2(n) \to 0$ as $n \to \infty$.

We claim that

$$\sup_{x \in \partial\mathcal{O}} \mathrm{dist}(x, \partial^{\mathrm{out}}\mathcal{O}_{n,0}) \leq \max\{\epsilon_1(n), \epsilon_2(n)\}. \tag{35}$$

To see this, let us fix $x \in \partial\mathcal{O}$. By the definition of $\epsilon_1(n)$, there exists $y_1 \in \partial F_n$ such that $|y_1 - x| \leq \epsilon_1(n)$. By the definition of $\epsilon_2(n)$, there exists $y_2 \in \partial^{\mathrm{out}} \mathrm{dil}_{l(n)}(\mathcal{O})$ such that $|y_2 - x| \leq \epsilon_2(n)$. y_1 lies in $\overline{\mathcal{O}}_{n,0}$ and, since $\mathrm{dil}_{l(n)}(\mathcal{O})$ contains $\mathcal{O}_{n,0}$, y_2 lies in the closure of the unbounded connected component of $\mathrm{int}((\mathcal{O}_{n,0})^c)$. Consequently, a path γ from y_1 to y_2, consisting of the union of a straight line from y_1 to x and a straight line y_2 to x, must intersect $\partial^{\mathrm{out}}\mathcal{O}_{n,0}$. Inequality (35) follows from the fact that any point $y \in \gamma$ satisfies $|y - x| \leq \max\{\epsilon_1(n), \epsilon_2(n)\}$.

Define

$$\epsilon(n) := \max\{\epsilon_1(n), \epsilon_2(n)\}. \tag{36}$$

Finally, (30) follows from (33) and (31) follows from (35), noting that

$$l(n) \leq \epsilon(n) \tag{37}$$

and that

$$\sup_{x\in\partial\mathcal{O}_n} \operatorname{dist}(x, \partial^{\mathrm{out}}\mathcal{O}_{n,0}) \leq l(n) + \sup_{x\in\partial\mathcal{O}} \operatorname{dist}(x, \partial^{\mathrm{out}}\mathcal{O}_{n,0}).$$

□

Theorem 3.11. *Suppose that $\mathcal{O} \subseteq \mathbb{R}^2$ is a connected, bounded, regular open set such that $\operatorname{vol}(\partial\mathcal{O}) = 0$, $\partial\mathcal{O}$ is path connected and $\operatorname{int}(\mathcal{O}^c)$ is connected. Then the pixelated domains $(\mathcal{O}_n)_{n\in\mathbb{N}}$ for $\mathcal{O}$ converge to $\mathcal{O}$ in the Mosco sense as $n \to \infty$.*

Proof. Since $\partial\mathcal{O}$ is compact, the boundaries $\partial\mathcal{O}_n$ of the pixelated domains are compact for all $n \in \mathbb{N}$. Hypothesis (1) of Proposition 3.2 holds by an application of Proposition 3.5 and hypothesis (3) holds by assumption hence it suffices to show that hypothesis (2) holds.

Let $f(n) := 2\epsilon(n)$ where $\epsilon(n)$ is defined as in Proposition 3.10. Then, $f(n) \to 0$ as $n \to \infty$ and, by (37), $2l(n) \leq f(n)$ for all $n \in \mathbb{N}$, where $l(n)$ is defined as in Proposition 3.2. The fact that $\mathcal{O}$ is a regular open set such that $\partial\mathcal{O}$ path-connected implies that $Q(\partial\mathcal{O}) > 0$ and hence that $4\sqrt{2}f(n) < Q(\partial\mathcal{O})$ for all sufficiently large n. The Poincaré-type inequality Theorem 2.14 therefore implies that the hypothesis (14) of Proposition 3.2 holds for $V = \mathcal{O}$ with $C = 5$, $\alpha = 1 + \sqrt{2}$ and all sufficiently large n.

It remains to show that

$$\|u\|_{L^2(\partial^{f(n)}\mathcal{O}_n)} \leq 10 f(n) \|\nabla u\|_{L^2(\partial^{2(1+\sqrt{2})f(n)}\mathcal{O}_n)} \tag{38}$$

for all sufficiently large n and all $u \in H^1_0(\mathcal{O}_n)$, which would establish that the hypothesis (14) of Proposition 3.2 holds for $V = \mathcal{O}_n$ with $C = 10$ and $\alpha = 2(1 + \sqrt{2})$. For large enough n let γ_n denote the connected component of $\partial\mathcal{O}_n$ that was constructed in Proposition 3.10. Let $\epsilon(n)$, $n \in \mathbb{N}$, be as in Proposition 3.10. Let

$$\mathcal{V}_n := \mathbb{R}^2 \backslash \gamma_n.$$

For large enough n, inequality (31) of Proposition 3.10 gives us

$$\partial^{4\epsilon(n)}\mathcal{V}_n = \operatorname{dil}_{4\epsilon(n)}(\partial\Gamma_n) \supseteq \partial^{2\epsilon(n)}\mathcal{O}_n$$

so, in particular, $H^1_0(\partial^{2\epsilon(n)}\mathcal{O}_n) \subseteq H^1_0(\partial^{4\epsilon(n)}\mathcal{V}_n)$. By inequality (30) and the fact that $\epsilon(n) \to 0$ as $n \to \infty$, we have that $Q(\partial\mathcal{V}_n) > 8\sqrt{2}\epsilon(n)$ for large enough n. Hence, we can apply the Poincaré-type inequality Theorem 2.14 to establish that

$$\begin{aligned} \|u\|_{L^2(\partial^{2\epsilon(n)}\mathcal{O}_n)} \leq \|u\|_{L^2(\partial^{4\epsilon(n)}\mathcal{V}_n)} &\leq 20\epsilon(n)\|\nabla u\|_{L^2(\partial^{4(1+\sqrt{2})\epsilon(n)}\mathcal{V}_n)} \\ &= 20\epsilon(n)\|\nabla u\|_{L^2(\partial^{4(1+\sqrt{2})\epsilon(n)}\mathcal{O}_n)} \end{aligned} \tag{39}$$

for all $u \in H^1_0(\mathcal{O}_n)$ and large enough n, proving (38).

□

4. Arithmetic algorithms for the spectral problem

4.1. **Algorithm.** First, we show that there exists a family of arithmetic algorithms capable of solving the *computational matrix pencil eigenvalue problem* to arbitrary specified precision. Let

$$\Omega_{\mathrm{mat},M} := \{(A, B) \in (\mathbb{R}^{M\times M})^2 : \ A, B \text{ symmetric}, B \text{ positive definite}\},$$

$$\Lambda_{\mathrm{mat},M} := \{(A, B) \mapsto A_{j,k} : A_{j,k} \text{ matrix element of } A\} \cup \{(A, B) \mapsto B_{j,k} : B_{j,k} \text{ matrix element of } B\}.$$

and

$$\Lambda^{\epsilon}_{\mathrm{mat},M} := \Lambda_{\mathrm{mat},M} \cup \{(A, B) \mapsto \epsilon\}.$$

Applying a-posteriori bounds of Oishi [23] for the matrix eigenvalue problem the following gives the following family of arithmetic algorithms:

Lemma 4.1. *For each $\epsilon > 0$, there exists an arithmetic algorithm $\Gamma^{\epsilon}_{\mathrm{mat},M} : \Omega_{\mathrm{mat},M} \to \mathbb{R}^M$ with input $\Lambda^{\epsilon}_{\mathrm{mat},M}$, such that*

$$|\Gamma^{\epsilon}_{\mathrm{mat},M}(A, B)_k - \lambda_k| \leq \epsilon \quad \textit{for all} \quad (A, B) \in \Omega_{\mathrm{mat},M} \quad \textit{and} \quad k \in \{1, ..., M\},$$

where λ_k denotes the eigenvalues of the matrix pencil (A, B).

Proof. Since $\Lambda^{\epsilon}_{\mathrm{mat},M}$ is a finite set, we can define the information available to the (arithmetic) algorithm $\Gamma^{\epsilon}_{\mathrm{mat},M}$ as $\Lambda_{\Gamma^{\epsilon}_{\mathrm{mat},M}} = \Lambda^{\epsilon}_{\mathrm{mat},M}$. By Gaussian elimination, the matrix elements of B^{-1} can be computed with a finite number of arithmetic operations. Then the eigenvalues of the matrix pencil (A, B) are exactly the eigenvalues $(\lambda_k)_{k=1}^M$ of $E := B^{-1}A$ and the matrix elements of E are accessible to the algorithm.

By the Jacobi eigenvalue algorithm (cf. [26]) there exists a family of approximations $(\tilde{\lambda}_k^m, \tilde{x}_k^m) \in \mathbb{R} \times \mathbb{R}^M$, $k \in \{1, ..., M\}$, $m \in \mathbb{N}$, such that $(\tilde{\lambda}_k^m, \tilde{x}_k^m)$ can be computed with finitely many arithmetic operations and

$$\|P_m^T D_m P_m - E\|_F + \|P_m^T P_m - I\|_F \to 0 \quad \text{as} \quad m \to \infty \tag{40}$$

where

$$D_m := \operatorname{diag}(\tilde{\lambda}_1^m, ..., \tilde{\lambda}_M^m) \quad \text{and} \quad P_m := (\tilde{x}_1^m, ..., \tilde{x}_M^m).$$

Here, $\|\cdot\|_F$ denotes the Frobenius matrix norm. Note that is equivalent to the statement that $(\tilde{\lambda}_k^m)_{k=1}^M$ tends to $(\lambda_k)_{k=1}^M$ as $m \to \infty$ and $(\tilde{x}_k^m)_{k=1}^M$ tends to the corresponding orthonormal basis of eigenvectors. By [23, Theorem 2],

$$|\lambda_k - \tilde{\lambda}_k^m| \le |\tilde{\lambda}_k^m| \|P_m^T P_m - I\|_F + \|P_m^T D_m P_m - E\|_F =: \mathcal{E}_k(m)$$

for all $k \in \{1, ..., M\}$. Let $m(\epsilon)$ denote the smallest positive integer such that $\mathcal{E}_k(m(\epsilon)) \le \epsilon$ for all $k \in \{1, ..., M\}$. $m(\epsilon)$ can be determined by the algorithm since, for each m, $\mathcal{E}_k(m)$ can be computed in finitely many arithmetic operations. The proof is completed by letting

$$\Gamma^\epsilon_{\mathrm{mat},M}(A, B) := (\tilde{\lambda}_k^{m(\epsilon)})_{k=1}^M.$$

□

Next, we show that there exists a family of arithmetic algorithms capable of computing, to arbitrary specified precision, the spectrum of the Dirichlet Laplacian on domains of the form

$$\mathcal{U} = \operatorname{int}\left(\bigcup_{j=1}^N (x_j + [-\frac{1}{2n}, \frac{1}{2n}]^2)\right) \tag{41}$$

with $n, N \in \mathbb{N}$ and $(x_1, ..., x_N) \in (\mathbb{Z}_n^2)^N$. Let

$$\Omega_{\mathrm{pix},n} := \{\mathcal{U} \subset \mathbb{R}^2 : \exists N \in \mathbb{N} \text{ and } (x_1, ..., x_N) \in (\mathbb{Z}_n^2)^N \text{ such that (41) holds}\},$$

$$\Lambda_{\mathrm{pix},n} := \{\mathcal{U} \mapsto \chi_{\mathcal{U}}(x) : x \in \mathbb{Z}_n^2\}.$$

and

$$\Lambda^\epsilon_{\mathrm{pix},n} := \Lambda_{\mathrm{pix},n} \cup \{\mathcal{U} \mapsto N(\mathcal{U}) := |\mathcal{U} \cap \mathbb{Z}_n^2|\} \cup \{\mathcal{U} \mapsto \epsilon\}. \tag{42}$$

The results of Liu and Oishi [20] combined with Lemma 4.1 yield the following:

Lemma 4.2. *For each $\epsilon > 0$, there exists an arithmetic algorithm $\Gamma^\epsilon_{\mathrm{pix},n} : \Omega_{\mathrm{pix}} \to \mathrm{cl}(\mathbb{C})$ with input $\Lambda_{\mathrm{pix},n}$, such that*

$$\mathrm{d_{AW}}\left(\Gamma^\epsilon_{\mathrm{pix},n}(\mathcal{U}), \sigma(\mathcal{U})\right) \le \epsilon \quad \textit{for all} \quad \mathcal{U} \in \Omega_{\mathrm{pix},n}.$$

Proof. First, we fix a finite subset $\Lambda_{\Gamma^\epsilon_{\mathrm{pix},n}}(\mathcal{U}) \subset \Lambda^\epsilon_{\mathrm{pix},n}$ which the defines the information available to the (arithmetic) algorithm $\Gamma^\epsilon_{\mathrm{pix},n}$, which we aim to construct. Choose

$$\Lambda_{\Gamma^\epsilon_{\mathrm{pix},n}}(\mathcal{U}) := \{\mathcal{U} \mapsto \chi_{\mathcal{U}}(x) : x \in \mathbb{Z}_n^2 \cap \overline{B}_{\kappa(\mathcal{U})}(0)\} \cup \{\mathcal{U} \mapsto N(\mathcal{U})\} \cup \{\mathcal{U} \mapsto \epsilon\},$$

where $\kappa(\mathcal{U})$ is defined as the smallest positive integer such that $|\overline{B}_{\kappa(\mathcal{U})}(0) \cap \mathcal{U} \cap \mathbb{Z}_n^2| = N(\mathcal{U})$. $\kappa(\mathcal{U})$ can be computed with a finite number of arithmetic computations on subsets of the finite collection of complex numbers $\{f(\mathcal{U}) : f \in \Lambda_{\Gamma^\epsilon_{\mathrm{pix},n}}(\mathcal{U})\}$, ensuring that the consistency hypothesis Definition 1.2 (ii) holds. With this choice, the algorithm has access to the list $(x_1, ..., x_N) \in (\mathbb{R}^2)^N$ for which (41) holds.

Using this list, we can construct, for each $m \in \mathbb{N}$, a uniform triangulation $\mathcal{T}^m$ of $\mathcal{U}$ such that the elements of $\mathcal{T}^m$ have diameter $1/nm$. Let $V^m \subset H_0^1(\mathcal{U})$ denote the piecewise-linear continuous finite element space for the triangulation $\mathcal{T}^m$. Let $\{\phi_k^m\}_{m=1}^{M_0}$ denote the basis of 'hat' functions for V^m, where $M_0 := \dim(V^m)$. Let $\{\lambda_k\}_{k=1}^\infty$ denote the eigenvalues of the operator $\mathcal{U}$, ordered such that $\lambda_k \le \lambda_{k+1}$ for all $k \in \mathbb{N}$.

The Ritz-Galerkin finite element approximations for $\{\lambda_k\}_{k=1}^\infty$ are the eigenvalues $\{\lambda_k^m\}_{k=1}^{M_0}$, $m \in \mathbb{N}$, of the matrix pencil (A^m, B^m), where the matrix elements of the matrices A^m and B^m read

$$A^m_{j,k} := \langle \nabla\phi_j^m, \nabla\phi_k^m \rangle_{L^2(\mathcal{U})} \quad \text{and} \quad B^m_{j,k} := \langle \phi_j^m,\ \phi_k^m \rangle_{L^2(\mathcal{U})} \qquad (j, k \in \{1, ..., M_0\})$$

respectively. These matrix elements can be computed from the information $(x_1, ..., x_N)$, n and m with a finite number of arithmetic computations. Note that A and B are symmetric and B is positive definite.

In [20], the authors introduce a quantity

$$q^m := l^m + (C_0/mn)^2,$$

where,

- $C_0 > 0$ can be bounded above by an explicit expression [20, Section 2] and,

- l^m is the maximum eigenvalue of a matrix pencil (D^m, E^m) [20, eq. (3.22)]. Here, the matrix elements of D^m and E^m are explicitly constructed from inner products between overlapping basis functions of piecewise-linear finite element spaces on $\mathcal{T}^m$ and hence can be computed with a finite number of arithmetic computations on $(x_1, ..., x_N)$, n and m. Also, E^m is diagonal and positive definite.

By [20, Remark 3.3], it holds that $q^m \to 0 \quad \text{as} \quad m \to \infty$. [20, Theorem 4.3] states that, for each $m \in \mathbb{N}$ and each $k \in \{1, ..., M_0\}$, if $q^m \lambda_k^m < 1$, then

$$\lambda_k^m/(1 + \lambda_k^m q^m) \leq \lambda_k \leq \lambda_k^m. \tag{43}$$

Since the matrix elements of A^m, B^m, D^m and E^m are available to the algorithm $\Gamma^\epsilon_{\mathrm{pix},n}$, by Lemma 4.1, the approximations

$$\lambda_k^{m,\delta} := \Gamma^\delta_{\mathrm{mat},M_0}(A^m, B^m)_k \qquad \text{and} \qquad q^{m,\delta} := \Gamma^\delta_{\mathrm{mat},M_0}(D^m, E^m)_{M_0}$$

are also available to $\Gamma^\epsilon_{\mathrm{pix},n}$, for any $\delta > 0$. These approximations provide upper and lower bounds for λ_k^m and q^m

$$\lambda_k^m \in [\lambda_k^{m,\delta} - \delta, \lambda_k^{m,\delta} + \delta], \qquad q^m \in [q^{m,\delta} - \delta, q^{m,\delta} + \delta]. \tag{44}$$

We claim that if $(M, m, \delta) \in \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+$ is such that $(\lambda_M^{m,\delta} + \delta)(q^{m,\delta} + \delta) < 1$, then

$$\mathrm{d_{AW}}\left(\{\lambda_k^{m,\delta}\}_{k=1}^M, \sigma(\mathcal{U})\right) \leq \delta + \mathcal{E}_1(M, m, \delta) + \mathcal{E}_2(M, m, \delta) \tag{45}$$

where

$$\mathcal{E}_1(M, m, \delta) := (q^{m,\delta} + \delta)(\lambda_M^{m,\delta} + \delta)^2/(1 + (q^{m,\delta} - \delta)_+(\lambda_M^{m,\delta} - \delta)_+)$$

and

$$\mathcal{E}_2(M, m, \delta) := 2^{-(\lambda_M^{m,\delta} - \delta)/2+1}.$$

To see this,first note that, using the formula

$$\mathrm{d_{AW}}(A, B) \leq \mathrm{d}_H(A, B) \qquad (A, B \subset \mathbb{C} \text{ bounded}),$$

we have

$$\begin{aligned}
\mathrm{d_{AW}}\left(\{\lambda_k^{m,\delta}\}_{k=1}^M, \sigma(\mathcal{U})\right) &\leq \mathrm{d}_H(\{\lambda_k^{m,\delta}\}_{k=1}^M, \{\lambda_k^m\}_{k=1}^M) + \mathrm{d_{AW}}\left(\{\lambda_k^m\}_{k=1}^M, \sigma(\mathcal{U})\right) \\
&\leq \delta + \sum_{j=1}^{\lfloor \lambda_M \rfloor} 2^{-j} \min\left\{1, \sup_{|x| \leq j} \left|\mathrm{dist}\big(x, \{\lambda_k^m\}_{k=1}^M\big) - \mathrm{dist}(x, \{\lambda_k\}_{k=1}^\infty)\right|\right\} + \sum_{j=\lceil \lambda_M \rceil}^\infty 2^{-j}.
\end{aligned} \tag{46}$$

Since $\mathrm{dist}(x, \{\lambda_k\}_{k=1}^\infty) = \mathrm{dist}(x, \{\lambda_k\}_{k=1}^M)$ for $|x| \leq \lfloor \lambda_M \rfloor$, the second term on the right hand side of (46) is bounded by

$$\sum_{j=1}^{\lfloor \lambda_M \rfloor} 2^{-j} \mathrm{d}_H(\{\lambda_k^m\}_{k=1}^M, \{\lambda_k\}_{k=1}^M) \leq \max\{|\lambda_k - \lambda_k^m| : k \in \{1, ..., M\}\} \leq \mathcal{E}_1(\delta, M, m)$$

where the final inequality follows from (43) and (44). Applying (43) and (44) again and noting that the condition $(\lambda_M^{m,\delta} + \delta)(q^{m,\delta} + \delta) < 1$ ensures that $\lambda_M^m q^m \leq 1$, we have

$$\sum_{j=\lceil \lambda_M \rceil}^\infty 2^{-j} \leq 2^{-\lambda_M + 1} \leq 2^{-\lambda_M^m/2+1} \leq \mathcal{E}_2(M, \delta, m),$$

bounding the third term on the right hand side of (46) and proving (45).

Let $\delta(M) := 1/M$. Define $m(M)$ as the smallest $m \in \mathbb{N}$ such that

$$(q^{m,\delta(M)} + \delta(M))(\lambda_M^{m,\delta(M)} + \delta(M))^2 \leq 1/M.$$

For each $M \in \mathbb{N}$, $m(M)$ can be determined by the algorithm since it can compute the quantities $q^{m,\delta(M)}$ and $\lambda_M^{m,\delta(M)}$. Define $M(\epsilon)$ as the smallest positive integer such that

$$\delta \circ M(\epsilon) + \mathcal{E}_1(M(\epsilon), m \circ M(\epsilon), \delta \circ M(\epsilon)) + \mathcal{E}_2(M(\epsilon), m \circ M(\epsilon), \delta \circ M(\epsilon)) \leq \epsilon.$$

For each $\epsilon > 0$, $M(\epsilon)$ can be determined by the algorithm since $\mathcal{E}_1$ and $\mathcal{E}_2$ can be computed. The proof of the lemma is completed by letting

$$\Gamma^\epsilon_{\mathrm{pix},n}(\mathcal{U}) := \{\lambda_k^{m \circ M(\epsilon), \delta \circ M(\epsilon)}\}_{k=1}^{M(\epsilon)}. \tag{47}$$

□

Remark 4.3. The results of [20] are formulated for connected domains only. While this assumption is not necessarily satisfied for domains in $\Omega_{\text{pix},n}$, the results from [20] can be applied to every connected component of a set $\mathcal{U} \in \Omega_{\text{pix},n}$ separately. This is justified, because the Dirichlet spectrum of $\mathcal{U}$ is simply the union of the Dirichlet spectra of all connected components of $\mathcal{U}$. Moreover, any $\mathcal{U} \in \Omega_{\text{pix},n}$ consists of only finitely many connected components, which can be determined in a finite number of steps from the information given in $\Lambda_{\text{pix},n}$.

Finally, we employ Mosco convergence for pixelated domains to prove the main result of the section.

Proposition 4.4. *Let*

$$\Omega_M := \Big\{\mathcal{O} \subset \mathbb{R}^2 : \mathcal{O} \text{ open, bounded and } \mathcal{O}_n \xrightarrow{\text{M}} \mathcal{O} \text{ where } \mathcal{O}_n \text{ pixelated domains for } \mathcal{O}\Big\}.$$

Then there exists a sequence of arithmetic algorithms $\Gamma_n : \Omega_M \to \mathrm{cl}(\mathbb{C})$, $n \in \mathbb{N}$, with input Λ_0 such that

$$\mathrm{d_{AW}}\left(\Gamma_n(\mathcal{O}), \sigma(\mathcal{O})\right) \to 0 \quad \textit{as} \quad n \to \infty \qquad \textit{for all} \qquad \mathcal{O} \in \Omega_M.$$

Proof. Let $\mathcal{O} \in \Omega_M$. Let $\mathcal{O}_n \subset \mathbb{R}^2$, $n \in \mathbb{N}$, denote the corresponding pixelated domains (cf. Definition 1.4).

We shall construct a family of (arithmetic) algorithms $\Gamma_n : \Omega_1 \to \mathrm{cl}(\mathbb{C})$ with input Λ_0. First, we define the information available to each algorithm Γ_n, by fixing a finite subset $\Lambda_{\Gamma_n} \subset \Lambda_0$. Let

$$\Lambda_{\Gamma_n} := \big\{\mathcal{O} \mapsto \chi_{\mathcal{O}}(x) : x \in \mathbb{Z}_n^2 \text{ with } |x| \le n\big\},$$

so that the algorithm Γ_n has access to the set $\{x_1, ..., x_N\} := \mathcal{O} \cap B_n(0) \cap \mathbb{Z}_n^2$.

Let

$$\tilde{\mathcal{O}}_n := \mathrm{int}\left(\bigcup_{j=1}^{N} (x_j + [-\frac{1}{2n}, \frac{1}{2n}]^2)\right).$$

It holds that $\tilde{\mathcal{O}}_n \in \Omega_{\text{pix},n}$ for each n and, since $\mathcal{O}$ is bounded, $\tilde{\mathcal{O}}_n = \mathcal{O}_n$ for all sufficiently large n. Hence using the hypothesis that $\mathcal{O}_n \xrightarrow{\text{M}} \mathcal{O}$, $\tilde{\mathcal{O}}_n$ converges to $\mathcal{O}$ in the Mosco sense as $n \to \infty$. Therefore, since $\tilde{\mathcal{O}}_n$ and $\mathcal{O}$ are bounded, the self-adjoint operators $-\Delta_{\tilde{\mathcal{O}}_n}$ converge to $-\Delta_{\mathcal{O}}$ in the norm-resolvent sense as $n \to \infty$ and so [12, Theorem 5.2.6][18, Chapter IV],

$$\mathrm{d_{AW}}\left(\sigma(\tilde{\mathcal{O}}_n), \sigma(\mathcal{O})\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $\tilde{O}_n$ is defined entirely by Λ_{Γ_n}, we can define

$$\Gamma_n(\mathcal{O}) = \Gamma^{1/n}_{\text{pix},n}(\tilde{O}_n)$$

for each $n \in \mathbb{N}$. Note that the consistency property Definition 1.2 (ii) holds trivially since Λ_{Γ_n} does not depend on $\mathcal{O}$. The theorem is proved by the fact that

$$\begin{aligned}\mathrm{d_{AW}}\left(\Gamma^{1/n}_{\text{pix},n}(\tilde{\mathcal{O}}_n), \sigma(\mathcal{O})\right) &\le \mathrm{d_{AW}}\left(\Gamma^{1/n}_{\text{pix},n}(\tilde{\mathcal{O}}_n), \sigma(\tilde{\mathcal{O}}_n)\right) + \mathrm{d_{AW}}\left(\sigma(\tilde{\mathcal{O}}_n), \sigma(\mathcal{O})\right)\\ &\le \frac{1}{n} + o(1) \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

□

4.2. **Counter-example.** Proposition 1.5 follows immediately from the following result.

Proposition 4.5. *Let $\Gamma_n : \Omega_0 \to \mathcal{M}$, $n \in \mathbb{N}$, be any family of arithmetic algorithms with input Λ_0. Then, for any $\mathcal{O} \in \Omega_0$ and any $\epsilon > 0$, there exists $\mathcal{O}_\epsilon \in \Omega_0$ with $\mathcal{O}_\epsilon \subseteq \mathcal{O}$ and $\mathrm{vol}(\mathcal{O}_\epsilon) \le \epsilon$ such that $\Gamma_n(\mathcal{O}) = \Gamma_n(\mathcal{O}_\epsilon)$ for all n and, for sufficiently small $\epsilon > 0$, $\sigma(\mathcal{O}) \neq \sigma(\mathcal{O}_\epsilon)$.*

Proof. Let Γ_n be as hypothesised, let $\epsilon > 0$ and let $\mathcal{O} \in \Omega_0$. Define the geometric quantity

$$r_{\text{int}}(\mathcal{O}) := \sup\{r > 0 \,:\, \exists \text{square } [s, s + r] \times [t, t + r] \subset \mathcal{O}\}.$$

Openness of $\mathcal{O}$ implies that $r_{\text{int}}(\mathcal{O}) > 0$. For any fixed n, $\Gamma_n(\mathcal{O})$ depends only on finitely many $\chi_{\mathcal{O}}(x)$, say $x_1^n, \dots, x_{k_n}^n$. We assume without loss of generality that the set $\{x_1^n, \dots, x_{k_n}^n\}$ is growing with n, i.e. that $\{x_1^n, \dots, x_{k_n}^n\} \subset \{x_1^{n+1}, \dots, x_{k_{n+1}}^{n+1}\}$ for all n. Thus, we may drop the superscript n and merely write $\{x_1, \dots, x_{k_n}\}$. Let us denote by $\{y_1, \dots, y_{l_n}\}$ the subset of points for which $\chi_{\mathcal{O}}(y_i) = 1$. Now, define a new domain $\mathcal{O}_\epsilon$ as follows. For $t > 0$ define the strips $S_t^k := \big(\big((y_k)_1 - \frac{t}{2},\, (y_k)_1 + \frac{t}{2}\big) \times \mathbb{R}\big) \cap \mathcal{O}$. Next, let

$$\mathcal{O}_\epsilon^n := \bigcup_{k=1}^{l_n} S^k_{2^{-k}\epsilon} \quad \text{and} \quad \mathcal{O}_\epsilon := \left(\bigcup_{n=1}^{\infty} \mathcal{O}_\epsilon^n\right) \cup \partial^\epsilon \mathcal{O}$$

where, recall that $\partial^\epsilon \mathcal{O} = \{x \in \mathcal{O} \,:\, \mathrm{dist}(x, \partial\mathcal{O}) < \epsilon\}$. Note that $\mathcal{O}_\epsilon$ is bounded, open and connected for any $\epsilon > 0$. One has $\chi_{\mathcal{O}}(x_k) = \chi_{\mathcal{O}_\epsilon}(x_k)$ for all $k \in \{1, \dots, k_n\}$ for all $n \in \mathbb{N}$, and therefore, by consistency of algorithms (cf. Definition 1.2 (ii)), $\Gamma_n(\mathcal{O}_\epsilon) = \Gamma_n(\mathcal{O})$ for all $n \in \mathbb{N}$.

However, it is easily seen from the min-max principle that for the lowest eigenvalue $\lambda_1(\mathcal{O})$, of $-\Delta_{\mathcal{O}}$, one has

$$\lambda_1(\mathcal{O}) \leq \frac{\pi^2}{r_{\text{int}}(\mathcal{O})^2},$$

Next, we use Poincaré's inequality [16, eq. (7.44)] to get

$$\|u\|_{L^2(\mathcal{O}_\epsilon)} \leq C\text{vol}(\mathcal{O}_\epsilon)^{\frac{1}{2}}\|\nabla u\|_{L^2(\mathcal{O}_\epsilon)} \leq C\left(\text{vol}(\partial^\epsilon\mathcal{O}) + \sum_{k=1}^\infty 2^{-k}\epsilon\,\text{diam}(\mathcal{O})\right)^{\frac{1}{2}}\|\nabla u\|_{L^2(\mathcal{O}_\epsilon)}$$

for some constant $C > 0$ independent of ϵ. Since $\text{vol}(\partial^\epsilon\mathcal{O}) \to 0$ as $\epsilon \to 0$, we conclude using the min-max principle that $\lambda_1(\mathcal{O}_\epsilon) \to \infty$ as $\epsilon \to 0$ and hence $\sigma(\mathcal{O}) \neq \sigma(\mathcal{O}_\epsilon)$ for small enough $\epsilon > 0$. □

Remark 4.6. The counterexample in the proof of Proposition 4.5 is pathological in the sense that the complement of the domain $\mathcal{O}_\epsilon$ has infinitely many connected components. This is not crucial. Indeed, one can easily construct a counterexample whose complement has only one connected component: Let $\mathcal{O} = (0,1)^2$ be the unit square and

$$\mathcal{O}_\epsilon := \left(\bigcup_{n=1}^\infty \bigcup_{k=1}^{k_n} S^k_{2^{-k}\epsilon}\right) \cup \big((0,1)\times(0,\epsilon)\big),$$

with the notation from the previous proof. Then $\Gamma_n(\mathcal{O}_\epsilon) = \Gamma_n(\mathcal{O})$ for all n and $\sigma((0,1)^2) \neq \sigma(\mathcal{O}_\epsilon)$ for sufficiently small $\epsilon > 0$ while $\mathcal{O}_\epsilon^c$ is connected.

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