Eigenvalues of Schrödinger operators perturbed by dissipative barriers

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Introduction

Consider the Schrödinger operator

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endowed with a Dirichlet boundary condition at 0, where:

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Spectral inclusion

We have (as $R \to \infty$)

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Theorem 1 ([1])

(a) *For any eigenvalue λ of H_∞, there exists eigenvalues λ_R of H_R and constants $C_0, \beta, R_0 > 0$ such that*

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(b) *If $\exists \varepsilon > 0 : \int e^{\varepsilon t}|q(t)|\,\mathrm{d}t < \infty$, then for (almost[1]) any $\mu \in \sigma_{\mathrm{ess}}(H_\infty)$ there exists eigenvalues λ_R of H_R and constants $C_0, R_0 > 0$ such that*

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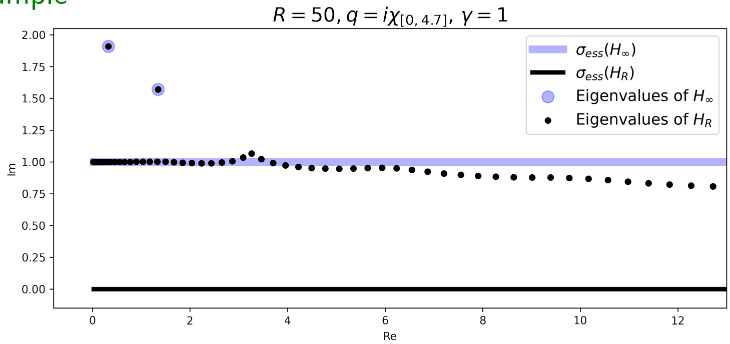

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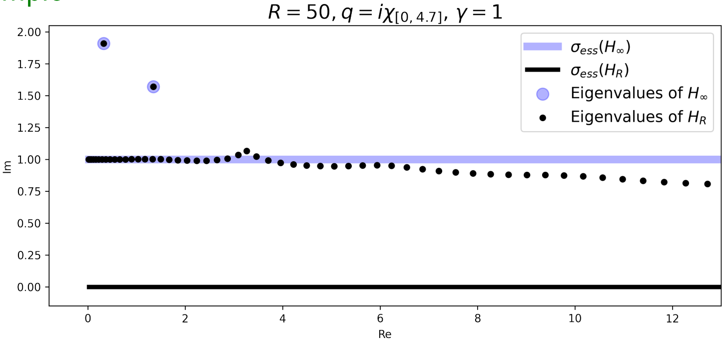

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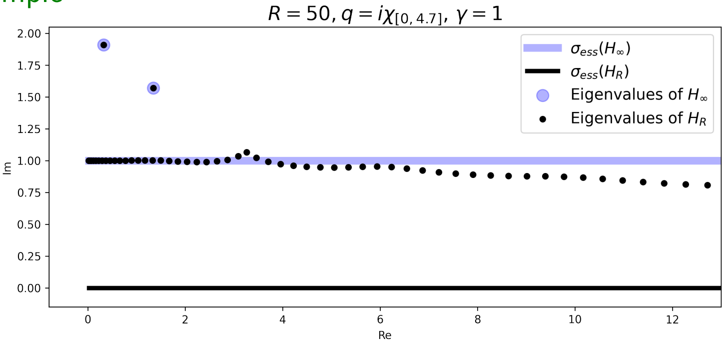

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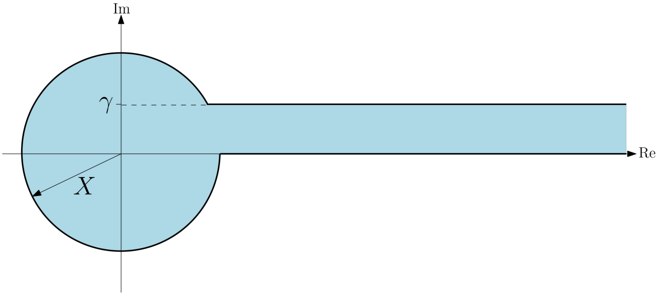
- Enclosures for the eigenvalues of H_R.
- Estimates for the number of eigenvalues $N(H_R)$.

Enclosures

Theorem 2 ([2])

(a) *$\exists$ constant $X = X(q,\gamma) > 0$ such that $\forall R > 0$ the eigenvalues of H_R lie in*

$$\Gamma := B_X(0) \cup ([0,\infty) + i[0,\gamma]).$$



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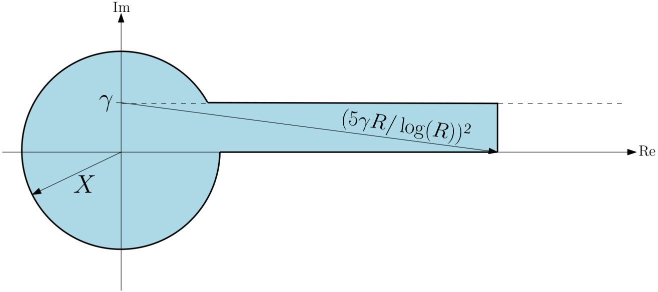

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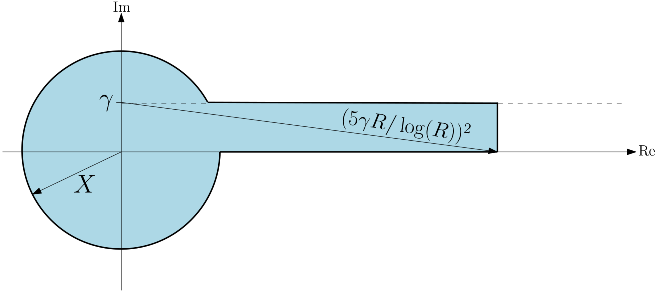
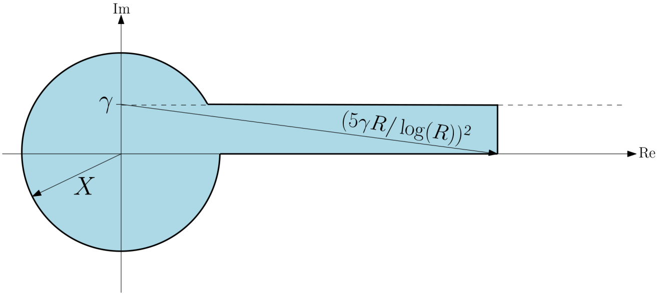
- To compare to (b), application of a *sharp* enclosure of Frank, Laptev and Seiringer (2011) gives $\sqrt{|\lambda_R|} = O(R)$.





Ideas in proof of Theorem 2.

1. **Use large-$|\lambda|$ Levinson asymptotics:** Solutions $\psi_\pm(\cdot,\lambda)$ to $-\psi'' + q\psi = \lambda\psi$ such that

$$\psi_\pm(x,\lambda) = e^{\pm i\sqrt{\lambda}x}(1 + E_\pm(x,\lambda))$$

where

$$|E_\pm(x,\lambda)| = O\left(\frac{1}{\sqrt{|\lambda|}}\right) \quad \text{as} \quad |\lambda| \to \infty.$$

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2. Construct analytic function f_R such that

$$\lambda \text{ eigenvalue of } H_R \quad \Longleftrightarrow \quad f_R(\lambda) = 0.$$

f_R has form

$$\begin{aligned} f_R(\lambda) = \psi_-(0,\lambda - i\gamma)\left(\sqrt{\lambda} - \sqrt{\lambda - i\gamma} + \mathcal{E}_1(R,\lambda)\right) e^{i\sqrt{\lambda - i\gamma}R} \\ - \psi_+(0,\lambda - i\gamma)\left(\sqrt{\lambda} + \sqrt{\lambda - i\gamma} + \mathcal{E}_2(R,\lambda)\right) e^{-i\sqrt{\lambda - i\gamma}R} \end{aligned}$$

where $|\mathcal{E}_1(R,\lambda)| + |\mathcal{E}_2(R,\lambda)| \leqslant C(q,\gamma)$.

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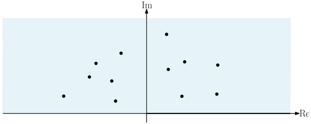
$$g_R(z) = 0 \iff z^2 \text{ eigval of } H_R.$$



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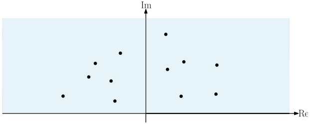

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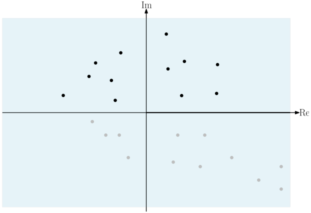

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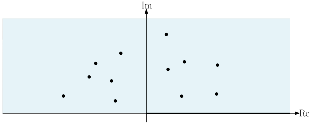
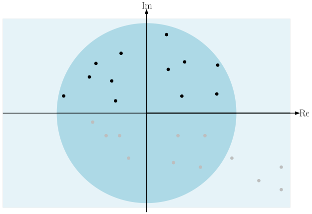


2. If q compactly supported, g_R admits analytic continuation to $\mathbb{C}$.
   - Apply Jensen's formula to bound $N(H_R)$ and prove Th. 3 (a).



Ideas in proof of Theorem 3. (cont.)

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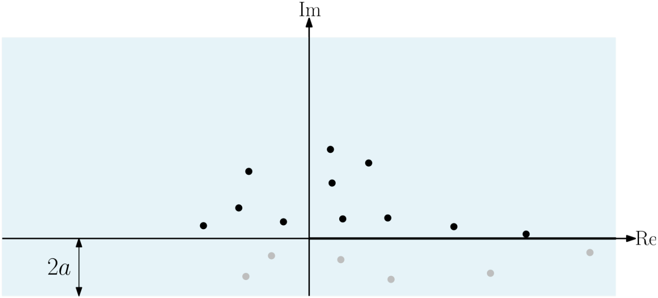
   $\Rightarrow$ g_R admits analytic continuation to $\{\Im z > -2a\}$.



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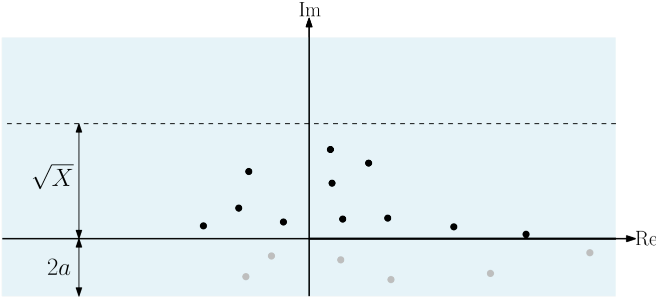

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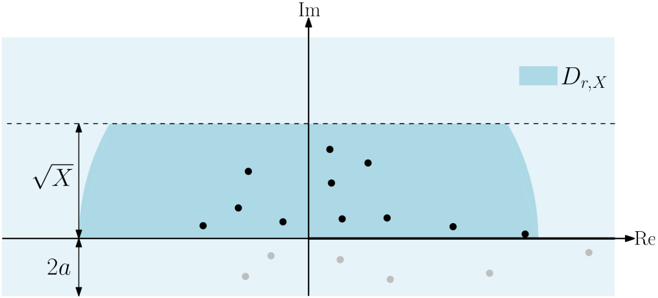

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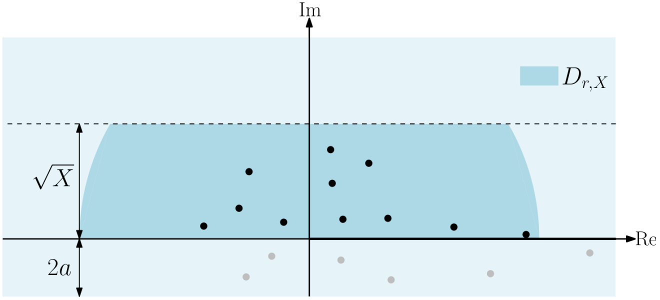
   - Apply this bound to g_R with $r = O(R/\log R)$.



Thanks for listening!

References

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Appendix

Proposition

Suppose that f is an analytic function defined on an open neighbourhood of the closed semi-disc $D_r := \overline{B}_r(0) \cap \overline{\mathbb{C}}_+$ for some $r > 0$. Let α and β be any numbers in the interval $(0,1)$ satisfying

$$\beta\left(\frac{1-\alpha}{\alpha+\beta}\right)^2 > \frac{Y}{\eta} \tag{1}$$

and let $N(\alpha r)$ denote the number of zeros in the region

$$D_{\alpha r,\eta,Y} := \{z \in \mathbb{C} : \eta \leqslant \Im z \leqslant Y,\ |z| \leqslant \alpha r\} \tag{2}$$

where $Y, \eta > 0$ are given parameters satisfying $\eta < Y < r$. Then,

$$N(\alpha r) \leqslant \frac{2}{\log \Lambda(r)} \log\left(\frac{1}{\min\{\beta, 1-\beta\}} \frac{\sup_{z\in\partial D_r} |f(z)|}{|f(i\beta r)|}\right) \tag{3}$$

where

$$\Lambda(r) := \frac{1 + \frac{4\beta\eta}{(\alpha+\beta)^2}\frac{1}{r}}{1 + \frac{4Y}{(1-\alpha)^2}\frac{1}{r}}. \tag{4}$$

Remark

One can always guarantee that condition (1) for α and β is satisfied by choosing, for instance,

$$\alpha = \beta = \frac{1}{4}\frac{\eta}{2Y+\eta}. \tag{5}$$