Periodic Orbits And Time Averages Of Observables (Phys3001 Project 2)

Alexei Stepanenko u6124616

May 28, 2017

Abstract

We introduce techniques to study time averages of dynamical observables. We then show that for certian kinds of systems, paricularly chaotic systems, these averages can be expressed in terms of periodic orbits.

1 Introduction

In physics, as well as in many other fields, many systems exhibit chaotic behavior. Chaotic behavior is characterized by an exponential sensitivity to initial conditions of trajectories while the trajectories still remaining bounded. Chaotic systems are non-linear and have no analytical solutions. These characteristics makes the long-time accurate prediction of chaotic systems impossible, there are always uncertainties created by, for example, limitations of experimental measurement or computer error. In this report, we study some techniques that can help us form predictions about the long time behavior of chaotic systems effectively. These techniques originated from an area of mathematics called *ergodic theory,* introduced originally by H. Poincaré. The basic philosophy is to study how entire chunks of phase space evolve instead of studying individual trajectories. Although individual trajectories, clearly do not behave in a regular manner, it turns out that if we wait long enough the percentage of time that the trajectory visits any certain area of phase tends to a constant value. This is evident in the strange attractors of chaotic systems. These values are encoded in the *natural invariant density.* As we shall see, using the natural invariant density we can calculate averages of dynamical observables in terms of the periodic orbits of a system..

2 Measure and densities

A dynamical system is defined by a state space $\mathcal{M}$, a time parameter t and an time evolution function f^t. Then given a state $x_0 \in \mathcal{M}$ at some time t_0, we can find the state at time $t + t_0$ by applying the time evolution function, $x(t+t_0) = f^t(x(t_0)) = f^t(x_0)$. The state space and time parameter can be either continuous or discrete, however, we assume that the state space is finite dimensional. An example of a dynamical system is the movements of N particles. Here the state space is $\mathcal{M} = \mathbb{R}^{6N}$ because for each particle we have to assign 3 position co-ordinates and 3 momenta coordinates. The time evolution function for the N particles is defined by the equations of motion which can be derived from classical mechanics and can include any sort of interaction (e.g. gravitational or electromagnetic) between the particles. Note that since the equations of motion for most systems are non-linear, most of the time we cannot obtain an analytical expression for f^t.

As mentioned in the introduction, for many systems studying individual trajectories can be an ineffective way of determining the behavior of a dynamical system, particularly when the dynamics are chaotic. To illustrate this, consider that in an experimental or engineering situation, there is always an uncertainty in our knowledge of initial conditions. In a chaotic system two trajectories initially separated by some distance less than than our uncertainty will eventually separate to a distance that is comparable to the size of the accessible phase space. Therefore, long time prediction is impossible in a chaotic system. What is possible (at least in principle) however is to estimate average dynamical quantity's - such as the average position, average time spent in a certain area of phase space, average distance moved in a given time etc. One way to estimate these quantities is to run many long trajectory's on the computer and estimate them directly. We explore a different approach here. In this approach, we no longer thing of individual states and trajectories but a continuum of states evolving in time according to the dynamics of the system.

Formally, we introduce a *measure* μ on the phase space. The measure assigns to any subset of the the state space a real number. The measure for an infinitesimal subset of the state space can be expressed as,

$$d\mu(x) = \rho(x)dx.$$

$\rho(x)$ is known as the *density.* Then the measure of a subset $\mathcal{S} \subset \mathcal{M}$ is,

$$\mu(\mathcal{S}) = \int_{\mathcal{S}} \rho(x)dx. \tag{1}$$

For example, a density $\rho(x) = \delta(x - y)$ corresponds to a single trajectory at y. Another example is a density of states equally distributed in some subset $\mathcal{S} \subset \mathcal{M}$. In this case the density function is given by $\rho(x) = \frac{1}{|\mathcal{S}|}\mathcal{X}_{\mathcal{S}}(x)$ where $|\mathcal{S}| := \int_{\mathcal{S}} dx$ is the volume of the the subset $\mathcal{S}$ and $\mathcal{X}_{\mathcal{S}}(x)$ is the characteristic function on $\mathcal{S}$ defined to be 1 on $\mathcal{S}$ and 0 otherwise. We always assume that the measure of the entire phase space can be normalized, so that,

$$\mu(\mathcal{M}) = \int_{\mathcal{M}} \rho(x)dx = 1. \tag{2}$$

Given an initial density in phase space $\rho(x, 0)$, the density at a later time is given by the Perron-Frobenius equation[3],

$$\rho(x, t) = \int_{\mathcal{M}} \delta(x - f^t(y))\rho(y)dy. \tag{3}$$

Intuitively, what the Perron-Frobenius says is that to obtain the density at time t, we take each point x of the density at time 0 to $f^t(x)$.

In order to relate the concepts of measure and density to dynamical averages we need to consider the dynamics of densities in the limit of $t \to \infty$. This motivates the definition of a natural invariant density. Given an initial statex_0, the natural density is defined as,

$$\rho_{x_0}(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \delta(x - f^\tau(x_0))d\tau.$$

Natural densities are special because they are invariant under the action of f^t, therefore are an *invariant density.*
.

3 Dynamical averages and ergodicity

In this section we discuss *dynamical averages,* the average value of dynamical quantities such as the ones discussed earlier and their connection to measures and densities. A dynamical quantity $a(x)$ can be any function of the state variables. Given an initial condition the *time average* of a dynamical quantity $a(x)$ is the average value of the dynamical variable along that trajectory in the limit $t \to \infty$,

$$\overline{a}_{x_0} = \lim_{t\to\infty} \frac{1}{t}\int_0^t d\tau a(f^\tau(x_0)). \tag{4}$$

Given a density $\rho(x)$ we can also define the *space average* with respect to ρ of a dynamical quantity as the average value of the dynamical quantity in space weighted with the value of the density

$$\langle a\rangle_\rho = \int_{\mathcal{M}} \rho(x)a(x)dx.$$

Ergodicity provides a condition for when we can connect these two types averages. A system is called *ergodic* if the time average of any observable (4) is constant with respect to the initial condition *almost everywhere.* Almost everywhere means that if T is the set of points for which the the time average $\overline{a}_{x_0}$ is not constant, then, T has zero volume, i.e $\int_T dx = 0$. The pointwise ergodic theorem [1] states that if a system is ergodic, then the time average for an observable of almost every trajectory is equal to the space average with respect to the natural density. So for an ergodic system, in order to determine the time average of a

To make the definition of ergodicity clearer let's consider examples of ergodic and non-ergodic systems.

First, consider a damped 1-D particle in a double well oscillator. The state space is $(x, p) \in \mathbb{R}^2$ where x is the position the and p is the momentum. The system evolves according to the equations,

$$\begin{aligned} \dot{x} &= p/m \\ \dot{p} &= -x - x^3 - kp, \end{aligned}$$

where $k > 0$ is the damping constant and m is the mass. The double well potential has three minima, one unstable minima at $x = 0$ and two stable minima x_+ and x_- (where $x_+ > 0$ and $x_- < 0$) corresponding to the bottom of the two wells. If the system starts with an initial condition (1,0) then it will fall into $x_+$and of the system starts with an initial condition (-1,0) the system will fall into x_-. Clearly the two trajectories will have different values for time average of dynamical quantities in general (for example the average position will be different), so the system in not ergodic.

An example of an ergodic system is the Lorenz system described by

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z. \end{aligned}$$

The Lorenz system is a chaotic system, nearby trajectories exponentially diverge and almost every initial condition (where almost every has the meaning defined earlier) will fall into a *strange attractor.* The strange attractor shown in figure 1 is a fractal set. Since almost all trajectories converge to the strange attractor, the strange attractor is the natural density. The amount of time any trajectories spends in any given region corresponds to the density of the strange attractor in that region, hence, the time average of any trajectory will be constant. Therefore the Lorenz system is ergodic. In general, chaotic systems are ergodic. If a system is chaotic, it is generally seen as undesirable in the sense that they are hard to analyse and control, however, ergodicity provides special properties that non-chaotic stems do not have. One of these special properties is *recurrence*; for any subset of the state space for which the natural more is non-zero,

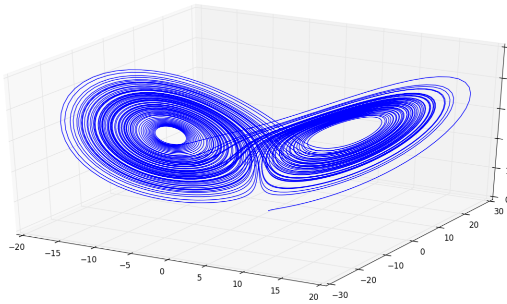

Figure 1: Lorenz attractor

a general trajectory will, given enough time, enter that region.

4 Unstable Periodic Orbits

For the Lorenz system, and in general for a chaotic system, within a strange attractor are an infinite number of *unstable periodic orbits*. These unstable To study these unstable periodic orbits, an indispensable tool is the Poincare section. Poincare sections are a mathematical tool that allows us to convert any continuous time dynamical system into an equivalent discrete map. Formally, a Poincare section is a hyper-surface of the state space. The map, called the *Poincare return map* is constructed by sampling the state every time the trajectory intersects the Poincare map. A periodic orbit then becomes a fixed point of some iterate of the Poincare map, vastly simplifying their analysis.

Suppose that we have found the location of a periodic orbit, corresponding to a fixed point of some iterate of the Poincare return map. The stability of the periodic orbit then corresponds to the stability of the corresponding fixed point in some iterate of the Poincare return map. To determine the stability of the fixed point in the map, we evaluate the Jacobian at the fixed point (this can be done by numerically evaluating the derivatives to the map with respect to the state variable). Then the eigenvalues of the Jacobian, called the Floquet multipliers, determine the stability of the fixed point. The eigenvalues are stable if their absolute value is less than 1 and unstable if their absolute value is greater than one. If the Jacobian evaluated at the fixed point has even one unstable eigenvalue then the fixed point is unstable. A set of unstable periodic orbits is called *hyperbolic* if it contains both stable and unstable Floquet multipliers. In general, the set of periodic orbits for a chaotic system will be hyperbolic. Also, a hyperbolic map is one whose fixed points have both stable and unstable eigenvalues. The Poincare return map of the Lorenz map, for example, is a two-dimensional hyperbolic map.

Now, let's see how unstable periodic orbits are linked with natural measures and hence dynamical averages. Let M^n be our equivalent of the time evolution function for a map, such that for any x_0 on the Poincare section $M^n(x_0) = M_n$. Suppose that the map is hyperbolic and strongly chaotic, the natural measure of a subset S is approximated by [9]

$$\mu(S) \approx \lim_{n\to\infty} \sum_{x_{jn}\in S\cap \mathrm{Fix} M^n} \frac{1}{|\Lambda(x_{jn})|}, \tag{5}$$

where x_{jn} are fixed points of M^n and $\Lambda(x_{jn})$ is the product of the unstable eigenvalues of the fixed point x_{jn}.

Now let's heuristically justify equation (5). The natural measure of a subset S is proportional to the probability that a point in S stays in S. Consider the an initial density $\rho_0(x) = \mathcal{X}_S(x)$ uniformly distributed over the subset S. Then, by the Perron-Frobenius equation, after n applications of the map, S is mapped to

$$\rho_n(x) = \int_S \delta(x - M^n(y))dy.$$

The volume of $\rho_n(x)$ that remains in S is given by $N_n := \int_S \rho_n(x)dx$. The volume of S is given by $N_0 = \int_S dz$. Then, the volume of points that stay in S after n applications of the map is given by $\Gamma_n := N_n/N_0$. The natural measure of a subset S is equal to the probability that a point in S stays in S in the limit of $n \to \infty$, so,

$$\mu(S) = \lim_{n\to\infty} \Gamma_n. \tag{6}$$

Now let's define a new quantity

$$A_n = \int_S \delta(x - M^n(x))dx. \tag{7}$$

In [8], it is argued and demonstrated numerically that for hyperbolic maps

$$\lim_{n\to\infty} A_n \approx \lim_{n\to\infty} \Gamma_n. \tag{8}$$

The argument in [8], is based off the Perron-Frobenious operator $\mathcal{L}^n(y,x) = \delta(x - M^n(y))$, which generates the Perron-Frobenius equation ($\rho_n(x) = \int dy \mathcal{L}^n(y,x)\rho_0(y)$), it's eigenfunctions (invariant densities such as natural densities) and eigenvalues. Here, we justify (7) in a simpler way. $\lim_{n\to\infty} A_n$ essentially is the volume of the points than tend to fixed points in S of any iterate of M^n. This is the set of the stable manifolds of the fixed points, $\{x \in S : \lim_{n\to\infty} M^n(x) = x_{jn} \text{ for some } j, n\}$. Because the map is assumed to be hyperbolic and chaotic, any points not in this set will be repelled from S and will wander chaotically around $\mathcal{M}$, returning to S only for finite time periods in the future, hence giving us equation (8).

Then, using (6) and (8),

$$\mu(S) \approx \lim_{n\to\infty} \int_S \delta(x - M^n(x))dx. \tag{9}$$

Using the identity $\int \delta(h(x))dx = \sum_{x\in \mathrm{Fix} h} \frac{1}{|\det h'(x)|}$, (9) becomes,

$$\mu(S) \approx \lim_{n\to\infty} \sum_{x_{jn}\in S\cap \mathrm{Fix} M^n} \frac{1}{|\det(\mathbf{1} - DM^n(x_{jn}))|}, \quad (10)$$

where DM^n is the Jacobian of M^n and $\mathbf{1}$ is the identity matrix (the dimension of these square matrices is equal to the dimension of $\mathcal{M}$). Using the characteristic polynomial we can expand the summand of (10),

$$\begin{aligned} |\det(\mathbf{1} - DM^n)|^{-1} &= |\prod_{i,j}(1-\Lambda_i^n)(1-\lambda_j^n)|^{-1} \\ &= \prod_i |1-\Lambda_i^n|^{-1} \prod_j |1-\lambda_j^n|^{-1}, \end{aligned} \quad (11)$$

where Λ_i are the unstable eigenvalues of M and λ_i are the stable eigenvalues of M. For strongly chaotic systems the magnitudes of Λ_i are much greater than one and the magnitude of λ_j are only slightly less than one, so, $|1-\Lambda_i^n|^{-1} \approx |\Lambda_i^n|^{-1}$ and $|1-\lambda_j^n|^{-1} \approx 1$, especially for large n. Therefore, $|\det(\mathbf{1}-DM^n)|^{-1} \approx \prod_i |\Lambda_i^n|^{-1} = |\Lambda_u|$, with the approximation becoming better for large n. Intuitively, what this approximation is saying is that for a strongly chaotic system, the proportion of points in a neighbourhood of a fixed point returning to that neighbourhood is proportional to the strength of the expanding direction $|\Lambda_u|$. Substituting into (10), we obtain (10), we obtain (5).

5 Conclusion

We have shown that dynamical averages of chaotic systems can be expressed in terms of the natural measure of the system and that the natural measure in turn can be expressed in terms of the Floquet multipliers of unstable periodic orbits of the system. For highly chaotic system, this 'cycle expansion' takes a particular simple approximate form (5). If the system is chaotic is chaotic but not strongly chaotic, a better approximatio for the natural measure can be found by substituting (11) into (10).

These cycle expansion can be used as a stepping stone to general theory of controlling chaos. Chaotic systems can be controlled using very small perturbations. This is due to the recurrence phenomena that chaotic system exhibit due to their ergodicity. Given enough time the state of the system will come close to the linear neighbourhood of a target unstable periodic orbit, at which point we can linearise the equations of motion and perform a small perturbation an accessible parameter in such a way to set the unstable eigenvalues to the UPO to a value of our choice. In the OGY method of control [10], we set the unstable eigenvalues of target UPO to zero, hence stabilising that UPO causing the state to follow the UPO. We observe that we can manipulate the eigenvalues of any UPO using small perturbations opening up the possibility of other types of control, such as causing the state to stay in one area of phase space [11].

The results shown in this report can be generalised to obtain highly convergent, exact expansions for dynamical quantities in terms of periodic orbits [4][2]. Recent work has also started to develop approximate methods for obtain the natural measure of stochastic systems [5][7]. This recent direction of research is particularly interesting because of the potential to apply these methods to chaotic semi-classical approximations of open quantum systems (semi-classical in the sense of the system being near the classical limit), undergoing noise due to interaction with the environment. An example of such a system is analysed in [6], where they found that parameters related to the measurement of the the system had an effect of the Lyapunov exponent of the system. Such results open the door for the possibility to develop algorithms extending OGY to semiclassical systems.

References

[1] Yves Coudene. *Ergodic Theory and Dynamic Systems*. Springer, 2016.

[2] P Cvitanovic. Dynamical averaging in terms of periodic orbits. *Physica D*, 1995.

[3] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay. *Chaos: Classical and Quantum*. Niels Bohr Institute, 2016.

[4] P Cvitanovic and B Eckhardt. Periodic orbit expansions for classical smooth flows. *Journal of Physics A*, 1991.

[5] Predrag Cvitanović and Domenico Lippolis. Knowing when to stop: How noise frees us from determinism. *AIP Conference Proceedings*, 2012.

[6] Jessica Eastman, Andre Carvhalo, and Joe Hope. Tuning quantum measurements to control chaos. *Nature*, 2017.

[7] Jeffrey Heninger, Domenico Lippolis, and Predrag Cvitanović. Neighborhoods of periodic orbits and the stationary distribution of a noisy chaotic system. *Phys. Rev. E*, 2015.

[8] Leo Kandanoff and Chao Tang. Escape from strange repellers. *Proceedings of the National Academy of Sciences of the United States of America*, 81(4), 1984.

[9] E. Ott. *Chaos in Dynamical Systems*. Cambridge Univ. Press, 1993.

[10] Edward Ott, Celso Grebogi, and James Yorke. Controlling chaos. *Physical Review Letters*, 1990.

[11] Troy Shinbrot, Edward Ott, Celso Grebogi, and James Yorke. Using chaos to direct trajectories to targets. *Physical Review Letters*, 1990.