CLASSICAL WHITE NOISE IN A QUANTUM HAMILTONIAN

ALEXEI STEPANENKO

Abstract. A generalisation for the Bloch vector to arbitrary dimensions is suggested. We use this co-ordinate chart to derive the Stochastic master equation describing a quantum system disturbed by classical white-noise in Ito form. The ensemble average master equation for the system is found to be in Linblad form. The filtering SME for the such a system undergoing homodyne measurement is written.

1. Generalised Bloch Vector

It is well-known that the density matrix ρ of a two-level quantum system can be expressed in terms of the co-ordinates of the Bloch vector $\mathbf{q} = (x, y, z)$,

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}. \tag{1.1}$$

Such an expression can be interpreted as a chart from the state-space $\mathcal{S}(\mathcal{H})$ for the two level system to a closed subset of $\mathbb{R}^3$. We label the chart $\iota : \mathcal{S}(\mathcal{H}) \to \mathbb{R}^3$, so that, $\iota(\rho) = \mathbf{q}$.

The Bloch vector has the desirable property that the 'state of no knowledge' $\phi = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}) \in \mathcal{S}(\mathcal{H})$ corresponds to the origin of $\mathbb{R}^3$ (the center of the Bloch Ball),

$$\iota(\phi) = \mathbf{0}. \tag{1.2}$$

Also, the purity $p := \mathbf{tr}\rho^2$ is related to the length of the Bloch vector in a simple way,

$$p = \frac{1}{2}(1 + |\mathbf{q}|^2), \tag{1.3}$$

so that ρ is pure if and only if $|\iota(\rho)|^2 = 1$.

In this section, we present (proof is omitted to a different document) a generalisation of the Bloch vector for density matrices of N-level (open) quantum systems. By this we simply mean that we find a global co-ordinate chart for the smooth manifold such that (1.2) & (1.3) hold and that the chart reduces to (1.1). It is not yet clear whether this generalisation is unique.

Proposition. *(Generalised Bloch Vector)*

Define

$$C := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

Let I denote the $N \times N$ identity matrix. Define,

$$A := \sqrt{C+I}^{-1}.$$

Our Generalised Bloch Vector and the associated chart ι is defined by,

$$\rho = \begin{pmatrix} \frac{1}{N} + (A\mathbf{z})_1 & \omega_{\varphi^{-1}(1)} & \cdots & \omega_{\varphi^{-1}(n-1)} \\ & \ddots & \ddots & \vdots \\ & & \frac{1}{N} + (A\mathbf{z})_{N-1} & \omega_{\varphi^{-1}(d_{xy})} \\ & H.C. & & \frac{1}{N} - \sum_{j=1}^{N-1} (A\mathbf{z})_j \end{pmatrix},$$

$$\iota(\rho) := [\mathbf{x}, \mathbf{y}, \mathbf{z}]$$

where

$$\begin{aligned} \omega_{\varphi^{-1}(j)} &= \frac{1}{2}(x_{\varphi^{-1}(j)} + i y_{\varphi^{-1}(j)}) \\ \varphi(j,k) &= \sum_{m=1}^{j-1} (N-m) + (k-j) \\ d_{xy} &= \frac{1}{2} N(N-1). \end{aligned}$$

It follows that,

$$\begin{aligned} \mathrm{tr}\rho^2 &= \frac{1}{N} + |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2, \\ \iota(\frac{1}{N} I) &= \mathbf{0}, \\ d = \dim \mathcal{S}(\mathcal{H}) &= (N+1)(N-1). \end{aligned}$$

2. White Noise Hamiltonian

Consider a Quantum system with Hamiltonian,

$$H = H_0 + H_1 \xi \tag{2.1}$$

where H_0 and H_1 are Hermitian and ξ is a Gaussian, white-noise process. In the absense of measurement, the density matrix of the system satisfies the following SME in Stratonivich form,

$$\dot{\rho} = -i[H_0, \rho] - i[H_1, \rho]\xi. \tag{2.2}$$

3. SME in Ito form (no measurement)

In this section, we derive the stochastic master equation (2.2) in Ito form. In Ito form, we can average over the white-noise process easily to obtain the master equation.

Applying ι on both sides of the SME in Stratonivich form, we arrive at a linear SDE in Stratonivich and differential form,

$$dq = Aqdt + BqdW. \tag{3.1}$$

In Ito form, (3.1) become [PLATEN,4.9.13],

$$dq_j = (Aq)_j dt + \frac{\Omega}{2}\sum_k (Nq)_k \frac{\partial}{\partial q_k}(Bq)_j dt + (Bq)_j dW, \tag{3.2}$$

where we assume that the spectral density of the white noise process is one. Applying ι^{-1} to both sides of (3.2),

$$d\rho = -i[H,\rho]dt + \frac{1}{2}\iota^{-1}\left(\sum_k (Bq)_k \frac{\partial}{\partial q_k}(Bq)\right)dt - i[H_1,\rho]dW. \tag{3.3}$$

Focusing on the middle term on the RHS of (3.3),

$$\begin{aligned}
\frac{\partial}{\partial q_k}(Bq)_j &= \frac{\partial}{\partial q_k}\sum_m B_{j,m}q_m \\
&= N_{j,k} \\
\Rightarrow \sum_k \frac{\partial}{\partial q_k}(Bq)_j(Bq)_k &= \sum_{k,n} B_{j,k}B_{k,n}q_n. \\
&= (N^2 q)_j. \\
\iota^{-1}(B^2 q) &= \iota^{-1}(B(Bq))) \\
&= -i[H_1, \iota^{-1}(Bq)] \\
&= -i[H_1, -i[H_1,\rho(t)]] \\
&= 2H_1\rho H_1 - H_1^2\rho - \rho H_2^2 \\
&= 2\mathcal{L}[H_1]\rho.
\end{aligned}$$

The stochastic master equation in Ito form follows,

$$d\rho(t) = (-i[H_0,\rho] + \mathcal{L}[H_1]\rho)dt - i[H_1,\rho]dW. \tag{3.4}$$

4. Linblad equation

The master equation can be now written by setting the differential Weiner process to zero in (3.4),

$$\dot{\rho} = -i[H_0,\rho] + \mathcal{L}[H_1]\rho. \tag{4.1}$$

The master equation (4.1) is in Linblad form.

5. Filtering Equations

A weak homodyned measurement on a system operator L, introduces another (non-linear) white-noise process and gives us a measurement signal I. Given estimates $\hat{H}_0$ & $\hat{L}$ for the Hamiltonian measurement we can process the measurement signal to obtain an estimate $\hat{\rho}$ for the density matrix state. We do not gain any information about the white noise process W_1 through the measurement, hence, we cannot account for the Hamiltonian white noise in the filter. The Filtering Equations for this process read (in Ito form) [BOUTEN, pg 36],

$$d\rho = (-i[H_0, \rho] + \mathcal{L}[H_1]\rho + \mathcal{L}[L]\rho)dt - i[H_1, \rho]dW_1 - \mathcal{F}[L](\rho)dW_2$$
$$d\hat{\rho} = (-i[\hat{H}_0, \hat{\rho}] + \mathcal{L}[\hat{L}]\hat{\rho})dt - \mathcal{F}[\hat{L}](\hat{\rho}) \left(dI - \mathrm{Tr}\left\{(\hat{L} + \hat{L}^\dagger)\hat{\rho}\right\}\right) dt$$

where the measurement signal is given by

$$dI = dW_2 - \mathrm{Tr}\left\{(L + L^\dagger)\rho\right\} dt$$

and

$$\mathcal{F}[L](\rho) := L\rho + \rho L^\dagger - \mathrm{Tr}\left\{(L + L^\dagger)\rho\right\} \rho.$$

References. [PLATEN] Numerical solutions of stochastic differential equations by Kloeden and Platen

[BOUTEN] Introduction to Quantum Filtering by Bouten, Von Handel and James