Two-dimensional representation of time for a quantum particle in a Bose-Einstein condensate

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A quantitative quantum field approach for interacting particles using a complex two dimensional representation of time is presented. The representation of a two-dimensional complex valued time arises from the constraint of particle number conservation and can be used to account simultaneously for both oscillations of the quantum state of a particle in a Bose-Einstein condensate as well as coherence times of the particle in the atomic cloud below the critical temperature. It is illustrated that, in contrast to so far established theories of purely imaginary complex time, two dimensional complex time has a preferred direction in positive direction of the imaginary axis below the critical temperature in agreement with the observation of a spontaneously broken phase gauge symmetry of the underlying fugacity spectrum. The results reduce to the standard scheme of purely imaginary time above the critical temperature.

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I. INTRODUCTION

As a fundamental property of quantum mechanics, coherence demonstrates the wave nature of quantum particles by the ability of the particles to interfere with themselves when the wave length becomes on the order of their average distance below the critical temperature [1]. In classical physics, when a particle is moved from a point A to a second point B, the particle has a well-defined position and velocity at each instant of time during the cycle from A to B, and indeed this time is well-defined by the path length between the two positions and the average velocity of the traveling particle [2].

Quantum mechanically, however, the dynamics of a particle is more complicated, because of quantization of position and velocity, thus in the setup as described above, the quantum particle may only be located at two different positions and therefore exhibit no probability to be located in the space between the two points A and B. As a consequence of second quantization, if the particle tunnels from point A to the point B, there remains a time interval during which it is quantum mechanically not well-defined wether the particle is located at position A or B, so that the quantum state of the particle may thus be in a coherent superposition of a state with localisation in point A and a state with localisation in point B, with a well-defined phase relation between these two states which allows to quantify the transition probability for the particle to move from point A to B, and vice versa. The time scale during which these two states are correlated and the particle can interfere with itself is called coherence time [3].

Calculation and analyses of coherence times typically relies on knowledge about the spectrum of the given quantum system. In a Bose-Einstein condensate below the critical temperature, all particles search to populate the ground state of the system due to energetic reasons,

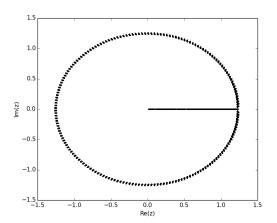


FIG. 1: (color online) Figure shows the fugacity spectrum of a Bose-Einstein condensate with N=1000 particles in a harmonic trap with trap frequencies $\omega_x, \omega_y = 2\pi \times 42.0$ Hz, and $\omega_z = 2\pi \times 120.0$ Hz for variable temperatures T. The relaxation time τ is defined by the logarithm of the fugacity's absolute value, i.e. by the radius of the ring in the complex plane.

and thus each particle can tunnel from the ground state to a (multimode) excited state of the single particle spectrum. As a consequence of single particle coherence as described above, each particle is thus in general in a superposition of a state with the localisation characteristics of the single particle ground state, and one or multiple of the excited single particle states. Since all particles tend to populate the ground state of the single particle spectrum within a given coherence time, in the many particle framework, a macroscopic occupation of the ground state mode occurs below the critical temperature for Bose-Einstein condensation with the characteristics of a coherent matter wave [4].

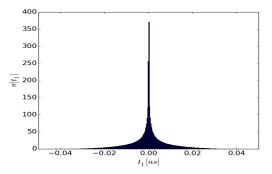
As a fundamental characteristic of quantum mechanics, the quantum state of a particle is only defined up to a global (coherent) phase, which may change in time due to interactions with an external source such as interactions with other particles. Hence, very generally, besides the coherence time as discussed before, there is a second important time scale for a quantum particle when interacting with an external environment, which is defined by the oscillation frequency of the particle's phase, i.e. the time scale during which the particle (which exhibits the characteristics of a coherent state) changes its phase due to interactions.

In order to connect the mathematical framework of Bose-Einstein condensates at thermal equilibrium to the physics of coherence and oscillation times of a single particle in a Bose gas at finite temperature, the concept of imaginary time turns out to be a suitable mathematical tool to describe both the quantitative scaling of coherence times as well as their uncertainty and broken gauge symmetry aspects induced by many-particle fluctuations below the critical temperature [5–7]. Whereas the definition of (purely) imaginary time often lacks a clear physical interpretation in other frameworks such as calculation of the Hawking radiation in black holes [7], or the definition and calculation of an Eukledian time in special relativity [8], in the present framework twodimensional complex time is derived from the underlying complex fugacity spectrum and can be interpreted as the time scale during which a quantum particle in the gas relaxes towards a quantum state corresponding to the many particle Boltzmann equilibrium (relaxation time), simultaneously exhibiting both coherence and oscillation times between two points A and B until it relaxes to the equilibrium state, following a well-defined distribution as a function of external as well as internal parameters with the fundamental characteristics of time uncertainty in quantum mechanics.

In the sequel of this study, in order to solve the equations for the relaxation time and their corresponding fluctuations numerically, a Metropolis algorithm for the calculation of both coherence and oscillation times of a particle in a Bose-Einstein condensate is applied [9], and it is shown that the relaxation time varies from a Gaussian profile distribution for the condition of maximum entropy, whereas it is randomly distributed when only constant temperature and particle number conservation is assumed. The quantitative scaling and relation of the relaxation, coherence and oscillation time in particular to the underlying spectral characteristics of the fugacity spectrum is discussed.

II. THEORY

The concept of imaginary time provides a solid starting point for analyses of relaxation time and its uncertainty.



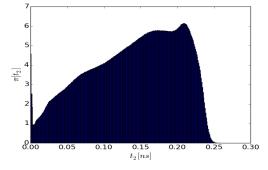


FIG. 2: (color online) Figure shows 10^5 realisations of the real parts and the imaginary parts of two-dimensional time $\tau=t_1+it_2$ as defined in Eq. (1) from random Markov sampling without a conditional probability condition for the chemical potential, but with the constraint of particle number conservation, for the same parameters as in Figure 1.

Whereas this concept is rather a pure mathematical convention in many quantum theoretical applications, in the current framework, the underlying mathematical relation (Wick rotation)

$$\beta \mu = i\tau \tag{1}$$

defines a relation between the absolute temperature, the chemical potential and the relaxation time of a quantum particle in the presence of the N-1 other particles in the Bose gas by its absolute value $|\tau| = \beta |\mu|$. The parameter τ can be interpreted as total relaxation time in units of unit time (defined by the energy of the quantum particle), or equivalently as the entropy in units of the Boltzmann constant k_B .

As a new ansatz in quantum field theory for Bose-Einstein condensation, in Eq. (1) τ is not a real, but a complex number, i.e. $\tau \in \mathbb{C}$. Hence, in the framework of this quantum field theory as presented, the definition of time consists of a real part t_1 which describes the coherent phase evolution of the system at the Boltzmann equilibrium (inverse oscillation frequency in the stationary state), which corresponds to the ring of the fugacity spectrum in Figure 1, whereas t_2 , the imaginary part (of τ) describes decay processes of the particle in the atomic

cloud at or close to the Boltzmann equilibrium (compare linear part of the fugacity spectrum in Figure 1), i.e.

$$\tau = t_1 + it_2,\tag{2}$$

which can be quantified by solving the equation for the constraint of particle number conservation numerically. Above the critical temperature, the real part of the fugacity spectrum has a gap between the ring of constant absolute time, indicating that the Boltzmann equilibrium can in principle only be perturbed by external perturbations which are larger than the gap between the outer ring and the inner linear (symmetry breaking) part of the fugacity spectrum [10].

Different from other theories of imaginary time known so far - the imaginary part of time t_2 (coherence time) has a preferred direction in one direction of the imaginary axis (which defines a direction of time evolution) below the critical temperature. Typically, the rotational direction of coherent time evolution scales as $\exp[i\tau]$ and thus the decay as $\exp[-t_2]$, so that the preferred direction of imaginary time is the positive imaginary axis with arbitrary real values (similar to the definition and direction of space time in Minkowski space). The directional preference in one of all possible time directions correlates with the broken gauge symmetry of the quantum field [6], which represents the fugacity spectrum weighted with phase factors of the corresponding quantum occupations rotated by an angle π . The directional preference of imaginary time and the corresponding breakdown of phase gauge symmetry is only present below the critical temperature for Bose-Einstein condensation, which means that both complex time and the fugacity spectrum tend (only for conventional reasons up to rotation of $\pi/2$) to the well-known scheme and characteristics of imaginary time without directional expression in the classical limit, i.e. for large particle numbers above the critical temperature with zero coherence times and small oscillation frequencies.

Probabilistic modeling of a dilute Bose gas of N particles below the critical is well described by the equation of conditional probability

$$\frac{p(\mu_A)}{p(\mu_B)} = \frac{e^{-\beta|\mu_A|}}{e^{-\beta|\mu_B|}} = \frac{e^{\tau_B}}{e^{\tau_A}},\tag{3}$$

which defines the relative probability of a quantum particle to move from a state A with corresponding chemical potential μ_A to a state B with chemical potential μ_B for a Bose gas at thermal equilibrium, where the chemical potential is defined by the intrinsic equation

$$\Delta N = \sum_{j \neq 0} z^{j}(\mu) \left[\prod_{k} \frac{1}{1 - e^{-j\beta\hbar\omega_{k}}} - 1 \right]. \tag{4}$$

In Eq. (4), ΔN is the difference between N the total number of particles, and N_{\perp} the number of non-condensate particles. Further, μ denotes the chemical potential of the condensate, and ω_k the trap frequency in mode direction k = x, y, z.

Equation (3) does not describe a coherent forward or backward propagation in time, but only the probability for a transition (or new event) from a state with (the characteristics) of a quantum state with relaxation time τ_A to a state with relaxation time τ_B , i.e. there is a non-vanishing probability that a transition with $\tau_A \geq \tau_B$ is accepted. Most likely, however, a transition is only accepted, if the absolute value $\tau_B \geq \tau_A$, which means that the particle tends to populate states with large relaxation (and coherence times) in the Markov evolution process of the present quantum theory.

Note that, in the present framework, the representation of two-dimensional time arises naturally from Eq. (4), rather than from purely mathematical definition [11] and the system is allowed to exchange energy with the external environment. Since τ can also be interpreted as the thermodynamic entropy in units of the Boltzmann constant, imposing Eq. (3) leads to a thermal equilibrium quantum state of maximum entropy. From the definition of the chemical potential(s) in Eq. (4), time is in principle quantized. Only the neglect energy discreteness in the thermodynamic limit (large particle number and small frequencies) leads to a variable $\tau = t_1 + it_2$ which gets continuous.

III. QUANTITATIVE ANALYSIS

In the following, we assume the Bose gas to be dilute and obey a constant number of particles at finite temperature in a harmonic trapping potential in order to numerically calculate the coherence time (real part of two-dimensional time as defined in Eq. (1)) of a quantum particle in the Bose gas. Interactions in the Bose gas are assumed to be captured by a single quantity, the s-wave scattering length of the particles in the gas. From Eq. (1), we learn that the Bose gas obeys a stable (Boltzmann) equilibrium for values of two dimensional time in the limit where $t_1 = 0$ and $t_2 = 0$. Indeed, from Figure 1, we can verify from the fugacity spectrum, which shows all possible values (poles) of the intrinsic Eq. (4), that the Boltzmann equilibrium can be related to the fugacity spectrum when the real part of the fugacity tends to one and the imaginary part to zero (corresponding to the case, where t_1 and t_2 are distributed around zero).

Applying only the constraint of particle number conservation as defined by the conservation equation in Eq. (4) numerical sampling of two dimensional complex time leads to a distribution of typical time scales for t_1 (oscillation time) and t_2 (coherence time), shown in Figure 2. From the sampling of two dimensional time, it is ob-

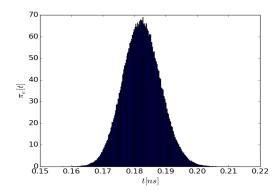


FIG. 3: (color online) Figure shows $5*10^5$ realisations of the absolute value of the distribution of two dimensional time τ (relaxation time) of a particle in a Bose-Einstein condensate from sampling with a conditional Boltzmann distribution. A clear Gaussian distribution is observed for non-isotropic trapping geometries with temperature T=10.0 nK when the trapping frequencies are $\omega_x, \omega_y=2\pi\times 120.0$ Hz and $\omega_z=42.0$ Hz below the critical temperature.

served that t_1 , the real part of complex two dimensional time, is distributed around zero (with a width of approximately 0.03 ns) in the given parameter range (as defined in Figure 1). The real part t_1 defines the average oscillation frequency in the fugacity spectrum of Figure 1, which means that the system (with constant particle number at finite temperature) mainly remains in many body quantum states which are related to $t_1 \sim 0 \pm 0.03$ ns (and fugacity with imaginary part zero). The distribution of the time scale t_2 , the directional imaginary part of two dimensional time (i.e. the coherence time for the particle to be in a coherent state in or out of the condensate fraction), is shown in the lower panel of Figure 2. The typical range of the time scale t_2 is around 0.00 ns to 0.20 ns in the present parameter range, with a slight extension to values between 0.10 ns and 0.20 ns. Comparing the distribution of the time scale t_2 to the fugacity spectrum in Figure 1, it turns out that from the conservation of particle number, the Bose gas below the critical temperature mainly remains in quantum states which are related to coherence times $t_2 = 0.00 - 0.20$ ns (and corresponding fugacity values of 0.80 - 1.00).

In order to study the relaxation time to the thermal (Boltzmann) equilibrium, the conditional probability condition in Eq. (3) is imposed additionally to the constraint of particle number conservation in Eq. (4) and the condition of constant temperature in the Monte Carlo sampling algorithm for the time scales t_1 and t_2 in Eq. (2). From these two constraints and the sampling of two dimensional time, we may calculate and analyse the absolute value of complex time (which corresponds to the relaxation time related to both the coherence time t_2 and the oscillation time t_1 , as shown in Figure 3. From

Figure 3, one can extract that the relaxation time τ is distributed around the mean value of approximately 0.18 \pm 0.01 ns. In contrast to the distributions shown in Figure 2, the relaxation time τ has a clearly shaped structure and tends to be distributed around larger values. This smoothing is due to the condition that the system has to converge to a (Boltzmann) equilibrium, where not only the temperature and conservation of particle number, but also the condition that the entropy of the system has to be maximized is to be fullfilled by and imposed onto the applied sampling method or algorithm, respectively.

IV. DISCUSSION

From a mathematical perspective, it is straight forward to define a variable describing complex two dimensional time, a complex number with two subspaces defined by the variables t_1 and t_2 . The interpretation, however, is not as trivial, and normally, purely imaginary time is only used as a formal mathematical concept in order to solve certain conceptual problems of theoretical physics. Physical processes usually depend on real time, which has a clear physical direction, the positive direction of real valued continuous positive numbers. The direction of purely imaginary time, in contrast, is usually not directionally expressed which means purely imaginary time can either pass in positive or negative imaginary direction and is used only for formal reasons and typically assumed to be a one-dimensional quantity.

In the present theory, it turns out that only the composition of real valued and purely imaginary valued time to a complex valued number which defines time in terms of a two dimensional variable (and actually arises from first principles) fully captures the system in its full complexity (interacting particles below the critical temperature), i.e. it is the spectrum of the fugacity which naturally defines two different and simultaneous time scales t_1 which describes oscillations of the quantum particle's wave function and t_2 which defines the coherence time of a quantum particle in the presence of the N-1 other particles. The absolute value of (two-dimensional) complex time defines the relaxation time of a quantum particle in the Bose-Einstein condensate. In the sequel of the analysis with two dimensional complex time, it turns out that the real part of time is typically distributed around zero, whereas the imaginary part of complex time mostly entails the systems relaxation properties. For temperatures much larger than the critical temperature, the coherence time tends to zero at thermal equilibrium and thus complex time tends to converge to the characteristics of nondirectional purely imaginary time.

The widths of the time distributions indicate the uncertainty and fluctuations in all time scales - which do vary with temperature and interaction strength. As a matter of fact, the real valued part of time t_1 first of all leads to phases of the particle's quantum state which are distributed all around a ring in the complex plane (fugacity spectrum as observed above the critical temperature), but mainly in the neighborhood of the intersection to the symmetry breaking purely real valued part of the fugacity spectrum (directional expression of time due to spontaneous symmetry breaking of the spectrum below the critical temperature). The distribution of the imaginary valued part t_2 of complex time further indicates that mainly quantum states in the neighbourhood of Re(z) = 1are populated at the Boltzmann equilibrium (quantum state of maximum entropy), however, the full range from Re(z) = 0 to Re(z) = 1 is maintained, if no probability condition is imposed. Finally, this indicates that the Bose gas highlights its quantum nature due to large coherence times in the limit of the Boltzmann equilibrium, whereas it behaves rather classically when only temperature and particle number conservation is assumed.

V. CONCLUSION

In conclusion, using a representation of two dimensional time, it was possible to show that the relaxation (and coherence) time of a quantum particle in a Bose-Einstein condensate is on the order of nanoseconds in the typical parameter regime of Bose gases at thermal equilibrium in harmonic traps below the critical temperature. In the presented quantum field theory, the definition of a two dimensional complex time arises naturally from the constraint of particle number conservation, mathematically defined by complex poles of an intrinsic equation for the fugacity (and the chemical potential) of the Bose-Einstein condensate. In the classical limit, fluctuations of the oscillation time tend to be distributed around a ring in the complex spectrum of the fugacity, i.e. purely imaginary time when coherence times tend to zero. A fundamentally new aspect for the concept of imaginary time is the directional preference of complex time to the positive half-plane of the imaginary axis (with arbitrary

real parts due to non-directional oscillations) below the critical temperature. Resulting coherence times show a clear Gaussian profile with quantum characteristics in the Boltzmann limit of maximum entropy. The characteristics of the fluctuations of two dimensional complex time show that the system at thermal equilibrium behaves rather classically with a uniform relaxation time distribution, when no constraint is imposed on the entropy of the system.

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- C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, 1998, Particle creation by black holes, Savoirs Actuels, Editions du CNRS Paris
- [2] W. Nolting, 2006, Grundkurs Theoretische Physik 1: Klassische Mechanik (Springer-Lehrbuch),
- [3] Roy J. Glauber, 1963, The Quantum Theory of Optical Coherence 130, 2529
- [4] M. R. Andrews, C. G. Townsend, G. Miesner, D., S. Durfee, D. M. Kurn, W. Ketterle, 1997, Science, 275, 637-641
- [5] A. A. Abrikosov, L. P. Gorkov, I. E. Dzyaloshinski, 1963, Courier Dover Publications 130.
- [6] A. Schelle, 2017, Fluctuations and Noise Letters 16, Spontaneously broken gauge symmetry in a Bose gas with constant particle number, 1
- [7] Stephen W. Hawking, 1975, Commun. Math. Phys. 43, Particle creation by black holes, 199-220
- [8] Stephen W. Hawking, 1978, Physical Review D. 18(6), Quantum gravity and path integrals (1978), 1747–1753
- [9] W. Krauth, 2006, Oxford Master Series in Statistical, Computational, and Theoretical Physics 18(6), Quantum gravity and path integrals
- [10] S. Elizur, 1975, Phys. Rev. D 12 3978-3982, Quantum gravity and path integrals
- [11] I. Dinov, V. Milen, 2021, Boston/Berlin: De Gruyter ISBN 9783110697803, Data Science - Time Complexity, Inferential Uncertainty, and Spacekime Analytics