

RANDOM WALK ON FINITE QUANTUM GROUPS

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ABSTRACT. In this paper we study convergence of random walks, on finite quantum groups, arising from linear combination of irreducible characters. We bound the distance to the Haar state and determine the asymptotic behavior, i.e. the limit state if it exists. We note that the possible limits are any central idempotent state. We also look at cut-off phenomenon in the Sekine finite quantum groups.

INTRODUCTION

Given a finite group G , and a current position h in G , choose randomly, according to a fixed probability distribution, an element g , and move to gh . If you repeat this procedure, with the same probability distribution π , you will get random positions $h_k = g_k g_{k-1} \dots g_1$ which define a walk on the Cayley graph of G . A frequently asked question is when the distribution $\pi^{\star k}$ of the current position h_k is close to the uniform distribution on G . Diaconis and Shahshahani answered this question in [4] for the case of permutation groups, by using the representation theory. They also observe that after less than t_n random transpositions a deck of n cards is not well mixed, but after more than t_n random transpositions it is well mixed, where $t_n = \frac{1}{2}n \log(n)$. This is called a cut-off. For more details on cut-off phenomenon in classical finite groups, the reader can look at the survey of Saloff-Coste [14].

Random walks on finite quantum groups were first studied by Franz and Gohm in [5]. We are now looking at convolution semigroups of states $\{\phi^{\star k}\}_{k \in \mathbb{N}}$ and their convergence to the Haar state. Recently, McCarthy develops the Diaconis-Shahshahani theory for finite quantum groups [10]. In particular, he gives upper and lower bounds for the distance to the Haar state, exploiting the corresponding tools, but he could not prove a cut-off in the Sekine family of finite quantum groups.

For a finite quantum group \mathbb{G} , let us denote by $I(\mathbb{G})$ the set of its irreducible representations, and, for a representation $\alpha \in I(\mathbb{G})$, d_α its dimension and χ_α the associated character.

We shall also use the Fourier transform. For x an element in $C(\mathbb{G})$, let us denote by $\mathcal{F}(x)$ the linear functional defined on $C(\mathbb{G})$ by

$$\mathcal{F}(x)(y) = \int_{\mathbb{G}} yx .$$

This allows us to define a convolution product on $C(\mathbb{G})$, also denoted \star like the product of linear functionals on $C(\mathbb{G})$, thanks to the formula

$$\mathcal{F}(a \star b) = \mathcal{F}(a) \star \mathcal{F}(b) := (\mathcal{F}(a) \otimes \mathcal{F}(b)) \circ \Delta .$$

Using Sweedler notation, we get $a \star b = \sum_{(b)} b_{(2)} \mathcal{F}(a) (S(b_{(1)}))$ in $C(\mathbb{G})$.

Here, we restrict ourselves to random walks arising from the Fourier transform of linear combination of irreducible characters, it means from elements of the central algebra. This work is separated into two parts. The first section is devoted to the study of the Kac-Paljutkin group \mathbb{KP} . In the second one, we look at the Sekine family \mathbb{KP}_n . We study cut-off phenomenon in subsection 2.3 and list all the possible types of asymptotic behavior for n fixed in Theorem 5. It allows us to determine all the central idempotent states on these finite quantum groups, by noting that the central algebra is the center of the algebra with respect to the convolution product.

1. RANDOM WALKS IN \mathbb{KP}

First let us look at random walks on the eight-dimensional Kac-Paljutkin finite quantum group \mathbb{KP} . This is the smallest Hopf-von Neumann algebra which is neither commutative nor cocommutative. As noted by Kac and Paljutkin in [9], the elements of $I(\mathbb{KP})$ are

$$\begin{aligned} \rho(1) &= 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \\ \rho(2) &= 1 \oplus -1 \oplus -1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \\ \rho(3) &= 1 \oplus -1 \oplus -1 \oplus 1 \oplus \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \\ \rho(4) &= 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and} \\ \mathfrak{X} &= \begin{pmatrix} 1 \oplus -1 \oplus 1 \oplus -1 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & e^{i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} & 0 \end{pmatrix} \\ 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{pmatrix} & 1 \oplus 1 \oplus -1 \oplus -1 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \end{aligned}$$

which gives the character $\chi(\mathfrak{X}) = 2 \oplus 0 \oplus 0 \oplus -2 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Note that $\rho(1)$ is the unit in $C(\mathbb{KP})$.

Let us fix an element $g = \sum_{u=1}^4 g_u \rho(u) + g_{\mathfrak{X}} \chi(\mathfrak{X})$ of \mathbb{KP} , with $g_{\alpha} \in \mathbb{C}$ for all $\alpha \in I(\mathbb{KP})$.

Let us note that $\mathcal{F}(e_i)(e_k) = \frac{1}{8} \delta_{ik}$, $\mathcal{F}(e_i)(E_{kl}) = 0$, $\mathcal{F}(E_{ij})(e_k) = 0$ and $\mathcal{F}(E_{ij})(E_{kl}) = \frac{1}{4} \delta_{il} \delta_{jk}$.

Then $\mathcal{F}(g)$ is a state on $C(\mathbb{KP})$ if and only the following conditions are satisfied:

$$(1) \quad \begin{aligned} g_1 &= 1 \\ 1 + g_2 + g_3 + g_4 &\geq 2|g_{\mathfrak{X}}| \\ 1 + g_2 - g_3 - g_4 &\geq 0 \\ 1 - g_2 + g_3 - g_4 &\geq 0 \\ 1 - g_2 - g_3 + g_4 &\geq 0 \end{aligned}$$

In particular, it implies that, for all $\alpha \in I(\mathbb{KP})$, g_α is real and moreover

$$(2) \quad |g_\alpha| \leq d_\alpha .$$

1.1. Upper and lower bounds.

Lemma 1. *By the Upper and Lower Bound Lemmas, for all $u \in \{2, 3, 4\}$ and all positive integer k ,*

$$(3) \quad \frac{1}{2} |g_u|^k \leq \left\| \mathcal{F}(g)^{\star k} - \int_{\mathbb{KP}} \right\|_{QTV} \leq \sqrt{\frac{1}{4} \sum_{v=2}^4 |g_v|^{2k} + \frac{|g_{\mathfrak{X}}|^{2k}}{4^k}}$$

Proof. The Quantum Diaconis-Shahshahani Upper Bound Lemma (Lemma 5.3.8 in [10]) gives

$$\left\| \mathcal{F}(g)^{\star k} - \int_{\mathbb{KP}} \right\|_{QTV}^2 \leq \frac{1}{4} \sum_{\substack{\beta \in I(\mathbb{KP}) \\ \beta \neq \mathbb{1}_{\mathbb{KP}}}} d_\beta \text{Tr} \left(\widehat{\mathcal{F}(g)}(\beta)^{\star k} \widehat{\mathcal{F}(g)}(\beta)^k \right)$$

with $\widehat{\mathcal{F}(g)}(\beta)$ the matrix composed by the $\mathcal{F}(g)(\beta_{ij})$'s for β_{ij} the matrix coefficients of β . On the other hand, by the Lower Bound Lemma (Lemma 5.3.9 in [10]), for all non trivial one-dimensional representation α ,

$$\left\| \mathcal{F}(g)^{\star k} - \int_{\mathbb{KP}} \right\|_{QTV} \geq \frac{1}{2} \left| \widehat{\mathcal{F}(g)}(\alpha) \right|^k .$$

Let us compute $\widehat{\mathcal{F}(g)}(\beta)$ for every $\beta \in I(\mathbb{KP})$. First of all, let us note that, by linearity of \mathcal{F} ,

$$\widehat{\mathcal{F}(g)}(\beta) = \sum_{\alpha \in I(\mathbb{KP})} g_\alpha \widehat{\mathcal{F}(\chi_\alpha)}(\beta) = \left(\sum_{\alpha \in I(\mathbb{KP})} g_\alpha \int_{\mathbb{KP}} \beta_{ij} \chi_\alpha \right)_{1 \leq i, j \leq d_\beta} .$$

Assume $d_\beta = 1$. If $d_\alpha = 1$, then $\gamma = \beta\alpha$ is again an irreducible one-dimensional representation of \mathbb{KP} , and $\int_{\mathbb{KP}} \gamma = \delta_{\gamma, \mathbb{1}_{\mathbb{KP}}}$. If $d_\alpha = 2$, then $\alpha = \mathfrak{X}$ and $\beta\chi_\alpha = \chi(\mathfrak{X})$. Since $\int_{\mathbb{KP}} \chi(\mathfrak{X}) = 0$, we get $\widehat{\mathcal{F}(g)}(\beta) = g_{\beta^{-1}}$ and

$$\text{Tr} \left(\widehat{\mathcal{F}(g)}(\beta)^{\star k} \widehat{\mathcal{F}(g)}(\beta)^k \right) = |g_{\beta^{-1}}|^{2k} .$$

Now, assume $d_\beta = 2$, which means that $\beta = \mathfrak{X}$. Then

$$\widehat{\mathcal{F}(g)}(\mathfrak{X}) = \begin{pmatrix} \mathcal{F}(g)(1 \oplus -1 \oplus 1 \oplus -1 \oplus 0_2) & \mathcal{F}(g)(e^{i\frac{\pi}{4}}E_{12} - e^{-i\frac{\pi}{4}}E_{21}) \\ \mathcal{F}(g)(e^{-i\frac{\pi}{4}}E_{12} - e^{i\frac{\pi}{4}}E_{21}) & \mathcal{F}(g)(1 \oplus 1 \oplus -1 \oplus -1 \oplus 0_2) \end{pmatrix}$$

is easy to calculate. We obtain $\widehat{\mathcal{F}(g)}(\mathfrak{X}) = \frac{1}{2}g_{\mathfrak{X}}I_2$. Therefore

$$\text{Tr} \left(\widehat{\mathcal{F}(g)}(\mathfrak{X})^{*k} \widehat{\mathcal{F}(g)}(\mathfrak{X})^k \right) = \frac{1}{2^{2k-1}} |g_{\mathfrak{X}}|^{2k}.$$

Hence, the Quantum Diaconis-Shahshahani Upper and Lower Bound Lemmas give the bounds announced in (3). \square

1.2. Asymptotic behavior.

Theorem 1. *The random walk defined by $g = \sum_{u=1}^4 g_u \rho(u) + g_{\mathfrak{X}} \chi(\mathfrak{X})$, satisfying conditions (1), converges to the Haar state if and only if*

$$(4) \quad \forall \alpha \in I(\mathbb{KP}) \setminus \{\mathbf{1}_{\mathbb{KP}}\}, \quad |g_\alpha| < d_\alpha.$$

In this case, the distance to the Haar state decreases exponentially fast.

Proof. By the upper bound (3) in Lemma 1, the condition (4) is sufficient.

By the lower bound (3) in Lemma 1, the convergence implies that $|g_u|$ is strictly less than 1, for each $u \in \{2, 3, 4\}$. Then, the second equation in (1) leads to $|g_{\mathfrak{X}}| < 2$, i.e. (4) holds.

In the case of convergence, let us denote by g_m the greatest element among $|g_2|$, $|g_3|$, $|g_4|$ and $\frac{|g_{\mathfrak{X}}|}{2}$. Then, $g_m < 1$ and, by Lemma 1,

$$\left\| \mathcal{F}(g)^{*k} - \int_{\mathbb{KP}} \right\|_{QTV} \leq \sqrt{\frac{1}{4} \sum_{u=2}^4 |g_u|^{2k} + \frac{|g_{\mathfrak{X}}|^{2k}}{4^k}} \leq \frac{\sqrt{7}}{2} g_m^k$$

which shows the exponential decrease. \square

By the inequality (2), the random walk does not converge to the Haar state if and only if there exists a non-trivial irreducible representation α such that $|g_\alpha| = d_\alpha$, i.e. $g_\alpha \in \{-d_\alpha, d_\alpha\}$. Let us note that it still might converge, but to some other state. Moreover, the limit state, if it exists, is clearly an idempotent state.

First of all, let us list the idempotent states on \mathbb{KP} [12], i.e. states on \mathbb{KP} satisfying $(\phi \otimes \phi) \circ \Delta =: \phi \star \phi = \phi$:

$$\begin{aligned}\phi_1 &= 8\mathcal{F}(e_1) \\ \phi_2 &= 4\mathcal{F}(e_1 + e_2) \\ \phi_3 &= 4\mathcal{F}(e_1 + e_3) \\ \phi_4 &= 4\mathcal{F}(e_1 + e_4) \\ \phi_5 &= 2\mathcal{F}(e_1 + e_2 + e_3 + e_4) \\ \phi_6 &= 2\mathcal{F}(e_1 + e_4 + E_{11}) \\ \phi_7 &= 2\mathcal{F}(e_1 + e_4 + E_{22}) \\ \phi_8 &= \mathcal{F}(\mathbf{1}_{\mathbb{KP}}) = \int_{\mathbb{KP}}\end{aligned}$$

Let us also note that $g^{*k} = \sum_{\alpha \in I(\mathbb{KP})} \frac{g_\alpha^k}{d_\alpha^{k-1}} \chi_\alpha$. Hence the random walk defined by such a g cannot converge to ϕ_2 or ϕ_3 . Then, by looking at the different cases, we get:

Proposition 1. *Assume that the random walk associated to g does not converge to the Haar state ϕ_8 . Then, we have six possibilities.*

If $g_{\mathfrak{X}} = 2$, then $g_u = 1$ for all $1 \leq u \leq 4$ and $\mathcal{F}(g) = \phi_1$.

If $g_\alpha = 1$ for at least two non trivial one-dimensional representations in $I(\mathbb{KP})$ and $|g_{\mathfrak{X}}| < 2$, then $g_u = 1$ for all $1 \leq u \leq 4$ and the random walk converges to ϕ_4 .

If $g_2 = 1$, then $g_3 = g_4$. Assume also that $|g_4| < 1$ and $|g_{\mathfrak{X}}| < 2$, then the random walk converges to ϕ_6 .

If $g_3 = 1$, then $g_2 = g_4$. Assume also that $|g_2| < 1$ and $|g_{\mathfrak{X}}| < 2$, then the random walk converges to ϕ_7 .

If $g_4 = 1$, then $g_2 = g_3$. Assume also that $|g_3| < 1$ and $|g_{\mathfrak{X}}| < 2$, then the random walk converges to ϕ_5 .

In all the other cases, it means if $g_{\mathfrak{X}} = -2$ or if there is $2 \leq u \leq 4$ such that $g_u = -1$, then the random walk is cyclic and does not converge.

2. RANDOM WALKS IN \mathbb{KP}_n

We use the definition and the representation theory of the Sekine finite quantum groups presented in [1]. Let us recall that for each n greater than or equal to 2, $C(\mathbb{KP}_n) = \bigoplus_{i,j \in \mathbb{Z}_n} \mathbb{C}e_{(i,j)} \oplus M_n(\mathbb{C})$ is endowed with the Haar

state $\int_{\mathbb{KP}_n}$ given by

$$\int_{\mathbb{KP}} \left(\sum_{i,j \in \mathbb{Z}_n} x_{(i,j)} e_{(i,j)} + \sum_{1 \leq i,j \leq n} X_{i,j} E_{i,j} \right) = \frac{1}{2n^2} \left(\sum_{i,j \in \mathbb{Z}_n} x_{(i,j)} + n \sum_{i=1}^n X_{i,i} \right)$$

where $E_{i,j}$ denotes a choice of matrix units in $M_n(\mathbb{C})$ and $x_{(i,j)}, X_{i,j} \in \mathbb{C}$.

The set of irreducible representations $I(\mathbb{KP}_n)$ depends on the parity of n . For n odd, its elements are, for $l \in \mathbb{Z}_n$, the one-dimensional representations

$$\rho_l^\pm = \sum_{i,j \in \mathbb{Z}_n} \eta^{il} e_{(i,j)} \pm \sum_{m=1}^n E_{m,m+l}$$

with $\eta = e^{\frac{2i\pi}{n}}$ a primitive n^{th} root of unity, and the two-dimensional representations $\mathfrak{X}^{u,v}$, whose characters are

$$\chi(\mathfrak{X}^{u,v}) = 2 \sum_{s,t \in \mathbb{Z}_n} \eta^{su} \cos\left(\frac{2tv\pi}{n}\right) e_{(s,t)}$$

for $u \in \mathbb{Z}_n$ and $v \in \{1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor\}$. If n is even, we also need to add, for each $l \in \mathbb{Z}_n$,

$$\sigma_l^\pm = \sum_{i,j \in \mathbb{Z}_n} (-1)^j \eta^{il} e_{(i,j)} \pm \sum_{m=1}^n (-1)^m E_{m,m+l}.$$

Note that ρ_0^+ is the unit in $C(\mathbb{KP}_n)$, for each n .

Let us fix an element $a = \sum_{\alpha \in I(\mathbb{KP}_n)} a_\alpha \chi_\alpha$ of $C(\mathbb{KP}_n)$, such that $a_{\rho_0^+} = 1$, and for all i, j in \mathbb{Z}_n , with $\mathbb{1}_{2\mathbb{N}}$ the indicator function of even integers,

$$(5) \quad \sum_{l=0}^{n-1} \left(a_{\rho_l^+} + a_{\rho_l^-} + \mathbb{1}_{2\mathbb{N}}(n) (-1)^j (a_{\sigma_l^+} + a_{\sigma_l^-}) + 2 \sum_{v=1}^{\lfloor \frac{n-1}{2} \rfloor} a_{\mathfrak{X}^{l,v}} \cos\left(\frac{2\pi jv}{n}\right) \right) \eta^{il} \geq 0$$

and such that the matrix $A = (A_{i,j})_{1 \leq i,j \leq n}$ is positive semi-definite, where $a_{\rho_{j-i}^+} - a_{\rho_{j-i}^-} + \mathbb{1}_{2\mathbb{N}}(n) (-1)^i (a_{\sigma_{j-i}^+} - a_{\sigma_{j-i}^-})$ is denoted $A_{i,j}$.

Then $\mathcal{F}(a)$ satisfies the first conditions in Lemma 6.4 in [6], which means that $\mathcal{F}(a)$ is a state on $C(\mathbb{KP}_n)$. To obtain this result, we need to note that

$$\begin{aligned} 2n^2 \mathcal{F}(e_{(s,t)})(e_{(i,j)}) &= \delta_{s,i} \delta_{t,j} & 2n^2 \mathcal{F}(e_{(s,t)})(E_{i,j}) &= 0 \\ 2n \mathcal{F}(E_{p,q})(e_{(i,j)}) &= 0 & 2n \mathcal{F}(E_{p,q})(E_{i,j}) &= \delta_{p,j} \delta_{q,i} \end{aligned}$$

and to express a in the canonical basis of $C(\mathbb{KP}_n)$.

Remark 1. By (5) and the property of A , $a_{\rho_0^-}$ is real and its absolute value is not greater than 1.

Example 1. Let p, q be integers such that $pq = n$, $q > 1$. Put $a_{\rho_{lp}^+} = \eta^{lp}$ for all $0 \leq l \leq q-1$ and $a_\alpha = 0$ for all the others $\alpha \in I(\mathbb{KP}_n)$. Then, for all $i, j \in \mathbb{Z}_n$, the left-hand-side term in (5) is

$$a_{(i,j)} = \sum_{l=0}^{q-1} \eta^{lp} \eta^{ilp} = q \mathbb{1}_{q\mathbb{Z}_n}(i+1)$$

which equals q if $i+1$ is a multiple of q in \mathbb{Z}_n and 0 otherwise. This is non negative, and the matrix A , defined by $A_{ij} = \eta^{lp}$ if $j = i + lp$ and 0

otherwise, is self-adjoint and its eigenvalues are 0 and q , i.e. A is semidefinite positive. Therefore $\mathcal{F}(a) = \phi_p$ is a state on $C(\mathbb{K}\mathbb{P}_n)$.

2.1. Conditions for convergence. First, note that, if we extend naturally the notation, $\mathfrak{X}^{u,0}$ is unitarily equivalent to the direct sum of the representations ρ_u^+ and ρ_u^- . Similarly, if n is even, $\mathfrak{X}^{u,\frac{n}{2}}$ is unitarily equivalent to the direct sum of σ_u^+ and σ_u^- , as representations of $\mathbb{K}\mathbb{P}_n$.

Moreover, the random walk associated to a converges if and only if the sequences of matrices $\left(\widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,v})^k\right)_{k \in \mathbb{N}}$ converge for all $u \in \mathbb{Z}_n$ and all $v \in \{0, 1, \dots, \lfloor \frac{n}{2} \rfloor\}$, since

$$\left(\widehat{\mathcal{F}(a)}^{\star k}(\mathfrak{X}^{u,v})\right) = \left(\widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,v})\right)^k.$$

Theorem 2. *The random walk associated to a converges if and only if every complex number $\frac{a_\alpha}{d_\alpha}$ is an element of the open unit disc \mathbb{D} or equals 1, for all $\alpha \in I(\mathbb{K}\mathbb{P}_n)$.*

Proof. First, let us fix (u, v) such that $0 \leq u \leq n-1$ and $1 \leq v \leq \lfloor \frac{n-1}{2} \rfloor$. Then

$$\widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,v}) = \begin{pmatrix} \sum_{s,t=0}^{n-1} \eta^{su+tv} \int_{\mathbb{K}\mathbb{P}_n} e_{(s,t)} a & \sum_{m=1}^n \eta^{mv} \int_{\mathbb{K}\mathbb{P}_n} E_{m,m+u} a \\ \sum_{m=1}^n \eta^{-mv} \int_{\mathbb{K}\mathbb{P}_n} E_{m,m+u} a & \sum_{s,t=0}^{n-1} \eta^{su-tv} \int_{\mathbb{K}\mathbb{P}_n} e_{(s,t)} a \end{pmatrix}$$

is easy to calculate if we note that $\int_{\mathbb{K}\mathbb{P}_n} e_{(i,j)} a$ equals $\frac{1}{2n^2}$ times $a_{(i,j)}$, the left-hand-side term in inequality (5), and $\int_{\mathbb{K}\mathbb{P}_n} E_{m,m+u} a = \frac{1}{2n} A_{m+u,m}$. By direct calculations, we obtain $\widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,v}) = \frac{1}{2} a_{\mathfrak{X}^{n-u,v}} I_2$. Hence the corresponding sequences of matrices converge if and only if

$$\frac{1}{2} |a_{\mathfrak{X}^{n-u,v}}| < 1 \text{ or } \frac{1}{2} a_{\mathfrak{X}^{n-u,v}} = 1.$$

The same computation for $v = 0$ gives

$$\begin{aligned} \widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,0}) &= \frac{1}{2} \begin{pmatrix} a_{\rho_{n-u}^+} + a_{\rho_{n-u}^-} & a_{\rho_{n-u}^+} - a_{\rho_{n-u}^-} \\ a_{\rho_{n-u}^+} - a_{\rho_{n-u}^-} & a_{\rho_{n-u}^+} + a_{\rho_{n-u}^-} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_{\rho_{n-u}^+} & 0 \\ 0 & a_{\rho_{n-u}^-} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \end{aligned}$$

This leads to the condition on $a_{\rho_i^+}$ and $a_{\rho_i^-}$.

If n is even, we get

$$\widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,\frac{n}{2}}) = \frac{1}{2} \begin{pmatrix} a_{\sigma_{n-u}^+} + a_{\sigma_{n-u}^-} & (-1)^u (a_{\sigma_{n-u}^+} - a_{\sigma_{n-u}^-}) \\ (-1)^u (a_{\sigma_{n-u}^+} - a_{\sigma_{n-u}^-}) & a_{\sigma_{n-u}^+} + a_{\sigma_{n-u}^-} \end{pmatrix}$$

whose eigenvalues are $a_{\sigma_{n-u}^+}$ and $a_{\sigma_{n-u}^-}$, and the condition on these coefficients follows.

□

Remark 2. Since the $\mathfrak{X}^{u,v}$'s are unitaries, the $\widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,v})$'s have norm not greater than 1, when $\mathcal{F}(a)$ is a state. In particular, for each irreducible representation α in $I(\mathbb{K}\mathbb{P}_n)$, the dimension bounds the modulus of the corresponding coefficient, i.e. $|a_\alpha| \leq d_\alpha$.

2.2. Distance to uniformity.

2.2.1. Upper bound.

Lemma 2. *By the Upper Bound Lemma*

$$(6) \quad \left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{K}\mathbb{P}_n} \right\|_{QTV} \leq \sqrt{\frac{1}{4} \sum_{\substack{\alpha \in I(\mathbb{K}\mathbb{P}_n) \\ \alpha \neq \rho_0^+, d_\alpha=1}} |a_\alpha|^{2k} + \frac{1}{4^k} \sum_{\substack{\alpha \in I(\mathbb{K}\mathbb{P}_n) \\ d_\alpha=2}} |a_{\mathfrak{X}^{u,v}}|^{2k}}$$

Proof. The Quantum Diaconis-Shahshahani Upper Bound Lemma (Lemma 5.3.8 in [10]) gives

$$\left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{K}\mathbb{P}_n} \right\|_{QTV}^2 \leq \frac{1}{4} \sum_{\substack{\beta \in I(\mathbb{K}\mathbb{P}_n) \\ \beta \neq \rho_0^+}} d_\beta \text{Tr} \left(\widehat{\mathcal{F}(a)}(\beta)^{\star k} \widehat{\mathcal{F}(a)}(\beta)^k \right).$$

Let us compute $\widehat{\mathcal{F}(a)}(\beta)$ for every $\beta \in I(\mathbb{K}\mathbb{P}_n)$. First of all, let us recall that, by linearity of \mathcal{F} ,

$$\widehat{\mathcal{F}(a)}(\beta) = \sum_{\alpha \in I(\mathbb{K}\mathbb{P}_n)} a_\alpha \widehat{\mathcal{F}(\chi_\alpha)}(\beta) = \left(\sum_{\alpha \in I(\mathbb{K}\mathbb{P}_n)} a_\alpha \int_{\mathbb{K}\mathbb{P}_n} \beta_{ij} \chi_\alpha \right)_{1 \leq i, j \leq d_\beta}.$$

Assume $d_\beta = 1$. If $d_\alpha = 1$, then $\gamma = \beta\alpha$ is again an irreducible one-dimensional representation of $\mathbb{K}\mathbb{P}_n$, and $\int_{\mathbb{K}\mathbb{P}_n} \gamma = \delta_{\gamma, \rho_0^+}$. If $d_\alpha = 2$, then

$$\beta \cdot \chi(\mathfrak{X}^{u,v}) = \begin{cases} \chi(\mathfrak{X}^{u+l,v}) & \text{if } \beta = \rho_l^\pm \\ \chi(\mathfrak{X}^{u+l, \frac{n}{2}-v}) & \text{if } \beta = \sigma_l^\pm \end{cases}.$$

Since $\int_{\mathbb{K}\mathbb{P}_n} \chi(\mathfrak{X}^{u,v}) = 0$, we get $\widehat{\mathcal{F}(a)}(\beta) = a_{\beta-1}$ and then

$$(7) \quad \text{Tr} \left(\widehat{\mathcal{F}(a)}(\beta)^{\star k} \widehat{\mathcal{F}(a)}(\beta)^k \right) = |a_{\beta-1}|^{2k}.$$

Now, assume $d_\beta = 2$, which means that there exists (u, v) such that $\beta = \mathfrak{X}^{u,v}$, $0 \leq u \leq n-1$ and $1 \leq v \leq \lfloor \frac{n-1}{2} \rfloor$. We already know that $\widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,v}) = \frac{1}{2} a_{\mathfrak{X}^{n-u,v}} I_2$. Therefore

$$(8) \quad \text{Tr} \left(\widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,v})^{\star k} \widehat{\mathcal{F}(a)}(\mathfrak{X}^{u,v})^k \right) = \frac{1}{2^{2k-1}} |a_{\mathfrak{X}^{n-u,v}}|^{2k}.$$

Finally, we get the bound (6) by replacing (7) and (8) in the Quantum Diaconis-Shahshahani Upper Bound Lemma.

□

Hence, the random walk associated to a converges to the Haar state if all the complex numbers cited in Theorem 2 are not equal to 1, except $a_{\rho_0^+}$, which means that

$$(9) \quad \forall \alpha \in I(\mathbb{KP}_n) \setminus \{\rho_0^+\}, \quad |a_\alpha| < d_\alpha$$

2.2.2. *Lower bound.*

Lemma 3. *For any non trivial irreducible representation α ,*

$$(10) \quad \frac{1}{2} \left| \frac{a_\alpha}{d_\alpha} \right|^k \leq \left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV}.$$

Proof. By the Lower Bound Lemma (Lemma 5.3.9 in [10]), for all non trivial one-dimensional representation α ,

$$\left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV} \geq \frac{1}{2} |a_\alpha|^k.$$

Moreover, the following equality [10] holds

$$(11) \quad \left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV} = \frac{1}{2} \sup_{x \in C(\mathbb{KP}_n), \|x\|_\infty \leq 1} \left| \mathcal{F}(a)^{\star k}(x) - \int_{\mathbb{KP}_n} x \right|$$

and any two-dimensional irreducible representation verifies $\left\| \frac{\chi(\mathfrak{X}^{u,v})}{2} \right\|_\infty \leq 1$. Thus

$$\left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV} \geq \frac{1}{4} \left| \mathcal{F}(a)^{\star k}(\chi(\mathfrak{X}^{u,v})) \right| = \frac{1}{2} \left| \frac{a_{\mathfrak{X}^{n-u,v}}}{2} \right|^k.$$

□

Remark 3. If (9) is not satisfied for some non trivial representation, we obtain that the sequence $(\mathcal{F}(a)^{\star k})_{k \in \mathbb{N}}$ stays "uniformly far away from randomness", since the norm is at least $\frac{1}{2}$ for all k .

Finally, we obtain

Theorem 3. *The random walk $(\mathcal{F}(a)^{\star k})_{k \in \mathbb{N}}$ converges to the Haar state if and only if the condition (9) holds.*

Remark 4. If the condition (9) is not satisfied for $\mathfrak{X}^{u,v}$, $a_{\mathfrak{X}^{u,v}} = 2$, by Theorem 2. Hence

$$\widehat{\mathcal{F}(a)}(\mathfrak{X}^{n-u,v}) = \frac{1}{2} a_{\mathfrak{X}^{u,v}} I_2 = I_2$$

and the limit of the sequence $(\widehat{\mathcal{F}(a)}(\mathfrak{X}^{n-u,v})^k)_{k \in \mathbb{N}}$ is the identity. But, $\widehat{\int_{\mathbb{KP}_n} (\mathfrak{X}^{p,q})}$ is the matrix null except if $p = q = 0$, it means that the limit state is not $\int_{\mathbb{KP}_n}$.

2.3. Cut-off phenomenon. In the classical case, we frequently observe cut-off phenomenon, i.e. the distance between $\mathcal{F}(a)^{\star k}$ and the Haar state is almost 1 for a time and then suddenly tends to 0, exponentially fast. There exist different definitions of the threshold (see for example [14],[3] and [13]), depending on how sharp you want the cut-off.

Note that, by upper and lower bounds, (6) and (10), we have roughly

$$\frac{1}{2}M_a^k \leq \left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV} \leq \frac{n}{\sqrt{2}}M_a^k$$

where M_a is the maximum of the $\frac{|a_\alpha|}{d_\alpha}$, α non-trivial. For n fixed, if M_a is less than 1, the random walk converges to the Haar state. In this subsection, we will look at what happens if a and k depend on n . To give a definition of a depending quite naturally on n , we look at some quantum version of the simple random walk on \mathbb{Z}_n , studied in Theorem 2, Chapter 3C of [2].

Since our lower bound is at most $\frac{1}{2}$, it will be ineffective to prove the cut-off. Following [14], we speak about pre-cut-off phenomenon when the distance between $\mathcal{F}(a)^{\star k}$ and the Haar state stays away from 0 for a time and then suddenly tends to 0, exponentially fast.

Unfortunately, if we only consider these bounds and want that nM_a^k vanishes when n goes to infinity, then the limit of M_a^k is also 0. Hence it will be also impossible to prove a pre-cut-off with these bounds. Actually, for our example, we are able to prove that there is no cut-off, as in its classical version.

Remark 5. In [11] there is another example of a random walk in Sekine family where there is no cut-off, but with a state which formally does not depend on n and which is not the Fourier transform of an element in the central algebra. Note that in [7] and [8], Freslon finds cut-off examples in compact quantum groups.

Theorem 4. *For n odd, define the element $a \in C(\mathbb{KP}_n)$ such that for any $l \in \mathbb{Z}_n$, $a_{\rho_l^+} = \cos\left(\frac{2l\pi}{n}\right)$ and $a_\alpha = 0$ otherwise. Then $\mathcal{F}(a)$ defines a state on $C(\mathbb{KP}_n)$, such that, for any non negative c , if $k = e^c n^2$,*

$$\frac{1}{2}e^{-k\left(\frac{\pi^2}{2n^2} + \frac{\pi^4}{4n^4}\right)} \leq \left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV} \leq e^{-k\frac{\pi^2}{2n^2}}.$$

Moreover the lower bound still holds when $k = e^{-c} n^2$.

Remark 6. If n is even, the same definition for $a_{\rho_l^+}$ leads to $a_{\rho_{\frac{n}{2}}^+} = -1$ and by Theorem 3 the random walk does not converge to the Haar state.

Proof. First let us prove that $\mathcal{F}(a)$ is a state on $C(\mathbb{KP}_n)$. We only need to check that every $a_{(i,j)}$ is non negative and that the matrix A is semidefinite positive, since $a_{\rho_0^+}$ is equal to 1 by definition.

Note that, for any i and any j in \mathbb{Z}_n ,

$$a_{(i,j)} = \sum_{l=0}^{n-1} \cos\left(\frac{2l\pi}{n}\right) \eta^{il} = \frac{n}{2} (\delta_{i,1} + \delta_{i,n-1})$$

is always non negative and the equality $A_{i,j} = \cos\left(\frac{2(j-i)\pi}{n}\right)$ defines a circulant matrix, whose eigenvalues are given by the same formula as the $a_{(i,j)}$'s, for i between 0 and $n-1$. Thus, the matrix A is semidefinite positive.

On the one hand, following the arguments of the proof of Theorem 2 in Chapter 3C of [2], the upper bound (6) gives

$$\begin{aligned} \left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV}^2 &\leq \frac{1}{4} \sum_{l=1}^{n-1} \cos\left(\frac{2l\pi}{n}\right)^{2k} \\ &\leq \frac{1}{2} \sum_{l=1}^{\frac{n-1}{2}} \cos\left(\frac{l\pi}{n}\right)^{2k} \\ &\leq \frac{1}{2} \sum_{l=1}^{\frac{n-1}{2}} e^{-\frac{kl^2\pi^2}{n^2}} \end{aligned}$$

where we use that $\cos(x) \leq e^{-\frac{x^2}{2}}$ for $0 \leq x < \frac{\pi}{2}$. Finally, since for every integer j we have $(j+1)^2 - 1 \geq 3j$,

$$\begin{aligned} \left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV}^2 &\leq \frac{e^{-\frac{k\pi^2}{n^2}}}{2} \sum_{j=0}^{+\infty} e^{-\frac{3kj\pi^2}{n^2}} \\ &= \frac{e^{-\frac{k\pi^2}{n^2}}}{2 \left(1 - e^{-\frac{3k\pi^2}{n^2}}\right)}. \end{aligned}$$

This works for any k and any odd n . Moreover, note that, for k greater than n^2 , $2 \left(1 - e^{-\frac{3k\pi^2}{n^2}}\right)$ is greater than 1, thus we obtain the announced upper bound.

On the other hand, the lower bound (10) leads to

$$\begin{aligned} \left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV} &\geq \frac{1}{2} \max_{0 \leq l \leq n-1} \left| \cos\left(\frac{2\pi l}{n}\right) \right|^k \\ &= \frac{1}{2} \cos\left(\frac{\pi}{n}\right)^k. \end{aligned}$$

Let us define $g(x) = \log \left(e^{\frac{x^2}{2} + \frac{x^4}{4}} \cos(x) \right)$. Since $g'(x) = x + x^3 - \tan(x)$ is positive on $]0 ; \frac{\pi}{3}]$, $g(x)$ is also positive on the same interval. Hence

$$\left\| \mathcal{F}(a)^{\star k} - \int_{\mathbb{KP}_n} \right\|_{QTV} \geq \frac{1}{2} \cos \left(\frac{\pi}{n} \right)^k \geq \frac{1}{2} e^{-\frac{k\pi^2}{2n^2} - \frac{k\pi^4}{4n^4}}.$$

This works for any k and any odd $n \geq 3$. \square

Remark 7. Thus, the limit of $\left\| \mathcal{F}(a)^{\star k(n)} - \int_{\mathbb{KP}_n} \right\|_{QTV}$, when n and $\frac{k(n)}{n^2}$ go to infinity, is null, and conversely, is always at least $\frac{1}{2} e^{-\frac{\pi^2}{2} - \frac{\pi^4}{36}}$ for $k(n)$ less than n^2 .

Remark 8. These bounds give in particular that for any positive integer C ,

$$0 < \liminf_{n \rightarrow +\infty} \left\| \mathcal{F}(a)^{\star Cn^2} - \int_{\mathbb{KP}_n} \right\|_{QTV} \leq \limsup_{n \rightarrow +\infty} \left\| \mathcal{F}(a)^{\star Cn^2} - \int_{\mathbb{KP}_n} \right\|_{QTV} < 1.$$

Thus k should satisfies $\lim_{n \rightarrow +\infty} \frac{k}{n^2} = +\infty$ in order to get the convergence of the upper bound to 0 when n goes to infinity. But, in this case, the lower bound vanishes too.

2.4. Asymptotic behavior. We go back to the setting where n and a are fixed, and k does not depend on them. We now study what happens when k goes to infinity.

2.4.1. Possible limit states. Let us note that, when the random walk converges, the limit state μ is an idempotent state. Thanks to [15], we know all the idempotent states on the Sekine quantum groups. There are four different types:

- $\int_{\mathbb{KP}_n}$, the Haar state,
- $h_\Gamma = \frac{2n^2}{\#\Gamma} \sum_{(i,j) \in \Gamma} \mathcal{F}(e_{(i,j)})$ where Γ is a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$,
- $h_{\Gamma,l} = \frac{n^2}{\#\Gamma} \sum_{(i,j) \in \Gamma} \mathcal{F}(e_{(i,j)}) + q \sum_{m \equiv l[q]} \mathcal{F}(E_{m,m})$ where $\Gamma = \mathbb{Z}_n \times q\mathbb{Z}_n$, $q \mid n$, $q > 1$ and $l \in \mathbb{Z}_q$,
- $h_{\Gamma,l,\tau} = \frac{n^2}{\#\Gamma} \sum_{(i,j) \in \Gamma} \mathcal{F}(e_{(i,j)}) + q \sum_{i,j \equiv l[q]} \tau_{j-i} \mathcal{F}(E_{i,j})$ where $\Gamma = p\mathbb{Z}_n \times q\mathbb{Z}_n$, $p > 1$, $pq = n$, $l \in \mathbb{Z}_q$ and $\tau \in \{-1, 1\}^{q\mathbb{Z}_n}$ such that $\sum_{j \in q\mathbb{Z}_n} \tau_j \eta^{ij} > 0$ for all $i \in \mathbb{Z}_p$.

Let us note that the two first types are Haar idempotent states, which means that they arise from the Haar state of a quantum subgroup.

Remark 9. We already know that

$$\widehat{\int_{\mathbb{KP}_n}} (\mathfrak{X}^{u,v}) = \delta_{(u,v), (0,0)} I_2.$$

The Remark 2.11 in [15] gives a useful characterization of types $h_{\Gamma,l}$ and $h_{\Gamma,l,\tau}$ by the $\hat{\mu}(\mathfrak{X}^{u,v})$'s:

$$(12) \quad \widehat{h_{\Gamma,l}}(\mathfrak{X}^{u,v}) = \begin{cases} \frac{1}{2} \begin{pmatrix} 1 & \eta^{-vl} \\ \eta^{vl} & 1 \end{pmatrix} & \text{if } u = 0 \text{ and } \frac{n}{q} \mid v \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise} \end{cases}$$

$$(13) \quad \widehat{h_{\Gamma,l,\tau}}(\mathfrak{X}^{u,v}) = \begin{cases} \frac{1}{2} \begin{pmatrix} 1 & \tau_{-v}\eta^{-vl} \\ \tau_v\eta^{vl} & 1 \end{pmatrix} & \text{if } q \mid u \text{ and } \frac{n}{q} \mid v \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise} \end{cases}$$

In particular, for μ of these types, and for v between 1 and $\lfloor \frac{n-1}{2} \rfloor$, the matrix $\hat{\mu}(\mathfrak{X}^{n-u,v})$ is either null or not diagonal. Thus, when some $a_{\mathfrak{X}^{u,v}}$ equals 2, by the proof of Theorem 3, the limit state should be of type h_{Γ} , if it exists.

Proposition 2. *If the random walk associated to a converges, the limit state μ is a central idempotent state.*

Proof. A central idempotent state is a state ϕ on $C(\mathbb{K}\mathbb{P}_n)$ such that $\phi \star \phi = \phi$ and $\phi \star \psi = \psi \star \phi$ for every bounded linear functional ψ on $C(\mathbb{K}\mathbb{P}_n)$.

Let us note that if $\mathcal{F}(a)^{\star k} \star \psi = \psi \star \mathcal{F}(a)^{\star k}$ holds for all k , then $\mu \star \psi = \psi \star \mu$ is verified, where μ is the limit state of the random walk $\mathcal{F}(a)^{\star k}$. Thus, we will study the commutativity for the random walk. Actually, we only need to check if $\mathcal{F}(a)$ is central, for a a linear combination of irreducible characters.

Let ψ be a linear functional on $C(\mathbb{K}\mathbb{P}_n)$. Then, there exists b in $C(\mathbb{K}\mathbb{P}_n)$ such that $\psi = \mathcal{F}(b)$. We want to prove the equality $\psi \star \mathcal{F}(a) = \mathcal{F}(a) \star \psi$. This is to show that if a is a linear combination of characters, $a \star b = b \star a$ holds for all b in $C(\mathbb{K}\mathbb{P}_n)$.

Using Sweedler notation, by definition of the convolution product on $C(\mathbb{K}\mathbb{P}_n)$, since the Haar state is tracial and S is involutive,

$$\begin{aligned} b \star a &= \sum_{(a)} a_{(2)} \int_{\mathbb{K}\mathbb{P}_n} S(a_{(1)})b \\ &= \left(\int_{\mathbb{K}\mathbb{P}_n} \otimes \rho_0^+ \right) [(b \otimes \rho_0^+) \cdot (S \otimes \rho_0^+) \Delta(a)] \\ &= \left(\int_{\mathbb{K}\mathbb{P}_n} \otimes S \right) [(b \otimes \rho_0^+) \cdot (S \otimes S) \Delta(a)] . \end{aligned}$$

Let us denote by σ the morphism $x \otimes y \mapsto y \otimes x$, and note that for any α in $I(\mathbb{K}\mathbb{P}_n)$, $\sigma(\Delta(\chi_\alpha)) = \Delta(\chi_\alpha)$. The property of the antipode gives $(S \otimes S) \Delta(a) = \sigma(\Delta(S(a)))$. Moreover the element $S(a)$ is also a linear

combination of characters, hence $\sigma(\Delta(S(a))) = \Delta(S(a))$. Thus, we get

$$\begin{aligned}
b \star a &= S \left(\int_{\mathbb{KP}_n} \otimes \rho_0^+ \right) [(b \otimes \rho_0^+) \cdot \Delta(S(a))] \\
&= S^2 \left(\int_{\mathbb{KP}_n} \otimes \rho_0^+ \right) [\Delta(b) \cdot (S(a) \otimes \rho_0^+)] \\
&= \left(\int_{\mathbb{KP}_n} \otimes \rho_0^+ \right) (S \otimes \rho_0^+) [(S \otimes \rho_0^+) \Delta(b) \cdot (a \otimes \rho_0^+)] \\
&= a \star b
\end{aligned}$$

where the second equality comes from the Kac property, and the last one from the fact that $\int_{\mathbb{KP}_n} S(x) = \int_{\mathbb{KP}_n} x$. \square

Remark 10. This result is true in any compact quantum group of Kac type, such that the Haar state is invariant with respect to the antipode.

2.4.2. Convergence to h_Γ .

Lemma 4. *Assume that the random walk associated to a converges, and denote by μ the limit state. The followings are equivalent:*

- i) $\forall l \in \mathbb{Z}_n$, $a_{\rho_l^+} = a_{\rho_l^-}$ and $a_{\sigma_l^+} = a_{\sigma_l^-}$ (when n is even)
 - ii) $a_{\rho_0^-} = 1$
 - iii) μ is a h_Γ for Γ a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$ such that
- $$(14) \quad (k, l) \in \Gamma \iff (k, -l) \in \Gamma$$

Proof. The implication $i) \Rightarrow ii)$ is clear. The converse follows from the fact that A is positive semidefinite.

Now, let us look at the equivalence between $ii)$ and $iii)$. First of all, let us note that

$$a^{\star k} = \sum_{\alpha \in I(\mathbb{KP}_n)} \frac{a_\alpha^k}{d_\alpha^{k-1}} \chi_\alpha.$$

Let us denote by $a_{(i,j)}^{(k)}$ and $A_{i,j}^{(k)}$ its coefficients in the canonical basis, and by $\mu_{(i,j)}$ and $M_{i,j}$ the coefficients of μ in basis $\{\mathcal{F}(e_{(i,j)}), \mathcal{F}(E_{i,j}), 1 \leq i, j \leq n\}$.

Then by [15], either $\sum_{i,j \in \mathbb{Z}_n} \mu_{(i,j)} = 2n^2$ and $\sum_{m=1}^n M_{m,m} = 0$ and μ is of type

h_Γ , or $\sum_{i,j \in \mathbb{Z}_n} \mu_{(i,j)} = n^2$ and $\sum_{m=1}^n M_{m,m} = n$ and μ is of another type.

On the other hand,

$$\begin{aligned}
\sum_{i,j \in \mathbb{Z}_n} \mu_{(i,j)} &= \lim_{k \rightarrow +\infty} \sum_{i,j \in \mathbb{Z}_n} a_{(i,j)}^{(k)} = \lim_{k \rightarrow +\infty} n^2 \left(1 + a_{\rho_0^-}^k \right) \\
\sum_{m=1}^n M_{m,m} &= \lim_{k \rightarrow +\infty} \sum_{i \in \mathbb{Z}_n} A_{i,i}^{(k)} = \lim_{k \rightarrow +\infty} n \left(1 - a_{\rho_0^-}^k \right)
\end{aligned}$$

Therefore the limit μ satisfies $\sum_{i,j \in \mathbb{Z}_n} \mu_{(i,j)} = 2n^2$ and $\sum_{m=1}^n M_{m,m} = 0$ if and only if $a_{\rho_0^-} = 1$.

Moreover, for a a linear combination of characters, $a_{(i,-j)}^{(k)} = a_{(i,j)}^{(k)}$ then $\mu_{(i,j)} = \mu_{(i,-j)}$. Thus the subgroup Γ of $\mathbb{Z}_n \times \mathbb{Z}_n$ satisfies the condition (14). \square

Remark 11. By Remark 9, we obtain that if the random walk converges and there exists $\alpha \in I(\mathbb{KP}_n)$ such that $a_\alpha = d_\alpha = 2$, then $a_{\rho_0^-} = 1$.

Lemma 5. *Assume that the random walk associated to the element a converges. If there exists a one-dimensional representation α^+ in $I(\mathbb{KP}_n)$ such that $a_{\alpha^+} = a_{\alpha^-} = 1$, then the limit state is of type h_Γ , with Γ a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$ satisfying (14).*

Proof. Let us denote by \mathfrak{X}^α the corresponding $\mathfrak{X}^{u,v}$ with $u \in \mathbb{Z}_n$ and $v = 0$ if $\alpha^+ = \rho_{n-u}^+$ or $v = \frac{n}{2}$ if $\alpha^+ = \sigma_{n-u}^+$. Then $\widehat{\mathcal{F}(a)^{*k}}(\mathfrak{X}^\alpha) = I_2$. Hence, by characterizations (12) and (13), the limit state is of type h_Γ . The same argument as in the preceding lemma gives the property of Γ . \square

2.4.3. Classification.

Theorem 5. *Denote by μ the limit state of the random walk associated to a , if it exists.*

If n is odd, there are only four possibilities:

- condition (9) holds, i.e. $\mu = \int_{\mathbb{KP}_n}$,
- $\forall \alpha \in I(\mathbb{KP}_n)$, $|a_\alpha| < d_\alpha$ or $a_\alpha = d_\alpha$, and $a_{\rho_0^-} = 1$, i.e. $\mu = h_\Gamma$, with Γ a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$ satisfying (14)
- $\forall \alpha \in I(\mathbb{KP}_n)$, $|a_\alpha| < d_\alpha$ or $a_\alpha = 1$, $a_{\rho_0^-} \neq 1$ but condition (9) does not hold, i.e. $\mu = h_{\Gamma,l,\tau}$ with $q = 1$ (and $p = n$), $l = 0$ and $\tau_i = 1$ for all $i \in \mathbb{Z}_n$,
- otherwise the random walk diverges.

If n is even, there are six possibilities:

- condition (9) holds, i.e. $\mu = \int_{\mathbb{KP}_n}$,
- $\forall \alpha \in I(\mathbb{KP}_n)$, $|a_\alpha| < d_\alpha$ or $a_\alpha = d_\alpha$, and $a_{\rho_0^-} = 1$, i.e. $\mu = h_\Gamma$, with Γ a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$ satisfying (14)
- $\forall \alpha \in I(\mathbb{KP}_n)$, $|a_\alpha| < d_\alpha$ or $a_\alpha = 1$, $a_{\rho_0^-} \neq 1$, $a_{\sigma_0^+} = 1$ or $a_{\sigma_0^-} = 1$ but $a_{\sigma_0^+} \neq a_{\sigma_0^-}$, $|a_{\sigma_2^+}| < 1$ and $|a_{\sigma_2^-}| < 1$, i.e. $\mu = h_{\Gamma,l}$ with $q = 2$ and l equals 0 if $a_{\sigma_0^+} = 1$, or 1 if $a_{\sigma_0^-} = 1$,
- $\forall \alpha \in I(\mathbb{KP}_n)$, $|a_\alpha| < d_\alpha$ or $a_\alpha = 1$, $a_{\rho_0^-} \neq 1$, $|a_{\sigma_0^+}| < 1$ and $|a_{\sigma_0^-}| < 1$ but condition (9) does not hold, i.e. $\mu = h_{\Gamma,l,\tau}$ with $q = 1$ (and $p = n$), $l = 0$ and $\tau_i = 1$ for all $i \in \mathbb{Z}_n$,

- $\forall \alpha \in I(\mathbb{KP}_n)$, $|a_\alpha| < d_\alpha$ or $a_\alpha = 1$, $a_{\rho_0^-} \neq 1$, $a_{\sigma_0^+} = a_{\sigma_2^+} = 1$ or $a_{\sigma_0^-} = a_{\sigma_2^-} = 1$ but $a_{\sigma_0^+} \neq a_{\sigma_0^-}$, i.e. $\mu = h_{\Gamma, l, \tau}$ with $q = 2$ (and $p = \frac{n}{2}$),
- otherwise the random walk diverges.

Remark 12. By Proposition 2, the limit states are central for the convolution product. One can check that all the central idempotent states arise as limit of such random walks.

Proof. We only need to look at the case when the limit μ exists but is not the Haar state or some h_Γ . We will again use the characterizations (12) and (13) of the idempotent states $h_{\Gamma, l}$ and $h_{\Gamma, l, \tau}$.

The limit of the $\widehat{\mathcal{F}(a)^{\star k}}(\mathfrak{X}^{u, v})$'s is null for all v between 1 and $\lfloor \frac{n-1}{2} \rfloor$, which is equivalent to q equals 1 or 2.

Since q divides n , if n is odd, the only possibility is $q = 1$. Hence, μ is the state $h_{\{0\} \times \mathbb{Z}_n, 0, (1)_{i=0}^{n-1}}$.

If n is even, $\mathfrak{X}^{0, \frac{n}{2}}$ exists, and

$$\widehat{\mathcal{F}(a)^{\star k}}(\mathfrak{X}^{0, \frac{n}{2}}) = \frac{1}{2} \begin{pmatrix} a_{\sigma_0^+}^k + a_{\sigma_0^-}^k & a_{\sigma_0^+}^k - a_{\sigma_0^-}^k \\ a_{\sigma_0^+}^k - a_{\sigma_0^-}^k & a_{\sigma_0^+}^k + a_{\sigma_0^-}^k \end{pmatrix}$$

tends to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ or $\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Let us note that the limit is null if and only if $a_{\sigma_0^+}$ and $a_{\sigma_0^-}$ have both modulus less than 1. Since, among the possible limit states, the equality $\widehat{\mu}(\mathfrak{X}^{0, \frac{n}{2}}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ characterizes $h_{\{0\} \times \mathbb{Z}_n, 0, (1)_{i=0}^{n-1}}$, we get the fourth case.

Assume now that the previous limit is not the matrix null. The case $u = n - 2$ and $v = \frac{n}{2}$ will determine the limit, since

$$\widehat{h_{\Gamma, l}}(\mathfrak{X}^{n-2, \frac{n}{2}}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{whereas} \quad \widehat{h_{\Gamma, l, \tau}}(\mathfrak{X}^{n-2, \frac{n}{2}}) = \widehat{h_{\Gamma, l, \tau}}(\mathfrak{X}^{0, \frac{n}{2}}).$$

We have

$$\widehat{\mathcal{F}(a)^{\star k}}(\mathfrak{X}^{n-2, \frac{n}{2}}) = \frac{1}{2} \begin{pmatrix} a_{\sigma_2^+}^k + a_{\sigma_2^-}^k & a_{\sigma_2^+}^k - a_{\sigma_2^-}^k \\ a_{\sigma_2^+}^k - a_{\sigma_2^-}^k & a_{\sigma_2^+}^k + a_{\sigma_2^-}^k \end{pmatrix}$$

whose limit is null, if and only if $|a_{\sigma_2^+}| < 1$ and $|a_{\sigma_2^-}| < 1$. This concludes the proof of the theorem. \square

Remark 13. If we look at the characterizations (12) and (13), we have also the following equivalences:

$$\begin{aligned} \forall l \in \mathbb{Z}_n, a_{\rho_l^+} = 1 \text{ and } |a_\alpha| < d_\alpha \text{ for all the others } \alpha \in I(\mathbb{KP}_n) \\ \iff \mu = h_{\{0\} \times \mathbb{Z}_n, 1, (1)_{i=0}^{n-1}} \end{aligned}$$

$$a_{\rho_0^+} = 1, a_{\sigma_0^+} = 1 \text{ and } |a_\alpha| < d_\alpha \text{ for all the others } \alpha \in I(\mathbb{KP}_n) \\ \iff \mu = h_{\mathbb{Z}_n \times 2\mathbb{Z}_n, 0}$$

$$a_{\rho_0^+} = 1, a_{\sigma_0^-} = 1 \text{ and } |a_\alpha| < d_\alpha \text{ for all the others } \alpha \in I(\mathbb{KP}_n) \\ \iff \mu = h_{\mathbb{Z}_n \times 2\mathbb{Z}_n, 1}$$

$$\forall i \in \mathbb{Z}_{\frac{n}{2}}, a_{\rho_{2i}^+} = 1, \forall i \in \mathbb{Z}_{\frac{n}{2}}, a_{\sigma_{2i}^+} = 1 \text{ or } \forall i \in \mathbb{Z}_{\frac{n}{2}}, a_{\sigma_{2i}^-} = 1 \text{ but } a_{\sigma_0^+} \neq a_{\sigma_0^-}, \\ \text{and } |a_\alpha| < d_\alpha \text{ for all the others } \alpha \in I(\mathbb{KP}_n) \\ \iff \mu = h_{\frac{n}{2}\mathbb{Z}_n \times 2\mathbb{Z}_n, l, \tau}$$

Remark 14. Thanks to Remark 2 we get that the random walk diverges if and only if there exists α in $I(\mathbb{KP}_n)$ such that $|a_\alpha| = d_\alpha$ but $a_\alpha \neq d_\alpha$. In this case, the random walk can be cyclic or not.

Example 2. Let us look again at the state ϕ_p defined above. By Theorem 2, the random walk diverges. Moreover, we have

$$\phi_p^{\star k} = \sum_{l=0}^{q-1} \eta^{klp} \mathcal{F}(\rho_{lp}^+),$$

for $q > 1$ such that $pq = n$. Thus the random walk is cyclic, with period q .

Let us note that $\rho_0^+ - \rho_0^- = 2 \sum_{m=1}^n E_{m,m}$ clearly satisfies the conditions of Lemma 6.4 in [6]. The associated random walk is also cyclic, with period 2, for all $n \geq 2$ (even for odd n). The state is the Haar state when k is even and $2 \sum_{m=1}^n \mathcal{F}(E_{m,m})$ for k odd.

Thus, we can define the random walk

$$\psi_p = \mathcal{F} \left((\rho_0^+ - \rho_0^-) \sum_{l=0}^{q-1} \eta^{lp} \rho_{lp}^+ \right) = \sum_{l=0}^{q-1} \left(\eta^{lp} \mathcal{F}(\rho_{lp}^+) - \eta^{lp} \mathcal{F}(\rho_{lp}^-) \right)$$

which is also cyclic, with period $\text{lcm}(2, q)$. In particular, it has period $2q$ when n is odd.

Example 3. Let us note that $\rho_0^+ - \rho_0^- + \frac{1}{2} \sum_{l=2}^{n-1} (\rho_l^+ - \rho_l^-)$ defines a state. One can check that the associated random walk is not cyclic and admits no limit.

2.5. Random walks in the dual of \mathbb{KP}_n . Let us denote the dual of \mathbb{KP}_n by $\widehat{\mathbb{KP}_n} = (C(\widehat{\mathbb{KP}_n}), \hat{\Delta})$, with the dual basis $\{e^{(i,j)}\}_{i,j \in \mathbb{Z}_n} \cup \{E^{ij}\}_{1 \leq i,j \leq n}$. Then its irreducible characters are the $e^{(i,j)}$'s and $\chi(\hat{\mathbf{x}}) = \sum_{i=1}^n E^{ii}$.

The Fourier transform, also denoted \mathcal{F} , on $C(\widehat{\mathbb{KP}_n})$ allows us to go back in the Sekine finite quantum group \mathbb{KP}_n , identifying $\mathcal{F}(e^{(i,j)})$ with $e_{(-i,-j)}$ and $\mathcal{F}(E^{ij})$ with $\frac{1}{n} E_{ij}$.

The Fourier transform of element $a = \sum_{\alpha \in I(\widehat{\mathbb{KP}}_n)} a_\alpha \chi_\alpha$ defines a state on $C(\widehat{\mathbb{KP}}_n)$ if and only if a_α is non negative for any α and $a_{(0,0)} = a_{e^{(0,0)}} = 1$.

The same type of computation as above gives us the following result:

Proposition 3. *The random walk associated to such an element a in $C(\widehat{\mathbb{KP}}_n)$ converges to the Haar state $\int_{\widehat{\mathbb{KP}}_n}$ if and only if*

$$\forall \alpha \in I(\widehat{\mathbb{KP}}_n) \setminus \{e^{(0,0)}\}, \quad a_\alpha < d_\alpha .$$

If this condition is not satisfied, but $a_\alpha \leq d_\alpha$ for any irreducible representation α , then the random walk converges to some central idempotent state, i.e.

an element of $C(\mathbb{KP}_n)$ of the form $\sum_{i,j \in \mathbb{Z}_n} \varepsilon_{(i,j)} e_{(i,j)} + \varepsilon_{\hat{x}} \sum_{i=1}^n E_{ii}$, with $\varepsilon_\alpha = 0$ or

1, for any α in $I(\widehat{\mathbb{KP}}_n)$. Moreover, every idempotent element in the center of $C(\mathbb{KP}_n)$ can be obtained this way.

Otherwise, there exists α such that $a_\alpha > d_\alpha$ and the random walk diverges.

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