# OPTIMAL ANGLE OF THE HOLOMORPHIC FUNCTIONAL CALCULUS FOR THE CLASSICAL ORNSTEIN-UHLENBECK OPERATOR ON $L^p$

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ABSTRACT. We give a simple proof of the fact that the classical Ornstein-Uhlenbeck operator L is R-sectorial of angle arcsin|1-2/p| on  $L^p(\mathbb{R}^n, \exp(-|x|^2/2)dx)$  (for 1 ). Applying the abstract holomorphic functional calculus theory of Kalton and Weis, this immediately gives a new proof of the fact that <math>L has a bounded  $H^{\infty}$  functional calculus with this optimal angle.

#### 1. Introduction

The Ornstein-Uhlenbeck operator appears in many areas of mathematics: as the number operator of quantum field theory, the analogue of the Laplacian in the Malliavin calculus, the generator of the transition semigroup associated with the simplest mean-reverting stochastic process (the Ornstein-Uhlenbeck process), or as the operator associated with the classical Dirichlet form on  $\mathbb{R}^d$  equipped with the Gaussian measure  $d\mu = e^{-|x|^2/2}dx$ . For the sake of this paper, the Ornstein-Uhlenbeck operator will be defined via the Ornstein-Uhlenbeck semigroup  $\{T_t\}_{t>0}$  given by

**Definition 1.** For t > 0 and  $f \in L^p(\mu)$ , define  $T_t f : \mathbb{R}^d \to \mathbb{C}$  as

$$x \mapsto \int_{\mathbb{R}^d} M_t(x, y) f(y) dy,$$

where  $M_t: \mathbb{R}^{2d} \to \mathbb{R}$  is given by

$$(x,y) \mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{1}{1 - e^{-2t}} \right)^{\frac{d}{2}} \exp\left( -\frac{1}{2} \frac{|e^{-t}x - y|^2}{(1 - e^{-2t})} \right),$$

the Mehler kernel.

Let us recall the basic properties of the Ornstein-Uhlenbeck semigroup used in this article.

**Theorem A.** For each  $p \in [1, \infty]$  and each t > 0, the map  $f \mapsto T_t f$  is bounded  $L^p(\mu) \to L^p(\mu)$ , with operator norm at most 1, and is a positive operator. For  $p \in [1, \infty)$ ,  $T_t : L^p(\mu) \to L^p(\mu)$  is a  $C_0$  semigroup, i.e. as  $t \to 0$ ,  $T_t \to I$  strongly and  $T_t T_s = T_{t+s}$  for all t, s > 0.

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For a proof of these preliminary facts, see for example Theorem 2.5 of [6]. It should be noted that although the Ornstein-Uhlenbeck semigroup arises from many different areas of mathematics, these basic properties can be proven solely with use of the explicit kernel and elementary techniques. It is a simple calculation to show that  $T_t$  is bounded with norm 1 on both  $L^{\infty}(\mu)$  and  $L^1(\mu)$ , from which interpolation can be used to deduce boundedness with norm 1 on  $L^p(\mu)$  for  $p \in [1, \infty]$ . Positivity follows from non-negativity of the Mehler kernel. The  $C_0$  nature follows as in typical proofs of the strong continuity of the classical heat semigroup, and the semigroup property follows from a tedious exercise in integrating Gaussian functions. Due to Theorem A, we can talk about the generator of the Ornstein-Uhlenbeck semigroup on  $L^p(\mu)$ ,  $p \in [1, \infty)$ , whose negative we shall call the Ornstein-Uhlenbeck operator and denote by L. Theorem 1.4 of [2], with the  $C_0$  nature of the Ornstein-Uhlenbeck semigroup, implies that L is a closed densely-defined unbounded operator on  $L^p(\mu)$ ,  $p \in [1, \infty)$ , which uniquely determines  $T_t$ . Thus from here on, we will use the notation  $\exp(-tL)$  for the operator  $T_t$ , on any of its possible domains (in arguments, p will have already been fixed so that there will be no confusion).

This paper presents a new proof of the following theorem

**Theorem 2.** For  $p \in (1, \infty)$ , the Ornstein-Uhlenbeck operator has a bounded  $H^{\infty}(\Sigma_{\theta_p})$  functional calculus on  $L^p(\mu)$ , where  $\sin(\theta_p) = \left|1 - \frac{2}{p}\right|$ .

See [5] for the theory of  $H^{\infty}$  functional calculus, and note that the difficulty here is to prove the boundedness of the calculus with precisely the angle  $\theta_p$  (which is known to be best possible).

This result was originally proven by García-Cuerva, Mauceri, Meda, Sjögren and Torrea in [4]. They use Mauceri's abstract multiplier theorem to reduce the problem to precisely estimating  $u \mapsto ||L^{iu}||$ . To do so, they express  $L^{iu}$  as an integral of the semigroup, using a carefully chosen contour of integration. They then consider the kernels of operators corresponding to different parts of the contour, and decompose them into a local and global part. To treat the global parts they then use a range of subtle kernel estimates.

In [1], Carbonaro and Dragicevic reproved and extended this result to treat arbitrary generators of symmetric contraction semigroups on an  $L^p$  space. To prove this striking result, they first reduce the problem to proving a bilinear embedding for the semigroup, with constants depending optimally on the angle  $\theta_p$ . They then use the Bellman function method, controlling the bilinear form by an optimally (depending on p) chosen function. This function turns out to be a known Bellman function introduced by Nazarov and Treil, but just proving that it has the right properties is a highly non-trivial task.

In contrast, the proof presented in this paper is mostly self-contained and completely transparent, requiring only simple manipulations of the kernel of the Ornstein-Uhlenbeck semigroup. It is based on an approach designed by van Neerven and Portal in [7], where they also recover classical results about the Ornstein-Uhlenbeck semigroup in a very direct manner. Their idea is to separate algebraic difficulties from analytic difficulties by considering a non-commutative functional calculus of the Gaussian position and momentum operators (the Weyl calculus). Using this calculus, one sees how to modify the kernels in a way that make their analysis straightforward. A posteriori, the use of

the Weyl calculus can be removed, and the proof can be read as a simple computation exploiting the change of time parameter  $t \mapsto \frac{1+e^{-t}}{1-e^{-t}}$  (which has been used by many authors before).

We shall use the following abstract result on the  $H^{\infty}$  functional calculus (Theorem 10.7.13 of [5])

**Theorem B.** Let  $(\Omega, m)$  be a measure space ( $\sigma$ -algebra omitted) and fix  $p \in (1, \infty)$ . If an unbounded operator T on  $L^p(\Omega, m)$  generates an analytic semigroup which is a positive contraction semigroup for real time, then T is R-sectorial and T has a bounded  $H^\infty$  functional calculus of the same angle as the angle of R-sectoriality.

See [5] for the theory of R-sectoriality. Theorem 2 then follows once we have proven that L generates an analytic semigroup and is R-sectorial of angle  $\theta_p$ , which we do in Theorem 5.

Throughout the paper, we make use of the following notation. The function  $\phi: \mathbb{R}^d \to \mathbb{R}$  will have action  $x \mapsto \frac{x^2}{2}$ . The Borel measure  $\mu$  on  $\mathbb{R}^d$  will have density  $d\mu = e^{-\phi(x)}dx$ . The Lebesgue measure on  $\mathbb{R}^d$  will be denoted by  $\lambda$ . As we only ever work over  $\mathbb{R}^d$  with Borel  $\sigma$ -algebra, the measurable space over which we consider Lebesgue spaces will be dropped from notation. For  $\theta \in [0, \pi]$ , we will write  $\Sigma_{\theta}$  for the sector  $\Sigma_{\theta} = \{z \in \mathbb{C} \setminus \{0\}; \Re(z) > \cos(\theta)|z|\}$ . We make use of alphabetical indexing for other's theorems, and numerical indexing for new or re-proven results.

## 2. R-Sectoriality of L

To simplify things, for the rest of the article we will assume that  $p \in (1, \infty)$  is fixed. Similarly, all concepts of boundedness and R-boundedness will be on either  $L^p(\mu)$  or  $L^p(\lambda)$  without explicit mention of the space, the measure being clear from context.

**Lemma 3.**  $M_t$  has the alternate form for t > 0 and  $x, y \in \mathbb{R}^d$ ,

$$M_t(x,y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{1}{1 - e^{-2t}} \right)^{\frac{d}{2}} \exp\left( -s_t \left( \frac{x + y}{2\sqrt{2}} \right)^2 - \frac{1}{4s_t} \left( \frac{x - y}{\sqrt{2}} \right)^2 \right) \exp\left( \frac{1}{2} \left( \phi(x) - \phi(y) \right) \right),$$

where  $s_t = \frac{1 - e^{-t}}{1 + e^{-t}}$ .

*Proof.* We will just rearrange the exponent from Definition 1 and show that it is equal to the exponent given above for all  $x, y \in \mathbb{R}^d$  and t > 0, as that is all that has changed between the two representations. For each  $t > 0, x, y \in \mathbb{R}^d$  we have

$$\begin{split} &-\frac{1}{2}\frac{|e^{-t}x-y|^2}{(1-e^{-2t})} = -\frac{1}{2}\frac{|e^{-t}x-y|^2}{(1-e^{-2t})} - \frac{1}{4}(x^2-y^2) + \frac{1}{4}(x^2-y^2) \\ &= -\frac{1}{2}\frac{|e^{-t}x-y|^2}{(1-e^{-2t})} - \frac{1}{4}(x^2-y^2) + \frac{1}{2}\left(\phi(x)-\phi(y)\right) \\ &= -\frac{1}{2(1-e^{-2t})}\left(|e^{-t}x-y|^2 + \frac{(1-e^{-2t})}{2}(x^2-y^2)\right) + \frac{1}{2}\left(\phi(x)-\phi(y)\right) \\ &= -\frac{1}{2(1-e^{-2t})}\left(e^{-2t}x^2 - 2e^{-t}xy + y^2 + \frac{(1-e^{-2t})}{2}(x^2-y^2)\right) + \frac{1}{2}\left(\phi(x)-\phi(y)\right) \\ &= -\frac{1}{2(1-e^{-2t})}\left(\frac{1}{2}\left(1+e^{-2t}\right)x^2 - 2e^{-t}xy + \frac{1}{2}\left(1+e^{-2t}\right)y^2\right) + \frac{1}{2}\left(\phi(x)-\phi(y)\right) \\ &= -\frac{1}{8(1-e^{-2t})} \\ &*\left(\left((1+e^{-t})^2 + (1-e^{-t})^2\right)x^2 + 2\left((1-e^{-t})^2 - (1+e^{-t})^2\right)xy + \left((1+e^{-t})^2 + (1-e^{-t})^2\right)y^2\right) \\ &+ \frac{1}{2}\left(\phi(x)-\phi(y)\right) \\ &= -\frac{1}{8(1-e^{-2t})}\left((1-e^{-t})^2(x+y)^2 + (1+e^{-t})^2(x-y)^2\right) + \frac{1}{2}\left(\phi(x)-\phi(y)\right) \\ &= -\left(\frac{1-e^{-t}}{1+e^{-t}}\left(\frac{x+y}{2\sqrt{2}}\right)^2 + \frac{1}{4}\frac{1+e^{-t}}{1-e^{-t}}\left(\frac{x-y}{\sqrt{2}}\right)^2\right) + \frac{1}{2}\left(\phi(x)-\phi(y)\right) \\ &= -\left(s_t\left(\frac{x+y}{2\sqrt{2}}\right)^2 + \frac{1}{4s_t}\left(\frac{x-y}{\sqrt{2}}\right)^2\right) + \frac{1}{2}\left(\phi(x)-\phi(y)\right). \end{split}$$

The next definition; albeit a simple one, forms the backbone of the rest of our arguments.

**Definition 4.** Define the isometry  $U_p: L^p(\mu) \to L^p(\lambda)$  by

$$U_p f = \left(x \mapsto f(x) \exp\left(-\frac{\phi(x)}{p}\right)\right).$$

To get to the proof of the critical result of this paper, that the Ornstein Uhlenbeck operator is R-sectorial of the known optimal angle, we shall apply the following Proposition 10.3.3 of [5]

**Theorem C.** Let A be a linear operator on a Banach space X. Then the following are equivalent.

- (1) A is R-sectorial of some angle  $\theta < \frac{\pi}{2}$ .
- (2) -A is the generator of an R-bounded analytic semigroup.

Moreover, the supremum of the angles of sectors for which  $\exp(-zA)$  is R-bounded is  $\frac{\pi}{2} - \theta$ .

Hence we need only show that the Ornstein-Uhlenbeck operator has an analytic extension to a sector of the correct angle, and that it is R-bounded on each smaller sector. We will in fact show a lot

more with no more effort. We shall work with the reparametrisation of the kernel of the semigroup in terms of  $s_t$  from Lemma 3. The function  $t \mapsto s_t$  is analytic and can clearly be extended to  $\mathbb{C}\setminus i\pi(2\mathbb{Z}+1)$ . We will consider the analytic extension  $z\mapsto s_z$  on domains of the form

(1) 
$$E := \{ z \in \mathbb{C}; s_z \in \Sigma_{\frac{\pi}{2} - \theta_p}; z \notin i\pi \mathbb{Z} \}$$

where  $\sin(\theta_p) = M_p := \left|1 - \frac{2}{p}\right|$ . We will show the Ornstein-Uhlenbeck semigroup extends to an analytic semigroup on the domain E. Moreover, we will simultaneously show that the Ornstein-Uhlenbeck semigroup is R-bounded on sets of the form

(2) 
$$E_{\epsilon,\delta} = \{z \in \mathbb{C}; |\Re(s_z)|^2/|s_z|^2 = \cos^2(\arg(s_z)) > M_p^2 + \epsilon; \operatorname{dist}(z, i\pi(2\mathbb{Z} + 1)) > \delta; z \notin 2i\pi\mathbb{Z}\}$$

for all  $\epsilon, \delta > 0$ . Note that, in terms of the reparametrisation  $s_z$ , these sets are just open sectors of angle  $\frac{\pi}{2} - \theta_p$  or less, with certain points removed. We claim that  $\sum_{\frac{\pi}{2} - \theta_p} \subset E$ , and that for all  $\epsilon' > 0$  there exists  $\epsilon, \delta > 0$  such that  $\sum_{\frac{\pi}{2} - \theta_p - \epsilon'} \subset E_{\epsilon, \delta}$  (see [7] for details of this calculation). These results combined will imply that the maximal domain of analyticity of the Ornstein-Uhlenbeck semigroup contains the sector  $\sum_{\frac{\pi}{2} - \theta_p}$ , and that it is R-bounded on each smaller sector, which combined with the quoted Theorem C will show at least that the Ornstein-Uhlenbeck operator is R-sectorial of the desired angle.

**Theorem 5.** For  $p \in (1, \infty)$ , the Ornstein-Uhlenbeck operator on  $L^p(\mu)$  is R-sectorial of angle  $\theta_p$ , where  $\sin(\theta_p) = M_p := \left|1 - \frac{2}{p}\right|$ .

Proof. To determine (R-)boundedness of the analytic extension of  $\exp(-tL)$  on  $L^p(\mu)$  we conjugate by the isometry  $U_p: L^p(\mu) \to L^p(\lambda)$ , and work with  $U_p \exp(-tL)U_p^{-1}$  on  $L^p(\lambda)$ . As isometries preserve (R-)boundedness,  $\exp(-tL)$  has an analytic extension to  $z \in \mathbb{C}$  if and only if  $U_p \exp(-tL)U_p^{-1}$  does, and both families of operators will be R-bounded on the same subdomains of the domain of analyticity. Using the integral kernel of Lemma 3 and the explicit form of the isometry  $U_p$  from Definition 4, we find the integral representation for  $f \in L^p(\lambda)$ :

$$U_p \exp(-tL)U_p^{-1}f = \left(x \mapsto \int_{\mathbb{R}^d} k_t(x,y)f(y)dy\right),$$

with

$$k_t(x,y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{1}{1 - e^{-2t}} \right)^{\frac{d}{2}} \exp\left( -s_t \left( \frac{x + y}{2\sqrt{2}} \right)^2 - \frac{1}{4s_t} \left( \frac{x - y}{\sqrt{2}} \right)^2 \right) \exp\left( \left( \frac{1}{2} - \frac{1}{p} \right) (\phi(x) - \phi(y)) \right)$$

and  $s_t = \frac{1-e^{-t}}{1+e^{-t}}$ . If  $U_p \exp(-tL)U_p^{-1}$  were to have an analytic extension  $U_p \exp(-zL)U_p^{-1}$  for z in some domains containing  $[0, \infty)$ , uniqueness theory of analytic functions implies that  $U_p \exp(-zL)U_p^{-1}$  would also have an integral representation, with kernel

$$k_z(x,y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{1}{1 - e^{-2z}} \right)^{\frac{d}{2}} \exp\left( -s_z \left( \frac{x+y}{2\sqrt{2}} \right)^2 - \frac{1}{4s_z} \left( \frac{x-y}{\sqrt{2}} \right)^2 \right) \exp\left( \left( \frac{1}{2} - \frac{1}{p} \right) (\phi(x) - \phi(y)) \right),$$

where  $s_z = \frac{1 - e^{-z}}{1 + e^{-z}}$ . We will now work on bounding this kernel. We start by assuming that  $z \in E$  (see Equation (1)). Note that this implies  $Re(s_z) > 0$  and  $1 - e^{-2z} \neq 0$ . Then we have:

$$\begin{aligned} |k_{z}(x,y)| &\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1-e^{-2z}} \right|^{\frac{d}{2}} \exp\left(-\Re(s_{z}) \left(\frac{x+y}{2\sqrt{2}}\right)^{2} - \frac{1}{4}\Re\left(\frac{1}{s_{z}}\right) \left(\frac{x-y}{\sqrt{2}}\right)^{2}\right) \exp\left(\left(\frac{1}{2} - \frac{1}{p}\right) (\phi(x) - \phi(y))\right) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1-e^{-2z}} \right|^{\frac{d}{2}} \exp\left(-\Re(s_{z}) \left(\frac{x+y}{2\sqrt{2}}\right)^{2} + M_{p} \frac{1}{4} \left(x^{2} - y^{2}\right) - \frac{1}{4}\Re\left(\frac{1}{s_{z}}\right) \left(\frac{x-y}{\sqrt{2}}\right)^{2}\right) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1-e^{-2z}} \right|^{\frac{d}{2}} \exp\left(-\Re(s_{z}) \left(\frac{x+y}{2\sqrt{2}}\right)^{2} + M_{p} \left(\frac{x+y}{2\sqrt{2}}\right) \left(\frac{x-y}{\sqrt{2}}\right) - \frac{1}{4}\Re\left(\frac{1}{s_{z}}\right) \left(\frac{x-y}{\sqrt{2}}\right)^{2}\right) \end{aligned}$$

For notational simplicity, let  $u = \frac{x+y}{2\sqrt{2}}$  and  $k = \frac{x-y}{\sqrt{2}}$ . Then rewriting in terms of u and k and completing the square in u gives

$$\begin{split} |k_z(x,y)| & \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \exp\left( -\Re(s_z)u^2 + M_p u k - \frac{1}{4}\Re\left(\frac{1}{s_z}\right)k^2\right) \\ & = \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \exp\left( -\left(\sqrt{\Re(s_z)}u - \frac{M_p}{2\sqrt{\Re(s_z)}}k\right)^2 - \frac{1}{4}\left(\Re\left(\frac{1}{s_z}\right) - \frac{M_p^2}{\Re(s_z)}\right)k^2\right). \end{split}$$

So

$$|k_{z}(x,y)| \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \exp\left(-\frac{1}{4} \left(\Re\left(\frac{1}{s_{z}}\right) - \frac{M_{p}^{2}}{\Re(s_{z})}\right) k^{2}\right)$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \exp\left(-\frac{1}{4} \left(\Re\left(\frac{1}{s_{z}}\right) - \frac{M_{p}^{2}}{\Re(s_{z})}\right) \left(\frac{x - y}{\sqrt{2}}\right)^{2}\right).$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \exp\left(-\frac{1}{8} \left(\Re\left(\frac{1}{s_{z}}\right) - \frac{M_{p}^{2}}{\Re(s_{z})}\right) (x - y)^{2}\right).$$

Let  $g_z: \mathbb{R}^d \to \mathbb{R}$  be the mapping

$$x \mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \exp\left(-\frac{1}{8} \left(\Re\left(\frac{1}{s_z}\right) - \frac{M_p^2}{\Re(s_z)}\right) x^2\right).$$

Then we have that for all  $z \in E$ ,  $f \in L^p(\lambda)$  and a.e.  $x \in \mathbb{R}^d$ 

$$\left| \left( U_p \exp(-tL) U_p^{-1} f \right)(x) \right| \le (g_z * |f|)(x)$$

Therefore, provided the family of convolution operators  $f \in L^p(\lambda) \mapsto g_z * f$  is (R-)bounded for z in (a subset of) E, we will have proven, by domination and isometry, that  $\exp(-zL)$  is (R-)bounded on (the same subset of) E (to see that domination implies R-boundedness, see Proposition 8.1.10

of [5], and note that in the proof of said proposition the fixed positive operator can be replaced by an R-bounded family of positive operators). For  $z \in E$ , we find

$$\Re\left(\frac{1}{s_z}\right) - \frac{M_p^2}{\Re(s_z)} = \frac{\Re(s_z)}{|s_z|^2} - \frac{M_p^2}{\Re(s_z)}$$

$$= \frac{1}{\Re(s_z)} \left(\frac{|\Re(s_z)|^2}{|s_z|^2} - M_p^2\right)$$

$$> 0,$$

since  $\Re(s_z) > 0$  and  $|\Re(s_z)|^2/|s_z|^2 = \cos^2(\arg(s_z)) > M_p^2$  by definition of E (since  $\cos\left(\frac{\pi}{2} - \theta_p\right) = \sin\left(\theta_p\right) = M_p$ ). So for  $z \in E$ ,  $g_z \in L^1(\lambda)$  and so by Young's convolution inequality, convolution by  $g_z$  is a bounded operator on  $L^p(\lambda)$  with operator norm at most  $||g_z||_{L^1(\lambda)}$ . Now we will focus on sets of the form  $E_{\epsilon,\delta}$  for some fixed  $\epsilon,\delta>0$  (see Equation (2)). We will show that

$$\sup_{z \in E_{\epsilon,\delta}} \int_{\mathbb{R}^d} \sup_{|y| > |x|} |g_z(y)| dx < \infty,$$

from which we can apply Proposition 8.2.3 of [5] to find that the family of convolution operators  $\{g_z*\}_{z\in E_{\epsilon,\delta}}$  is R-bounded on  $L^p(\lambda)$ . Noting that each  $g_z$  is radially decaying and positive, the quantity to bound is

$$\sup_{z \in E_{\epsilon, \delta}} \int_{\mathbb{R}^d} \sup_{|y| > |x|} |g_z(y)| dx = \sup_{z \in E_{\epsilon, \delta}} \int_{\mathbb{R}^d} g_z(x) dx \\
= \sup_{z \in E_{\epsilon, \delta}} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \exp\left( -\frac{1}{8} \left( \Re\left( \frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} \right) x^2 \right) dx \\
= \sup_{z \in E_{\epsilon, \delta}} \frac{1}{(2)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \left( \frac{1}{8} \left( \Re\left( \frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} \right) \right)^{-\frac{d}{2}} \\
\leq \sup_{z \in E_{\epsilon, \delta}} 2^d \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \left( \frac{\epsilon}{\Re(s_z)} \right)^{-\frac{d}{2}} \\
= \sup_{z \in E_{\epsilon, \delta}} \epsilon^{-\frac{d}{2}} 2^d \left( \left| \frac{\Re(s_z)}{1 - e^{-2z}} \right| \right)^{\frac{d}{2}} \\
\leq \sup_{z \in E_{\epsilon, \delta}} \epsilon^{-\frac{d}{2}} 2^d \left( \frac{|s_z|}{|1 - e^{-z}||1 + e^{-z}|} \right)^{\frac{d}{2}} \\
= \sup_{z \in E_{\epsilon, \delta}} \epsilon^{-\frac{d}{2}} 2^d \left( \frac{\left| \frac{1 - e^{-z}}{1 + e^{-z}} \right|}{|1 - e^{-z}||1 + e^{-z}|} \right)^{\frac{d}{2}} \\
\leq \sup_{z \in E_{\epsilon, \delta}} \epsilon^{-\frac{d}{2}} 2^d \left( \frac{1}{|1 + e^{-z}|} \right)^d \\
\leq \exp_{z \in E_{\epsilon, \delta}} \epsilon^{-\frac{d}{2}} 2^d \left( \frac{1}{|1 + e^{-z}|} \right)^d \\
\leq \exp_{z \in E_{\epsilon, \delta}} \epsilon^{-\frac{d}{2}} 2^d \left( \frac{1}{|1 + e^{-z}|} \right)^d$$

since z is bounded away from  $(2\mathbb{Z}+1)i\pi$ . So the family of convolution operators  $\{g_z*\}_{z\in E_{\epsilon,\delta}}$  is R-bounded. By pointwise domination,  $U_p\exp(-zL)U_p^{-1}$  is bounded for  $z\in E$ , and is R-bounded

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on subsets  $E_{\epsilon,\delta} \subset E$  of the form (2). Hence by isometric equivalence,  $\exp(-zL)$  shares the same properties. Hence the claim follows from the discussion precluding this proof.

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