Electromagnetic surface wave propagation in a metallic wire and the Lambert W function

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We revisit the solution due to Sommerfeld of a problem in classical electrodynamics, namely, that of the propagation of an electromagnetic axially symmetric surface wave (a low-attenuation single TM_{01} mode) in a cylindrical metallic wire, and his iterative method to solve the transcendental equation that appears in the determination of the propagation wave number from the boundary conditions. We present an elementary analysis of the convergence of Sommerfeld's iterative solution of the approximate problem and compare it both with the numerical solution of the exact transcendental equation and the solution of the approximate problem by means of the Lambert W function.

Keywords: Electromagnetic wave propagation; surface waves; special functions; approximate solutions

I. INTRODUCTION

The propagation of electromagnetic (EM) waves across metallic wires was a possibility of considerable interest in the second half of the XIX century, due both to the advent of the modern theory of electromagnetism and the deployment of the early electric telegraph networks.[1,2] First attempts to investigate EM waves on wires were made, among others, by Hertz about 1886–1887 using spark gaps as sources of EM radiation. Some of the main questions at that time were whether EM waves propagate with a finite velocity, as predicted by Maxwell's equations, and, if so, whether they propagate at the same speed on air and other materials. Underlying these questions was the larger issue of the similarity of EM waves to light. Hertz could solve neither the theoretical nor the experimental problems involving wires, despite his many other outstanding contributions to the field.[3,4]

In 1899, Sommerfeld tackled the problem of the propagation of a circular symmetric EM wave on a thin conducting wire using the full set of Maxwell's equations, demonstrating that such a cylindrical geometry could support what has become known as a surface wave.[5] He showed, by means of an example (see Sec. II), that the attenuation of the surface EM waves at high frequencies was too large to make them useful for telecommunications. In the ensuing years, Sommerfeld's work was extended to include noncircular symmetric modes and to treat coated wires and dielectric cylinders,[6–9] untill by mid-1930s the general interest shifted to the study of EM cavities (metal boxes) and waveguides (hollow metal tubes) as more practical and efficient devices to store and transmit EM energy across space.[10–15]

In 1950, a German physicist by the name of Georg J. E. Goubau, brought to USA from Germany after World War II as part of Operation Paperclip, resumed the investigation of the "Sommerfeld waves" with regard to their practical application to transmission lines.[16] Goubau was able to show, both theoretically and experimentally, that EM surface waves can be generated on wires by means of coaxial flared horn-shaped launchers, making the otherwise theoretical waves practical. His main result, though, was the demonstration that a sheathed or subwavelength corrugated wire would perform better than a naked wire in the transmission of low frequency ($\leqslant 1$ THz) EM energy along the line. This is not because of substantially less ohmic or dielectric losses, but because the modified surface of the wire has a much lower effective conductivity than the metal alone, reducing the extension of the radial electric field outside the wire (field confinement), that for low frequencies and large wire diameters can be large. For example, a wire of radius $a = 10$ mm carrying a surface wave of 1 GHz propagates 75% of its power into an area of radius ~ 1.5 m around the wire, not only diverting energy but also potentially coupling with the environment and compromising the entire engineering solution by requiring too much clearance around the transmission line. Surface EM waves on wires have since been known in radio engineering as Sommerfeld-Goubau waves.[17–20] Such waves have repeatedly been considered as a cheap alternative to long-distance, beyond-the-horizon transportation of EM energy—for example, for last mile simplex or half-duplex data communication over existing overhead electric power lines—in conditions under which microwave and coaxial links become inneficient or costly.[21–24]

In this paper we revisit Sommerfeld's solution of the problem of an EM axially symmetric surface wave (to wit, a pure transverse magnetic, TM_{01} mode) propagating in a cylindrical metallic wire. Of particular interest to us is the solution of the boundary condition for the EM fields that lead to a transcendental equation that Sommerfeld solved approximately by means of a practical and effective device. We explore the iterative procedure devised by Sommerfeld to solve the transcendental equation and briefly investigate its convergence. As we shall see, the solution of the approximate physical system can immediately be given in terms of the (complex-valued) Lambert W function, a most interesting function that gained a lot of attention in the last few decades. Both solutions (Sommerfeld's and the one in terms of the Lambert W function) are compared with the exact numerical solution of the boundary condition in a concrete case. Some remarks on the physical significance of the solutions are made.

II. CIRCULAR SYMMETRIC EM WAVE IN A METALLIC WIRE

Comprehensive treatments of EM waves particularly suited to our discussion can be found in [25, Chap. IX] and [26, §20–§25], while for definitions and properties of special functions we refer the reader to [28–30]. Except for one mention to electron-volts in Sec. III, we use SI units throughout.

Maxwell's equations in a conducting medium free of free charges lead to the following wave equation for the electric field $\mathbf{E} = \mathbf{E}(\mathbf{r},t)$,

$$\nabla^2\mathbf{E} - \sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = 0, \tag{1}$$

with an identical equation for the magnetic field intensity $\mathbf{H} = \mathbf{H}(\mathbf{r},t)$, where ε and μ are, respectively, the time-independent electric permittivity and magnetic permeability of the material and σ its electrical conductivity. Outside of the conductor $\sigma = 0$, and in vacuum (and to a very good approximation also dry air) $\varepsilon = \varepsilon_0 = 1/\mu_0 c^2$ F m^{-1} $\simeq 8.854\times10^{-12}$ F m^{-1} and $\mu = \mu_0 = 4\pi\times10^{-7}$ H m^{-1}, with $c = 1/\sqrt{\varepsilon_0\mu_0} = 299\,792\,458$ m s^{-1} the exact speed of light.

Let us consider a long and thin metallic wire of cylindrical symmetry and constant circular cross section of radius a embedded in a dielectric medium, and assume that there is only a harmonic oscillating electric field along the direction of propagation z, i. e., a pure transverse magnetic (TM) mode. This kind of EM wave can be excited on one end of the wire provided that its source is infinitely far removed, or by some device specifically designed for the task, as the coaxial horn-shaped launchers devised by Goubau in the 1950s (essentially mode transducers transforming the TEM wave of a coaxial cable to the TM_{01} wave of the surface waveguide).[16,31] Following the usual practice, we introduce the complex wave number h (to be determined from the boundary conditions) to express both the propagation and the attenuation of the EM wave across the wire in the z-direction through the time-dependent phase factor $\phi(z,t) = \exp[i(hz-\omega t)]$ with a real angular frequency ω, such that

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}^0(r,\theta)\phi(z,t) = \mathbf{E}^0(r,\theta)\exp[i(hz-\omega t)] \tag{2}$$

and similarly for $\mathbf{H}(\mathbf{r},t)$. The imaginary part of h must be positive if the wave is to attenuate; we shall also take the real part of h positive, corresponding to the phase of the wave propagating in the positive z-direction. Because of the cylindrical symmetry of the problem, the time-independent field vector $\mathbf{E}^0(r,\theta)$ depends only on the r-coordinate, $\mathbf{E}^0(r,\theta) = \mathbf{E}^0(r)$, and similarly for $\mathbf{H}^0(\mathbf{r})$. Within these settings, Maxwell's equations

$$\nabla\times\mathbf{H} = \sigma\mathbf{E} + \frac{\partial(\varepsilon\mathbf{E})}{\partial t} \quad \text{(Ampère's circuital law)} \tag{3a}$$

and

$$\nabla\times\mathbf{E} = -\frac{\partial(\mu\mathbf{H})}{\partial t} \quad \text{(Faraday's law of induction)} \tag{3b}$$

imply that an electric field E_z varying along the z-direction induces (by Ampère's law) a circulating magnetic field H_θ that, in turn, induces (by Faraday's law) an electric field with components in the r and z directions—and the three fields E_z, H_θ, and E_r suffice to describe the physics of the problem. The other components E_θ, H_z, and H_r vanish together. The geometry of these fields are depicted schematically in Figure II.

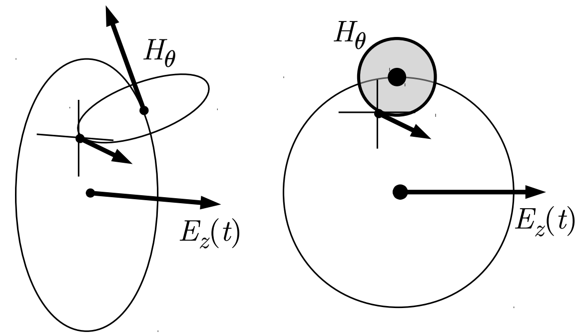

FIG. 1. Schematic geometry of the fields inside the wire, in perspective (left panel) and in the z-r plane (right panel), where H_θ is depicted coming out of the page.

The physics of the problem indicates that everything must depend on E_z, the sole forcing field. Indeed, perusal of equations (3) shows that E_r^0 and H_θ^0 can be obtained from E_z^0 as

$$E_r^0(\hat\rho) = \frac{ih}{\sqrt{\hat k^2 - h^2}}\frac{\partial E_z^0(\hat\rho)}{\partial\hat\rho}, \tag{4a}$$

$$\sqrt{\frac{\mu}{\hat\varepsilon}}H_\theta^0(\hat\rho) = \frac{i\hat k}{\sqrt{\hat k^2 - h^2}}\frac{\partial E_z^0(\hat\rho)}{\partial\hat\rho}, \tag{4b}$$

where we have introduced the complex quantities

$$\hat k^2 = \varepsilon\mu\omega^2 + i\sigma\mu\omega, \tag{5a}$$

$$\hat\rho = \sqrt{\hat k^2 - h^2}\,r, \tag{5b}$$

$$\hat\varepsilon = \frac{\hat k^2}{\mu\omega^2} = \varepsilon + i\frac{\sigma}{\omega}. \tag{5c}$$

The hats indicate the quantities inside the wire ($0 < r < a$), where $\sigma \neq 0$; outside of the wire ($r > a$) they become real and lose their hat. For definiteness, we always take the imaginary part of $\hat\rho$ to be positive, which unambiguously defines the sign of the square root in (5b). The introduction of the complex dielectric constant $\hat\varepsilon = \hat\varepsilon(\omega)$ allows us to unify the treatment of the problem for dielectric media, conducting or not. In particular, it allows us to write the complex dispersion relation

$$\hat k(\omega) = \sqrt{\hat\varepsilon\mu}\,\omega \tag{6}$$

everywhere. If the medium is nonconducting, $\hat\varepsilon = \varepsilon$ and the usual dispersion relation $k(\omega) = \sqrt{\varepsilon\mu}\,\omega = (n/c)\omega$ recovers, with $n \geqslant 1$ the refractive index of the medium. Definition (5c) was employed in the simplification of expression (4b).

We must find $E_z^0(\hat\rho)$ to complete the description of the system. It follows from (1) that E_z^0 must obey

$$\frac{d^2E_z^0}{d\hat\rho^2} + \frac{1}{\hat\rho}\frac{dE_z^0}{d\hat\rho} + E_z^0 = 0. \tag{7}$$

Equation (7) is a particular case ($\nu = 0$) of Bessel's differential equation, for which the last term on the l. h. s. in the general case would read $(1-\nu^2/\hat{\rho}^2)E_z^0$, with ν a complex number. The physical solutions of (7) are given by

$$E_z^0(\hat{\rho}) = A\frac{\sqrt{\hat{k}^2-h^2}}{ih} J_0(\hat{\rho}), \quad 0<r<a, \tag{8a}$$

$$E_z^0(\rho) = B\frac{\sqrt{k^2-h^2}}{ih} H_0^{(1)}(\rho), \quad a<r<\infty, \tag{8b}$$

where A and B are arbitrary real constants and $J_0(\hat{\rho})$ and $H_0^{(1)}(\rho)$ are, respectively, Bessel and Hankel functions of the first kind. The other field components follow from (4).

Solution (8a) contains only the regular Bessel function, because the other linearly independent solution—the Neumann function—is singular at the origin and thus unphysical. Solution (8b), however, could in principle read as a linear combination of the Hankel functions of the first and second kinds,

$$E_z^0(\rho) \stackrel{?}{=} B_1 H_0^{(1)}(\rho) + B_2 H_0^{(2)}(\rho), \quad a<r<\infty. \tag{9}$$

We dismiss this possibility on the basis of two arguments. First, note that the leading asymptotic form ($|\rho|\gg 1$, $\nu \ll |\rho|$) of the Hankel functions is

$$H_\nu^{(1,2)}(\rho) \sim \sqrt{\frac{2}{\pi\rho}} \exp\left[\pm i(\rho - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\right], \tag{10}$$

where the $+$ $(-)$ sign in the exponent belongs to $H_\nu^{(1)}$ $(H_\nu^{(2)})$. If $\rho = \sqrt{k^2-h^2}$ is complex, then, as per our sign convention (see discussion following (5)), we must discard $H_0^{(2)}(\rho)$ because it diverges as $\operatorname{Im}\rho \to \infty$. If, otherwise, ρ is real, an energy condition rules $H_0^{(2)}(\rho)$ out from the solution as follows. For an infinite wire sustaining a longitudinal EM wave that draws its energy from a source at $z=-\infty$, the energy flux across a cylindrical surface coaxial with the wire must vanish. This energy flux is given by $U_r(r,t) = 2\pi r S_r(r,t)$, where S_r is the radial component of the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$, namely,

$$S_r(r,t) = -\operatorname{Re}\left\{E_z^0(r)\phi(z,t)\right\} \operatorname{Re}\left\{H_\theta^0(r)\phi(z,t)\right\}. \tag{11}$$

Taking (10) into account and doing the algebra we arrive at

$$U_r(r,t) = 4\frac{r}{\rho}\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{k}{\sqrt{k^2-h^2}}\Big[B_1^2\cos^2(\rho - \tfrac{1}{4}\pi + (hz-\omega t)) - B_2^2\cos^2(\rho - \tfrac{1}{4}\pi - (hz-\omega t))\Big]. \tag{12}$$

It is a good exercise to verify that $U_r(r,t)$ has dimensions of energy flux per unit length, $[U_r] = \mathrm{MLT}^{-3} = \mathrm{ML}^2\mathrm{T}^{-2}/\mathrm{TL}$ (in SI units, $\mathrm{J\,s^{-1}\,m^{-1}}$). Since r/ρ remains finite at arbitrarily large r (it is in fact constant) and the phase factor $(hz-\omega t)$ can assume any value we conclude that, for real ρ, the condition $U_r(r \gg a, t) = 0$ leads to $B_1 = B_2 = 0$. Expressions (8) are then justified.

Finally, fields E_z^0 and H_θ^0 must be continuous at the surface $(r=a)$ of the wire. This boundary condition leads to the transcendental equation

$$\begin{aligned}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{\sqrt{k^2-h^2}}{k}\,\frac{H_0^{(1)}\big(a\sqrt{k^2-h^2}\big)}{H_0^{(1)\prime}\big(a\sqrt{k^2-h^2}\big)} &= \\ = \sqrt{\frac{\mu}{\hat{\varepsilon}}}\,\frac{\sqrt{\hat{k}^2-h^2}}{\hat{k}}\,\frac{J_0\big(a\sqrt{\hat{k}^2-h^2}\big)}{J_0'\big(a\sqrt{\hat{k}^2-h^2}\big)}&,\end{aligned} \tag{13}$$

where the primes indicate differentiation with respect to the given argument. Equation (13) determines the wave number h, from which most of the electromagnetic properties of the system can be derived. In what follows we solve (13) both numerically and by an approximate method due to Sommerfeld and compare the results.

III. EXACT NUMERICAL SOLUTIONS OF THE BOUNDARY CONDITION

Before embarking on the solution of (13), let us look at the magnitude of the physical quantities involved in a typical case.

Consider a microwave propagating in a hard-drawn copper wire of radius $a = 1$ mm (approximately wire nr. 12 in the American Wire Gauge system, for which $2a = 2.053$ mm). The following parameters are taken from [32 and 33]. The dc electrical conductivity of copper at 293 K (~ 20 °C) is 5.96×10^7 $\Omega^{-1}\,\mathrm{m}^{-1}$, while for $f = 1$ GHz we have $ka \simeq 0.021$, where $k = \omega/c = 2\pi f/c = 2\pi/\lambda$ is the free-space EM wave number and $\lambda = c/f \simeq 30$ cm its wavelength. Note that for most metals the electrical conductivity is essentially static up to about the infrared region $f \sim 10$ THz or, more meaningfully, photon energies about $E = hf \sim 40$ meV (where Planck's constant $h \simeq 4.136\times10^{-15}$ eV s);[34] microwaves ($f \sim$ 1–100 GHz, $\lambda \sim$ 300–3 mm, $E \sim$ 0.004–0.4 meV) are way below this energy scale. As we have already mentioned, for copper $\mu \simeq \mu_0$ (the more precise current value is $\mu_{\mathrm{Cu}} = 0.999\,994\,\mu_0$), but the electric permittivity is a complex-valued function of the frequency, $\hat{\varepsilon} = \hat{\varepsilon}(\omega)$. For most metals, however, the conduction current $\mathbf{J} = \sigma\mathbf{E}$ is much larger than the displacement current $\mathbf{J}_D = \varepsilon\partial_t\mathbf{E} = -i\omega\varepsilon\mathbf{E}$, since $\varepsilon/\sigma \ll 1$. In fact, for most metals $\varepsilon/\sigma \sim 10^{-18}$, and as long as $f \lesssim 10^{16}$ Hz the displacement current can be ignored. This corresponds to the low-frequency limit of an ac Drude model for copper. We thus obtain from (5a) that $\hat{k}^2 \approx i\sigma\mu\omega$, or

$$\hat{k} \approx \frac{1+i}{\sqrt{2}}\sqrt{\sigma\mu\omega} = (1+i)\kappa(\omega). \tag{14}$$

The real quantity $\delta(\omega) = 1/\kappa(\omega)$ measures the depth in the material at which the current density decreases to a fraction $e^{-1} \simeq 0.368$ of its value at the surface—the so-called skin-depth. Note that $\delta(\omega) \sim 1/\sqrt{\omega}$ can become a really small quantity at high frequencies, sometimes just a few nanometers, with the current mostly confined to the periphery of the conductor. If we plug in the figures of our example we obtain $\kappa \simeq 4.851 \times 10^5\ \mathrm{m}^{-1}$, that is, $\delta \simeq 2\ \mu\mathrm{m}$.

To find the wave number h, we solve (13) numerically with the help of the software package Mathematica[35] using the figures given above for the physical quantities, as well as approximation (14) for $\hat{k}$. Graphing the expressions in (13) is unilluminating, so we just quote the solution,

$$h = 20.960 + 7.516 \times 10^{-4}\, i. \tag{15}$$

Two conclusions of physical significance can be immediately drawn from this number. First, that the wave is damped, since the phase factor $\exp(ihz)$ contains an attenuation factor given by $\exp(-7.516 \times 10^{-4} z)$, and the amplitude of the wave decays by a factor e^{-1} every ~ 1330 m. This attenuation can be considered small, and such a naked copper wire nr. 12 at 1 GHz indeed qualifies as a viable candidate for last mile data carrier! The second conclusion is that the wave travels along the wire at phase speed

$$\frac{\omega}{h} = \frac{\omega}{k}\cdot\frac{k}{h} = c[1-(7.438+3.586i)\times 10^{-5}], \tag{16}$$

i. e., at about $0.999\,926c$, just a tad below the speed of light. The "surface wave" is thus able to travel without much damping for a reasonable distance—but not telegraphic distances, much to the chagrin of the telecommunications engineers at the turn of the XXth century—while keeping the phase.

IV. SOMMERFELD'S APPROXIMATE SOLUTION

Sommerfeld approached the solution of (13) by making well informed assumptions about the physical nature of the system at hand and then approximating the functions appearing there accordingly. He was then led to a much simpler transcendental equation that he solved by means of a practical and effective device. In what follows we review the rationale leading to the approximate equation for the boundary condition and Sommerfeld's iterative solution to the problem.

A. The physical approximations

For a metallic wire, σ is large and $\hat{k}$ and $\hat{\rho}$ are large complex numbers. Given that $J_0'(\hat{\rho}) = -J_1(\hat{\rho})$ and that for large $\hat{\rho}$ in the complex upper half-plane (remember that we are using $\operatorname{Im}\hat{\rho} \geqslant 0$ by convention) it holds that

$$J_\nu(\hat{\rho}) \sim \sqrt{\frac{2}{\pi\hat{\rho}}}\cos(\hat{\rho} - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi), \tag{17}$$

the r. h. s.of (13) can be simplified by taking $\sqrt{\hat{k}^2-h^2}/\hat{k} \approx 1$ and $J_0(\hat{\rho})/J_0'(\hat{\rho}) \approx -e^{-i\frac{\pi}{2}} = i$ (we always take the principal branch), such that

$$\sqrt{\frac{\mu}{\hat{\varepsilon}}}\,\frac{\sqrt{\hat{k}^2-h^2}}{\hat{k}}\,\frac{J_0\big(a\sqrt{\hat{k}^2-h^2}\big)}{J_0'\big(a\sqrt{\hat{k}^2-h^2}\big)} \simeq i\sqrt{\frac{\mu}{\hat{\varepsilon}}}. \tag{18}$$

In metals widely used for electric wiring like copper, silver or aluminium, $|\mu| \simeq \mu_0$ to a very good approximation but $|\hat{\varepsilon}| \gg \varepsilon_0$, and we see from (18) that the l. h. s. of (13) must also be a small number, i. e., $|\sqrt{k^2-h^2}|$ must be small. The expansion of the Hankel functions about the origin gives

$$H_0^{(1)}(\rho) \approx \frac{2i}{\pi}\ln\Big(\frac{e^\gamma\rho}{2i}\Big), \tag{19a}$$

$$H_\nu^{(1)}(\rho) \approx \frac{\Gamma(\nu)}{i\pi}\Big(\frac{2}{\rho}\Big)^\nu,\ \nu > 0, \tag{19b}$$

where $\gamma = 0.577\cdots$ is the Euler-Mascheroni constant and $\Gamma(\nu)$ is the usual gamma function. Since $H_0'(\rho) = -H_1(\rho)$, the l. h. s. of (13) becomes

$$\begin{aligned}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{\sqrt{k^2-h^2}}{k}\,\frac{H_0\big(a\sqrt{k^2-h^2}\big)}{H_0'\big(a\sqrt{k^2-h^2}\big)} &\simeq \\ \simeq \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{a(k^2-h^2)}{k}\,\ln\Big(\frac{e^\gamma a\sqrt{k^2-h^2}}{2i}\Big)&.\end{aligned} \tag{20}$$

Now we put everything together. Defining the variables

$$u = \Big(\frac{e^\gamma a\sqrt{k^2-h^2}}{2i}\Big)^2 \quad \text{and} \quad v = \frac{e^{2\gamma}}{2i}\sqrt{\frac{\varepsilon_0\mu}{\hat{\varepsilon}\mu_0}}\,ka, \tag{21}$$

equation (13) for the wave number h of a relatively low frequency pure TM mode propagating in a cylindrical metallic wire becomes, within the aforementioned approximations,

$$u\ln u = v. \tag{22}$$

This equation corresponds to equations (22)–(22′) in [5], (IX.23–24) in [25], and (22.15) in [26]. It is interesting to realize that while on the l. h. s. of (22) we have all the geometric factors of the problem (a, k, and h), the r. h. s. concentrates the physical factors ($\hat{\varepsilon}/\varepsilon_0$ and μ/μ_0), providing a striking illustration of the role played by boundary conditions in electrodynamics.

B. The iterative solution

Sommerfeld solved (22) by "a method of successive approximations [that] proved to be very suitable."[36] Several references repeat his solution, sometimes second-hand from

[25], while Goubau offers an interesting approximate graphical solution of his own.[16] In what follows we present Sommerfeld's method according to more modern standards (and also a lot of hindsight).

Let $f(x) = x \ln x$ be a real function of a real argument $x \geqslant 0$. Routine inspection reveals that $f(x)$ has an absolute minimum at $x_{\min} = e^{-1} \simeq 0.367\,879$, at which $f(x_{\min}) = -e^{-1}$, and since $\lim_{x\to 0} f(x) = 0$ and $f(1) = 0$, it is decreasing for $0 \leqslant x < e^{-1}$ and increasing for $x > e^{-1}$. We thus conclude that the equation

$$x \ln x = a \tag{23}$$

does not have real solutions for $a < -e^{-1}$, has two solutions for $-e^{-1} \leqslant a < 0$, one in the interval $(0, e^{-1})$ and the other in the interval $(e^{-1}, 1)$ (the two coinciding when $a = -e^{-1}$), and has a single solution for $a \geqslant 0$. These facts are summarized in Figure IV B.

1. *The smaller solution*

Based on the previous analysis, Sommerfeld's devised two closely related iterative procedures to find the solutions of (23) depending on the location of the solution. Let us first consider the case $-e^{-1} < a < 0$. The smaller solution of (23), say, x_l^*, is located to the left of $x_{\min} = e^{-1}$. Sommerfeld then observes that if we write (23) as $x(-\ln x) = -a$ and take logarithms we obtain

$$\ln x = \ln(-a) - \ln(-\ln x), \tag{24}$$

and since $\ln x$ varies slowly in comparison with x—an error of δ in the value of x gives an error of roughly δ/x in the value of $\ln x$—, we can ignore the second term on the r. h. s. of (24) and take $x_1 \simeq -a$ as a first approximation to x_l^*. Once we have this first approximation to x_l^*, we can substitute it on the r. h. s. of (24) to obtain a new approximation, say, x_2, hopefully better than the former, given by

$$\ln x_2 = \ln(-a) - \ln(-\ln x_1), \quad \text{or} \quad x_2 = \frac{a}{\ln x_1}. \tag{25}$$

Note that since $a < 0$ and $\ln(-a) < -1$ we have $0 < x_2 < x_1$. Plugging x_2 in (24) leads to $x_3 = a/\ln x_2$ with $0 < x_3 < x_2$.

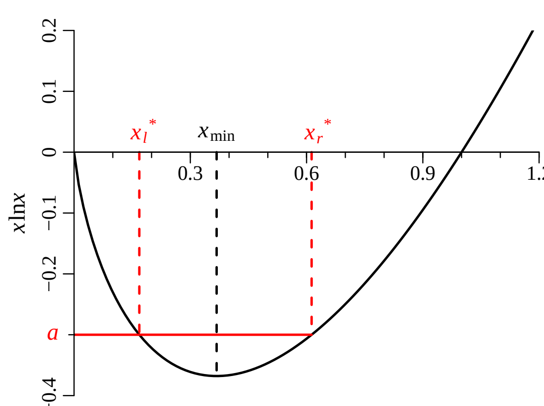

FIG. 2. Location of the solutions of $x \ln x = a$ for $-e^{-1} < a < 0$.

The iterative procedure is clear:

$$x_1 = -a, \quad x_{n+1} = \frac{a}{\ln x_n}, \quad n \geqslant 1 \tag{26}$$

Stratton states that "the convergence [of (26)] can be demonstrated for complex as well as real values of a" ([25, p. 529]) but neither offers a proof nor refers to one. He was certainly quoting from [5]. The mathematical work [36] of Sommerfeld on the iterative solution of (22) or (23) is rarely (if ever) mentioned. Sommerfeld proved that $\lim x_n = x_l^*$ in (26) based on the ratio test and the fact that $0 < x_{n+1} < x_n$ to conclude that $\{x_n\}$ forms a Cauchy sequence. Anticipating generalizations—namely, to the complex domain—in what follows we adopt a more general approach.

From elementary mathematical analysis,[37,38] we know that the "discrete dynamical system" (26) has a fixed point $x^* = T(x^*)$ if the map $T(x) = a/\ln x$ is a contraction, i. e., if it satisfies the so-called Lipschitz condition

$$\|T(y) - T(x)\| \leqslant c\|y - x\| \tag{27}$$

with $c < 1$ for any two points x, y in the domain of T. The constant c is called the Lipschitz constant of the map. The nice thing about contractions is that a powerful theorem guarantees, under certain conditions, that they have a unique fixed point: the Banach contraction principle. Its statement reads:

> A contraction $T : X \to X$ on a complete metric space X has a unique fixed point x^*.

Remember that a complete metric space is a space endowed with a distance function in which every Cauchy sequence is convergent. An easy corollary of the Banach contraction principle is that $\lim_{n\to\infty} T^{(n)}(x) = x^*$ for all $x \in X$, where $T^{(n)}(x)$ denotes the n-fold composition $(T \circ \cdots \circ T)(x)$.

We now revisit (26) in the light of the above knowledge. First, note that the interval $X = [0, e^{-1}]$ is a complete metric space (in the usual Euclidean metric $\|y - x\| = \sqrt{(y-x)^2} = |y - x|$) because it is a closed subset of $\mathbb{R}$, which is complete. To check whether $T(x) = a/\ln x$ is a contraction on X we take

$$|T(y) - T(x)| = \left|\frac{a}{\ln y} - \frac{a}{\ln x}\right| \leqslant |a| \max_{x,y\in X} \left|\frac{1}{\ln y} - \frac{1}{\ln x}\right|. \tag{28}$$

The largest possible value of the r. h. s. of (28) occurs when one of its terms becomes zero (for instance, when x or $y \to 0$) and the other becomes $1/\ln(e^{-1}) = -1$, such that $|T(y) - T(x)| \leqslant |a|$ in X. Putting everything together we get

$$|T(y) - T(x)| \leqslant |a| \leqslant c \max_{x,y\in X} |y - x| = ce^{-1}. \tag{29}$$

It thus suffices to have $|a| < e^{-1}$ to make T a contraction. But this is exactly the range of values that we are considering for a. We thus conclude that $T(x)$ is indeed a contraction on $X = [0, e^{-1}]$ and that its unique fixed point $x^* = T(x^*)$ is, by construction, the solution of (23) on X.

2. The larger solution

If we try to find the second solution of (23) in the interval $e^{-1} < x < 1$ with $-e^{-1} < a < 0$ using the recursion relation (26), it may or may not converge and, if it converges, it will do so to the smaller solution. From (23), however, we see that there are two options to draw an iterative procedure of it: either $x_{n+1} \ln x_n = a$, corresponding to (26), or $x_n \ln x_{n+1} = a$. Sommerfeld realized that the second option provides a scheme to find the second solution of (23) in the interval $e^{-1} < x < 1$ when $-e^{-1} < a < 0$. The recursion relation reads

$$x_1 = \exp(a), \quad x_{n+1} = \exp(a/x_n), \ n \geqslant 1 \tag{30}$$

We employ again the contraction principle to prove that this recursion relation indeed converges to the right solution.

Let $X = [e^{-1}, 1]$ and $T : X \to X$ be the map $T(x) = \exp(a/x)$. We want to check whether

$$|T(y) - T(x)| = |e^{a/y} - e^{a/x}| \leqslant c|y - x| \tag{31}$$

with $c < 1$. The function $c(x, y, a) = |e^{a/y} - e^{a/x}|/|y - x|$ is decreasing in all of its arguments, so its maximum should be at the point $Q = (e^{-1}, e^{-1}, -e^{-1})$. Unfortunately, $c(x, y, a)$ is undefined for $y = x$. However, since $c(x, y, a) = c(y, x, a)$, we argue that approaching Q along the x-axis or the y-axis should give the same result, making $\lim_{P \to Q} c(P)$ well defined (if it exists at all). We then fix x and let $y \to x$ to obtain

$$\lim_{y \to x} c(x, y, a) = \frac{|ae^{a/x}|}{|x^2|}, \tag{32}$$

where we resolved the indeterminate form $0/0$ by L'Hôpital's rule. Setting $x = e^{-1}$ and $a = -e^{-1}$ we find that $c(x, y, a) \leqslant 1$, with strict inequality for $a > -e^{-1}$. We thus conclude that $T(x)$ satisfies the contraction principle on X for $-e^{-1} < a < 0$ such that its unique fixed point on X indeed solves (23).

It is useful to examine an example. Let $a = -0.2$. Application of (26) provides the sequence $x_1 = 0.2 \to x_2 = 0.124267 \to \cdots \to x_{20} = 0.0786584 \to x_{21} = 0.0786584$, converging to seven decimal places in 20 iterations, with $x_{20} \ln x_{20} \simeq -0.200\,000\,061$. The solution is $x_l^* = 0.078\,658\,360\cdots$. Now let us apply (30) to find the righmost solution of (23). We obtain $x_1 = 0.8187308 \to x_2 = 0.7832679 \to \cdots \to x_{11} = 0.7716910 \to x_{12} = 0.7716910$, and the procedure converges to seven decimal places in just 11 iterations, with $x_{11} \ln x_{11} \simeq -0.199\,999\,981$. The precise value for the rightmost solution is $x_r^* = 0.771\,690\,974\cdots$. The sequence converges faster because the Lipschitz constant for (30) is smaller than the one for (26), bringing separate points close together faster.

3. Cases $a > 0$

When $a > 0$, Sommerfeld splits the resolution of (23) again into two cases. For $0 < a < e$, the proper convergent iterative scheme reads the same as (30), namely,

$$x_1 = \exp(a), \quad x_{n+1} = \exp(a/x_n), \ n \geqslant 1, \tag{33}$$

while for $a \geqslant e$ the scheme that converges coincides with (26) except that now we start with $x_1 = a$, to wit,

$$x_1 = a, \quad x_{n+1} = \frac{a}{\ln x_n}, \ n \geqslant 1 \tag{34}$$

We invite the reader to repeat the analysis of the above maps along the lines of what we did for the other cases to prove their convergence, as well as to examine some examples. We will not make use of these procedures in our study.

C. Solution of the approximate boundary condition

Now we apply what we have learned to the solution of (22). We have a problem, though: equation (22) involves complex quantities. The Banach contraction principle, however, only requires that the underlying space be complete in some norm, and this is the case of the complex plane—in fact, the complex plane $\mathbb{C}$ is isometric with the real plane $\mathbb{R}^2$, with the usual complex norm $\sqrt{z^* z}$ in the former corresponding naturally to the Euclidean norm $\sqrt{x^2 + y^2}$ in the later. Of course, before employing (26) or (30) one must analyse the moduli and the arguments of x and a to make a better informed trial.

Using the approximations $\mu \simeq \mu_0$ and $\varepsilon\omega \ll \sigma$ valid for good metal conductors at low-frequencies (see the discussion preceding (14)) we obtain for v (eq. (21))

$$v \simeq \frac{e^{2\gamma}}{2i} \sqrt{\frac{\varepsilon_0 \omega}{i\sigma}}\, ka = -(1+i) \frac{e^{2\gamma}}{2\sqrt{2}} \sqrt{\frac{\varepsilon_0 \omega}{\sigma}}\, ka \tag{35}$$

and inserting the values of Sec. III for copper at 1 GHz we get

$$v \simeq -7.181 \times 10^{-7}(1+i), \tag{36}$$

a number with a small negative real part. We shall thus try to apply procedure (26) to find the solution to (22). We obtain, to three decimal places, the following sequence of numbers, where z_{-e} reads $z \times 10^{-e}$: $u_1 = (7.181 + 7.181\,i)_{-7} \to u_2 = (4.892 + 5.482\,i)_{-8} \to \cdots \to u_6 = (4.095 + 4.529\,i)_{-8} \to u_7 = (4.095 + 4.529\,i)_{-8}$, and the procedure converged after just 6 iterations to the solution

$$u = (4.095 + 4.529\,i) \times 10^{-8}. \tag{37}$$

This solution is slightly at variance with that found by Sommerfeld because he uses $v = -7.2 \times 10^{-7}(1+i)$) and also because we keep more digits in the intermediate calculations.

From the value for u above we obtain from (21) that

$$h = \sqrt{k^2 + \frac{4e^{-2\gamma}}{a^2} u} = 20.960 + 1.362 \times 10^{-3}\, i. \tag{38}$$

Compare this number with the one in (15). The real parts are virtually identical (the difference starts from the fourth decimal place, not shown), but the imaginary parts differ almost by a factor of two. The root of this difference is likely the somewhat uncontrolled approximation $\sqrt{\hat{k}^2 - h^2}/\hat{k} \approx 1$ that we made in passing from (13) to (18); the approximation is valid in absolute value, but not componentwise. Following the analysis presented after (15), we conclude that the wave is damped by a factor e^{-1} every ~ 734 m and that it travels along the wire at phase speed

$$\frac{\omega}{h} = \frac{\omega}{k} \cdot \frac{k}{h} = c[1 - (5.878 + 6.500\,i) \times 10^{-5}], \tag{39}$$

i. e., at about $0.999\,941\,c$. While the physical conclusions in this case remain the same, namely, that the wave goes reasonably undamped and can keep its phase, the attenuation factor appears larger than it really is as a result of the approximations that were made to make the problem more tractable.

In his textbook,[26] Sommerfeld compares the above case with that of a very thin platinum wire of radius $a = 2 \times 10^{-6}$ m forced at $f = 300$ MHz. The conductivity of platinum at 293 K is $\sigma = 9.52 \times 10^{6}\ \Omega^{-1}\,\mathrm{m}^{-1}$, approximately 6.3 times smaller than that of copper.[33] Note that in this case $\hat{\rho}(a) = \sqrt{\hat{k}^2 - h^2}a$ is a small quantity and the approximation (18) no longer holds—one has to employ the series expansion of the Bessel functions around zero instead. The reader is invited to perform the analysis of the EM wave propagation characteristics with these physical parameters to conclude that in this case the wire behaves almost like a resistor, something that can be guessed already by a simple estimate of the skin-depth $\delta(\omega) \sim \sqrt{2/\sigma\mu_0\omega}$.

V. THE SOLUTION IN TERMS OF THE LAMBERT W FUNCTION

The Lambert W function has been around since at least the XVIII century, when it was introduced in disguise by Lambert in 1758 in the series solution of the trinomial equation $x = q + x^m$, a solution which Euler revisited and expanded in 1776. As applied mathematics and mathematical physics became more diverse in the XIX and XX centuries, the Lambert W function started to pop up in several areas, from statistical mechanics and astrophysics to population dynamics, materials science, combinatorics, and the analysis of algorithms, among others.[39–48] The fact that numerical routines to evaluate the Lambert W function started to appear in the early 1970s also attests its ubiquitous occurrence in many areas of science and technology.[49–54] By mid-1990s, notation—in particular the use of the letter W for the function after its first implementation in the Maple computer algebra software—and basic properties were settled, with reference [40] having become the standard reference on the Lambert W function.

The Lambert W function is defined to be the inverse of the map $z \mapsto ze^z$, with z a complex number. Put equivalently, the values of $W(z)$ are given by the solutions of the equation

$$W(z)e^{W(z)} = z. \tag{40}$$

Since the map $z \mapsto ze^z$ is not injective, the solutions of (40) are multivalued. The map cannot be inverted easily—in fact, this is the whole point of defining the "special function" $W(z)$. A simple strategy to plot $W(x)$ for real x is to plot $y = xe^x$ and interchange the roles of axis x and y: look at the plot in a mirror or from behind and rotate it $\pi/2$ clockwise. A plot of $W(z)$ for real z is displayed in Figure V. From this picture it is clear that the point $z = -e^{-1}$, at which $W(z) = -1$, is a branching point. The two real branches of $W(z)$ are $W_0(z)$, also called the principal branch, defined for $z \geqslant -e^{-1}$, and $W_{-1}(z)$, defined for $-e^{-1} \leqslant z < 0$ with a singularity $\lim_{z\to 0^-} W_{-1}(z) = -\infty$. If $z < -e^{-1}$ or z is complex, then $W(z)$ becomes complex. The asymptotic behavior of $W_0(z)$ for small z is given by

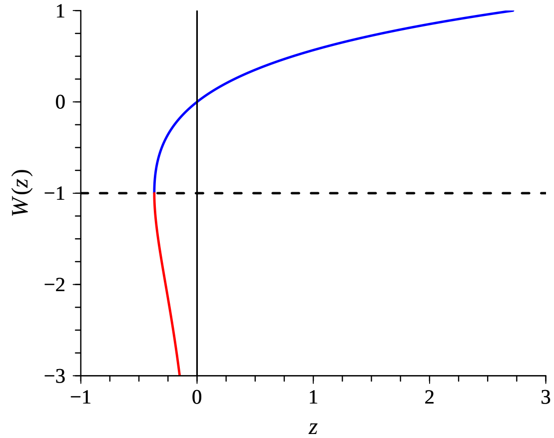

FIG. 3. Lambert W function of a real argument z. Values lying above $W(z) \geqslant -1$ define the principal branch $W_0(z)$, while values lying below $W(z) \leqslant -1$ define $W_1(z)$.

$$W_0(z) = z - z^2 + \frac{3}{2}z^3 + \cdots, \quad |z| \ll 1, \tag{41}$$

while for large z it is given by

$$W_0(z) = \ln z - \ln\ln z + \frac{\ln\ln z}{\ln z} + \cdots, \quad z \gg 1. \tag{42}$$

Back to our mathematical physics problem, if we put $z = \ln u$, equation (22) becomes $ze^z = v$, which immediately furnishes the solution $u = e^{W(v)}$, or, equivalently,

$$u = \frac{v}{W(v)}. \tag{43}$$

Unbeknownst to Sommerfeld, the solution of (22) is "simply" given in terms of the Lambert W function! This is not really helpful unless you have a procedure to compute $W(v)$—exactly what Sommerfeld devised with his iterative method described in Sec. IV B. By comparing Figs. IV B and V and noting that the two are connected by the relationship $z = \ln x$ one can easily conclude that the leftmost solution of (23) corresponds to $\exp[W_{-1}(a)]$, while the righmost solution corresponds to $\exp[W_0(a)]$. In the software package Mathematica,[35] one can calculate $W_0(z)$ with the command `ProductLog[z]`, while the calculation of $W_{-1}(z)$ is achieved with the command `ProductLog[1,z]`. When we insert $v = -7.181 \times 10^{-7}(1+i)$ (see (36)) into (43) we obtain, using Mathematica, the solution

$$u = (4.095 + 4.529\,i) \times 10^{-8}, \tag{44}$$

exactly the same value as before, equation (37).

VI. SUMMARY AND CONCLUSIONS

We have shown that a certain approximate equation arising from the boundary condition for EM surface waves between a cylindrical metallic wire and a dielectric can be solved by means of the Lambert W function. While one can solve the exact condition (13) numerically without much fuss, the physical rationale leading to the approximate condition (22) are very illuminating and, as we have seen, provides a reasonably accurate description of the system. In fact, it is quite satisfying to verify that the answer obtained by Sommerfeld through his apparently offhand iterative method and the solution in terms of the Lambert W function agree. While the comparison between Sommerfeld's solution and the numerical solution of the exact boundary condition (13) do not quite match, they are qualitatively compatible and predict the same physics. We can trace the somewhat uncontrolled approximation $\sqrt{\hat{k}^2 - h^2}/\hat{k} \approx 1$ as the probable reason for the mismatch between the exact and the approximate solutions.

Equation (40) is the simplest possible exponential polynomial equation, and this is arguably one of the reasons for its ubiquity. Slighly generalized equations, however, sometimes appear that can also be solved in terms of $W(z)$.[55] For example, in the study of the capture of a diffusing prey by diffusing predators in one dimension, the authors in [56] obtain the equation $ye^{y^2} = M$, where $y = x_{\text{last}}/\sqrt{4Dt}$ is the rescaled position of the last predator (the record position of the predators' one-dimensional random walks) and solve it iteratively as $y = \sqrt{\ln M}(1 - \frac{1}{4}\ln\ln M/\ln M + \cdots)$. Clearly, squaring the original equation and then multiplying it by 2 immediately leads to the solution $y^2 = \frac{1}{2}W_0(2M^2)$, from which good asymptotics, explicit numbers, and graphs can be obtained with a couple of commands in more or less any modern computer algebra system or programming language through their scientific libraries.

The Lambert W function can be computed to arbitrary precision by iterative root-finding of $we^w = z$, with the usual trade off between complexity of implementation and number of iterations to converge at the required precision. Newton's simple first-order method, for example, is appropriate, but converges too slowly for modern standards. The Maple software package implements Halley's third-order variant of Newton's method that attains high precision in affordable time.[39,57] The investigation of the recursion relations (26) and (30) on the complex plane in the language of dynamical systems—their basins of attraction, stability, bifurcations, etc.[38]—is not entirely without interest and may even prove useful in the implementation of numerical routines to calculate $W(z)$. An accessible and comprehensive presentation of discrete dynamical systems based on the ideas of inverse functions ("root-finding") and fixed point theorems is given in [38].

In the introduction to Volume III of his Lectures on Theoretical Physics, Sommerfeld states that "Heinrich Hertz's greatest paper on the 'Fundamental Equations of Electrodynamics for Bodies at Rest' has served as model for my lectures on electrodynamics since my student days."[26] This is a remarkable note and advice, namely, that even the masters acknowledge that there is much to be learned from the masters. There is a little bit—usually much more—for everyone. Maybe it is because they were closer to sparking gaps, vacuum tubes, flaring chemicals, and gears than the average contemporary student will ever be. We encourage the reader to browse through references [25–28] looking for historical perspective, foundations, applications, and classical problems to solve.

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1 A. A. Huurdeman, *The Worldwide History of Telecommunications* (Wiley, Hoboken, 2003).
2 P. J. Nahim, *Oliver Heaviside: The Life, Work, and Times of an Electrical Genius of the Victorian Age* (Johns Hopkins University, Baltimore, 2002).
3 H. Hertz, *Electric Waves: Being Researches on the Propagation of Electric Action with Finite Velocity through Space* (MacMillan, London, 1893).
4 D. L. Sengupta, T. K. Sarkar, "Maxwell, Hertz, the Maxwellians, and the early history of electromagnetic waves," IEEE Antennas Propag. Mag. **45** (2), 13–19 (2003).
5 A. Sommerfeld, "Über die Fortpflanzung elektrodynamischer Wellen längs eines Drahtes," Ann. d. Phys. u. Chem. (N. F.) **67** (2), 233–290 (1899).
6 F. Harms, "Elektromagnetische Wellen an einem Draht mit isolierender zylindrischer Hülle," Ann. d. Phys. **328** (6), 44–60 (1907).
7 D. Hondros, "Über elektromagnetische Drahtwellen," Ann. d. Phys. **335** (15), 905–950 (1909).
8 D. Hondros, P. Debye, "Elektromagnetische Wellen an dielektrischen Drähten," Ann. d. Phys. **337** (8), 465–476 (1910).
9 O. Schriever, "Elektromagnetische Wellen an dielektrischen Drähten," Ann. d. Phys. **368** (23), 645–673 (1920).
10 G. C. Southworth, "Hyper-frequency wave guides—general considerations and experimental results," Bell Syst. Tech. J. **15** (2), 284–309 (1936).
11 G. C. Southworth, "Some fundamental experiments with waveguides," Proc. IRE **25** (7), 807–822 (1937).
12 R. E. Collin, "Introduction to 'Some fundamental experiments with waveguides'," Proc. IEEE **87** (3), 508–514 (1999).
13 J. R. Carson, S. P. Mead, S. A. Schelkunoff, "Hyper-frequency waveguides—mathematical theory," Bell Sys. Tech. J. **15** (2), 310–333 (1936).
14 W. L. Barrow, "Transmission of electromagnetic waves in hollow tubes of metal," Proc. IRE **24**, 1298–1328 (1937).
15 K. S. Packard, "The origin of waveguides: A case of multiple rediscovery," IEEE Trans. Microw. Theory Techn. **32** (9), 961–969 (1984).

[16] G. Goubau, “Surface waves and their application to transmission lines,” J. Appl. Phys. **21** (11), 1119–1128 (1950).
[17] H. M. Barlow, A. E. Karbowiak, “An investigation of the characteristics of cylindrical surface waves,” Proc. IEE – Part III: Radio Commun. Eng. **100** (68), 321–328 (1953).
[18] H. M. Barlow, A. L. Cullen, “Surface waves,” in: Proc. IEE – Part III: Radio Commun. Eng. **100** (68), 329–341 (1953).
[19] T. E. Roberts, “Theory of the single-wire transmission line,” J. Appl. Phys. **24** (1), 57–67 (1953).
[20] H. M. Barlow, “Surface waves,” Proc. IRE **46** (7), 1413–1417 (1958).
[21] M. Friedman, R. F. Fernsler, “Low-loss RF transport over long distance,” IEEE Trans. Microw. Theory Techn. **49** (2), 341–348 (2001).
[22] K. Wang, D. M. Mittleman, “Metal wires for terahertz wave guiding,” Nature **432** (7015), 376–379 (2004).
[23] T.-I. Jeon, J. Zhang, D. Grischkowsky, “THz Sommerfeld wave propagation on a single metal wire,” Appl. Phys. Lett. **86** (16), 161904 (2005).
[24] T. Akalin, A. Treizebre, B. Bocquet, “Single-wire transmission lines at terahertz frequencies,” IEEE Trans. Microw. Theory Techn. **54** (6), 2762–2767 (2006).
[25] J. A. Stratton, *Electromagnetic Theory* (MGraw-Hill, New York, 1941).
[26] A. Sommerfeld, *Electrodynamics* (Academic, New York, 1952).
[27] R. E. Collin, *Field Theory of Guided Waves*, 2nd ed. (Wiley-IEEE Press, New York, 1991).
[28] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic, New York, 1949).
[29] G. B. Arfken, H. J. Weber, F. E. Harris, *Mathematical Methods for Physicists*, 7th ed. (Academic, Waltham, 2013).
[30] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders (eds.), *NIST Digital Library of Mathematical Functions*, Release 1.0.20. Available at https://dlmf.nist.gov/.
[31] Surface wave launchers and the associated transmission lines for UHF antennas were actually manufactured and marketed in USA during the 1950s as “G-Line” antennas. See L. Lieberman, “The G-Line antenna lead-in” in the early consumer electronics magazine *Radio and Television News* of April 1955 (Ziff-Davis Publishing, New York, 1955), pp. 124–125. Available at https://www.americanradiohistory.com/Radio_News_Master_Page_Guide.htm.
[32] N. W. Ashcroft, N. D. Mermin, *Solid State Physics* (Saunders College, Philadelphia, 1976).
[33] Wikipedia contributors, “Electrical resistivities of the elements (data page),” Wikipedia, The Free Encyclopedia (2018).
[34] On November 2018, the Bureau International des Poids et Mesures decided to fix the value of Planck’s constant at exactly $h = 6.626\,070\,15 \times 10^{-34}$ J s as part of its efforts to redefine the *système international d’unités* (SI) in terms of physical constants, effectively redefining (from May 2019 on) the kilogram in terms of the second and the speed of light. See B. Skuse, “SI gets a makeover,” Phys. World **31** (11), 32–35 (2018).
[35] Wolfram Research, Inc., *Mathematica*, Version 11.3 (Champaign, IL, 2018).
[36] A. Sommerfeld, “Ueber die numerische Auflösung transcendenter Gleichungen durch successive Approximationen,” Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. **1898** (4), 360–369 (1898).
[37] K. R. Davidson, A. P. Donsig, *Real Analysis and Applications: Theory in Practice* (Springer, New York, 2010).
[38] S. Sternberg, *Dynamical Systems*, rev. ed. (Dover, New York, 2013).
[39] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, “Lambert’s W function in Maple,” Maple Tech. Newsl. **9** (1), 12–22 (1993).
[40] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, “On the Lambert W function,” Adv. Comput. Math. **5** (1), 329–359 (1996).
[41] S. R. Valluri, D. J. Jeffrey, R. M. Corless, “Some applications of the Lambert W function to physics,” Can. J. Phys. **78** (9), 823–831 (2000).
[42] J.-M. Caillol, “Some applications of the Lambert W function to classical statistical mechanics,” J. Phys. A: Math. Gen. **36** (42), 10431–10442 (2003).
[43] S. R. Cranmer, “New views of the solar wind with the Lambert W function,” Am. J. Phys. **72** (11), 1397–1403 (2004).
[44] R. D. H. Warburton, J. Wang, “Analysis of asymptotic projectile motion with air resistance using the Lambert W function,” Am. J. Phys. **72** (11), 1404–1407 (2004).
[45] E. W. Packel, D. S. Yuen, “Projectile motion with resistance and the Lambert W function,” College Math. J. **35** (5), 337–350 (2004).
[46] B. Hayes, “Why W?,” Amer. Sci. **93**, 104–108 (2005).
[47] T. C. Scott, R. Mann, R. E. Martinez II, “General relativity and quantum mechanics: towards a generalization of the Lambert W function,” Appl. Algebra Eng. Commun. Comput. **17** (1), 41–47 (2006).
[48] D. Veberic, “Lambert W function for applications in physics,” Comput. Phys. Commun. **183** (12), 2622–2628 (2012).
[49] E. M. Wright, “Solution of the equation $ze^z = a$,” Proc. Roy. Soc. Edinb. A **65** (2), 193–203 (1959).
[50] F. N. Fritsch, R. E. Shafer, W. P. Crowley, “Algorithm 443: Solution of the transcendental equation $we^w = x$,” Commun. ACM **16** (2), 123–124 (1973).
[51] D. A. Barry, P. J. Culligan-Hensley, S. J. Barry, “Real values of the W-function,” ACM Trans. Math. Software **21** (2), 161–171 (1995).
[52] D. A. Barry, S. J. Barry, P. J. Culligan-Hensley, “Algorithm 743: WAPR–a Fortran routine for calculating real values of the W-function,” ACM Trans. Math. Software **21** (2), 172–181 (1995).
[53] J. P. Boyd, “Global approximations to the principal real-valued branch of the Lambert W-function,” Appl. Math. Lett. **11** (6), 27–31 (1998).
[54] D. A. Barry, J. Y. Parlange, L. Li, H. Prommer, C. J. Cunningham, F. Stagnitti, “Analytical approximations for real values of the Lambert W-function,” Math. Comput. Simul. **53** (1), 95–103 (2000).
[55] R. W. Gosper Jr, “The solutions of $ye^{y^2} = x$ and $ye^y = x$,” ACM SIGSAM Bull. **32** (1), 8–10 (1998).
[56] S. Redner, P. L. Krapivsky, “Capture of the lamb: Diffusing predators seeking a diffusing prey,” Am. J. Phys. **67** (12), 1277–1283 (1999).
[57] D. Veberič, “Having fun with Lambert W(x) function,” arXiv:1003.1628v2 (2018).