Differentiability of the Evolution Map and Mackey Continuity

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Abstract

We solve the differentiablity problem for the evolution map in Milnor's infinite dimensional setting. We first show that the evolution map of each C^k-semiregular Lie group admits a particular kind of sequentially continuity – called Mackey continuity – and then prove that this continuity property is strong enough to ensure differentiability of the evolution map. In particular, this drops any continuity presumptions made in this context so far. Remarkably, Mackey continuity rises directly from the regularity problem itself – which makes it particular among the continuity conditions traditionally considered. As a further application of the introduced notions, we discuss the strong Trotter property in the sequentially-, and the Mackey continuous context.

Contents

1 Introduction **2**

2 Preliminaries **4**
2.1 Conventions . . . 4
2.1.1 Sets of Curves . . . 5
2.1.2 Lie Groups . . . 5
2.1.3 Uniform Limits . . . 6
2.2 The Right Logarithmic Derivative . . . 6
2.2.1 The Evolution Map . . . 7
2.2.2 The Product Integral . . . 7
2.2.3 The Exponential Map . . . 8
2.2.4 Standard Topologies . . . 8
2.3 The Riemann Integral . . . 8
2.4 Standard Facts and Estimates . . . 9
2.5 Continuity Statements . . . 11

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3 Auxiliary Results **11**
3.1 Sets of Curves . . . 12
3.2 Weak Continuity . . . 12
3.3 Mackey Continuity . . . 14
3.4 Sequentially Continuity . . . 15
3.5 Piecewise Integrable Curves . . . 16
3.5.1 Basic Facts and Definitions . . . 16
3.5.2 A Continuity Statement . . . 17

4 Mackey Continuity **18**
4.1 Some Estimates . . . 18
4.2 Proof of Theorem 1 . . . 20

5 The Strong Trotter Property **21**

6 Differentiation **23**
6.1 Some Technical Statements . . . 24
6.2 The Differential of the Evolution Map . . . 24
6.2.1 The Non-Constant Case . . . 25
6.2.2 The Exponential Map . . . 26
6.3 Integrals with Parameters . . . 27
6.4 Differentiation at Zero . . . 29

1 Introduction

Differentiability of the evolution map is one of the key components of the regularity problem in Milnor's infinite dimensional setting. In [2, 6], this issue has been discussed in the standard topological context – implicitly meaning that continuity of the evolution maps w.r.t. to the C^k-topology was presumed. In this paper, we solve the differentiability problem in full generality, as we drop any continuity presumption made in this context so far. We furthermore generalize the results obtained in [3, 7] concerning the strong Trotter property, by weakening the continuity presumptions made there. More specifically, let G be an infinite dimensional Lie group in Milnor's sense [1, 5, 9, 12] that is modeled over the Hausdorff locally convex vector space E. We denote the Lie algebra of G by $\mathfrak{g}$; and denote the evolution maps by

$$\mathrm{Evol}\colon C^0([0,1],\mathfrak{g}) \supseteq \mathrm{D} \to \{\mu \in C^1([0,1],G) \mid \mu(0)=e\}$$
$$\mathrm{evol}\colon \mathrm{D} \ni \phi \mapsto \mathrm{Evol}(\phi)(1)$$

as well as

$$\mathrm{Evol_k} := \mathrm{Evol}|_{\mathrm{D_k}} \qquad \text{and} \qquad \mathrm{evol_k} := \mathrm{evol}|_{\mathrm{D_k}},$$

with $\mathrm{D_k} \equiv \mathrm{D} \cap C^k([0,1],\mathfrak{g})$ for each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$.[1] We say that G is C^k-semiregular *iff* $C^k([0,1],\mathfrak{g}) \subseteq \mathrm{D}$ holds. It was shown in [2] that if G is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\infty\}$, then $\mathrm{Evol_k}$ (thus, $\mathrm{evol_k}$) is smooth *iff* $\mathrm{evol_k}$ is of class C^1. Then, it was proven in [6] that $\mathrm{evol_k}$ is of class C^1 *iff* it is continuous, with $\mathfrak{g}$ Mackey complete for $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ (as well as integral complete for $k \equiv 0$). All these statement have been established in the standard topological context – specifically meaning that $\mathrm{evol_k}$ (and $\mathrm{Evol_k}$) was presumed to be continuous w.r.t. the C^k-topology. In this paper, we more generally show that, cf. (the more comprehensive) Theorem 2 in Sect. 6.2.1

[1] Here, $C^{\mathrm{lip}}([0,1],\mathfrak{g})$ denotes the set of Lipschitz curves, and $C^{\mathrm{c}}([0,1],\mathfrak{g})$ denotes the set of constant curves.

Theorem A. *Suppose that G is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$. Then,* evol_k *is differentiable iff* $\mathfrak{g}$ *is* k-*complete. In this case,* $\mathrm{evol}^k_{[r,r']}$ *is differentiable for each $[r,r'] \in \mathfrak{K}$, with*

$$\mathrm{d}_\phi \mathrm{evol}^k_{[r,r']}(\psi) = \mathrm{d}_e \mathrm{L}_{\int \phi}\big(\textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s \big) \qquad \forall\, \phi, \psi \in C^k([r,r'],\mathfrak{g}).$$

This theorem will be derived from significantly more fundamental results established in this paper:

Let $\Xi\colon \mathfrak{U} \to \mathcal{V} \subseteq E$ be a fixed chart around e, and $\mathfrak{P}$ the set of continuous seminorms on E. A pair $(\phi,\psi) \in C^0([0,1],\mathfrak{g}) \times C^0([0,1],\mathfrak{g})$ is said to be

- *admissible iff* $\phi + (-\delta,\delta)\cdot\psi \subseteq \mathrm{D}$ holds for some $\delta > 0$,
- *regulated iff* it is admissible, with

$$\mathrm{lim}^\infty_{h\to 0}\, \Xi\big(\mathrm{Evol}(\phi)^{-1}\cdot \mathrm{Evol}(\phi + h\cdot\psi)\big) = 0.$$

Here, the limit is understood to be uniform – In general, we write $\mathrm{lim}^\infty_{h\to 0}\, \alpha = \beta$ for $\alpha\colon (-\delta,0)\cup(0,\delta)\times[0,1] \to E$ with $\delta > 0$ and $\beta\colon [0,1] \to \overline{E}$ *iff*

$$\mathrm{lim}_{h\to 0}\, \sup\{\overline{\mathfrak{p}}(\alpha(h,t) - \beta(t)) \,|\, t \in [0,1]\} = 0 \qquad \forall\, \mathfrak{p} \in \mathfrak{P}$$

holds, where $\overline{\mathfrak{p}}\colon \overline{E} \to \mathbb{R}_{\geq 0}$ denotes the continuous extension of the seminorm $\mathfrak{p} \in \mathfrak{P}$ to the completion $\overline{E}$ of E. Then, the first result we want to mention is, cf. Proposition 3 in Sect. 6.2

Proposition B. *Suppose that (ϕ,ψ) is admissible.*

1) The pair (ϕ,ψ) is regulated iff we have

$$\mathrm{lim}^\infty_{h\to 0}\, 1/h \cdot \Xi\big(\mathrm{Evol}(\phi)^{-1}\cdot \mathrm{Evol}(\phi + h\cdot\psi)\big) = \textstyle\int_r^\bullet (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{\mathrm{Evol}(\phi)(s)^{-1}})(\psi(s))\, \mathrm{d}s \in \overline{E}.$$

2) If (ϕ,ψ) is regulated, then $(-\delta,\delta) \ni h \mapsto \mathrm{evol}(\phi + h\cdot\psi) \in G$ is differentiable at $h = 0$ (for $\delta > 0$ suitably small) iff $\int \mathrm{Ad}_{\mathrm{Evol}(\phi)(s)^{-1}}(\psi(s))\, \mathrm{d}s \in \mathfrak{g}$ holds. In this case, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0} \mathrm{evol}(\phi + h\cdot\psi) = \mathrm{d}_e \mathrm{L}_{\mathrm{evol}(\phi)}\big(\textstyle\int \mathrm{Ad}_{\mathrm{Evol}(\phi)(s)^{-1}}(\psi(s))\, \mathrm{d}s \big).$$

Evidently, each $(\phi,\psi) \in C^k([0,1],\mathfrak{g}) \times C^k([0,1],\mathfrak{g})$ is admissible *iff* G is C^k-semiregular; and, in Sect. 4, we furthermore prove that, cf. Theorem 1

Theorem C. *If G is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, then G is Mackey* k-*continuous.*

Here, Mackey k-continuity is a specific kind of sequentially continuity (cf. Sect. 3.3) that, in particular implies that each admissible $(\phi,\psi) \in C^k([0,1],\mathfrak{g}) \times C^k([0,1],\mathfrak{g})$ is regulated (cf. Lemma 16 in Sect. 3.3) – Theorem A thus follows immediately from Proposition B and Theorem C.

Now, Proposition B is actually a consequence of a more fundamental differentiability result (Proposition 2 in Sect. 6) that we will also use to generalize Theorem 5 in [6] – Specifically, we will prove that, cf. Theorem 3 in Sect. 6.3

Theorem D. *Suppose that G is Mackey* k-*continuous for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$ – additionally abelian if $k \equiv \mathrm{c}$ holds. Let $\Phi\colon I \times [0,1] \to \mathfrak{g}$ ($I \subseteq \mathbb{R}$ open) be given with $\Phi(z,\cdot) \in \mathrm{D}_\mathrm{k}$ for each $z \in I$. Then,*

$$\mathrm{lim}^\infty_{h\to 0}\, 1/h \cdot \Xi\big(\mathrm{Evol}(\Phi(x,\cdot))^{-1}\cdot \mathrm{Evol}(\Phi(x+h,\cdot))\big) = \textstyle\int_r^\bullet (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{\mathrm{Evol}(\Phi(x,\cdot))(s)})(\partial_z\Phi(x,s))\, \mathrm{d}s \in \overline{E}$$

holds for $x \in I$, provided that

a) We have $(\partial_z\Phi)(x,\cdot) \in C^k([0,1],\mathfrak{g})$.

b) For each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, there exists $L_{\mathfrak{p},\mathrm{s}} \geq 0$, as well as $I_{\mathfrak{p},\mathrm{s}} \subseteq I$ open with $x \in I_{\mathfrak{p},\mathrm{s}}$, such that

$$1/|h| \cdot \mathfrak{p}^{\mathrm{s}}_{\infty}(\Phi(x+h,\cdot) - \Phi(x,\cdot)) \leq L_{\mathfrak{p},\mathrm{s}} \qquad \forall\, h \in I_{\mathfrak{p},\mathrm{s}} - x.$$

In particular, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0} \mathrm{evol}(\Phi(x+h,\cdot)) = \mathrm{d}_e \mathrm{L}_{\mathrm{evol}(\Phi(x,\cdot))}\big(\textstyle\int \mathrm{Ad}_{\mathrm{Evol}(\Phi(x,\cdot))(s)}(\partial_z \Phi(x,s))\, \mathrm{d}s\big)$$

iff the Riemann integral on the right side exists in $\mathfrak{g}$.

We explicitly recall at this point that, by Theorem C, for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, Mackey k-continuity is automatically given if G is C^k-semiregular. We finally want to mention that we also discuss the strong Trotter property [3,7,10,13] in the sequentially/Mackey continuous context, cf. Proposition 1 in Sect. 5.

This paper is organized as follows:

- In Sect. 2, we provide the basic definitions; and discuss the most elementary properties of the core mathematical objects of this paper.
- In Sect. 3, we discuss the continuity notions considered in this paper.
- In Sect. 4, we prove Theorem 1 ($\equiv$ Theorem C).
- In Sect. 5, we discuss the strong Trotter property in the sequentially/Mackey continuous context.
- In Sect. 6, we establish the differentiability results for the evolution map.

2 Preliminaries

In this section, we fix the notations, and discuss the properties of the product integral (evolution map) that we will need in the main text. The proofs of the facts mentioned but not verified in this section, can be found, e.g., in Sect. 3 and Sect. 4 in [6].

2.1 Conventions

In this paper, Manifolds and Lie groups are always understood to be in the sense of [1,5,9,12]; in particular, smooth, Hausdorff, and modeled over a Hausdorff locally convex vector space.[2] If $f\colon M \to N$ is a C^1-map between the manifolds M and N, then $\mathrm{d}f\colon TM \to TN$ denotes the corresponding tangent map between their tangent manifolds – we write $\mathrm{d}_x f \equiv \mathrm{d}f(x,\cdot)\colon T_xM \to T_{f(x)}N$ for each $x \in M$. A curve is a continuous map $\gamma\colon D \to M$, for M a manifold and D a proper interval (i.e., $D \subseteq \mathbb{R}$ is connected, non-empty, and non-singleton). If $D \equiv I$ is open, then γ is said to be of class C^k for $k \in \mathbb{N} \sqcup \{\infty\}$ *iff* it is of class C^k when considered as a map between the manifolds I and M. If D is an arbitrary interval, then γ is said to be of class C^k for $k \in \mathbb{N} \sqcup \{\infty\}$ *iff* $\gamma = \gamma'|_D$ holds for a C^k-curve $\gamma'\colon I \to M$ that is defined on an open interval I containing D – we write $\gamma \in C^k(D, M)$ in this case. If $\gamma\colon D \to M$ is of class C^1, then we denote the corresponding tangent vector at $\gamma(t) \in M$ by $\dot\gamma(t) \in T_{\gamma(t)}M$. The above conventions also hold if $M \equiv F$ is a Hausdorff locally convex vector space with system of continuous seminorms $\mathfrak{Q}$. In this case, we let $\overline{F}$ denote the completion of F; as well as $\overline{\mathfrak{q}}\colon \overline{F} \to \mathbb{R}_{\geq 0}$ the continuous extension of $\mathfrak{q}$ to $\overline{F}$, for each $\mathfrak{q} \in \mathfrak{Q}$. If X, Y are sets, then $\mathrm{Map}(X,Y) \equiv Y^X$ denotes the set of all mappings $X \to Y$.

[2] We explicitly refer to Definition 3.1 and Definition 3.3 in [1] – A review of the corresponding differential calculus – including the standard differentiation rules used in this paper – can be found, e.g., in Appendix A that essentially equals Sect. 3.3.1 in [6].

2.1.1 Sets of Curves

Let F be a Hausdorff locally convex vector space with system of continuous seminorms $\mathfrak{Q}$. We denote by

- $C^{\mathrm{lip}}([r,r'],F)$, the set of all Lipschitz curves on $[r,r'] \in \mathfrak{K}$; i.e., all curves $\gamma\colon [r,r'] \to F$, such that

$$\mathrm{Lip}(\mathfrak{q},\gamma) := \sup\left\{ \tfrac{\mathfrak{q}(\gamma(t)-\gamma(t'))}{|t-t'|} \;\middle|\; t,t' \in [r,r'],\ t \neq t' \right\} \in \mathbb{R}_{\geq 0}$$

  exists for each $\mathfrak{q} \in \mathfrak{Q}$ – i.e., we have

$$\mathfrak{q}(\gamma(t)-\gamma(t')) \leq \mathrm{Lip}(\mathfrak{q},\gamma) \cdot |t-t'| \qquad \forall\, t,t' \in [r,r'],\ \ \mathfrak{q} \in \mathfrak{Q}. \tag{1}$$

- $C^{\mathrm{c}}([r,r'],F)$, the set of all constant curves on $[r,r'] \in \mathfrak{K}$; i.e., all curves of the form

$$\gamma_X \colon [r,r'] \to F, \qquad t \mapsto X$$

  for some $X \in F$.

We define $\mathrm{c}+1 := \infty$, $\infty + 1 := \infty$, $\mathrm{lip}+1 := 1$; as well as

$$\begin{aligned}
&\mathfrak{q}_\infty(\gamma) := \mathfrak{q}^0_\infty(\gamma) && \forall\, \gamma \in C^0([r,r'],F)\\
&\mathfrak{q}^{\mathrm{lip}}_\infty(\gamma) := \max(\mathfrak{q}_\infty(\gamma), \mathrm{Lip}(\mathfrak{q},\gamma)) && \forall\, \gamma \in C^{\mathrm{lip}}([r,r'],F)\\
&\mathfrak{q}^{\mathrm{s}}_\infty(\gamma) := \sup\left\{\mathfrak{q}\big(\gamma^{(m)}(t)\big) \;\middle|\; 0 \leq m \leq \mathrm{s},\ \ t \in [r,r']\right\} && \forall\, \gamma \in C^k([r,r'],F)
\end{aligned}$$

for each $\mathfrak{q} \in \mathfrak{Q}$, $k \in \mathbb{N} \sqcup \{\infty\}$, $s \preceq k$, and $[r,r'] \in \mathfrak{K}$ – Here, $s \preceq k$ means

- $\mathrm{s} \equiv \mathrm{lip}$ for $k \equiv \mathrm{lip}$,
- $\mathrm{s} \leq k$ for $k \in \mathbb{N}$,
- $\mathrm{s} \in \mathbb{N}$ for $k \equiv \infty$,
- $\mathrm{s} = 0$ for $k \equiv \mathrm{c}$.

The C^k-topology on $C^k([r,r'],F)$ is the Hausdorff locally convex topology that is generated by the seminorms $\mathfrak{q}^{\mathrm{s}}_\infty$ for all $\mathfrak{q} \in \mathfrak{Q}$ and $\mathrm{s} \preceq k$.

Remark 1. *In the Lipschitz case, the above conventions deviate from the conventions used, e.g., in [6, 8] as there the $\mathfrak{p}_\infty$-seminorms, i.e., the C^0-topology is considered on $C^{\mathrm{lip}}([r,r'],F)$.* ‡

Finally, we let $\mathrm{CP}^0([r,r'],F)$ denote the set of piecewise C^0-curves on $[r,r'] \in \mathfrak{K}$; i.e., all $\gamma\colon [r,r'] \to F$, such that there exist $r = t_0 < \ldots < t_n = r'$ and

$$\gamma[p] \in C^0([t_p,t_{p+1}],F) \qquad \text{with} \qquad \gamma|_{(t_p,t_{p+1})} = \gamma[p]|_{(t_p,t_{p+1})} \qquad \text{for} \qquad p = 0,\ldots,n-1. \tag{2}$$

2.1.2 Lie Groups

In this paper, G will always denote an infinite dimensional Lie group in Milnor's sense [1, 5, 9, 12] that is modeled over the Hausdorff locally convex vector space E, with corresponding system of continuous seminorms $\mathfrak{P}$. We define

$$\overline{\mathrm{B}}_{\mathfrak{p},\varepsilon} := \{X \in E \mid \mathfrak{p}(X) \leq \varepsilon\} \qquad \forall\, \mathfrak{p} \in \mathfrak{P},\ \ \varepsilon > 0,$$

and denote the Lie algebra of G by $(\mathfrak{g},[\cdot,\cdot])$. We fix a chart

$$\Xi \colon G \supseteq \mathcal{U} \to \mathcal{V} \subseteq E,$$

with $\mathcal{V}$ convex, $e \in \mathcal{U}$ and $\Xi(e) = 0$; and define

$$\boldsymbol{.}\mathfrak{p} := \mathfrak{p} \circ \mathrm{d}_e\Xi \qquad \forall\, \mathfrak{p}, \in \mathfrak{P}.$$

We let $\mathrm{m}\colon G \times G \to G$ denote the Lie group multiplication, $\mathrm{R}_g := \mathrm{m}(\cdot, g)$ the right translation by $g \in G$, and $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$ the adjoint action – i.e., we have

$$\mathrm{Ad}(g, X) \equiv \mathrm{Ad}_g(X) := \mathrm{d}_e\mathrm{Conj}_g(X) \qquad \text{with} \qquad \mathrm{Conj}_g\colon G \ni h \mapsto g \cdot h \cdot g^{-1} \in G$$

for each $g \in G$ and $X \in \mathfrak{g}$. We furthermore recall the product rule

$$\mathrm{d}_{(g,h)}\mathrm{m}(v, w) = \mathrm{d}_g\mathrm{R}_h(v) + \mathrm{d}_h\mathrm{L}_g(w) \qquad \forall\, g, h \in G,\;\; v \in T_gG,\;\; w \in T_hG. \tag{3}$$

2.1.3 Uniform Limits

Let $\mu \in \mathrm{Map}([r, r'], G)$, $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Map}([r, r'], G)$, and $\{\mu_h\}_{h\in(-\delta,0)\cup(0,\delta)} \subseteq \mathrm{Map}([r, r'], G)$ for $\delta > 0$ be given. We write

- $\lim^\infty_n \mu_n = \mu$ *iff* for each open neighbourhood $U \subseteq G$ of e, there exists some $n_U \in \mathbb{N}$ with $\mu^{-1} \cdot \mu_n \in U$ for each $n \geq n_U$.
- $\lim^\infty_{h\to 0} \mu_h = \mu$ *iff* for each open neighbourhood $U \subseteq G$ of e, there exists some $0 < \delta_U < \delta$ with $\mu^{-1} \cdot \mu_h \in U$ for each $0 < |h| < \delta_U$.

Evidently, then we have

Lemma 1. *Let $D \equiv (-\delta, 0) \cup (0, \delta)$ for $\delta > 0$, and $\{\mu_h\}_{h\in D} \subseteq C^0([r, r'], G)$ be given. If $\lim^\infty_n \mu_{h_n} = e$ holds for each sequence $D \supseteq \{h_n\}_{n\in\mathbb{N}} \to 0$, then we have $\lim^\infty_{h\to 0} \mu_h = e$.*

Proof. If the claim is wrong, then there exists a neighbourhood $U \subseteq G$ of e, such that the following holds: For each $n \in \mathbb{N}$, there exists some $h_n \neq 0$ with $|h_n| \leq \frac{1}{n}$ as well as some $\tau_n \in [r, r']$, such that $\mu^{-1}(\tau_n) \cdot \mu_{h_n}(\tau_n) \notin U$ holds. Since we have $\{h_n\}_{n\in\mathbb{N}} \to 0$, this contradicts the presumptions. □

The same conventions (and Lemma 1) also hold if $(G, \cdot) \equiv (F, +)$ is a Hausdorff locally convex vector space (or its completion) – In this case, we use the following convention:

Let $D \equiv (-\delta, 0) \cup (0, \delta)$ for $\delta > 0$; and $\alpha\colon D \times [r, r'] \to F$ be given, with $\alpha(h, \cdot) \in \mathrm{Map}([r, r'], F)$ for each $h \in D$ as well as $\alpha(h, \cdot) = 0$. Then, for $\beta \in \mathrm{Map}([r, r'], F)$, we write

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|^\infty_{h=0}\, \alpha = \beta \qquad \stackrel{\text{def.}}{\Longleftrightarrow} \qquad \lim^\infty_{h\to 0} \big[1/h \cdot \alpha(h, \cdot)\big] = \beta.$$

Remark 2. *In the following, the above convention will mainly be used in the following form: $F = \overline{E}$ will be the completion of a Hausdorff locally convex vector space E; and we will have $\alpha\colon D \times [r, r'] \to E \subseteq \overline{E}$ as well as $\beta \in \mathrm{Map}([r, r'], \overline{E})$.* ‡

2.2 The Right Logarithmic Derivative

In this subsection, we provide the relevant facts and definitions concerning the right logarithmic derivative.

2.2.1 The Evolution Map

We let $\mathfrak{K} \equiv \{[r,r'] \subseteq \mathbb{R} \mid r < r'\}$ denote the set of all (proper) compact intervals in $\mathbb{R}$; and define

$$C^k_*([r,r'],G) := \{\mu \in C^k([r,r'],G) \mid \mu(r) = e\} \qquad \forall\, [r,r'] \in \mathfrak{K},\ \ k \in \mathbb{N} \sqcup \{\infty\}.$$

The right logarithmic derivative is given by

$$\delta^r \colon C^1([r,r'],G) \to C^0([r,r'],\mathfrak{g}), \qquad \mu \mapsto \mathrm{d}_\mu \mathrm{R}_{\mu^{-1}}(\dot\mu)$$

for each $[r,r'] \in \mathfrak{K}$; and we define $\mathfrak{D}_{[r,r']} := \delta^r(C^1([r,r'],G))$ for each $[r,r'] \in \mathfrak{K}$, as well as

$$\mathfrak{D}^k_{[r,r']} := \mathfrak{D}_{[r,r']} \cap C^k([r,r'],\mathfrak{g}) \qquad \forall\, [r,r'] \in \mathfrak{K},\ \ k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty,\mathrm{c}\}.$$

Then, δ^r restricted to $C^1_*([r,r'],G)$ is injective for each $[r,r'] \in \mathfrak{K}$; so that

$$\mathrm{Evol} \colon \textstyle\bigsqcup_{[r,r']\in\mathfrak{K}} \mathfrak{D}_{[r,r']} \to \bigsqcup_{[r,r']\in\mathfrak{K}} C^1_*([r,r'],G)$$

is well defined by

$$\mathrm{Evol} \colon \mathfrak{D}_{[r,r']} \to C^1_*([r,r'],G), \qquad \delta^r(\mu) \mapsto \mu \cdot \mu(r)^{-1}$$

for each $[r,r'] \in \mathfrak{K}$. Here,

$$\mathrm{Evol}|_{\mathfrak{D}^k_{[r,r']}} \colon \mathfrak{D}^k_{[r,r']} \to C^{k+1}([r,r'],G)$$

holds for each $[r,r'] \in \mathfrak{K}$, and each $k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty,\mathrm{c}\}$.

2.2.2 The Product Integral

The product integral is given by

$$\textstyle\int_s^t \phi := \mathrm{Evol}\big(\phi|_{[s,t]}\big)(t) \in G \qquad \forall\, [s,t] \subseteq \mathrm{dom}[\phi],\ \ \phi \in \bigsqcup_{[r,r']\in\mathfrak{K}} \mathfrak{D}_{[r,r']};$$

and we let $\int \phi \equiv \int_r^{r'} \phi$ as well as $\int_c^c \phi := e$ for $\phi \in \mathfrak{D}_{[r,r']}$ and $c \in [r,r']$. We furthermore denote

$$\mathrm{evol}^k_{[r,r']} \equiv \textstyle\int|_{\mathfrak{D}^k_{[r,r']}} \qquad \forall\, k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty,\mathrm{c}\},\ \ [r,r'] \in \mathfrak{K};$$

and let $\mathrm{evol_k} \equiv \mathrm{evol}^k_{[0,1]}$ as well as $\mathrm{D_k} \equiv \mathfrak{D}^k_{[0,1]}$ for each $k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty,\mathrm{c}\}$. We furthermore define

$$\mathrm{evol} \equiv \mathrm{evol}_0 \colon \mathrm{D} \equiv \mathrm{D}_0 \to G.$$

Then, we have the following elementary identities, cf. Sect. 3.5.2 in [6]:

a) For each $\phi,\psi \in \mathfrak{D}_{[r,r']}$, we have $\phi + \mathrm{Ad}_{\int_r^\bullet \phi}(\psi) \in \mathfrak{D}_{[r,r']}$, with

$$\textstyle\int_r^t \phi \cdot \int_r^t \psi = \int_r^t \phi + \mathrm{Ad}_{\int_r^\bullet \phi}(\psi).$$

b) For each $\phi,\psi \in \mathfrak{D}_{[r,r']}$, we have $\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi) \in \mathfrak{D}_{[r,r']}$, with

$$\textstyle\big[\int_r^t \phi\big]^{-1}\big[\int_r^t \psi\big] = \int_r^t \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi).$$

c) For $r = t_0 < \ldots < t_n = r'$ and $\phi \in \mathfrak{D}_{[r,r']}$, we have

$$\textstyle\int_r^t \phi = \int_{t_p}^t \phi \cdot \int_{t_{p-1}}^{t_p} \phi \cdot \ldots \cdot \int_r^{t_1} \phi \qquad \forall\, t \in (t_p, t_{p+1}],\ \ p = 0,\ldots,n-1.$$

d) For $\varrho \colon [\ell,\ell'] \to [r,r']$ of class C^1 and $\phi \in \mathfrak{D}_{[r,r']}$, we have $\dot\varrho \cdot \phi \circ \varrho \in \mathfrak{D}_{[\ell,\ell']}$, with

$$\textstyle\int_r^\varrho \phi = \big[\int_\ell^\bullet \dot\varrho \cdot \phi \circ \varrho\big] \cdot \big[\int_r^{\varrho(\ell)} \phi\big].$$

e) For each Lie group homomorphism $\Psi \colon G \to H$ that is of class C^1, we have

$$\textstyle\Psi \circ \int_r^\bullet \phi = \int_r^\bullet \mathrm{d}_e\Psi \circ \phi \qquad \forall\, \phi \in \mathfrak{D}_{[r,r']}.$$

We say that G is ***C^k*-semiregular** for $k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty\}$ *iff* $\mathrm{D_k} = C^k([0,1],\mathfrak{g})$ holds; which, by d), is equivalent to that $\mathfrak{D}^k_{[r,r']} = C^k([r,r'],\mathfrak{g})$ holds for each $[r,r'] \in \mathfrak{K}$, cf. e.g., Lemma 12 in [6].

2.2.3 The Exponential Map

The exponential map is defined by

$$\exp\colon \mathrm{dom}[\exp] \equiv \mathfrak{i}^{-1}(\mathrm{D_c}) \to G, \qquad X \mapsto \textstyle\int \phi_X|_{[0,1]} \equiv (\mathrm{evol_c} \circ \mathfrak{i})(X)$$

with $\mathfrak{i}\colon \mathfrak{g} \ni X \to \phi_X|_{[0,1]} \in C^{\mathrm{c}}([0,1], \mathfrak{g})$.

Then, instead of saying that G is C^{c}-semiregular, we will rather say that G admits an exponential map in the following. We furthermore remark that d) implies $\mathbb{R} \cdot \mathrm{dom}[\exp] \subseteq \mathrm{dom}[\exp]$; and that $t \mapsto \exp(t \cdot X)$ is a smooth Lie group homomorphism for each $X \in \mathrm{dom}[\exp]$, with

$$\exp(t \cdot X) \equiv \textstyle\int t \cdot \phi_X|_{[0,1]} \stackrel{\mathrm{d)}}{=} \int_0^t \phi_X|_{[0,1]} \qquad \forall\, t \geq 0, \tag{4}$$

cf., e.g., Remark 2.1) in [6]. Finally, if G is abelian, then $X + Y \in \mathrm{dom}[\exp]$ holds for all $X, Y \in \mathrm{dom}[\exp]$, because we have

$$\exp(X) \cdot \exp(Y) \stackrel{\mathrm{a)}}{=} \textstyle\int \phi_X|_{[0,1]} \cdot \int \phi_Y|_{[0,1]} = \int \phi_{X+Y}|_{[0,1]}.$$

2.2.4 Standard Topologies

We say that G is $\boldsymbol{C^k}$**-continuous** for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$ *iff* $\mathrm{evol_k}$ is continuous w.r.t. the C^k-topology. We explicitly remark that under the identification $\mathfrak{i}\colon \mathfrak{g} \to \{\phi_X|_{[0,1]} \mid X \in \mathfrak{g}\}$ the C^{c}-topology just equals the subspace topology on $\mathrm{dom}[\exp]$ that is inherited by the locally convex topology on $\mathfrak{g}$. So, instead of saying that G is C^{c}-continuous *iff* $\mathrm{evol_c}$ is continuous w.r.t. this topology, we will rather say that the exponential map is continuous.

2.3 The Riemann Integral

Let F be a Hausdorff locally convex vector space with system of continuous seminorms $\mathfrak{Q}$, and completion $\overline{F}$. For each $\mathfrak{q} \in \mathfrak{P}$, we let $\overline{\mathfrak{q}}\colon \overline{F} \to \mathbb{R}_{\geq 0}$ denote the continuous extension of $\mathfrak{q}$ to $\overline{F}$. The Riemann integral of $\gamma \in C^0([r, r'], F)$ (for $[r, r'] \in \mathfrak{K}$) is denoted by $\int \gamma(s)\, \mathrm{d}s \in \overline{F}$; and we define

$$\textstyle\int_a^b \gamma(s)\, \mathrm{d}s := \int \gamma|_{[a,b]}(s)\, \mathrm{d}s, \qquad \int_b^a \gamma(s)\, \mathrm{d}s := -\int_a^b \gamma(s)\, \mathrm{d}s, \qquad \int_c^c \gamma(s)\, \mathrm{d}s := 0 \tag{5}$$

for $r \leq a < b \leq r'$, $c \in [r, r']$. Clearly, the Riemann integral is linear, with

$$\textstyle\int_a^c \gamma(s)\, \mathrm{d}s = \int_a^b \gamma(s)\, \mathrm{d}s + \int_b^c \gamma(s)\, \mathrm{d}s \qquad \forall\, r \leq a < b < c \leq r' \tag{6}$$

$$\gamma - \gamma(r) = \textstyle\int_r^\bullet \dot{\gamma}(s)\, \mathrm{d}s \tag{7}$$

$$\mathfrak{q}(\gamma - \gamma(r)) \leq \textstyle\int_r^\bullet \mathfrak{q}(\dot{\gamma}(s))\, \mathrm{d}s \qquad \forall\, \mathfrak{q} \in \mathfrak{Q},\ \gamma \in C^1([r, r'], F), \tag{8}$$

as well as

$$\overline{\mathfrak{q}}\big(\textstyle\int_r^\bullet \gamma(s)\, \mathrm{d}s\big) \leq \int_r^\bullet \mathfrak{q}(\gamma(s))\, \mathrm{d}s \qquad \forall\, \mathfrak{q} \in \mathfrak{Q},\ \gamma \in C^0([r, r'], F). \tag{9}$$

We furthermore have the substitution formula

$$\textstyle\int_r^{\varrho(t)} \gamma(s)\, \mathrm{d}s = \int_\ell^t \dot{\varrho}(s) \cdot (\gamma \circ \varrho)(s)\, \mathrm{d}s \tag{10}$$

for each $\gamma \in C^0([r, r'], F)$, and each $\varrho\colon \mathfrak{K} \ni [\ell, \ell'] \to [r, r']$ of class C^1 with $\varrho(\ell) = r$ and $\varrho(\ell') = r'$. Moreover, if E is a Hausdorff locally convex vector space, and $\mathfrak{L}\colon F \to E$ a continuous linear map, then we have

$$\textstyle\int \gamma(s)\, \mathrm{d}s \in F \quad \text{for} \quad \gamma \in C^0([r, r'], F) \qquad \Longrightarrow \qquad \mathfrak{L}(\int \gamma(s)\, \mathrm{d}s) = \int \mathfrak{L}(\gamma(s))\, \mathrm{d}s. \tag{11}$$

Finally, for $\gamma \in \mathrm{CP}^0([r,r'],F)$ with $\gamma[0],\dots,\gamma[n-1]$ as in (2), we define

$$\textstyle\int \gamma(s)\,\mathrm{d}s := \sum_{p=0}^{n-1} \int \gamma[p](s)\,\mathrm{d}s. \tag{12}$$

A standard refinement argument in combination with (6) then shows that this is well defined; i.e., independent of any choices we have made. We define $\int_a^b \gamma(s)\,\mathrm{d}s$, $\int_b^a \gamma(s)\,\mathrm{d}s$ and $\int_c^c \gamma(s)\,\mathrm{d}s$ as in (5); and observe that (12) is linear and fulfills (6).

2.4 Standard Facts and Estimates

Let us first recall that

Lemma 2. *Let X be a topological space; and let $\Phi\colon X \times F_1 \times \ldots \times F_n \to E$ be continuous with $\Phi(x,\cdot)$ n-multilinear for each $x \in X$. Then, for each compact $\mathrm{K} \subseteq X$ and each $\mathfrak{p} \in \mathfrak{P}$, there exist seminorms $\mathfrak{q}_1 \in \mathfrak{Q}_1,\dots,\mathfrak{q}_n \in \mathfrak{Q}_n$ as well as $O \subseteq X$ open with $\mathrm{K} \subseteq O$, such that*

$$(\mathfrak{p}\circ\Phi)(y,X_1,\dots,X_n) \leq \mathfrak{q}_1(X_1)\cdot\ldots\cdot\mathfrak{q}_n(X_n) \qquad \forall\, y \in O$$

holds for all $X_1 \in F_1,\dots,X_n \in F_n$.

Proof. Confer, e.g., Corollary 1 in [6]. □

Lemma 3. *Let F, E be Hausdorff locally convex vector spaces. If $\Phi\colon F \to E$ is continuous and linear, then Φ extends uniquely to a continuous linear map $\overline{\Phi}\colon \overline{F} \to \overline{E}$.*

Lemma 4. *Let $f\colon F \supseteq U \to E$ be of class C^2; and suppose that $\gamma\colon D \to F \subseteq \overline{F}$ is continuous at $t \in D$, such that $\lim_{h\to 0} 1/h\cdot(\gamma(t+h)-\gamma(t)) =: X \in \overline{F}$ exists. Then, we have*

$$\lim_{h\to 0} 1/h\cdot(f(\gamma(t+h)) - f(\gamma(t))) = \overline{\mathrm{d}_{\gamma(t)} f}(X).$$

Proof. Confer, e.g., Lemma 7 in [6]. □

Remark 3. *Let F be a Hausdorff locally convex vector space, and $U \subseteq F$ open. A map $f\colon U \to G$ is said to be*

- *differentiable at $x \in U$ iff there exists a chart $(\Xi',\mathcal{U}')$ with $f(x) \in \mathcal{U}'$, such that*

$$(D_v^{\Xi'} f)(x) := \lim_{h\to 0} 1/h\cdot((\Xi'\circ f)(x+h\cdot v) - (\Xi'\circ f)(x)) \in E \qquad \forall\, v \in F \tag{13}$$

  exists. Then, Lemma 4 applied to coordinate changes shows that (13) holds for one chart around $f(x)$ iff it holds for each chart around $f(x)$ – and that

$$\mathrm{d}_x f(v) := \big(\mathrm{d}_{\Xi'(f(x))}\Xi'^{-1}\circ(D_v^{\Xi'} f)\big)(x) \in T_{f(x)}G \qquad \forall\, v \in F$$

  is independent of the explicit choice of (Ξ',U').
- *differentiable iff f is differentiable at each $x \in U$.* ‡

We furthermore recall that, cf. Sect. 3.4.1 in [6]

I) For each compact $\mathrm{C} \subseteq G$ and each $\mathfrak{v} \in \mathfrak{P}$, there exists some $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$, such that $\boldsymbol{.}\mathfrak{v}\circ\mathrm{Ad}_g \leq \boldsymbol{.}\mathfrak{w}$ holds for each $g \in \mathrm{C}$.

II) For each $\mathfrak{m} \in \mathfrak{P}$, there exists some $\mathfrak{m} \leq \mathfrak{q} \in \mathfrak{P}$, as well as $O \subseteq G$ symmetric open with $e \in O$, such that $\boldsymbol{.}\mathfrak{m}\circ\mathrm{Ad}_g \leq \boldsymbol{.}\mathfrak{q}$ holds for each $g \in O$.

III) Suppose that $\mathrm{im}[\mu] \subseteq \mathcal{U}$ holds for $\mu \in C^1([r,r'],G)$. Then, we have

$$\delta^r(\mu) = \omega(\Xi \circ \mu, \partial_t(\Xi \circ \mu)), \tag{14}$$

for the smooth map $\omega\colon \mathcal{V} \times E \ni (x,X) \mapsto \mathrm{d}_{\Xi^{-1}(x)}\mathrm{R}_{[\Xi^{-1}(x)]^{-1}}(\mathrm{d}_x\Xi^{-1}(X)) \in \mathfrak{g}$. Since ω is linear in the second argument, for each $\mathfrak{q} \in \mathfrak{P}$, there exists some $\mathfrak{q} \leq \mathfrak{m} \in \mathfrak{P}$ with

$$(\boldsymbol{.}\mathfrak{q} \circ \omega)(x,X) \leq \mathfrak{m}(X) \qquad \forall\, x \in \overline{\mathrm{B}}_{\mathfrak{m},1},\ \ X \in E. \tag{15}$$

IV) Suppose that $\mathrm{im}[\mu] \subseteq \mathcal{U}$ holds for $\mu \in C^1([r,r'],G)$. Then, we have

$$\partial_t(\Xi \circ \mu) = \upsilon(\Xi \circ \mu, \delta^r(\mu)), \tag{16}$$

for the smooth map $\upsilon\colon \mathcal{V} \times \mathfrak{g} \ni (x,X) \mapsto \big(\mathrm{d}_{\Xi^{-1}(x)}\Xi \circ \mathrm{d}_e\mathrm{R}_{\Xi^{-1}(x)}\big)(X) \in E$. Since υ is linear in the second argument, for each $\mathfrak{q} \in \mathfrak{P}$, there exists some $\mathfrak{u} \leq \mathfrak{m} \in \mathfrak{P}$ with

$$(\mathfrak{u} \circ \upsilon)(x,X) \leq \boldsymbol{.}\mathfrak{m}(X) \qquad \forall\, x \in \overline{\mathrm{B}}_{\mathfrak{m},1},\ \ X \in \mathfrak{g}.$$

For each $\mu \in C^1([r,r'],G)$ with $\mathrm{im}[\Xi \circ \mu] \subseteq \overline{\mathrm{B}}_{\mathfrak{m},1}$, we thus obtain from (16), (7), and (8) that

$$\mathfrak{u}(\Xi \circ \mu) = \mathfrak{u}\big(\textstyle\int_r^\bullet \upsilon((\Xi\circ\mu)(s), \delta^r(\mu)(s))\,\mathrm{d}s\big) \leq \int_r^\bullet \boldsymbol{.}\mathfrak{m}(\delta^r(\mu)(s))\,\mathrm{d}s. \tag{17}$$

For instance, we immediately obtain from (17) that

Lemma 5. *For each $\mathfrak{u} \in \mathfrak{P}$, there exist $\mathfrak{u} \leq \mathfrak{m} \in \mathfrak{P}$, and $U \subseteq G$ open with $e \in U$, such that*

$$(\mathfrak{u} \circ \Xi)(\textstyle\int_r^\bullet \chi) \leq \int_r^\bullet \mathfrak{m}(\chi(s))\,\mathrm{d}s$$

holds, for each $\chi \in \mathfrak{D}_{[r,r']}$ with $\int_r^\bullet \chi \in U$; for all $[r,r'] \in \mathfrak{K}$.

Moreover,

Lemma 6. *We have $\mathrm{Ad}_\mu(\phi) \in C^k([r,r'],\mathfrak{g})$ for each $\mu \in C^{k+1}([r,r'],G)$, $\phi \in C^k([r,r'],\mathfrak{g})$, and $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$.*

Proof. Confer, e.g., Lemma 13 in [6]. □

Lemma 7. *Let $[r,r'] \in \mathfrak{K}$, $k \in \mathbb{N} \sqcup \{\infty\}$, and $\phi \in \mathfrak{D}^k_{[r,r']}$ be fixed. Then, for each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, there exists some $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ with*

$$\boldsymbol{.}\mathfrak{p}^{\mathrm{p}}_\infty\big(\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi)\big) \leq \boldsymbol{.}\mathfrak{q}^{\mathrm{p}}_\infty(\psi) \qquad \forall\, \psi \in C^k([r,r'],\mathfrak{g}),\ \ 0 \leq \mathrm{p} \leq \mathrm{s}.$$

Proof. Confer, e.g., Lemma 14 in [6]. □

Then, modifying the argumentation used in the proof of the Lipschitz case in Lemma 13 in [6] to our deviating convention concerning the topology on the set of Lipschitz curves, we also obtain

Lemma 8. *Let $[r,r'] \in \mathfrak{K}$, and $\phi \in \mathfrak{D}_{[r,r']}$ be fixed. Then, for each $\mathfrak{p} \in \mathfrak{P}$, there exists some $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ with*

$$\boldsymbol{.}\mathfrak{p}^{\mathrm{lip}}_\infty\big(\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi)\big) \leq \boldsymbol{.}\mathfrak{q}^{\mathrm{lip}}_\infty(\psi) \qquad \forall\, \psi \in C^{\mathrm{lip}}([r,r'],\mathfrak{g}).$$

Proof. Confer Appendix B. □

2.5 Continuity Statements

For $h \in G$, we define $\Xi_h(g) := \Xi(h^{-1} \cdot g)$ for each $g \in h \cdot \mathcal{U}$; and recall that, cf. Lemma 8 in [6]

Lemma 9. *Let* $\mathrm{C} \subseteq \mathcal{U}$ *be compact. Then, for each* $\mathfrak{p} \in \mathfrak{P}$*, there exists some* $\mathfrak{p} \leq \mathfrak{u} \in \mathfrak{P}$*, and a symmetric open neighbourhood* $V \subseteq \mathcal{U}$ *of* e *with* $\mathrm{C} \cdot V \subseteq \mathcal{U}$ *and* $\overline{\mathrm{B}}_{\mathfrak{u},1} \subseteq \Xi(V)$*, such that*

$$\mathfrak{p}(\Xi(q) - \Xi(q')) \leq \mathfrak{u}(\Xi_{g \cdot h}(q) - \Xi_{g \cdot h}(q')) \qquad \forall\, q, q' \in g \cdot V,\ \ h \in V$$

holds for each $g \in \mathrm{C}$.

Now, combining Lemma 5 with Lemma 9, we obtain the following variation of Proposition 1 in [6]:

Lemma 10. *For each* $\mathfrak{p} \in \mathfrak{P}$*, there exist* $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ *and* $V \subseteq G$ *open with* $e \in V$*, such that*

$$\mathfrak{p}\big(\Xi\big(\textstyle\int_r^\bullet \phi\big) - \Xi\big(\textstyle\int_r^\bullet \psi\big)\big) \leq \textstyle\int_r^\bullet \mathbf{.}\mathfrak{q}(\phi(s) - \psi(s))\, \mathrm{d}s$$

holds for all $\phi, \psi \in \mathfrak{D}_{[r,r']}$ *with* $\int_r^\bullet \phi,\ \int_r^\bullet \psi \in V$*; for each* $[r, r'] \in \mathfrak{K}$.

Proof. We let $\mathfrak{p} \leq \mathfrak{u} \in \mathfrak{P}$ and V be as in Lemma 9 for $\mathrm{C} \equiv \{e\}$ there (i.e., V is symmetric with $\overline{\mathrm{B}}_{\mathfrak{u},1} \subseteq \Xi(V)$). We choose $U \subseteq G$ and $\mathfrak{u} \leq \mathfrak{m} \in \mathfrak{P}$ as in Lemma 5. We furthermore let $\mathfrak{m} \leq \mathfrak{q} \in \mathfrak{P}$ and $O \subseteq G$ be as in II). Then, shrinking V if necessary, we can assume that $V^{-1} \cdot V \subseteq U$ as well as $V \subseteq O$ holds. Then, for ϕ, ψ as in the presumptions, Lemma 9 applied to $q \equiv \int_r^\bullet \phi$, $q' \equiv \int_r^\bullet \psi$, $h \equiv \int_r^\bullet \phi \in V$, and $g \equiv e$ gives

$$\mathfrak{p}\big(\Xi\big(\textstyle\int_r^\bullet \phi\big) - \Xi\big(\textstyle\int_r^\bullet \psi\big)\big) \leq \mathfrak{u}\big(\Xi_{\int_r^\bullet \phi}\big(\textstyle\int_r^\bullet \phi\big) - \Xi_{\int_r^\bullet \phi}\big(\textstyle\int_r^\bullet \psi\big)\big) = (\mathfrak{u} \circ \Xi)\big([\textstyle\int_r^\bullet \phi]^{-1}[\textstyle\int_r^\bullet \psi]\big).$$

By assumption, for each $t \in [r, r']$, we have

$$U \supseteq V^{-1} \cdot V \ni [\textstyle\int_r^t \phi]^{-1}[\textstyle\int_r^t \psi] \stackrel{\mathrm{b)}}{=} \textstyle\int_r^t \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi) \qquad \text{with} \qquad [\textstyle\int_r^\bullet \phi]^{-1} \in V^{-1} = V \subseteq O;$$

obtain from Lemma 5 and II) that

$$(\mathfrak{u} \circ \Xi)\big([\textstyle\int_r^t \phi]^{-1}[\textstyle\int_r^t \psi]\big) \leq \textstyle\int_r^t \mathbf{.}\mathfrak{m}\big(\mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s) - \phi(s))\big)\, \mathrm{d}s \leq \textstyle\int_r^t \mathbf{.}\mathfrak{q}(\psi(s) - \phi(s))\, \mathrm{d}s$$

holds for each $t \in [r, r']$; which proves the claim. □

We furthermore observe that

Lemma 11. *Suppose that* $\exp\colon \mathrm{dom}[\exp] \to G$ *is continuous; and let* $X \in \mathrm{dom}[\exp]$ *be fixed. Then, for each open neighbourhood* $V \subseteq G$ *of* e*, there exists some* $\mathfrak{m} \in \mathfrak{P}$*, such that*

$$\mathbf{.}\mathfrak{m}(Y - X) \leq 1 \quad \text{for} \quad Y \in \mathrm{dom}[\exp] \qquad \Longrightarrow \qquad \textstyle\int_0^\bullet \phi_Y|_{[0,1]} \in \int_0^\bullet \phi_X|_{[0,1]} \cdot V.$$

Proof. By assumption, $\alpha\colon [0,1] \times \mathrm{dom}[\exp] \ni (t, Y) \mapsto \exp(t \cdot X)^{-1} \cdot \exp(t \cdot Y)$ is continuous; and we have $\alpha(\cdot, X) = e$. For $\tau \in [0,1]$ fixed, there thus exists an open interval $I_\tau \subseteq \mathbb{R}$ containing τ, as well as an open neighbourhood $O_\tau \subseteq \mathfrak{g}$ of X, such that we have

$$\exp(t \cdot X)^{-1} \cdot \exp(t \cdot Y) \in V \qquad \forall\, t \in I_\tau \cap [0,1],\ \ Y \in O_\tau \cap \mathrm{dom}[\exp]. \tag{18}$$

We choose $\tau_1, \dots, \tau_n \in [0,1]$ with $[0,1] \subseteq I_{\tau_1} \cup \ldots \cup I_{\tau_n}$; and define $O := O_{\tau_1} \cap \ldots \cap O_{\tau_n}$. Then, (18) holds for each $t \in [0,1]$ and $Y \in O \cap \mathrm{dom}[\exp]$; so that the claim holds for each fixed $\mathfrak{m} \in \mathfrak{P}$ with $\overline{\mathrm{B}}_{\mathfrak{m},1} \subseteq O$. □

3 Auxiliary Results

In this section, we introduce the continuity notions that we will need to formulate our main results. We furthermore provide some elementary continuity statements that we will need in the main text.

3.1 Sets of Curves

Let $[r,r'] \in \mathfrak{K}$ be fixed. We will tacitly use in the following that $C^k([r,r'],\mathfrak{g})$ in a real vector space for each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$. We will furthermore use that:

A) For each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, $\phi \in \mathfrak{D}^k_{[r,r']}$, and $\psi \in C^k([r,r'],\mathfrak{g})$, we have $\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi) \in C^k([r,r'],\mathfrak{g})$ by Lemma 6. Evidently, the same statement also holds for $k \equiv \mathrm{c}$ if G is abelian.

B) For each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$, $\phi \in \mathfrak{D}^k_{[r,r']}$, $[\ell,\ell'] \in \mathfrak{K}$, and

$$\varrho\colon [\ell,\ell'] \to [r,r'], \qquad t \mapsto r + (t-\ell)\cdot |r'-r|/|\ell'-\ell|,$$

we have $\dot\varrho \cdot \phi \circ \varrho = |r'-r|/|\ell'-\ell| \cdot \phi\circ\varrho \in \mathfrak{D}^k_{[\ell,\ell']}$ by d); with

$$\begin{aligned} \mathbin{\bullet}\mathfrak{p}^{\mathrm{s}}_\infty(\dot\varrho\cdot\phi\circ\varrho) &= \left[\tfrac{|r'-r|}{|\ell'-\ell|}\right]^{\mathrm{s}+1} \cdot \mathbin{\bullet}\mathfrak{p}^{\mathrm{s}}_\infty(\phi) \quad \text{with} \quad \mathrm{s} \preceq k \quad \text{for} \quad k \in \mathbb{N}\sqcup\{\infty,\mathrm{c}\}, \\ \mathrm{Lip}(\mathbin{\bullet}\mathfrak{p}, \dot\varrho\cdot\phi\circ\varrho) &= \left[\tfrac{|r'-r|}{|\ell'-\ell|}\right]^{2} \cdot \mathrm{Lip}(\mathbin{\bullet}\mathfrak{p},\phi) \quad \text{for} \quad k \equiv \mathrm{lip}. \end{aligned}$$

We say that $\mathfrak{g}$ is **k-complete** for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$ *iff*

$$\textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\chi(s))\,\mathrm{d}s \in \mathfrak{g} \tag{19}$$

holds for all $\phi, \chi \in \mathfrak{D}^k_{[r,r']}$, for each $[r,r'] \in \mathfrak{K}$. Then,

Remark 4.

- $\mathfrak{g}$ *is* c*-complete if G is abelian.*
- $\mathfrak{g}$ *is k-complete for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$ iff* (19) *holds for* $[r,r'] \equiv [0,1]$.

  For this, let $\phi,\chi \in \mathfrak{D}^k_{[r,r']}$ be given. Then, for $\varrho\colon [0,1] \to [r,r']$ as in B) with $[\ell,\ell'] \equiv [0,1]$ there, we have $\dot\varrho\cdot\phi\circ\varrho,\ \ \dot\varrho\cdot\chi\circ\varrho \in \mathfrak{D}^k_{[0,1]}$ with

$$\begin{aligned} \textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\chi(s))\,\mathrm{d}s &= \textstyle\int_r^{\varrho(1)} \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\chi(s))\,\mathrm{d}s \\ &\overset{(10)}{=} \textstyle\int_0^1 \dot\varrho(s)\cdot \mathrm{Ad}_{[\int_r^{\varrho(s)} \phi]^{-1}}(\chi(\varrho(s)))\,\mathrm{d}s \\ &= \textstyle\int_0^1 \mathrm{Ad}_{[\int_r^{\varrho(s)} \phi]^{-1}}((\dot\varrho\cdot\chi\circ\varrho)(s))\,\mathrm{d}s \\ &\overset{\mathrm{d})}{=} \textstyle\int_0^1 \mathrm{Ad}_{[\int_0^s \dot\varrho\cdot\phi\circ\varrho]^{-1}}((\dot\varrho\cdot\chi\circ\varrho)(s))\,\mathrm{d}s. \end{aligned}$$

In particular, Point A) then shows:

- *If G is C^0-semiregular, then $\mathfrak{g}$ is 0-complete iff $\mathfrak{g}$ is integral complete – i.e., iff $\int \phi(s)\,\mathrm{d}s \in \mathfrak{g}$ holds for each $\phi \in C^0([0,1],\mathfrak{g})$.*
- *If G is C^k-semiregular for $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip},\infty\}$, then $\mathfrak{g}$ is* k*-complete iff $\mathfrak{g}$ is Mackey-complete.*[3] ‡

3.2 Weak Continuity

A pair $(\phi,\psi) \in C^0([r,r'],\mathfrak{g}) \times C^0([r,r'],\mathfrak{g})$ is said to be

- **admissible** *iff* $\phi + (-\delta,\delta)\cdot\psi \subseteq \mathfrak{D}_{[r,r']}$ holds for some $\delta > 0$.

[3]Recall that $\mathfrak{g}$ is Mackey complete *iff* $\int \phi(s)\,\mathrm{d}s \in \mathfrak{g}$ holds for each $\phi \in C^k([0,1],\mathfrak{g})$, for any $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip},\infty\}$.

- **regulated** *iff* it is admissible with

$$\lim^\infty_{h\to 0} \textstyle\int_r^\bullet \phi + h\cdot\psi = \int_r^\bullet \phi.$$

Then,

Remark 5.

1) It follows from c) that (ϕ,χ) is admissible/regulated iff $(\phi|_{[\ell,\ell']},\chi|_{[\ell,\ell']})$ is admissible/regulated for each $r\leq \ell<\ell'\leq r'$.

2) Each $(0,\mathfrak{i}(X))$ with $X\in \mathrm{dom}[\exp]$ is regulated; because we have

$$\textstyle\int_0^t h\cdot\phi_X|_{[0,1]} \stackrel{(4)}{=} \int th\cdot\phi_X|_{[0,1]} \stackrel{(4)}{=} \int_0^{th}\phi_X|_{[0,1]}$$

for each $t\in[0,1]$, and each $h\in\mathbb{R}$. ‡

We say that G is **weakly k-continuous** for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty,\mathrm{c}\}$ *iff* each admissible $(\phi,\psi)\in C^k([0,1],\mathfrak{g})\times C^k([0,1],\mathfrak{g})$ is regulated.

Lemma 12. *If G is weakly* k*-continuous for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty,\mathrm{c}\}$, then each admissible $(\phi,\psi)\in C^k([r,r'],\mathfrak{g})\times C^k([r,r'],\mathfrak{g})$ (for each $[r,r']\in\mathfrak{K}$) is regulated.*

Proof. We define $\varrho\colon [0,1]\ni t\mapsto r+t\cdot|r'-r|\in[r,r']$; and observe that

$$\textstyle\int_r^\varrho \phi \stackrel{\mathrm{d)}}{=} \int_0^\bullet \dot\varrho\cdot\phi\circ\varrho,$$
$$\textstyle\int_r^\varrho [\phi+h\cdot\psi] \stackrel{\mathrm{d)}}{=} \int_0^\bullet [\dot\varrho\cdot\phi\circ\varrho + h\cdot\dot\varrho\cdot\psi\circ\varrho]$$

holds for $h>0$ suitably small. Since we have $\varrho\cdot\phi\circ\varrho,\ \dot\varrho\cdot\psi\circ\varrho\in C^k([0,1],\mathfrak{g})$ by Point B), the claim is clear from the presumptions. □

Lemma 13. *G is weakly* k*-continuous for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$ iff*

$$\lim^\infty_{h\to 0}\textstyle\int_0^\bullet h\cdot\chi = e \tag{20}$$

holds, for each $\chi\in\mathrm{D}_\mathrm{k}$ with $(-\delta,\delta)\cdot\chi\subseteq\mathrm{D}_\mathrm{k}$ for some $\delta>0$. The same statement also holds for $k\equiv\mathrm{c}$ if G is abelian.

Proof. The one direction is evident. For the other direction, we suppose that $(\phi,\psi)\in C^k([0,1],\mathfrak{g})\times C^k([0,1],\mathfrak{g})$ is admissible. Since $\phi\in\mathfrak{D}^k_{[0,1]}$ holds, we have

$$[\textstyle\int_0^t\phi]^{-1}[\int_0^t\phi+h\cdot\psi] \stackrel{\mathrm{b)}}{=} \int_0^t h\cdot\mathrm{Ad}_{[\int_0^\bullet\phi]^{-1}}(\psi) \qquad \forall\, t\in[0,1]$$

with $\chi:=\mathrm{Ad}_{[\int_0^\bullet\phi]^{-1}}(\psi)\in C^k([0,1],\mathfrak{g})$ by Point A). The claim is thus clear from (20). □

Corollary 1. *If G is abelian, then G is weakly* c*-continuous.*

Proof. This is clear from Lemma 13 and Remark 5.2). □

3.3 Mackey Continuity

We write $\{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi$ for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$, $\{\phi_n\}_{n\in\mathbb{N}} \subseteq C^k([r,r'],\mathfrak{g})$, and $\phi \in C^k([r,r'],\mathfrak{g})$ *iff*

$$\mathfrak{p}^{\mathrm{s}}_\infty(\phi - \phi_n) \leq \mathfrak{c}^{\mathrm{s}}_\mathfrak{p} \cdot \lambda_n \qquad \forall\, n \geq \mathfrak{l}_\mathfrak{p},\ \ \mathfrak{p} \in \mathfrak{P},\ \ \mathrm{s} \preceq k \tag{21}$$

holds, for certain $\{\mathfrak{c}^{\mathrm{s}}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\{\mathfrak{l}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{N}$, and $\mathbb{R}_{\geq 0} \supseteq \{\lambda_n\}_{n\in\mathbb{N}} \to 0$.

Remark 6. *Suppose that $\mathfrak{D}^k_{[r,r']} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi \in \mathfrak{D}^k_{[r,r']}$ holds for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$. Then,*

$$\{\phi_{\iota(n)}\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi$$

holds for each strictly increasing $\iota\colon \mathbb{N} \to \mathbb{N}$. ‡

We say that G is **Mackey k-continuous** *iff*

$$\mathrm{D}_k \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi \in \mathrm{D}_k \qquad \Longrightarrow \qquad \lim_n^\infty \textstyle\int_r^\bullet \phi_n = \int_r^\bullet \phi. \tag{22}$$

In analogy to Lemma 12, we obtain

Lemma 14. *G is Mackey* k*-continuous for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$ iff*

$$\mathfrak{D}^k_{[r,r']} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi \in \mathfrak{D}^k_{[r,r']} \qquad \Longrightarrow \qquad \lim_n^\infty \textstyle\int_r^\bullet \phi_n = \int_r^\bullet \phi, \tag{23}$$

for each $[r,r'] \in \mathfrak{K}$.

Proof. The one direction is evident. For the other direction, we suppose that (22) holds. Then, for $[r,r'] \in \mathfrak{K}$ fixed, we let $\varrho\colon [0,1] \ni t \mapsto r + t \cdot |r' - r| \in [r,r']$; and obtain

$$\begin{aligned}
\mathfrak{D}^k_{[r,r']} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi \in \mathfrak{D}^k_{[r,r']} \quad &\stackrel{\mathrm{B)}}{\Longrightarrow} \quad \mathrm{D_k} \supseteq \{\dot\varrho \cdot \phi_n \circ \varrho\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \dot\varrho \cdot \phi \circ \varrho \in \mathrm{D_k} \\
&\Longrightarrow \quad \lim_n^\infty \textstyle\int_r^\bullet \dot\varrho \cdot \phi_n \circ \varrho = \int_r^\bullet \dot\varrho \cdot \phi \circ \varrho \\
&\stackrel{\mathrm{d)}}{\Longrightarrow} \quad \lim_n^\infty \textstyle\int_r^\bullet \phi_n = \int_r^\bullet \phi,
\end{aligned}$$

whereby the second step is due to the presumptions. □

In analogy to Lemma 13, we obtain

Lemma 15. *G is Mackey* k*-continuous for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ iff*

$$\mathrm{D_k} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} 0 \qquad \Longrightarrow \qquad \lim_n^\infty \textstyle\int_0^\bullet \phi_n = e. \tag{24}$$

The statement also holds for $k \equiv \mathrm{c}$ if G is abelian.

Proof. The one direction is evident. For the other direction, we suppose that $\mathrm{D_k} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi \in \mathrm{D_k}$ holds; and observe that

$$[\textstyle\int_0^t \phi]^{-1}[\int_0^t \phi_n] \stackrel{\mathrm{b)}}{=} \int_0^t \underbrace{\mathrm{Ad}_{[\int_0^\bullet \phi]^{-1}}(\phi_n - \phi)}_{\psi_n \in \mathrm{D_k}} \qquad \forall\, n \in \mathbb{N},\ \ t \in [0,1]$$

holds by Point A). Then, by Lemma 7 and Lemma 8, we have $\mathrm{D_k} \supseteq \{\psi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} 0$; from which the claim is clear. □

We furthermore observe that

Lemma 16. *If G is Mackey* k*-continuous for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$, then G is weakly* k*-continuous.*

Proof. If G is not weakly k-continuous, then there exists an admissible $(\phi,\psi)\in C^k([r,r'],\mathfrak{g})\times C^k([r,r'],\mathfrak{g})$, an open neighbourhood $U\subseteq G$ of e, as well as sequences $\{\tau_n\}_{n\in\mathbb{N}}\subseteq[r,r']$ and $\mathbb{R}_{\neq 0}\supseteq\{h_n\}\rightarrow 0$, such that

$$[\textstyle\int_r^{\tau_n}\phi]^{-1}[\textstyle\int_r^{\tau_n}\phi+h_n\cdot\psi]\notin U\qquad\forall\, n\in\mathbb{N}$$

holds. Then, G cannot be Mackey k-continuous, because we have $\{\phi+h_n\cdot\psi\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{m}.\mathrm{k}}\phi$. □

Remarkably, we also have

Lemma 17. *G is Mackey* k*-continuous for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$ iff*

$$\mathrm{D}_{\mathrm{k}}\supseteq\{\phi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{m}.\mathrm{k}}0\qquad\Longrightarrow\qquad\lim_n\textstyle\int\phi_n=e.\tag{25}$$

The statement also holds for $k\equiv\mathrm{c}$ if G is abelian.

Proof. The one direction is evident. For the other direction, we suppose that (25) holds; and that G is not Mackey k-continuous. By Lemma 15, there exist $\mathrm{D}_{\mathrm{k}}\supseteq\{\phi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{m}.\mathrm{k}}0$, $U\subseteq G$ open with $e\in G$, a sequence $\{\tau_n\}_{n\in\mathbb{N}}\subseteq[0,1]$, and $\iota\colon\mathbb{N}\rightarrow\mathbb{N}$ strictly increasing, such that

$$\textstyle\int_0^{\tau_n}\phi_{\iota_n}\notin U\qquad\forall\, n\in\mathbb{N}\tag{26}$$

holds. For each $n\in\mathbb{N}$, we define

$$\mathrm{D}_{\mathrm{k}}\ni\chi_n:=\dot{\varrho}_n\cdot\phi_{\iota_n}\circ\varrho_n\qquad\text{with}\qquad\varrho_n\colon[0,1]\ni t\mapsto t\cdot\tau_n\in[0,\tau_n];$$

and conclude from Remark 6 and Point B) that $\{\chi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{m}.\mathrm{k}}0$ holds. Then, (25) implies

$$\lim_n\textstyle\int_0^{\tau_n}\phi_{\iota_n}\overset{\mathrm{d})}{=}\lim_n\textstyle\int\chi_n=e,$$

which contradicts (26). □

3.4 Sequentially Continuity

We write $\{\phi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{s}.\mathrm{k}}\phi$ for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty,\mathrm{c}\}$, $\{\phi_n\}_{n\in\mathbb{N}}\subseteq C^k([r,r'],\mathfrak{g})$, and $\phi\in C^k([r,r'],\mathfrak{g})$ *iff*

$$\lim_n\,\boldsymbol{.}\mathfrak{p}^{\mathrm{s}}_\infty(\phi-\phi_n)=0\qquad\forall\,\mathfrak{p}\in\mathfrak{P},\ \ \mathrm{s}\preceq k\tag{27}$$

holds. We say that G is **sequentially k-continuous** *iff*

$$\mathrm{D}_{\mathrm{k}}\supseteq\{\phi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{s}.\mathrm{k}}\phi\in\mathrm{D}_{\mathrm{k}}\qquad\Longrightarrow\qquad\lim_n^\infty\textstyle\int_r^\bullet\phi_n=\textstyle\int_r^\bullet\phi.$$

Remark 7.

1) *Suppose that G is sequentially* k*-continuous for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty,\mathrm{c}\}$. Evidently, then G is Mackey* k*-continuous; thus, weakly* k*-continuous by Lemma 16.*
2) *G is sequentially* k*-continuous for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty,\mathrm{c}\}$ iff*

$$\mathfrak{D}^k_{[r,r']}\supseteq\{\phi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{s}.\mathrm{k}}\phi\in\mathfrak{D}^k_{[r,r']}\qquad\Longrightarrow\qquad\lim_n^\infty\textstyle\int_r^\bullet\phi_n=\textstyle\int_r^\bullet\phi$$

   holds for each $[r,r']\in\mathfrak{K}$. This just follows as in Lemma 14.
3) *Let $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty,\mathrm{c}\}$, with G abelian for $k\equiv\mathrm{c}$. Then, the same arguments as in Lemma 15 show that G is sequentially* k*-continuous iff*

$$\mathrm{D}_{\mathrm{k}}\supseteq\{\phi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{s}.\mathrm{k}}0\qquad\Longrightarrow\qquad\lim_n\textstyle\int_0^\bullet\phi_n=e.$$

4) Let $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$, with G abelian for $k \equiv \mathrm{c}$. Then, the same arguments as in Lemma 17 show that G is sequentially k*-continuous iff*

$$\mathrm{D_k} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} 0 \qquad\qquad \Longrightarrow \qquad\qquad \lim_n^\infty \textstyle\int \phi_n = e.$$

5) If G is C^k-continuous for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$, then G is sequentially k*-continuous – This is clear*

- *for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ from 4),*
- *for $k \equiv \mathrm{c}$ from Lemma 11.* ‡

3.5 Piecewise Integrable Curves

We now finally discuss piecewise integrable curves. Specifically, we provide the basic facts and definitions[4]; and furthermore show that Sequentially 0-continuity and Mackey 0-continuity carry over to the piecewise integrable category. This will be used in Sect. 5 to generalize Theorem 1 in [7].

3.5.1 Basic Facts and Definitions

For $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$ and $[r, r'] \in \mathfrak{K}$, we let $\mathfrak{DP}^k([r, r'], \mathfrak{g})$ denote the set of all $\psi\colon [r, r'] \to \mathfrak{g}$, such that there exist $r = t_0 < \ldots < t_n = r'$ and

$$\psi[p] \in \mathfrak{D}^k_{[t_p, t_{p+1}]} \qquad \text{with} \qquad \psi|_{(t_p, t_{p+1})} = \psi[p]|_{(t_p, t_{p+1})} \qquad \text{for} \qquad p = 0, \ldots, n-1. \tag{28}$$

In this situation, we define $\int_r^r \psi := e$, as well as

$$\textstyle\int_r^t \psi := \int_{t_p}^t \psi[p] \cdot \int_{t_{p-1}}^{t_p} \psi[p-1] \cdot \ldots \cdot \int_{t_0}^{t_1} \psi[0] \qquad \forall\, t \in (t_p, t_{p+1}]. \tag{29}$$

A standard refinement argument in combination with c) then shows that this is well defined; i.e., independent of any choices we have made. It is furthermore not hard to see that for $\phi, \psi \in \mathfrak{DP}^k([r, r'], \mathfrak{g})$, we have $\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi) \in \mathfrak{DP}^k([r, r'], \mathfrak{g})$ with

$$\big[\textstyle\int_r^t \phi\big]^{-1} \big[\int_r^t \psi\big] = \int_r^t \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi) \qquad \forall\, t \in [r, r']. \tag{30}$$

We write

- $\{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{s}} \phi$ for $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{DP}^0([r, r'], \mathfrak{g})$ and $\phi \in \mathfrak{DP}^0([r, r'], \mathfrak{g})$ *iff*

  $$\lim_n \mathfrak{p}_\infty(\phi - \phi_n) = 0 \qquad \forall\, \mathfrak{p} \in \mathfrak{P}$$

  holds.
- $\{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}} \phi$ for $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{DP}^0([r, r'], \mathfrak{g})$ and $\phi \in \mathfrak{DP}^0([r, r'], \mathfrak{g})$ *iff*

  $$\mathfrak{p}_\infty(\phi - \phi_n) \leq \mathfrak{c}_{\mathfrak{p}} \cdot \lambda_n \qquad \forall\, n \geq \mathfrak{l}_{\mathfrak{p}},\ \ \mathfrak{p} \in \mathfrak{P}$$

  holds, for certain $\{\mathfrak{c}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\{\mathfrak{l}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{N}$, and $\mathbb{R}_{\geq 0} \supseteq \{\lambda_n\}_{n\in\mathbb{N}} \to 0$.

[4]Confer Sect. 4.3 in [6] for the statements mentioned but not proven here.

3.5.2 A Continuity Statement

We recall the construction made in Sect. 4.3 in [6].

i) We fix (a bump function) $\boldsymbol{\rho}\colon [0,1] \to [0,2]$ smooth with

$$\boldsymbol{\rho}|_{(0,1)} > 0, \ \int_0^1 \boldsymbol{\rho}(s)\, \mathrm{d}s = 1 \qquad \text{as well as} \qquad \boldsymbol{\rho}^{(k)}(0) = 0 = \boldsymbol{\rho}^{(k)}(1) \qquad \forall\, k \in \mathbb{N}. \tag{31}$$

Then, $[r,r'] \in \mathfrak{K}$, and $r = t_0 < \ldots < t_n = r'$ given, we let

$$\rho_p\colon [t_p, t_{p+1}] \to [0,2], \qquad t \mapsto \boldsymbol{\rho}(|t-t_p|/|t_{p+1}-t_p|) \qquad \forall\, p = 0,\ldots,n-1;$$

and define $\rho\colon [r,r'] \to [0,2]$ by

$$\rho|_{[t_p,t_{p+1}]} := \rho_p \qquad \forall\, p = 0,\ldots,n-1.$$

Then, ρ is smooth, with $\rho^{(k)}(t_p) = 0$ for each $k \in \mathbb{N}$, $p = 0,\ldots,n$; and (10) shows that

$$\varrho\colon [r,r'] \to [r,r'], \qquad t \mapsto r + \int_r^t \rho(s)\, \mathrm{d}s$$

holds, with $\varrho(t_p) = t_p$ for $p = 0,\ldots,n-1$.

ii) For $\psi \in \mathfrak{DP}^0([r,r'],\mathfrak{g})$ with $r = t_0 < \ldots < t_n = r'$ as well as $\psi[0],\ldots,\psi[n-1]$ as in (28), we let $\varrho\colon [r,r'] \to [r,r']$ and $\rho \equiv \dot{\varrho}\colon [r,r'] \to [0,2]$ be as in the previous point. Then, it is straightforward from the definitions that

$$\chi := \rho \cdot \psi \circ \varrho \in \mathfrak{D}^0_{[r,r']} \qquad \text{holds, with} \qquad \int_r^\varrho \psi = \int_r^\bullet \chi \qquad \text{and} \qquad \boldsymbol{.}\mathfrak{p}_\infty(\chi) \leq 2 \cdot \boldsymbol{.}\mathfrak{p}_\infty(\psi)$$

for each $\mathfrak{p} \in \mathfrak{P}$.[5]

We obtain

Lemma 18.

1) If G is sequentially 0-continuous, then

$$\mathfrak{DP}^0([r,r'],\mathfrak{g}) \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{s}} \phi \in \mathfrak{DP}^0([r,r'],\mathfrak{g}) \qquad \Longrightarrow \qquad \lim_n^\infty \int_r^\bullet \phi_n = \int_r^\bullet \phi.$$

2) If G is Mackey 0-continuous, then

$$\mathfrak{DP}^0([r,r'],\mathfrak{g}) \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}} \phi \in \mathfrak{DP}^0([r,r'],\mathfrak{g}) \qquad \Longrightarrow \qquad \lim_n^\infty \int_r^\bullet \phi_n = \int_r^\bullet \phi.$$

Specifically, in both situations, for each $\mathfrak{p} \in \mathfrak{P}$, there exists some $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ and $n_\mathfrak{p} \in \mathbb{N}$ with

$$(\mathfrak{p} \circ \Xi)([\textstyle\int_r^\bullet \phi]^{-1}[\int_r^\bullet \phi_n]) \leq \int \boldsymbol{.}\mathfrak{q}(\phi_n(s) - \phi(s))\, \mathrm{d}s \qquad \forall\, n \geq n_\mathfrak{p}.$$

Proof. Let $\phi \in \mathfrak{DP}^0([r,r'],\mathfrak{g})$ and $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{DP}^0([r,r'],\mathfrak{g})$ be given. For $\mathfrak{p} \equiv \mathfrak{u} \in \mathfrak{P}$ fixed, we choose $U \subseteq G$ and $\mathfrak{u} \leq \mathfrak{m} \in \mathfrak{P}$ as in Lemma 5. We furthermore let $\mathfrak{m} \leq \mathfrak{q} \equiv \mathfrak{w} \in \mathfrak{P}$ be as in I), for $\mathrm{C} \equiv \mathrm{im}[[\int_r^\bullet \phi]^{-1}]$ and $\mathfrak{v} \equiv \mathfrak{m}$ there. Then, for each $n \in \mathbb{N}$, we let $\varrho_n \equiv \varrho$, $\rho_n \equiv \rho$, and $\chi_n \equiv \chi$ be as in ii), for

$$\psi \equiv \psi_n := \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\phi_n - \phi) \in \mathfrak{DP}^0([r,r'],\mathfrak{g})$$

there. Then,

- we have

$$[\textstyle\int_r^{\varrho_n(t)} \phi]^{-1}[\int_r^{\varrho_n(t)} \phi_n] \overset{(30)}{=} \int_r^{\varrho_n(t)} \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\phi_n - \phi) \overset{\text{ii)}}{=} \int_r^t \chi_n \qquad \forall\, n \in \mathbb{N},\ t \in [r,r']. \tag{32}$$

[5] In the proof of Lemma 24 in [6], this statement was more generally verified for the case that $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ holds.

- we have $\mathbin{\text{.}}\mathfrak{m}_\infty(\chi_n) \leq 2\cdot \mathbin{\text{.}}\mathfrak{m}_\infty(\psi_n) \leq 2 \cdot \mathbin{\text{.}}\mathfrak{q}_\infty(\phi_n - \phi)$ for each $n\in\mathbb{N}$ by Lemma 7, which shows that
  - $\mathfrak{D}^0_{[r,r']} \supseteq \{\chi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{s}.0} 0$ holds if we are in the situation of *1)*,
  - $\mathfrak{D}^0_{[r,r']} \supseteq \{\chi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.0} 0$ holds if we are in the situation of *2)*.

  In both situations, there thus exists some $n_\mathfrak{p}\in\mathbb{N}$ with $\int_r^\bullet \chi_n \in U$ for each $n \geq n_\mathfrak{p}$.

We obtain from Lemma 5 (second step), and I) (last step) that[6]

$$\begin{aligned}(\mathfrak{p}\circ\Xi)([\textstyle\int_r^{\varrho_n(t)}\phi]^{-1}[\int_r^{\varrho_n(t)}\phi_n]) &\overset{(32)}{=} (\mathfrak{p}\circ\Xi)(\textstyle\int_r^t \chi_n)\\ &\leq \textstyle\int_r^t (\mathbin{\text{.}}\mathfrak{m}\circ\chi_n)(s)\,\mathrm{d}s\\ &\overset{(10)}{=} \textstyle\int_r^{\varrho_n(t)} (\mathbin{\text{.}}\mathfrak{m}\circ\psi_n)(s)\,\mathrm{d}s\\ &\leq \textstyle\int_r^{\varrho_n(t)} \mathbin{\text{.}}\mathfrak{q}(\phi_n(s)-\phi(s))\,\mathrm{d}s\end{aligned}$$

holds for all $n\geq n_\mathfrak{p}$ and $t\in[r,r']$; which proves the claim. □

Remark 8. *Besides the application in Sect. 5, it is likely that Lemma 18 also serves to generalize the integrability results obtained in Theorem 3 in [6] in the locally μ-convex setting, to the sequentially* 0*-continuous-, as well as the Mackey* 0*-continuous context. Since Theorem 1 shows that G is Mackey* 0*-continuous if G is C^0-semiregular - this would substitute, e.g., "locally μ-convexity" in Theorem 3.2) in [6] by a necessary continuity condition – namely, Mackey* 0*-continuity. A detailed analysis of this issue, however, would go beyond the scope of this article – so that we will present it in a separate paper.* ‡

4 Mackey Continuity

In this section, we show that

Theorem 1. *If G is C^k-semiregular for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$, then G is Mackey* k*-continuous.*

The proof of Theorem 1 is based on a bump function argument similar to that one used in the proof of Theorem 2 in [6]. It furthermore makes use of the fact that $[0,1]\ni t\mapsto \int_0^\bullet \in G$ is continuous for each $\phi\in \mathrm{D}_\mathrm{k}$. However, before we can provide the proof, we need some technical preparation first.

4.1 Some Estimates

Let $\boldsymbol{\rho}\colon [0,1]\to[0,2]$ be smooth with

$$\boldsymbol{\rho}|_{(0,1)} > 0,\ \textstyle\int_0^1 \boldsymbol{\rho}(s)\,\mathrm{d}s = 1 \qquad\text{and}\qquad \boldsymbol{\rho}^{(k)}(0) = 0 = \boldsymbol{\rho}^{(k)}(1) \qquad \forall\, k\in\mathbb{N}.$$

Now, suppose that we are given $\varrho\colon [r,r']\to[r,r']$; and define $\rho\equiv\dot{\varrho}$ as well as

$$C[\rho,\mathrm{s}] := \max\big(1, \max_{0\leq m,n\leq \mathrm{s}}(\sup\{|\rho^{(m)}(t)|^{n+1}\mid t\in[r,r']\})\big) \qquad \forall\,\mathrm{s}\in\mathbb{N}.$$

We observe the following:

- Let $\psi\in C^k([r,r'],\mathfrak{g})$ for $k\in\mathbb{N}\sqcup\{\infty\}$ and $\mathrm{s}\preceq k$ be given. By **c)**, **d)** in Appendix A, we have

$$(\rho\cdot\psi\circ\varrho)^{(\mathrm{s})} = \textstyle\sum_{q,m,n=0}^{\mathrm{s}} h_\mathrm{s}(q,m,n)\cdot\big(\rho^{(m)}\big)^{n+1}\cdot(\psi^{(q)}\circ\varrho),$$

  for a map $h_\mathrm{s}\colon (0,\dots,\mathrm{s})^3\to\{0,1\}$ that is independent of ϱ,ρ,ϕ. We obtain

$$\mathbin{\text{.}}\mathfrak{p}\big((\rho\cdot\psi\circ\varrho)^{(\mathrm{s})}\big) \leq (\mathrm{s}+1)^3\cdot C[\rho,\mathrm{s}]\cdot \mathbin{\text{.}}\mathfrak{p}^{\mathrm{s}}_\infty(\psi) \tag{33}$$

  for each $\mathfrak{p}\in\mathfrak{P}$, $0\leq\mathrm{s}\preceq k$, and $\psi\in C^k([r,r'],\mathfrak{g})$.

[6]For the third step observe that $\rho_n\geq 0$ holds for each $n\in\mathbb{N}$.

- Let $\psi \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$ be given. Then we have

$$
\begin{aligned}
&\mathfrak{p}((\rho\cdot\psi\circ\varrho)(t)-(\rho\cdot\psi\circ\varrho)(t'))\\
&\qquad\qquad \leq |\rho(t)-\rho(t')|\cdot\mathfrak{p}(\psi(\varrho(t))) \;+\; |\rho(t')|\cdot\mathfrak{p}(\psi(\varrho(t))-\psi(\varrho(t')))\\
&\qquad\qquad \leq |t-t'|\cdot C[\rho,1]\cdot\mathfrak{p}_\infty(\psi) \;+\; C[\rho,0]\cdot\mathrm{Lip}(\mathfrak{p},\psi)\cdot\underbrace{|\varrho(t)-\varrho(t')|}_{\leq C[\rho,0]\cdot|t-t'|}\\
&\qquad\qquad \leq 2\cdot C[\rho,1]^2\cdot\mathfrak{p}^{\mathrm{lip}}_\infty(\psi)\cdot|t-t'|
\end{aligned}
$$

  for each $t,t'\in[r,r']$; thus,

$$\mathrm{Lip}(\mathfrak{p},\rho\cdot\psi\circ\varrho)\leq 2\cdot C[\rho,1]^2\cdot\mathfrak{p}^{\mathrm{lip}}_\infty(\psi). \tag{34}$$

Let now $\{\phi_n\}_{n\in\mathbb{N}}\subseteq\mathfrak{D}^k_{[0,1]}$ with $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$ be given; as well as $\{t_n\}_{n\in\mathbb{N}}\subseteq[0,1]$ strictly decreasing with $t_0=1$. For each $n\in\mathbb{N}$, we define $0<\delta_n:=t_n-t_{n+1}<1$, as well as

$$\rho_n:=\delta_n^{-1}\cdot\boldsymbol{\rho}\circ\kappa_n \qquad\text{for}\qquad \kappa_n\colon[t_{n+1},t_n]\ni t\mapsto\delta_n^{-1}\cdot|t-t_{n+1}|\in[0,1].$$

We obtain from (10) that

$$\varrho_n\colon[t_{n+1},t_n]\ni t\mapsto\int_{t_{n+1}}^t\rho_n(s)\,\mathrm{d}s\in[0,1]\qquad\forall\,n\in\mathbb{N}$$

holds; and furthermore observe that

$$C[\rho_n,\mathrm{s}]\leq\delta_n^{-(\mathrm{s}+1)^2}\cdot C[\boldsymbol{\rho},\mathrm{s}]\qquad\forall\,n\in\mathbb{N}. \tag{35}$$

Then,

- We obtain from (33) and (35) that

$$
\begin{aligned}
\mathfrak{p}\big((\rho_n\cdot\phi_n\circ\varrho_n)^{(\mathrm{s})}\big) &\overset{(33)}{\leq} (\mathrm{s}+1)^3\cdot C[\rho_n,\mathrm{s}]\cdot\mathfrak{p}^{\mathrm{s}}_\infty(\phi_n)\\
&\overset{(35)}{\leq} (\mathrm{s}+1)^3\cdot\delta_n^{-(\mathrm{s}+1)^2}\cdot C[\boldsymbol{\rho},\mathrm{s}]\cdot\mathfrak{p}^{\mathrm{s}}_\infty(\phi_n)
\end{aligned}
\tag{36}
$$

  holds, for each $\mathfrak{p}\in\mathfrak{P}$, $\mathrm{s}\preceq k$, and $n\in\mathbb{N}$.
- We obtain from (34) and (35) that

$$\mathrm{Lip}(\mathfrak{p},\rho_n\cdot\phi_n\circ\varrho_n)\overset{(34)}{\leq}2\cdot C[\rho_n,1]^2\cdot\mathfrak{p}^{\mathrm{lip}}_\infty(\phi_n)\overset{(35)}{\leq}2\cdot\delta_n^{-8}\cdot C[\boldsymbol{\rho},1]^2\cdot\mathfrak{p}^{\mathrm{lip}}_\infty(\phi_n) \tag{37}$$

  holds, for each $\mathfrak{p}\in\mathfrak{P}$ and $n\in\mathbb{N}$.

We define $\phi\colon[0,1]\to\mathfrak{g}$, by $\phi(0):=0$ and

$$\phi|_{[t_{n+1},t_n]}:=\rho_n\cdot\phi_n\circ\varrho_n\qquad\forall\,n\in\mathbb{N}.$$

Then, it is straightforward to see that[7] $\phi|_{[t_{n+1},1]}\in\mathfrak{D}^k_{[t_{n+1},1]}$ holds for each $n\in\mathbb{N}$, with

$$\int_{t_{n+1}}^{\varrho_n}\phi_n\overset{\mathrm{d)}}{=}\int_{t_{n+1}}^{\bullet}\phi|_{[t_{n+1},t_n]}\qquad\forall\,n\in\mathbb{N}. \tag{38}$$

Moreover, for $k\equiv\mathrm{lip}$, we obtain from (37) that

$$\mathrm{Lip}(\mathfrak{p},\phi|_{[t_{n+1},1]})\leq 2\cdot C[\boldsymbol{\rho},1]^2\cdot\max\big(\delta_0^{-8}\cdot\mathfrak{p}^{\mathrm{lip}}_\infty(\phi_0),\dots,\delta_n^{-8}\cdot\mathfrak{p}^{\mathrm{lip}}_\infty(\phi_n)\big) \tag{39}$$

holds, cf. Appendix C.

[7] The technical details can be found, e.g., in the proof of Lemma 24 in [6].

4.2 Proof of Theorem 1

We are ready for the

Proof of Theorem 1. Suppose that the claim is wrong, i.e., that G is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ but not Mackey k-continuous. Then, by Lemma 15, there exists a sequence $\mathrm{D_k} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathrm{m.k}} 0$ (with $\{\mathfrak{c}^{\mathrm{s}}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\{\mathfrak{l}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{N}$, and $\mathbb{R}_{\geq 0} \supseteq \{\lambda_n\}_{n\in\mathbb{N}} \to 0$ as in (21) for $\phi \equiv 0$ there), as well as $U \subseteq G$ open with $e \in U$, such that $\mathrm{im}[\int_0^\bullet \phi_n] \not\subseteq U$ holds for infinitely many $n \in \mathbb{N}$. Passing to a subsequence if necessary, we thus can assume that

$$\mathrm{im}[\textstyle\int_0^\bullet \phi_n] \not\subseteq U \qquad \text{and} \qquad \lambda_n \leq 2^{-(n+1)^2} \qquad \forall\, n \in \mathbb{N} \tag{40}$$

holds. We let $t_0 := 1$, and $t_n := 1 - \sum_{k=1}^n 2^{-k}$ for each $n \in \mathbb{N}_{\geq 1}$; so that $\delta_n = t_n - t_{n+1} = 2^{-(n+1)}$ holds for each $n \in \mathbb{N}$. We construct $\phi\colon [0,1] \to \mathfrak{g}$ as described above; and fix $V \subseteq U$ open with $e \in V$ and $V \cdot V^{-1} \subseteq U$.

Suppose now that we have shown that ϕ is of class C^k; i.e., that $\phi \in \mathrm{D_k}$ holds as G is C^k-semiregular. Since $[0,1] \ni t \mapsto \int_0^t \phi \in G$ is continuous, there exists some $\ell \geq 1$ with $\int_0^t \phi \in V$ for each $t \in [0, t_\ell]$; thus,

$$\textstyle\int_{t_{n+1}}^{\varrho_n(t)} \phi_n \overset{(38)}{=} \int_{t_{n+1}}^{t} \phi|_{[t_{n+1},t_n]} \overset{\mathrm{c)}}{=} [\int_0^t \phi] \cdot [\int_0^{t_{n+1}} \phi]^{-1} \in V \cdot V^{-1} \subseteq U$$

for each $t \in [t_{n+1}, t_n]$ with $n \geq \ell$, which contradicts (40).

To prove the claim, it thus suffices to show that ϕ is of class C^k:

- Suppose first that $k \in \mathbb{N} \sqcup \{\infty\}$ holds. Then, it suffices to show that

  $$\lim_{(0,1]\ni h\to 0} 1/h \cdot \phi^{(\mathrm{s})}(h_p) = 0 \qquad \forall\, \mathrm{s} \preceq k$$

  holds, because ϕ is of class C^k on $(0,1]$.

  For this, we let $h \in [t_{n+1}, t_n]$ for $n \in \mathbb{N}$ be given; and observe that

  $$h \geq t_{n+1} = t_n - t_{n+1} + t_{n+1} = 2^{-(n+1)} + 1 - \textstyle\sum_{k=1}^{n+1} 2^{-k} \geq 2^{-(n+1)}$$

  holds. For $\mathfrak{p} \in \mathfrak{P}$ fixed and $n \geq \mathfrak{l}_{\mathfrak{p}}$, we obtain from (36) (with $\delta_n = 2^{-(n+1)}$) as well as (40) that

  $$\begin{aligned} 1/h \cdot \mathfrak{p}\big(\phi^{(\mathrm{s})}(h)\big) \;&\leq\; 2^{n+1} \cdot \mathfrak{p}\big((\rho_n \cdot \phi_n \circ \varrho_n)^{(\mathrm{s})}\big) \\ &\overset{(36)}{\leq} (\mathrm{s}+1)^3 \cdot 2^{((\mathrm{s}+1)^2+1)\cdot(n+1)} \cdot C[\boldsymbol{\rho}, \mathrm{s}] \cdot \mathfrak{p}^{\mathrm{s}}_\infty(\phi_n) \\ &\overset{(40)}{\leq} (\mathrm{s}+1)^3 \cdot C[\boldsymbol{\rho}, \mathrm{s}] \cdot \mathfrak{c}^{\mathrm{s}}_{\mathfrak{p}} \cdot 2^{((\mathrm{s}+1)^2+1)\cdot(n+1)-(n+1)^2} \\ &= (\mathrm{s}+1)^3 \cdot C[\boldsymbol{\rho}, \mathrm{s}] \cdot \mathfrak{c}^{\mathrm{s}}_{\mathfrak{p}} \cdot 2^{(((\mathrm{s}+1)^2+1)-(n+1))\cdot(n+1)} \end{aligned}$$

  holds; which clearly tends to zero for $n \to \infty$.
- Suppose now that $k \equiv \mathrm{lip}$ holds. The previous point then shows $\phi \in C^0([0,1], \mathfrak{g})$. For $\mathfrak{p} \in \mathfrak{P}$ fixed, we thus have $\mathfrak{p}_\infty(\phi) < \infty$. We define

  $$D_{\mathfrak{p}} := \max\big(2 \cdot \delta_0^{-8} \cdot \mathfrak{p}^{\mathrm{lip}}_\infty(\phi_0), \dots, 2 \cdot \delta_{\mathfrak{l}_{\mathfrak{p}}}^{-8} \cdot \mathfrak{p}^{\mathrm{lip}}_\infty(\phi_{\mathfrak{l}_{\mathfrak{p}}})\big),$$

and obtain for $n \geq \mathfrak{l}_\mathfrak{p}$ that

$$
\begin{aligned}
\mathrm{Lip}(\raisebox{0.5ex}{.}\mathfrak{p}, \phi|_{[t_{n+1},1]}) &\overset{(39)}{\leq} C[\boldsymbol{\rho}, 1]^2 \cdot \max\big(2 \cdot \delta_0^{-8} \cdot \raisebox{0.5ex}{.}\mathfrak{p}^{\mathrm{lip}}_\infty(\phi_0), \dots, 2 \cdot \delta_n^{-8} \cdot \raisebox{0.5ex}{.}\mathfrak{p}^{\mathrm{lip}}_\infty(\phi_n)\big) \\
&\overset{(21)}{\leq} C[\boldsymbol{\rho}, 1]^2 \cdot \max\big(D_\mathfrak{p}, \mathfrak{c}^{\mathrm{lip}}_\mathfrak{p} \cdot \max\big(2^{1+8(\mathfrak{l}_\mathfrak{p}+2)} \cdot \lambda_{\mathfrak{l}_\mathfrak{p}+1}, \dots, 2^{1+8(n+1)} \cdot \lambda_n\big)\big) \\
&\overset{(40)}{\leq} C[\boldsymbol{\rho}, 1]^2 \cdot \max\big(D_\mathfrak{p}, \mathfrak{c}^{\mathrm{lip}}_\mathfrak{p} \cdot \max\big\{2^{1+8(\ell+1)} \cdot 2^{-(\ell+1)^2} \,\big|\, \mathfrak{l}_\mathfrak{p} + 1 \leq \ell \leq n\big\}\big) \\
&= C[\boldsymbol{\rho}, 1]^2 \cdot \max\big(D_\mathfrak{p}, \mathfrak{c}^{\mathrm{lip}}_\mathfrak{p} \cdot \max\big\{2^{-\ell^2+6\ell+8} \,\big|\, \mathfrak{l}_\mathfrak{p} + 1 \leq \ell \leq n\big\}\big) \\
&= C[\boldsymbol{\rho}, 1]^2 \cdot \max\big(D_\mathfrak{p}, \mathfrak{c}^{\mathrm{lip}}_\mathfrak{p} \cdot \max\big\{2^{17-(\ell-3)^2} \,\big|\, \mathfrak{l}_\mathfrak{p} + 1 \leq \ell \leq n\big\}\big) \\
&\leq C[\boldsymbol{\rho}, 1]^2 \cdot \max\big(D_\mathfrak{p}, \mathfrak{c}^{\mathrm{lip}}_\mathfrak{p} \cdot 2^{17}\big) =: \mathfrak{L}
\end{aligned}
$$

holds; thus,

$$\raisebox{0.5ex}{.}\mathfrak{p}(\phi(t) - \phi(t')) \leq \mathfrak{L} \cdot |t - t'| \qquad \forall\, t, t' \in (0, 1].$$

Since ϕ is continuous with $\phi(0) = 0$, for each $t \in [0, 1]$, we also have

$$
\begin{aligned}
\raisebox{0.5ex}{.}\mathfrak{p}(\phi(0) - \phi(t)) &= \lim_{\ell\to\infty} \raisebox{0.5ex}{.}\mathfrak{p}(\phi(0) - \phi(1/\ell) + \phi(1/\ell) - \phi(t)) \\
&\leq \lim_{\ell\to\infty} \raisebox{0.5ex}{.}\mathfrak{p}(\phi(1/\ell)) + \lim_{\ell\to\infty} \raisebox{0.5ex}{.}\mathfrak{p}(\phi(1/\ell) - \phi(t)) \\
&= \lim_{\ell\to\infty} \raisebox{0.5ex}{.}\mathfrak{p}(\phi(1/\ell) - \phi(t)) \\
&\leq \lim_{\ell\to\infty} \mathfrak{L} \cdot |1/\ell - t| \\
&= \mathfrak{L} \cdot |0 - t|.
\end{aligned}
$$

This shows $\mathrm{Lip}(\raisebox{0.5ex}{.}\mathfrak{p}, \phi) \leq \mathfrak{L}$, i.e., $\phi \in C^{\mathrm{lip}}([0, 1], \mathfrak{g})$. □

5 The Strong Trotter Property

In this Section, we want to give a brief application of the notions introduced so far. For this, we recall that a Lie group G is said to have the strong Trotter property *iff* for each $\mu \in C^1_*([0, 1], G)$ with $\dot{\mu}(0) \in \mathrm{dom}[\exp]$, we have

$$\lim_n \mu(\tau/n)^n = \exp(\tau \cdot \dot{\mu}(0)) \qquad \forall\, \tau \in [0, \ell] \tag{41}$$

uniformly[8] for each $\ell > 0$. We now will show that

Proposition 1. *1) If G is sequentially* 0*-continuous, then G has the strong Trotter property.*

2) If G is Mackey 0*-continuous, then* (41) *holds for each $\mu \in C^1_*([0, 1], G)$ with $\dot{\mu}(0) \in \mathrm{dom}[\exp]$ and $\delta^r(\mu) \in C^{\mathrm{lip}}([0, 1], \mathfrak{g})$.*

Here,

- By Remark 7.5) (and Theorem 1 in [6]), Proposition 1.1) generalizes Theorem 1 in [7].
- By Theorem 1, the presumptions made in Proposition 1.2) are always fulfilled, e.g., if G is C^0-semiregular, and μ is of class C^2 with $\dot{\mu}(0) \in \mathrm{dom}[\exp]$.

[8] Thus, for each neighbourhood $U \subseteq G$ of e, there exists some $n_U \in \mathbb{N}$ with $\exp(-\tau \cdot \dot{\mu}(0)) \cdot \mu(\tau/n)^n \in U$ for each $n \geq n_U$ and $\tau \in [0, \ell]$.

We will need the following observations: Let $\ell > 0$, $\mu \in C^1_*([0,1],\mathcal{U})$ be given; and define $\phi \equiv \delta^r(\mu)$, $X := \dot{\mu}(0) = \phi(0)$, as well as

$$\mu_\tau\colon [0,1/\ell] \ni t \mapsto \mu(\tau\cdot t) \in G \qquad \forall\, \tau \in [0,\ell].$$

Then, for each $t \in [0,1/\ell]$, we have

$$\begin{aligned}\delta^r(\mu_\tau)(t) - \tau\cdot X &\overset{(14)}{=} \omega((\Xi\circ\mu_\tau)(t), \partial_t(\Xi\circ\mu_\tau)(t)) - \tau\cdot X\\ &= \tau\cdot\omega((\Xi\circ\mu)(\tau\cdot t), \partial_t(\Xi\circ\mu)(\tau\cdot t)) - \tau\cdot X\\ &= \tau\cdot\delta^r(\mu)(\tau\cdot t) - \tau\cdot X\\ &= \tau\cdot(\phi(\tau\cdot t) - X).\end{aligned}$$

For each $\mathfrak{p} \in \mathfrak{P}$, $\tau \in [0,\ell]$, and $s \leq 1/\ell$, we thus obtain

$$\mathbin{.}\mathfrak{p}_\infty(\delta^r(\mu_\tau)|_{[0,s]} - \tau\cdot X) = \ell\cdot\sup\{\mathbin{.}\mathfrak{p}(\phi(\tau\cdot t) - \phi(0)) \mid t \in [0,s]\}; \tag{42}$$

whereby, for the case that $\phi \in C^{\mathrm{lip}}([0,1],\mathfrak{g})$ holds, we additionally have

$$\sup\{\mathbin{.}\mathfrak{p}(\phi(\tau\cdot t) - \phi(0)) \mid t \in [0,s]\} \leq s\cdot\ell\cdot\mathrm{Lip}(\mathbin{.}\mathfrak{p},\phi). \tag{43}$$

We are ready for the

Proof of Proposition 1. Let $\phi \equiv \delta^r(\mu)$ and $X := \dot{\mu}(0) = \phi(0)$, for

1) $\mu \in C^1_*([0,1],G)$ with $\dot{\mu}(0) \in \mathrm{dom}[\exp]$ if G is sequentially 0-continuous.
2) $\mu \in C^1_*([0,1],G)$ with $\dot{\mu}(0) \in \mathrm{dom}[\exp]$ and $\phi \in C^{\mathrm{lip}}([0,1],\mathfrak{g})$ if G is Mackey 0-continuous.

We suppose that (41) is wrong; i.e., that there exists some $\ell > 0$, an open neighbourhood $U \subseteq G$ of e, a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subseteq [0,\ell]$, and a strictly increasing sequence $\{\iota_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}_{\geq 1}\cap[\ell,\infty)$ with

$$\exp(-\tau_n\cdot X)\cdot\mu(\tau_n/\iota_n)^{\iota_n} \notin U \qquad \forall\, n \in \mathbb{N}. \tag{44}$$

Passing to a subsequence if necessary, we can additionally assume that $\lim_n \tau_n = \tau \in [0,\ell]$ exists. We choose $V \subseteq G$ open with $e \in V$ and $V\cdot V \subseteq U$, and fix some $n_V \in \mathbb{N}$ with

$$\exp((\tau - \tau_n)\cdot X) \in V \qquad \forall\, n \geq n_V. \tag{45}$$

Moreover, for each $n \in \mathbb{N}$:

- We define

$$\chi_n := \delta^r(\mu_{\tau_n})|_{[0,1/\iota_n]} \in \mathfrak{D}^k_{[0,1/\iota_n]} \qquad\text{with}\qquad \mu_{\tau_n}\colon [0,1/\ell] \ni t \mapsto \mu(t\cdot\tau_n).$$

- We define $t_{n,m} := m/\iota_n$ for $m = 0,\dots,\iota_n$; as well as $\phi_n[m]\colon [t_{n,m}, t_{n,m+1}] \to \mathfrak{g},\ t \mapsto \chi_n(\cdot - t_{n,m})$ for $m = 0,\dots,\iota_{n-1}$. Then, we have[9]

$$\textstyle\int \phi_n[m] \overset{\mathrm{d)}}{=} \int \chi_n = \mu_{\tau_n}(1/\iota_n) = \mu(\tau_n/\iota_n) \qquad \forall\, m = 0,\dots,\iota_{n-1}. \tag{46}$$

- We define $\phi_n \in \mathfrak{D}\mathrm{P}^0([0,1],\mathfrak{g})$ by

$$\begin{aligned}\phi_n|_{[t_{n,m},t_{n,m+1})} &:= \phi_n[m]|_{[t_{n,m},t_{n,m+1})} \qquad \forall\, m = 0,\dots,\iota(n)-2,\\ \phi_n|_{[t_{n,\iota_n-1},t_{n,\iota_n}]} &:= \phi_n[\iota_n - 1];\end{aligned}$$

  and obtain

$$\textstyle\int \phi_n \overset{(29)}{=} \int_{t_{n,\iota_n-1}}^{t_{n,\iota_n}} \phi_n[\iota_n - 1]\cdot\ldots\cdot\int_{t_{n,0}}^{t_{n,1}} \phi_n[0] \overset{(46)}{=} \mu(\tau_n/\iota_n)^{\iota_n} \qquad \forall\, n \in \mathbb{N}. \tag{47}$$

[9]In the first step below, d) is applied with $\varrho\colon [t_{n,m}, t_{n,m+1}] \mapsto t - t_{n,m} \in [0,1/\iota_n]$.

Then, for each $n \in \mathbb{N}$ and $\mathfrak{p} \in \mathfrak{P}$, we have

$$\begin{aligned}
\boldsymbol{.}\mathfrak{p}_\infty(\phi_n - \tau \cdot \phi_X) &\leq \boldsymbol{.}\mathfrak{p}_\infty(\phi_n - \tau_n \cdot X) + \boldsymbol{.}\mathfrak{p}(\tau_n \cdot X - \tau \cdot X) \\
&= \boldsymbol{.}\mathfrak{p}_\infty\big(\delta^r(\mu_{\tau_n})|_{[0,1/\iota_n]} - \tau_n \cdot X\big) + |\tau - \tau_n| \cdot \boldsymbol{.}\mathfrak{p}(X) \\
&\overset{(42)}{\leq} \ell \cdot \sup\{\boldsymbol{.}\mathfrak{p}(\phi(\tau_n \cdot t) - \phi(0)) \,|\, t \in [0, 1/\iota_n]\} + |\tau - \tau_n| \cdot \boldsymbol{.}\mathfrak{p}(X).
\end{aligned} \tag{48}$$

For the case that $\phi \in C^{\mathrm{lip}}([0,1], \mathfrak{g})$ holds, we furthermore obtain

$$\begin{aligned}
\boldsymbol{.}\mathfrak{p}_\infty(\phi_n - \tau \cdot \phi_X) &\overset{(48),(43)}{\leq} \ell^2/\iota_n \cdot \mathrm{Lip}(\boldsymbol{.}\mathfrak{p}, \phi) + |\tau - \tau_n| \cdot \boldsymbol{.}\mathfrak{p}(X) \\
&\leq \underbrace{(\mathrm{Lip}(\boldsymbol{.}\mathfrak{p}, \phi) + \boldsymbol{.}\mathfrak{p}(X))}_{\mathfrak{c}_\mathfrak{p}} \cdot \underbrace{(\ell^2/\iota_n + |\tau - \tau_n|)}_{\lambda_n}
\end{aligned}$$

for each $n \in \mathbb{N}$ and $\mathfrak{p} \in \mathfrak{P}$; whereby $\lim_n \lambda_n = 0$ holds. We thus have

$$\begin{aligned}
&\mathfrak{D}\mathrm{P}^0([0,1], \mathfrak{g}) \supseteq \{\phi_n\}_{n \in \mathbb{N}} \rightharpoonup_{\mathfrak{s}} \tau \cdot \phi_X \\
\text{as well as} \qquad &\mathfrak{D}\mathrm{P}^0([0,1], \mathfrak{g}) \supseteq \{\phi_n\}_{n \in \mathbb{N}} \rightharpoonup_{\mathfrak{m}} \tau \cdot \phi_X \qquad \text{if} \qquad \phi \in C^{\mathrm{lip}}([0,1], \mathfrak{g}) \qquad \text{holds.}
\end{aligned}$$

In both cases, by Lemma 18, there exists some $\mathbb{N} \ni n'_V \geq n_V$ with

$$[\textstyle\int \tau \cdot \phi_X|_{[0,1]}]^{-1} \cdot [\textstyle\int \phi_n] \in V \qquad \forall\, n \geq n'_V; \tag{49}$$

and we obtain for $n \geq n'_V$ that

$$\exp(-\tau_n \cdot X) \cdot \mu(\tau_n/\iota_n)^{\iota_n} = \overbrace{\exp\big((\tau - \tau_n) \cdot X\big)}^{\overset{(45)}{\in} V} \cdot \underbrace{\Big(\overbrace{\exp(-\tau \cdot X)}^{\equiv [\int \tau \cdot \phi_X|_{[0,1]}]^{-1}} \cdot \overbrace{\mu(\tau_n/\iota_n)^{\iota_n}}^{\overset{(47)}{=} \int \phi_n}\Big)}_{\overset{(49)}{\in} V} \in V \cdot V \subseteq U$$

holds, which contradicts (44). $\square$

6 Differentiation

In this section, we discuss several differentiability properties of the evolution map. The whole discussion is based on the following generalization of Proposition 7 in [6].

Proposition 2. *Let* $\{\varepsilon_n\}_{n \in \mathbb{N}} \subseteq C^0([r, r'], \mathfrak{g})$, $\chi \in C^0([r, r'], \mathfrak{g})$, *and* $\mathbb{R}_{\neq 0} \supseteq \{h_n\}_{n \in \mathbb{N}} \to 0$ *be given with* $\{h_n \cdot \chi + h_n \cdot \varepsilon_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{D}_{[r,r']}$, *such that*

i) $\lim_n \varepsilon_n(t) = 0$ *holds for each* $t \in [r, r']$,

ii) $\sup\{\mathfrak{p}_\infty(\varepsilon_n) \,|\, n \in \mathbb{N}\} < \infty$ *holds for each* $\mathfrak{p} \in \mathfrak{P}$.

Then, the following two conditions are equivalent:

a) $\lim_n^\infty \Xi(\int_r^\bullet h_n \cdot \chi + h_n \cdot \varepsilon_n) = 0$.

b) $\lim_n^\infty 1/h_n \cdot \Xi\big(\int_r^\bullet h_n \cdot \chi + h_n \cdot \varepsilon_n\big) = \int_r^\bullet (\mathrm{d}_e\Xi \circ \chi)(s)\, \mathrm{d}s \in \overline{E}$.

The proof of Proposition 2 will be established in Sect. 6.4. We now first use this proposition, to discuss the differential of the evolution maps as well as the differentiation of parameter-dependent integrals.

6.1 Some Technical Statements

We will need the following variation of Proposition 2:

Corollary 2. *Let* $D \equiv (-\delta, 0) \cup (0, \delta)$ *for* $\delta > 0$, $\{\varepsilon_h\}_{h\in D} \subseteq C^0([r, r'], \mathfrak{g})$, *and* $\chi \in C^0([r, r'], \mathfrak{g})$ *be given with* $\{h \cdot \chi + h \cdot \varepsilon_h\}_{h\in D} \subseteq \mathfrak{D}_{[r,r']}$, *such that*

i) $\lim_{h\to 0} \varepsilon_h(t) = 0$ *holds for each* $t \in [r, r']$,

ii) $\sup\{\mathfrak{p}_\infty(\varepsilon_h) \,|\, h \in (-\delta_\mathfrak{p}, 0) \cup (0, \delta_\mathfrak{p})\} < \infty$ *holds for some* $0 < \delta_\mathfrak{p} \leq \delta$, *for each* $\mathfrak{p} \in \mathfrak{P}$.

Then, the following conditions are equivalent:[10]

a) $\lim^\infty_{h\to 0} \Xi(\int_r^\bullet h \cdot \chi + h \cdot \varepsilon_h) = 0$.

b) $\lim^\infty_n \Xi(\int_r^\bullet h_n \cdot \chi + h_n \cdot \varepsilon_{h_n}) = 0$ *for each sequence* $D \supseteq \{h_n\}_{n\in\mathbb{N}} \to 0$.

c) $\lim^\infty_n 1/h_n \cdot \Xi\big(\int_r^\bullet h_n \cdot \chi + h_n \cdot \varepsilon_{h_n}\big) = \int_r^\bullet (\mathrm{d}_e\Xi \circ \chi)(s)\, \mathrm{d}s \in \overline{E}$ *for each sequence* $D \supseteq \{h_n\}_{n\in\mathbb{N}} \to 0$.

d) $\frac{\mathrm{d}}{\mathrm{d}h}\big|^\infty_{h=0} \Xi\big(\int_r^\bullet h \cdot \chi + h \cdot \varepsilon_h\big) = \int_r^\bullet (\mathrm{d}_e\Xi \circ \chi)(s)\, \mathrm{d}s \in \overline{E}$.

Proof. By Lemma 1 (applied to $(G, +) \equiv (\overline{E}, +)$ there), a) is equivalent to b). Moreover, by Proposition 2, b) is equivalent to c), because

– Condition i) implies Condition *i)* in Proposition 2, for $\varepsilon_n \equiv \varepsilon_{h_n}$ there,

– Condition ii) implies Condition *ii)* in Proposition 2, for $\varepsilon_n \equiv \varepsilon_{h_n}$ there.

Finally, by Lemma 1 (applied to $(G, +) \equiv (\overline{E}, +)$ there), c) is equivalent to d). □

Lemma 19. *Suppose that* G *is Mackey* k*-continuous for* $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$. *Suppose furthermore that we are given* $\mathfrak{D}^k_{[r,r']} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi \in \mathfrak{D}^k_{[r,r']}$ *as well as* $\mathfrak{D}^k_{[r,r']} \supseteq \{\psi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{s}.0} \psi \in \mathfrak{D}^k_{[r,r']}$ *for* $[r, r'] \in \mathfrak{K}$, *such that the expressions*

$$\begin{aligned} \xi(\phi, \psi) &:= \mathrm{d}_e\mathrm{L}_{\int \phi}\big(\textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s\big) \\ \xi(\phi_m, \phi_n) &:= \mathrm{d}_e\mathrm{L}_{\int \phi_m}\big(\textstyle\int \mathrm{Ad}_{[\int_r^s \phi_m]^{-1}}(\psi_n(s))\, \mathrm{d}s\big) \qquad \forall\, n \in \mathbb{N} \end{aligned}$$

are well defined; i.e., such that the occurring Riemann integrals exist in $\mathfrak{g}$. *Then, we have*

$$\lim_{(m,n)\to(\infty,\infty)} \xi(\phi_n, \phi_m) = \xi(\phi, \psi). \tag{50}$$

Proof. This follows by the same arguments as in Corollary 13 and Lemma 41 in [6]. For completeness, the adapted argumentation is provided in Appendix D. □

6.2 The Differential of the Evolution Map

We now discuss the differential of the evolution map – for which we recall the conventions fixed in Remark 3. Then, Corollary 2 (with $\varepsilon_h \equiv 0$ there) provides us with

Proposition 3. *Suppose that* (ϕ, ψ) *is admissible, with* $\mathrm{dom}[\phi], \mathrm{dom}[\psi] = [r, r']$.

1) The pair (ϕ, ψ) *is regulated iff we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|^\infty_{h=0} \Xi\big([\textstyle\int_r^\bullet \phi]^{-1}[\int_r^\bullet \phi + h \cdot \psi]\big) = \int_r^\bullet (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{[\int_r^s \phi]^{-1}})(\psi(s))\, \mathrm{d}s \in \overline{E}.$$

2) If (ϕ, ψ) *is regulated, then* $(-\delta, \delta) \ni h \mapsto \int \phi + h \cdot \psi \in G$ *is differentiable at* $h = 0$ *(for* $\delta > 0$ *suitably small) iff* $\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s \in \mathfrak{g}$ *holds. In this case, we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0} \textstyle\int \phi + h \cdot \psi = \mathrm{d}_e\mathrm{L}_{\int \phi}\big(\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s\big). \tag{51}$$

[10]Recall Remark 2 for the notation used in d).

Proof. 1) For $|h| < \delta$, with $\delta > 0$ suitably small, we have

$$\Xi\big([\textstyle\int_r^t \phi]^{-1}[\int_r^t \phi + h \cdot \psi]\big) \stackrel{\mathrm{b)}}{=} \Xi\big(\int_r^t h \cdot \overbrace{\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi)}^{=:\,\chi} \big) \qquad \forall\, t \in [r, r']. \tag{52}$$

We obtain from the Equivalence of a) and d) in Corollary 2 for $\varepsilon_h \equiv 0$ there that (third step)

$$\begin{aligned}
& \lim^\infty_{h\to 0} \textstyle\int_r^\bullet \phi + h \cdot \psi = \int_r^\bullet \phi \\
\Longleftrightarrow\quad & \lim^\infty_{h\to 0} \Xi\big([\textstyle\int_r^\bullet \phi]^{-1}[\int_r^\bullet \phi + h \cdot \psi]\big) = 0 \\
\stackrel{(52)}{\Longleftrightarrow}\quad & \lim^\infty_{h\to 0} \Xi(\textstyle\int_r^\bullet h \cdot \chi) = 0 \\
\Longleftrightarrow\quad & \tfrac{\mathrm{d}}{\mathrm{d}h}\big|^\infty_{h=0} \Xi(\textstyle\int_r^\bullet h \cdot \chi) = \int_r^\bullet (\mathrm{d}_e\Xi \circ \chi)(s)\, \mathrm{d}s \in \overline{E} \\
\stackrel{(52)}{\Longleftrightarrow}\quad & \tfrac{\mathrm{d}}{\mathrm{d}h}\big|^\infty_{h=0} \Xi\big([\textstyle\int_r^\bullet \phi]^{-1}[\int_r^\bullet \phi + h \cdot \psi]\big) = \int_r^\bullet (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{[\int_r^s \phi]^{-1}})(\psi(s))\, \mathrm{d}s \in \overline{E}.
\end{aligned}$$

2) The statement is straightforward from Remark 3, cf. Appendix E. □

6.2.1 The Non-Constant Case

Combining Proposition 3 with Theorem 1 and Lemma 16, we obtain

Theorem 2. *Suppose that G is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$. Then,* $\mathrm{evol_k}$ *is differentiable iff $\mathfrak{g}$ is* k*-complete. In this case,* $\mathrm{evol}^k_{[r,r']}$ *is differentiable for each $[r, r'] \in \mathfrak{K}$, with*

$$\mathrm{d}_\phi \mathrm{evol}^k_{[r,r']}(\psi) = \mathrm{d}_e\mathrm{L}_{\int\phi}\big(\textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s\big) \qquad \forall\, \phi, \psi \in C^k([r, r'], \mathfrak{g}).$$

In particular,

a) $\mathrm{d}_\phi \mathrm{evol}^k_{[r,r']} \colon C^k([r, r'], \mathfrak{g}) \to T_{\int\phi}G$ *is linear and C^0-continuous for each $\phi \in C^k([r, r'], \mathfrak{g})$.*

b) we have

$$\lim_{(m,n)\to(\infty,\infty)} \mathrm{d}_{\phi_m} \mathrm{evol}^k_{[r,r']}(\psi_n) = \mathrm{d}_\phi \mathrm{evol}^k_{[r,r']}(\psi)$$

for all sequences $\{\phi_m\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} \phi$, $\{\psi_n\}_{m\in\mathbb{N}} \rightharpoonup_{\mathfrak{s}.0} \psi$.

Proof. The first part is clear from Theorem 1, Lemma 16, Remark 4, and Proposition 3.2). Then, *b)* is clear from Lemma *19*. Moreover, (by the first part) $\mathrm{d}_\phi \mathrm{evol}^k_{[r,r']}$ is linear; with (cf. (3))

$$\mathrm{d}_\phi \mathrm{evol}^k_{[r,r']}(\psi) = \mathrm{d}_{(\int\phi, e)}\, \mathrm{m}(0, \Gamma_\phi(\psi)) \qquad \text{for} \qquad \Gamma_\phi \colon C^k([0, 1], \mathfrak{g}) \ni \psi \to \textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s \in \mathfrak{g}.$$

Then, since Γ_ϕ is C^0-continuous by (9) and I), *a)* is clear from smoothness of the Lie group multiplication. □

Corollary 3. *Suppose that G is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, and that $\mathfrak{g}$ is* k*-complete. Then, $\mu \colon \mathbb{R} \ni h \mapsto \int \phi + h \cdot \psi$ is of class C^1 for each $\phi, \psi \in C^k([r, r'], \mathfrak{g})$ and $[r, r'] \in \mathfrak{K}$.*

Proof. Theorem 1 and Lemma 16 show that μ is continuous. Moreover, for each $t \in \mathbb{R}$, and each sequence $\{h_n\}_{n\in\mathbb{N}} \to 0$, we have $\{\phi + (t + h_n) \cdot \psi\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{m}.\mathrm{k}} (\phi + t \cdot \psi)$; thus,

$$\lim_n \dot\mu(t + h_n) = \lim_n \mathrm{d}_{\phi+(t+h_n)\cdot\psi}\, \mathrm{evol}^k_{[r,r']}(\psi) = \mathrm{d}_{\phi+t\cdot\psi}\, \mathrm{evol}^k_{[r,r']}(\psi) = \dot\mu(t)$$

by Theorem 2.*b)*. This shows that $\dot\mu$ is continuous, i.e., that μ is of class C^1. □

Remark 9.

1) *It is straightforward from Corollary 3, the differentiation rules* **d)** *and* **c)**, *as well as* (3), (12), c), *and* e) *(for* $\Psi \equiv \mathrm{Conj}_g$ *there) that* (51) *also holds for all* $\phi, \psi \in \mathfrak{DP}^k([r,r'],\mathfrak{g})$, *for each* $[r,r'] \in \mathfrak{K}$.

2) *Expectably,* μ *as defined Corollary 3 is even of class* C^∞. *A detailed proof of this fact, however, would require further technical preparation – which we don't want to carry out at this point.*

3) *Expectably, the equivalence*

$$\lim^\infty_{h\to 0} \Xi(\textstyle\int_r^\bullet h\cdot\chi) = 0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}h}\big|^\infty_{h=0}\, \Xi\big(\textstyle\int_r^\bullet h\cdot\chi\big) = \int_r^\bullet (\mathrm{d}_e\Xi\circ\chi)(s)\,\mathrm{d}s \in \overline{E}$$

also holds for $\chi \in \mathfrak{DP}^0([r,r'],\mathfrak{g})$ *– and thus, Proposition 3 in an adapted form. This might be shown by the same arguments (Taylor expansion) as used in the proof of Lemma 7 in [6] (cf. Lemma 4) additionally efforting the fact that for* $n \in \mathbb{N}$ *fixed,*

$$f\colon G^n \to G, \qquad (g_1,\dots,g_n) \mapsto g_1\cdot\ldots\cdot g_n$$

is smooth, with $\mathrm{d}_{(e,\dots,e)} f(X_1,\dots,X_n) = X_1+\dots+X_n$ *for all* $X_1,\dots,X_n \in \mathfrak{g}$. *The details, however, appear to be quite technical, so that we leave this issue to another paper.* ‡

6.2.2 The Exponential Map

We recall the conventions fixed in Sect. 2.2.3 – Specifically, that $\exp = \mathrm{evol}^{\mathrm{c}}_{[0,1]} \circ \mathfrak{i}|_{\mathrm{dom}[\exp]}$ holds. Then, Proposition 3.2), for $k \equiv \mathrm{c}$ and $[r,r'] \equiv [0,1]$ there, reads

Corollary 4. *Suppose that* $(\mathfrak{i}(X),\mathfrak{i}(Y))$ *is regulated for* $X, Y \in \mathfrak{g}$. *Then,* $(-\delta,\delta) \ni h \mapsto \exp(X + h\cdot Y)$ *is differentiable at* $h = 0$ *(for* $\delta > 0$ *suitably small) iff* $\int \mathrm{Ad}_{\exp(-s\cdot X)}(Y)\,\mathrm{d}s \in \mathfrak{g}$ *holds. In this case, we have*

$$\frac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0} \exp(X+h\cdot Y) = \mathrm{d}_e\mathrm{L}_{\exp(X)}\big(\textstyle\int \mathrm{Ad}_{\exp(-s\cdot X)}(Y)\,\mathrm{d}s\big).$$

Remark 10.

1) *Suppose that* G *admits an exponential map; and that* G *is weakly* c*-continuous. Then, Corollary 4 shows that*

$$\frac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0} \exp(X+h\cdot Y) = \mathrm{d}_e\mathrm{L}_{\exp(X)}\big(\textstyle\int \mathrm{Ad}_{\exp(-s\cdot X)}(Y)\,\mathrm{d}s\big) \qquad \forall\, X, Y \in \mathfrak{g} \tag{53}$$

holds iff $\mathfrak{g}$ *is* c*-complete. For instance,* G *is weakly* c*-continuous, and* $\mathfrak{g}$ *is* c*-complete if*

- $\exp\colon \mathfrak{g} \to G$ *is of class* C^1, *by Remark 7.1), Remark 7.5), and Corollary 4.*
- G *is abelian, by Corollary 1 and Remark 4.*

2) *Suppose that* $\mathfrak{g}$ *is* c*-complete; and that* G *admits a continuous exponential map. Then,* G *is* C^{c}*-semiregular; as well as* G *is weakly* c*-continuous by Remark 7.1) and Remark 7.5). More formally,* (53) *then reads*

$$\mathrm{d}_\phi\mathrm{evol}_{\mathrm{c}}(\psi) = \mathrm{d}_e\mathrm{L}_{\int\phi}\big(\textstyle\int \mathrm{Ad}_{[\int_r^s\psi]^{-1}}(\psi(s))\,\mathrm{d}s\big) \qquad \forall\,\phi,\psi \in C^{\mathrm{c}}([0,1],\mathfrak{g}). \tag{54}$$

The same arguments as in [6] then show that $\mathrm{evol}_{\mathrm{c}}$ *(thus* $\exp$*) is of class* C^1. *More specifically, one has to replace Lemma 23 by Lemma 11 in the proof of Lemma 41 in [6]. Then, subsituting Equation (95) in [6] by* (54), *the proof of Corollary 13 in [6] just carries over to the case where* $k \equiv \mathrm{c}$ *holds (a similar adaption has been done in the proof of Lemma 19).*

As in the Lipschitz case, cf. Remark 7 in [6], it is to be expected that a (quite elaborate and technical) induction shows that $\exp$ *is even smooth if* $\mathfrak{g}$ *is Mackey complete (or, more generally, if all the occurring iterated Riemann integrals exist in* $\mathfrak{g}$*).* ‡

6.3 Integrals with Parameters

The following theorem generalizes Theorem 5 in [6] – with significantly simplified proof.

Theorem 3. *Suppose that G is Mackey* k*-continuous for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$ – additionally abelian if $k \equiv \mathrm{c}$ holds. Let $\Phi\colon I \times [r,r'] \to \mathfrak{g}$ ($I \subseteq \mathbb{R}$ open) be given with $\Phi(z,\cdot) \in \mathfrak{D}^k_{[r,r']}$ for each $z \in I$. Then,*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|^{\infty}_{h=0}\, \Xi\big([\textstyle\int_r^\bullet \Phi(x,\cdot)]^{-1}[\int_r^\bullet \Phi(x+h,\cdot)]\big) = \int_r^\bullet (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{[\int_r^s \Phi(x,\cdot)]^{-1}})(\partial_z\Phi(x,s))\,\mathrm{d}s \in \overline{E}$$

holds for $x \in I$, provided that

a) We have $(\partial_z\Phi)(x,\cdot) \in C^k([r,r'],\mathfrak{g})$.[11]

b) For each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, there exists $L_{\mathfrak{p},\mathrm{s}} \geq 0$, as well as $I_{\mathfrak{p},\mathrm{s}} \subseteq I$ open with $x \in I_{\mathfrak{p},\mathrm{s}}$, such that

$$1/|h| \cdot \mathbf{.}\mathfrak{p}^{\mathrm{s}}_\infty(\Phi(x+h,\cdot) - \Phi(x,\cdot)) \leq L_{\mathfrak{p},\mathrm{s}} \qquad \forall\, h \in I_{\mathfrak{p},\mathrm{s}} - x.$$

In particular, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0} \textstyle\int \Phi(x+h,\cdot) = \mathrm{d}_e\mathrm{L}_{\int\Phi(x,\cdot)}\big(\int \mathrm{Ad}_{[\int_r^s \Phi(x,\cdot)]^{-1}}(\partial_z\Phi(x,s))\,\mathrm{d}s\big)$$

iff the Riemann integral on the right side exists in $\mathfrak{g}$.

Proof. The last statement follows from the first statement and Lemma 4 – just as in the proof of Proposition 3.2). Now, for $x + h \in I$, we have

$$\Phi(x+h,t) = \Phi(x,t) + h \cdot \partial_z\Phi(x,t) + h \cdot \varepsilon(x+h,t) \qquad \forall\, t \in [r,r'],$$

with $\varepsilon\colon I \times [r,r'] \to \mathfrak{g}$ such that

i) $\lim_{h\to 0} \varepsilon(x+h,t) = \varepsilon(x,t) = 0 \qquad \forall\, t \in [r,r']$,

ii) $\mathbf{.}\mathfrak{p}^{\mathrm{s}}_\infty(\varepsilon(x+h,\cdot)) \leq L_{\mathfrak{p},\mathrm{s}} + \mathbf{.}\mathfrak{p}^{\mathrm{s}}_\infty((\partial_z\Phi)(x,\cdot)) =: C_{\mathfrak{p},\mathrm{s}} < \infty \qquad \forall\, h \in I_{\mathfrak{p},\mathrm{s}} - x$ for all $\mathfrak{p} \in \mathfrak{P}$, $\mathrm{s} \preceq k$.

We let $\alpha := \int_r^\bullet \Phi(x,\cdot)$; and obtain

$$[\textstyle\int_r^\bullet \Phi(x,\cdot)]^{-1}[\int_r^\bullet \Phi(x+h,\cdot)] = \int_r^\bullet \overbrace{h \cdot \underbrace{\mathrm{Ad}_{\alpha^{-1}}(\partial_z\Phi(x,\cdot))}_{\chi} + h \cdot \underbrace{\mathrm{Ad}_{\alpha^{-1}}(\varepsilon(x+h,\cdot))}_{\varepsilon_h}}^{\psi_h} \tag{55}$$

with $\psi_h \in \mathfrak{D}^k_{[r,r']}$, because our presumptions ensure that $\chi, \varepsilon_h \in C^k([r,r'],\mathfrak{g})$ holds. By Lemma 7 and Lemma 8, for each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, there exists some $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ with[12]

$$\mathbf{.}\mathfrak{p}^{\mathrm{s}}_\infty(\psi_h) \leq |h| \cdot \mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty(\partial_z\Phi(x,\cdot) + \varepsilon(x+h,\cdot)) \overset{b)}{\leq} |h| \cdot L_{\mathfrak{q},\mathrm{s}} \qquad \forall\, h \in I_{\mathfrak{q},\mathrm{s}} - x.$$

For each fixed sequence $I_{\mathfrak{q},\mathrm{s}} - x \supseteq \{h_n\}_{n\in\mathbb{N}} \to 0$, we thus have $\psi_{h_n} \rightharpoonup_{\mathrm{m.k}} 0$. Since G is Mackey k-continuous, this implies

$$\lim^\infty_n \Xi(\textstyle\int_r^\bullet h_n \cdot \chi + h_n \cdot \varepsilon_{h_n}) \equiv \lim^\infty_n \Xi(\int_r^\bullet \psi_{h_n}) = 0. \tag{56}$$

Now, for $\delta > 0$ such small that $D \equiv (-\delta,0) \cup (0,\delta) \subseteq I - \{x\}$ holds, by i) and ii), $\{\varepsilon_h\}_{h\in D}$ fulfills the presumptions in Corollary 2. We thus have

$$\lim^\infty_{h\to 0} \Xi\big([\textstyle\int_r^\bullet \Phi(x,\cdot)]^{-1}[\int_r^\bullet \Phi(x+h,\cdot)]\big) = \int_r^\bullet (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{[\int_r^s \Phi(x,\cdot)]^{-1}})(\partial_z\Phi(x,s))\,\mathrm{d}s \in \overline{E}$$

by (55), (56), as well as the equivalence of b) and d) in Corollary 2. □

[11] More precisely, this means that for each $t \in [r,r']$, the map $I \ni z \mapsto \Phi(z,t)$ is differentiable at $z = x$ with derivative $(\partial_z\Phi)(x,t)$, such that $(\partial_z\Phi)(x,\cdot) \in C^k([r,r'],\mathfrak{g})$ holds. The latter condition in particular ensures that $\mathbf{.}\mathfrak{p}^{\mathrm{s}}_\infty((\partial_z\Phi)(x,\cdot)) < \infty$ holds for each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, cf. ii).

[12] If $k \equiv \mathrm{c}$ holds, we can just choose $\mathrm{s} \equiv 0$ and $\mathfrak{q} \equiv \mathfrak{p}$, because G is presumed to be abelian in this case.

We immediately obtain

Corollary 5. *Suppose that G is C^k-semiregular for $k \in \mathbb{N}\sqcup\{\mathrm{lip},\infty\}$; and that $\mathfrak{g}$ is* k*-complete. Let $\Phi\colon I\times[r,r']\to\mathfrak{g}$ ($I\subseteq\mathbb{R}$ open) be given with $\Phi(z,\cdot)\in\mathfrak{D}^k_{[r,r']}$ for each $z\in I$. Then,*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\textstyle\int\Phi(x+h,\cdot)=\mathrm{d}_e\mathrm{L}_{\int\Phi(x,\cdot)}\big(\int\mathrm{Ad}_{[\int_r^s\Phi(x,\cdot)]^{-1}}(\partial_z\Phi(x,s))\,\mathrm{d}s\big)$$

holds for $x\in I$, provided that the conditions a) and b) in Theorem 3 are fulfilled.

Proof. This is clear from Theorem 1 and Theorem 3. □

We furthermore obtain the following generalization of Corollary 11 in [6].

Corollary 6. *Suppose that G is Mackey* k*-continuous for $k\in\{\infty,\mathrm{c}\}$ – additionally abelian if $k\equiv\mathrm{c}$ holds. Suppose furthermore that $\mathfrak{X}\colon I\to\mathrm{dom}[\exp]\subseteq\mathfrak{g}$ is of class C^1; and define $\alpha:=\exp\circ\mathfrak{X}$. Then, for $x\in I$, we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\,\alpha(x+h)=\mathrm{d}_e\mathrm{L}_{\exp(\mathfrak{X}(x))}\big(\textstyle\int_0^1\mathrm{Ad}_{\exp(-s\cdot\mathfrak{X}(x))}(\dot{\mathfrak{X}}(x))\,\mathrm{d}s\big),$$

provided that the Riemann integral on the right side exists in $\mathfrak{g}$. If this is the case for each $x\in I$, then α is of class C^1.

Proof. We let $\Phi\colon I\times[0,1]\ni(z,t)\mapsto\mathfrak{X}(z)$; and observe that $\alpha(z)=\int\Phi(z,\cdot)$ holds for each $z\in I$. Then, the first statement is clear from Theorem 3. For the second statement, we suppose that

$$\dot\alpha(x)=\mathrm{d}_e\mathrm{L}_{\exp(\mathfrak{X}(x))}\big(\textstyle\int_0^1\mathrm{Ad}_{\exp(-s\cdot\mathfrak{X}(x))}(\dot{\mathfrak{X}}(x))\,\mathrm{d}s\big)\qquad\forall\,x\in I$$

is well defined; i.e., that the Riemann integral on the right side exists for each $x\in I$. We fix $x\in I$ and $\delta>0$ with $[x-\delta,x+\delta]\subseteq I$; and observe that

$$\bullet\mathfrak{p}(\mathfrak{X}(x+h)-\mathfrak{X}(x))\leq|h|\cdot\sup\{\bullet\mathfrak{p}(\dot{\mathfrak{X}}(z))\,|\,z\in[x-\delta,x+\delta]\}\qquad\forall\,\mathfrak{p}\in\mathfrak{P},\;\;|h|\leq\delta$$

holds by (8). For each sequence $I\supseteq\{h_n\}_{n\in\mathbb{N}}\to 0$, we thus have

$$\mathrm{D_c}\supseteq\{\phi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{m}.k}\phi\in\mathrm{D_c}\qquad\text{for}\qquad\phi:=\mathfrak{i}(\mathfrak{X}(x))\quad\text{and}\quad\phi_n:=\mathfrak{i}(\mathfrak{X}(x+h_n))\quad\forall\,n\in\mathbb{N}.$$

Moreover, since $\mathfrak{X}$ is of class C^1, we have

$$\mathrm{D_c}\supseteq\{\psi_n\}_{n\in\mathbb{N}}\rightharpoonup_{\mathfrak{s}.0}\psi\in\mathrm{D_c}\qquad\text{for}\qquad\psi:=\mathfrak{i}(\dot{\mathfrak{X}}(x))\quad\text{and}\quad\psi_n:=\mathfrak{i}(\dot{\mathfrak{X}}(x+h_n))\quad\forall\,n\in\mathbb{N};$$

so that Lemma 19 shows

$$\lim_{n\to\infty}\dot\alpha(x+h_n)=\lim_{(n,n)\to(\infty,\infty)}\xi(\phi_n,\psi_n)=\xi(\phi,\psi)=\dot\alpha(x).$$

This shows that $\dot\alpha$ is continuous at x. Since $x\in I$ was arbitrary, it follows that α is of class C^1. □

For instance, we obtain the following generalization of Remark 2.3) in [6].

Example 1. *Suppose that G is Mackey* c*-continuous and abelian. Then, for each $\phi\in C^0([r,r'],\mathfrak{g})$ with $[r,r']\ni t\mapsto\int_r^t\phi(s)\,\mathrm{d}s\in\mathrm{dom}[\exp]$, we have, cf. Appendix F*

$$\textstyle\int\phi=\exp\big(\int\phi(s)\,\mathrm{d}s\big).\tag{57}$$

In particular, if G admits an exponential map, then G is C^k-semiregular for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$ if $\mathfrak{g}$ is k*-complete.* ‡

6.4 Differentiation at Zero

In this subsection, we prove Proposition 2. We start with some general remarks:

Let $[r,r'] \in \mathfrak{K}$ and $\chi \in C^0([r,r'],\mathfrak{g})$ be given. For $m \geq 1$ fixed, we define $t_k := r + k/m \cdot |r'-r|$ for $k = 0,\dots,m$; as well as $X_k := \chi(t_k)$ for $k = 0,\dots,m-1$. We furthermore define $\chi_m \in C^0([r,r'],\mathfrak{g})$ by $\chi_m(r) := X_0$ and

$$\chi_m(t) = X_k + |t-t_k|/|t_{k+1}-t_k| \cdot (X_{k+1}-X_k) \qquad \forall\, t \in (t_k,t_{k+1}],\ \ k = 0,\dots,m-1.$$

Then, $\{\chi_m\}_{m\geq 1} \subseteq C^0([r,r'],\mathfrak{g})$ constructed in this way, admits the following properties:

a) We have $\lim_m \mathfrak{p}_\infty(\chi - \chi_m) = 0$ for each $\mathfrak{p} \in \mathfrak{P}$.

b) We have $\gamma_{h,m} := h \cdot \mathrm{d}_e\Xi(\int_r^\bullet \chi_m(s)\,\mathrm{d}s) \in E$ for each $h \in \mathbb{R}$ and $m \geq 1$.

c) Since $\mathrm{im}[\chi] \subseteq \mathfrak{g}$ is bounded, also $\{\mathrm{im}[\chi_m]\}_{m\geq 1} \subseteq \mathfrak{g}$, $\{\mathrm{im}[\mathrm{d}_e\Xi(\int_r^\bullet \chi_m(s)\,\mathrm{d}s)]\}_{m\geq 1} \subseteq E$ are bounded. Thus,

- For each $\mathfrak{p} \in \mathfrak{P}$, there exists some $C_\mathfrak{p} > 0$ with

$$\mathfrak{p}_\infty(\gamma_{h,m}) \leq |h| \cdot C_\mathfrak{p} \qquad \forall\, h \in \mathbb{R},\ \ m \geq 1. \tag{58}$$

- For $\delta > 0$ suitably small,

$$\mu_{h,m} := \Xi^{-1} \circ \gamma_{h,m} \in C^1([r,r'],G)$$

  is well defined for each $|h| \leq \delta$, $m \geq 1$; and we define

$$\chi_{h,m} := \delta^r(\mu_{h,m}) \overset{(14)}{=} h \cdot \omega(\gamma_{h,m}, \mathrm{d}_e\Xi(\chi_m)) \qquad \forall\, |h| \leq \delta,\ \ m \geq 1. \tag{59}$$

  Moreover, for each fixed open neighbourhood $V \subseteq G$ of e, there exists some $0 < \delta_V \leq \delta$ with

$$V \ni \mu_{h,m} \equiv \textstyle\int_r^\bullet \chi_{h,m} \qquad \forall\, |h| \leq \delta_V,\ \ m \geq 1. \tag{60}$$

Modifying the proof of Proposition 7 in [6], we obtain the

Proof of Proposition 2. Suppose first that *b)* holds; and let

$$A := \Xi(\textstyle\int_r^\bullet h_n \cdot \chi + h_n \cdot \varepsilon_n) \qquad\qquad B := \int_r^\bullet (\mathrm{d}_e\Xi \circ \chi)(s)\,\mathrm{d}s.$$

Then, *a)* is clear from $\mathfrak{p}_\infty(A) \leq |h_n| \cdot \mathfrak{p}_\infty(1/h_n \cdot A - B) + |h_n| \cdot \mathfrak{p}_\infty(B)$.

Suppose now that *a)* holds – i.e. that we have $\lim_n^\infty \Xi(\int_r^\bullet \psi_n) = 0$ with

$$\psi_n := h_n \cdot \chi + h_n \cdot \varepsilon_n \qquad \forall\, n \in \mathbb{N}.$$

We now have to show that for $\mathfrak{p} \in \mathfrak{P}$ fixed, the expression

$$\Delta_n := 1/|h_n| \cdot \overline{\mathfrak{p}}_\infty\big(\Xi(\textstyle\int_r^\bullet \psi_n) - h_n \cdot \int_r^\bullet \mathrm{d}_e\Xi(\chi(s))\,\mathrm{d}s\big)$$

tends to zero for $n \to \infty$. For this, we choose $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ and $e \in V \subseteq G$ as in Lemma 10; and let

$$\{\chi_m\}_{m\geq 1}, \quad \{\chi_{h,m}\}_{m\geq 1}, \quad \{\gamma_{h,m}\}_{m\geq 1}, \quad \{\mu_{h,m}\}_{m\geq 1}, \quad \delta_V > 0$$

be as above – with δ_V additionally such small that $\int_r^\bullet \psi_n \in V$ holds for each $n \in \mathbb{N}$ with $|h_n| < \delta_V$. We choose $\ell \in \mathbb{N}$ such large that $\{h_n\}_{n\geq \ell} \subseteq (-\delta_V, \delta_V)$ holds. Then, for each $n \geq \ell$ and $m \geq 1$, we obtain from (9) (second step), (60) and Lemma 10 (fifth step), as well as (59) (last step) that

$$\begin{aligned}
\Delta_n &\leq 1/|h_n| \cdot \mathfrak{p}_\infty\big(\Xi(\textstyle\int_r^\bullet \psi_n) - h_n \cdot \mathrm{d}_e\Xi(\int_r^\bullet \chi_m(s)\,\mathrm{d}s)\big) &&+\ \overline{\mathfrak{p}}_\infty\big(\textstyle\int_r^\bullet \mathrm{d}_e\Xi(\chi(s))\,\mathrm{d}s - \int_r^\bullet \mathrm{d}_e\Xi(\chi_m(s))\,\mathrm{d}s\big)\\
&\leq 1/|h_n| \cdot \mathfrak{p}_\infty\big(\Xi(\textstyle\int_r^\bullet \psi_n) - \gamma_{h_n,m}\big) &&+\ \textstyle\int (\mathfrak{p}\circ \mathrm{d}_e\Xi)(\chi(s) - \chi_m(s))\,\mathrm{d}s\\
&= 1/|h_n| \cdot \mathfrak{p}_\infty\big(\Xi(\textstyle\int_r^\bullet \psi_n) - \Xi(\mu_{h_n,m})\big) &&+\ \textstyle\int \mathbf{.}\mathfrak{p}(\chi(s) - \chi_m(s))\,\mathrm{d}s\\
&= 1/|h_n| \cdot \mathfrak{p}_\infty\big(\Xi(\textstyle\int_r^\bullet \psi_n) - \Xi(\int_r^\bullet \chi_{h_n,m})\big) &&+\ \textstyle\int \mathbf{.}\mathfrak{p}(\chi(s) - \chi_m(s))\,\mathrm{d}s\\
&\leq 1/|h_n| \cdot \textstyle\int \mathbf{.}\mathfrak{q}\big(\psi_n(s) - \chi_{h_n,m}(s)\big)\,\mathrm{d}s &&+\ \textstyle\int \mathbf{.}\mathfrak{p}(\chi(s) - \chi_m(s))\,\mathrm{d}s\\
&\leq\ \textstyle\int \mathbf{.}\mathfrak{q}(\varepsilon_n(s))\,\mathrm{d}s \ +\ \int \mathbf{.}\mathfrak{q}\big(\chi(s) - \omega(\gamma_{h_n,m}(s), \mathrm{d}_e\Xi(\chi_m(s)))\big)\,\mathrm{d}s &&+\ |r'-r| \cdot \mathbf{.}\mathfrak{p}_\infty(\chi - \chi_m)
\end{aligned}$$

holds. By Lebesgue's dominated convergence theorem and *i)*, *ii)*, the first term tends to zero for $n \to \infty$; and, by **a)**, the third term tends to zero for $m \to \infty$. Thus, $\varepsilon > 0$ given, there exists some $\ell_\varepsilon \geq \ell$, such that both the first-, and the third term is bounded by $\varepsilon/4$ for all $m, n \geq \ell_\varepsilon$. Moreover, since $\chi = \omega(0, \mathrm{d}_e\Xi(\chi))$ holds (second step), we can estimate the second term by

$$\begin{aligned}
\textstyle\int \mathbf{.}\mathfrak{q}\big(\chi(s) &- \omega(\gamma_{h_n,m}(s), \mathrm{d}_e\Xi(\chi_m(s)))\big)\,\mathrm{d}s\\
&\leq |r'-r| \cdot \mathbf{.}\mathfrak{q}_\infty\big(\chi - \omega(\gamma_{h_n,m}, \mathrm{d}_e\Xi(\chi_m))\big)\\
&= |r'-r| \cdot \mathbf{.}\mathfrak{q}_\infty\big(\omega(0, \mathrm{d}_e\Xi(\chi)) - \omega(\gamma_{h_n,m}, \mathrm{d}_e\Xi(\chi_m))\big)\\
&\leq |r'-r| \cdot \mathbf{.}\mathfrak{q}_\infty\big(\omega(0, \mathrm{d}_e\Xi(\chi)) - \omega(\gamma_{h_n,m}, \mathrm{d}_e\Xi(\chi))\big)\\
&\quad + |r'-r| \cdot \mathbf{.}\mathfrak{q}_\infty\big(\omega(\gamma_{h_n,m}, \mathrm{d}_e\Xi(\chi - \chi_m))\big).
\end{aligned} \tag{61}$$

- Since $\mathrm{im}[\chi]$ is compact, increasing ℓ_ε if necessary, by (58), we can achieve that the fourth line in (61) is bounded by $\varepsilon/4$ for each $n, m \geq \ell_\varepsilon$.
- To estimate the last line in (61), we choose $\mathfrak{q} \leq \mathfrak{m} \in \mathfrak{P}$ as in (15); and increase ℓ_ε in such a way (use (58)) that $\mathfrak{m}_\infty(\gamma_{h_n,m}) \leq 1$ holds for all $n, m \geq \ell_\varepsilon$; thus,

  $$\mathbf{.}\mathfrak{q}_\infty\big(\omega(\gamma_{h_n,m}, \mathrm{d}_e\Xi(\chi - \chi_m))\big) \stackrel{(15)}{\leq} \mathbf{.}\mathfrak{m}_\infty(\chi - \chi_m).$$

  Then, it is clear from **a)** that for $\ell'_\varepsilon \geq \ell_\varepsilon$ suitably large, the third line in (61) is bounded by $\varepsilon/4$ for all $m, n \geq \ell'_\varepsilon$.

We thus have $\Delta_n \leq \varepsilon$ for each $n \geq \ell'_\varepsilon \in \mathbb{N}$; which shows $\lim_n \Delta_n = 0$. $\square$

APPENDIX

A Appendix

In this Appendix, we recall the differential calculus from [1, 5, 9, 12], cf. also Sect. 3.3.1 in [6].

Let E and F be Hausdorff locally convex vector spaces. A map $f\colon U \to E$, with $U \subseteq F$ open, is said to be differentiable at $x \in U$ *iff*

$$(D_v f)(x) := \lim_{h\to 0} 1/h \cdot (f(x + h \cdot v) - f(x)) \in E$$

exists for each $v \in F$. Then, f is said to be differentiable *iff* it is differentiable at each $x \in U$. More generally, f is said to be k-times differentiable for $k \geq 1$ *iff*

$$D_{v_k,\dots,v_1} f \equiv D_{v_k}(D_{v_{k-1}}(\dots(D_{v_1}(f))\dots))\colon U \to E$$

is well defined for each $v_1, \ldots, v_k \in F$ – implicitly meaning that f is p-times differentiable for each $1 \leq p \leq k$. In this case, we define

$$\mathrm{d}_x^p f(v_1, \ldots, v_p) \equiv \mathrm{d}^p f(x, v_1, \ldots, v_p) := D_{v_p, \ldots, v_1} f(x) \qquad \forall\, x \in U, \;\; v_1, \ldots, v_p \in F$$

for $p = 1, \ldots, k$. Then, f is said to be

- of class C^0 *iff* it is continuous – In this case, we let $\mathrm{d}^0 f \equiv f$.
- of class C^k for $k \geq 1$ *iff* it is k-times differentiable, such that

  $$\mathrm{d}^p f \colon U \times F^p \to E, \qquad (x, v_1, \ldots, v_p) \mapsto D_{v_p, \ldots, v_1} f(x)$$

  is continuous for each $p = 0, \ldots, k$. In this case, $\mathrm{d}_x^p f$ is symmetric and p-multilinear for each $x \in U$ and $p = 1, \ldots, k$, cf. [1].
- of class C^∞ *iff* it is of class C^k for each $k \in \mathbb{N}$.

We have the following differentiation rules [1]:

a) A map $f \colon F \supseteq U \to E$ is of class C^k for $k \geq 1$ *iff* $\mathrm{d}f$ is of class C^{k-1} when considered as a map $F' \supseteq U' \to E$ for $F' \equiv F \times F$ and $U' \equiv U \times F$.

b) If $f \colon U \to F$ is linear and continuous, then f is smooth; with $\mathrm{d}_x^1 f = f$ for each $x \in E$, as well as $\mathrm{d}^k f = 0$ for each $k \geq 2$.

c) Suppose that $f \colon F \supseteq U \to U' \subseteq F'$ and $f' \colon F' \supseteq U' \to F''$ are of class C^k for $k \geq 1$, for Hausdorff locally convex vector spaces F, F', F''. Then, $f' \circ f \colon U \to F''$ is of class C^k with

$$\mathrm{d}_x(f' \circ f) = \mathrm{d}_{f(x)} f' \circ \mathrm{d}_x f \qquad \forall\, x \in U.$$

d) Let $F_1, \ldots, F_m, E$ be Hausdorff locally convex vector spaces, and $f \colon F_1 \times \ldots \times F_m \supseteq U \to E$ of class C^0. Then, f is of class C^1 *iff* for $p = 1, \ldots, m$, the "partial derivative"

$$\partial_p f \colon U \times F_p \ni ((x_1, \ldots, x_m), v_p) \mapsto \lim_{h \to 0} 1/h \cdot (f(x_1, \ldots, x_p + h \cdot v_p, \ldots, x_m) - f(x_1, \ldots, x_m))$$

exists in E, and is continuous. In this case, we have

$$\begin{aligned} \mathrm{d}_{(x_1, \ldots, x_m)} f(v_1, \ldots, v_m) &= \textstyle\sum_{p=1}^m \partial_p f((x_1, \ldots, x_m), v_p) \\ &= \textstyle\sum_{p=1}^m \mathrm{d}f((x_1, \ldots, x_m), (0, \ldots, 0, v_p, 0, \ldots, 0)) \end{aligned}$$

for each $(x_1, \ldots, x_m) \in U$, and $v_p \in F_p$ for $p = 1, \ldots, m$.

B Appendix

Proof of Lemma 8. By definition, there exists some $\mu \in C^1(I, G)$, for I an open interval containing $[r, r']$, with $\delta^r(\mu)|_{[r,r']} = \phi$ and $\mu(r) = e$. We now have to show that

$$C^{\mathrm{lip}}([r, r'], \mathfrak{g}) \ni \chi \colon [r, r'] \ni t \mapsto \mathrm{Ad}_{\mu^{-1}(t)}(\psi(t))$$

holds, for each fixed $\psi \in C^{\mathrm{lip}}([r, r'], \mathfrak{g})$. For this, we let $\mathfrak{p} \in \mathfrak{P}$ be fixed; and obtain

$$\boldsymbol{\cdot}\mathfrak{p}(\chi(t) - \chi(t')) \leq \boldsymbol{\cdot}\mathfrak{p}\big(\mathrm{Ad}_{\mu^{-1}(t)}(\psi(t) - \psi(t'))\big) + \boldsymbol{\cdot}\mathfrak{p}\big(\big(\mathrm{Ad}_{\mu^{-1}(t)} - \mathrm{Ad}_{\mu^{-1}(t')}\big)(\psi(t'))\big). \tag{62}$$

- We let $\mathrm{C} := \mathrm{im}[\mu^{-1}]$, choose $\mathfrak{p} \leq \mathfrak{w} \in \mathfrak{P}$ as in I) for $\mathfrak{v} \equiv \mathfrak{p}$ there; and obtain

$$\boldsymbol{\cdot}\mathfrak{p}(\chi(t)) \leq \boldsymbol{\cdot}\mathfrak{w}(\psi(t)) \qquad \forall\, t \in [r, r'], \tag{63}$$

$$\boldsymbol{\cdot}\mathfrak{p}\big(\mathrm{Ad}_{\mu^{-1}(t)}(\psi(t) - \psi(t'))\big) \leq \boldsymbol{\cdot}\mathfrak{w}(\psi(t) - \psi(t')) \leq \mathrm{Lip}(\boldsymbol{\cdot}\mathfrak{w}, \psi) \cdot |t - t'| \qquad \forall\, t, t' \in [r, r']. \tag{64}$$

- The map $\alpha\colon I\times\mathfrak{g}\ni(s,X)\rightarrow\partial_s\mathrm{Ad}_{\mu^{-1}(s)}(X)$ is well defined, continuous, and linear in the second argument. By Lemma 2 applied to $\mathrm{K}\equiv\mathrm{C}$, there thus exists some $\mathfrak{p}\leq\mathfrak{m}\in\mathfrak{P}$ with

$$(\bullet\mathfrak{p}\circ\alpha)(s,X)\leq\bullet\mathfrak{m}(X)\qquad\forall\, s\in[r,r'],\;\; X\in\mathfrak{g}.$$

  Then, we obtain from (8) that

$$\begin{aligned}\bullet\mathfrak{p}\big(\big(\mathrm{Ad}_{\mu^{-1}(t)}-\mathrm{Ad}_{\mu^{-1}(t')}\big)(\psi(t'))\big)&\leq\textstyle\int_{t'}^{t}\bullet\mathfrak{p}\big(\partial_s\mathrm{Ad}_{\mu^{-1}(s)}(\psi(t'))\big)\,\mathrm{d}s\\&=\textstyle\int_{t'}^{t}(\bullet\mathfrak{p}\circ\alpha)(s,\psi(t'))\,\mathrm{d}s\\&\leq\bullet\mathfrak{m}_\infty(\psi)\cdot|t-t'|\end{aligned}\tag{65}$$

  holds, for each $t,t'\in[r,r']$ with $t'\leq t$.

We choose $\mathfrak{q}\in\mathfrak{P}$ with $\mathfrak{q}\geq 2\cdot\max(\mathfrak{m},\mathfrak{w})$ (i.e., $\mathfrak{p},\mathfrak{m},\mathfrak{w}\leq\mathfrak{q}$); and obtain

$$\bullet\mathfrak{p}_\infty(\chi)\overset{(63)}{\leq}\bullet\mathfrak{w}_\infty(\psi)\leq\bullet\mathfrak{q}_\infty(\psi).\tag{66}$$

We furthermore obtain from (62), (64), (65) that

$$\bullet\mathfrak{p}(\chi(t)-\chi(t'))\leq\mathrm{Lip}(\bullet\mathfrak{w},\psi)\cdot|t-t'|+\bullet\mathfrak{m}_\infty(\psi)\cdot|t-t'|\leq\bullet\mathfrak{q}^{\mathrm{lip}}_\infty(\psi)\cdot|t-t'|$$

holds for each $t,t'\in[r,r']$; thus,

$$\mathrm{Lip}(\bullet\mathfrak{p},\chi)\leq\bullet\mathfrak{q}^{\mathrm{lip}}_\infty(\psi)\qquad\overset{(66)}{\Longrightarrow}\qquad\bullet\mathfrak{p}^{\mathrm{lip}}_\infty(\chi)\leq\bullet\mathfrak{q}^{\mathrm{lip}}_\infty(\psi),$$

which proves the claim. □

C Appendix

Proof of Equation (39). We let $\boldsymbol{\varphi}_n:=\rho_n\cdot\phi_n\circ\varrho_n$ for each $n\in\mathbb{N}$; so that

$$\mathrm{Lip}(\bullet\mathfrak{p},\boldsymbol{\varphi}_n)\leq 2\cdot\delta_n^{-8}\cdot C[\boldsymbol{\rho},1]^2\cdot\bullet\mathfrak{p}^{\mathrm{lip}}_\infty(\phi_n)$$

holds by (37). Then, for $t,t'\in[t_{\ell+1},t_\ell]$ with $\ell\in\mathbb{N}$, we have

$$\begin{aligned}\bullet\mathfrak{p}(\phi(t)-\phi(t'))&=\bullet\mathfrak{p}(\boldsymbol{\varphi}_\ell(t)-\boldsymbol{\varphi}_\ell(t'))\\&\leq\mathrm{Lip}(\bullet\mathfrak{p},\boldsymbol{\varphi}_\ell)\cdot|t-t'|\\&\leq 2\cdot C[\boldsymbol{\rho},1]^2\cdot\delta_\ell^{-8}\cdot\bullet\mathfrak{p}^{\mathrm{lip}}_\infty(\phi_\ell)\cdot|t-t'|.\end{aligned}\tag{67}$$

Moreover, for $t \in [t_{(\ell+1)+m}, t_{\ell+m}]$ and $t' \in [t_{\ell+1}, t_\ell]$, with $m \geq 1$ and $\ell \in \mathbb{N}$, we have

$$
\begin{aligned}
\mathfrak{p}(\phi(t) - \phi(t')) &\leq \mathfrak{p}(\phi(t) - \phi(t_{\ell+m})) \\
&\quad + \textstyle\sum_{k=m-1}^{1} \mathfrak{p}(\phi(t_{(\ell+1)+k}) - \phi(t_{\ell+k})) \\
&\quad + \mathfrak{p}(\phi(t_{\ell+1}) - \phi(t')) \\
&\leq \mathfrak{p}(\boldsymbol{\varphi}_{\ell+m}(t) - \boldsymbol{\varphi}_{\ell+m}(t_{\ell+m})) \\
&\quad + \textstyle\sum_{k=m-1}^{1} \mathfrak{p}(\boldsymbol{\varphi}_{\ell+k}(t_{(\ell+1)+k}) - \boldsymbol{\varphi}_{\ell+k}(t_{\ell+k})) \\
&\quad + \mathfrak{p}(\boldsymbol{\varphi}_{\ell}(t_{\ell+1}) - \boldsymbol{\varphi}_{\ell}(t')) \\
&\leq \mathrm{Lip}(\mathfrak{p}, \boldsymbol{\varphi}_{\ell+m}) \cdot |t - t_{\ell+m}| \\
&\quad + \textstyle\sum_{k=m-1}^{1} \mathrm{Lip}(\mathfrak{p}, \boldsymbol{\varphi}_{\ell+k}) \cdot |t_{(\ell+1)+k} - t_{\ell+k}| \\
&\quad + \mathrm{Lip}(\mathfrak{p}, \boldsymbol{\varphi}_{\ell}) \cdot |t_{\ell+1} - t'| \\
&\leq 2 \cdot \delta_{\ell+m}^{-8} \cdot C[\boldsymbol{\rho}, 1]^2 \cdot \mathfrak{p}_\infty^{\mathrm{lip}}(\phi_{\ell+m}) \cdot |t - t_{\ell+m}| \\
&\quad + \textstyle\sum_{k=m-1}^{1} 2 \cdot \delta_{\ell+k}^{-8} \cdot C[\boldsymbol{\rho}, 1]^2 \cdot \mathfrak{p}_\infty^{\mathrm{lip}}(\phi_{\ell+k}) \cdot |t_{(\ell+1)+k} - t_{\ell+k}| \\
&\quad + 2 \cdot \delta_{\ell}^{-8} \cdot C[\boldsymbol{\rho}, 1]^2 \cdot \mathfrak{p}_\infty^{\mathrm{lip}}(\phi_{\ell}) \cdot |t_{\ell+1} - t'| \\
&\leq 2 \cdot C[\boldsymbol{\rho}, 1]^2 \cdot \max\big(\delta_\ell^{-8} \cdot \mathfrak{p}_\infty^{\mathrm{lip}}(\phi_\ell), \dots, \delta_{\ell+m}^{-8} \cdot \mathfrak{p}_\infty^{\mathrm{lip}}(\phi_{\ell+m})\big) \cdot |t - t'|.
\end{aligned} \tag{68}
$$

Combining (67) with (68), we obtain (39). □

D Appendix

In this section, we prove Lemma 19. For this, we first show the following analogue to Lemma 41 in [6].

Lemma 20. *Suppose that G is Mackey* k*-continuous for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty, \mathrm{c}\}$; and let $[r, r'] \in \mathfrak{K}$ be fixed. Let $\Gamma\colon G \times \mathfrak{g} \to \mathfrak{g}$ be continuous; and define*

$$\widehat{\Gamma}\colon \mathfrak{D}^k_{[r,r']} \times C^k([r,r'], \mathfrak{g}) \to \overline{\mathfrak{g}}, \qquad (\phi, \psi) \mapsto \textstyle\int \Gamma\big(\int_r^s \phi, \psi(s)\big)\, \mathrm{d}s.$$

Then, we have

$$\lim\nolimits_{(m,n)\to(\infty,\infty)} \widehat{\Gamma}(\phi_m, \psi_n) = \widehat{\Gamma}(\phi, \psi),$$

for sequences $\mathfrak{D}^k_{[r,r']} \supseteq \{\phi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathrm{m.k}} \phi \in \mathfrak{D}^k_{[r,r']}$ and $C^k([r,r'], \mathfrak{g}) \supseteq \{\psi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{s}.0} \psi \in C^k([r,r'], \mathfrak{g})$.

Proof. By (9), it suffices to show that for

$$\widetilde{\Gamma}\colon \mathfrak{D}^k_{[r,r']} \times C^k([r,r'], \mathfrak{g}) \to C^0([r,r'], \mathfrak{g}), \qquad (\phi, \psi) \mapsto \big[t \mapsto \Gamma\big(\textstyle\int_r^t \phi, \psi(t)\big)\big],$$

we have $\lim_{(m,n)\to(\infty,\infty)} \widetilde{\Gamma}(\phi_m, \psi_n) = \widetilde{\Gamma}(\phi, \psi)$ w.r.t. the C^0-topology; i.e., that for $\mathfrak{p} \in \mathfrak{P}$ and $\varepsilon > 0$ fixed, there exists some $N_\varepsilon \in \mathbb{N}$ with

$$\mathfrak{p}_\infty\big(\widetilde{\Gamma}(\phi_m, \psi_n) - \widetilde{\Gamma}(\phi, \psi)\big) < \varepsilon \qquad \forall\, m, n \geq N_\varepsilon. \tag{69}$$

For this, we let $\mu := \int_r^\bullet \phi$, and consider the continuous map

$$\alpha\colon G \times \mathfrak{g} \times G \times \mathfrak{g} \to \mathfrak{g}, \qquad ((g, X), (g', X')) \mapsto \mathfrak{p}(\Gamma(g, X) - \Gamma(g', X')).$$

Then, for $t \in [r, r']$ fixed, there exists an open neighbourhood $W[t] \subseteq G$ of e, as well as $U[t] \subseteq \mathfrak{g}$ open with $0 \in U[t]$, such that

$$\alpha((\mu(t), \psi(t)), (g', Z')) < \varepsilon \qquad \forall\, (g', Z') \in \big[\mu(t) \cdot W[t]\big] \times \big[\psi(t) + U[t]\big] \tag{70}$$

holds. We choose

a) $V[t] \subseteq G$ open with $e \in V[t]$ and $V[t] \cdot V[t] \subseteq W[t]$.

b) $O[t] \subseteq \mathfrak{g}$ open with $0 \in O[t]$ and $O[t] + O[t] \subseteq U[t]$.

c) $J[t] \subseteq \mathbb{R}$ open with $t \in J$, such that for $D[t] := J[t] \cap [r, r']$, we have

$$\mu(D[t]) \subseteq \mu(t) \cdot V[t] \qquad \text{and} \qquad \psi(D[t]) \subseteq \psi(t) + O[t]. \tag{71}$$

Since $[r, r']$ is compact, there exist $t_0, \dots, t_n \in [r, r']$, such that $[r, r'] \subseteq D_0 \cup \dots \cup D_n$ holds.

- We define $V := V[t_0] \cap \dots \cap V[t_n]$.

  Since G is Mackey k-continuous, there exists some $N_\varepsilon \in \mathbb{N}$ with

$$\textstyle\int_r^\bullet \phi_n \in \int_r^\bullet \phi \cdot V \qquad \forall\, n \geq N_\varepsilon. \tag{72}$$

- We define $O := O[t_0] \cap \dots \cap O[t_n]$.

  Since $\{\psi_n\}_{n\in\mathbb{N}} \rightharpoonup_{\mathfrak{s}.0} \psi$ holds, increasing N_ε if necessary, we can achieve that

$$(\psi_n(t) - \psi(t)) \in O \qquad \forall\, t \in [r, r'],\ \ n \geq N_\varepsilon. \tag{73}$$

Then, for $\tau \in D_p$ with $0 \leq p \leq n$, and $n \geq N_\varepsilon$, we obtain from (72), (73), as well as (71) for $t \equiv t_p$ there that

- $\mu(t_p)^{-1} \cdot \int_r^\tau \phi_n = \big(\mu(t_p)^{-1} \cdot \mu(\tau)\big) \cdot \big([\int_r^\tau \phi]^{-1}[\int_r^\tau \phi_n]\big) \in V \cdot V \subseteq W[t_p]$,
- $\psi_n(\tau) - \psi(t_p) = (\psi_n(\tau) - \psi(\tau)) + (\psi(\tau) - \psi(t_p)) \in O + O \subseteq U[t_p]$.

The claim is thus clear from (70). $\square$

Proof of Lemma 19. For each $\chi, \chi' \in \mathfrak{D}^k_{[r,r']}$, we have, cf. (3)

$$\xi(\chi, \chi') = \mathrm{d}_{(\int \chi, e)}\, \mathrm{m}\big(0, \widehat{\Gamma}(\chi, \chi')\big) \qquad \text{for} \qquad \Gamma \equiv \mathrm{Ad}(\mathrm{inv}(\cdot), \cdot);$$

so that (50) holds by Lemma 20, because the Lie group multiplication is smooth. $\square$

E Appendix

Proof of Proposition 3.2). Let (ϕ, ψ) be regulated; and $\mu\colon (-\delta, \delta) \ni h \mapsto \int \phi + h \cdot \psi \in G$.

- Suppose that $\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s \in \mathfrak{g}$ holds; and let (shrink $\delta > 0$ if necessary)

$$\gamma\colon (-\delta, \delta) \to \mathcal{V}, \qquad h \mapsto \Xi([\textstyle\int \phi]^{-1}[\int \phi + h \cdot \psi]).$$

  Then, we have

$$\dot{\gamma}(0) \stackrel{\text{Part 1)}}{=} \textstyle\int (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{[\int_r^s \phi]^{-1}})(\psi(s))\, \mathrm{d}s \stackrel{(11)}{=} \mathrm{d}_e\Xi\big(\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s\big). \tag{74}$$

  Since $\mu = \Xi'^{-1} \circ \gamma$ (thus, $\gamma = \Xi' \circ \mu$) holds for the chart

$$\Xi'\colon \textstyle\int \phi \cdot \mathcal{U} =: \mathcal{U}' \to \mathcal{V}, \qquad g \mapsto \Xi([\int \phi]^{-1} \cdot g), \tag{75}$$

  (74) shows that μ is differentiable at 0 – Specifically, we have, cf. Remark 3

$$\begin{aligned}
\dot{\mu}(0) \ &\equiv\ \mathrm{d}_0\Xi'^{-1}\big(\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0} (\Xi' \circ \mu)(h)\big) \\
&\stackrel{(74)}{=} (\mathrm{d}_0\Xi'^{-1} \circ \mathrm{d}_e\Xi)\big(\textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s\big) \\
&=\ \big(\mathrm{d}_e\mathrm{L}_{\int \phi} \circ \mathrm{d}_0\Xi^{-1} \circ \mathrm{d}_e\Xi\big)\big(\textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s\big) \\
&=\ \mathrm{d}_e\mathrm{L}_{\int \phi}\big(\textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s\big);
\end{aligned}$$

  which shows (51).

- Suppose that μ is differentiable at $h=0$. Then, for Ξ' as in (75) we have, cf. Remark 3

$$E \ni \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} (\Xi' \circ \mu)(h) = \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} \Xi\big([\textstyle\int \phi]^{-1}[\int \phi + h\cdot\psi]\big) \stackrel{\text{Part 1)}}{=} \int (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{[\int_r^s \phi]^{-1}})(\psi(s))\, \mathrm{d}s.$$

We obtain

$$\mathfrak{g} \ni \mathrm{d}_0\Xi^{-1}\big(\textstyle\int (\mathrm{d}_e\Xi \circ \mathrm{Ad}_{[\int_r^s \phi]^{-1}})(\psi(s))\, \mathrm{d}s\big) \stackrel{(11)}{=} \int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s.$$

In particular, (51) holds by the previous point. □

F Appendix

Proof of Equation (57). We fix $I \equiv (\iota, \iota')$ with $\iota < r < r' < \iota'$, define $\psi \in C^0([\iota,\iota'],\mathfrak{g})$ by

$$\psi|_{[\iota,r)} := \phi(r) \qquad\qquad \psi|_{[r,r']} := \phi \qquad\qquad \psi|_{(r',\iota']} := \phi(r');$$

and observe that[13]

$$\mathfrak{X}\colon I \to \mathrm{dom}[\exp], \qquad x \mapsto -|r-\iota|\cdot\phi(r) + \textstyle\int_\iota^x \psi(s)\, \mathrm{d}s$$

fulfills the presumptions in Corollary 6, with

$$\exp\big(\textstyle\int_r^\bullet \phi(s)\, \mathrm{d}s\big) = \alpha|_{[r,r']} \qquad\qquad \text{for} \qquad\qquad \alpha \equiv \exp\circ\mathfrak{X}.$$

By Corollary 6, we thus have $\alpha \in C^1(I,G)$, with

$$\begin{aligned}
\delta^r(\alpha)(x) &= \tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\alpha(x+h)\cdot\alpha(x)^{-1}\\
&= \tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\alpha(x)^{-1}\cdot\alpha(x+h)\\
&= \mathrm{d}_{\exp(\mathfrak{X}(x))}\mathrm{L}_{\exp(-\mathfrak{X}(x))}\big(\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\alpha(x+h)\big)\\
&= \big(\mathrm{d}_{\exp(\mathfrak{X}(x))}\mathrm{L}_{\exp(-\mathfrak{X}(x))}\circ \mathrm{d}_e\mathrm{L}_{\exp(\mathfrak{X}(x))}\big)\big(\textstyle\int_0^1 \mathrm{Ad}_{\exp(-s\cdot\mathfrak{X}(x))}(\dot{\mathfrak{X}}(x))\, \mathrm{d}s\big)\\
&= \textstyle\int_0^1 \dot{\mathfrak{X}}(x)\, \mathrm{d}s = \dot{\mathfrak{X}}(x) = \psi(x) = \phi(x)
\end{aligned}$$

for each $x \in [r,r']$. Here, we have used in the second-, and the fifth step that G is abelian. □

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