Universal Dynamics of a Degenerate Bose Gas Quenched to Unitarity

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(Dated: December 18, 2018)

Motivated by an unexpected experimental observation from the Cambridge group, [Eigen *et al.*, Nature **563**, 221 (2018)], we study the evolution of the momentum distribution of a degenerate Bose gas quenched from the weakly interacting to the unitarity regime. For the two-body problem, we establish a relation that connects the momentum distribution at long time to a sub-leading term in the initial wave function. For the many-body problem, we employ the time-dependent Bogoliubov variational wave function and find that, in certain momentum regimes, the momentum distribution at long times displays the same exponential behavior found by the experiment. Moreover, we find that this behavior is universal and independent of the short-range details of the interaction potential. Consistent with the relation found in the two-body problem, we also numerically show that this exponential form is hidden in the same sub-leading term of the Bogoliubov wave function in the initial stages. Our results establish a consistent picture to understand the universal dynamics observed in the Cambridge experiment.

Because interactions in cold atomic systems are usually controlled by optical and magnetic fields, these can be tuned in a time scale that is much shorter than the relaxation time. Cold atomic systems are also very clean and the microscopic interactions between atoms can be understood very well in terms of universal low-energy effective interactions. Because of these two reasons, cold atoms are ideal for studying far-from-equilibrium dynamics in the many-body sector from a microscopic point of view. In equilibrium, there exists a lot of phenomena that are universally applicable, independently of the details of interactions at the microscopic scale. A major question for the non-equilibrium physics is whether such universal phenomena can also be found in far-from-equilibrium situations.

A recent experiment on strongly interacting Bose gases reveals a great surprise [1]. The system is initially prepared as a nearly pure Bose condensate at very low temperature and with weak interactions. Then the interaction is changed abruptly and the system is quenched to unitarity with the s-wave scattering length being infinite. The subsequent many-body dynamics was monitored by observing the evolution of the momentum distribution $n_{\mathbf{k}}$. A prethermalization stage was found where $n_{\mathbf{k}}$ remains a constant for a long time. The most surprising finding in the experiment is that $n_{\mathbf{k}}$ has a functional form

$$n_{\mathbf{k}} \sim e^{-\Lambda k/k_n}, \tag{1}$$

where $k = |\mathbf{k}|$, $k_n = (6\pi^2 n)^{1/3}$ is a momentum scale, and $\Lambda = 3.62$ is obtained from a fit to the experimental data [1]. This functional form is seen to be valid for k ranging from $\sim k_n$ to a few times k_n. There are a number of previous theoretical works that have studied weakly interacting Bose gases quenched to the strongly interacting regime [2–12], with either finite or infinite scattering lengths. However, this phenomenon has not been predicted by any theory before.

Here we focus on understanding of the origin of the emergent exponential behavior of $n_{\mathbf{k}}$ and answering whether this functional form is universal or not. Here we should note that at unitarity, because there is no other length scale, both the two-body collisional rate and the three-body loss rate are proportional to E_n, where E_n is given by $\hbar^2 k_n^2/(2m)$. However, it has been shown previously that the coefficient for the two-body collisional rate is much larger than that for the three-body loss rate [13]. That is to say, the many-body dynamics is governed by two-body collisions for long time before the three-body loss takes over and heats the system up. Therefore, it is very reasonable to view both the prethermalization and processes occurring before that as caused by the two-body collisions, while temperature increasing at later times is due to three-body losses. This separation of time scales allows us to safely ignore the three-body loss and only focus on two-body collisional effects when analyzing the prethermal dynamics. In other words, we can view the prethermalization regime as the long time limit of the dynamics governed by the two-body collisions. This is also the spirit of our analysis below.

In this letter we address this issue from both two-body and many-body perspectives. The main results can be summarized as follows:

I. For the two-body problem, we prove a relation between the long time behavior of the momentum distribution and the properties of the initial wave function. This relation works for arbitrary short-range potentials. With this theorem, we can determine which property of the initial wave function is responsible for the exponential form of Eq. (1) in the momentum distribution at long time.

II. For the many-body problem, we employ a variational time-dependent Bogoliubov wave function and by solving the time-dependent equation, we find that for a certain range of momentum, the averaged $n_{\mathbf{k}}$ for long time evolution indeed obeys the form of Eq. (1). We use three different potentials tuned near the vicinity of a scattering resonance: the square well, the Gaussian potential and the Yukawa potential and find that this behavior is independent of the short-range details.

At the end, we also discuss the connection between the two-body and the many-body results.

Two-body Problem. Let us first start with the two-body problem whose Hamiltonian can be written in terms of the relative coordinate $\mathbf{r}$ as $\hat{H} = \hat{H}_0 + \hat{V}(\mathbf{r})$, where $\hat{H}_0 = -\hbar^2\nabla^2/m$ is the kinetic energy with m being the mass, and $\hat{V}(\mathbf{r})$ is a short-range potential. Here we choose $t = 0$ as the time right after the quench of interactions and we denote the initial wave function $|\phi^i\rangle$. During the evolution, we focus on the situation where $V(\mathbf{r})$ is at an s-wave resonance and we only consider the s-wave interaction. The momentum distribution $n_{\mathbf{k}}(t)$ at momentum $\mathbf{k}$ and time t is given by

$$n_{\mathbf{k}}(t) = |\langle\mathbf{k}|e^{-\frac{i}{\hbar}\hat{H}t}|\phi^i\rangle|^2 = |\langle\mathbf{k}|e^{\frac{i}{\hbar}\hat{H}_0 t}e^{-\frac{i}{\hbar}\hat{H}t}|\phi^i\rangle|^2, \quad (2)$$

where $|\mathbf{k}\rangle$ is a plane wave state. The second equality follows from the fact that $|\mathbf{k}\rangle$ is an eigenstate of $\hat{H}_0$ and $e^{\frac{i}{\hbar}\hat{H}_0 t}$ only gives rise to a phase factor that does not change $n_{\mathbf{k}}$. Furthermore, making use of the properties of the Møller operator [14]

$$\hat{\Omega}^{(-)} = \lim_{t\to+\infty} e^{\frac{i}{\hbar}\hat{H}t}e^{-\frac{i}{\hbar}\hat{H}_0 t} \quad (3)$$

in scattering theory, we can derive the following relation

$$n_{\mathbf{k}}(t\to+\infty) = |\langle\mathbf{k}|\hat{\Omega}^{(-)\dagger}|\phi^i\rangle|^2 = |\langle\mathbf{k}^{(-)}|\phi^i\rangle|^2. \quad (4)$$

Here $|\mathbf{k}^{(-)}\rangle$ is the inward scattering wave function defined as [14]

$$|\mathbf{k}^{(-)}\rangle = |\mathbf{k}\rangle + \frac{1}{\epsilon_{\mathbf{k}} + i0^- - \hat{H}}\hat{V}|\mathbf{k}\rangle, \quad (5)$$

where $\epsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/m$. For a short-range potential, it is straightforward to show that outside the range of interaction, $\langle\mathbf{r}|\mathbf{k}^{(-)}\rangle$ behaves as [14]

$$\langle\mathbf{r}|\mathbf{k}^{(-)}\rangle = \frac{1}{(2\pi)^{3/2}}\left(e^{i\mathbf{kr}} + \frac{1}{ik}\frac{e^{-ikr}}{r}\right). \quad (6)$$

where $r = |\mathbf{r}|$ and $k = |\mathbf{k}|$.

With the relation Eq. (4), we can determine the requirement for the initial wave function $|\phi^i\rangle$ that can lead to the long-time behavior of Eq. (1) in $n_{\mathbf{k}}$. It is important to know that $\{|\mathbf{k}^{(-)}\rangle\}$ form a complete and orthogonal basis, and we can expand the wave function in terms of this basis. Let us introduce

$$\psi(\mathbf{k}) = e^{i\theta(\mathbf{k})}\sqrt{n_{\mathbf{k}}(t\to+\infty)}, \quad (7)$$

then the exponential form of $n_{\mathbf{k}}$ will translate to the same kind of exponential dependence for $\psi(\mathbf{k})$ up to a phase factor. We can then write the initial wave function ϕ^i as

$$|\phi^i\rangle = \int d^3\mathbf{p}\psi(\mathbf{p})|\mathbf{p}^{(-)}\rangle, \quad (8)$$

and in the momentum space

$$\phi^i(\mathbf{k}) = \langle\mathbf{k}|\phi^i\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{r}d^3\mathbf{p}e^{-i\mathbf{kr}}\psi(\mathbf{p})\langle\mathbf{r}|\mathbf{p}^{(-)}\rangle. \quad (9)$$

Considering the situation where $\psi(\mathbf{p})$ is isotropic, i.e. it can be written as $\psi(p)$ with $p = |\mathbf{p}|$, we can substitute Eq. (6) into Eq. (9) and integrate out the azimuthal degrees of freedom, we find

$$\phi^i(k) = \frac{1}{2\pi}\left(-\frac{i}{k}\right)\lim_{\epsilon\to0^+}\int_0^{+\infty} dpp\psi(p) \times\left[\sum_{\sigma=\pm}\frac{\sigma}{p+\sigma(k+i\epsilon)} + \sum_{\sigma'=\pm}\frac{\sigma'}{p+\sigma'(k-i\epsilon)}\right]. \quad (10)$$

Let us introduce an auxiliary function $\Psi(z)$ in the complex plane, such that it satisfies the requirement at the positive side of the real axis $\Psi(z=p>0) = \psi(p)$ and at its negative side $\Psi(z=p<0) = -\psi(-p)$ [15]. With the help of this auxiliary function, it can be shown that

$$\phi^i(k) = -\psi(k) - \frac{1}{k}\sum_j\frac{\mathrm{Res}\,[2z\Psi(z)]_{z=z_j}}{z_j - k}, \quad (11)$$

where $\mathrm{Res}[f(z)]_{z=z_j}$ denotes the residue of the function $f(z)$ at its pole z_j.

Eq. (11) is a very interesting result. Here we should note that the amplitude of $\psi(k)$ obeys this universal exponential form only at the momentum $\gtrsim k_n$, however, the auxiliary function $\Psi(z)$ will certainly depend on the small momentum behavior of $\psi(k)$. Therefore, the residues of $\Psi(z)$, as well as the coefficient for this second term in the r.h.s. of Eq. (11), are non-universal. When $\psi(k)$ is a regular function as in Eq. (1) and Eq. (7), the second term recovers the well-known $1/k^4$ behavior of the momentum distribution at large k. Hence, Eq. (11) tells us that, one can subtract the leading order $1/k^2$ term from fitting the large momentum, and the remaining regular sub-leading term reveals the momentum distribution at long time. That is to say that, in order for the long time behavior of $n_{\mathbf{k}}$ to obey Eq. (1), the sub-leading term of the initial wave function has to obey the form given by Eq. (1) and Eq. (7).

Many-Body Problem. Now we turn into the many-body problem whose Hamiltonian can be written in second quantized form as

$$\hat{\mathcal{H}} = \sum_{\boldsymbol{k}}\epsilon_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \frac{1}{2L^3}\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}}\hat{a}^\dagger_{\boldsymbol{k}+\boldsymbol{q}}\hat{a}^\dagger_{\boldsymbol{k}'-\boldsymbol{q}}V(\boldsymbol{q})\hat{a}_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}}. \quad (12)$$

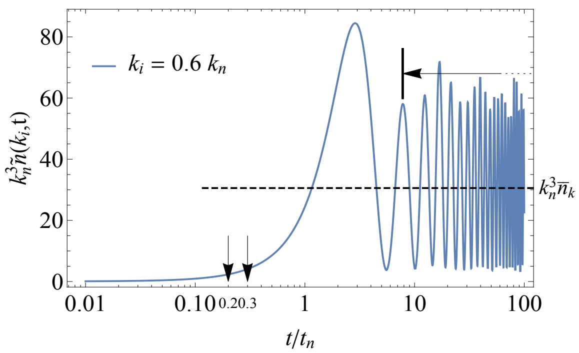

FIG. 1: A typical value of the normalized $\tilde{n}_{\mathbf{k}}(t)$ (in unit of $1/k_n^3$) is plotted as a function of t. Here we take $k = 0.6k_n$ and the interaction potential is the Yukawa potential. The horizontal line with arrow indicates the time domain in which we take average of $\tilde{n}_{\mathbf{k}}(t)$ to obtain $\bar{n}_{\mathbf{k}}$. The two vertical arrows indicates two time slots where the wave function is plotted in Fig. 3.

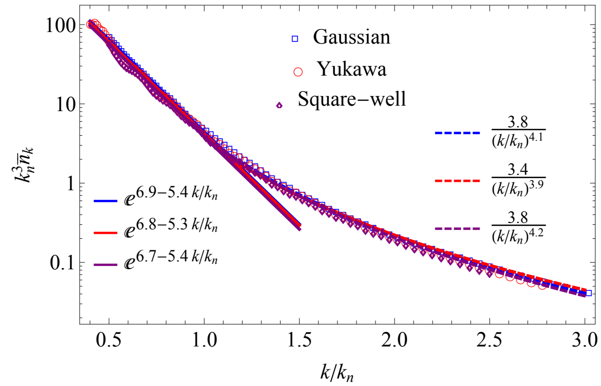

FIG. 2: (color online) $\bar{n}_{\mathbf{k}}$ (in unit of $1/k_n^3$) is plotted as a function of k/k_n. Three different microscopic potentials are used in the calculation.

Here $\hat{a}^\dagger_{\boldsymbol{k}}(\hat{a}_{\boldsymbol{k}})$ is the creation (annihilation) operator for bosons with momentum $\mathbf{k}$. L^3 is the system's volume. $V(\boldsymbol{q}) = \int d^3\boldsymbol{r} e^{i\boldsymbol{q}\boldsymbol{r}} V(\boldsymbol{r})$ is the Fourier transform of the interaction potential $V(\mathbf{r})$. We implement the Bogoliubov-type variational ansatz

$$|\Phi(t)\rangle = \mathcal{A}(t) \exp\left[g_0(t)\hat{a}_0^\dagger + \sum_{\boldsymbol{k}\cdot\hat{z}>0} g_{\boldsymbol{k}}(t) \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}^\dagger_{-\boldsymbol{k}} \right] |0\rangle. \tag{13}$$

Here $\mathcal{A}(t)$ is a normalization factor, $|0\rangle$ is the particle vacuum; g_0 and $g_{\boldsymbol{k}}$ are variational parameters, which can determine $N_0(t) = |g_0|^2$ and $N_{\mathbf{k}}(t) = |g_{\boldsymbol{k}}|^2/(1-|g_{\boldsymbol{k}}|^2)$ as the particle number at zero-momentum and finite momentum $\mathbf{k}$ modes. The Bogoliubov ansatz assumes that the system remains as a Bose condensate during the entire dynamics, which is indeed the case for this experiment.

The dynamical equations for $g_0(t)$ and $g_{\mathbf{k}}(t)$ can be obtained from the Euler-Lagrange equation for the Lagrangian $\mathcal{L} = i\hbar \left[\langle \Phi|\dot{\Phi}\rangle - \langle \Phi|\dot{\Phi}\rangle \right] - \langle \Phi|\hat{H}|\Phi\rangle$. It results in coupled equations for g_0 and $g_{\mathbf{k}}$ as [3]

$$\begin{aligned} i\hbar \dot{g}_0 &= nV(0)g_0 + \frac{1}{L^3}\sum_{\boldsymbol{k}\neq \boldsymbol{0}} V(\boldsymbol{k}) \frac{g_0^* g_{\boldsymbol{k}} + g_0|g_{\boldsymbol{k}}|^2}{1-|g_{\boldsymbol{k}}|^2}, \\ i\hbar \dot{g}_{\boldsymbol{p}} &= 2[\epsilon_{\boldsymbol{p}} + nV(0)]g_{\boldsymbol{p}} + \frac{V(\boldsymbol{p})}{L^3}[g_0^2 + g_0^{*2} g_{\boldsymbol{p}}^2 + 2|g_0|^2 g_{\boldsymbol{p}}] \\ &\quad + \frac{1}{L^3}\sum_{\boldsymbol{k}\neq \boldsymbol{0}} V(\boldsymbol{p}-\boldsymbol{k}) \frac{2|g_{\boldsymbol{k}}|^2 g_{\boldsymbol{p}} + g_{\boldsymbol{k}} + g_{\boldsymbol{k}}^* g_{\boldsymbol{p}}^2}{1-|g_{\boldsymbol{k}}|^2}. \end{aligned} \tag{14}$$

The total number $N = N_0 + \sum_{\mathbf{k}\neq 0} N_{\mathbf{k}}(t)$ is a conserved quantity, and $n = N/L^3$ is the total density. Making use of the spherical symmetry of this system, we consider that $g_{\mathbf{k}}$ only depends on k and can be simplified as g_k, and we can further simplify this equation by performing the azimuthal integration first. Without loss of generality, we take the initial state to be a pure BEC, i.e. $g_0(0) = \sqrt{N}$ and all $g_{\boldsymbol{k}}(0) = 0$. As in the two-body case, we start the time evolution right after the interaction quench, and therefore we set the interaction at scattering resonance.

Here, to verify whether the dynamics is universal, that is to say, whether it depends on the short-range details, we consider three different short-range potentials:

(i) The square well potential: $V_{\rm SW}(\boldsymbol{r}) = -\frac{\hbar^2\gamma_{\rm s}}{mr_0^2}\Theta(r_0 - r)$, where Θ is the Heaviside step function. The s-wave resonance occurs at $\gamma_{\rm s} = (\pi/2)^2$.

(ii) The Gaussian potential: $V_{\rm GW}(\boldsymbol{r}) = -\frac{\hbar^2\gamma_{\rm g}}{mr_0^2} e^{-r^2/r_0^2}$, and the s-wave resonance occurs at $\gamma_{\rm g} \approx 2.68$.

(iii) The Yukawa potential: $V_{\rm YW}(\boldsymbol{r}) = -\frac{\hbar^2\gamma_{\rm y}}{mr_0}\frac{e^{-r/r_0}}{r}$, and the s-wave resonance occurs at $\gamma_{\rm y} \approx 21.1$.

We numerically solve the coupled equations Eq. (14) with these three potentials by discretizing both the radial momentum and the time, from which we can obtain $g_{\mathbf{k}}(t)$ and $N_{\mathbf{k}}(t)$. Following Ref. [1], we introduce a normalized momentum distribution

$$\tilde{n}_{\mathbf{k}}(t) = \frac{L^3 N_{\mathbf{k}}(t)}{N_{\rm ex}(t)}, \tag{15}$$

such that $\frac{1}{L^3}\sum_{\mathbf{k}\neq 0} \tilde{n}_{\mathbf{k}}(t) = 1$, where $N_{\rm ex}(t) = \sum_{\mathbf{k}\neq 0} N_{\mathbf{k}}(t)$ is the total number of the excited atoms. In Fig. 1 we plot $\tilde{n}_{\mathbf{k}}(t)$ as a function of t. One can see that following a growth at the initial stage, $\tilde{n}_{\mathbf{k}}(t)$ exhibits an oscillatory behavior for $t \gg t_n$, where t_n is a typical time scale defined as $t_n = \hbar/E_n$. We take a long time average of $\tilde{n}_{\mathbf{k}}(t)$ starting from the second peak in the oscillation, as indicated in Fig. 1. The average is denoted by $\bar{n}_{\mathbf{k}}$, which is taken as the long time saturated value of the momentum distribution.

In Fig. 2, we plot the dimensionless quantity $k_n^3 \bar{n}_{\mathbf{k}}$ as a function of k/k_n. The fit shows a regime around

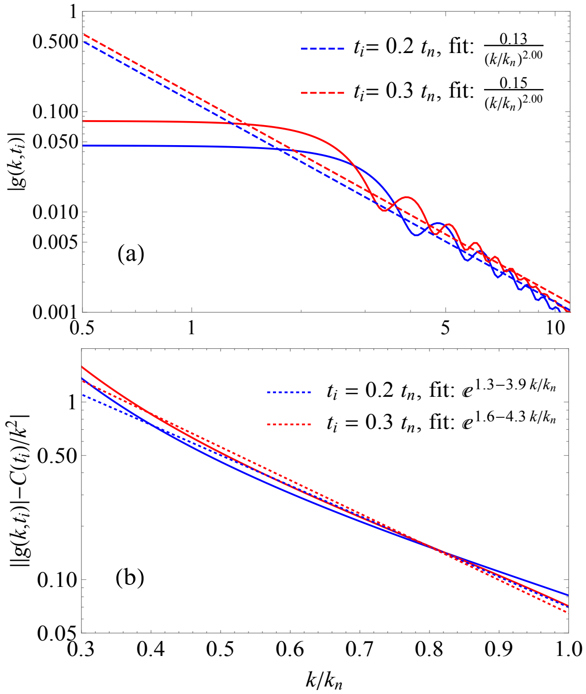

FIG. 3: (color online) The momentum space Bogoliubov wave function $g_{\mathbf{k}}$ for two time slots at the early time $t = 0.2t_n$ (blue lines) and $t = 0.3t_n$ (red lines) as marked in Fig. 1. In (a), the log-log plot shows that the large k part of $|g_k|$ can be well fitted by $\sim 1/k^2$. In (b), the log plot shows that for the intermediate $k \sim k_n$, after subtracting $1/k^2$ part, the sub-leading term can be well fitted by $\sim e^{-\Lambda k/k_n}$.

$k \sim k_n$, where $\bar{n}_{\mathbf{k}}$ behaves as Eq. (1), consistent with the experimental observation in Ref. [1]. This fitting yields a coefficient $\Lambda = 5.3 - 5.4$. For large k, the fitting yields a $1/k^4$ behavior. Most importantly, we note that the curves obtained using the three different potentials defined above collapse onto one another, which shows that this emergent exponential behavior of the momentum distribution is independent of the short-range details of the interaction potential.

Connection between the Two- and Many-Body Problems. To summarize the results above, on one hand, our discussion on the two-body problem has established a relation between $n_{\mathbf{k}}$ at the long time and the wave function behavior at the initial time; and on the other hand, our Bogoliubov calculation for the many-body problem has discovered the exponential form for $n_{\mathbf{k}}$ at the long time as observed in the experiment reported in Ref. [1]. Now a natural question is whether the same relation also holds in the Bogoliubov wave function, namely, whether the exponential form is also hidden in the sub-leading term of the Bogoliubov wave function at short times. To check this conjecture, we look into the wave function g_k at times $t < t_n$, far before the saturation of the momentum distribution, as indicated by arrows in Fig. 1. This early stage wave function is reminiscent of the initial wave function in the two-body case. In Fig. 3(a), we plot $|g_k|$ as a function of k/k_n, which does not show any exponential behavior, and the large k part can be well fitted by a $1/k^2$ tail. In Fig. 3(b), following the same spirt of Eq. (11) discovered in the two-body problem, we subtract the $1/k^2$ part in $|g_k|$, and plot the sub-leading term as a function of k/k_n. Interestingly, in the same momentum range where the long time $\bar{n}_{\mathbf{k}}$ plotted in Fig. 2 shows an exponential behavior, this sub-leading term in the early time wave function plotted in Fig. 3(b) also shows an exponential behavior. Notice that we start from an initial state with all atoms in the zero-momentum state, the exponential behavior at the early stage wave function may originate from the pair production process, as discussed in Ref. [16].

Comments on Comparison with Experiment. Finally we want to emphasize that the agreement between our Bogoliubov theory and the experiment is only qualitative. Experimentally, this exponential behavior of $n_{\mathbf{k}}$ is valid up to $\sim 3k_n$ and they do not find $1/k^4$ behavior, but in our case it is only up to $\sim k_n$ and is followed by a $1/k^4$ tail at higher momenta. The value of Λ is also somewhat different between our calculation and the experimental result. However, since the system is strongly interacting, we do not expect the mean-field type Bogoliubov theory to be quantitatively accurate anyway. Moreover, our calculation leads to a very fast oscillation of $n_{\mathbf{k}}$ at long times, although its mean value saturates. We expect that such a fast oscillation can be smeared out by decoherence mechanism in reality. In experiment, the momentum distribution eventually takes off again after a prethermalization plateau, and this is caused by the heating due to the three-body loss which we do not include in our theory. Our results offer valuable insight for understanding this observation but more involved theory is required for a more quantitative comparison with experiment.

Note Added. When finishing this paper, we became aware of another preprint, Ref. [17], which also did the many-body calculation with the Bogoliubov wave function.

Acknowledgment. We thank Wei Zheng, Manuel Valiente, Xin Chen, Zeng-Qiang Yu, Shizhong Zhang, Ran Qi for inspiring discussion. We are particularly grateful to Manuel Valiente for carefully reading our manuscript and helpful suggestions. This work is supported by NSFC Grant No. 11734010, 11604300, 11835011, 11774315, and MOST under Grant No. 2016YFA0301600.

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