Toeplitz operators with analytic symbols

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Abstract

We provide asymptotic formulas for the Bergman projector and Berezin-Toeplitz operators on a compact Kähler manifold. These objects depend on an integer N and we study, in the limit $N \to +\infty$, situations in which one can control them up to an error $O(e^{-cN})$ for some $c > 0$.

We develop a calculus of Toeplitz operators with real-analytic symbols, which applies to Kähler manifolds with real-analytic metrics. In particular, we prove that the Bergman kernel is controlled up to $O(e^{-cN})$ on any real-analytic Kähler manifold as $N \to +\infty$. We also prove that Toeplitz operators with analytic symbols can be composed and inverted up to $O(e^{-cN})$. As an application, we study eigenfunction concentration for Toeplitz operators if both the manifold and the symbol are real-analytic. In this case we prove exponential decay in the classically forbidden region.

1 Introduction

Toeplitz quantization associates, to a real-valued function f on a compact Kähler manifold M, a family of *Toeplitz operators*, which are self-adjoint linear operators $(T_N(f))_{N\in\mathbb{N}}$ acting on holomorphic sections over M. Examples of Toeplitz operators are spin operators (where $M = \mathbb{S}^2$), which are indexed by the total spin $S = \frac{N}{2} \in \frac{1}{2}\mathbb{N}$. In this paper we study *exponential estimates*, that is, approximate expressions with $O(e^{-cN})$ remainder for some $c > 0$.

The family of holomorphic section spaces in Toeplitz quantization is described by a sequence of *Bergman projectors* $(S_N)_{N\geq 1}$. The operators S_N can be written as integral operators (the integral kernels are sections of suitable bundles over $M \times M$), and a first step toward understanding Toeplitz Toeplitz is the asymptotic study, in the limit $N \to +\infty$, of the Bergman kernel.

We show (Theorem A) that the Bergman kernel admits an asymptotic expansion in decreasing powers of N, up to an error $O(e^{-cN})$, as soon as the Kähler manifold is real-analytic. To study the Bergman projector, as well as compositions of Toeplitz operators (Theorem B), it is useful to interpret the $N \to +\infty$ limit as a semiclassical limit (with semiclassical parameter $\hbar = \frac{1}{N}$), and to use tools which were developed for the study of Partial Differential Equations (PDEs) with small parameters. We build new semiclassical tools in real-analytic regularity (in particular, new analytic symbol classes, see Definition 3.3), which can be of more general use.

This study of the calculus of Toeplitz operators allows us to state results concerning sequences of eigenfunctions of Toeplitz operators $(T_N(f))_{N\geq 1}$ for a real-analytic f. We prove the following (Theorem C): if

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$(u_N)_{N\geq 1}$ is a sequence of normalised eigenfunctions with energy near $E \in \mathbb{R}$, that is,

$$T_N(f)u_N = \lambda_N u_N, \qquad \lambda_N \underset{N\to+\infty}{\to} E, \qquad \|u_N\|_{L^2(M,L^{\otimes N})} = 1,$$

and if $V \subset M$ is an open set at positive distance from $\{x \in M, f(x) = E\}$, then u_N is uniformly controlled by $O(e^{-cN})$ on V; We say that $(u_N)_{N\in\mathbb{N}}$ has an *exponential decay rate* on V.

In a forthcoming paper, we provide an asymptotic expansion, with $O(e^{-cN})$ error, for the ground state of a Toeplitz operator $T_N(f)$, for f real-analytic and Morse.

1.1 Bergman kernels and Toeplitz operators

This article is devoted to the study of exponential estimates concerning the Bergman kernel and Toeplitz operators on Kähler manifolds with real-analytic data. In this subsection we quickly recall the framework of Toeplitz quantization. We refer the reader to more detailed introductions [2, 5, 8].

Definition 1.1.

- A compact Kähler manifold (M, J, ω) is said to be quantizable when the symplectic form ω has integer cohomology: there exists a unique Hermitian line bundle (L, h) over M such that the curvature of h is $-2i\pi\omega$. This line bundle is called the prequantum line bundle over (M, J, ω). The manifold (M, J, ω) is said to be *real-analytic* when ω (or, equivalently, h) is real-analytic on the complex manifold (M, J).

- Let (M, J, ω) be a quantizable compact Kähler manifold with (L, h) its prequantum bundle and let $N \in \mathbb{N}$.
  - The Hardy space $H_0(M, L^{\otimes N})$ is the space of holomorphic sections of the N-th tensor power of L. It is a finite-dimensional, closed subspace of the space $L^2(M, L^{\otimes N})$ of all square-integrable sections of $L^{\otimes N}$.
  - The Bergman projector S_N is the orthogonal projector from $L^2(M, L^{\otimes N})$ to $H_0(M, L^{\otimes N})$.
  - The contravariant Toeplitz operator associated with a symbol $f \in L^\infty(M, \mathbb{C})$ is defined as

$$\begin{aligned} T_N(f) : H_0(M, L^{\otimes N}) &\to H_0(M, L^{\otimes N}) \\ u &\mapsto S_N(fu). \end{aligned}$$

In a related way, one can define *covariant* Toeplitz operators, which are kernel operators acting on $H_0(M, L^{\otimes N})$ (see Definition 4.1). We are interested the Bergman projector and both types of Toeplitz operators in the *semiclassical limit* $N \to +\infty$.

A particular motivation for the study of Toeplitz operators is the quantization, on $M = (\mathbb{S}^2)^d$, of polynomials in the coordinates (in the standard immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$). The operators obtained are spin operators, with total spin $\frac{N}{2}$. Tunnelling effects in spin systems, in the large spin limit, are widely studied in the physics literature (see [25] for a review). This article also aims at giving a mathematical ground to this study.

We will use the following estimate on the operator $\overline{\partial}$ acting on $L^2(M, L^{\otimes N})$ and the Bergman projector S_N:

Proposition 1.2. *Let (M, ω, J) be a compact quantizable Kähler manifold and $(S_N)_{N\geq 1}$ be the associated sequence of Bergman projectors. There exists $C > 0$ such that, for every $N \geq 1$ and $u \in L^2(M, L^{\otimes N})$, one has:*

$$\|\overline{\partial}u\|_{L^2} \geq C\|u - S_N u\|_{L^2}. \tag{1}$$

This estimate follows from the work of Kohn [16, 17], and relies on the basic theory of unbounded operators on Hilbert spaces; it is widely used in the asymptotic study of the Bergman kernel, where it is sometimes named after Hörmander or Kodaira.

The Bergman projector S_N admits a kernel, in a sense which we make precise here. The space $H_0(M, L^{\otimes N})$ is finite-dimensional, so that it is spanned by a Hilbert base $s_1, \ldots, s_{d_N}$ of holomorphic sections of $L^{\otimes N}$. The following section of $L^{\otimes N} \boxtimes \overline{L}^{\otimes N}$ is the integral kernel of the Bergman projector:

$$S_N(x,y) = \sum_{i=1}^{d_N} s_i(x) \otimes \overline{s_i(y)}.$$

Here $\overline{L}$ is the complex conjugate bundle of L, and $\boxtimes$ stands for pointwise direct product: $L^{\otimes N} \boxtimes \overline{L}^{\otimes N}$ is a bundle over $M \times M$.

More generally, any section of $L^{\otimes N} \boxtimes \overline{L}^{\otimes N}$ gives rise to an operator on $L^2(M, L^{\otimes N})$.

1.2 Statement of the main results

We begin with the definition of what will be the phase of the Bergman kernel. We use the standard notion of holomorphic extensions of real-analytic functions and manifolds, under a notation convention which is recalled in detail in Section 2.3.

Definition 1.3 (A section of $L^{\otimes N} \boxtimes \overline{L}^{\otimes N}$)**.** Let M be a real-analytic Kähler manifold and let $U \subset M$ be a contractible open set.

Let s denote a non-vanishing, bounded, holomorphic section of L on U. Then $\phi = -\frac{1}{2}\log(|s|_h^2)$ is called a *Kähler potential* on U. The function ϕ is real-analytic on U since h is real-analytic, so that there is a unique function $\widetilde{\phi}$ on a neighbourhood of the diagonal of $U \times U$, which is holomorphic in the first variable and anti-holomorphic in the second variable, and such that $\widetilde{\phi}(x,x) = \phi(x)$. We call *holomorphic extension* such a $\widetilde{\phi}$. (This coincides with the usual notion of holomorphic extension, see Subsections 2.2 and 2.3 for details).

The function $(x,y) \mapsto e^{2N\widetilde{\phi}(x,y)}$ is well-defined in a neighbourhood of the diagonal in $U \times U$, so that the following section of $(L \boxtimes \overline{L})^{\otimes N}$

$$\Psi_s^N : (x,y) \mapsto (s(x))^{\otimes N} \otimes (\overline{s(y)})^{\otimes N} e^{2N\widetilde{\phi}(x,y)}.$$

is well-defined in a neighbourhood of the diagonal of $U \times U$, holomorphic in the first variable and anti-holomorphic in the second variable.

The section Ψ_s^N is independent of the holomorphic chart on U. It is also independent of the choice of s. Indeed, if s' is another non-vanishing holomorphic section of L on U, one has $s' = e^f s$ where f is a holomorphic function on U. In particular, the associated Kähler potential $\phi' = -\frac{1}{2}\log(|s'|_h^2)$ satisfies

$$\phi' = \phi + \frac{1}{2}(f + \overline{f}),$$

so that

$$\widetilde{\phi}'(x,y) = \widetilde{\phi}(x,y) + \frac{1}{2}(f(x) + \overline{f(y)});$$

hence

$$\Psi_{s'}^N(x,y) = \Psi_s^N(x,y) e^{-N(f(x) - f(x) + \overline{f(y) - f(y)})} = \Psi_s^N(x,y).$$

As this section does not depend on s, we call it now Ψ_U^N. In particular, given two contractible open sets $U \cap V$, one has $\Psi_U^N = \Psi_V^N$ near the diagonal of $U \cap V$. Hence, there exists a section Ψ^N of $L^{\otimes N} \boxtimes \overline{L}^{\otimes N}$ on a neighbourhood of the diagonal in $M \times M$, whose restriction to each open set U is $\Psi_U^{\otimes N}$.

Note that the domain of definition of Ψ^N is independent of N.

In the general setting of a Kähler manifold with real-analytic data, it has been conjectured [12] that the Bergman kernel takes the following form: for some $c > 0$, for all $(x,y) \in M^2$,

$$S_N(x,y) = \Psi^N(x,y) \sum_{k=0}^{cN} N^{d-k} a_k(x,y) + O(e^{-cN}),$$

where the a_k are, in a neighbourhood of the diagonal in $M \times M$, holomorphic in the first variable and anti-holomorphic in the second variable, with

$$\|a_k\|_{C^0} \leq CR^k k!.$$

The well-behaviour of such sequences of functions when the sum $\sum N^{-k} a_k$ is computed up to the rank cN with $c < e/2R$ was first observed in [28] and was the foundation for a theory of analytic pseudodifferential operators and Fourier Integral Operators. Here, we rely on more specific function classes, where we control successive derivatives of the a_k's. Without giving a precise definition at this stage let us call "analytic symbols" such well-controlled sequences of real-analytic functions. See Definition 3.3 about the analytic symbol spaces $S^{r,R}_m(X)$ and the associated summation. This allows us to prove the conjecture above:

Theorem A. *Let M be a quantizable compact real-analytic Kähler manifold of complex dimension d. There exists positive constants r, R, m, c, c', C, a neighbourhood U of the diagonal in $M \times M$, and an analytic symbol $a \in S^{r,R}_m(U)$, holomorphic in the first variable, anti-holomorphic in the second variable, such that the Bergman kernel S_N on M satisfies, for each $x, y \in M \times M$ and $N \geq 1$:*

$$\left\| S_N(x,y) - \Psi^N(x,y) \sum_{k=0}^{cN} N^{d-k} a_k(x,y) \right\|_{h^{\otimes N}} \leq Ce^{-c'N}.$$

Equivalently, the operator with kernel given by $\Psi^N(x,y)\sum_{k=0}^{cN} N^{d-k} a_k(x,y)$ is exponentially close (in the $L^2 \mapsto L^2$ operator sense) to the Bergman projector.

Theorem A also appears in recent and independent work [27], where the authors use Local Bergman kernels as developed in [1] to study locally the Bergman kernel as an analytic Fourier Integral Operator.

In order to study contravariant Toeplitz operators of Definition 1.1, as well as the Bergman kernel itself, it is useful to consider *covariant* Toeplitz operators, which are the object of the next Theorem. Recalling the section Ψ^N of Definition 1.3, for f an analytic symbol on $M \times M$, which is, near the diagonal, holomorphic in the first variable and anti-holomorphic in the second variable, the associated covariant Toeplitz operator is defined as the operator with kernel:

$$T_N^{cov}(f)(x,y) = \Psi^N(x,y) \left(\sum_{k=0}^{cN} N^{d-k} f_k(x,y) \right),$$

for some small $c > 0$; see Definition 4.1.

Theorem B. *Let M be a quantizable compact real-analytic Kähler manifold. Let f and g be analytic symbols on a neighbourhood U of the diagonal in $M \times M$, which are holomorphic in the first variable and anti-holomorphic in the second variable.*

Then there exists $c' > 0$ and an analytic symbol $f \sharp g$ on the same neighbourhood U, holomorphic in the first variable and anti-holomorphic in the second variable, and such that

$$T_N^{cov}(f) T_N^{cov}(g) = T_N^{cov}(f \sharp g) + O(e^{-c'N}).$$

For any r, R, m large enough, the product $\sharp$ is a continuous bilinear application from $S^{r,R}_m(U) \times S^{2r,2R}_m(U)$ to $S^{2r,2R}_m(U)$ (see Definition 3.3); the constant c' depends only on r, R, m.

If the principal symbol of f does not vanish on M then there is an analytic symbol $f^{\sharp-1}$ such that, for some $c' > 0$, one has

$$T_N^{cov}(f)T_N^{cov}(f^{\sharp-1}) = S_N + O(e^{-c'N}).$$

Given an analytic symbol $f \in S^{r_0,R_0}_{m_0}(U)$ with non-vanishing subprincipal symbol, there exists $C > 0$ such that for every r, R, m large enough (depending on f, r_0, R_0, m_0), one has

$$\|f^{\sharp-1}\|_{S^{r,R}_m(U)} \leq C\|f\|_{S^{r,R}_m(U)}.$$

As an application of composition and inversion properties, one can study the concentration rate of eigenfunctions, in the general case (exponential decay in the forbidden region) as well as in the particular case where the principal symbol has a non-degenerate minimum.

Theorem C. *Let M be a quantizable compact real-analytic Kähler manifold. Let f be a real-analytic, real-valued function on M and $E \in \mathbb{R}$. Let $(u_N)_{N\geq 1}$ be a normalized sequence of $(\lambda_N)_{N\geq 1}$-eigenstates of $T_N(f)$ with $\lambda_N \underset{N\to+\infty}{\to} E$. Then, for every open set V at positive distance from $\{f = E\}$ there exist positive constants c, C such that, for every $N \geq 1$, one has*

$$\int_V \|u_N(x)\|_h^2 \frac{\omega^{\wedge n}}{n!}(dx) \leq Ce^{-cN}.$$

We say informally that, in the forbidden region $\{f \neq E\}$, the sequence $(u_N)_{N\in\mathbb{N}}$ has an exponential decay rate.

1.3 Exponential estimates in semiclassical analysis

Exact or approximate eigenstates of quantum Hamiltonians are often searched for in the form of a Wentzel-Kramers-Brillouin (WKB) ansatz:

$$e^{\frac{\phi(x)}{\hbar}}(a_0(x) + \hbar a_1(x) + \hbar^2 a_2(x) + \ldots),$$

where $\hbar$ is the Planck constant (approximately $1.05 \cdot 10^{-34} Js$ in standard units). In the formula above, $\Re(\phi) \leq 0$ so that this expression is extremely small outside the set $\{\Re(\phi) = 0\}$ where it concentrates.

From this intuition, an interest developed towards decay rates for solutions of PDEs with small parameters. The most used setting in the mathematical treatment of quantum mechanics is the Weyl calculus of pseudodifferential operators [35]. Typical decay rates in this setting are of order $O(\hbar^\infty)$. Indeed, the composition of two pseudodifferential operators (or, more generally, Fourier Integral Operators) associated with smooth symbols can only be expanded in negative powers of $\hbar$ up to an error $O(\hbar^\infty)$.

In the particular case of a Schrödinger operator $P_\hbar = -\hbar^2\Delta + V$ where V is a smooth function, one can obtain an *Agmon estimate* [10], which is an $O(e^{\frac{\phi(x)}{\hbar}})$ pointwise control of eigenfunctions of $P_\hbar$ with eigenvalues close to E. Here, $\phi < 0$ on $\{V > 0\}$. In this setting one can easily conjugate $P_\hbar$ with multiplication operators of the form $e^{-\frac{\phi}{\hbar}}$, which allows to prove the control above. This conjugation property is not true for more general pseudodifferential operators. Moreover, Agmon estimates yield exponential decay in space variables, and give no information about the concentration rate of the semiclassical Fourier transform, which is only known to decay at $O(h^\infty)$ speed outside zero.

In the setting of pseudodifferential operators on $\mathbb{R}^d$ with *real-analytic* symbols, following analytic microlocal techniques [28], exponential decay rates in phase space (that is, exponential decay of the FBI or Bargmann transform) were obtained in [20, 21, 22, 23]. Exponential estimates in semiclassical analysis have important applications in physics [6] where they validate the WKB ansatz which, in turn, yields precise results on spectral gaps or dynamics of quantum states (quantum tunnelling). Moreover, on the mathematical level, these techniques can be used to study non-self-adjoint perturbations [14, 15] and resonances [11, 29, 24, 30, 9].

Since exponential decay in phase space for pseudodifferential operators is defined by means of the FBI or Bargmann transform, it seems natural to formulate these questions in terms of Bargmann quantization, which then generalises to Berezin-Toeplitz quantization on Kähler manifolds, where the semiclassical parameter is the inverse of an integer: $\hbar = N^{-1}$. Yet, for instance, the validity of the WKB ansatz for a Toeplitz operator, at the bottom of a non-degenerate real-analytic well, was only performed when the underlying manifold is $\mathbb{C}$ (see [32]), and some results were recently obtained for non-self-adjoint perturbations of Toeplitz operators on complex one-dimensional tori [26].

The analysis of Toeplitz operators depends on the knowledge of the Bergman projector, which encodes the geometrical data of the manifold on which the quantization takes place. The original microlocal techniques for the study of this projector [3, 34, 5] allow for a direct control of the Bergman kernel up to $O(N^{-\infty})$, from which one can deduce $O(N^{-\infty})$ estimates for composition and eigenpairs of Toeplitz operators with smooth symbols [19, 7, 8]. Based on analytic pseudodifferential techniques, the tools of *Local Bergman kernels* make it possible to show, under real-analyticity hypothesis, exponential (that is, $O(e^{-cN})$) decay of the coherent states in Toeplitz quantization [1]. Recently, this method was used to show an $O(e^{-c\sqrt{N}})$ control of the Bergman kernel under the same hypothesis [12]. Another recent paper [18] establishes an $O(e^{-c\sqrt{N}})$ decay rate for eigenfunctions of Toeplitz operators with smooth symbols.

Pseudodifferential operators, on which exponential estimates were originally studied, also satisfy a "well-balanced" condition: in the term of order k of the composition of two symbols f and g (which is, a priori, a bidifferential operator on f and g of total order $2k$), both symbols are differentiated at most k times. We believe that the techniques developped in this paper can be extended to more general "well-balanced" Fourier Integral Operators with real-analytic regularity. This method is somewhat elementary, since the only technical part consists in estimating quotients of factorials and powers by writing them as binomial or multinomial coefficients. This method sheds some light on the difficulty to formulate equivalence of quantizations in real-analytic settings without a loss of regularity. This fact is of little importance if one is concerned with spectral theory, but precise results (without loss of regularity) about the composition and inversion properties in a given analytic class, such as Theorem B, cannot be passed from one quantization to another if there is a loss of regularity inbetween.

Remark 1.4 (Gevrey case)**.** The methods and symbol classes developed in this paper can be easily applied to the Gevrey case. s-Gevrey symbol classes are defined, and studied, by putting all factorials to the power $s > 1$. s-Gevrey functions have almost holomorphic extensions with controlled error near the real locus, so that all results in this paper should be valid in the Gevrey case under the two following modifications:

- The summation of s-Gevrey symbols is performed up to $k = cN^{\frac{1}{s}}$.
- All $O(e^{-c'N})$ controls are replaced with $O(e^{-c'N^{\frac{1}{s}}})$.

For instance, we conjecture that the Bergman kernel on a quantizable compact Gevrey Kähler manifold is known up to $O(e^{-c'N^{\frac{1}{s}}})$. Its kernel decays at speed $N^{\dim(M)}e^{-\left(\frac{1}{2}-\varepsilon\right)N\,\mathrm{dist}(x,y)^2}$ as long as $\mathrm{dist}(x,y) \leq cN^{-\frac{s-1}{2s}}$. This would improve recent results [13].

1.4 Outline

In Section 2 we recall the basic properties of holomorphic extensions of real-analytic functions. Then, in Section 3, we define analytic symbol classes for sequences of functions $(f_k)_{k\geq 0}$ and we give a meaning to the sum $\sum N^{-k} f_k$ up to exponential precision. These symbol classes are more precise than the ones appearing in the literature since [28]. In Section 4 we show Theorems A and B: the Bergman kernel on a compact quantizable real-analytic Kähler manifold, and the composition of analytic covariant Toeplitz operators, are known up to $O(e^{-cN})$ precision, in terms of analytic symbols, from which we deduce, in Subsection 4.5, general exponential decay (Theorem C) in the forbidden region, for covariant as well as contravariant Toeplitz operators with analytic symbols. Section 5 contains a few useful combinatorial inequalities.

In Sections 3 and those that follow, the fundamental tool is a version in real-analytic regularity of the stationary phase lemma (Lemma 3.13). The various proofs in the second part have a common denominator: the general strategy consists in applying the complex stationary phase lemma and controlling the growth of the derivatives of the successive terms. We rely systematically on a "well-balanced" condition in the expansions in the stationary phase, which corresponds, in the setting of Toeplitz operators, to the Wick or anti-Wick quantization rules for contravariant or covariant symbols. This particular information allows us to bound non-trivial quotients of factorials which appear in the expansions, each time in a slightly different manner, but in every case based on discrete convex analysis and elementary combinatorial properties.

2 Holomorphic extensions

In this section we provide a general formalism for holomorphic extensions of various real-analytic data, which we use throughout this paper. The constructions of holomorphic extensions of real-analytic functions and manifolds is somewhat standard. We refer to [33] for details on these constructions. In particular, we study in Subsection 2.4 a specific class of analytic function spaces, which is a prerequisite to the Definition 3.3 of analytic symbol classes.

2.1 Combinatorial notations

In this subsection we recall some basic combinatorial notation. Analytic functions and analytic symbol spaces are defined using sequences which grow as fast as a factorial (see Definitions 2.10 and 3.3) so that we will frequently need to bound expressions involving binomial or multinomial coefficients.

Definition 2.1. Let $0 \leq i \leq j$ be integers. The associated *binomial* coefficient is

$$\binom{j}{i} = \frac{j!}{i!(j-i)!}.$$

Let more generally $(i_k)_{1\leq k\leq n}$ be a family of non-negative integers and let $j \geq \sum_{k=1}^n i_k$. The associated *multinomial* coefficient[1] is

$$\binom{j}{i_1, \ldots, i_k} = \frac{j!}{(j - \sum_{k=1}^n i_k)! \prod_{k=1}^n i_k!}$$

Definition 2.2.

1. A *polyindex* (plural: polyindices) μ is an ordered family $(\mu_1, \ldots, \mu_d)$ of non-negative integers. The cardinal d of the family is called the *dimension* of the polyindex (we will only consider the case where d is finite).

2. The norm $|\mu|$ of the polyindex $\mu = (\mu_1, \ldots, \mu_d)$ is defined as $\sum_{i=1}^d \mu_i$.

[1] An alternative definition of multinomial coefficient assumes $j = i_1 + \ldots + i_n$, in which case one defines $\binom{j}{i_1,\ldots,i_n} = \dfrac{j!}{i_1!\ldots i_n!}$. The definition we give contains this one, and is more consistent with the notation for binomial coefficients.

3. The partial order $\leq$ on polyindices of same dimension is defined as follows: $\nu \leq \mu$ when, for every $1 \leq i \leq d$, one has $\nu_i \leq \mu_i$.

4. The factorial $\mu!$ is defined as $\prod_{i=1}^{d} \mu_i!$. Together with the partial order, this allows to extend the notation for binomial coefficients. If $\nu \leq \mu$, then we define the associated binomial coefficient as

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu-\nu)!}$$

A few useful inequalities about binomial coefficients are proved in Section 5. We will use extensively the following inequality:

Lemma 2.3. *Let $(i_1 \ldots, i_n)$ with $\sum_{i=1}^{n} i_k \leq j$. Then*

$$\binom{j}{i_1, \ldots, i_n} \leq (n+1)^j.$$

Proof. One has

$$(n+1)^j = (\underbrace{1+1+\ldots+1}_{n+1})^j = \sum_{\substack{(i_1,\ldots,i_n) \\ \sum i_k \leq j}} \binom{j}{i_1, \ldots, i_n}.$$

As each term in the sum is positive, the sum is greater than any of its terms. $\square$

2.2 Extensions of real-analytic functions

The fundamental object that one is allowed to extend in a holomorphic way is a real-analytic function.

Definition 2.4. Let $f : U \mapsto V$ be a real-analytic function on an open set $U \in \mathbb{R}^n$, which takes values into a real or complex Banach space E. A *holomorphic extension* of f is a couple $(\widetilde{f}, \widetilde{U})$, where $\widetilde{U}$ is an open set of $\mathbb{C}^n$ and $\widetilde{f} : \widetilde{U} \mapsto E \otimes \mathbb{C}$, such that

- $\overline{\partial} \widetilde{f} = 0$.
- $U \subset \widetilde{U}$,
- $\widetilde{f}|_U = f$

Naturally, two holomorphic extensions coincide on the connected components of their intersections which intersect U since, on a connected open set of $\mathbb{C}^d$, a holomorphic function which vanishes on a real set vanishes everywhere.

If E is a real Banach space then $E \otimes \mathbb{C}$ is the complexification of E; if E is complex to begin with then $E \otimes \mathbb{C} = E$.

The following Proposition gives a natural choice of holomorphic extension:

Proposition 2.5. *Let U be an open set of $\mathbb{R}^d$, E be a Banach space and $f : U \mapsto E$ be a real-analytic function.*

Let $x \in U$. There exists a radius $r(x)$ such that the series

$$\sum_{\nu \in \mathbb{N}^d} \frac{\partial^\nu f}{\nu!} (y-x)^\nu$$

is absolutely convergent for all $y \in \overline{B(x, r(x))}$, with limit $f(y)$: we choose $r(x)$ smaller than half of the suprema of all r such that the power series above converge on $B(x, r)$, and such that $B_{\mathbb{R}^d}(x, r(x)) \subset U$.

Then, with

$$\widetilde{U} = \bigcup_{x \in U} B_{\mathbb{C}^d}(x, r(x)),$$

one can define $\widetilde{f}$ on $\widetilde{U}$ as the limit of the series above. Then $(\widetilde{f}, \widetilde{U})$ is a holomorphic extension of (f, U).

From now on, we will only use the term "holomorphic extension" for extensions whose domains are contained in the set $\widetilde{U}$ constructed in Proposition 2.5. In particular, the function $\widetilde{f}$ is unique up to restriction of its domain.

Proposition 2.6. *Let U and V be open sets of $\mathbb{R}^m$ and let $f : U \mapsto V$ be a real-analytic (local) diffeomorphism, then $\widetilde{f}$ is a (local) biholomorphism up to restriction of the domain.*

Proof. On the extended domain $\widetilde{U}$ one has

$$\widetilde{(df)} = \partial \widetilde{f},$$

so that, if $\det(df)$ does not vanish on U, then $\det(\partial \widetilde{f})$ does not vanish on a neighbourhood of U in $\widetilde{U}$; if moreover f is is a global diffeomorphism, that is, if f is injective on U, then $\widetilde{f}$ is injective on a neighbourhood of U in $\widetilde{U}$, which concludes the proof. □

2.3 Extensions of manifolds

Proposition 2.6 allows us to extend real-analytic manifolds into complex manifolds.

Proposition 2.7. *Let M be a real-analytic manifold. There is a complex manifold $(\widetilde{M}, J_e)$ with boundary, such that M is a totally real submanifold of $\widetilde{M}$. Then $\widetilde{M}$ is called a* holomorphic extension *of M.*

In this setting, "totally real" means that

$$\forall x \in M, T_xM \cap J_e(T_xM) = \{0\}.$$

Proof. The proof consists in extending all charts of M in the complex space; the standard complex structure J_{st} of every chart is preserved by the change of charts, which are biholomorphic by construction. This gives the complex structure J_e of $\widetilde{M}$; see [33], Proposition 1 for details.

By construction, in the local charts above, the submanifold M of $\widetilde{M}$ is mapped to $\mathbb{R}^{\dim(M)}$, which is totally real for the standard complex structure. Hence, M is totally real in $\widetilde{M}$. □

The extension of real-analytic manifolds is naturally associated with an extension of their real-analytic functions.

Proposition 2.8. *Let f be a real-analytic function on a real-analytic manifold M. Then there exists a holomorphic function $\widetilde{f}$ on a holomorphic extension $\widetilde{M}$ of M such that $\widetilde{f}|_M = f$.*

Proof. Any real-analytic function on M can be extended on a holomorphic extension $\widetilde{M}$ by extending the domain of its power series as in Proposition 2.5.

□

In the body of this article we will frequently extend real-analytic functions on holomorphic manifolds. We introduce a convenient notation to this end. Locally, a real-analytic function f on a complex manifold of dimension d can be written as

$$f : z \mapsto \sum_{\nu, \rho \in \mathbb{N}^d} c_{\nu,\rho} z^\nu \overline{z}^\rho.$$

As the function f is not holomorphic, we specifically write $f(z,\overline{z})$. There is then a natural notion of an extension

$$\widetilde{f} : (z,w) \mapsto \sum_{\nu,\rho\in\mathbb{N}^d} c_{\nu,\rho} z^\nu w^\rho.$$

This function is holomorphic on a neighbourhood of 0 in $\mathbb{C}^{2d}$. It coincides with $\widetilde{f}$, but the totally real manifold of interest is not $\{\Im(z) = 0\}$ anymore but rather $\{(z,w), w = \overline{z}\}$.

Let M be a complex manifold; using the convention above let us treat local charts for M and its holomorphic extension $\widetilde{M}$. A change of charts in M is a biholomorphism ϕ which, in the convention above, depends only on z as a function on $\widetilde{M}$. The extended biholomorphism $\widetilde{\phi}$ constructed in the previous subsection can be written as

$$(z,w) \mapsto (\phi(z), \overline{\phi(\overline{w})}).$$

Gluing open sets along the charts $\overline{\phi}$ (defined by $\overline{\phi}(z) = \overline{\phi(\overline{z})}$) yields a manifold $\overline{M}$, and there is a natural identification $M \ni z \mapsto \overline{z} \in \overline{M}$, so that $\overline{M}$ is simply M with reversed complex structure.

The expression of $\widetilde{\phi}$ above yields

$$\widetilde{M} = M \times \overline{M},$$

and M sits in $\widetilde{M}$ as the totally real submanifold

$$\{(z,w) \in M \times \overline{M}, \overline{z} = w\}.$$

This copy of M is said to be the codiagonal of $M \times \overline{M}$.

Any real-analytic function on M can be extended as a holomorphic function in a neighbourhood of the codiagonal of $\widetilde{M}$. If the function was holomorphic (on a small open set of M) to begin with, then its extension depends only on the first variable (on a small open set of $M \times \overline{M}$).

2.4 Analytic functional spaces

In this subsection we derive a few tools about the study of holomorphic functions near a compact totally real set. We first fix a notion of convenient open sets on which our analysis can take place.

Definition 2.9. A *domain* of $\mathbb{R}^d$ is an open, relatively compact set U with piecewise smooth boundary.

Recall that a holomorphic function f near zero can be written as

$$f(z) = \sum_{\nu\in\mathbb{N}^d} \frac{f_\nu}{\nu!} z^\nu.$$

Then, in particular $f_\nu = \partial^\nu f(0)$. Since f is holomorphic, the sum above congerges for $|z|$ sufficiently small. In other terms, there exists $r > 0$ and $C > 0$ such that, for every $\nu \in \mathbb{N}^d$, one has

$$|f_\nu| \leq C\nu! r^{|\nu|}.$$

Definition 2.10. For $j \in \mathbb{N}$ and f a function on a domain of $\mathbb{R}^d$ of class C^j, we denote by $\nabla^j f$ the function $(\partial^\alpha f(x))_{|\alpha|=j}$, which maps U to $\mathbb{R}^{\binom{j+d-1}{d-1}}$. For $n \in \mathbb{N}$ and $v \in \mathbb{R}^n$, we denote $\|v\|_{\ell^1} = \sum_{j=1}^n |v_1| + \ldots + |v_n|$.

Let $m \in \mathbb{N}$ and $r > 0$. Let U be a domain in $\mathbb{R}^d$. The space $H(m,r,U)$ is defined as the set of real-analytic functions on U such that there exists a constant C satisfying, for every $j \in \mathbb{N}$,

$$\sup_{x\in U} \|\nabla^j f(x)\|_{\ell^1} \leq \frac{Cr^j j!}{(j+1)^m}.$$

The space $H(m,r,U)$ is a Banach space for the norm $\|\cdot\|_{H(m,r,U)}$ defined as the smallest constant C such that the inequality above is true for every j.

Such functions can be extended to a neighbourhood of U in $\mathbb{C}^d$, with imaginary part bounded by r^{-1} (and by the distance to the boundary of U). The spaces $H(m,r,U)$ are compactly embedded in each other for the lexicographic order on $(r,-m)$: if either $r<r'$ or $r=r', m>m'$, then

$$H(m,r,U) \subset H(m',r',U).$$

Introducing a parameter m will allow us to control polynomial quantities which appear when one manipulates these holomorphic function spaces, using Lemmas 2.12 and 3.7. They correspond to a regularity condition at the boundary of a maximal holomorphic extension: for instance, the function $x \mapsto x\log(x)$ belongs to $H(1,1,(1/2,3/2))$ but not to $H(m,1,(1/2,3/2))$ for $m>1$.

It will be useful in the course of this paper to consider various analytic norms for the same function while maintaining a fixed norm. The definition of the spaces $H(m,r,U)$ immediately imply the following fact.

Proposition 2.11. *Let $m_0 \in \mathbb{N}$ and ,$r_0>0$. Let U be a domain in $\mathbb{R}^d$. Let $f \in H(m_0,r_0,U)$. Then, for all $m \geq m_0$, for all $r \geq r_0 2^{m-m_0}$, one has $f \in H(m,r,U)$ with*

$$\|f\|_{H(m,r,U)} \leq \|f\|_{H(m_0,r_0,U)}.$$

The following lemma will be used several times in what follows.

Lemma 2.12. *Let $d \in \mathbb{N}$. There exists $C>0$ such that, for any $m \geq \max(d+2, 2(d+1))$, for any $j \in \mathbb{N}$, one has*

$$\sum_{i=0}^{j} \frac{\min(i+1,j-i+1)^d (j+1)^m}{(i+1)^m (j-i+1)^m} \leq 2 + C\frac{3^m}{4^m}.$$

Proof. If $j=1$ then this sum is exactly 2. We now suppose $j \geq 2$.

let us first prove that, if $1 \leq i \leq j-1$ and $m \geq d$, then

$$\frac{\min(i+1,j-i+1)^d (j+1)^m}{(i+1)^m (j-i+1)^m} \leq 2^d \frac{3^m}{4^m}.$$

Since $x \mapsto -\log(x)$ is convex on $(0,+\infty)$, the function of i above is log-convex on $[1,j/2]$ as well as on $[j/2,j-1]$. By symmetry, it is then sufficient to prove the bound above for $i=1$ and $i=j/2$.

For $i=1$, since $j \geq 2$ one can bound

$$\frac{2^d (j+1)^m}{2^m j^m} = 2^d 2^{-m} \left(\frac{j+1}{j}\right)^m \leq 2^d \frac{3^m}{4^m}.$$

For $i=j/2$ the expression becomes

$$2^{-d} \left(\frac{4(j+1)}{(j+2)(j+2)}\right)^m \leq 2^{-d} \frac{3^m}{4^m}.$$

We are now ready to prove the claim. Let us decompose the sum into

$$2 + 2\sum_{i=1}^{\lfloor j/3 \rfloor} \frac{(i+1)^d (j+1)^m}{(i+1)^m (j-i+1)^m} + \sum_{i=\lfloor j/3 \rfloor +1}^{\lceil 2j/3 \rceil -1} \frac{(i+1)^d (j+1)^m}{(i+1)^m (j-i+1)^m}.$$

1. If $j-i\geq \frac{2j}{3}$ then
$$\frac{(j+1)^m}{(j-i+1)^m}\leq \frac{3^m}{2^m}.$$
Hence, the sum
$$2\sum_{i=1}^{\lfloor j/3\rfloor}\frac{(i+1)^d(j+1)^m}{(i+1)^m(j-i+1)^m}$$
is smaller than
$$2\cdot\frac{3^m}{2^m}\sum_{i=1}^{\lfloor j/3\rfloor}\frac{1}{(i+1)^{m-d}}\leq 2\cdot\frac{3^m}{2^m}(\zeta(m-d)-1),$$
where ζ denotes the Riemann zeta function. If $m-d\geq 2$ one has $\zeta(m-d)\leq 1+3\cdot 2^{-(m-d)}$. Hence, this sum is smaller than $6\cdot 2^d\dfrac{3^m}{4^m}$.

2. The sum
$$\sum_{i=\lfloor j/3\rfloor+1}^{\lceil 2j/3\rceil-1}\frac{(i+1)^d(j+1)^m}{(i+1)^m(j-i+1)^m}$$
is smaller than
$$2\frac{(9/4)^m(j+1)^{d+1}}{(j+1)^m},$$
since for each index i between the bounds one has
$$\frac{(j+1)^m}{(i+1)^m(j-i+1)^m}\leq\frac{(j+1)^m}{(2(j+1)/3)^m(2(j+1)/3)^m}\leq\frac{(9/4)^m}{(j+1)^m}.$$
Suppose $m\geq 2(d+1)$, so that
$$2\frac{(9/4)^m(j+1)^{d+1}}{(j+1)^m}\leq 2\frac{(9/4)^m}{(\sqrt{j+1})^m}.$$
Hence, if $j\geq 10$ then this sum is smaller than $2\cdot\frac{3^m}{4^m}$. In the other case we have at most 4 terms, each of them smaller than $2^d\dfrac{3^m}{4^m}$.

The total sum is then controlled by
$$2+\left(10\cdot 2^d\right)\frac{3^m}{4^m},$$
hence the claim. ☐

Analytic function classes form an algebra and nonvanishing functions can be inverted:

Proposition 2.13. *There exists $C>0$ such that the following is true. Let $m\geq 2$. Let $r>0$ and let U be a domain in $\mathbb{R}^n$. Let $f,g\in H(m,r,U)$. Then $fg\in H(m,r,U)$, and*
$$\|fg\|_{H(m,r,U)}\leq C\|f\|_{H(m,r,U)}\|g\|_{H(m,r,U)}.$$
The constant C is universal.

If f is bounded away from zero on U, then $f^{-1}\in H(m,r,U)$, with
$$\|f^{-1}\|_{H(m,r,U)}\leq\frac{\|f\|_{H(m,r,U)}}{\inf_U(|f|)^2}.$$

Proof. Let $f, g \in H(m,r,U)$ and $j \in \mathbb{N}$. Then

$$\sum_{|\alpha|=j} |\partial^\alpha (fg)| \leq \sum_{|\beta+\gamma|=j} \binom{\beta+\gamma}{\beta} |\partial^\beta f|\,|\partial^\gamma g|$$

By Lemma 5.2, one has, for every β and γ such that $|\beta+\gamma| = j$,

$$\binom{\beta+\gamma}{\beta} \leq \binom{|\beta+\gamma|}{|\beta|} = \binom{j}{|\beta|}.$$

Hence,

$$\sum_{|\alpha|=j} |\partial^\alpha (fg)| \leq \sum_{i=0}^{|j|} \binom{j}{i} \|\nabla^i f\|_{\ell^1} \|\nabla^{|\alpha|-i} g\|_{\ell^1},$$

so that, for any $j \geq 0$, one has

$$\|\nabla^j (fg)\|_{\ell^1} \leq \|f\|_{H(m,r,U)} \|g\|_{H(m,r,U)} \frac{r^j j!}{(j+1)^m} \sum_{i=0}^{j} \binom{j}{i}^{-1} \binom{j}{i} \frac{(j+1)^m}{(i+1)^m (j-i+1)^m}.$$

Hence,

$$\|\nabla^j (fg)\|_{\ell^1} \leq \|f\|_{H(m,r,U)} \|g\|_{H(m,r,U)} \frac{r^j j!}{(j+1)^m} \sum_{i=0}^{j} \frac{(j+1)^m}{(i+1)^m (j-i+1)^m}.$$

Let us use Lemma 2.12 with $d = 0$. If $m \geq 2$, this quantity is bounded independently of j and m, so that

$$\|\nabla^j (fg)\|_{\ell^1} \leq C \|f\|_{H(m,r,U)} \|g\|_{H(m,r,U)} \frac{r^j j!}{(j+1)^m}.$$

This concludes the first part of the proof.

Let now $f \in H(m,r,U)$ which does is bounded away from zero on U. We introduce the modified product $f \cdot g = \frac{fg}{C}$, for which $H(m,r,U)$ is a Banach algebra.

First, $|f|^2$ is real-valued and strictly positive; moreover $|f|^2 = f\overline{f} \in H(m,r,U)$ and, by the property above,

$$\||f|^2\|_{H(m,r,U)} \leq C \|f\|^2_{H(m,r,U)}.$$

Let $g = \frac{|f|^2}{2\||f|^2\|_{H(m,r,U)}}$. Then

$$\|1-g\|_{H(m,r,U)} \leq 1 - \frac{\inf_U(|f|^2)}{2\||f|^2\|_{H(m,r,U)}} < 1.$$

In particular, $g = 1 - (1-g)$ so that, letting h be such that $g \cdot h = 1$, one has

$$h = \sum_{k=0}^{+\infty} (1-g)^{\cdot k}.$$

Hence, one can control

$$\|h\|_{H(m,r,U)} \leq \frac{2\||f|^2\|_{H(m,r,U)}}{\inf_U(|f|^2)}.$$

Now $|f|^{-2} = \dfrac{h}{2C\||f|^2\|_{H(m,r,U)}}$ so that

$$\||f|^{-2}\|_{H(m,r,U)} \leq \frac{1}{C\inf_U(|f|^2)}.$$

We now turn to $f^{-1} = \overline{f}|f|^{-2}$, which is controlled as follows:

$$\|f^{-1}\|_{H(m,r,U)} \leq \frac{\|f\|_{H(m,r,U)}}{\inf_U(|f|^2)}.$$

This concludes the proof. $\square$

The spaces $H(r,m,U)$ contain all holomorphic functions.

Proposition 2.14. *Let $d \in \mathbb{N}$. For every $T > 0$ we let $P(0,T)$ be the polydisk of center 0 and of radius T in $\mathbb{C}^d$.*

Let f be a holomorphic, bounded function on $P(0,2T)$, continuous up to the boundary. Then

$$\|f\|_{H(-d,dT^{-1},P(0,T))} \leq C \sup_{P(0,2T)} |f|.$$

Proof. The proof relies on the Cauchy formula. For all $z \in P(0,T)$ and $\nu \in \mathbb{N}^d$, there holds

$$\partial^\nu f(z) = C\int_{|\xi_1|=\ldots=|\xi_d|=2T} \frac{\nu! f(\xi)}{(\xi_1-z_1)^{\nu_1}(\xi_2-z_2)^{\nu_2}\ldots(\xi_d-z_d)^{\nu_d}}\mathrm{d}\xi.$$

As $z \in P(0,r)$ and $|\xi_1| = \ldots = |\xi_d| = 2T$, for every $1 \leq i \leq d$ there holds $|\xi_i - z_i| \geq T$, so that

$$\sup_{P(0,T)} |\partial^\nu(f)| \leq CT^{-|\nu|}\nu! \sup_{P(0,2T)} |f|.$$

In particular, since $\nu! \leq |\nu|!d^{|\nu|}$, by summing over ν's with same norm we obtain

$$\sup_{x\in P(0,T)} \|\nabla^j f(x)\|_{\ell^1} \leq C(j+1)^d(dT^{-1})^j j!,$$

hence the claim. $\square$

3 Calculus of analytic symbols

In this section we define and study (formal) *analytic symbols*, which we will show to be well suited to the study of stationary phases with complex, real-analytic phases.

3.1 Analytic symbols

We begin with an explicit definition of C^j-norms on compact manifolds.

Definition 3.1. Let X be a compact manifold (with smooth boundary). We fix a finite set $(\rho_V)_{V\in\mathcal{V}}$ of local charts on open sets V which cover X.

Let $j \geq 0$. The C^j norm of a function $f : X \mapsto \mathbb{C}$ which is continuously differentiable j times is defined as

$$\|f\|_{C^j(X)} = \max_{V\in\mathcal{V}} \sup_{x\in V} \sum_{|\mu|=j} |\partial^\mu(f\circ\rho_V)(x)|.$$

This definition is adapted to the multiplication of two functions:

Proposition 3.2. *Let X be a compact manifold (with smooth boundary) with fixed local charts, and $f, g \in C^j(X, \mathbb{R})$.*

Then $fg \in C^j(X, \mathbb{R})$ with

$$\|fg\|_{C^j(X)} \leq \sum_{i=0}^{j} \binom{j}{i} \|f\|_{C^i(X)} \|g\|_{C^{j-i}(X)}.$$

Proof. One has, in local coordinates,

$$\partial^\mu (fg) = \sum_{\nu \leq \mu} \binom{\mu}{\nu} \partial^\nu f \partial^{\mu-\nu} g,$$

with, by Lemma 5.2,

$$\binom{\mu}{\nu} \leq \binom{|\mu|}{|\nu|}.$$

Hence,

$$\begin{aligned}
\sum_{|\mu|=j} |\partial^\mu (fg)(x)| &\leq \sum_{|\mu|=j} \sum_{\nu \leq \mu} \binom{j}{|\nu|} |\partial^\nu f(x)| |\partial^{\mu-\nu} g| \\
&= \sum_{i=0}^{j} \binom{j}{i} \sum_{|\nu|=i} |\partial^\nu f| \left(\sum_{|\mu|=j, \nu \leq \mu} |\partial^{\mu-\nu} g| \right) \\
&= \sum_{i=0}^{j} \binom{j}{i} \left(\sum_{|\nu|=i} |\partial^\nu f| \right) \left(\sum_{|\rho|=j-i} |\partial^\rho g| \right),
\end{aligned}$$

hence the claim. □

Using the convention above, let us generalise Definition 2.10, in order to define analytic symbols.

Definition 3.3. Let X be a compact manifold (with boundary), with a fixed set of covering local charts.

Let r, R, m be positive real numbers. The space of analytic symbols $S_m^{r,R}(X)$ consists of sequences $(a_k)_{k \geq 0}$ of real-analytic functions on X, such that there exists $C \geq 0$ such that, for every $j \geq 0, k \geq 0$, one has

$$\|a_k\|_{C^j(X)} \leq C \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

The norm of an element $a \in S_m^{r,R}(X)$ is defined as the smallest C as above; then $S_m^{r,R}(X)$ is a Banach space.

We are interested in symbols which have an expansion in increasing powers of the semiclassical parameter. We will use the term "symbols" while, in the usual semiclassical vocabulary, we are dealing with formal symbols to which we associate classical symbols by a summation process in Proposition 3.6.

As for the analytic function classes $H(m, r, U)$ of Definition 2.10, the spaces $S_m^{r,R}(X)$ are included in each other for a lexicographic order, and the constants of injection are controlled as follows:

Proposition 3.4. *Let X be a compact manifold (with boundary) with a fixed finite set of covering charts. Let r_0, R_0, m_0 positive. Let $f \in S_{m_0}^{r_0,R_0}(X)$. For every $m \geq m_0$, for every $r \geq r_0 2^{m-m_0}$ and $R \geq R_0 2^{m-m_0}$, one has $f \in S_m^{r,R}$ with*

$$\|f\|_{S_m^{r,R}(X)} \leq \|f\|_{S_{m_0}^{r_0,R_0}(X)}.$$

The notion of sum of a formal series in N^{-1} is well-defined up to $O(N^{-\infty})$, by a process known as *Borel summation*. In a similar but more explicit way, formal series corresponding to analytic symbols can be summed up to an exponentially small error.

Definition 3.5. Let X be a compact Riemannian manifold (with boundary) and let $f \in S_m^{r,R}(X)$. Let $c_R = \frac{e}{3R}$. The *summation* of f is defined as

$$X \times \mathbb{N} \ni (x, N) \mapsto f(N)(x) = \sum_{k=0}^{c_R N} N^{-k} f(x).$$

Proposition 3.6. *Let X be a compact Riemannian manifold with boundary and let $f \in S_m^{r,R}(X)$. Let $c_R = \frac{e}{3R}$. Then*

1. *The function $f(N)$ is bounded on X uniformly for $N \in \mathbb{N}$.*

2. *For every $0 < c_1 < c_R$, there exists $c_2 > 0$ such that*

$$\sup_{x\in X} \left| \sum_{k=c_1 N}^{c_R N} N^{-k} f_k(x) \right| = O(e^{-c_2 N}).$$

Proof.

1. Since

$$\sup_{x\in X} |f_k(x)| \leq \|f\|_{S_m^{r,R}(X)} R^k k!,$$

it remains to control

$$\sum_{k=0}^{c_R N} N^{-k} R^k k!.$$

In this series, the first term is 1, and the ratio between two consecutive terms is

$$\frac{N^{-k} R^k k!}{N^{-k+1} R^{k-1} (k-1)!} = \frac{Rk}{N} \leq R c_R = \frac{e}{3} < 1.$$

Hence,

$$\sup_{x\in X} |f(x, N)| \leq \|f\|_{S_m^{r,R}(X)} \sum_{k=0}^{c_R N} (e/3)^k \leq \|f\|_{S_m^{r,R}(X)} \frac{3}{3-e}.$$

2. The claim reduces to a control on

$$\sum_{k=c_1 N}^{c_R N} N^{-k} R^k k!.$$

In this series, on which each term is smaller than $(e/3)^k$, the first term is controlled by

$$(e/3)^{c_1 N} = \exp(c_1 \log(e/3) N).$$

Hence the claim, with $c_2 = c_1 \log(e/3)$.

□

From the second point of Proposition 3.6, we see that the constant $c_R = \frac{e}{3R}$ is quite arbitrary (using the Stirling formula to control factorials, one could in fact consider any constant smaller than $\frac{e}{R}$). We use it in Definition 3.5 to avoid dealing with equivalence classes of sequences whose difference is $O(e^{-c'N})$ for some c', as in [28].

Before studying further the space $S_m^{r,R}(X)$, let us generalize Lemma 2.12.

Lemma 3.7. *Let $d \in \mathbb{N}$ and $n \geq 2$. There exists $C(n,d) > 0$ such that, for any $m \geq \max(d+2, 2(d+n-1))$, for any $\ell \in \mathbb{N}$, one has*

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ i_1 + \ldots + i_n = \ell}} \frac{(i_{n-1}+1)^d (\ell+1)^m}{(i_1+1)^m \ldots (i_n+1)^m} \leq 1 + C\frac{3^m}{4^m}.$$

This is indeed, up to a factor 2, a generalisation of Lemma 2.12 which corresponds to the case $n = 2$.

Proof. As before, the case $\ell = 1$ is trivial, so we assume $\ell \geq 2$. The only term in the sum such that $i_{n-1} = 0$ is equal to 1; let us control the sum restricted on $\{i_{n-1} \geq 1\}$. Let us first show that, if $i_{n-1} \geq 1$, then

$$\frac{(i_{n-1}+1)^d (\ell+1)^m}{(i_1+1)^m \ldots (i_n+1)^m} \leq (\ell+1)^d \frac{3^m}{4^m}. \tag{2}$$

One has directly $(i_{n-1}+1)^d \leq (\ell+1)^d$.

We are left with

$$\frac{(\ell+1)^m}{(i_1+1)^m \ldots (i_n+1)^m},$$

which is a symmetric expression of $(i_1, \ldots, i_n)$, log-convex as soon as $m \geq 0$, and which we wish to bound on the symmetrised set

$$\left\{(i_1, \ldots, i_n) \in \mathbb{N}_0^n, \sum_{k=1}^{n} i_k = \ell, \text{ at least two of them are } \geq 1\right\}.$$

By Lemma 5.4, it is sufficient to control the quantity above at the permutations of $(\ell-1, 1, 0, \ldots, 0)$. At each of those points, since $\ell \geq 2$, one has

$$\frac{(\ell+1)^m}{(i_1+1)^m \ldots (i_n+1)^m} = \left(\frac{\ell+1}{2\ell}\right)^m \leq \frac{3^m}{4^m}.$$

We are now in position to prove the claim. Let us first restrict our attention to $\{i_1 \geq \frac{\ell+1}{3(n-1)}\}$. There are less than $(\ell+1)^{n-1}$ such terms (since there are less than $(\ell+1)^{n-1}$ terms in total), and each of these terms is smaller than

$$\frac{(\ell+1)^d (\ell+1)^m}{\left(\frac{\ell+1}{3(n-1)}\right)^{mn}} = \frac{(\ell+1)^d \, (3(n-1))^{mn}}{(\ell+1)^{m(n-1)}}.$$

Hence, this sum is controlled by

$$\frac{(\ell+1)^{n+d-1} \, (3(n-1))^{mn}}{(\ell+1)^{m(n-1)}}$$

We now consider the sum on $\{i_1 \leq \frac{\ell+1}{3n-1} \leq i_2\}$. There are again less than $(\ell+1)^{n-1}$ such terms, each of them smaller than

$$\frac{(\ell+1)^d (\ell+1)^m}{\left(\frac{\ell+1}{3(n-1)}\right)^{m(n-1)}} = \frac{(\ell+1)^d \, (3(n-1))^{m(n-1)}}{(\ell+1)^{m(n-2)}}.$$

Thus, this sum is smaller than

$$\frac{(\ell+1)^{n+d-1}\,(3(n-1))^{m(n-1)}}{(\ell+1)^{m(n-2)}}.$$

Similarly, we are able to control the sum restricted on $\{i_k \leq \frac{\ell+1}{3(n-1)} \leq i_{k+1}\}$, for $k \leq n-2$, by

$$\frac{(\ell+1)^{n+d-1}\,(3(n-1))^{m(n-k)}}{(\ell+1)^{m(n-k-1)}}.$$

If $m \geq 2(d+n-1)$, then $(\ell+1)^{n+d-1+m} \leq (\ell+1)^{3m/2}$, so that, for any $k \leq n-2$, if $\ell+1 \geq 3n$, one has

$$\frac{(\ell+1)^{n+d-1}\,(3(n-1))^{m(n-k)}}{(\ell+1)^{m(n-k-1)}} \leq (\ell+1)^{\frac{3m}{2}}\left(\frac{3(n-1)}{\ell+1}\right)^{m(n-k)} \leq (\ell+1)^{3m/2}\left(\frac{3(n-1)}{\ell+1}\right)^{2m} = \left(\frac{9(n-1)^2}{\sqrt{\ell+1}}\right)^{m}.$$

Thus, for ℓ large enough (depending on n), this quantity is smaller than $\frac{3^m}{4^m}$; for ℓ small we have a number of terms bounded by a function of n, each term being smaller than $C(n,d)\frac{3^m}{4^m}$ by (2).

It remains to control the sum restricted on $\{1 \leq i_{n-1} \leq \frac{\ell+1}{3(n-1)}\}$. In this case, $i_n + 1 \geq \frac{2(\ell+1)}{3}$, so that the sum is smaller than

$$\frac{3^m}{2^m}\sum_{\substack{0\leq i_1\leq\cdots\leq i_{n-1}\leq\frac{\ell+1}{3(n-1)}\\ i_{n-1}\geq 1}}\frac{(i_{n-1}+1)^d}{(i_1+1)^m(i_2+1)^m\ldots(i_{n-1}+1)^m} \leq \frac{3^m}{2^m}(\zeta(m))^{n-2}(\zeta(m-d)-1).$$

The Riemann zeta function is decreasing, and if $m \geq d+2$, then $\zeta(m-d) \leq 1 + 3\cdot 2^{-(m-d)}$, so that the expression above is controlled by $C(n,d)\frac{3^m}{4^m}$. This concludes the proof. $\square$

Analytic symbols behave well with respect to the Cauchy product, which corresponds to the product of their summations.

Proposition 3.8. *There exists $C_0 \in \mathbb{R}$ and a function $C : \mathbb{R}^2 \mapsto \mathbb{R}$ such that the following is true.*

Let X be a compact Riemannian manifold (with boundary) and with a fixed finite set of covering charts. Let $r, R \geq 0$ and $m \geq 4$. For $a, b \in S^{r,R}_m(X)$, let us define the Cauchy product of a and b as

$$(a*b)_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

1. *The space $S^{r,R}_m(X)$ is an algebra for this Cauchy product, that is,*

$$\|a*b\|_{S^{r,R}_m} \leq C_0\|a\|_{S^{r,R}_m}\|b\|_{S^{r,R}_m},$$

   Moreover, there exists $c > 0$ depending only on R such that as $N \to +\infty$, one has

$$(a*b)(N) = a(N)b(N) + O(e^{-cN}).$$

2. *Let r_0, R_0, m_0 positive and $a \in S^{r_0,R_0}_{m_0}(X)$ with a_0 nonvanishing. Then, for every m large enough depending on a, for every $r \geq r_0 2^{m-m_0}, R \geq R_0 2^{m-m_0}$, a is invertible (for the Cauchy product) in $S^{r,R}_m(X)$, and its inverse $a^{\star-1}$ satisfies:*

$$\|a^{*-1}\|_{S^{r,R}_m(X)} \leq C(\|a\|_{S^{r_0,R_0}_{m_0}(X)}, \min(|a|)).$$

Proof.

1. From Proposition 3.2, one has, for every $0 \le i \le k$ and $j \ge 0$,

$$\|a_i b_{k-i}\|_{C^j} \le \sum_{\ell=0}^{j} \binom{j}{\ell} \|a_i\|_{C^\ell} \|b_{k-i}\|_{C^{j-\ell}}.$$

In particular,

$$\|(a*b)_k\|_{C^j} \le \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}} \frac{r^j R^k (j+k)!}{(j+k+1)^m} \sum_{i=0}^{k} \sum_{\ell=0}^{j} \binom{j+k}{i+\ell}^{-1} \binom{j}{\ell} \frac{(j+k+1)^m}{(i+\ell+1)^m (j+k-i-\ell+1)^m}.$$

By Lemma 5.1, one has

$$\binom{j}{\ell} \le \binom{j+i}{\ell+i} \le \binom{j+k}{\ell+i}.$$

This yields

$$\begin{aligned}
\|(a*b)_k\|_{C^j} &\le \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}} \frac{r^j R^k (j+k)!}{(j+k+1)^m} \sum_{i=0}^{k} \sum_{\ell=0}^{j} \frac{(j+k+1)^m}{(i+\ell+1)^m (j+k-i-\ell+1)^m} \\
&\le \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}} \frac{r^j R^k (j+k)!}{(j+k+1)^m} \sum_{i'=0}^{k+j} \frac{\min(i'+1, j+k-i'+1)(j+k+1)^m}{(i'+1)^m (j+k-i'+1)^m}.
\end{aligned}$$

Here, we let $i' = i + \ell$.

We are reduced to Lemma 2.12 with $d = 1$. If $m \ge 4$, this sum is smaller than a universal constant C independently of j, k, so that

$$\|a * b\|_{S_m^{r,R}} \le C \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}}.$$

Let us control the product of the associated analytic series. By Proposition 3.6, for some $c > 0$ depending only on R, one has

$$a(N) = \sum_{k=0}^{\frac{eN}{12R}} N^{-k} a_k + O(e^{-cN}),$$

and similar controls for $b(N)$ and $(a \star b)(N)$.

The first $\dfrac{eN}{12R}$ terms of the expansion in decreasing powers of $(a * b)(N)$ and $a(N)b(N)$ then coincide by definition of the Cauchy product. It remains to control

$$\sum_{\frac{eN}{12R} \le i+j \le \frac{eN}{6R}} N^{-(i+j)} a_i b_j.$$

From

$$\sup(|a_i b_j|) \le C R^{i+j} i! j! \le C (2R)^{i+j} (i+j)!,$$

one has, as in Proposition 3.6,

$$\left| \sum_{\frac{eN}{12R} \le i+j \le \frac{eN}{6R}} N^{-(i+j)} a_i b_j \right| \le \sum_{\frac{eN}{12R} \le i+j \le \frac{eN}{6R}} N^{-(i+j)} (2R)^{i+j} (i+j)! \le e^{-cN},$$

hence the claim.

2. The unit element of the Cauchy product is $(1,0,0,\ldots)$, which belongs to $S^{r,R}_m(X)$. Let $a \in S^{r_0,R_0}_{m_0}(X)$ be such that a_0 does not vanish on X, and let us try to find b such that $(a*b)_0 = 1$ and $(a*b)_k = 0$ whenever $k \neq 0$.

   The first condition yields $b_0 = a_0^{-1}$, which is a function with real-analytic regularity and same radius as a_0, by Proposition 2.13, so that

$$\|b_0\|_{C^j} \leq C_0 \frac{r_0^j j!}{(j+1)^{m_0}}.$$

   In particular, by Lemma 2.11, for all $m \geq m_0, r \geq r_0 2^{m-m_0}$, one has

$$\|b_0\|_{C^j} \leq C_0 \frac{r^j j!}{(j+1)^{m}}.$$

   The coefficients b_k are then determined by induction:

$$b_k = a_0^{-1} \sum_{i=1}^{k} a_i b_{k-i} = b_0 \sum_{i=1}^{k} a_i b_{k-i}.$$

   Let us control $\|b\|_{S^{r,R}_m(X)}$ by $\|a\|_{S^{r,R}_m(X)}$ by induction, for some r, R, m which will be chosen later.

   We now proceed by induction on k. Suppose that, for all $\ell \leq k-1$ and $j \geq 0$, one has

$$\|b_\ell\|_{C^j} \leq C_b \frac{r^j R^\ell (j+\ell)!}{(j+\ell+1)^m},$$

   We wish to prove the same control for $\ell = k$. The constant C_b will be chosen later.

   By induction hypothesis,

$$\begin{aligned}\|b_k\|_{C^j} &\leq C_0 C_b \|a\|_{S^{r,R}_m} \sum_{j_1=0}^{j}\sum_{i=1}^{k}\sum_{j_2=0}^{j-j_1} \binom{j}{j_1, j_2} \frac{r^{j_1} j_1!}{(j_1+1)^m} \frac{r^{j_2} R^i (j_2+i)! r^{j-j_1-j_2} R^{k-i} (j-j_1-j_2+k-i)!}{(i+j_2+1)^m (j-j_1-j_2+k-i+1)^m} \\ &\leq C_b C_0 \|a\|_{S^{r,R}_m} \frac{r^j R^k (j+k)!}{(j+k+1)^m} \sum_{j_1=0}^{j}\sum_{i=1}^{k}\sum_{j_2=0}^{j-j_1} \binom{j}{j_1, j_2} \binom{j+k}{j_1, j_2+i}^{-1} \\ &\qquad \times \frac{(j+k+1)^m}{(j_1+1)^m (j_2+i+1)^m (j-j_1-j_2+k-i+1)^m}.\end{aligned}$$

   Let us prove that, for every i, j, j_1, j_2, k in the range above, one has

$$\binom{j+k}{j_1, j_2+i} \geq \binom{j}{j_1, j_2}.$$

   There holds

$$\binom{j+1}{j_1, j_2+1} = \binom{j}{j_1, j_2} \frac{j+1}{j-j_1-j_2} \geq \binom{j}{j_1, j_2},$$

   so that

$$\binom{j+k}{j_1, j_2+i} \geq \binom{j+i}{j_1, j_2+i} \geq \binom{j}{j_1, j_2}.$$

Hence,

$$\begin{aligned}\|b_k\|_{C^j} &\leq C_bC_0\|a\|_{S_m^{r,R}}\frac{r^jR^k(j+k)!}{(j+k+1)^m}\sum_{j_1=0}^{j}\sum_{i=1}^{k}\sum_{j_2=0}^{j-j_1}\frac{(j+k+1)^m}{(j_1+1)^m(j_2+i+1)^m(j-j_1-j_2+k-i+1)^m}\\ &\leq C_bC_0\|a\|_{S_m^{r,R}}\frac{r^jR^k(j+k)!}{(j+k+1)^m}\sum_{\substack{j_1+i_1+i_2=j+k\\ i_1\geq 1}}\frac{\min(i_1+1,i_2+1)(j+k+1)^m}{(j_1+1)^m(i_1+1)^m(i_2+1)^m}.\end{aligned}$$

From Lemma 3.7 with $n=3$ and $d=1$, the sum

$$\sum_{\substack{j_1+i_1+i_2=j+k\\ i_1\geq 1}}\frac{\min(i_1+1,i_2+1)(j+k+1)^m}{(j_1+1)^m(i_1+1)^m(i_2+1)^m}$$

is bounded independently of j and k for $m\geq 6$. However this control is not enough since it yields a constant in front of $\dfrac{r^jR^k(j+k)!}{(j+k+1)^m}$ which is a priori $CC_0C_b\|a\|_{S_m^{r,R}}\geq C_b$.

However, the only term in this expansion which contributes as 1 is $j_1=0, i_1=k+j, i_2=0$, which corresponds to $j_1=0, i=k, j_2=j$. One can control this term independently of C_b since

$$|a_0^{-1}|\|a_k\|_{C^j}|b_0|\leq C_0^2\frac{r^jR^k(j+k)!}{(j+k+1)^m}.$$

The sum over all other terms is smaller than $CC_bC_0\|a\|_{S_m^{r,R}}(3/4)^m$ for some C, by Lemma 3.7.

We can conclude: if m is large with respect to $\|a\|_{S_m^{r,R}}$ (which can be done using Proposition 3.4 by setting $r\geq r_0 2^{m-m_0}$ and $R\geq R_0 2^{m-m_0}$) and if $C_b\geq 2C_0^2$ (recall from Proposition 2.13 that $C_0^2=\min(|a|)^{-4}\|a\|^2_{S_m^{r,R}}$), one has, by induction,

$$\|b_k\|_{C^j}\leq C_b\frac{r^jR^k(j+k)!}{(j+k+1)^m}.$$

This concludes the proof.

□

Remark 3.9. The method of proof for Proposition 3.8 will be used again in Section 4. This method consists in an induction, in which quotients of factorials must be bounded; this reduces the control by induction to Lemma 3.7. Constants which appear must be carefully chosen so that the induction can proceed. In particular, given a fixed object in an analytic class, it will be useful to change the parameters (typically m,r,R) in its control, while maintaining a fixed norm.

The classes $H(m,r,V)$ of real-analytic functions introduced in Section 2 contain all holomorphic functions. In a similar manner, the symbol classes $S_m^{r,R}$ contain all classical analytic symbols in the sense of Sjöstrand [28]:

Proposition 3.10. *Let U be an open set of $\mathbb{C}^n$ and let $a=(a_k)_{k\geq 0}$ be a sequence of bounded holomorphic functions on U such that there exists $C>0$ and $R>0$ satisfying, for all $k\geq 0$,*

$$\sup_U |a_k|\leq CR^kk!.$$

Then for every $V\subset\subset U$ there exists $r>0$ such that $a\in S_0^{r,R}(V)$.

In particular, given an analytic symbol a and a biholomorphism κ, then $a\circ\kappa$ is an analytic symbol.

Proof. By Proposition 2.14, there exists $C_1 > 0$ and $r > 0$ such that, for every $k \geq 0$, one has $a_k \in H(r, 0, V)$ with

$$\|a_k\|_{H(0,r,V)} \leq C_1 \sup_U |a_k|.$$

In other terms, for every $k \geq 0, j \geq 0$, one has

$$\|a_k\|_{C^j(V)} \leq C_1 C r^j R^k j!k! \leq C_1 C r^j R^k (j+k)!.$$

Hence $a \in S_0^{r,R}(V)$. $\square$

3.2 Complex stationary phase lemma

In this subsection we present the tools of stationary phase in the context of real-analytic regularity, as developed by Sjöstrand [28]. We wish to study integrals of the form

$$\int_\Omega e^{N\Phi(x)} a(x) \mathrm{d}x,$$

as $N \to +\infty$. If Φ is purely imaginary, then by integration by parts, this integral is $O(N^{-\infty})$ away from the points where $d\Phi$ vanishes. At such points, if Φ is Morse, a change of variables leads to the usual case where Φ is quadratic nondegenerate; then there is a full expansion of the integral in decreasing powers of N. If Φ is real-valued, a similar analysis (Laplace method) yields a related expansion.

On one hand, we wish to study such an integral, in the more general case where $i\Phi$ is complex-valued. On the other hand we want to improve the $O(N^{-\infty})$ estimates into $O(e^{-cN})$. This is done via a complex change of variables; to this end we have to impose real-analytic regularity on Φ and a.

Let us introduce a notion of analytic phase, which generalises positive phase functionsa as appearing in [28].

Definition 3.11. Let $d, k \in \mathbb{N}$. Let Ω be a domain of $\mathbb{R}^d$. Let Φ be a real-analytic function on $\Omega \times \mathbb{R}^k$. For each $\lambda \in \mathbb{R}^k$ we let $\Phi_\lambda = \Phi(\cdot, \lambda)$. Then Φ is said to be an *analytic phase* on Ω under the following conditions.

- There exists an open set $\widetilde{\Omega} \subset \mathbb{C}^d$ such that, for every $\lambda \in \mathbb{R}^k$, the function Φ_λ extends to a holomorphic function $\widetilde{\Phi}_\lambda$ on $\widetilde{\Omega}$.
- For every $\lambda \in \mathbb{R}^k$, there exists exactly one point $\tilde{x}_\lambda \in \widetilde{\Omega}$ such that $d\widetilde{\Phi}_\lambda(\tilde{x}_\lambda) = 0$; this critical point is non-degenerate. There holds $\widetilde{\Phi}_\lambda(\tilde{x}_\lambda) = 0$.
- One has $\tilde{x}_0 = 0$ and moreover $\Re\Phi_0 < 0$ on $\Omega \setminus \{0\}$.

Under the conditions of Definition 3.11, the function $\lambda \mapsto \tilde{x}_\lambda$ is real-analytic. A first change of integration paths leads to the usual definition of positive phase functions [28]. That is, one can assume, without loss of generality, that $\tilde{x}_\lambda = 0$.

Proposition 3.12. *Let Φ_λ be an analytic phase in the sense of Definition 3.11, and $\widetilde{\Phi}_\lambda$ its extension on the domain $\widetilde{\Omega}$. We let $\Omega_\lambda = \left(\mathbb{R}^d + \tilde{x}_\lambda\right) \cap \widetilde{\Omega}$. There exists $c' > 0, C > 0$, and a small neighbourhood $\Lambda \subset \mathbb{R}^k$ of zero, such that the following is true.*

Let a_λ be a family of real-analytic functions on Ω which extend to holomorphic functions $\tilde{a}_\lambda$ on $\widetilde{\Omega}$. Then, for every $\lambda \in \Lambda$ and every $N \in \mathbb{N}$,

$$\left| \int_\Omega e^{N\Phi_\lambda} a_\lambda - \int_{\Omega_\lambda} e^{N\widetilde{\Phi}_\lambda} \tilde{a}_\lambda \right| \leq C \sup_{\widetilde{\Omega}} |\tilde{a}_\lambda| e^{-c'N}.$$

Moreover, for $\lambda \in \Lambda$, one has $\Re\widetilde{\Phi}_\lambda < 0$ on $\Omega_\lambda \setminus \{\tilde{x}_\lambda\}$.

Proof. The proof proceeds in two steps. In the first step, we apply the Morse lemma and show that, for some analytic symbol b_λ, one has

$$\int_\Omega e^{N\Phi_\lambda} a_\lambda = \int_\Omega e^{-N|y|^2} b_\lambda(y)\mathrm{d}y + O(e^{-cN}).$$

In the second step, we provide an expansion, up to an exponentially small error, for the right-hand term above. We let V be an open subset of Ω containing 0. Then, for every $\lambda \in \mathbb{R}^k$, either $\widetilde{x}_\lambda \in \mathbb{R}^d$, in which case there is nothing to prove, or the set $V + [0,1]\widetilde{x}_\lambda$ has real dimension $d+1$. In the latter case, the boundary of $V + [0,1]\widetilde{x}_\lambda$ can be decomposed as follows:

$$\partial(V + [0,1]\widetilde{x}_\lambda) = V \cup (V + \widetilde{x}_\lambda) \cup (\partial V + [0,1]\widetilde{x}_\lambda).$$

By hypothesis, there exists $c' > 0$ such that $\Re\Phi_0 < -2c'$ on ∂V. By continuity (and since $\widetilde{x}_\lambda$ has real-analytic dependence on λ), for λ in a small neighbourhood Λ of zero, one has $V + [0,1]\widetilde{x}_\lambda \subset \widetilde{\Omega}$ and $\Re\widetilde{\Phi}_\lambda < -c'$ on $\partial V + [0,1]\widetilde{x}_\lambda$.

Then, the contour integral of $e^{N\widetilde{\Phi}_\lambda} a_\lambda$ on $\partial(V + [0,1]\widetilde{x}_\lambda)$ is zero, so that

$$\left| \int_V e^{N\Phi_\lambda} a_\lambda - \int_{V+\widetilde{x}_\lambda} e^{N\widetilde{\Phi}_\lambda} \tilde{a}_\lambda \right| \leq C \sup_{\widetilde{\Omega}} |\tilde{a}_\lambda| e^{-c'N}.$$

Since $\Omega \setminus V \Subset U$, the first integral is exponentially close to the integral over Ω. In the same way, one can replace the second integral by an integral over $\widetilde{\Omega}_\lambda$. This ends the proof. □

We are now in position to prove an analytic stationary phase Lemma.

Proposition 3.13. *Let Φ be an analytic phase on a domain Ω. There exists $c > 0, c' > 0, C' > 0$, a neighbourhood $\Lambda \subset \mathbb{R}^k$ of zero, and a biholomorphism $\widetilde{\kappa}_\lambda$, with real-analytic dependence*[2] *on $\lambda \in \Lambda$, such that the associated Laplace operator $\widetilde{\Delta}(\lambda) = \kappa_\lambda \circ \Delta \circ \kappa_\lambda^{-1}$ satisfies, for every function a_λ holomorphic on $\widetilde{\Omega}$:*

$$\int_\Omega e^{N\Phi_\lambda} a_\lambda = \sum_{k=0}^{c\lambda} \left(k! N^{\frac{d}{2}+k}\right)^{-1} \widetilde{\Delta}(\lambda)^k (\tilde{a}_\lambda J_\lambda^{-1})(\widetilde{x}_\lambda) + R_\lambda(N),$$

where, uniformly in $\lambda \in \Lambda$,

$$|R_\lambda(N)| \leq C e^{-c'N} \sup_{\widetilde{\Omega}} |\tilde{a}_\lambda|,$$

and J_λ is the Jacobian determinant associated with the change of variables.

Proof. For $y = (y_1, \ldots, y_d) \in \mathbb{C}^d$ we denote $y \cdot y = \sum_{i=1}^d y_i^2$. If in particular $y \in \mathbb{R}^d$, we denote

$$|y| = \sqrt{y \cdot y} = |y|_{\ell^2} = (y_1^2 + \ldots + y_d^2)^{\frac{1}{2}}.$$

By Proposition 3.12, without loss of generality $\widetilde{x}_\lambda = 0$ so that $\Re(\widetilde{\Phi}_\lambda) < 0$ on $\Omega \setminus \{0\}$.

The holomorphic Morse lemma [31] states that there is a biholomorphism κ_λ of neighbourhoods of 0 in $\mathbb{C}^d$, with real-analytic dependence on λ, such that, for every x in the domain of κ,

$$\widetilde{\Phi}_\lambda(\kappa_\lambda(x), \kappa_\lambda(x)) = -\kappa_\lambda(x) \cdot \kappa_\lambda(x).$$

[2]By this we mean: a real-analytic function κ on $U \times \Lambda$, where U is a neighbourhood of 0 in $\widetilde{\Omega}$, holomorphic in the first variable, such that there exists σ with the same properties, satisfying $\sigma(\kappa(x,\lambda),\lambda) = \kappa(\sigma(x,\lambda),\lambda) = x$ for all $(x,\lambda) \in U \times \Lambda$.

Let V be a small neighbourhood of 0 in $\mathbb{C}^d$ such that κ_λ is well-defined on V, and let $V_{\mathbb{R}} = V \cap \mathbb{R}^d$. Since $\Re(\widetilde{\Phi}_\lambda(x)) < 0$ for $0 \neq x \in \Omega$, uniformly in λ close to 0, one can restrict the domain of integration: for some small $c' > 0$ and C depending only on Φ, one has

$$\left| \int_\Omega e^{N\Phi_\lambda} a_\lambda - \int_{V_{\mathbb{R}}} e^{N\Phi_\lambda} a_\lambda \right| \leq C \sup(|a_\lambda|) e^{-c'N}.$$

Applying the change of variables κ_λ yields

$$\int_{V_{\mathbb{R}}} e^{N\Phi_\lambda} a_\lambda = \int_{W_\lambda} e^{-Ny\cdot y} (\widetilde{a}_\lambda \circ \kappa_\lambda^{-1})(y) J_\lambda(y) \mathrm{d}y,$$

where $W_\lambda = \kappa_\lambda(V_{\mathbb{R}})$, and J_λ is the appropriate Jacobian.

We let $b_\lambda = (\widetilde{a}_\lambda \circ \kappa_\lambda^{-1}) J_\lambda$. Then, by Proposition 2.14, the function $\widetilde{a}_\lambda \circ \kappa_\lambda^{-1}$, which is bounded and holomorphic on a small open neighbourhood of 0, belongs to some analytic space $H(2, r_1, \kappa_\lambda(V))$ for r_1 large depending only on r and Φ_λ if V is chosen small enough. Without loss of generality, $J_\lambda \in H(2, r_1, \kappa_\lambda(V))$ as well. Then, by Proposition 2.13, b_λ belongs to $H(r_1, \kappa_\lambda(V))$, with r_1 depending only on r and Φ_λ, and the norm of b_λ is controlled as follows: there exists C which depends only on $\widetilde{\Phi}_\lambda$ and $\widetilde{\Omega}$ such that

$$\|b\|_{H(r_1, \kappa_\lambda(V))} \leq C \sup_{\widetilde{\Omega}} |\widetilde{a}_\lambda|.$$

The biholomorphism κ_λ does not preserve $\mathbb{R}^d$ (unless Φ_λ is real-valued). We now wish to change contours so that

$$\int_{W_\lambda} e^{-Ny\cdot y} b_\lambda(y) \mathrm{d}y = \int_{V_{\mathbb{R}}} e^{-Ny\cdot y} b_\lambda(y) \mathrm{d}y + O(e^{-c'N} \sup |b_\lambda|).$$

Consider the following homotopy of functions on $\mathbb{C}^d$:

$$\sigma_t(z) = \Re(z) + (1-t)\Im(z).$$

Then $\sigma_0 = Id$ while σ_1 is the projection on the real locus. If $y \in W_\lambda$ is not zero, then $y \cdot y > 0$, so that $\sigma_t(y) \cdot \sigma_t(y) \geq y \cdot y > 0$. Hence, the set $U \cup_{t\in[0,1]} \sigma_t(W_\lambda)$, of real dimension $d+1$, is contained in $\{y \cdot y > 0\} \cup \{0\}$. Then, since

$$\partial U = W_\lambda \cup \sigma_1(W_\lambda) \cup U'$$

with U' far from zero, and since the contour integral over ∂U is zero, one has, for some $c' > 0$ and $C > 0$ depending only on Φ,

$$\left| \int_{W_\lambda} e^{-Ny\cdot y} b_\lambda(y) \mathrm{d}y - \int_{\sigma_1(W_\lambda)} e^{-Ny\cdot y} b_\lambda(y) \mathrm{d}y \right| \leq C e^{-c'N} \sup_{\widetilde{\Omega}} |b_\lambda|.$$

Applying again a domain restriction, there holds

$$\left| \int_{\sigma_1(W_\lambda)} e^{-Ny\cdot y} b_\lambda(y) \mathrm{d}y - \int_{V_{\mathbb{R}}} e^{-Ny\cdot y} b_\lambda(y) \mathrm{d}y \right| \leq C e^{-cN} \sup_{\widetilde{\Omega}} |b_\lambda|.$$

To conclude the first part of the proof, for some $C > 0$ and $c' > 0$, there holds

$$\left| \int_\Omega e^{-N\Phi_\lambda(y)} a_\lambda(y) \mathrm{d}y - \int_{V_{\mathbb{R}}} e^{-Ny\cdot y} b_\lambda(y) \mathrm{d}y \right| \leq C e^{-c'N} \sup_{\widetilde{\Omega}} |b_\lambda|.$$

We now pass to the second step of the proof. Let us prove that, for some $c > 0$ and $c' > 0$, there holds

$$\left| \int_{V_{\mathbb{R}}} e^{-N|y|^2} b_\lambda(y) \mathrm{d}y - N^{-d/2} \sum_{k=0}^{cN} \frac{\Delta^k b_\lambda}{N^k k!}(0) \right| \leq C \|b\|_{H(r_1,V)} e^{-c'N}.$$

Let us first replace b_λ by its Taylor series up to $2cN$:

$$\left|b_\lambda(y) - \sum_{|\nu|\le 2cN} \frac{b_{\lambda,\nu}}{\nu!}\,y^\nu\right| \le \frac{\|b_\lambda\|_{C^{2cN+1}}|y|_{\ell^1}^{2cN+1}}{(2cN+1)!} \le Cr_1^{2cN}|y|^{2cN}\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|.$$

The integral of the remainder is then controlled as follows, by the Stirling formula:

$$\begin{aligned}
Cr_1^{2cN}\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|\int_{V_{\mathbb{R}}} e^{-N|y|^2}|y|^{2cN}\mathrm{d}y &\le Cr_1^{2cN}\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|\int_{\mathbb{R}^d} e^{-N|y|^2}|y|^{2cN+1}\mathrm{d}y\\
&\le CN^{-\frac{d}{2}-1}r_1^{2cN}N^{-cN}\Gamma(cN+d/2+1)\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|\\
&\le CN^{-1}r_1^{2cN}N^{-cN}\Gamma(cN+1)\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|\\
&\le CN^{-1}\exp(cN\log(r_1^2)-cN\log(N)+cN\log(cN)-cN)\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|\\
&\le CN^{-1}\exp(N\log(r_1^2c/e))\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|.
\end{aligned}$$

Thus, as long as $c<\frac{e}{r_1^2}$, for some $c'>0$ one has

$$\left|\int_{V_{\mathbb{R}}} e^{-Ny^2}\left(b_\lambda(y) - \sum_{|\nu|\le 2cN}\frac{b_{\lambda,\nu}}{\nu!}y^\nu\right)\mathrm{d}y\right| = O(e^{-c'N}\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|).$$

It remains to estimate, for every $0\le j\le 2cN$, the integral

$$\int_{V_{\mathbb{R}}} e^{-N|y|^2}\sum_{|\nu|=j}\frac{b_{\lambda,\nu}}{\nu!}y^\nu\mathrm{d}y.$$

Let us first show that one can replace the integral over $V_{\mathbb{R}}$ by an integral over $\mathbb{R}^d$, up to an exponentially small error.

One has, as $b_\lambda\in H(0,r_1,V)$ with controlled norm,

$$\sum_{|\nu|=j}|b_{\lambda,\nu}|\le Cr_1^j j!\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|.$$

Moreover,

$$|y|^j = (y_1^2+\ldots+y_d^2)^{j/2}\ge d^{-\frac{j}{2}}(|y_1|+\ldots+|y_d|)^j = d^{-\frac{j}{2}}\sum_{|\nu|=j}\frac{j!}{\nu!}|y|^\nu \ge j!d^{-\frac{j}{2}}\max_{|\nu|=j}\frac{|y|^\nu}{\nu!}.$$

Hence,

$$\left|\sum_{|\nu|=j}\frac{b_{\lambda,\nu}}{\nu!}y^\nu\right|\le C(\sqrt{d}r_1)^j|y|^j\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|.$$

Let $T>0$ be such that $B(0,T)\subset V_{\mathbb{R}}$. Then

$$\begin{aligned}
\left|\int_{\mathbb{R}^d\setminus V_{\mathbb{R}}} e^{-N|y|^2}\sum_{|\nu|=j}\frac{b_{\lambda,\nu}}{\nu!}y^\nu\mathrm{d}y\right| &\le C(\sqrt{d}r_1)^j\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|\int_{T^2}^{+\infty}e^{-Nr}r^{j+d-1}\mathrm{d}r\\
&\le CN^{-d}(\sqrt{d}r_1)^jN^{-j}\sup_{\widetilde{\Omega}}|\widetilde{a}_\lambda|\int_{NT^2}^{+\infty}e^{-r}r^{j+d-1}\mathrm{d}r
\end{aligned}$$

The function $r \mapsto e^{-r/2}r^{j+d-1}$ reaches its maximum at $r = 2(j+d-1)$. If $c < T^2$, then for N large enough $2NT^2 > 2cN + d - 1 \geq j + d - 1$, so that

$$\int_{NT^2}^{+\infty} e^{-r} r^{j+d-1} \mathrm{d}r \leq e^{-NT^2/2}(NT^2)^{j+d-1} \int_{NT^2}^{+\infty} e^{-r/2} \mathrm{d}r \leq Ce^{-NT^2}(NT^2)^{j+d-1}.$$

Hence, for every $N \in \mathbb{N}$,

$$\begin{aligned}\left| \int_{\mathbb{R}^d \setminus V_{\mathbb{R}}} e^{-Ny^2} \sum_{|\nu|=j} \frac{b_{\lambda,\nu}}{\nu!} y^\nu \mathrm{d}y \right| &\leq CN^{-1}(r_1\sqrt{d}T^2)^j e^{-NT^2} \sup_{\widetilde{\Omega}} |\widetilde{a}_\lambda| \\ &\leq CN^{-1}(r_1\sqrt{d}T^2)^{2cN} e^{-NT^2} \sup_{\widetilde{\Omega}} |\widetilde{a}_\lambda| \\ &\leq CN^{-1} \sup_{\widetilde{\Omega}} |\widetilde{a}_\lambda| \exp(N(-T^2 + 2c\log(r_1\sqrt{d}T^2))).\end{aligned}$$

In particular, if $c < \frac{T^2}{2\log(r_1\sqrt{d}T^2)}$ then there exists $c' > 0$ such that

$$\left| \int_{\mathbb{R}^d \setminus V_{\mathbb{R}}} e^{-Ny^2} \sum_{|\nu|=j} \frac{b_{\lambda,\nu}}{\nu!} y^\nu \mathrm{d}y \right| \leq CN^{-1} e^{-c'N} \sup_{\widetilde{\Omega}} |\widetilde{a}_\lambda|.$$

Summing over $0 \leq j \leq 2cN$ yields

$$\left| \int_{\mathbb{R}^d \setminus V_{\mathbb{R}}} e^{-Ny^2} \sum_{|\nu|\leq 2cN} \frac{b_{\lambda,\nu}}{\nu!} y^\nu \mathrm{d}y \right| \leq Ce^{-c'N} \sup_{\widetilde{\Omega}} |\widetilde{a}_\lambda|.$$

We are left with

$$\sum_{j \leq 2cN} \int_{\mathbb{R}^d} e^{-Ny^2} \sum_{|\nu|=j} \frac{b_{\lambda,\nu}}{\nu!} y^\nu \mathrm{d}y = N^{-\frac{d}{2}} \sum_{k=0}^{cN} \frac{\Delta^k b_\lambda(0)}{N^k k!}.$$

This concludes the proof. ☐

Remark 3.14. In what follows, we will apply the complex stationary phase lemma in situations where, for λ belonging to a compact Z, one has $\widetilde{x}_\lambda = 0$ and $\Re\Phi_\lambda < 0$ on $\Omega \setminus \{0\}$. In this setting, Proposition 3.13 is true uniformly for λ in a small, N-independent neighbourhood of Z.

4 Calculus of covariant Toeplitz operators

In this section we prove our three main Theorems.

We begin in Subsection 4.1 with the definition, and the first properties, of covariant Toeplitz operators. Then, in Subsections 4.2 to 4.4, we study them. We prove that they can be composed (Proposition 4.7), and inverted (Propositions 4.8 and 4.9), with a precise control on the analytic classes involved. This allows us to prove Theorem A: see the beginning of Section 4.4 for a detailed proof strategy for Theorems A and B. To conclude, in Subsection 4.5 we prove Theorem C.

Until the end of Section 4, M is a compact real-analytic quantizable Kähler manifold of dimension d.

4.1 Covariant Toeplitz operators

Definition 4.1. Let U denote a small, smooth neighbourhood of the codiagonal in $M \times M$; for instance $U = \{(x,y) \in M \times M, \mathrm{dist}(x,y) < \epsilon\}$ with ϵ small enough so that the section Ψ^N of Definition 1.3 is defined on a neighbourhood of U. The space $T_m^{-,r,R}(U)$ of *covariant analytic Toeplitz operators* consists of operators with kernel

$$T_N^{cov}(f) : (x,y) \mapsto N^d \mathbb{1}_{(x,y)\in U} \Psi^N(x,y) f(N)(x,y),$$

where $f(N)$ is the summation of an analytic symbol $f \in S_m^{r,R}(U)$, with f holomorphic in the first variable and anti-holomorphic in the second variable.

Proposition 4.2. *There exists $c > 0$ such that, for all $(x,y) \in U$, there holds*

$$|\Psi^1(x,y)| \leq e^{-c\,\mathrm{dist}(x,y)^2}.$$

Proof. If $x = y$ then $\Psi^1(x,y) = |s(x)|^2 e^{-2\phi(x)} = 1$. In a holomorphic chart ρ for M around x (which sends 0 to x), one can choose ϕ such that the Taylor expansion of $\phi \circ \rho$ at zero is $\phi \circ \rho(z) = |z|^2 + O(|z|^3)$. Then $\mathrm{dist}(x, \rho(z)) = |z|^2 + O(|z|^3)$ as well, so that

$$|\Psi^1(x, \rho(z))| = e^{-\phi(x) - \phi(\rho(z)) + 2\widetilde{\phi}(x,\rho(z))} = e^{-|z|^2 + O(|z|^3)}$$

is smaller than $e^{-c\,\mathrm{dist}(x,\rho(z))^2}$ on a neighbourhood of 0. $\square$

Covariant Toeplitz operators are almost endomorphisms of $H_0(M, L^{\otimes N})$.

Proposition 4.3. *Let U denote a small, smooth neighbourhood of the diagonal in $M \times M$. There exists $c > 0$ such that the following is true. Let $f \in S_m^{r,R}(U)$ be holomorphic in the first variable and anti-holomorphic in the second variable, and S_N denote the Bergman kernel on M.*

Then, as $N \to +\infty$,

$$S_N T_N^{cov}(f) = T_N^{cov}(f) + O_{L^2 \mapsto L^2}(e^{-cN}).$$

Proof. We apply the Kohn estimate (Proposition 1.2) to the kernel of $T_N^{cov}(f)$. Let χ be a smooth function on $M \times M$, which is equal to 1 on a neighbourhood of the diagonal and is supported inside U. Then, since $|\Psi| < 1$ outside the diagonal there exists c such that

$$\sup_{y \in M} \|x \mapsto N^d \Psi^N(x,y)(1 - \chi(x,y))\|_{L^2} = O(e^{-cN}).$$

In particular, since $f(N)(x,y)$ is bounded independently on x, y, N by Proposition 3.6, one has

$$\sup_{y \in M} \|x \mapsto (1 - \chi(x,y)) T_N^{cov}(f)(x,y)\|_{L^2} = O(e^{-cN}).$$

Since S_N is an orthogonal projector, it reduces the L^2 norm, so that

$$\sup_{y \in M} \|S_N(x \mapsto (1 - \chi(x,y)) T_N^{cov}(f)(x,y))\|_{L^2} = O(e^{-cN}).$$

Moreover, $x \mapsto \chi(x,y) T_N^{cov}(f)(x,y)$ is holomorphic except on $\{x \in M, 0 < \chi(x,y) < 1\}$ where $T_N^{cov}(f)(x,y)$ is exponentially small. Then

$$\sup_{y \in M} \|\overline{\partial}(x \mapsto \chi(x,y) T_N^{cov}(f)(x,y))\|_{L^2} \leq \|\overline{\partial}\chi\|_{L^\infty} O(e^{-cN}) = O(e^{-cN}).$$

Hence, by (1),

$$\sup_{y\in M}\|(I-S_N)(x\mapsto \chi(x,y)T_N^{cov}(f)(x,y))\|_{L^2}=O(e^{-cN}).$$

In particular,

$$\sup_{y\in M}\|(I-S_N)(x\mapsto T_N^{cov}(f)(x,y))\|_{L^2}=O(e^{-cN}).$$

Since M is compact, its volume is finite, so that one can conclude:

$$\begin{aligned}\|(I-S_N)T_N^{cov}(f)\|^2_{L^2\to L^2} &\leq \iint_{M\times M}|((I-S_N)T_N^{cov}(f))(x,y)|^2\mathrm{d}x\mathrm{d}y\\ &\leq Vol(M)\sup_{y\in M}\|(I-S_N)(x\mapsto T_N^{cov}(f)(x,y))\|_{L^2}=O(e^{-cN}).\end{aligned}$$

□

4.2 Study of an analytic phase

In this work, covariant Toeplitz operators of Definition 4.1 have the following integral kernels:

$$T_N^{cov}(f):(x,y)\mapsto \Psi^N(x,y)\left(\sum_{k=0}^{cN}N^{d-k}f_k(x,y)\right).$$

The integral kernel of the composition of two covariant Toeplitz is of particular interest, so let us study its phase.

If f and g are analytic symbols, then $T_N^{cov}(f)T_N^{cov}(g)$ has the following kernel:

$$(x,z)\mapsto \Psi^N(x,z)\int_M e^{N(2\widetilde{\phi}(x,y)-2\phi(y)+2\widetilde{\phi}(y,z)-2\widetilde{\phi}(x,z))}\left(\sum_{k=0}^{cN}N^{d-k}f_k(x,y)\right)\left(\sum_{j=0}^{cN}N^{d-j}g_j(y,z)\right)\mathrm{d}y.$$

Indeed, if s is a local holomorphic non-vanishing section of L, with $\langle s,s\rangle_h=e^{-2\phi}$, and $\widetilde{\phi}$ denotes the complex extension of ϕ, then for every $(x,y,z)\in M^3$ one has

$$\begin{aligned}\langle \Psi^N(x,y),\Psi^N(y,z)\rangle_h &= s(x)^{\otimes N}\otimes\overline{s(z)}^{\otimes N}e^{2N\widetilde{\phi}(x,y)+2N\widetilde{\phi}(y,z)}\langle s(y),s(y)\rangle_h^N\\ &=\Psi^N(x,z)e^{2N\widetilde{\phi}(x,y)-2N\phi(y)+2N\widetilde{\phi}(y,z)-2N\widetilde{\phi}(x,z)}.\end{aligned}$$

We let Φ_1 be the complex extension (with respect to the middle variable) of the phase appearing in the last formula:

$$\Phi_1:(x,y,\overline{w},\overline{z})\mapsto 2\widetilde{\phi}(x,\overline{w})-2\widetilde{\phi}(y,\overline{w})+2\widetilde{\phi}(y,\overline{z})-2\widetilde{\phi}(x,\overline{z}).$$

We write $\Phi_1(x,y,\overline{w},\overline{z})$ to indicate anti-holomorphic dependence on the two last variables. In particular, Φ_1 is holomorphic on the open set $U\times U$ of $M\times\widetilde{M}\times\overline{M}=M_x\times(M_y\times\overline{M}_{\overline{w}})\times\overline{M}_{\overline{z}}$.

Proposition 4.4. *There exists a smooth neighbourhood U of $\{(x,\overline{z})\in M\times\overline{M},\overline{x}=\overline{z}\}$ such that function Φ_1, on the open set*

$$\{(x,y,\overline{y},\overline{z}),(x,\overline{w})\in U,(y,\overline{w})\in U,(x,\overline{z})\in U\},$$

is an analytic phase of $(y,\overline{w})$, with parameter $\lambda=(x,\overline{z})$. The critical point is $(x,\overline{z})$.

In particular, after a trivialisation of a tubular neighbourhood of

$$\{(x,y,\overline{w},\overline{z})\in M\times\widetilde{M}\times\overline{M},(x,\overline{z})\in U,(y,\overline{w})=(x,\overline{z})\}$$

in

$$\{(x,y,\overline{w},\overline{z})\in M\times\widetilde{M}\times\overline{M},(x,\overline{z})\in U\}$$

as a vector bundle over the former, the analytic phase Φ_1 satisfies the assumptions of Remark 3.14.

Proof. On the diagonal $x = z$, the Taylor expansion of Φ_1 near $(x, \overline{x})$ with respect to the variables $(y, \overline{w})$ is

$$(y, \overline{w}) \mapsto -(x - y)(\overline{x} - \overline{w}) + O(|x - y|^3 + |\overline{x} - \overline{w}|^3),$$

so that there is a critical point at $(x, \overline{x})$ in $\widetilde{M}$, where the real part of Φ_1 reaches zero as nondegenerate maximum. Hence, for z close to x there is only one critical point near $(x, \overline{x})$.

This critical point is explicit: it solves the following two equations:

$$0 = \overline{\partial}_{\overline{w}}\Phi_1 = -\overline{\partial}_2\widetilde{\phi}(x, \overline{w}) + \overline{\partial}_2\widetilde{\phi}(y, \overline{w})$$
$$0 = \partial_y\Phi_1 = -\partial_1\widetilde{\phi}(y, \overline{z}) + \partial_1\widetilde{\phi}(y, \overline{w}).$$

These equations are satisfied if $y = x, \overline{w} = \overline{z}$, which concludes the proof. □

4.3 Composition of covariant Toeplitz operators

In this subsection we study the composition rules for operators with kernels of the form

$$T_N^{cov}(f)(x, y) = \Psi^N(x, y)\left(\sum_{k=0}^{cN} N^{d-k} f_k(x, y)\right).$$

Here, for a small, smooth neighbourhood U of the diagonal in $M \times M$, one has $f \in S_m^{r,R}(U)$, and f is holomorphic in the first variable and anti-holomorphic in the second variable.

It is well-known that such operators can be formally composed, that is $T_N^{cov}(f)T_N^{cov}(g) = T)N^{cov}(f\sharp g) + O(N^{-\infty})$ where $f\sharp g$ is a classical symbol. We first study this formal calculus by proving a weak form of the Wick rule in Proposition 4.5. Then in Lemma 4.6 we control, in an analytic norm, differential operators as the ones relating $f\sharp g$ to f and g. This allows us, in Proposition 4.7, to prove that, if f and g are analytic symbols, then $f\sharp g$ is also an analytic symbol, so that one can perform an analytic summation (as in Proposition 3.6), and the error in the composition becoms $O(e^{-cN})$.

Proposition 4.5. *(See also [4], Lemme 2.33, for the normalised covariant version) The composition of two covariant Toeplitz operators can be written as a formal series in N^{-1}. More precisely, if f and g are functions on a neighbourhood of the diagonal in $M \times M$, holomorphic in the first variable, anti-holomorphic in the second variable, then*

$$T_N^{cov}(f)T_N^{cov}(g) = T_N^{cov}(h) + O(N^{-\infty}),$$

where h is a formal series $h \sim \sum_{k\geq 0} N^{-k}h_k$, holomorphic in the first variable, anti-holomorphic in the second variable. The composition law can be written as

$$h_k = B_k(f, g),$$

where B_k is a bidifferential operator of degree at most k in f and at most k in g.

Proof. It is well-known (see [5], Theorem 2) that there exists an invertible formal series a of functions defined on a neighbourhood of the diagonal in $M \times M$, holomorphic in the first variable and anti-holomorphic in the second variable, which correspond to the Bergman kernel, that is, such that

$$T_N^{cov}(a) = S_N + O(N^{-\infty}).$$

In Theorem A, we will prove that a is in fact an analytic symbol; for the moment, it is sufficient to know that a exists as a formal series.

Let us deform covariant Toeplitz operators by this formal symbol a, into *normalised* covariant Toeplitz operators of the form $T_N^{cov}(f * a)$. Here $*$ denotes the Cauchy product of symbols (Proposition 3.8). Since in this case f and g are simply holomorphic functions one has $f * a = fa$ and $g * a = ga$.

We will first prove our claim for this modified quantization: that is, there exists a sequence of bidifferential operators $(C_k)_{k\geq 0}$ acting on functions on a neighbourhood of the diagonal in $M \times M$, such that, given two such functions f and g, if we let

$$h = \sum_{k=0}^{+\infty} N^{-k} C_k(f,g) + O(N^{-\infty}),$$

then

$$T_N^{cov}(h * a) = T_N^{cov}(fa) T_N^{cov}(ga) + O(N^{-\infty}).$$

Moreover, C_k is of order at most k in each of its arguments. Then, we will relate the coefficients C_k with the coefficients B_k in the initial claim.

The claim is easier to prove for the coefficients C_k because normalised covariant Toeplitz quantization follows the Wick rule. Indeed, if the function f, near a point x_0, depends only on the first variable (that is, the restriction of f to the diagonal is, near this point, a holomorphic function on M), then the kernel $T_N^{cov}(af)(x,y)$, for x close to x_0, can be written as $f(x)T_N^{cov}(a)(x,y) = f(x)S_N(x,y)+O(N^{-\infty})$. In particular, for x close to x_0 the Wick rule holds:

$$T_N^{cov}(af)T_N^{cov}(ag)(x,y) = T_N^{cov}(afg)(x,y) + O(N^{-\infty}),$$

since by Proposition 4.3 the kernel of $T_N^{cov}(ag)$ is almost holomorphic in the first variable, up to an $O(N^{-\infty})$ error. Thus, locally where f depends only on the first variable, there holds

$$\forall k \geq 1,\ C_k(f,g) = 0.$$

More generally, we wish to compute

$$N^{2d}\Psi^N(x,z)\int_M \exp(N\Phi_1(x,y,\overline{y},\overline{z}))(fa)(N)(x,\overline{y})(ga)(N)(y,\overline{z})\mathrm{d}y,$$

where we recall that

$$\Phi_1(x,y,\overline{w},\overline{z}) = -2\widetilde{\phi}(x,\overline{w}) + 2\widetilde{\phi}(y,\overline{w}) - 2\widetilde{\phi}(y,\overline{z}) + 2\widetilde{\phi}(x,\overline{z}).$$

Here, we write $(fa)(N)(x,\overline{y})$ to indicate that fa is holomorphic in the first variable and anti-holomorphic in the second variable. Similarly, we write $\Phi_1(x,y,\overline{w},\overline{z})$ to indicate that Φ_1 is a function on $M_x \times \widetilde{M}_{y,\overline{w}} \times M_z$, holomorphic in its two first arguments and anti-holomorphic in the third argument; we integrate over M which is the subset of $\widetilde{M}$ such that $\overline{w} = \overline{y}$.

First of all, since for any $(x,z) \in U$ one has $|\Psi^N(x,z)| \leq e^{-cN\,\mathrm{dist}(x,z)^2}$, then there exists $C > 0$ such that, for any analytic symbol b on $U \times U$, there holds

$$\begin{aligned}
&N^{2d}\sup_x \int_M \left|\Psi^N(x,z)\int_M \exp(N\Phi_1(x,y,\overline{y},\overline{z}))b(N)(x,y,\overline{y},z)\mathrm{d}y\right|\mathrm{d}z\\
\leq & N^{2d}\sup_{U\times U}|b(N)|\sup_x \int_M\int_M |\Psi^N(x,y)||\Psi^N(y,z)|\mathrm{d}y\mathrm{d}z\\
\leq & \sup_{U\times U}|b(N)| N^{2d}\sup_x \int_{M\times M} e^{-Nc\,\mathrm{dist}(x,y)^2 - Nc\,\mathrm{dist}(y,z)^2}\mathrm{d}y\mathrm{d}z\\
\leq & C\sup_{U\times U}|b(N)|.
\end{aligned}$$

In particular, by the Schur test, the operator with kernel

$$(x,z) \mapsto N^{2d} \int_M \exp(N\Phi_1(x,y,\overline{y},\overline{z}))b(x,y,\overline{y},z)\mathrm{d}y$$

is bounded from $L^2(M, L^{\otimes N})$ to itself, independently on N.

As $\partial_y \Phi_1$ vanishes in a non-degenerate way at $\overline{w} = \overline{z}$, one can write

$$f(x,\overline{w}) = f(x,\overline{z}) - \partial_y\Phi_1 \cdot F_1(x,\overline{z},y,\overline{w}).$$

Thus,

$$\begin{aligned}
N^{2d}\Psi^N(x,z) \int_M &\exp(N\Phi_1(x,y,\overline{y},\overline{z}))(fa)(N)(x,\overline{y})(ga)(N)(y,\overline{z})\mathrm{d}y \\
&= N^{2d}\Psi^N(x,z)f(x,\overline{z}) \int_M \exp(N\Phi_1(x,y,\overline{y},\overline{z}))a(N)(x,\overline{y})(ga)(N)(y,\overline{z})\mathrm{d}y \\
&+ N^{-1}N^{2d}\Psi^N(x,z) \int_M \exp(N\Phi_1(x,y,\overline{y},\overline{z}))a(N)(x,\overline{y})\partial_M \left[F_1(x,\overline{z},y,\overline{y})(ga)(N)(y,\overline{z})\right] \mathrm{d}y.
\end{aligned}$$

The first term in the right-hand side above is equal to

$$f(x,\overline{z}) \int_M T_N^{cov}(a)(x,\overline{y})T_N^{cov}(ga)(y,\overline{z})\mathrm{d}y = f(x,\overline{z})T_N^{cov}(ga)(x,\overline{z}) + O(N^{-\infty}),$$

since $T_N^{cov}(a) = S_N + O(N^{-\infty})$.

In the second line, which is of order N^{-1} by a Schur test, derivatives of g of order at most 1 appear. This remainder can be written as

$$\begin{aligned}
N^{-1}N^{2d}\Psi^N(x,z) \int_M &\exp(N\Phi_1(x,y,\overline{y},\overline{z}))a(N)(x,\overline{y}) \left[\partial_y F_1(x,\overline{z},y,\overline{y})\right] (ga)(N)(y,\overline{z})\mathrm{d}y \\
&+ N^{-1}N^{2d}\Psi^N(x,z) \int_M \exp(N\Phi_1(x,y,\overline{y},\overline{z}))a(N)(x,\overline{y})F_1(x,\overline{z},y,\overline{y})[\partial_y(ga)(N)(y,\overline{z})\mathrm{d}y.
\end{aligned}$$

We recover the initial expression, where f has been replaced with either F_1 or $\partial_y F_1$, and g has potentially been differentiated once. Thus, by induction, the coefficient $C_k(f,g)$ only differentiates at most k times on g. By duality, $C_k(f,g)$ only differentiates at most k times on f.

Let us now relate the coefficients C_k and B_k. Let a^{*-1} denote the inverse of a for the Cauchy product. One has

$$T_N^{cov}(f)T_N^{cov}(g) = T_N^{cov}((fa^{*-1}) * a)T_N^{cov}((ga^{*-1}) * a) + O(N^{-\infty}) = T_N^{cov}((C_k(f,g))_{k\geq 0} * a) + O(N^{-\infty}),$$

so that the coefficients B_k in the initial claim are recovered as

$$B_k(f,g) = \sum_{j+l+m\leq k} a_j C_{k-j-l-m}(fa_l^{*-1}, ga_m^{*-1}),$$

thus B_k itself differentiates at most k times on f and at most k times on g. □

The covariant normalised version of the result above is shown in [4], using a different computational method for the stationary phase.

The previous proposition predicts that, when applying a stationary phase lemma to Φ_1 in order to study $T_N^{cov}(f)T_N^{cov}(g)$, at order n, only derivatives of f and g at order n will appear. However, in the stationary phase (Lemma 3.13), these derivatives appear in the form of an usual Laplace operator, conjugated by a change of variables. Let us then prove the following technical lemma.

Lemma 4.6. *Let U, V, Λ be domains in $\mathbb{C}^d$ containing 0. Let κ_λ be a biholomorphism from V to U,with real-analytic dependence on $\lambda \in \Lambda$, and such that $\kappa_\lambda(0) = 0$ for all $\lambda \in \Lambda$. Let $\kappa(\lambda, v) \mapsto \kappa_\lambda(v)$, and suppose that there exists C_κ, r_0, m_0 such that, for all $j \in \mathbb{N}$, one has*

$$\|\kappa\|_{C^j(V\times\Lambda)} \leq C\frac{r_0^j j!}{(j+1)^{m_0}}.$$

Then the following is true for all $m \geq m_0, r \geq 8r_0 2^{m-m_0}$.

Let f be a real-analytic function on $U \times \Lambda$, and suppose that there exists C_f and $k \geq 0$ such that

$$\|f\|_{C^j(U\times\Lambda)} \leq C_f\frac{r^j(j+k)!}{(j+k+1)^m}.$$

Let $n \leq k$ and $i \leq 2n$; let ∇_v^i denote the i-th gradient (as in Definition 2.10) over the first set of variables, acting on $V \times \Lambda$; then

$$g \mapsto (\lambda \mapsto \nabla_v^i g(\kappa_\lambda(v), \lambda)_{v=0})$$

is a differential operator of degree i, from functions on $U \times \Lambda$ to vector-valued functions on Λ. Let $(\nabla_\kappa^i)^{[\leq n]}$ denote the truncation of this differential operator to a differential operator of degree less than n.

Then, with

$$\gamma = 4Cr,$$

one has, for every $j \geq 0$,

$$\|(\nabla_\kappa^i)^{[\leq n]}f\|_{\ell^1(C^j(\Lambda))} \leq i^{d+1}j^{d+1}\gamma^i C_f\frac{r^{j+i}}{(i+j+l+1)^m}\begin{cases}(i+j+k)! & \text{if } i \leq n\\ \max((n+j+k)!(i-n)!, (j+k)!i!) & \text{otherwise.}\end{cases}$$

Proof. Let us make explicit the operator $(\nabla_\kappa^i)^{[\leq n]}$. Given a polyindex μ with $|\mu| = i$, the Faà di Bruno formula states:

$$\partial_v^\mu(f(\kappa_\lambda(v), \lambda))_{v=0} = \sum_{P\in\Pi(\{1,\ldots,i\})} f^{|P|}(0,\lambda)\prod_{E\in P}(\partial^E\kappa_\lambda)(0),$$

where the sum runs among all partitions $P = \{E_1, \ldots, E_{|P|}\}$ of $\{1, \ldots, i\}$.

When considering the operator $(\nabla_\kappa^i)^{[\leq n]}$, we only need to consider partitions P such that $|P| \leq n$. If the sizes $|E_1| = s_1, \ldots, |E_{|P|}| = s_{|P|}$ of the elements of P are fixed, the number of possible partitions is simply

$$\frac{i!}{(|P|)!s_1!\ldots s_{|P|}!}.$$

Then, since there are less than i^d polyindices μ with $|\mu| = i$, one has, for all $\rho \in \mathbb{N}^d$ with $|\rho| = j$, by differentiation of the Faà di Bruno formula and Proposition 3.2,

$$\|\partial^\rho((\nabla_\kappa^i)^{[\leq n]}f)\|_{\ell^1} \leq i^d \sum_{|P|=1}^{\min(n,i)} \sum_{\substack{e_0+\ldots+e_{|P|}=j\\ s_1+\ldots+s_{|P|}=|P|}} \frac{j!}{e_0!e_1!\ldots e_{|P|}!}\frac{i!}{(|P|)!s_1!\ldots s_{|P|}!}\|f\|_{C^{|P|+e_0}}\prod_{i=1}^{|P|}\|\kappa\|_{C^{s_i+e_i}}.$$

Here κ denotes the real-analytic function $(\lambda, v) \mapsto \kappa_\lambda(v)$.

In particular, since there are less than j^d polyindices ρ such that $|\rho| = j$, one has

$$\|\partial^\rho((\nabla_\kappa^i)^{[\leq n]}f)\|_{\ell^1} \leq i^d j^d \sum_{|P|=1}^{\min(n,i)} \sum_{\substack{e_0+\ldots+e_{|P|}=j\\ s_1+\ldots+s_{|P|}=|P|}} \left(\frac{j!}{e_0!e_1!\ldots e_{|P|}!}\frac{i!}{(|P|)!s_1!\ldots s_{|P|}!}\|f\|_{C^{|P|+e_0}}\prod_{i=1}^{|P|}\|\kappa\|_{C^{s_i+e_i}}\right). \quad (3)$$

Since, for all $j \geq 0$, one has

$$\|\kappa\|_{C^j(V\times\Lambda)} \leq C\frac{r_0^j j!}{(j+1)^{m_0}},$$

by Lemma 2.11, for all $m \geq m_0, r \geq 8r_0 2^{m-m_0}$, one has

$$\|\kappa\|_{C^j} \leq C\frac{(r/8)^j j!}{(j+1)^m}.$$

In particular, if $j \geq 1$, there holds

$$\|\kappa\|_{C^j} \leq C\frac{(r/4)^j(j-1)!}{j^m} j\left(\frac{j}{j+1}\right)^m 2^{-j} \leq C\frac{(r/4)^j(j-1)!}{j^m},$$

since

$$j\left(\frac{j}{j+1}\right)^m 2^{-j} \leq j2^{-j} \leq 1.$$

Let us suppose further that

$$\|f\|_{C^j(U\times\Lambda)} \leq C_f\frac{r^j R^l (j+l)!}{(j+l+1)^m}.$$

Then, the contribution of one term in the sum (3) is

$$\begin{aligned}
&\frac{j!}{e_0!e_1!\dots e_{|P|}!}\frac{i!}{(|P|)!s_1!\dots s_{|P|}!}\|f\|_{C^{|P|+e_0}}\prod_{i=1}^{|P|}\|\kappa\|_{C^{s_i+e_i}} \\
&\qquad \leq C_f C^{|P|}\frac{r^{|P|+e_0}(r/4)^{i+j-e_0}R^l(|P|+e_0+l)!i!}{(|P|+e_0+l+1)^m(|P|)!s_1!\dots s_{|P|}!}\frac{j!(s_1+e_1-1)!\dots(s_{|P|}+e_{|P|}-1)!}{e_0!e_1!\dots e_{|P|}!(s_1+e_1)^m\dots(s_{|P|}+e_{|P|})^m}.
\end{aligned}$$

As $e_0+\ldots+e_{|P|}=j$ and $s_1+\ldots+s_{|P|}=i$, and since, as soon as $x \geq 0$, $y \geq 0$, there holds

$$(1+x)(1+y) = 1+x+y+xy \geq 1+x+y,$$

one has

$$(|P|+e_0+l+1)^m(s_1+e_1)^m\dots(s_{|P|}+e_{|P|})^m \geq (|P|+j+i+l-|P|+1)^m = (j+i+l+1)^m,$$

so that one can simplify

$$\begin{aligned}
&C_f C^{|P|}\frac{r^{|P|+e_0}(r/4)^{i+j-e_0}R^l(|P|+e_0+l)!i!}{(|P|+e_0+l+1)^m(|P|)!s_1!\dots s_{|P|}!}\frac{j!(s_1+e_1-1)!\dots(s_{|P|}+e_{|P|}-1)!}{e_0!e_1!\dots e_{|P|}!(s_1+e_1)^m\dots(s_{|P|}+e_{|P|})^m} \\
&\qquad \leq C_f C^{|P|}\frac{r^{|P|+e_0}(r/4)^{i+j-e_0}R^l(|P|+e_0+l)!}{(j+i+l+1)^m}\frac{i!j!(s_1+e_1-1)!\dots(s_{|P|}+e_{|P|}-1)!}{e_0!(|P|)!s_1!\dots s_{|P|}!e_1!\dots e_{|P|}!}.
\end{aligned}$$

By Lemma 5.3, one has

$$\frac{(s_1+e_1-1)!\dots(s_{|P|}+e_{|P|}-1)!}{s_1!\dots s_{|P|}!e_1!\dots e_{|P|}!} \leq \frac{(i-|P|+j-e_0)!}{(i-|P|+1)!(j-e_0)!}.$$

Hence, the contribution of one term in the sum (3) is smaller than

$$C_f C^{|P|}\frac{i!}{(|P|)!(i-|P|+1)!}\frac{r^{|P|+e_0}(r/4)^{i+j-e_0}R^l(|P|+e_0+l)!j!(i-|P|+j-e_0)!}{(j+i+l+1)^m e_0!(j-e_0)!}.$$

As $(i-|P|+j-e_0)! \leq (j-e_0)!(i-|P|)!2^{i+j-e_0}$ and $i! \leq 2^i(|P|)!(i-|P|)!$, we control each term in the sum (3) with

$$C_f 2^{e_0-j} C^{|P|} r^i \frac{r^{j+|P|}R^l(|P|+e_0+l)!}{(j+i+l+1)^m}\frac{j!(i-|P|)!}{e_0!} \leq C_f 2^{e_0-j}(Cr)^i \frac{r^{j+i}R^l(|P|+e_0+l)!}{(j+i+l+1)^m}\frac{j!(i-|P|)!}{e_0!}.$$

There are $\binom{i}{|P|} \leq 2^i$ choices for positive $s_1,\ldots,s_{|P|}$ such that their sum is i; similarly, there are $\binom{j-e_0+|P|}{|P|} \leq 2^{j-e_0+|P|}$ choices for non-negative $e_1,\ldots,e_{|P|}$ such that their sum is $j-e_0$. Hence

$$\begin{aligned}\|(\nabla_\kappa^i)^{[\leq n]}f\|_{\ell^1(C^j)} &\leq i^d j^d \sum_{|P|=1}^{\min(n,i)}\sum_{e_0=0}^{j} 2^{j+|P|-e_0}2^i C_f 2^{e_0-j}(Cr)^i \frac{r^{j+i}R^l(|P|+e_0+l)!}{(j+i+l+1)^m}\frac{j!(i-|P|)!}{e_0!}\\ &\leq i^d j^d \sum_{|P|=1}^{\min(n,i)}\sum_{e_0=0}^{j} C_f(4Cr)^i \frac{r^{j+i}R^l(|P|+e_0+l)!}{(j+i+l+1)^m}\frac{j!(i-|P|)!}{e_0!}.\end{aligned}$$

By Lemma 5.1, the terms in the sum above are increasing with respect to e_0, so that

$$\|\nabla_v^i f(x,\kappa(x,v,\overline{z}))_{v=0}\|_{\ell^1(C^j)} \leq i^d j^{d+1}\sum_{|P|=1}^{\min(n,i)} C_f(4Cr)^i\frac{r^{j+i}R^l(|P|+j+l)!}{(i+j+l+1)^m}(i-|P|)!.$$

Observe that the quantity in the sum above is log-convex with respect to $|P|$ as it is a product of factorials, so that

$$\|(\nabla_\kappa^i)^{[\leq n]}f\|_{\ell^1(C^j)} \leq i^{d+1}j^{d+1}C_f\frac{r^{j+i}R^l}{(i+j+l+1)^m}(4Cr)^i\max((n+j+l)!(i-n)!\ ,\ (j+l)!i!)$$

if $i \geq n$, and

$$\|(\nabla_\kappa^i)^{[\leq n]}f\|_{\ell^1(C^j)} \leq i^{d+1}j^{d+1}C_f\frac{r^{j+i}R^l}{(i+j+l+1)^m}(4Cr)^i(i+j+l)!$$

if $i \leq n$. This concludes the proof, with $\gamma = 4Cr$. □

We are in position to prove the first part of Theorem B, which does not use the structure of the Bergman kernel. Let us prove that the composition of two covariant Toeplitz operators with analytic symbols also admits an analytic symbol, up to an exponentially small error.

Proposition 4.7. *There exists a small neighbourhood U of the diagonal in $M\times M$, and constants C, m_0, r_0 such that, for every $m \geq m_0, r \geq r_0, R \geq Cr^3$, there exists $c' > 0$ such that, for every $f \in S_m^{r,R}(U)$ and $g \in S_m^{2r,2R}(U)$, holomorphic in the first variable, anti-holomorphic in the second variable, there exists $f\sharp g \in S_m^{2r,2R}(U)$ with the same properties, such that*

$$\|T_N^{cov}(f)T_N^{cov}(g) - T_N^{cov}(f\sharp g)\|_{L^2\mapsto L^2} \leq Ce^{-cN}\|g\|_{S_m^{2r,2R}(U)}\|f\|_{S_m^{r,R}(U)}.$$

Moreover

$$\|f\sharp g\|_{S_m^{2r,2R}(U)} \leq C\|g\|_{S_m^{2r,2R}(U)}\|f\|_{S_m^{r,R}(U)}.$$

Proof. The kernel of $T_N^{cov}(f)T_N^{cov}(g)$ can be written as

$$(x,z)\mapsto \Psi^N(x,z)\int_{y\in M} e^{N\Phi_1(x,y,\overline{y},z)}\left(\sum_{k=0}^{cN}N^{d-k}f_k(x,\overline{y})\right)\left(\sum_{j=0}^{cN}N^{d-j}g_j(y,\overline{z})\right)\mathrm{d}y.$$

Here, and until the end of the proof, we write $f_k(x,\overline{y})$ to indicate that f_k is holomorphic in the first variable and anti-holomorphic in the second variable. We similarly write $g_j(y,\overline{z})$.

Since Φ_1 is an analytic phase (Proposition 4.4), let us apply the stationary phase lemma (Proposition 3.13). There exists a biholomorphism on a neighbourhood of x in $\widetilde{M}$, of the form

$$\kappa_{(x,\overline{z})} : (y,\overline{y}) \mapsto v(x,y,\overline{y},\overline{z}),$$

with holomorphic dependence on $(x,\overline{z})$ (that is, holomorphic in x and anti-holomorphic in z), in which the phase Φ_1 can be written as $-|v|^2$. In particular,

$$v(x,x,\overline{z},\overline{z}) = 0.$$

Let J denote the Jacobian of this change of variables. Then

$$T_N^{cov}(f)T_N^{cov}(g)(x,z) = \Psi^N(x,z)\sum_{k=0}^{\dots}\sum_{j=0}^{\dots}\sum_{n=0}^{\dots} N^{d-k-j-n}\frac{\Delta_v^n}{n!}\left(f_k(x,\overline{y}(x,v,\overline{z}))g_j(y(x,v,\overline{z}),\overline{z})J(x,v,\overline{z})\right)_{v=0} + \dots$$

We will make sense of this sum later on; that is, prove that one can sum until k,j or n is equal to cN, up to an exponentially small error. For the moment, let us treat this formula in decreasing powers of N. Writing

$$T_N^{cov}(f)T_N^{cov}(g)(x,z) = T_N^{cov}(f\sharp g)(x,\overline{z}) = \Psi^N(x,z)\sum_{k=0}^{\dots} N^{d-k}(f\sharp g)_k(x,\overline{z}) + \dots$$

the symbol $f\sharp g$ must be holomorphic in the first variable, anti-holomorphic in the second variable, and such that

$$(f\sharp g)_k(x,\overline{z}) = \sum_{n=0}^{k}\frac{\Delta_v^n}{n!}\left(\sum_{l=0}^{k-n} f_l(x,\overline{y}(x,v,\overline{z}))g_{k-n-l}(y(x,v,\overline{z}),\overline{z})J(x,v,\overline{z})\right)_{v=0}.$$

Here the Laplace operator acts on v.

The proof proceeds now in three steps. In the first step, we write a control of the formal symbol $f\sharp g$ using the analytic symbol structure of f and g and Lemma 4.6. This control involves a complicated quotient of factorials as well as a rational expression similar to the one appearing in Lemma 3.7. The second step is a control the quotients of factorials, thus reducing the proof that $f\sharp g \in S_m^{2r,2R}$ to Lemma 3.7. In the third step we prove that, when identifying between $T_N^{cov}(f)T_N^{cov}(g)$ and $T_N^{cov}(f\sharp g)$, one can perform analytic sums, so that the remainder is exponentially small.

First step.

We wish to control $\|(f\sharp g)_k\|_{C^j(U)}$, which amounts to control, for each $0 \leq n \leq k, 0 \leq l \leq k-n$, the C^j-norm of

$$(x,z) \mapsto \Delta_v^n\left(f_l(x,\overline{y}(x,v,\overline{z}))g_{k-n-l}(y(x,v,\overline{z}),\overline{z})J(x,v,\overline{z})\right)_{v=0}.$$

This bidifferential operator acting on f_l and g_{k-n-l} coincides, up to a multiplicative factor, with the operator B_n considered in Proposition 4.5. Indeed, if $f = f_0$ and $g = g_0$, then

$$(f\sharp g)_k(x,\overline{z}) = \frac{\Delta_v^k}{k!}\left(f_0(x,\overline{y}(x,v,\overline{z}))g_0(y(x,v,\overline{z}))J(x,v,\overline{z})\right)_{v=0} = B_k(f_0,g_0),$$

where $(B_k)_{k\geq 0}$ is the sequence of bidifferential operators appearing in Proposition 4.5. In particular, when expanding

$$\Delta_v^n\left(f_l(x,\overline{y}(x,v,\overline{z}))g_{k-n-l}(y(x,v,\overline{z}),\overline{z})J(x,v,\overline{z})\right)_{v=0},$$

using the Leibniz and Faà di Bruno formulas, no derivative of f_l and g_{k-n-l} of order greater than n will appear. Let us write this expansion.

Until the end of the proof, C^j or analytic norms of functions are implicitly on the domain U or $U \times U$. For every $n \in \mathbb{N}$, by the multinomial formula, there holds

$$\Delta_v^n = \left(\sum_{i=1}^{2d} \frac{\partial^2}{\partial v_j^2} \right)^n = \sum_{\substack{\mu \in \mathbb{N}^{2d} \\ |\mu|=n}} \frac{n!}{\mu!} \partial_v^{2\mu}.$$

Applying the generalised Leibniz rule twice, one has then

$$\begin{aligned}&\Delta_v^n \left(f_l(x, \overline{y}(x, v, \overline{z})) g_{k-n-l}(y(x, v, \overline{z}), \overline{z}) J(x, v, \overline{z}) \right)_{v=0} \\ &\quad = \sum_{\substack{|\mu|=n \\ \nu_1+\nu_2 \leq 2\mu}} \frac{n!(2\mu)!}{\mu!\nu_1!\nu_2!(2\mu-\nu_1-\nu_2)!} \partial_v^{\nu_1} f_l(x, \overline{y}(x, v, \overline{z}))_{v=0} \partial_v^{\nu_2} g_{k-n-l}(y(x, v, \overline{z}), \overline{z})_{v=0} \partial_v^{2\mu-\nu_1-\nu_2} J_{v=0}.\end{aligned}$$

By Proposition 4.5, in the formula above one can replace $\partial_v^{\nu_1} f(x, \overline{y}(x, v, \overline{z}))_{v=0}$ by its truncation into a differential operator of degree less than n, applied on f, which we denote $(\partial_\kappa^{\nu_1})^{[\leq n]} f(x, \overline{z})$ (similarly as in Lemma 4.6). Similarly one can replace $\partial_v^{\nu_2} g(y(x, v, \overline{z}), \overline{z})_{v=0}$ by $(\partial_\kappa^{\nu_2})^{[\leq n]} g(x, \overline{z})$. Then

$$\begin{aligned}&\Delta_v^n \left(f_l(x, \overline{y}(x, v, \overline{z})) g_{k-n-l}(y(x, v, \overline{z}), \overline{z}) J(x, v, \overline{z}) \right)_{v=0} \\ &\quad = \sum_{\substack{|\mu|=n \\ \nu_1+\nu_2 \leq 2\mu}} \frac{n!(2\mu)!}{\mu!\nu_1!\nu_2!(2\mu-\nu_1-\nu_2)!} (\partial_\kappa^{\nu_1})^{[\leq n]} f_l(x, \overline{z}) (\partial_\kappa^{\nu_2})^{[\leq n]} g_{k-n-l}(x, \overline{z}) \partial_v^{2\mu-\nu_1-\nu_2} J_{v=0},\end{aligned}$$

with, by Lemma 5.2,

$$\begin{aligned}\frac{n!\mu_1!}{\nu_1!\nu_2!(2\mu-\nu_1-\nu_2)!} &= \frac{n!}{\mu!} \frac{(2\mu)!}{\nu_1!(2\mu-\nu_1)!} \frac{(2\mu-\nu_1)!}{\nu_2!(2\mu-\nu_1-\nu_2)!} \\ &\leq \frac{n!}{\mu!} \frac{(2n)!}{|\nu_1|!(2n-|\nu_1|)!} \frac{(2n-|\nu_1|)!}{|\nu_2|!(2n-|\nu_1|-|\nu_2|)!} \\ &= \frac{n!}{\mu!} \binom{2n}{|\nu_1|, |\nu_2|} \leq (2d)^n \binom{2n}{|\nu_1|, |\nu_2|}.\end{aligned}$$

Moreover, applying Proposition 3.2 twice,

$$\begin{aligned}&\|(\partial_\kappa^{\nu_1})^{[\leq n]} f_l(x, \overline{z}) (\partial_\kappa^{\nu_2})^{[\leq n]} g_{k-n-l}(x, \overline{z}) \partial_v^{2\mu-\nu_1-\nu_2} J_{v=0}\|_{C^j} \\ &\quad \leq \sum_{j_1+j_2 \leq j} \binom{j}{j_1, j_2} \|(\partial_\kappa^{\nu_1})^{[\leq n]} f_l(x, \overline{z})\|_{C^{j_1}} \|(\partial_\kappa^{\nu_2})^{[\leq n]} g_{k-n-l}(x, \overline{z})\|_{C^{j_2}} \|\partial_v^{2\mu-\nu_1-\nu_2} J_{v=0}\|_{C^{j-j_1-j_2}}.\end{aligned}$$

In particular, using the notation $(\nabla_\kappa^j)^{[\leq n]}$ as introduced in Lemma 4.6, one has

$$\begin{aligned}\|n!B_n(f_l, g_{k-n-l})\|_{C^j} &= \|\Delta_v^n \left(f_l(x, \overline{y}(x, v, \overline{z})) g_{k-n-l}(y(x, v, \overline{z}), \overline{z}) J(x, v, \overline{z}) \right)_{v=0}\|_{C^j} \\ &\leq (2d)^n \sum_{\substack{j_1+j_2 \leq j \\ i_1+i_2 \leq 2n}} \binom{j}{j_1, j_2} \binom{2n}{i_1, i_2} \|(\nabla_\kappa^{i_1})^{[\leq n]} f_l(x, \overline{z})\|_{\ell^1(C^{j_1})} \\ &\qquad \|(\nabla_\kappa^{i_2})^{[\leq n]} g_{k-n-l}(x, \overline{z})\|_{\ell^1(C^{j_2})} \|\nabla_v^{2n-i_1-i_2} J\|_{\ell^1(C^{j-j_1-j_2})}.\end{aligned}$$

By Lemma 4.6, for some γ_r depending linearly on r (but independent of R,m), one has

$$\|(\nabla_\kappa^{i_1})^{[\leq n]} f_l(x,\overline{z})\|_{\ell^1(C^{j_1})} \leq i_1^{d+1} j_1^{d+1} \|f\|_{S_m^{r,R}} \gamma_r^{i_1} \frac{r^{j_1+i_1} R^l}{(i_1+j_1+l+1)^m} A(i_1,j_1,l,n),$$
$$\|(\nabla_\kappa^{i_2})^{[\leq n]} g_{k-n-l}(x,\overline{z})\|_{\ell^1(C^{j_2})} \leq i_2^{d+1} j_2^{d+1} \|g\|_{S_m^{2r,2R}} \gamma_r^{i_2} \frac{(2r)^{j_2+i_2}(2R)^{k-n-l}}{(i_2+j_2+l+1)^m} A(i_2,j_2,k-n-l,n),$$

where

$$A(i,j,l,n) = \begin{cases} (i+j+l)! & \text{if } i \leq n, \\ \max((n+j+l)!(i-n)!, (j+l)!i!) & \text{otherwise,} \end{cases}$$

The real-analytic function J belongs to some fixed analytic space, so that there exists r_0, m_0 such that.

$$\|J\|_{C^j} \leq C_J \frac{r_0^j j!}{(j+1)^{m_0}},$$

If $r \geq 2r_0 2^{m-m_0}$, by Proposition 2.11, one has

$$\|J\|_{C^j} \leq C_J \frac{(r/2)^j j!}{(j+1)^m},$$

hence

$$\|(f\sharp g)_k\|_{C^j} \leq C_J \|f\|_{S_m^{r,R}} \|g\|_{S_m^{2r,2R}} \frac{(2r)^j (2R)^k (j+k)!}{(k+j+1)^m} \sum_{n=0}^{k} \left(\frac{\gamma_r r^2}{R}\right)^n \sum_{l=0}^{k-n} \sum_{i_1+i_2\leq 2n} \sum_{j_1+j_2\leq j}$$
$$\frac{(2n)! j! A(i_1,j_1,l,n) A(i_2,j_2,k-l,n)(2n+j-j_1-j_2-i_1-i_2)!}{2^{2n+j-j_1-j_2-i_1-i_2} 2^{j_1+i_1+l} i_1! i_2! j_1! j_2! (2n-i_1-i_2)!(j-j_1-j_2)! n! (k+j)!}$$
$$\frac{i_1^d i_2^d j_1^d j_2^d (k+j+1)^m}{(j_1+i_1+l+1)^m (j_2+i_2+k-n-l+1)^m (j+2n-i_1-i_2-j_1-j_2+1)^m}.$$

Second step.
Let us control the quotient of factorials above. There holds

$$\frac{(2n+j-j_1-j_2-i_1-i_2)!}{2^{2n+j-j_1-j_2-i_1-i_2}(j-j_1-j_2)!(2n-i_1-i_2)!} = \frac{\binom{2n+j-j_1-j_2-i_1-i_2}{j-j_1-j_2}}{2^{2n+j-j_1-j_2-i_1-i_2}} \leq 1.$$

Thus, the middle line in the control on $\|(f\sharp g)_k\|_{C^j}$ is smaller than

$$\frac{(2n)! j! A(i_1,j_1,l,n) A(i_2,j_2,k-l,n)}{2^{j_1+i_1+l} i_1! i_2! j_1! j_2! n! (k+j)!}.$$

Let us prove that, if $i_1 \leq 2n$, $i_2 \leq 2n$, $0 \leq l \leq k-n$, $j_1+j_2 \leq j$, then

$$\frac{(2n)! j! A(i_1,j_1,l,n) A(i_2,j_2,k-l,n)}{2^{j_1+i_1+l} i_1! i_2! j_1! j_2! n! (k+j)!} \leq 4^n.$$

For the moment, let us focus on the $i_1 \leq n, i_2 \leq n$ case. As $i_1 \geq 0$ one has $\frac{1}{2^{i_1}} \leq 1$ and it remains to control

$$\frac{(2n)! j! (j_1+i_1+l)! (j_2+i_2+k-n-l)!}{2^{j_1+l} i_1! i_2! j_1! j_2! n! (k+j)!}.$$

This expression is increasing with respect to i_1 and i_2, so that we only need to control the $i_1 = i_2 = n$ case, which is

$$\frac{(2n)!j!(j_1+n+l)!(j_2+k-l)!}{2^{j_1+l}(n!)^3 j_1!j_2!(k+j)!}$$

Moreover, the expression above is log-convex with respect to l, so that we only need to control the $l = 0$ and $l = k - n$ case.

If $l = 0$ we are left with

$$\frac{(2n)!j!(j_1+n)!(k+j_2)!}{2^{j_1}(n!)^3 j_1!j_2!(k+j)!} = 2^n\binom{2n}{n}\frac{\binom{j_1+n}{n}}{2^{j_1+n}}\frac{\binom{k+j+j_2}{j_2}}{\binom{k+j+j_2}{j}} \leq 4^n\frac{\binom{k+j+j_2}{j_2}}{\binom{k+j+j_2}{j}}.$$

To conclude, j is closer from $\dfrac{k+j+j_2}{2}$ than j_2 since $j \geq j_2$, so that $\frac{\binom{k+j+j_2}{j_2}}{\binom{k+j+j_2}{j}} \leq 1$, hence the claim.

If $l = k - n$, one has

$$\frac{(2n)!j!(j_1+k)!(j_2+n)!}{2^{j_1+k-n}(n!)^3 j_1!j_2!(k+j)!} = 2^n\binom{2n}{n}\frac{\binom{j_1+k}{k}}{2^{j_1+k}}\frac{\binom{j_2+n}{n}}{\binom{j+k}{k}} \leq 4^n.$$

We now consider the case $i_1 \geq n$ or $i_2 \geq n$. We need to replace $(i_1+j_1+l)!$ with either $(j_1+l)!i_1!$ or $(j_1+l+n)!(i_1-n)!$. By Proposition 5.1, one has

$$\begin{aligned}\frac{(j_1+l)!i_1!}{i_1!} &= (j_1+l)! \leq \frac{(j_1+l+n)!}{n!}\\ \frac{(j_1+l+n)!(i_1-n)!}{i_1!} &\leq \frac{(j_1+l+n)!i_1!}{i_1!n!} = \frac{(j_1+l+n)!}{n!}.\end{aligned}$$

The same inequalities apply with i_1, j_1 replaced with i_2, j_2. Hence, in all cases, we are left with

$$\frac{(2n)!j!(j_1+n+l)!(j_2+k-l)!}{2^{j_1+l}(n!)^3 j_1!j_2!(k+j)!},$$

which we just proved to be smaller than 4^n.

This yields

$$\|(f\sharp g)_k\|_{C^j} \leq C_J\|f\|_{S_m^{r,R}}\|g\|_{S_m^{2r,2R}}\frac{(2r)^j(2R)^k(j+k)!}{(k+j+1)^m}\sum_{n=0}^{k}\left(\frac{4\gamma_r r^2}{R}\right)^n\sum_{l=0}^{k-n}\sum_{i_1,i_2=0}^{n}\sum_{j_1+j_2\leq j}$$
$$\frac{(k+j+1)^m i_1^d i_2^d j_1^d j_2^d}{(j_1+i_1+l+1)^m(j_2+i_2+k-n-l+1)^m(j+2n-i_1-i_2-j_1-j_2+1)^m}.$$

We are almost in position to apply Lemma 3.7; since

$$(k+j+n+1)^m \geq (k+j+1)^m,$$

one has

$$\|(f\sharp g)_k\|_{C^j} \leq C_J\|f\|_{S_m^{r,R}}\|g\|_{S_m^{2r,2R}}\frac{(2r)^j(2R)^k(j+k)!}{(k+j+1)^m}\sum_{n=0}^{k}\left(\frac{4\gamma_r r^2}{R}\right)^n\sum_{l=0}^{k-n}\sum_{i_1,i_2=0}^{n}\sum_{j_1+j_2\leq j}$$
$$\frac{i_1^d i_2^d j_1^d j_2^d (k+j+n+1)^m}{(j_1+i_1+l+1)^m(j_2+i_2+k-n-l+1)^m(j+2n-i_1-i_2-j_1-j_2+1)^m}.$$

Applying Lemma 3.7 yields, for m large enough depending on d,

$$\|(f\sharp g)_k\|_{C^j} \leq C_J \|f\|_{S_m^{r,R}} \|g\|_{S_m^{2r,2R}} \frac{(2r)^j(2R)^k(j+k)!}{(k+j+1)^m} \sum_{n=0}^{k} \left(\frac{4\gamma_r r^2}{R}\right)^n.$$

As long as $R \geq 4\gamma_r r^2$, which is possible if R is chosen large enough since γ_r depends only on r, one can conclude:

$$\|(f\sharp g)_k\|_{C^j} \leq 2^m C_J \|f\|_{S_m^{r,R}} \|g\|_{S_m^{2r,2R}} \frac{(2r)^j(2R)^k(j+k)!}{(k+j+1)^m}.$$

At this stage, we are almost done with the proof: we obtained that the formal series which corresponds, in the C^∞ class, to the composition $T_N^{cov}(f)T_N^{cov}(g)$, belongs to the same analytic symbol class than g.

Third step.

It remains to prove that computing symbol sums in decreasing powers of N, up to an order cN for $c > 0$ small, yields an exponentially small error.

Let $c > 0$ be small enough depending on r, R, m. The analytic sums $f(N)$ and $g(N)$ appearing in $T_N^{cov}(f)$ and $T_N^{cov}(g)$ can be replaced, by Proposition 3.6, by a sum until cN, up to a small error $O(e^{-c'N})$ with $c' > 0$. Then, by construction,

$$\begin{aligned}
&\left[T_N^{cov}\left(\sum_{k=0}^{cN} N^{d-k} f_k\right) T_N^{cov}\left(\sum_{k=0}^{cN} N^{d-k} g_k\right) - T_N^{cov}\left(\sum_{k=0}^{cN} N^{d-k} (f\sharp g)_k\right)\right](x,z) \\
&\qquad = \int_M \Psi^N(x,y)\Psi^N(y,z) \sum_{j=0}^{cN} \sum_{k=cN-j}^{cN} N^{2d-j-k} f_j(x,\overline{y}) g_k(y,\overline{z}) \mathrm{d}y + \sum_{j+k\leq cN} N^{-k-j} R(j,k,N).
\end{aligned}$$

Here, $R(j,k,N)$ is the remainder at order $cN-k-j$ in the stationary phase Lemma applied to

$$N^{2d}\Psi^N(x,z) \int_{y\in M} e^{-N\Phi_1(x,y,\overline{y},\overline{z})} f_j(x,\overline{y}) g_k(y,\overline{z}) \mathrm{d}y.$$

As

$$\begin{aligned}
\|f_j\|_{C^l} &\leq C_f (4R)^j j! \frac{(4r)^l l!}{(j+l+1)^m} \leq C_f (4R)^j j! \frac{(4r)^l l!}{(l+1)^m} \\
\|g_k\|_{C^l} &\leq C_g (4R)^k k! \frac{(4r)^l l!}{(k+l+1)^m} \leq C_g (4R)^k k! \frac{(4r)^l l!}{(l+1)^m},
\end{aligned}$$

one has, by Lemma 2.13,

$$\|f_j g_k\|_{C^l} \leq C C_f C_g (4R)^{j+k} j! k! \frac{(4r)^l l!}{(l+1)^m}.$$

In other terms,

$$\|f_j g_k\|_{H(m,4r,U\times U)} \leq C C_f C_g (8R)^{j+k} (j+k)!,$$

so that, by Proposition 3.13, for some $c' > 0$ depening on r, one has

$$\begin{aligned}
N^{-k-j}|R(j,k,N)| &\leq N^{2d} C C_f C_g N^{-k-j} (4R)^{j+k} j! k! e^{-c'(cN-j-k)} \\
&\leq N^{2d} C C_f C_g N^{-k-j} (8R)^{j+k} (j+k)! e^{-c'(cN-j-k)}.
\end{aligned}$$

We must estimate this quantity in the range $0 \leq j+k \leq cN$. Observe that, if $j+k-1$ is replaced with $j+k$, then the right-hand term is multiplied by

$$\frac{8R}{N}(j+k)e^{c'} \leq 8Rce^{c'}.$$

If $c>0$ is chosen small enough then this ratio is smaller than 1, so that it suffices to estimate the $k+j=0$ case, for which it is $O(\exp(-(c'-\epsilon)cN))$.

Since $|\Psi^N| \leq 1$ on U, it remains to estimate

$$\sum_{j=0}^{cN} \sum_{k=cN-j}^{cN} N^{2d-j-k} \sup(|f_j|) \sup(|g_k|),$$

which is smaller than (with $l=k+j$):

$$C_f C_g N^{2d+1} \sum_{l=cN}^{2cN} N^{-l}(2R)^l l!.$$

Let $l/N = \tilde{c} \in [c, 2c]$. Then, by the Stirling formula, one has

$$N^{-l}(2R)^l l! \leq C\sqrt{l}\exp\left[-l\log(N) + l\log(2R) + l\log(l) - l\right] = C\sqrt{l}\exp\left[-\frac{e}{2R}N\left(-\frac{2Rl}{eN}\log\left(\frac{2Rl}{eN}\right)\right)\right].$$

If $c>0$ is small enough then $\frac{4Rc}{e}<1$, so that $-\frac{2Rl}{eN}\log(\frac{2Rl}{eN})$ is bounded away from zero independently of N for $l \in [cN, 2cN]$. In particular, there exists $c'>0$ such that

$$N^{-l}(2R)^l l! \leq C\sqrt{N}\exp(-c'N).$$

Hence, if $c'' < c'$, then

$$N^{2d}\sum_{j=0}^{cN} \sum_{k=cN-j}^{cN} N^{-j-k} \sup(|f_j|) \sup(|g_k|) = O(e^{-c''N}).$$

This concludes the proof. □

4.4 Inversion of covariant Toeplitz operators and the Bergman kernel

In this subsection we prove Theorem A as well as the second part of Theorem B. To do so, we first show in Proposition 4.8, as a reciprocal to Proposition 4.7, that if f and h are analytic symbols of covariant Toeplitz operators with f_0 non-vanishing, then there exists an analytic symbol g such that

$$T_N^{cov}(f)T_N^{cov}(g) = T_N^{cov}(h) + O(e^{-cN}).$$

We then prove in Proposition 4.9 that, under the same hypotheses, $T_N^{cov}(f)$, whose image is almost contained in $H_0(M, L^{\otimes N})$ by Proposition 4.3, is invertible on this space up to an exponentially small error. Thus, one can conclude that, on $H_0(M, L^{\otimes N})$, there holds

$$T_N^{cov}(g) = T_N^{cov}(h)(T_N^{cov}(f))^{-1} + O(e^{-cN}).$$

This allows us to prove Theorem A, since by setting $h=f$ one recovers that the Bergman kernel can be written as $T_N^{cov}(f)(T_N^{cov}(f))^{-1} = T_N(a)$. Then, the second part of Theorem B follows from Proposition 4.8 by setting $h=a$.

Following the lines of Proposition 4.7, let us try to construct inverses for analytic symbols.

Proposition 4.8. *Let U denote a small neighbourhood of the diagonal in $M \times M$ and let $f, h \in S^{r_0,R_0}_{m_0}(U)$ be analytic symbols, holomorphic in the first variable and anti-holomorphic in the second variable, for some r_0, R_0, m_0. Suppose that the principal symbol f_0 of f is bounded away from zero on U.*

Then there exists r, R, m as well as $g \in S^{r,R}_m(U)$, holomorphic in the first variable, anti-holomorphic in the second variable, such that

$$T_N^{cov}(f)T_N^{cov}(g) = T_N^{cov}(h) + O(e^{-cN}).$$

Proof. Recalling the proof of Proposition 4.7, let us recover g from f and $h = f\sharp g$. By definition of h_k, one has

$$g_k(x,\overline{z})f_0(x,\overline{z})J(x,x,\overline{z},\overline{z}) = h_k(x,\overline{z}) - \sum_{n=0}^{k}\frac{\Delta_v^n}{n!}\left(\sum_{\substack{l=0\\ l+n>0}}^{k-n} f_l(x,\overline{y}(x,v,\overline{z}))g_{k-n-l}(y(x,v,\overline{z}),\overline{z})J(x,v,\overline{z})\right)_{v=0}. \tag{4}$$

As f_0 is bounded away from zero, this indeed defines g_k by induction. Let us try to control g in an analytic space.

We first let m large enough, and $r \geq 2r_0 2^{m-m_0}$ as well as $R \geq 2R_0 2^{m-m_0}$. Then, by Lemma 3.4, there exist C_f, C_h, C_J independent of m, r, R such that, for every $k \geq 0, j \geq 0$,

$$\|f_k\|_{C^j(U)} \leq C_f \frac{(r/2)^j (R/2)^k (j+k)!}{(j+k+1)^m}$$

$$\|h_k\|_{C^j(U)} \leq C_h \frac{r^j r^k (j+k)!}{(j+k+1)^m}$$

$$\|J\|_{C^j(U\times U)} \leq C_J \frac{(r/2)^j j!}{(j+1)^m}.$$

Here J denotes again the Jacobian in the change of variables corresponding to the Morse lemma for the phase Φ_1.

We first note that

$$g_0(x,\overline{z}) = f_0(x,\overline{z})^{-1}h_0(x,\overline{z})J(x,x,\overline{z},\overline{z}),$$

so that, by Lemma 2.13, there exists C_0 such that, for every $r \geq 2r_0 2^{m-m_0}$ and $R \geq 2R_0 2^{m-m_0}$, for every $j \geq 0$,

$$\|g_0\|_{C^j(U)} \leq C_0 \frac{r^j j!}{(j+1)^m}.$$

Let us prove by induction on $l \geq 1$ that, for some fixed C_g, m, r, R, for every $j \geq 0$, one has

$$\|g_l\|_{C^j} \leq C_g \frac{r^j R^l (j+l)!}{(j+l+1)^m}.$$

Over the course of the induction, we will fix the values of C_g, m, r, R.

Suppose that a control above is true for indices up to $l = k-1$. Then, from the recursive formula (4), if we repeated the proof of Proposition 4.7, we would obtain

$$\|g_k\|_{C^j} \leq C(C_h + C_g C_f C_J)\frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

This is not enough, as the constant $C(C_h + C_g C_f C_J)$ appearing here might be greater than C_g. However, as we will see, the constant can be made arbitrarily small by choosing C_g large enough, as well as m large enough, depending on f, and R/r^2 large enough.

Let $C_1 = C\|(f_0 J)^{-1}\|_{H(m,r,U)}$ where C is the constant appearing in Proposition 2.13.

There holds

$$C_h \leq \frac{C_g}{4C_1}$$

if C_g is large enough with respect to C_h, C_f, C_J, C_0. It remains to estimate the second term on the right-hand side of (4).

Let us isolate the $n = 0, l = k$ term in (4). This term is $-g_0 J f_k$, and the $S_m^{r,R}(U)$-norm of $g_0 J f$ is again smaller than $\frac{C_g}{4C_1}$ if C_g is large enough with respect to $C_f C_0 C_J$.

Repeating the proof of Proposition 4.7, the $n = 0, l < k$ terms in (4) are bounded in C^j-norm by

$$CC_J C_f C_g \frac{r^j R^k (j+k)!}{(j+k+1)^m} \sum_{l=1}^{k-1} \sum_{j_1+j_2 \leq j} \frac{(j+k+1)^m}{(j_1+l+1)^m (j_2+k-l+1)^m (j+k-j_1-j_2+1)^m}.$$

By Lemma 3.7, since no term in the sum

$$\sum_{\substack{1 \leq l \leq k-1 \\ j_1+j_2 \leq j}} \frac{(j+k+1)^m}{(j_1+l+1)^m (j_2+k-l+1)^m (j+k-j_1-j_2+1)^m} = \sum_{\substack{i_1+i_2+i_3=j+k \\ i_1 \geq 1 \\ i_2 \geq 1}} \frac{(j+k+1)^m}{(i_1+1)^m (i_2+1)^m (i_3+1)^m}$$

contribute as 1, by Lemma 3.7 (with $d = 0$ and $n = 3$), this sum is smaller than $C(3/4)^m$ for some $C > 0$. Hence, if m is large enough, this contribution is also smaller than $\frac{C_g}{4C_1}$. Now m is fixed.

It remains to control the $n \geq 1$ terms in (4). From the proof of Proposition 4.7, their sum is smaller than

$$CC_J C_f C_g \sum_{n=1}^{k} \frac{r^j R^k (j+k)!}{(j+k+1)^m} \left(\frac{4\gamma_r r^2}{R} \right)^n.$$

As long as R/r^2 is large enough with respect to $\gamma_r C_J C_f$, (which is possible if R is large enough since $\gamma_r = Cr$ for some fixed C), this is again smaller than $\frac{C_g}{4C_1}$.

In conclusion,

$$\|g_k f_0 J\|_{C^j} \leq \frac{C_g}{C_1} \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

In particular, by Lemma 2.13, and since $\|(f_0 J)^{-1}\|_{H(m,r,U)} = C_1/C$, one has

$$\|g_k\|_{C^j} = \|g_k f_0 J (f_0 J)^{-1}\|_{C^j} \leq C_g \frac{r_j R^k (j+k)!}{(j+k+1)^m}.$$

This concludes the induction.

Once the formal series g is controlled in an analytic symbol space, the composition $T_N(g)T_N(f)$ coincides with $T_N(h)$ up to an exponentially small error as in the end of the proof of Proposition 4.7, hence the claim. □

Proposition 4.9. *Let f be a function on U, holomorphic with respect to the first variable, anti-holomorphic with respect to the second variable. If f is nonvanishing then $S_N T_N^{cov}(f)$ has an inverse on $H_0(M, L^{\otimes N})$, with operator norm bounded independently of N.*

Proof. One can invert $S_N T_N^{cov}(f)$ by a formal covariant symbol, that is, up to an $O(N^{-K})$ error for any fixed K. In particular, there exists an operator A_N on $H_0(M, L^{\otimes N})$ such that $A_N S_N T_N^{cov}(f) = S_N + O(N^{-1})$, and such that the operator norm of A_N is bounded independently on N.

Since $A_N S_N T_N^{cov}(f)$ is invertible on $H_0(M, L^{\otimes N})$, so is $S_N T_N^{cov}(f)$, and the operator norm of this inverse is $\|A_N\|_{L^2 \to L^2}(1 + O(N^{-1}))$, which is bounded independently on N, hence the claim. □

Let us now conclude the proofs of Theorems A and B.

Let U be a small neighbourhood of the diagonal in $M \times M$ and let f be any function on U bounded away from zero, holomorphic in the first variable, anti-holomorphic in the second variable. From Proposition 4.8 there exists an analytic symbol a with the same properties, such that

$$T_N^{cov}(f)T_N^{cov}(a) = T_N^{cov}(f) + O(e^{-cN}).$$

Let $A_N = (S_N T_N^{cov}(f))^{-1}$ on $H_0(M, L^{\otimes N})$; we know from Proposition 4.9 that A_N is well-defined and bounded independently on N. Then, for any $u \in H_0(M, L^{\otimes N})$, one has

$$T_N^{cov}(a)u = u + O(e^{-cN}).$$

Moreover, by Proposition 4.3, there holds

$$(I - S_N)T_N^{cov}(a) = O(e^{-cN}).$$

To conclude, one has $T_N^{cov}(a) = S_N + O(e^{-cN})$. In other terms,

$$S_N(x, \overline{y}) = \Psi^N(x, y) \sum_{k=0}^{cN} N^{d-k} a_k(x, \overline{y}) + O(e^{-cN}).$$

This concludes the proof of Theorem A.

Let us complete the proof of Theorem B. Its first part is Proposition 4.7. For the second part, we apply Proposition 4.8 with $h = a$, the symbol of the Bergman kernel.

Remark 4.10 (Normalised covariant Toeplitz operators)**.** Let $T_N^{cov}(a)$ denote the approximate Bergman kernel constructed in the previous proposition. Once the symbol a is known, one can study, as in the proof of Proposition 4.5, *normalised covariant* Toeplitz operators, of the form

$$\Psi^N(x, y) \left(\sum_{k=0}^{cN} N^{-k} a_k(x, \overline{y}) \right) \left(\sum_{k=0}^{cN} N^{d-k} f_k(x, \overline{y}) \right).$$

Under this convention, the operator associated with the function $f = 1$ is $S_N + O(e^{-cN})$, as in contravariant Toeplitz quantization.

Propositions 4.7 and 4.8 can be adapted to normalised covariant Toeplitz operators, for which the algebra product is

$$(f, g) \mapsto ((f * a)\sharp(g * a)) * a^{*-1}.$$

For instance, since the Cauchy product is continuous on each symbol class, there holds, for m large enough, $r > 2^m$ and $R > Cr^3$,

$$\|((f * a)\sharp(g * a)) * a^{*-1}\|_{S_m^{2r,2R}(U)} \leq C_a \|f\|_{S_m^{r,R}(U)} \|g\|_{S_m^{2r,2R}(U)}.$$

To conclude this section, we prove that analytic contravariant Toeplitz opeartors are contained within analytic covariant Toeplitz operators.

Proposition 4.11. *Let f be a real-analytic function on M. There exists an analytic symbol g and $c > 0$ such that*

$$T_N(f) = T_N^{cov}(g) + O(e^{-cN}).$$

Proof. Recall from Theorem A that there exists an analytic symbol a such that

$$S_N = T_N^{cov}(a) + O(e^{-cN}).$$

Letting $\widetilde{f}$ be a holomorphic extension of f, the kernel of $T_N(f) = S_N f S_N$ is then

$$(x,z) \mapsto \Psi^N(x,z) \int_{y\in M} e^{-N\Phi_1(x,y,\overline{y},\overline{z})} \left(\sum_{k=0}^{cN} N^{d-k} a_k(x,\overline{y})\right) \left(\sum_{k=0}^{cN} N^{d-k} a_k(y,\overline{z})\right) \widetilde{f}(y,\overline{y})\mathrm{d}y + O(e^{-cN}).$$

One can then repeat the proof of Proposition 4.7 with J replaced with $(x,y,\overline{y},\overline{z}) \mapsto J(x,y,\overline{y},z)\widetilde{f}(y,\overline{y})$. This yields an analytic symbol g such that

$$g_k(x,\overline{z}) = \sum_{n=0}^{k} \frac{\widetilde{\Delta}_v^n}{n!} \left(\sum_{l=0}^{k-n} a_l(x,\overline{y}(x,v,\overline{z}) a_{k-n-l}(y(x,v,\overline{z}),\overline{z}) J(x,(y,\overline{y})(x,v,\overline{z}),\overline{z}) \widetilde{f}((y,\overline{y})(x,v,\overline{z})) \right)_{v=0},$$

that is,

$$T_N^{cov}(g) = S_N f S_N + O(e^{-cN}).$$

$\square$

4.5 Exponential decay of low-energy states

Since covariant analytic Toeplitz operators form an algebra up to exponentially small error terms (Theorem B), and since contravariant Toeplitz operators are a subset of covariant analytic Toeplitz operators (Proposition 4.11), one can study exponential localisation for eigenfunctions of contravariant analytic Toeplitz operators. In this subsection we prove Theorem C.

Let h be a real-analytic, real-valued fnuction on M, let $E \in \mathbb{R}$ and let $(u_N)_{N\geq 1}$ be a normalized family of eigenstates of $T_N(h)$ with eigenvalue $\lambda_N = E + o(1)$. Let V be an open set at positive distance from $\{f = E\}$. Let $a \in C^\infty(M,\mathbb{R}^+)$ be such that $\mathrm{supp}(a) \cap \{f = E\} = \emptyset$ and $a = 1$ on V. The function a is of course not real-analytic; we will nevertheless prove that

$$T_N(a)u_N = O(e^{-cN}).$$

This implies Theorem C, since

$$\int_V |u_N|^2 = \langle u_N, \mathbb{1}_V u_N\rangle \leq \langle u_N, a u_N\rangle = \langle u_N, T_N(a) u_N\rangle = O(e^{-cN}).$$

Let W be an open set of M such that

$$\mathrm{supp}(a) \subset\subset W \subset\subset \{f \neq E\}.$$

On W, the function $b - E$ is bounded away from zero. Let us consider, on a neighbourhood of $\mathrm{diag}(W)$ in $M \times M$, the analytic covariant symbol g which is such that $T_N^{cov}(g)$ is the analytic inverse (on this neighbourhood) of $T_N(f - \lambda(N))$. This symbol is well-defined: one can check that the construction of an inverse symbol in Proposition 4.8 only relies on local properties. The function $f - \lambda(N)$ might not be a classical analytic symbol, since we made no assumption on the eigenvalue $\lambda(N)$. However, for every t close to E one can define the microlocal inverse g_t of $f - t$ near W, in an analytic class independent of t, so that we define the microlocal inverse of $T_N(f - \lambda(N))$ as the operator with kernel

$$T_N^{cov}(g) : (x,y) \mapsto \Psi^N(x,y) g_{\lambda(N)}(N)(x,y).$$

We arbitrarily cut off g outside a neighbourhood of $\operatorname{diag}(W_1)$, where $W \subset\subset W_1 \subset\subset \{f \neq E\}$ so that $T_N^{cov}(g)$ is a well-defined operator. Let us prove that, for some $c > 0$ small, one has

$$T_N(a)T_N^{cov}(g)T_N(f-\lambda_N) = T_N(a) + O(e^{-cN}).$$

By construction, uniformly on $x \in W_1$ and $z \in M$, one has

$$\int_{y\in M} T_N^{cov}(g)(x,y)T_N(f-\lambda_N)(y,z) = S_N(x,z) + O(e^{-cN}).$$

In particular, since $T_N(a)$ is bounded by $O(e^{-cN})$ on $W \times (M \setminus W_1)$, for $x \in W$ one has

$$\begin{aligned}
&\int_{y_1\in M, y_2\in M} T_N(a)(x,y_1)T_N^{cov}(g)(y_1,y_2)T_N(f)(y_2,z)\\
&\qquad = \int_{y_1\in W_1, y_2\in M} T_N(a)(x,y_1)T_N^{cov}(g)(y_1,y_2)T_N(f-\lambda_N)(y_2,z) + O(e^{-cN})\\
&\qquad\qquad = \int_{y_1\in W_1} T_N(a)(x,y_1)S_N(y_1,z) + O(e^{-cN})\\
&\qquad\qquad\qquad = \int_{y_1\in M} T_N(a)(x,y_1)S_N(y_1,z) + O(e^{-cN}) = T_N(a)(x,z) + O(e^{-cN}).
\end{aligned}$$

Moreover, uniformly on $(x \notin W, y \in M)$ there holds $T_N(a)(x,y_1) = O(e^{-cN})$ so that, finally,

$$T_N(a)T_N^{cov}(g)T_N(f-\lambda_N) = T_N(a) + O(e^{-cN}).$$

In particular,

$$0 = T_N(a)T_N^{cov}(g)T_N(f-\lambda(N))u_N = T_N(a)u_N + O(e^{-cN}),$$

which concludes the proof.

5 Combinatorial inequalities

In this section we prove several inequalities which appear throughout this paper.

We denote by Γ the Gamma function, which is the only log-convex function on $(0,+\infty)$ such that $\Gamma(n+1) = n!$ for every integer n. We denote by ψ the Digamma function, defined as the log-derivative of Γ. The letters i, j, k, l, n represent integers, and the letters μ, ν represent polyindices.

Lemma 5.1. *Let $c > 0$. The function $\Gamma(x+c)/\Gamma(x)$ is increasing on $(0,+\infty)$.*

In particular, if $i \leq j \leq k$ then

$$\binom{j}{i} \leq \binom{k}{i}.$$

Proof. The log-derivative of $x \mapsto \Gamma(x+c)/\Gamma(x)$ is $\psi(x+c) - \psi(x)$. Since Γ is log-convex, ψ is increasing so that $\psi(x+c) - \psi(x) > 0$, hence the claim.

For the second part of the claim, we consider the function $x \mapsto \binom{x}{i} = \frac{\Gamma(x+i+1)}{\Gamma(x+1)\Gamma(i+1)}$. This function is increasing as we have just shown, so that its value at j is smaller than its value at $k \geq j$. $\square$

Lemma 5.2. *If $\nu \leq \mu$ then*

$$\binom{\mu}{\nu} \leq \binom{|\mu|}{|\nu|}.$$

Proof. Let us prove the following inequality, from which one can deduce the original claim by induction:

$$\binom{j}{i}\binom{l}{k} \leq \binom{j+l}{i+k}.$$

The well-known identity

$$\binom{j+l}{i+k} = \binom{j+l-1}{i+k-1} + \binom{j+l-1}{i+k} = \binom{1}{1}\binom{j+l-1}{i+k-1} + \binom{1}{1}\binom{j+l-1}{i+k}$$

can be generalised by induction:

$$\binom{j+l}{i+k} = \sum_{n=0}^{j} \binom{j}{n}\binom{l}{i+k-n}.$$

All terms in the sum are positive so that the sum is greater than any of its terms. In particular,

$$\binom{j+l}{i+k} \geq \binom{j}{i}\binom{l}{k}.$$

□

Lemma 5.3. *If $0 \leq i \leq j$ and $1 \leq k \leq l-1$, then*

$$\frac{(i+k-1)!(j+l-i-k-1)!}{i!k!(j-i)!(l-k)!} \leq \frac{(j+l-2)!}{j!(l-1)!}.$$

In particular, if $a_1, \dots, a_n$ are nonnegative integers and $b_1, \dots, b_n$ are positive integers, with $\sum_{i=1}^n a_i = j$ and $\sum_{i=1}^n b_i = l$, then

$$\frac{(a_1+b_1-1)! \dots (a_n+b_n-1)!}{a_1!b_1! \dots a_n!b_n!} \leq \frac{(j+l-n)!}{j!(l-n+1)!}.$$

Proof. For the first part, let $k' = k-1$, then

$$\frac{(i+k-1)!(j+l-i-k-1)!}{i!k!(j-i)!(l-k)!} = \frac{1}{k(l-k)}\binom{i+k'}{i}\binom{j+l-2-i-k'}{j-i}.$$

Since $1 \leq k \leq l-1$ there holds $\frac{1}{k(l-k)} \leq \frac{1}{l-1}$. Moreover, from Lemma 5.2, one has

$$\binom{i+k'}{i}\binom{j+l-2-i-k'}{j-i} \leq \binom{j+l-2}{j} = \frac{(j+l-1)!}{j!(l-2)!}.$$

Hence,

$$\frac{(i+k-1)!(j+l-i-k-1)!}{i!k!(j-i)!(l-k)!} \leq \frac{(j+l-2)}{j!(l-1)!}.$$

The second part is deduced from the first part by induction. Indeed, we just proved that, denoting $a'_{n-1} = a_{n-1} + a_n$ and $b'_{n-1} = b_{n-1} + b_n - 1$, one has

$$\frac{(a_1+b_1-1)! \dots (a_n+b_n-1)!}{a_1!b_1! \dots a_n!b_n!} \leq \frac{(a_1+b_1-1)! \dots (a_{n-2}+b_{n-2}-1)!(a'_{n-1}+b'_{n-1}-1)!}{a_1!b_1! \dots a_{n-2}!b_{n-2}!a'_{n-1}!b'_{n-1}!}.$$

Here, the sum of the a_i's has not changed but the sum of the b_i's has been reduced by one. By induction,

$$\frac{(a_1+b_1-1)! \dots (a_n+b_n-1)!}{a_1!b_1! \dots a_n!b_n!} \leq \frac{(j+l-n)!}{j!(l-n+1)!}.$$

□

Lemma 5.4. *Let $\ell \geq 2$ and $n \geq 2$ be integers. The set*

$$\left\{(i_1, \ldots, i_n) \in \mathbb{N}_0^n, \sum_{k=1}^{n} i_k = \ell, \text{ at least two of them are} \geq 1\right\}.$$

is contained in the convex hull of all permutations of $(\ell - 1, 1, 0, \ldots, 0)$.

Proof. Let us call *support* of a tuple $(i_1, \ldots, i_n)$ the number of its elements which are non-zero. We will prove by induction on $2 \leq k \leq \min(n, \ell)$ that the convex hull S of the permutations of $(\ell - 1, 1, 0, \ldots, 0)$ contain all tuples of support k such that the sum of all elements is ℓ.

For $k = 2$, we can indeed recover all elements of the form $(\ell - x, x, 0, \ldots, 0)$ for all $1 \leq x \leq \ell - 1$ by a convex combination of $(\ell - 1, 1, 0, \ldots, 0)$ and $(1, \ell - 1, 0, \ldots, 0)$.

We now proceed to the induction. Suppose that S contains all elements of the form $(i_1, \ldots, i_{k-1}, 0, \ldots, 0)$ and their permutations. Then, in particular, it contains $a_0 = (\ell - k + 2, 1, \ldots, 1, 0, \ldots, 0)$. For every $1 \leq j \leq k-2$, S also contains the image of a_0 by the transposition $(k, k-j)$, which we denote by a_j. Moreover, S contains $(\frac{\ell}{k-1}, \ldots, \frac{\ell}{k-1}, 0, \ldots, 0)$ and its permutations. From the $(a_j)_{0 \leq j \leq k-2}$ and $(\frac{\ell}{k-1}, \ldots, \frac{\ell}{k-1}, 0, \ldots, 0)$, one can form the convex combination

$$\frac{\ell - k + 1}{(\ell - k + 2)(k - 2)} \sum_{j=0}^{k-2} a_j + \frac{1}{\ell - k + 2}\left(0, \frac{\ell}{k-1}, \ldots, \frac{\ell}{k-1}, 0, \ldots, 0\right) = (\ell - k + 1, \underbrace{1, \ldots, 1}_{k-1}, 0, \ldots, 0).$$

In particular, S contains all permutations of $(l - k + 1, 1, \ldots, 1, 0, \ldots, 0)$. Thus, S contains all elements of support k, since the k-uple $(l - k, 0, \ldots, 0)$ and its permutations are the extremal points of the convex $\{\sum_{j=1}^{k} i_j = \ell - k\}$). This concludes the induction. $\square$

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