

# AN INVITATION TO GABOR ANALYSIS

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ABSTRACT. In this paper, we give an introduction to Gabor analysis by discussing three open problems.

## 1. INTRODUCTION

In his celebrated work [32], Dennis Gabor suggested that any signal  $f \in L^2(\mathbb{R})$  has the following (non orthogonal) expansion

$$(1) \quad f(x) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{nk} e^{-\frac{\pi(x-n\alpha)^2}{2\alpha^2}} e^{2\pi i k x / \alpha}$$

where  $\alpha > 0$ , and argued on how to find the coefficients  $(c_{nk})_{n,k \in \mathbb{Z}} \in \mathbb{C}$  using successive local approximations by Fourier series. In fact, in 1932, John von Neumann already made a related claim, when he stipulated that the system of functions

$$(2) \quad \mathcal{G}(\varphi, 1, 1) = \{\varphi_{mn}(\cdot) := e^{2\pi i n \cdot} \varphi(\cdot - m) : n, m \in \mathbb{Z}\}$$

where  $\varphi(x) = e^{-\pi x^2}$  spans a dense subspace of  $L^2(\mathbb{R})$  [69].

These two statements were positively established in 1971 independently by V. Bargmann, P. Butera, L. Girardelo, and J. R. Klauder [5] and A. M. Perelomov [61]. Taken together, they could be thought of as the foundations of what is known today as Gabor analysis, an active research field at the intersection of (quantum) physics, signal processing, mathematics, and engineering. The goal of this paper is to give an overview of some interesting open problems in Gabor analysis that are in need of solutions. For a more complete introduction to the theory and applications of Gabor analysis we refer to [13, 20, 27, 28, 33, 42, 45].

## 2. GABOR FRAME THEORY

We start with a motivating example based on the  $L^2$  theory of Fourier series. Let  $g(x) = \chi_{[0,1)}(x)$ , where  $\chi_I$  denotes the indicator function of the measurable set  $I$ . Any  $f \in L^2(\mathbb{R})$  can be written as

$$(3) \quad f(x) = \sum_{n=-\infty}^{\infty} f(x) g(x - n)$$

with convergence  $L^2$ . For each  $n \in \mathbb{Z}$ ,  $f(\cdot) g(\cdot - n) \in L^2([n, n+1))$  is the restriction of  $f$  to the interval  $[n, n+1)$ . As such, it can be expanded into its  $L^2$  convergent Fourier series

$$(4) \quad f(x) g(x - n) = \sum_{k \in \mathbb{Z}} c_{nk} e^{2\pi i k x}$$

where for each  $k \in \mathbb{Z}$ ,

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$$c_{nk} = \langle f(\cdot)g(\cdot - n), e^{2\pi i k \cdot} \rangle_{L^2([n, n+1])} = \int_{-\infty}^{\infty} f(x)g(x - n)e^{-2\pi i k x} dx = \langle f, g_{nk} \rangle$$

with  $g_{nk}(x) = g(x - n)e^{2\pi i k x}$ . Substituting this in (4) and (3) leads to

$$(5) \quad f(x) = \sum_{n=-\infty}^{\infty} f(x)g(x - n) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{nk}e^{2\pi i k x}g(x - n) = \sum_{k, n=-\infty}^{\infty} \langle f, g_{nk} \rangle g_{nk}(x)$$

This expansion of  $f$  is similar to Gabor's claim (1), with the the following key differences:

- The coefficients in (5) are explicitly given and are linear in  $f$ .
- (1) is based on the gaussian while (5) is based on the indicator function of  $[0, 1)$ .
- Finally, one expansion is an orthonormal decomposition while the other is not.

One of the goals of this section is to elucidate the difference in behavior of these two systems by elaborating on the existence of orthonormal bases of the above type, and in their absence, discuss some of their substitutes, see Section 4.

The two systems of functions in (1) and (2) are examples of Gabor (or Weyl-Heisenberg) systems. More specifically, for  $a, b \in \mathbb{R}$  and a function  $g$  defined on  $\mathbb{R}$ , let  $M_b f(x) = e^{2\pi i b x} f(x)$  and  $T_a f(x) = f(x - a)$  be respectively the modulation operator, and the translation operator. The *Gabor system* generated by a function  $g \in L^2(\mathbb{R})$ , and parameters  $\alpha, \beta > 0$ , is the set of functions [33]

$$\mathcal{G}(g, \alpha, \beta) = \{M_{k\beta}T_{n\alpha}g(\cdot) = e^{2\pi i k \beta \cdot}g(\cdot - n\alpha) : k, n \in \mathbb{Z}\}.$$

Given  $g \in L^2(\mathbb{R})$ , and  $\alpha, \beta > 0$ , the Gabor system  $\mathcal{G}(g, \alpha, \beta)$  is called a *frame* for  $L^2(\mathbb{R})$  if there exist constants  $0 < A \leq B < \infty$  such that

$$(6) \quad A\|f\|^2 \leq \sum_{k, n \in \mathbb{Z}} |\langle f, M_{k\beta}T_{n\alpha}g \rangle|^2 \leq B\|f\|^2 \quad \forall f \in L^2(\mathbb{R}).$$

When  $A = B$  we say that the Gabor frame is *tight*. A tight Gabor frame for which  $A = B = 1$  is called a *Parseval* frame. Clearly, if  $\mathcal{G}(g, \alpha, \beta)$  is an ONB then it is a Parseval frame, and conversely, if  $\mathcal{G}(g, \alpha, \beta)$  is a Parseval frame and  $\|g\| = 1$ , then it is a Gabor ONB. We refer to [13, 20, 33, 45, 42] for more background on Gabor frames, and summarize below some results needed in the sequel.

The (*Gabor*) *frame operator* associated to the Gabor system  $\mathcal{G}(g, \alpha, \beta)$  is defined by

$$(7) \quad Sf := S_{g, \alpha, \beta}f = \sum_{n, k \in \mathbb{Z}} \langle f, M_{k\beta}T_{n\alpha}g \rangle M_{k\beta}T_{n\alpha}g$$

Given  $f \in L^2(\mathbb{R})$ , we can (formally) write that

$$\langle Sf, f \rangle = \sum_{k, n \in \mathbb{Z}} |\langle f, M_{k\beta}T_{n\alpha}g \rangle|^2.$$

It follows that  $\mathcal{G}(g, \alpha, \beta)$  is a frame for  $L^2$  if and only if there exist constants  $0 < A \leq B < \infty$  such that

$$A\|f\|_2^2 \leq \langle Sf, f \rangle \leq B\|f\|_2^2 \quad \forall f \in L^2(\mathbb{R}).$$

In particular,  $\mathcal{G}(g, \alpha, \beta)$  is a frame for  $L^2$  if and only if the frame operator  $S$  is bounded and positive definite. Furthermore, the optimal upper frame bound  $B$  is the largest eigenvalue of  $S$  while the optimal lower bound  $A$  is its smallest eigenvalue. In addition,  $\mathcal{G}(g, \alpha, \beta)$  is a tight frame for  $L^2$  if and only if  $S$  is a multiple of the identity.

Suppose that  $\mathcal{G}(g, \alpha, \beta)$  is a Gabor frame for  $L^2$ , and let  $f \in L^2$ . For all  $(\ell, m) \in \mathbb{Z}^2$  we have

$$S(M_{\ell\beta}T_{m\alpha}f) = M_{\ell\beta}T_{m\alpha}(S(f)),$$

that is  $S$  and  $M_{\ell\beta}T_{m\alpha}$  commute for all  $(\ell, m) \in \mathbb{Z}^2$ . It follows that  $S^{-1}$  and  $M_{\ell\beta}T_{m\alpha}$  also commute for all  $(\ell, m) \in \mathbb{Z}^2$ . As a consequence, given  $f \in L^2(\mathbb{R})$  we have

$$\begin{aligned} f &= S(S^{-1}f) = \sum_{k,n \in \mathbb{Z}} \langle S^{-1}f, M_{k\beta}T_{n\alpha}g \rangle M_{k\beta}T_{n\alpha}g \\ &= \sum_{k,n \in \mathbb{Z}} \langle f, S^{-1}M_{k\beta}T_{n\alpha}g \rangle M_{k\beta}T_{n\alpha}g = \sum_{k,n \in \mathbb{Z}} \langle f, M_{k\beta}T_{n\alpha}\tilde{g} \rangle M_{k\beta}T_{n\alpha}g \end{aligned}$$

where  $\tilde{g} = S^{-1}g \in L^2(\mathbb{R})$  is called the *canonical dual* of  $g$ . Similarly, by writing  $f = S^{-1}(Sf)$  we get that

$$f = \sum_{k,n \in \mathbb{Z}} \langle f, M_{k\beta}T_{n\alpha}g \rangle M_{k\beta}T_{n\alpha}\tilde{g}.$$

The role of the canonical dual can be seen as follows. For  $f \in L^2$ , let  $\tilde{c} = (\langle f, M_{k\beta}T_{n\alpha}\tilde{g} \rangle)_{k,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ . Given any (other) sequence  $(c_{k,n})_{k,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$  such that

$$f = \sum_{k,n \in \mathbb{Z}} \tilde{c}_{k,n} M_{k\beta}T_{n\alpha}g = \sum_{k,n \in \mathbb{Z}} c_{k,n} M_{k\beta}T_{n\alpha}g,$$

we have

$$\|\tilde{c}\|_2^2 = \sum_{k,n \in \mathbb{Z}} |\langle f, M_{k\beta}T_{n\alpha}\tilde{g} \rangle|^2 = \langle S^{-1}f, f \rangle = \sum_{k,n \in \mathbb{Z}} c_{k,n} \langle S^{-1}M_{k\beta}T_{n\alpha}g, f \rangle = \sum_{k,n \in \mathbb{Z}} c_{k,n} \overline{\tilde{c}_{k,n}} = \langle c, \tilde{c} \rangle.$$

Consequently,  $\langle c - \tilde{c}, \tilde{c} \rangle = 0$ , leading to

$$\|c\|^2 = \|c - \tilde{c}\|^2 + \|\tilde{c}\|^2 \geq \|\tilde{c}\|^2$$

with equality if and only if  $c = \tilde{c}$ . In other words, for a Gabor frame  $\mathcal{G}(g, \alpha, \beta)$ , and given  $f \in L^2$ , among all expansions  $f = \sum_{k,n \in \mathbb{Z}} c_{k,n} M_{k\beta}T_{n\alpha}g$ , with  $c = (c_{k,n})_{k,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ , the coefficient  $\tilde{c} = (\langle f, M_{k\beta}T_{n\alpha}\tilde{g} \rangle)_{k,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$  has the least norm.

Because the frame operator  $S$  is positive definite,  $S^{1/2}$  is well defined and positive definite as well. Thus, we can write

$$f = S^{-1/2}SS^{-1/2}f = \sum_{k,n} \langle f, S^{-1/2}M_{k\beta}T_{n\alpha}g \rangle S^{-1/2}M_{k\beta}T_{n\alpha}g = \sum_{k,n} \langle f, M_{k\beta}T_{n\alpha}g^\dagger \rangle M_{k\beta}T_{n\alpha}g^\dagger$$

where  $g^\dagger = S^{-1/2}g \in L^2$ . In other words,  $\mathcal{G}(g^\dagger, \alpha, \beta)$  is a Parseval frame.

Finally, assume that  $A, B$  are the optimal frame bounds for  $\mathcal{G}(g, \alpha, \beta)$ . Then, for all  $f \in L^2$ , we have

$$\sum_{k,n \in \mathbb{Z}} |\langle f, M_{k\beta}T_{n\alpha}\tilde{g} \rangle|^2 = \langle S^{-1}f, f \rangle = \langle S^{-1}f, S(S^{-1}f) \rangle \leq B\|S^{-1}f\|^2 \leq \tilde{B}\|f\|_2^2$$

and similarly, we have the lower bound

$$\langle S^{-1}f, f \rangle = \langle S^{-1}f, S(S^{-1}f) \rangle \geq A\|S^{-1}f\|^2 \geq \tilde{A}\|f\|_2^2$$

Therefore, if  $\mathcal{G}(g, \alpha, \beta)$  is Gabor frame for  $L^2(\mathbb{R})$ , then so is  $\mathcal{G}(\tilde{g}, \alpha, \beta)$  where  $\tilde{g} = S^{-1}g \in L^2(\mathbb{R})$ . We summarize all these facts in the following result.

**Proposition 1** (Reconstruction formulas for Gabor frame). *Let  $g \in L^2(\mathbb{R})$  and  $\alpha, \beta > 0$ . Suppose that  $\mathcal{G}(g, \alpha, \beta)$  is a frame for  $L^2(\mathbb{R})$  with frame bounds  $A, B$ . Then the following statements hold.*

- (a) *The Gabor system  $\mathcal{G}(\tilde{g}, \alpha, \beta)$  with  $\tilde{g} = S^{-1}g \in L^2$ , is also a frame for  $L^2$  with frame bounds  $1/B, 1/A$ . Furthermore, for each  $f \in L^2$  we have the following reconstruction formulas:*

$$f = \sum_{k,n \in \mathbb{Z}} \langle f, M_{k\beta}T_{n\alpha}\tilde{g} \rangle M_{k\beta}T_{n\alpha}g = \sum_{k,n \in \mathbb{Z}} \langle f, M_{k\beta}T_{n\alpha}g \rangle M_{k\beta}T_{n\alpha}\tilde{g}.$$

In addition, among all sequences  $c = (c_{k,n})_{k,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$  such that  $f = \sum_{k,n \in \mathbb{Z}} c_{k,n} M_{k\beta} T_{n\alpha} g$ , the sequence  $\tilde{c} = (\langle f, M_{k\beta} T_{n\alpha} \tilde{g} \rangle)_{k,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$  satisfies

$$\|\tilde{c}\|^2 = \sum_{k,n \in \mathbb{Z}} |\langle f, M_{k\beta} T_{n\alpha} \tilde{g} \rangle|^2 \geq \sum_{k,n \in \mathbb{Z}} |c_{k,n}|^2 = \|c\|^2$$

with equality if and only if  $c = \tilde{c}$ .

- (b) The Gabor system  $\mathcal{G}(g^\dagger, \alpha, \beta)$  where  $g^\dagger = S^{-1/2}g \in L^2$ , is a Parseval frame. In particular, each  $f \in L^2$  has the following expansion

$$f = \sum_{k,n} \langle f, M_{k\beta} T_{n\alpha} g^\dagger \rangle M_{k\beta} T_{n\alpha} g^\dagger.$$

It is worth pointing out that the coefficients  $(\langle f, M_{k\beta} T_{n\alpha} g \rangle)_{k,n}$  appearing in (6) or in (7) are samples of the *Short-Time Fourier Transform* (STFT) of  $f$  with respect to  $g$ . This is the function  $V_g$  defined on  $\mathbb{R}^2$  by

$$V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i t \xi} dt.$$

$V_g$  is an isometry from  $L^2(\mathbb{R})$  onto a closed subspace of  $L^2(\mathbb{R}^2)$  and for all  $f \in L^2(\mathbb{R})$

$$(8) \quad \int_{\mathbb{R}} |f(t)|^2 dt = \iint_{\mathbb{R}^2} |V_g f(x, \xi)|^2 dx d\xi$$

Furthermore, for any  $h \in L^2$  such that  $\langle g, h \rangle \neq 0$

$$(9) \quad f(t) = \frac{1}{\langle g, h \rangle} \iint_{\mathbb{R}^2} V_g f(x, \xi) M_\xi T_x h(t) dx d\xi$$

where the integral is interpreted in the weak sense. We refer to [30, Chapter 1] and [33, Chapter 3] for more on the STFT and related phase-space or time-frequency transformations.

The reconstruction formulas in Proposition 1 can be viewed as discretizations of the inversion formula for the STFT (9). In particular, sampling the STFT on the lattice  $\alpha\mathbb{Z} \times \beta\mathbb{Z}$  and using the weights  $\tilde{c} = (\langle f, M_{k\beta} T_{n\alpha} \tilde{g} \rangle)_{k,n \in \mathbb{Z}} = (V_g f(\alpha k, \beta n))_{k,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$  perfectly reconstructs  $f$ . As such one can expect that in addition to the quality of the window  $g$  (and hence  $\tilde{g}$ ), the density of the lattice must play a role in establishing these formulas. Thus, it must not come as a surprise that the following results hold.

**Proposition 2** (Density theorems for Gabor frames). *Let  $g \in L^2(\mathbb{R})$  and  $\alpha, \beta > 0$ .*

- (a) *If  $\mathcal{G}(g, \alpha, \beta)$  is a Gabor frame for  $L^2(\mathbb{R})$  then  $0 < \alpha\beta \leq 1$*
- (b) *If  $\alpha\beta > 1$ , then  $\mathcal{G}(g, \alpha, \beta)$  is incomplete in  $L^2(\mathbb{R})$ .*
- (c)  *$\mathcal{G}(g, \alpha, \beta)$  is an orthonormal basis for  $L^2(\mathbb{R})$  if and only if  $\mathcal{G}(g, \alpha, \beta)$  is a tight frame for  $L^2(\mathbb{R})$ ,  $\|g\| = 1$ , and  $\alpha\beta = 1$ .*

These results were proved using various techniques ranging from operator theory to signal analysis illustrating the multi-origin of Gabor frame theory. For more on these density results we refer to [3, 19, 23, 31, 62], and for a complete historical perspective see [42].

At this point some questions arise naturally. For example, can one classify  $g \in L^2(\mathbb{R})$  and the parameters  $\alpha, \beta > 0$ , such that  $\mathcal{G}(g, \alpha, \beta)$  generates a frame, or an ONB for  $L^2(\mathbb{R})$ ? Despite some spectacular results both in the theory and the applications of Gabor frames [27, 28], these problems have not been completely resolved. Section 3 will be devoted to addressing the *frame set problem for Gabor frames*. That is given  $g \in L^2(\mathbb{R})$  characterize the set of all  $(a, b) \in \mathbb{R}_+^2$  such that  $\mathcal{G}(g, a, b)$  is a frame. On the other hand, and as seen from part (c) of Proposition 2, Gabor ONB can only occur when  $\alpha\beta = 1$ . In addition to this restriction, there does not exist a Gabor ONB with  $g \in L^2$  such that

$$\int_{-\infty}^{\infty} |x|^2 |g(x)|^2 dx \int_{-\infty}^{\infty} |\xi|^2 |\hat{g}(\xi)|^2 d\xi < \infty.$$

This uncertainty principle-type result known as the Balian-Low Theorem (BLT) precludes the existence of Gabor ONBs with well-localized windows [4, 6, 19, 58]. To be precise we have

**Proposition 3** (The Balian-Low Theorem). *Let  $g \in L^2(\mathbb{R})$  and  $\alpha > 0$ . If  $\mathcal{G}(g, \alpha, 1/\alpha)$  is an orthonormal basis for  $L^2(\mathbb{R})$  then*

$$\int_{-\infty}^{\infty} |x|^2 |g(x)|^2 dx \int_{-\infty}^{\infty} |\xi|^2 |\hat{g}(\xi)|^2 d\xi = \infty.$$

We refer to [8] for a survey on the BLT. In Section 4 we will introduce a modification of Gabor frames that will result in an ONB called *Wilson basis* with very regular window functions  $g$ . These ONBs were introduced by K. G. Wilson [70] under the name of Generalized Wannier functions. The fact that these are indeed ONBs was later established by Daubechies, Jaffard and Journé [22] who developed a systematic construction method for these kinds of systems. A very interesting application involving the Wilson bases is the recent detection of the gravitational waves [51, 52]. We refer to the short survey [12], and to [26] for some historical perspectives. For more on the Wilson bases we refer to [10, 54, 71, 72].

### 3. THE FRAME SET PROBLEM FOR GABOR FRAMES

The *frame set* of a function  $g \in L^2(\mathbb{R})$  is defined as

$$\mathcal{F}(g) = \{(a, b) \in \mathbb{R}_+^2 : \mathcal{G}(g, a, b) \text{ is a frame}\}.$$

In general, determining  $\mathcal{F}(g)$  for a given function  $g$  is still an open problem. One of the most known general result proved by Feichtinger and Kaiblinger [29] states that  $\mathcal{F}(g)$  is an open subset of  $\mathbb{R}_+^2$  if  $g \in L^2(\mathbb{R})$  belongs to the modulation space  $M^1(\mathbb{R})$  ([33]), i.e.,

$$\iint_{\mathbb{R}^2} |V_g g(x, \xi)| dx d\xi < \infty.$$

Examples of functions in this space include  $g(x) = e^{-\pi|x|^2}$  or  $g(x) = \frac{1}{\cosh x}$ . In fact, for these specific functions more is known. Indeed,  $\mathcal{G} = \mathbb{R}_+^2$  if  $g \in \{e^{-\pi x^2}, \frac{1}{\cosh x}, e^{-x} \chi_{[0, +\infty)}(x), e^{-|x|}\}$ , [46, 48, 50, 59, 67, 68]. On the other hand when  $g(x) = \chi_{[0, c]}(x)$ ,  $c > 0$ ,  $\mathcal{F}(g)$  is a rather complicated set that has only been fully described in recent years by Dai and Sun [18], see also [40, 49] for earlier work on this example.

Let  $g(x) = e^{-|x|}$  and observe that  $\hat{g}(\xi) = \frac{2}{1 + 4\pi^2 \xi^2}$ , which makes  $g(x) = e^{-|x|}$  an example of a totally positive function of order 2. More generally,  $g \in L^2(\mathbb{R})$  is a *totally positive function of type  $M$* , where  $M$  is a natural number, if its Fourier transform has the form  $\hat{g}(\xi) = \prod_{k=1}^M (1 + 2\pi i \delta_k \xi)^{-1}$  where  $\delta_k \neq \delta_\ell \in \mathbb{R}$  for  $k \neq \ell$ . It was proved that for all such functions  $g$ ,  $\mathcal{F}(g) = \mathbb{R}_+^2$  [39, 34]. A similar result holds for the class of totally positive functions of Gaussian type, [38], which are functions whose Fourier transforms have the form  $\hat{g}(\xi) = \prod_{k=1}^M (1 + 2\pi i \delta_k \xi)^{-1} e^{-c\xi^2}$  where  $\delta_1, \dots, \delta_M \in \mathbb{R}$  and  $c > 0$ . We refer to [34] for a survey of the structure of  $\mathcal{F}(g)$  not only for the rectangular lattices we consider here, but more general Gabor frame on discrete (countable) sets  $\Lambda \subset \mathbb{R}^2$ .

However, there are other “simple” functions  $g$  for which determining  $\mathcal{F}(g)$  remains largely a mystery. In the rest of this section we consider the frame set for the  $B$  splines  $g_N$  given by

$$\begin{cases} g_1(x) = \chi_{[-1/2, 1/2]}, & \text{and} \\ g_N(x) = g_1 * g_{N-1}(x) & \text{for } N \geq 2. \end{cases}$$

Christensen lists the characterization of  $\mathcal{F}(g_N)$  for  $N \geq 2$  as one of the six main problems in frame theory [14]. Due to the fact that  $g_N \in M^1(\mathbb{R})$  for  $N \geq 2$ , we know that  $\mathcal{F}(g_N)$  is an open subset of  $\mathbb{R}_+^2$ . The current description of points in this set can be found in [1, 15, 16, 36, 53, 55].

For example, consider the case  $N = 2$  where

$$g_2(x) = \chi_{[-1/2, 1/2]} * \chi_{[-1/2, 1/2]}(x) = \max(1 - |x|, 0) = \begin{cases} 1 + x & \text{if } x \in [-1, 0] \\ 1 - x & \text{if } x \in [0, 1] \end{cases}$$

The known results on  $\mathcal{F}(g_2)$  can be summarized as follows.

**Proposition 4** (Frame set of the 2-spline,  $g_2$ ). *The following statements hold.*

- (a) *If  $(\alpha, \beta) \in \mathcal{F}(g_2)$ , then  $\alpha\beta < 1$  and  $\alpha < 2$ .*
- (b) *Assume that  $1 \leq \alpha < 2$  and  $0 < \beta < \frac{1}{\alpha}$ . Then,  $(\alpha, \beta) \in \mathcal{F}(g_2)$ .*
- (c) *Assume that  $0 < \alpha < 2$ , and  $0 < \beta \leq \frac{2}{2+\alpha}$ . Then,  $(\alpha, \beta) \in \mathcal{F}(g_2)$ , and there is a unique dual  $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$  such that  $\text{supp } h \subseteq [-\frac{\alpha}{2}, \frac{\alpha}{2}]$ .*
- (d) *Assume that  $0 < \alpha < 2$ , and  $\frac{2}{2+\alpha} < \beta \leq \frac{4}{2+3\alpha}$ . Then,  $(\alpha, \beta) \in \mathcal{F}(g_2)$ , and there is a unique dual  $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$  such that  $\text{supp } h \subseteq [-\frac{3\alpha}{2}, \frac{3\alpha}{2}]$ .*
- (e) *Assume that  $0 < \alpha < 1/2$ , and  $\frac{4}{2+3\alpha} < \beta \leq \frac{2}{1+\alpha}$ . Then,  $(\alpha, \beta) \in \mathcal{F}(g_2)$ , and there is a unique dual  $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$  such that  $\text{supp } h \subseteq [-\frac{5\alpha}{2}, \frac{5\alpha}{2}]$ .*
- (f) *Assume that  $\frac{1}{2} \leq \alpha \leq \frac{4}{5}$ , and  $\frac{4}{2+3\alpha} < \beta \leq \frac{6}{2+5\alpha}$ , with  $\beta > 1$ . Then,  $(\alpha, \beta) \in \mathcal{F}(g_2)$ , and there is a unique dual  $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$  such that  $\text{supp } h \subseteq [-\frac{5\alpha}{2}, \frac{5\alpha}{2}]$ .*
- (g) *Assume that  $\frac{2}{3} \leq \alpha \leq 1$ , and  $\frac{4}{2+3\alpha} < \beta < 1$ . Then,  $(\alpha, \beta) \in \mathcal{F}(g_2)$ , and there is a unique compactly supported dual  $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ .*
- (h) *If  $0 < \alpha < 2$ ,  $\beta = 2, 3, \dots$ , and  $\alpha\beta < 1$ , then,  $(\alpha, \beta) \notin \mathcal{F}(g_2)$ .*

These results are illustrated in Figure 1, where except for the red regions, all other regions are contained in  $\mathcal{F}(g_2)$ . For the proofs we refer to [1, 2, 16, 15, 13, 37, 55, 53]. But we point out that the main idea in establishing parts (c–g) is based on the following result due to Janssen [47]. Before stating it we recall that for  $\alpha, \beta > 0$  and  $g \in L^2(\mathbb{R})$ , the Gabor system  $\mathcal{G}(g, \alpha, \beta)$  is called a *Bessel sequence* if only the upper bound in (6) is satisfied for some  $B > 0$ .

**Proposition 5** (Sufficient and necessary condition for dual Gabor frames). [47] *Let  $\alpha, \beta > 0$  and  $g, h \in L^2(\mathbb{R})$ . The Bessel sequences  $\mathcal{G}(g, \alpha, \beta)$  and  $\mathcal{G}(h, \alpha, \beta)$  are dual Gabor frames if and only if*

$$\sum_{k \in \mathbb{Z}} \overline{g(x - n/\beta - k\alpha)} h(x - k\alpha) = \beta \delta_{n,0} \quad \text{a.e. } x \in [0, \alpha].$$

Using this result with  $g = g_N$  and imposing that  $h$  is also compactly supported, leads one to seek an appropriate (finite) square matrix from the (infinite) linear system

$$\sum_{k \in \mathbb{Z}} g_N(x - \frac{\ell}{\beta} + k\alpha) h(x + k\alpha) = \beta \delta_\ell \quad \text{a.e. } x \in [-\frac{\alpha}{2}, \frac{\alpha}{2}].$$

In particular, [1] shows that the region  $\{(\alpha, \beta) \in \mathbb{R}_+^2 : 0 < \alpha\beta < 1\}$  can be partitioned in subregions  $R_m$ ,  $m \geq 1$ , such that a  $(2m-1) \times (2m-1)$  matrix  $G_m$  can be extracted from the above system leading to

$$G_m(x) \begin{bmatrix} h(x + (1-m)\alpha) \\ \vdots \\ h(x) \\ \vdots \\ h(x + (m-1)\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \beta \\ \vdots \\ 0 \end{bmatrix} \quad \text{a.e. } x \in [-\alpha/2, \alpha/2].$$

Choosing  $N = 2$  results in parts (c–g) of Proposition 4, for the cases  $m = 1, 2$ , and 3. For these cases, one proves that the matrix  $G_m(x)$  is invertible for a.e.  $x \in [-\alpha/2, \alpha/2]$ . However, only a subregion for the case  $m = 3$  has been settled in [2]. It is also known that the remaining part of this subregion contains some obstruction points, for example the line  $\beta = 2$  in Figure 1. Nonetheless, it seems one should be able to prove that the region

$$\{(\alpha, \beta) : \frac{1}{2} \leq \alpha < 1, \frac{6}{2+5\alpha} \leq \beta < \frac{2}{1+\alpha}, \beta > 1\}$$

is also contained in  $\mathcal{F}(g_2)$ . But this is still open.

We end this section by observing that the frame set problem is a special case of the more general question of characterizing the *full frame set*  $\mathcal{F}_{\text{full}}(g)$  of a function  $g$ , where

$$\mathcal{F}_{\text{full}}(g) = \{\Lambda \subset \mathbb{R}^2 : \mathcal{G}(g, \Lambda) \text{ is a frame}\}$$

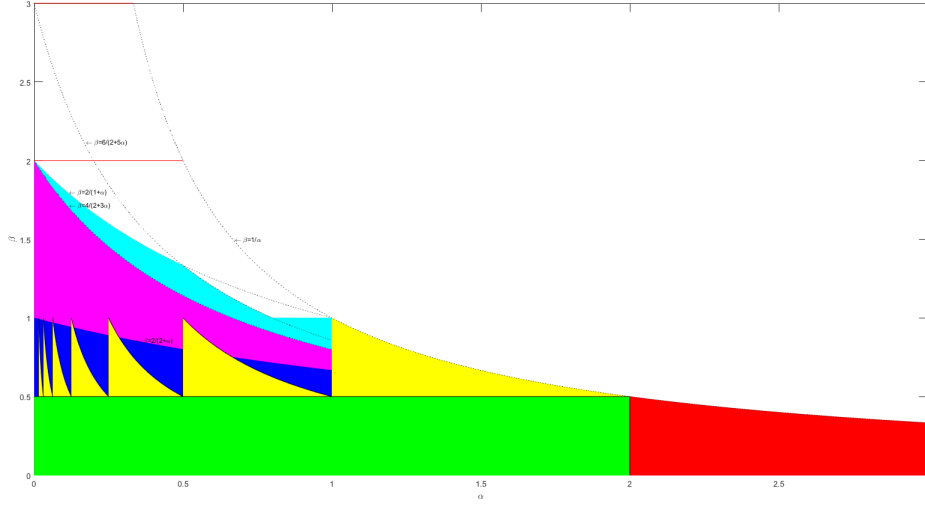


FIGURE 1. A sketch of  $\mathcal{F}(g_2)$ . The red region contains points  $(\alpha, \beta)$  for which  $\mathcal{G}(g_2, \alpha, \beta)$  is not a frame. All other colors indicate the frame property. The green region is the classical: "painless expansions" [21], and the yellow region is the result from [15]. The blue and the magenta regions are respectively from [55] and [16]. The cyan region is from [2].

where  $\Lambda$  is the lattice  $\Lambda = A\mathbb{Z}^2 \subset \mathbb{R}^2$  with  $\det A \neq 0$ . The only general result known in this case is for  $g(x) = e^{-a|x|^2}$  with  $a > 0$  in which case [59, 67]

$$\mathcal{F}_{\text{full}}(g) = \{\Lambda \subset \mathbb{R}^2 : \text{Vol} \Lambda < 1\},$$

where the volume of  $\Lambda$  is defined by  $\text{Vol}(\Lambda) = |\det A|$ .

#### 4. WILSON BASES

By the BLT (Proposition 3) and Proposition 2(iii), we know that  $\mathcal{G}(g, \alpha, 1/\alpha)$  cannot be an ONB if  $g$  is well-localized in the time-frequency plane. To overcome the BLT, K. G. Wilson introduced an ONB  $\{\psi_{n,\ell}, n \in \mathbb{N}_0, \ell \in \mathbb{Z}\}$ , where  $\psi_{0,\ell}(x) = \psi_\ell(x)$  and for  $n \geq 1$ ,  $\psi_{n,\ell}(x) = \psi_\ell(x - n)$ , and such that  $\hat{\psi}_{n,\ell}$  is localized around  $\pm n$ , that is,  $\psi_{n,\ell}$  is a bimodal function. Wilson presented numerical evidences that this system of functions is an ONB for  $L^2(\mathbb{R})$ . In 1992, Daubechies, Jaffard, and Journé formalized Wilson's ideas and constructed examples of bimodal Wilson bases generated by smooth functions. To be specific, the Wilson system associated with a given function  $g \in L^2$ , is  $\mathcal{W}(g) = \{\psi_{j,m} : j \in \mathbb{Z}, m \in \mathbb{N}_0\}$  where

$$(10) \quad \psi_{j,m}(x) = \begin{cases} g(x - j) & \text{if } j \in \mathbb{Z} \\ \frac{1}{\sqrt{2}} T_{\frac{j}{2}}(M_m + (-1)^{j+m} M_{-m})g(x) & \text{if } (j, m) \in \mathbb{Z} \times \mathbb{N}, \end{cases}$$

which can simply be rewritten as

$$\psi_{j,m}(x) = \begin{cases} \sqrt{2} \cos 2\pi m x g(x - \frac{j}{2}), & \text{if } j + m \text{ is even} \\ \sqrt{2} \sin 2\pi m x g(x - \frac{j}{2}), & \text{if } j + m \text{ is odd.} \end{cases}$$

It is not hard to see  $\{\psi_{j,m}\}$  is an ONB for  $L^2(\mathbb{R})$  if and only if

$$\begin{cases} \|\psi_{j,m}\| = 1 \text{ for all } (j,m) \in \mathbb{N}_0 \times \mathbb{Z} \\ \langle f, h \rangle = \sum_{j,m} \langle f, \psi_{j,m} \rangle \overline{\langle h, \psi_{j,m} \rangle} \text{ for all } f, h \in L^2. \end{cases}$$

Assuming that  $g$  and  $\hat{g}$  are smooth enough,  $\hat{g}$  real-valued, one can show that this is equivalent to

$$\sum_{m \in \mathbb{Z}} \hat{g}(\xi - m) \hat{g}(\xi - m + 2j) = \delta_{j,0} \text{ a.e. for each } j \in \mathbb{Z}.$$

It follows that one can construct compactly supported  $\hat{g}$  that will solve this system of equations. On the other hand, one can convert these equations into a single one by using another time-frequency analysis tool, the Zak transform which we now define. For  $f \in L^2(\mathbb{R})$  we let  $Zf : [0, 1) \times [0, 1) \rightarrow \mathbb{C}$  be given by

$$Zf(x, \xi) = \sqrt{2} \sum_{j \in \mathbb{Z}} f(2(x - j)) e^{2\pi i j \xi}$$

$Z$  is a unitary map from  $L^2(\mathbb{R})$  onto  $L^2([0, 1)^2)$  and enjoys some periodicity-like properties [33, Chapter 8]. Using the Zak transform, and under suitable regularity assumptions on  $g$  and  $\hat{g}$ , one can show that  $\{\psi_{j,m}\}$  is an ONB if and only if

$$|Z\hat{g}(x, \xi)|^2 + |Z\hat{g}(x, \xi + \frac{1}{2})|^2 = 2 \text{ a.e. } (x, \xi) \in [0, 1]^2.$$

Real-valued functions  $g$  solving this equation, can be constructed with the additional requirement that both  $g$  and  $\hat{g}$  have exponential decay.

To connect this Wilson system to Gabor frame, we use once again the Zak transform, and observe that the frame operator of the Gabor system  $\mathcal{G}(g, 1, 1/2)$  is a multiplication operator in the Zak transform domain, that is

$$ZS_g f(x, \xi) = M(x, \xi) Zf(x, \xi)$$

where  $M(x, \xi) = |Zg(x, \xi)|^2 + |Zg(x, \xi - \frac{1}{2})|^2$ . Consequently,  $\mathcal{G}(g, 1, 1/2)$  is a tight frame if and only if

$$M(x, \xi) = |Zg(x, \xi)|^2 + |Zg(x, \xi - \frac{1}{2})|^2 = A \text{ a.e.,}$$

where  $A$  is a constant. These ideas were used in [22] resulting in the following.

**Proposition 6** ([22]). *There exist unit-norm real-valued functions  $g \in L^2(\mathbb{R})$  with the property that both  $g$  and  $\hat{g}$  have exponential decay and such that the Gabor system  $\mathcal{G}(g, 1, 1/2)$  is a tight frame for  $L^2(\mathbb{R})$  if and only if the associated Wilson system  $\mathcal{W}(g)$  is an orthonormal basis for  $L^2(\mathbb{R})$ .*

Proposition 6 also provides an alternate view of the Wilson ONB. Indeed, each function in (10) is a linear combination of at most two Gabor functions from a tight Gabor frame  $\mathcal{G}(g, 1, 1/2)$  of redundancy 2. Furthermore, such Gabor systems can be constructed so as the generators are well-localized in the time-frequency plane. Suppose now that we are given a tight Gabor system  $\mathcal{G}(g, \alpha, \beta)$  where  $(\alpha\beta)^{-1} = N \in \mathbb{N}$  where  $N > 2$ . Hence, the redundancy of this tight frame is  $N$ . Can a Wilson-type ONB (generated by well-localized window) be constructed from this system by taking appropriate linear combinations? This problem was posed by Gröchenig for the case  $\alpha = 1$  and  $\beta = 1/3$  [33, Section 8.5], and to the best of our knowledge it is still open. If one is willing to give up on the orthogonality, Wojdyło [71] proved the existence of Parseval Wilson-type frames for  $L^2(\mathbb{R})$  from Gabor tight frames of redundancy 3. More recently, explicit examples have been constructed starting from Gabor tight frames of redundancy  $\frac{1}{\beta} \in \mathbb{N}$  where  $N \geq 3$ .

**Proposition 7.** [9] *For any  $\beta \in [1/4, 1/2)$  there exists a real-valued function  $g \in S(\mathbb{R})$  such that the following equivalent statements hold.*

- (i)  $\mathcal{G}(g, 1, \beta)$  is a tight Gabor frame of redundancy  $\beta^{-1}$ .



(ii) *The associated Wilson system given by*

$$(11) \quad \mathcal{W}(g, \beta) = \{\psi_{j,m} : j \in \mathbb{Z}, m \in \mathbb{N}_0\}$$

where

$$(12) \quad \psi_{j,m}(x) = \begin{cases} \sqrt{2\beta} g_{2j,0}(x) = \sqrt{2\beta} g(x - 2\beta j) & \text{if } j \in \mathbb{Z}, m = 0, \\ \sqrt{\beta} [e^{-2\pi i \beta j m} g_{j,m}(x) + (-1)^{j+m} e^{2\pi i \beta j m} g_{j,-m}(x)] & \text{if } (j, m) \in \mathbb{Z} \times \mathbb{N} \end{cases}$$

is a Parseval frame for  $L^2(\mathbb{R})$ .

If in addition  $\beta = \frac{1}{2n}$  where  $n$  is any odd natural number, then we can choose  $g$  to be real-valued such that both  $g$  and  $\hat{g}$  have exponential decay.

To turn these Parseval (Wilson) systems in ONBs, one needs to ensure that  $\|\psi_{j,m}\|_2 = 1$  for all  $j, m$ . This requires in particular that  $\|g\| = \frac{1}{\sqrt{2\beta}}$ , which seems to be incompatible with all the other conditions imposed  $g$ . It has then been suggested in [9] that to obtain a Wilson ONB with redundancy different from 2, one must modify in a fundamental way (12). For example, if we want to have a Wilson ONB with  $\alpha = 1, \beta = 1/3$ , it seems that one should take linear combinations of three Gabor atoms instead of the two in Proposition 7. While we have no proof of this claim, it seems to be supported by a recent construction of multivariate Wilson ONBs which is not a tensor products on 1-Wilson ONBs. This new approach was introduced in [10], where a relationship between these bases and the theory of Generalized Shift Invariant Spaces (GSIS) [63, 64, 65] was used to construct (non-separable) well-localized Wilson ONB for  $L^2(\mathbb{R}^d)$  starting from tight Gabor frames of redundancy  $2^k$  where  $k = 0, 1, 2, \dots, d-1$ . In particular, the functions in the corresponding Wilson systems are linear combinations of  $2^k$  elements from the tight Gabor frame.

## 5. HRT

In this last section we present a fascinating open problem that was posed in 1990 by C. Heil, J. Ramanathan, and P. Topiwala, and is now referred to as the HRT conjecture [41, 43].

**Conjecture 1** (The HRT Conjecture). *Given any  $0 \neq g \in L^2(\mathbb{R})$  and  $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$ ,  $\mathcal{G}(g, \Lambda)$  is a linearly independent set in  $L^2(\mathbb{R})$ , where*

$$\mathcal{G}(g, \Lambda) = \{e^{2\pi i b_k \cdot} g(\cdot - a_k), k = 1, 2, \dots, N\}.$$

To be more explicit, the conjecture claims the following: Given  $c_1, c_2, \dots, c_N \in \mathbb{C}$  such that

$$(13) \quad \sum_{k=1}^N c_k M_{b_k} T_{a_k} g(x) = \sum_{k=1}^N c_k e^{2\pi i b_k x} g(x - a_k) = 0 \text{ a.e.}, \implies c_1 = c_2 = \dots = c_N = 0.$$

The conjecture is still generally open even if one assumes that  $g \in S(\mathbb{R})$ .

Observe that for a given  $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$ , and  $g \in L^2(\mathbb{R})$ , we can always assume that  $(a_1, b_1) = (0, 0)$ , if not, applying  $M_{-b_1} T_{-a_1}$  to  $\mathcal{G}(g, \Lambda)$  results in  $\mathcal{G}(M_{-b_1} T_{-a_1} g, \Lambda')$  where  $\Lambda'$  will include the origin. In addition, by rotating and scaling if necessary, we may also assume that  $\Lambda$  contains  $(0, 1)$ . This will result in unitarily changing  $g$ . Finally, by applying a shear matrix, we may assume that  $\Lambda$  contains  $(a, 0)$  for some  $a \neq 0$ . Consequently, given  $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$  with  $N \geq 3$ , we shall assume that  $\{(0, 0), (0, 1), (a, 0)\} \subseteq \Lambda$ , for some  $a \neq 0$ .

To illustrate some of the difficulties arising in investigating this problem, we would like to give some ideas of the proof of the conjecture when  $N \leq 3$  and  $0 \neq g \in L^2(\mathbb{R})$ . Let us first consider the case  $N = 2$ , and from the above observations we can assume that  $\Lambda = \{(0, 0), (0, 1)\}$ . Suppose that  $c_1, c_2 \in \mathbb{C}$  such that  $c_1 g + c_2 M_1 g = 0$ . This is equivalent to

$$(c_1 + c_2 e^{2\pi i x}) g(x) = 0$$

Since  $g \neq 0$  and  $c_1 + c_2 e^{2\pi i x}$  is a trigonometric polynomial, we see that  $c_1 = c_2 = 0$ .

Now consider the case  $N = 3$ , and assume that  $\Lambda = \{(0, 0), (0, 1), (a, 0)\}$  where  $a > 0$  is such that  $\mathcal{G}(g, \Lambda)$  is linearly dependent. Thus there are non-zero complex numbers  $c_1, c_2$  such that

$$g(x - a) = (c_1 + c_2 e^{2\pi i x})g(x) = P(x)g(x) \quad \forall x \in S$$

where  $S \subset \text{supp}(g) \cap (0, 1)$  has positive Lebesgue measure. Note that  $P(x)$  is a 1-periodic trigonometric polynomial, that is nonzero a.e.. We can now iterate (5) along  $\pm na$  for  $n > 0$  to obtain

$$\begin{cases} g(x - na) = g(x) \prod_{j=0}^{n-1} P(x - ja) = g(x)P_n(x) \\ g(x + na) = g(x - a) \prod_{j=0}^n P(x + ja)^{-1} = g(x)Q_n(x) \end{cases}$$

Consequently,  $g(x + na) = g(x)Q_n(x) = g(x)P_n(x + na)^{-1}$  implying that

$$(14) \quad Q_n(x) = P_n(x + na)^{-1} \quad x \in S$$

In addition, using the fact that  $g \in L^2(\mathbb{R})$  one can conclude that

$$(15) \quad \lim_{n \rightarrow \infty} P_n(x) = \lim_{n \rightarrow \infty} Q_n(x) = 0 \quad a.e. x \in S$$

However, one can show that (14) and (15) cannot hold simultaneously by distinguishing the case  $a \in \mathbb{Q}$  and the case  $a$  is irrational. Hence, the HRT conjecture holds when  $\#\Lambda = 3$ . We refer to [43] for details.

In addition to the fact that the HRT conjecture is true for any set of 3 distinct points, the known results generally fall into the following categories.

**Proposition 8** (HRT for arbitrary set  $\Lambda \subset \mathbb{R}^2$ ). *Suppose that  $\Lambda \subset \mathbb{R}^2$  is a finite subset of distinct points. Then the HRT conjecture holds in each of the following cases.*

- (a)  $g$  is compactly supported, or just supported within a half-interval  $(-\infty, a]$ , or  $[a, \infty)$  [43].
- (b)  $g(x) = p(x)e^{-\pi x^2}$  where  $p$  is a polynomial [43].
- (c)  $g$  is such that  $\lim_{x \rightarrow \infty} |g(x)|e^{cx^2} = 0$  for all  $c > 0$  [11].
- (d)  $g$  is such that  $\lim_{x \rightarrow \infty} |g(x)|e^{cx \log x} = 0$  for all  $c > 0$  [11].

**Proposition 9** (HRT for arbitrary  $g \in L^2(\mathbb{R})$ ). *Suppose that  $0 \neq g \in L^2(\mathbb{R})$  is arbitrary. Then the HRT conjecture holds in each of the following cases.*

- (a)  $\Lambda$  is a finite set with  $\Lambda \subset A(\mathbb{Z}^2) + z$  where  $A$  is a full rank  $2 \times 2$  matrix and  $z \in \mathbb{R}^2$  [56]. In particular, Conjecture 1 holds when  $\#\Lambda \leq 3$  [43].
- (b)  $\#\Lambda = 4$  where two of the four points in  $\Lambda$  lie on a line and the remaining two points lie on a second parallel line [24, 25]. Such set  $\Lambda$  is called a  $(2, 2)$  configuration.
- (c)  $\Lambda$  consists of collinear points [43].
- (d)  $\Lambda$  consists of  $N - 1$  collinear and equi-spaced points, with the last point located off this line [43].

We observe that when  $\Lambda$  consists of collinear points, the HRT conjecture reduces to the question of linear independence of (finite) translates of  $L^2$  functions that was investigated by Rosenblatt [66]. Recently, Currey and Oussa showed that the HRT conjecture is equivalent to the question of linear independence of finite translates of square integrable functions on the Heisenberg group [17].

To date and to the best of our knowledge Proposition 8 and Proposition 9 are the most general known results on the HRT conjecture. Nonetheless, we give a partial list of known results when one makes restrictions on both the function  $g$  and the set  $\Lambda$ . For an extensive survey on the state of the HRT conjecture we refer to [41, 44], and to [35] for some perspectives from a numerical point of view.

**Proposition 10** (HRT in special cases). *The HRT conjecture holds in each of the following cases.*

- (a)  $g \in S(\mathbb{R})$ , and  $\#\Lambda = 4$  where three of the four points in  $\Lambda$  lie on a line and the fourth point is off this line [24]. Such set  $\Lambda$  is called a  $(1, 3)$  configuration.
- (b)  $g \in L^2(\mathbb{R})$  is ultimately positive, and  $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$  is such that  $\{b_k\}_{k=1}^N$  are independent over the rationals  $\mathbb{Q}$  [7].

- (c)  $\#\Lambda = 4$ , when  $g \in L^2(\mathbb{R})$  is ultimately positive,  $g(x)$  and  $g(-x)$  are ultimately decreasing [7].
- (d)  $g \in L^2(\mathbb{R})$  is real-valued, and  $\#\Lambda = 4$  and three of the four points in  $\Lambda$  lie on a line and the fourth point is off this line [60].
- (e)  $g \in S(\mathbb{R})$  is real-valued functions in  $\mathcal{S}(R)$  and  $\#\Lambda = 4$  [60].

Recently, some of the techniques used to establish the HRT for  $(2, 2)$  configurations were extended to deal with some special  $(3, 2)$  configurations [60]. From these results, and when restricting to real-valued functions, it was concluded that the HRT hold for certain sets of four points. We briefly describe this method here.

Let  $\Lambda = \{(0, 0), (0, 1), (a_0, 0), (a, b)\}$ , and assume that  $\Lambda$  is neither a  $(1, 3)$ , nor a  $(2, 2)$  configuration. Let  $0 \neq g \in L^2(\mathbb{R})$  be a real-valued function. Suppose that  $\mathcal{G}(g, \Lambda)$  is linearly dependent. Then there exist  $0 \neq c_k \in \mathbb{C}$ ,  $k = 1, 2, 3$  such that

$$T_{a_0}g = c_1g + c_2M_1g + c_3M_bT_ag$$

Taking the complex conjugate of this equation leads to

$$\bar{T}_{a_0}g = \bar{c}_1g + \bar{c}_2M_{-1}g + \bar{c}_3M_{-b}T_ag$$

Taking the difference of these two equations gives

$$(c_1 - \bar{c}_1)g + c_2M_1g - \bar{c}_2g + c_3M_bT_ag - \bar{c}_3M_{-b}T_ag = 0.$$

Since  $c_2, c_3 \neq 0$  we conclude that  $\mathcal{G}(g, \Lambda')$  where  $\Lambda' = \{(0, 0), (0, 1), (0, -1), (a, b), (a, -b)\}$  is a (symmetric)  $(3, 2)$  configuration, is linearly dependent. Consequently, we have proved the following result.

**Proposition 11.** *Let  $0 \neq g \in L^2(\mathbb{R})$  be a real-valued function. Suppose that  $(a, b) \in \mathbb{R}^2$  is such that  $\mathcal{G}(g, \Lambda_0)$  is linearly independent where  $\Lambda_0 = \{(0, 0), (0, 1), (0, -1), (a, b), (a, -b)\}$ . Then for all  $0 \neq c \in \mathbb{R}$ ,  $\mathcal{G}(g, \Lambda)$  is linearly independent where  $\Lambda = \{(0, 0), (0, 1), (c, 0), (a, b)\}$ .*

In [60, Theorem 6, Theorem 7] it was proved that the hypothesis of Proposition 11 is satisfied when  $g \in L^2(\mathbb{R})$  (not necessarily real-valued) for certain values of  $a$ , and  $b$ . These results were viewed as a *restriction principle* for the HRT, whereby proving the conjecture for special sets of  $N+1$  points one can establish it for certain related sets of  $N$  points. In addition, a related *extension principle*, which can be viewed as an induction-like technique was introduced. The premise of this principle is based on the following question. Suppose that the HRT conjecture holds for all  $g \in L^2(\mathbb{R})$  and a set  $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$ . For which points  $(a, b) \in \mathbb{R}^2 \setminus \Lambda$  will the conjecture remain true for the same function  $g$  and the new set  $\Lambda' = \Lambda \cup \{(a, b)\}$ ?

We elaborate on of this method for  $\#\Lambda = 3$ . Let  $g \in L^2(\mathbb{R})$  with  $\|g\|_2 = 1$  and suppose that  $\Lambda = \{(0, 0), (0, 1), (a_0, 0)\}$ . We denote  $\Lambda' = \Lambda \cup \{(a, b)\} = \{(0, 0), (0, 1), (a_0, 0), (a, b)\}$ . Since,  $\mathcal{G}(g, \Lambda)$  is linearly independent, the Gramian of this set of function is a positive definite matrix. We recall that the Gramian of a set of  $N$  vectors  $\{f_k\}_{k=1}^N \subset L^2(\mathbb{R})$  is the (positive semi-definite  $N \times N$  matrix  $(\langle f_k, f_\ell \rangle)_{k, \ell=1}^N$ . In the case at hand, the  $4 \times 4$  Gramian matrix  $G := G_g(a, b)$  of  $\mathcal{G}(g, \Lambda')$  can be written in the following block structure:

$$(16) \quad G = \begin{bmatrix} A & u(a, b) \\ u(a, b)^* & 1 \end{bmatrix}$$

where  $A$  is the  $3 \times 3$  Gramian of  $\mathcal{G}(g, \Lambda)$  and

$$u(a, b) = \begin{bmatrix} V_g g(a, b) \\ V_g g(a, b-1) \\ e^{-2\pi i a_0 b} V_g g(a - a_0, b) \end{bmatrix}$$

and  $u(a, b)^*$  is the adjoint of  $u(a, b)$ . By construction  $G$  is positive semi-definite for all  $(a, b) \in \mathbb{R}^2$  and we seek the set of points  $(a, b) \in \mathbb{R}^2 \setminus \Lambda$  such that  $G$  is positive definite. We can encode this

information into the determinant of this matrix, or into a related function  $F : \mathbb{R}^2 \rightarrow [0, \infty)$  given by

$$(17) \quad F(a, b) = \langle A^{-1}u(a, b), u(a, b) \rangle.$$

The following was proved in [60].

**Proposition 12** (The HRT Extension function). *Given the above notations the function  $F$  satisfies the following properties.*

- (i)  $0 \leq F(a, b) \leq 1$  for all  $(a, b) \in \mathbb{R}^2$ , and moreover,  $F(a, b) = 1$  if  $(a, b) \in \Lambda$ .
- (ii)  $F$  is uniformly continuous and  $\lim_{|(a, b)| \rightarrow \infty} F(a, b) = 0$ .
- (iii)  $\iint_{\mathbb{R}^2} F(a, b) da db = 3$ .
- (iv)  $\det G_g(a, b) = (1 - F(a, b)) \det A$ .

Consequently, there exists  $R > 0$  such that the HRT conjecture holds for  $g$  and  $\Lambda' = \Lambda \cup \{(a, b)\} = \{(0, 0), (0, 1), (a_0, 0), (a, b)\}$  whenever  $|(a, b)| > R$ .

We conclude the paper by elaborating on the case  $\Lambda = 4$ . Let  $\Lambda \subset \mathbb{R}^2$  contain 4 distinct points, and assume without loss of generality that  $\Lambda = \{(0, 0), (0, 1), (a_0, 0), (a, b)\}$ .

When  $b = 0$  and  $a = -a_0$  or  $a = 2a_0$ , then  $\Lambda$  is a  $(1, 3)$  configuration with the additional fact that its three collinear points are equi-spaced. This case is handled by Fourier methods as was done in [43]; see Proposition 9 (d). But, for general  $(1, 3)$  configurations, the Fourier methods are ineffective. Nonetheless, this case was considered by Demeter [24], who proved that the HRT conjecture holds for all  $(1, 3)$  configurations when  $g \in \mathcal{S}(\mathbb{R})$ , and for a family of  $(1, 3)$  configurations when  $g \in L^2(\mathbb{R})$ . It was latter proved by Liu that in fact, the HRT holds for all functions  $g \in L^2(\mathbb{R})$  and for almost all (in the sense of Lebesgue measure)  $(1, 3)$  configurations [57]. In fact, more is true, in the sense that for  $g \in L^2(\mathbb{R})$ , there exists at most one (equivalence class of)  $(1, 3)$  configuration  $\Lambda_0$  such that  $\mathcal{G}(g, \Lambda_0)$  is linearly dependent [60]. Here, we say that two sets  $\Lambda_1$  and  $\Lambda_2$  are equivalent if there exists a symplectic matrix  $A \in SL(2, \mathbb{R})$  (the determinant of  $A$  is 1) such that  $\Lambda_2 = A\Lambda_1$ . However, it still not known if the HRT holds for all  $(1, 3)$  configurations when  $g \in L^2$ .

Next if  $b = 1$  with  $a \notin \{0, a_0\}$ , or if  $a = a_0$  with  $b \neq 0$  then  $\Lambda$  is a  $(2, 2)$  configuration. Demeter and Zaharescu [25] established the HRT for all  $g \in L^2$  and all such configurations.

Consequently, to establish the HRT conjecture for all sets of four distinct points and all  $L^2$  function, one needs to focus on

- showing that there is no equivalence class of  $(1, 3)$  configurations for which the HRT fails; and
- proving the HRT for sets of four points that are neither  $(1, 3)$  configurations nor  $(2, 2)$  configurations.

For illustrative purposes we pose the following question.

**Question 1.** *Let  $0 \neq g \in L^2(\mathbb{R})$ . Prove that  $\mathcal{G}(G, \Lambda)$  is linearly independent in each of the following cases*

- (a)  $\Lambda = \{(0, 0), (0, 1), (1, 0), (\sqrt{2}, \sqrt{2})\}$ , see [41, Conjecture 9.2].
- (b)  $\Lambda = \{(0, 0), (0, 1), (1, 0), (\sqrt{2}, \sqrt{3})\}$ .

To be more explicit, the question is to prove that each of the following two sets are linearly independent

$$\{g(x), g(x-1), e^{2\pi i x} g(x), e^{2\pi i \sqrt{2} x} g(x - \sqrt{2})\}$$

and

$$\{g(x), g(x-1), e^{2\pi i x} g(x), e^{2\pi i \sqrt{3} x} g(x - \sqrt{2})\}$$

When  $g$  is real-valued, then part (a) was proved in [60], but nothing can be said for part (b). On the other hand, [60, Theorem 7] establishes part (b) when  $g \in \mathcal{S}(\mathbb{R})$ .

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