MARTINGALE INEQUALITIES FOR SPLINE SEQUENCES

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ABSTRACT. We show that D. Lépingle's $L_1(\ell_2)$ -inequality

$$\left\| \left(\sum_{n} \mathbb{E}[f_n | \mathscr{F}_{n-1}]^2 \right)^{1/2} \right\|_1 \le 2 \cdot \left\| \left(\sum_{n} f_n^2 \right)^{1/2} \right\|_1, \quad f_n \in \mathscr{F}_n,$$

extends to the case where we substitute the conditional expectation operators with orthogonal projection operators onto spline spaces and where we can allow that f_n is contained in a suitable spline space $\mathscr{S}(\mathscr{F}_n)$. This is done provided the filtration (\mathscr{F}_n) satisfies a certain regularity condition depending on the degree of smoothness of the functions contained in $\mathscr{S}(\mathscr{F}_n)$. As a by-product, we also obtain a spline version of H_1 -BMO duality under this assumption.

1. Introduction

This article is part of a series of papers that extend martingale results to polynomial spline sequences of arbitrary order (see e.g. [17, 15, 12, 10, 14, 13, 8]). In order to explain those martingale type results, we have to introduce a little bit of terminology: Let k be a positive integer, (\mathscr{F}_n) an increasing sequence of σ -algebras of sets in [0,1] where each \mathscr{F}_n is generated by a finite partition of [0,1] into intervals of positive length. Moreover, define the spline space

$$\mathscr{S}_k(\mathscr{F}_n) = \{ f \in C^{k-2}[0,1] : f \text{ is a polynomial of order } k \text{ on each atom of } \mathscr{F}_n \}$$

and let $P_n^{(k)}$ be the orthogonal projection operator onto $\mathscr{S}_k(\mathscr{F}_n)$ with respect to the L_2 inner product on [0,1] with the Lebesgue measure $|\cdot|$. The space $\mathscr{S}_1(\mathscr{F}_n)$ consists of piecewise constant functions and $P_n^{(1)}$ is the conditional expectation operator with respect to the σ -algebra \mathscr{F}_n . Similarly to the definition of martingales, we introduce the following notion: let $(f_n)_{n\geq 0}$ be a sequence of integrable functions. We call this sequence a k-martingale spline sequence (adapted to (\mathscr{F}_n)) if, for all n,

$$P_n^{(k)} f_{n+1} = f_n.$$

For basic facts about martingales and conditional expectations, we refer to [11].

Classical martingale theorems such as Doob's inequality or the martingale convergence theorem in fact carry over to k-martingale spline sequences corresponding to arbitrary filtrations (\mathscr{F}_n) of the above type, just by replacing conditional expectation operators by the projection operators $P_n^{(k)}$. Indeed, we have

(i) (Shadrin's theorem) there exists a constant C_k depending only on k such that

$$\sup_{n} \|P_n^{(k)} : L_1 \to L_1\| \le C_k,$$

(ii) (Doob's weak type inequality for splines) there exists a constant C_k depending only on k such that for any k-martingale spline sequence (f_n) and any $\lambda > 0$,

$$|\{\sup_{n} |f_n| > \lambda\}| \le C_k \frac{\sup_{n} ||f_n||_1}{\lambda},$$

(iii) (Doob's L_p inequality for splines) for all $p \in (1, \infty]$ there exists a constant $C_{p,k}$ depending only on p and k such that for all k-martingale spline sequences (f_n) ,

$$\left\| \sup_{n} |f_n| \right\|_p \le C_{p,k} \sup_{n} \|f_n\|_p,$$

- (iv) (Spline convergence theorem) if (f_n) is an L_1 -bounded k-martingale spline sequence, then (f_n) converges almost surely to some L_1 -function,
- (v) (Spline convergence theorem, L_p -version) for $1 , if <math>(f_n)$ is an L_p -bounded k-martingale spline sequence, then (f_n) converges almost surely and in L_p .

Property (i) is proved in [17], properties (ii) and (iii) in [15] and properties (iv) and (v) in [10], but see also [14].

Here, we continue this line of transferring martingale results to k-martingale spline sequences and extend D. Lépingle's $L_1(\ell_2)$ -inequality [9], which reads

(1.1)
$$\left\| \left(\sum_{n} \mathbb{E}[f_{n} | \mathscr{F}_{n-1}]^{2} \right)^{1/2} \right\|_{1} \leq 2 \cdot \left\| \left(\sum_{n} f_{n}^{2} \right)^{1/2} \right\|_{1},$$

provided the sequence of (real-valued) random variables f_n is adapted to the filtration (\mathscr{F}_n) , i.e., each f_n is \mathscr{F}_n -measurable. The spline version of this inequality is contained in Theorem 4.1.

This inequality is an L_1 extension of the following result for 1 , proved by E. M. Stein [19], that holds for*arbitrary* $integrable functions <math>f_n$:

(1.2)
$$\left\| \left(\sum_{n} \mathbb{E}[f_{n} | \mathscr{F}_{n-1}]^{2} \right)^{1/2} \right\|_{p} \leq a_{p} \left\| \left(\sum_{n} f_{n}^{2} \right)^{1/2} \right\|_{p},$$

for some constant a_p depending only on p. This can be seen as a dual version of Doob's inequality $\|\sup_{\ell} |\mathbb{E}[f|\mathscr{F}_{\ell}]|\|_p \le c_p \|f\|_p$ for p > 1, see [1]. Once we know Doob's inequality for spline projections, which is point (iii) above, the same proof as in [1] works for spline projections if we use suitable positive operators T_n instead of $P_n^{(k)}$ that also satisfy Doob's inequality and dominate the operators $P_n^{(k)}$ pointwise (cf. Sections 3.1 and 3.2).

The usage of those operators T_n is also necessary in the extension of inequality (1.1) to splines. D. Lépingle's proof of (1.1) rests on an idea by C. Herz [7] of splitting $\mathbb{E}[f_n \cdot h_n]$ (for f_n being \mathscr{F}_n -measurable) by Cauchy-Schwarz after introducing the square function $S_n^2 = \sum_{\ell \le n} f_\ell^2$:

$$(\mathbb{E}[f_n \cdot h_n])^2 \le \mathbb{E}[f_n^2/S_n] \cdot \mathbb{E}[S_n h_n^2]$$

and estimating both factors on the right hand side separately. A key point in estimating the second factor is that S_n is \mathscr{F}_n -measurable, and therefore, $\mathbb{E}[S_n|\mathscr{F}_n] = S_n$. If we want to allow $f_n \in \mathscr{S}_k(\mathscr{F}_n)$, S_n will not be contained in $\mathscr{S}_k(\mathscr{F}_n)$ in general. Under certain conditions on the filtration (\mathscr{F}_n) , we will show in this article how to substitute S_n in

estimate (1.3) by a function $g_n \in \mathscr{S}_k(\mathscr{F}_n)$ that enjoys similar properties to S_n and allows us to proceed (cf. Section 3.4, in particular Proposition 3.4 and Theorem 3.6). As a byproduct, we obtain a spline version (Theorem 4.2) of C. Fefferman's theorem [4] on H^1 -BMO duality. For its martingale version, we refer to A. M. Garsia's book [5] on Martingale Inequalities.

2. Preliminaries

In this section, we collect all tools that are needed subsequently.

2.1. Properties of polynomials. We will need Remez' inequality for polynomials:

Theorem 2.1. Let $V \subset \mathbb{R}$ be a compact interval in \mathbb{R} and $E \subset V$ a measurable subset. Then, for all polynomials p of order k (i.e. degree k-1) on V,

$$||p||_{L_{\infty}(V)} \le \left(4\frac{|V|}{|E|}\right)^{k-1} ||p||_{L_{\infty}(E)}.$$

Applying this theorem with the set $E = \{x \in V : |p(x)| \le 8^{-k+1} ||p||_{L_{\infty}(V)}\}$ immediately yields the following corollary:

Corollary 2.2. Let p be a polynomial of order k on a compact interval $V \subset \mathbb{R}$. Then

$$\left|\left\{x \in V : |p(x)| \ge 8^{-k+1} \|p\|_{L_{\infty}(V)}\right\}\right| \ge |V|/2.$$

2.2. **Properties of spline functions.** For an interval σ -algebra \mathscr{F} (i.e., \mathscr{F} is generated by a finite collection of intervals having positive length), the space $\mathscr{S}_k(\mathscr{F})$ is spanned by a very special local basis (N_i) , the so called B-spline basis. It has the properties that each N_i is non-negative and each support of N_i consists of at most k neighboring atoms of \mathscr{F} . Moreover, (N_i) is a partition of unity, i.e., for all $x \in [0,1]$, there exist at most k functions N_i so that $N_i(x) \neq 0$ and $\sum_i N_i(x) = 1$. In the following, we denote by E_i the support of the B-spline function N_i . The usual ordering of the B-splines (N_i) -which we also employ here—is such that for all i, inf $E_i \leq \inf E_{i+1}$ and $\sup E_i \leq \sup E_{i+1}$.

We write $A(t) \lesssim B(t)$ to denote the existence of a constant C such that for all t, $A(t) \leq CB(t)$, where t denote all implicit and explicit dependencies the expression A and B might have. If the constant C additionally depends on some parameter, we will indicate this in the text. Similarly, the symbols \gtrsim and \simeq are used.

Another important property of B-splines is the following relation between B-spline coefficients and the L_p -norm of the corresponding B-spline expansions.

Theorem 2.3 (B-spline stability, local and global). Let $1 \le p \le \infty$ and $g = \sum_j a_j N_j$. Then, for all j,

$$(2.1) |a_j| \lesssim |J_j|^{-1/p} ||g||_{L_p(J_j)},$$

where J_j is an atom of \mathscr{F} contained in E_j having maximal length. Additionally,

(2.2)
$$||g||_p \simeq ||(a_j|E_j|^{1/p})||_{\ell_p},$$

where in both (2.1) and (2.2), the implied constants depend only on the spline order k.

Observe that (2.1) implies for $g \in \mathcal{S}_k(\mathcal{F})$ and any measurable set $A \subset [0,1]$

(2.3)
$$||g||_{L_{\infty}(A)} \lesssim \max_{j:|E_j \cap A| > 0} ||g||_{L_{\infty}(J_j)}.$$

We will also need the following relation between the B-spline expansion of a function and its expansion using B-splines of a finer grid.

Theorem 2.4. Let $\mathscr{G} \subset \mathscr{F}$ be two interval σ -algebras and denote by $(N_{\mathscr{G},i})_i$ the B-spline basis of the coarser space $\mathscr{S}_k(\mathscr{G})$ and by $(N_{\mathscr{F},i})_i$ the B-spline basis of the finer space $\mathscr{S}_k(\mathscr{F})$. Then, given $f = \sum_j a_j N_{\mathscr{G},j}$, we can expand f in the basis $(N_{\mathscr{F},i})_i$

$$\sum_{j} a_{j} N_{\mathscr{G},j} = \sum_{i} b_{i} N_{\mathscr{F},i},$$

where for each i, b_i is a convex combination of the coefficients a_j with supp $N_{\mathscr{G},j} \supseteq \text{supp } N_{\mathscr{F},i}$.

For those results and more information on spline functions, in particular B-splines, we refer to [16] or [3]. We now use the B-spline basis of $\mathscr{S}_k(\mathscr{F})$ and expand the orthogonal projection operator P onto $\mathscr{S}_k(\mathscr{F})$ in the form

(2.4)
$$Pf = \sum_{i,j} a_{ij} \left(\int_0^1 f(x) N_i(x) \, \mathrm{d}x \right) \cdot N_j$$

for some coefficients (a_{ij}) . Denoting by E_{ij} the smallest interval containing both supports E_i and E_j of the B-spline functions N_i and N_j respectively, we have the following estimate for a_{ij} [15]: there exist constants C and 0 < q < 1 depending only on k so that for each interval σ -algebra \mathscr{F} and each i, j,

$$(2.5) |a_{ij}| \le C \frac{q^{|i-j|}}{|E_{ij}|}.$$

2.3. **Spline square functions.** Let (\mathscr{F}_n) be a sequence of increasing interval σ -algebras in [0,1] and we assume that each \mathscr{F}_{n+1} is generated from \mathscr{F}_n by the subdivision of exactly one atom of \mathscr{F}_n into two atoms of \mathscr{F}_{n+1} . Let P_n be the orthogonal projection operator onto $\mathscr{S}_k(\mathscr{F}_n)$. We denote $\Delta_n f = P_n f - P_{n-1} f$ and define the spline square function

$$Sf = \left(\sum_{n} |\Delta_n f|^2\right)^{1/2}.$$

We have Burkholder's inequality for the spline square function, i.e., for all $1 ([12]), the <math>L_p$ -norm of the square function Sf is comparable to the L_p -norm of f:

$$(2.6) ||Sf||_p \simeq ||f||_p, f \in L_p$$

with constants depending only on p and k. Moreover, for p = 1, it is shown in [6] that

(2.7)
$$||Sf||_1 \simeq \sup_{\varepsilon \in \{-1,1\}^{\mathbb{Z}}} ||\sum_n \varepsilon_n \Delta_n f||_1, \qquad Sf \in L_1,$$

with constants depending only on k and where the proof of the \lesssim -part only uses Khint-chine's inequality whereas the proof of the \gtrsim -part uses fine properties of the functions $\Delta_n f$.

2.4. $L_p(\ell_q)$ -spaces. For $1 \leq p, q \leq \infty$, we denote by $L_p(\ell_q)$ the space of sequences of measurable functions (f_n) on [0,1] so that the norm

$$||(f_n)||_{L_p(\ell_q)} = \left(\int_0^1 \left(\sum_n |f_n(t)|^q\right)^{p/q} dt\right)^{1/p}$$

is finite (with the obvious modifications if $p = \infty$ or $q = \infty$). For $1 \le p, q < \infty$, the dual space (see [2]) of $L_p(\ell_q)$ is $L_{p'}(\ell_{q'})$ with p' = p/(p-1), q' = q/(q-1) and the duality pairing

$$\langle (f_n), (g_n) \rangle = \int_0^1 \sum_n f_n(t) g_n(t) dt.$$

Hölder's inequality takes the form $|\langle (f_n), (g_n) \rangle| \leq ||(f_n)||_{L_p(\ell_q)} ||(g_n)||_{L_{p'}(\ell_{q'})}$.

3. Main Results

In this section, we prove our main results. Section 3.1 defines and gives properties of suitable positive operators that dominate our (non-positive) operators P_n pointwise. In Section 3.2, we use those operators to give a spline version of Stein's inequality (1.2). A useful property of conditional expectations is the tower property $\mathbb{E}_{\mathscr{G}}\mathbb{E}_{\mathscr{F}}f=\mathbb{E}_{\mathscr{G}}f$ for $\mathscr{G}\subset\mathscr{F}$. In this form, it extends to the operators (P_n) , but not to the operators T from Section 3.1. In Section 3.3 we prove a version of the tower property for those operators. Section 3.4 is devoted to establishing a duality estimate using a spline square function, which is the crucial ingredient in the proofs of the spline versions of both Lépingle's inequality (1.1) and H_1 -BMO duality in Section 4.

3.1. The positive operators T. As above, let \mathscr{F} be an interval σ -algebra on [0,1], (N_i) the B-spline basis of $\mathscr{S}_k(\mathscr{F})$, E_i the support of N_i and E_{ij} the smallest interval containing both E_i and E_j . Moreover, let q be a positive number smaller than 1. Then, we define the linear operator $T = T_{\mathscr{F},q,k}$ by

$$Tf(x) := \sum_{i,j} \frac{q^{|i-j|}}{|E_{ij}|} \langle f, \mathbb{1}_{E_i} \rangle \mathbb{1}_{E_j}(x) = \int_0^1 K(x,t) f(t) dt,$$

where the kernel $K = K_T$ is given by

$$K(x,t) = \sum_{i,j} \frac{q^{|i-j|}}{|E_{ij}|} \mathbb{1}_{E_i}(t) \cdot \mathbb{1}_{E_j}(x).$$

We observe that the operator T is selfadjoint (w.r.t the standard inner product on L_2) and

(3.1)
$$k \le K_x := \int_0^1 K(t, x) \, \mathrm{d}t \le \frac{2(k+1)}{1-q}, \qquad x \in [0, 1],$$

which, in particular, implies the boundedness of the operator T on L_1 and L_{∞} :

$$||Tf||_1 \le \frac{2(k+1)}{1-q} ||f||_1, \qquad ||Tf||_{\infty} \le \frac{2(k+1)}{1-q} ||f||_{\infty}.$$

Another very important property of T is that it is a positive operator, i.e. it maps non-negative functions to non-negative functions and that T satisfies Jensen's inequality in the form

(3.2)
$$\varphi(Tf(x)) \le K_x^{-1} T(\varphi(K_x \cdot f))(x), \qquad f \in L_1, x \in [0, 1],$$

for convex functions φ . This is seen by applying the classical Jensen inequality to the probability measure $K(t,x) dt/K_x$.

Let $\mathcal{M}f$ denote the Hardy-Littlewood maximal function of $f \in L_1$, i.e.,

$$\mathscr{M}f(x) = \sup_{I \ni x} \frac{1}{|I|} \int_{I} |f(y)| \, \mathrm{d}y,$$

where the supremum is taken over all subintervals of [0,1] that contain the point x. This operator is of weak type (1,1), i.e.,

$$|\{\mathcal{M}f > \lambda\}| \le C\lambda^{-1}||f||_1, \qquad f \in L_1, \lambda > 0$$

for some constant C. Since trivially we have the estimate $\|\mathcal{M}f\|_{\infty} \leq \|f\|_{\infty}$, by Marcinkiewicz interpolation, for any p > 1, there exists a constant C_p depending only on p so that

$$||\mathscr{M}f||_p \le C_p ||f||_p.$$

For those assertions about \mathcal{M} , we refer to (for instance) [18].

The significance of T and \mathcal{M} at this point is that we can use formula (2.4) and estimate (2.5) to obtain the pointwise bound

$$(3.3) |Pf(x)| \le C_1(T|f|)(x) \le C_2 \mathcal{M}f(x), f \in L_1, x \in [0,1],$$

where $T = T_{\mathscr{F},q,k}$ with q given by (2.5) and C_1, C_2 are two constants solely depending on k. In other words, the positive operator T dominates the non-positive operator P pointwise, but at the same time, T is dominated by \mathscr{M} pointwise independently of \mathscr{F} .

3.2. Stein's inequality for splines. We now use this pointwise dominating, positive operator T to prove Stein's inequality for spline projections. For this, let (\mathscr{F}_n) be an interval filtration on [0,1] and P_n be the orthogonal projection operator onto the space $\mathscr{S}_k(\mathscr{F}_n)$ of splines of order k corresponding to \mathscr{F}_n . Working with the positive operators $T_{\mathscr{F}_n,q,k}$ instead of the non-positive operators P_n , the proof of Stein's inequality (1.2) for spline projections can be carried over from the martingale case (cf. [19, 1]). For completeness, we include it here.

Theorem 3.1. Suppose that (f_n) is a sequence of arbitrary integrable functions on [0,1]. Then, for $1 \le r \le p < \infty$ or 1 ,

where the implied constant depends only on p, r and k.

Proof. By (3.3), it suffices to prove this inequality for the operators $T_n = T_{\mathscr{F}_n,q,k}$ with q given by (2.5) instead of the operators P_n . First observe that for r = p = 1, the assertion follows from Shadrin's theorem ((i) on page 1). Inequality (3.3) and the $L_{p'}$ -boundedness of \mathscr{M} for $1 < p' < \infty$ imply that

(3.5)
$$\|\sup_{1 \le n \le N} |T_n f|\|_{p'} \le C_{p',k} \|f\|_{p'}, \qquad f \in L_{p'}$$

with a constant $C_{p',k}$ depending on p' and k. Let $1 \leq p < \infty$ and $U_N : L_p(\ell_1^N) \to L_p$ be given by $(g_1, \ldots, g_N) \mapsto \sum_{j=1}^N T_j g_j$. Inequality (3.5) implies the boundedness of the adjoint $U_N^* : L_{p'} \to L_{p'}(\ell_\infty^N)$, $f \mapsto (T_j f)_{j=1}^N$ for p' = p/(p-1) by a constant independent of N and a fortiori the boundedness of U_N . Since $|T_j f| \leq T_j |f|$ by the positivity of T_j , letting $N \to \infty$ implies (3.4) for T_n instead of P_n in the case r = 1 and outer parameter $1 \leq p < \infty$.

If $1 < r \le p$, we use Jensen's inequality (3.2) and estimate (3.1) to obtain

$$\sum_{j=1}^{N} |T_{j}g_{j}|^{r} \lesssim \sum_{j=1}^{N} T_{j}(|g_{j}|^{r})$$

and apply the result for r=1 and the outer parameter p/r to get the result for $1 \le r \le p < \infty$. The cases $1 now just follow from this result using duality and the self-adjointness of <math>T_i$.

3.3. Tower property of T. Next, we will prove a substitute of the tower property $\mathbb{E}_{\mathscr{G}}\mathbb{E}_{\mathscr{F}}f=\mathbb{E}_{\mathscr{G}}f$ ($\mathscr{G}\subset\mathscr{F}$) for conditional expectations that applies to the operators T.

To formulate this result, we need a suitable notion of regularity for σ -algebras which we now describe. Let \mathscr{F} be an interval σ -algebra, let (N_j) be the B-spline basis of $\mathscr{S}_k(\mathscr{F})$ and denote by E_j the support of the function N_j . The k-regularity parameter $\gamma_k(\mathscr{F})$ is defined as

$$\gamma_k(\mathscr{F}) := \max_i \max(|E_i|/|E_{i+1}|, |E_{i+1}|/|E_i|),$$

where the first maximum is taken over all i so that E_i and E_{i+1} are defined. The name k-regularity is motivated by the fact that each B-spline support E_i of order k consists of at most k (neighboring) atoms of the σ -algebra \mathscr{F} .

Proposition 3.2 (Tower property of T). Let $\mathcal{G} \subset \mathcal{F}$ be two interval σ -algebras on [0,1]. Let $S = T_{\mathcal{G},\sigma,k}$ and $T = T_{\mathcal{F},\tau,k'}$ for some $\sigma,\tau \in (0,1)$ and some positive integers k,k'. Then, for all $q > \max(\tau,\sigma)$, there exists a constant C depending on q,k,k' so that

$$(3.6) |STf(x)| \le C \cdot \gamma^k \cdot (T_{\mathscr{G},q,k}|f|)(x), f \in L_1, x \in [0,1],$$

where $\gamma = \gamma_k(\mathcal{G})$ denotes the k-regularity parameter of \mathcal{G} .

Proof. Let (F_i) be the collection of B-spline supports in $\mathscr{S}_{k'}(\mathscr{F})$ and (G_i) the collection of B-spline supports in $\mathscr{S}_k(\mathscr{G})$. Moreover, we denote by F_{ij} the smallest interval containing F_i and F_j and by G_{ij} the smallest interval containing G_i and G_j .

We show (3.6) by showing the following inequality for the kernels K_S of S and K_T of T (cf. (3.1))

(3.7)
$$\int_0^1 K_S(x,t) K_T(t,s) \, \mathrm{d}t \le C \gamma^k \sum_{i,j} \frac{q^{|i-j|}}{|G_{ij}|} \mathbb{1}_{G_i}(x) \mathbb{1}_{G_j}(s), \qquad x, s \in [0,1]$$

for all $q > \max(\tau, \sigma)$ and some constant C depending on q, k, k'. In order to prove this inequality, we first fix $x, s \in [0, 1]$ and choose i such that $x \in G_i$ and ℓ such that $s \in F_\ell$. Moreover, based on ℓ , we choose j so that $s \in G_j$ and $G_j \supset F_\ell$. There are at most k choices for each of the indices i, ℓ, j and without restriction, we treat those choices separately, i.e., we only have to estimate the expression

$$\sum_{m,r} \frac{\sigma^{|m-i|}\tau^{|r-\ell|}|G_m \cap F_r|}{|G_{im}||F_{\ell r}|}.$$

Since, for each r, there are also at most k + k' - 1 indices m so that $|G_m \cap F_r| > 0$ (recall that $\mathscr{G} \subset \mathscr{F}$), we choose one such index m = m(r) and estimate

$$\Sigma = \sum_{r} \frac{\sigma^{|m(r)-i|} \tau^{|r-\ell|} |G_{m(r)} \cap F_r|}{|G_{i,m(r)}| |F_{\ell r}|}.$$

Now, observe that for any parameter choice of r in the above sum,

$$G_{i,m(r)} \cup F_{\ell r} \supseteq (G_{ij} \setminus G_j) \cup G_i$$

and therefore, since also $G_{m(r)} \cap F_r \subset G_{i,m(r)} \cap F_{\ell r}$,

$$\Sigma \le \frac{2}{|(G_{ij} \setminus G_j) \cup G_i|} \sum_r \sigma^{|m(r)-i|} \tau^{|r-\ell|},$$

which, using the k-regularity parameter $\gamma = \gamma_k(\mathcal{G})$ of the σ -algebra \mathcal{G} and denoting $\lambda = \max(\tau, \sigma)$, we estimate by

$$\Sigma \leq \frac{2\gamma^k}{|G_{ij}|} \sum_{m} \lambda^{|m-i|} \sum_{r:m(r)=m} \lambda^{|r-\ell|} \lesssim \frac{\gamma^k}{|G_{ij}|} \sum_{m} \lambda^{|i-m|+|m-j|}$$
$$\lesssim \frac{\gamma^k}{|G_{ij}|} (|i-j|+1) \lambda^{|i-j|},$$

where the implied constants depend on λ, k, k' and the estimate $\sum_{r:m(r)=m} \lambda^{|r-\ell|} \lesssim \lambda^{|m-j|}$ used the fact that, essentially, there are more atoms of \mathscr{F} between F_r and F_ℓ (for r as in the sum) than atoms of \mathscr{G} between G_m and G_j . Finally, we see that for any $q > \lambda$,

$$\Sigma \lesssim C \gamma^k \frac{q^{|i-j|}}{|G_{ij}|}$$

for some constant C depending on q, k, k', and, as $x \in G_i$ and $s \in G_j$, this shows inequality (3.7).

As a corollary of Proposition 3.2, we have

Corollary 3.3. Let (f_n) be functions in L_1 . We denote by P_n the orthogonal projection onto $\mathscr{S}_k(\mathscr{F}_n)$ and by P'_n the orthogonal projection onto $\mathscr{S}_{k'}(\mathscr{F}_n)$ for some positive integers k, k'. Moreover, let T_n be the operator $T_{\mathscr{F}_n,q,k}$ from (3.3) dominating P_n pointwise.

Then, for any integer n and for any $1 \le p \le \infty$,

$$\left\| \sum_{\ell \ge n} P_n \left((P'_{\ell-1} f_{\ell})^2 \right) \right\|_p \lesssim \left\| \sum_{\ell \ge n} T_n \left((P'_{\ell-1} f_{\ell})^2 \right) \right\|_p \lesssim \gamma_k (\mathscr{F}_n)^k \cdot \left\| \sum_{\ell \ge n} f_{\ell}^2 \right\|_p,$$

where the implied constants only depend on k and k'.

We remark that by Jensen's inequality and the tower property, this is trivial for conditional expectations $\mathbb{E}(\cdot|\mathscr{F}_n)$ instead of the operators $P_n, T_n, P'_{\ell-1}$ even with an absolute constant on the right hand side.

Proof. We denote by T_n the operator $T_{\mathscr{F}_n,q,k}$ and by T'_n the operator $T_{\mathscr{F}_n,q',k'}$, where the parameters q,q'<1 are given by inequality (3.3) depending on k and k' respectively. Setting $U_n:=T_{\mathscr{F}_n,\max(q,q')^{1/2},k}$, we perform the following chain of inequalities, where we

use the positivity of T_n and (3.3), Jensen's inequality for $T'_{\ell-1}$, the tower property for $T_n T'_{\ell-1}$ and the L_p -boundedness of U_n , respectively:

$$\left\| \sum_{\ell \geq n} T_n \left((P'_{\ell-1} f_{\ell})^2 \right) \right\|_p \lesssim \left\| \sum_{\ell \geq n} T_n \left((T'_{\ell-1} | f_{\ell}|)^2 \right) \right\|_p$$

$$\lesssim \left\| \sum_{\ell \geq n} T_n \left(T'_{\ell-1} f_{\ell}^2 \right) \right\|_p$$

$$\leq \left\| T_n \left(T'_{n-1} f_n^2 \right) \right\|_p + \left\| \sum_{\ell > n} T_n \left(T'_{\ell-1} f_{\ell}^2 \right) \right\|_p$$

$$\lesssim \left\| f_n^2 \right\|_p + \gamma_k (\mathscr{F}_n)^k \cdot \left\| \sum_{\ell > n} U_n (f_{\ell}^2) \right\|_p$$

$$\lesssim \gamma_k (\mathscr{F}_n)^k \cdot \left\| \sum_{\ell > n} f_{\ell}^2 \right\|_p,$$

where the implied constants only depend on k and k'.

3.4. A duality estimate using a spline square function. In order to give the desired duality estimate contained in Theorem 3.6, we need the following construction of a function $g_n \in \mathscr{S}_k(\mathscr{F}_n)$ based on a spline square function.

Proposition 3.4. Let (f_n) be a sequence of functions with $f_n \in \mathscr{S}_k(\mathscr{F}_n)$ for all n and set

$$X_n := \sum_{\ell \le n} f_\ell^2.$$

Then, there exists a sequence of non-negative functions $g_n \in \mathscr{S}_k(\mathscr{F}_n)$ so that for each n,

- (1) $g_n \le g_{n+1}$, (2) $X_n^{1/2} \le g_n$
- (3) $\mathbb{E}g_n \lesssim \mathbb{E}X_n^{1/2}$, where the implied constant depends on k and on $\sup_{m \leq n} \gamma_k(\mathscr{F}_m)$.

For the proof of this result, we need the following simple lemma.

Lemma 3.5. Let c_1 be a positive constant and let $(A_j)_{j=1}^N$ be a sequence of atoms in \mathscr{F}_n . Moreover, let $\ell: \{1, \ldots, N\} \to \{1, \ldots, n\}$ and, for each $j \in \{1, \ldots, N\}$, let B_j be a subset of an atom D_i of $\mathscr{F}_{\ell(i)}$ with

(3.8)
$$|B_j| \ge c_1 \sum_{\substack{i:\ell(i) \ge \ell(j), \\ D_i \subseteq D_j}} |A_i|.$$

Then, there exists a set-valued mapping φ on $\{1,\ldots,N\}$ so that

- (1) $|\varphi(j)| = c_1 |A_j|$ for all j,
- (2) $\varphi(j) \subseteq B_i$ for all j,
- (3) $\varphi(i) \cap \varphi(j) = \emptyset$ for all $i \neq j$.

Proof. Without restriction, we assume that the sequence (A_i) is enumerated such that $\ell(j+1) \leq \ell(j)$ for all $1 \leq j \leq N-1$. We first choose $\varphi(1)$ as an arbitrary (measurable) subset of B_1 with measure $c_1|A_1|$, which is possible by assumption (3.8). Next, we assume that for $1 \leq j \leq j_0$, we have constructed $\varphi(j)$ with the properties

$$(1) |\varphi(j)| = c_1 |A_j|,$$

- (2) $\varphi(j) \subseteq B_i$,
- $(3) \varphi(i) \cap \bigcup_{i \leq i} \varphi(i) = \emptyset.$

Based on that, we now construct $\varphi(j_0+1)$. Define the index sets $\Gamma=\{i:\ell(i)\geq\ell(j_0+1)\}$ 1), $D_i \subseteq D_{j_0+1}$ and $\Lambda = \{i : i \leq j_0+1, D_i \subseteq D_{j_0+1}\}$. Since we assumed that ℓ is decreasing, we have $\Lambda \subseteq \Gamma$ and by the nestedness of the σ -algebras \mathscr{F}_n , we have for $i \leq j_0 + 1$ that either $D_i \subset D_{j_0+1}$ or $|D_i \cap D_{j_0+1}| = 0$. This implies

$$\begin{vmatrix} B_{j_0+1} \setminus \bigcup_{i \le j_0} \varphi(i) \end{vmatrix} = |B_{j_0+1}| - \left| B_{j_0+1} \cap \bigcup_{i \le j_0} \varphi(i) \right|
\geq c_1 \sum_{i \in \Gamma} |A_i| - \left| D_{j_0+1} \cap \bigcup_{i \le j_0} \varphi(i) \right|
\geq c_1 \sum_{i \in \Lambda} |A_i| - \left| \bigcup_{i \in \Lambda \setminus \{j_0+1\}} \varphi(i) \right|
\geq c_1 \sum_{i \in \Lambda} |A_i| - \sum_{i \in \Lambda \setminus \{j_0+1\}} c_1 |A_i| = c_1 |A_{j_0+1}|.$$

Therefore, we can choose $\varphi(j_0+1)\subseteq B_{j_0+1}$ that is disjoint to $\varphi(i)$ for any $i\leq j_0$ and $|\varphi(j_0+1)|=c_1|A_{j_0+1}|$ which completes the proof.

Proof of Proposition 3.4. Fix n and let $(N_{n,j})$ be the B-spline basis of $\mathscr{S}_k(\mathscr{F}_n)$. Moreover, for any j, set $E_{n,j} = \text{supp } N_{n,j}$ and $a_{n,j} := \max_{\ell \le n} \max_{r: E_{\ell,r} \supset E_{n,j}} \|X_{\ell}\|_{L_{\infty}(E_{\ell,r})}^{1/2}$ and we define $\ell(j) \leq n$ and r(j) so that $E_{\ell(j),r(j)} \supseteq E_{n,j}$ and $a_{n,j} = ||X_{\ell(j)}||_{L_{\infty}(E_{\ell(j),r(j)})}^{1/2}$. Set

$$g_n := \sum_j a_{n,j} N_{n,j} \in \mathscr{S}_k(\mathscr{F}_n)$$

and it will be proved subsequently that this g_n has the desired properties.

PROPERTY (1): In order to show $g_n \leq g_{n+1}$, we use Theorem 2.4 to write

$$g_n = \sum_j a_{n,j} N_{n,j} = \sum_j \beta_{n,j} N_{n+1,j},$$

where $\beta_{n,j}$ is a convex combination of those $a_{n,r}$ with $E_{n+1,j} \subseteq E_{n,r}$, and thus

$$g_n \le \sum_{j} \left(\max_{r: E_{n+1,j} \subseteq E_{n,r}} a_{n,r} \right) N_{n+1,j}.$$

By the very definition of $a_{n+1,i}$, we have

$$\max_{r:E_{n+1,j}\subseteq E_{n,r}} a_{n,r} \le a_{n+1,j},$$

and therefore, $g_n \leq g_{n+1}$ pointwise, since the B-splines $(N_{n+1,j})_j$ are nonnegative functions. PROPERTY (2): Now we show that $X_n^{1/2} \leq g_n$. Indeed, for any $x \in [0,1]$,

$$g_n(x) = \sum_{j} a_{n,j} N_{n,j}(x) \ge \min_{j: E_{n,j} \ni x} a_{n,j} \ge \min_{j: E_{n,j} \ni x} \|X_n\|_{L_{\infty}(E_{n,j})}^{1/2} \ge X_n(x)^{1/2},$$

since the collection of B-splines $(N_{n,j})_j$ forms a partition of unity.

PROPERTY (3): Finally, we show $\mathbb{E}g_n \lesssim \mathbb{E}X_n^{1/2}$, where the implied constant depends only on k and on $\sup_{m \leq n} \gamma_k(\mathscr{F}_m)$. By B-spline stability (Theorem 2.3), we estimate the integral of g_n by

(3.9)
$$\mathbb{E}g_n \lesssim \sum_{j} |E_{n,j}| \cdot ||X_{\ell(j)}||_{L_{\infty}(E_{\ell(j),r(j)})}^{1/2},$$

where the implied constant only depends on k. In order to continue the estimate, we next show the inequality

(3.10)
$$||X_{\ell}||_{L_{\infty}(E_{\ell,r})} \lesssim \max_{s:|E_{\ell,r} \cap E_{\ell,s}| > 0} ||X_{\ell}||_{L_{\infty}(J_{\ell,s})},$$

where by $J_{\ell,s}$ we denote an atom of \mathscr{F}_{ℓ} with $J_{\ell,s} \subset E_{\ell,s}$ of maximal length and the implied constant depends only on k. Indeed, we use Theorem 2.3 in the form of (2.3) to get $(f_m \in \mathscr{S}_k(\mathscr{F}_{\ell}))$ for $m \leq \ell$

$$(3.11) \qquad \|X_{\ell}\|_{L_{\infty}(E_{\ell,r})} \leq \sum_{m \leq \ell} \|f_m\|_{L_{\infty}(E_{\ell,r})}^2$$

$$\lesssim \sum_{m \leq \ell} \sum_{s:|E_{\ell,s} \cap E_{\ell,r}| > 0} \|f_m\|_{L_{\infty}(J_{\ell,s})}^2 = \sum_{s:|E_{\ell,s} \cap E_{\ell,r}| > 0} \sum_{m \leq \ell} \|f_m\|_{L_{\infty}(J_{\ell,s})}^2.$$

Now observe that for atoms I of \mathscr{F}_{ℓ} , due to the equivalence of p-norms of polynomials (cf. Corollary 2.2),

$$\sum_{m < \ell} \|f_m\|_{L_{\infty}(I)}^2 \lesssim \sum_{m < \ell} \frac{1}{|I|} \int_I f_m^2 = \frac{1}{|I|} \int_I X_{\ell} \le \|X_{\ell}\|_{L_{\infty}(I)},$$

which means that, inserting this in estimate (3.11),

$$||X_{\ell}||_{L_{\infty}(E_{\ell,r})} \lesssim \sum_{s:|E_{\ell,s}\cap E_{\ell,r}|>0} ||X_{\ell}||_{L_{\infty}(J_{\ell,s})},$$

and, since there are at most k indices s so that $|E_{\ell,s} \cap E_{\ell,r}| > 0$, we have established (3.10).

We define $K_{\ell,r}$ to be an interval $J_{\ell,s}$ with $|E_{\ell,r} \cap E_{\ell,s}| > 0$ so that

$$\max_{s:|E_{\ell,r}\cap E_{\ell,s}|>0} \|X_{\ell}\|_{L_{\infty}(J_{\ell,s})} = \|X_{\ell}\|_{L_{\infty}(K_{\ell,r})}.$$

This means, after combining (3.9) with (3.10), we have

(3.12)
$$\mathbb{E}g_n \lesssim \sum_{j} |J_{n,j}| \cdot ||X_{\ell(j)}||_{L_{\infty}(K_{\ell(j),r(j)})}^{1/2}.$$

We now apply Lemma 3.5 with the function ℓ and the choices

$$A_j = J_{n,j}, D_j = K_{\ell(j),r(j)},$$

$$B_j = \left\{ t \in D_j : X_{\ell(j)}(t) \ge 8^{-2(k-1)} ||X_{\ell(j)}||_{L_{\infty}(D_j)} \right\}.$$

In order to see assumption (3.8) of Lemma 3.5, fix the index j and let i be such that $\ell(i) \geq \ell(j)$. By definition of $D_i = K_{\ell(i),r(i)}$, the smallest interval containing $J_{n,i}$ and D_i contains at most 2k-1 atoms of $\mathscr{F}_{\ell(i)}$ and, if $D_i \subset D_j$, the smallest interval containing $J_{n,i}$ and D_j contains at most 2k-1 atoms of $\mathscr{F}_{\ell(j)}$. This means that, in particular, $J_{n,i}$ is a subset of the union V of 4k atoms of $\mathscr{F}_{\ell(j)}$ with $D_j \subset V$. Since each atom of \mathscr{F}_n

occurs at most k times in the sequence (A_j) , there exists a constant c_1 depending on k and $\sup_{u < \ell(j)} \gamma_k(\mathscr{F}_u) \le \sup_{u < n} \gamma_k(\mathscr{F}_u)$ so that

$$|D_j| \ge c_1 \sum_{\substack{i:\ell(i) \ge \ell(j) \\ D_i \subset D_j}} |A_i|,$$

which, since $|B_j| \ge |D_j|/2$ by Corollary 2.2, shows the assumption of Lemma 3.5 and we get a set-valued function φ so that $|\varphi(j)| = c_1 |J_{n,j}|/2$, $\varphi(j) \subset B_j$, $\varphi(i) \cap \varphi(j) = \emptyset$ for all i, j. Using these properties of φ , we continue the estimate in (3.12) and write

$$\mathbb{E}g_{n} \lesssim \sum_{j} |J_{n,j}| \cdot ||X_{\ell(j)}||_{L_{\infty}(D_{j})}^{1/2} \leq 8^{k-1} \cdot \sum_{j} \frac{|J_{n,j}|}{|\varphi(j)|} \int_{\varphi(j)} X_{\ell(j)}^{1/2}$$

$$= \frac{2}{c_{1}} \cdot 8^{k-1} \cdot \sum_{j} \int_{\varphi(j)} X_{\ell(j)}^{1/2}$$

$$\lesssim \sum_{j} \int_{\varphi(j)} X_{n}^{1/2} \leq \mathbb{E}X_{n}^{1/2},$$

with constants depending only on k and $\sup_{u \leq n} \gamma_k(\mathscr{F}_u)$.

Employing this construction of g_n , we now give the following duality estimate for spline projections (for the martingale case, see for instance [5]). The martingale version of this result is the essential estimate in the proof of both Lépingle's inequality (1.1) and the H^1 -BMO duality.

Theorem 3.6. Let (\mathscr{F}_n) be such that $\gamma := \sup_n \gamma_k(\mathscr{F}_n) < \infty$ and $(f_n)_{n \geq 1}$ a sequence of functions with $f_n \in \mathscr{S}_k(\mathscr{F}_n)$ for each n. Additionally, let $h_n \in L_1$ be arbitrary. Then, for any N,

$$\sum_{n\leq N} \mathbb{E}[|f_n\cdot h_n|] \lesssim \mathbb{E}\left[\left(\sum_{\ell\leq N} f_\ell^2\right)^{1/2}\right] \cdot \sup_{n\leq N} \|P_n\left(\sum_{\ell=n}^N h_\ell^2\right)\|_{\infty}^{1/2},$$

where the implied constant depends only on k and γ .

Proof. Let $X_n := \sum_{\ell \le n} f_\ell^2$ and $f_\ell \equiv 0$ for $\ell > N$ and $\ell \le 0$. By Proposition 3.4, we choose an increasing sequence (g_n) of functions with $g_0 = 0$, $g_n \in \mathscr{S}_k(\mathscr{F}_n)$ and the properties $X_n^{1/2} \le g_n$ and $\mathbb{E}g_n \lesssim \mathbb{E}X_n^{1/2}$ where the implied constant depends only on k and γ . Then, apply Cauchy-Schwarz inequality by introducing the factor $g_n^{1/2}$ to get

$$\sum_{n} \mathbb{E}[|f_n \cdot h_n|] = \sum_{n} \mathbb{E}\left[\left|\frac{f_n}{g_n^{1/2}} \cdot g_n^{1/2} h_n\right|\right] \le \left[\sum_{n} \mathbb{E}[f_n^2/g_n]\right]^{1/2} \cdot \left[\sum_{n} \mathbb{E}[g_n h_n^2]\right]^{1/2}.$$

We estimate each of the factors on the right hand side separately and set

$$\Sigma_1 := \sum_n \mathbb{E}[f_n^2/g_n], \qquad \Sigma_2 := \sum_n \mathbb{E}[g_n h_n^2].$$

The first factor is estimated by the pointwise inequality $X_n^{1/2} \leq g_n$:

$$\Sigma_1 = \mathbb{E}\left[\sum_n \frac{f_n^2}{g_n}\right] \le \mathbb{E}\left[\sum_n \frac{f_n^2}{X_n^{1/2}}\right]$$

$$= \mathbb{E}\left[\sum_{n} \frac{X_n - X_{n-1}}{X_n^{1/2}}\right] \le 2\mathbb{E}\sum_{n} (X_n^{1/2} - X_{n-1}^{1/2}) = 2\mathbb{E}X_N^{1/2}.$$

We continue with Σ_2 :

$$\Sigma_{2} = \mathbb{E}\left[\sum_{\ell=1}^{N} g_{\ell} h_{\ell}^{2}\right] = \mathbb{E}\left[\sum_{\ell=1}^{N} \sum_{n=1}^{\ell} (g_{n} - g_{n-1}) h_{\ell}^{2}\right]$$

$$= \mathbb{E}\left[\sum_{n=1}^{N} (g_{n} - g_{n-1}) \cdot \sum_{\ell=n}^{N} h_{\ell}^{2}\right]$$

$$= \mathbb{E}\left[\sum_{n=1}^{N} P_{n}(g_{n} - g_{n-1}) \cdot \sum_{\ell=n}^{N} h_{\ell}^{2}\right]$$

$$= \mathbb{E}\left[\sum_{n=1}^{N} (g_{n} - g_{n-1}) \cdot P_{n}\left(\sum_{\ell=n}^{N} h_{\ell}^{2}\right)\right]$$

$$\leq \mathbb{E}\left[\sum_{n=1}^{N} (g_{n} - g_{n-1})\right] \cdot \sup_{1 \leq n \leq N} \|P_{n}\left(\sum_{\ell=n}^{N} h_{\ell}^{2}\right)\|_{\infty},$$

where the last inequality follows from $g_n \geq g_{n-1}$. Noting that by the properties of g_n , $\mathbb{E}\left[\sum_{n=1}^N (g_n - g_{n-1})\right] = \mathbb{E}g_N \lesssim \mathbb{E}X_N^{1/2}$ and combining the two parts Σ_1 and Σ_2 , we obtain the conclusion.

4. Applications

We give two applications of Theorem 3.6, (i) D. Lépingle's inequality and (ii) an analogue of C. Fefferman's H_1 -BMO duality in the setting of splines. Once the results from Section 3 are known, the proofs of the subsequent results proceed similarly to their martingale counterparts in [9] and [5] by using spline properties instead of martingale properties.

4.1. Lépingle's inequality for splines.

Theorem 4.1. Let k, k' be positive integers. Let (\mathscr{F}_n) be an interval filtration with $\sup_n \gamma_k(\mathscr{F}_n) < \infty$ and, for any $n, f_n \in \mathscr{S}_k(\mathscr{F}_n)$ and P'_n be the orthogonal projection operator on $\mathscr{S}_{k'}(\mathscr{F}_n)$. Then,

$$\|(P'_{n-1}f_n)\|_{L_1(\ell_2)} = \left\| \left(\sum_n (P'_{n-1}f_n)^2 \right)^{1/2} \right\|_1 \lesssim \left\| \left(\sum_n f_n^2 \right)^{1/2} \right\|_1 = \|(f_n)\|_{L_1(\ell_2)},$$

where the implied constant depends only on k, k' and $\sup_{n} \gamma_{k}(\mathscr{F}_{n})$.

Proof. We first assume that $f_n = 0$ for n > N and begin by using duality

$$\mathbb{E}\left[\left(\sum_{n} (P'_{n-1}f_{n})^{2}\right)^{1/2}\right] = \sup_{(H_{n})} \mathbb{E}\left[\sum_{n} (P'_{n-1}f_{n}) \cdot H_{n}\right],$$

where sup is taken over all $(H_n) \in L_{\infty}(\ell_2)$ with $||(H_n)||_{L_{\infty}(\ell_2)} = 1$. By the self-adjointness of P'_{n-1} ,

$$\mathbb{E}\big[(P'_{n-1}f_n)\cdot H_n\big] = \mathbb{E}\big[f_n\cdot (P'_{n-1}H_n)\big].$$

Then we apply Theorem 3.6 for f_n and $h_n = P'_{n-1}H_n$ to obtain (denoting by P_n the orthogonal projection operator onto $\mathscr{S}_k(\mathscr{F}_n)$)

(4.1)
$$\sum_{n < N} |\mathbb{E}[f_n \cdot h_n]| \lesssim \mathbb{E}\left[\left(\sum_{\ell < N} f_{\ell}^2\right)^{1/2}\right] \cdot \sup_{n \le N} \left\| P_n\left(\sum_{\ell = n}^N (P_{\ell-1}' H_{\ell})^2\right) \right\|_{\infty}^{1/2}.$$

To estimate the right hand side, we note that for any n, by Corollary 3.3,

$$\left\| P_n \left(\sum_{\ell=n}^N (P'_{\ell-1} H_{\ell})^2 \right) \right\|_{\infty} \lesssim \left\| \sum_{\ell=n}^N H_{\ell}^2 \right\|_{\infty}.$$

Therefore, (4.1) implies

$$\mathbb{E}\left[\left(\sum_{n} (P'_{n-1} f_n)^2\right)^{1/2}\right] = \sup_{(H_n)} \mathbb{E}\left[\sum_{n} f_n \cdot (P'_{n-1} H_n)\right] \lesssim \mathbb{E}\left[\left(\sum_{\ell \le N} f_\ell^2\right)^{1/2}\right],$$

with a constant depending only on k,k' and $\sup_{n\leq N} \gamma_k(\mathscr{F}_n)$. Letting N tend to infinity, we obtain the conclusion.

4.2. H_1 -BMO duality for splines. We fix an interval filtration $(\mathscr{F}_n)_{n=1}^{\infty}$, a spline order k and the orthogonal projection operators P_n onto $\mathscr{S}_k(\mathscr{F}_n)$ and additionally, we set $P_0 = 0$. For $f \in L_1$, we introduce the notation

$$\Delta_n f := P_n f - P_{n-1} f, \qquad S_n(f) := \left(\sum_{\ell=1}^n (\Delta_\ell f)^2\right)^{1/2}, \qquad S(f) = \sup_n S_n(f).$$

Observe that for $\ell < n$ and $f, g \in L_1$,

(4.2)
$$\mathbb{E}[\Delta_{\ell} f \cdot \Delta_n g] = \mathbb{E}[P_{\ell}(\Delta_{\ell} f) \cdot \Delta_n g] = \mathbb{E}[\Delta_{\ell} f \cdot P_{\ell}(\Delta_n g)] = 0.$$

Let V be the L_1 -closure of $\cup_n \mathscr{S}_k(\mathscr{F}_n)$. Then, the uniform boundedness of P_n on L_1 implies that $P_n f \to f$ in L_1 for $f \in V$. Next, set

$$H_{1,k} = H_{1,k}((\mathscr{F}_n)) = \{ f \in V : \mathbb{E}(S(f)) < \infty \}$$

and equip $H_{1,k}$ with the norm $||f||_{H_{1,k}} = \mathbb{E}S(f)$. Define

$$BMO_k = BMO_k((\mathscr{F}_n)) = \{ f \in V : \sup_n \| \sum_{\ell > n} T_n((\Delta_{\ell} f)^2) \|_{\infty} < \infty \}$$

and the corresponding quasinorm

$$||f||_{\text{BMO}_k} = \sup_{n} ||\sum_{\ell > n} T_n((\Delta_{\ell} f)^2)||_{\infty}^{1/2},$$

where T_n is the operator from (3.3) that dominates P_n pointwise.

Let us now assume $\sup_n \gamma_k(\mathscr{F}_n) < \infty$. In this case we identify, similarly to H_1 -BMO-duality (cf. [4, 7, 5]), BMO_k as the dual space of $H_{1,k}$.

First, we use the duality estimate Theorem 3.6 and (4.2) to prove, for $f \in H_{1,k}$ and $h \in BMO_k$,

$$\left| \mathbb{E} \left[(P_n f) \cdot (P_n h) \right] \right| \leq \sum_{\ell \leq n} \mathbb{E} \left[|\Delta_{\ell} f| \cdot |\Delta_{\ell} h| \right] \lesssim \|S_n(f)\|_1 \cdot \|h\|_{\text{BMO}_k}.$$

This estimate also implies that the limit $\lim_n \mathbb{E}[(P_n f) \cdot (P_n h)]$ exists and satisfies

$$\left|\lim_{n} \mathbb{E}\left[(P_n f) \cdot (P_n h) \right] \right| \lesssim \|f\|_{H_{1,k}} \cdot \|h\|_{\mathrm{BMO}_k}.$$

On the other hand, similarly to the martingale case (see [5]), given a continuous linear functional L on $H_{1,k}$, we extend L norm-preservingly to a continuous linear functional Λ on $L_1(\ell_2)$, which, by Section 2.4, has the form

$$\Lambda(\eta) = \mathbb{E}\left[\sum_{\ell} \sigma_{\ell} \eta_{\ell}\right], \qquad \eta \in L_1(\ell_2)$$

for some $\sigma \in L_{\infty}(\ell_2)$. The k-martingale spline sequence $h_n = \sum_{\ell \leq n} \Delta_{\ell} \sigma_{\ell}$ is bounded in L_2 and therefore, by the spline convergence theorem ((v) on page 2), has a limit $h \in L_2$ with $P_n h = h_n$ and which is also contained in BMO_k. Indeed, by using Corollary 3.3, we obtain $||h||_{\text{BMO}_k} \lesssim ||\sigma||_{L_{\infty}(\ell_2)} = ||\Lambda|| = ||L||$ with a constant depending only on k and $\sup_n \gamma_k(\mathscr{F}_n)$. Moreover, for $f \in H_{1,k}$, since L is continuous on $H_{1,k}$,

$$L(f) = \lim_{n} L(P_{n}f) = \lim_{n} \Lambda \left((\Delta_{1}f, \dots, \Delta_{n}f, 0, 0, \dots) \right)$$
$$= \lim_{n} \sum_{\ell=1}^{n} \mathbb{E}[\sigma_{\ell} \cdot \Delta_{\ell}f] = \lim_{n} \mathbb{E}[(P_{n}f) \cdot (P_{n}h)].$$

This yields the following

Theorem 4.2. If $\sup_{n} \gamma_k(\mathscr{F}_n) < \infty$, the linear mapping

$$u: \mathrm{BMO}_k \to H_{1,k}^*, \qquad h \mapsto \left(f \mapsto \lim_n \mathbb{E} \left[(P_n f) \cdot (P_n h) \right] \right)$$

is bijective and satisfies

$$||u(h)||_{H_{1}^*} \simeq ||h||_{\mathrm{BMO}_k},$$

where the implied constants depend only on k and $\sup_{n} \gamma_{k}(\mathscr{F}_{n})$.

Remark 4.3. We close with a few remarks concerning the above result and we assume that (\mathscr{F}_n) is a sequence of increasing interval σ -algebras with $\sup_n \gamma_k(\mathscr{F}_n) < \infty$.

(1) By Khintchine's inequality, $||Sf||_1 \lesssim \sup_{\varepsilon \in \{-1,1\}^{\mathbb{Z}}} ||\sum_n \varepsilon_n \Delta_n f||_1$. Based on the interval filtration (\mathscr{F}_n) , we can generate an interval filtration (\mathscr{G}_n) that contains (\mathscr{F}_n) as a subsequence and each \mathscr{G}_{n+1} is generated from \mathscr{G}_n by dividing exactly one atom of \mathscr{G}_n into two atoms of \mathscr{G}_{n+1} . Denoting by $P_n^{\mathscr{G}}$ the orthogonal projection operator onto $\mathscr{S}_k(\mathscr{G}_n)$ and $\Delta_j^{\mathscr{G}} = P_j^{\mathscr{G}} - P_{j-1}^{\mathscr{G}}$, we can write

$$\sum_{n} \varepsilon_{n} \Delta_{n} f = \sum_{n} \varepsilon_{n} \sum_{i=a_{n}}^{a_{n+1}-1} \Delta_{j}^{\mathscr{G}} f$$

for some sequence (a_n) . By using inequalities (2.7) and (2.6) and writing $(S^{\mathscr{G}}f)^2 = \sum_j |\Delta_j^{\mathscr{G}}f|^2$, we obtain for p > 1

$$||Sf||_1 \lesssim ||S^{\mathscr{G}}f||_1 \leq ||S^{\mathscr{G}}f||_p \lesssim ||f||_p.$$

This implies $L_p \subset H_{1,k}$ for all p > 1 and, by duality, $BMO_k \subset L_p$ for all $p < \infty$.

(2) If (\mathscr{F}_n) is of the form that each \mathscr{F}_{n+1} is generated from \mathscr{F}_n by splitting exactly one atom of \mathscr{F}_n into two atoms of \mathscr{F}_{n+1} and under the condition $\sup_n \gamma_{k-1}(\mathscr{F}_n) < \infty$ (which is stronger than $\sup_n \gamma_k(\mathscr{F}_n) < \infty$), it is shown in [6] that

$$||Sf||_1 \simeq ||f||_{H_1},$$

where H_1 denotes the atomic Hardy space on [0, 1], i.e. in this case, $H_{1,k}$ coincides with H_1 .

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