POINTWISE MULTIPLIERS OF MUSIELAK–ORLICZ SPACES AND FACTORIZATION

KAROL LEŚNIK AND JAKUB TOMASZEWSKI

Abstract. We prove that the space of pointwise multipliers between two distinct Musielak–Orlicz spaces is another Musielak–Orlicz space and the function defining it is given by an appropriately generalized Legendre transform. In particular, we obtain characterization of pointwise multipliers between Nakano spaces. We also discuss factorization problem for Musielak–Orlicz spaces and exhibit some differences between Orlicz and Musielak–Orlicz cases.

1. Introduction

Given two function spaces X and Y (over the same domain), the space of pointwise multipliers $M(X,Y)$ is the space of all functions f such that $fg \in Y$ for each $g \in X$. $M(X,Y)$ may be regarded as a generalized Köthe dual space (cf. [15, 3]) and a basic question is to identify $M(X,Y)$ for a given spaces X and Y. Many authors have investigated this problem for Orlicz spaces and many characterizations (mainly partial) have been given – see for example Shragin [21], Ando [1], O'Neil [20], Zabreiko–Rutickii [22], Maurey [16], Maligranda–Persson [15] and Maligranda–Nakai [14]. In 2000 Djakov and Ramanujan settled the problem for Orlicz sequence spaces and, recently, in [12] the authors established an analogous characterization for Orlicz function spaces. In both cases, the space of pointwise multipliers $M(L^{\varphi_1}, L^{\varphi})$ between Orlicz spaces is proved to be just another Orlicz space, i.e.

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi\ominus\varphi_1}, \tag{1.1}$$

where the function $\varphi \ominus \varphi_1$ is generalized Young conjugate (generalized Legendre transform) of φ_1 with respect to φ. Observe that the above characterization generalizes, in the evident way, the classical Kőthe duality formula for Orlicz spaces, this is

$$(L^{\varphi_1})' = M(L^{\varphi_1}, L^1) = L^{\varphi_1^*}, \tag{1.2}$$

where φ_1^* is the Young conjugate of φ_1 (i.e. $\varphi_1^* = id \ominus \varphi_1$). Let us also mention here, that the identification as in (1.1) seems to be the most desirable, since the function $\varphi \ominus \varphi_1$ is given in an explicit and constructive way, in contrast to theorems from [14] and [10], which have rather existential character (cf. [21, 1, 20, 22, 16]).

In the paper we focus on the multipliers of Musielak–Orlicz spaces. Such investigations have been already initiated by Nakai [18] (cf. [19]). Under a number of assumptions on functions φ, φ_1 he generalized results of [14] to the Musielak–Orlicz setting. Since this method is not constructive (see discussion in [12]), we are not going to employ it. Instead of that we will use ideas of [5] and [12] to prove that the representation (1.1) holds also in the Musielak–Orlicz case, for an arbitrary σ - finite measure space and without any additional assumptions on Musielak–Orlicz functions φ, φ_1.

The paper is organized as follows. In Section 2 we give necessary definitions on Banach function space and Musielak–Orlicz spaces. We also define the function $\varphi\ominus\varphi_1$ (Young conjugate of φ_1 with respect to φ) for Musieak–Orlicz functions φ, φ_1.

The next section contains a number of technical lemmas concerning Musielak–Orlicz spaces and multipliers. Consequently, we are ready to prove the representation theorem in the third section. Finally, the last section is devoted to discussion on factorization and differences between

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Orlicz and Musielak–Orlicz cases. In particular, we give an example showing that inequality $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \succ \varphi^{-1}$ is not necessary condition for factorization of Musielak–Orlicz spaces, unlike in the Orlicz spaces case (cf. [12, Theorem 2]).

2. Notation and preliminaries

Trough the paper we will assume that (Ω, Σ, μ) is a σ-finite, complete measure space. For a given set $A \in \Sigma$ we will denote the non-atomic part and purely atomic part of A by A^c and A^a, respectively. When ω is an atom we will write $\omega \in \Omega^a$ and use the convention that ω will always denote atoms.

Let $L^0 = L^0(\Omega, \Sigma, \mu)$ be the space of classes of equivalence of μ-measurable, real valuable and finite μ-a.e. functions. A Banach space $X \subset L^0$ is called the *Banach ideal space* if it satisfies the ideal property, i.e. $x \in L^0, y \in X$ and $|x| \leq |y|$ implies $x \in X$ and $\|x\|_X \leq \|y\|_X$ ($|x| \leq |y|$ means that $|x(t)| \leq |y(t)|$ for μ-a.e. $t \in \Omega$).

For $x \in L^0$ we define its support as $\operatorname{supp}(x) := \{t \in \Omega : x(t) \neq 0\}$. A support $\operatorname{supp} X$ of a Banach ideal space X is defined as a measurable subset of Ω such that:
i) for each $x \in X$ there is $A \in \Sigma$ with $\mu(A) = 0$ such that $\operatorname{supp}(x) \subset \operatorname{supp} X \cup A$,
ii) there is $x \in X$ such that $\mu(\operatorname{supp} X - \operatorname{supp}(x)) = 0$.
Notice that according to the above definition $\operatorname{supp} X$ is not unique, thus we rather write a support, than the support of X.

For any measurable $F \subset \Omega$ and a Banach ideal space X we define

$$X[F] := \{x \in X : \mu(\operatorname{supp}(x) \setminus F) = 0\} \text{ with the norm } \|x\|_{X[F]} = \|x\chi_F\|_X.$$

Given a Banach ideal space X on Ω and a positive measurable weight function v, the weighted space $X(v)$ is defined as

$$X(v) := \{x \in L^0 : xv \in X\} \text{ with the norm } \|x\|_{X(v)} = \|xv\|_X.$$

Writing $X = Y$ for two Banach lattices X, Y we mean that they are equal as set, but norms are just equivalent. Recall also that for Banach ideal spaces X, Y the inclusion $X \subset Y$ is always continuous, i.e. there is $c > 0$ such that $\|x\|_Y \leq c\|x\|_X$ for each $x \in X$.

A Banach ideal space X satisfies the *Fatou property* ($X \in (FP)$ for short) if for each sequence $(x_n) \subset X$ satisfying $x_n \uparrow x$ μ-a.e. and $\sup_n \|x_n\|_X < \infty$, there holds $x \in X$ and $\|x\|_X \leq \sup_n \|x_n\|_X$.

Given two Banach ideal spaces X, Y over the same measure space (Ω, Σ, μ) we define their *pointwise product space*

$$X \odot Y = \{x \cdot y \in L^0 : x \in X, y \in Y\},$$

with a quasi–norm

$$\|z\|_{X \odot Y} = \inf\{\|x\|_X \|y\|_Y : z = xy\}.$$

If additionally $\operatorname{supp} X = \Omega$, then *the space of pointwise multipliers* from X to Y is defined as

$$M(X, Y) = \{y \in L^0 : xy \in Y \text{ for all } y \in X\},$$

with the natural operator norm

$$\|y\|_M = \sup_{\|x\|_X \leq 1} \|xy\|_Y.$$

When there is no risk of confusion we will just write $\|\cdot\|_M$ for the norm of $M(X, Y)$. If Banach lattices X and Y have the Fatou property then both spaces $M(X, Y)$ and $X \odot Y$ have the Fatou property [15, 10, 11].

We will need the following easy observation concerning the space of pointwise multipliers. Let $A, B \subset \Omega$ be measurable sets such that $A \cup B = \Omega$. Given a Banach ideal space X over Ω, we can decompose it as

$$X = X[A] \oplus X[B],$$

with the (equivalent) norm given by $\|x\|_{X[A]\oplus X[B]} = \|x\chi_A\|_{X[A]} + \|x\chi_B\|_{X[B]}$. It is easy to see that the space of pointwise multipliers respects such a "decomposition", i.e. $M(X,Y)$ may be written as follows

$$M(X,Y) = M(X[A] \oplus X[B], Y[A] \oplus Y[B]) = M(X[A], Y[A]) \oplus M(X[B], Y[B]). \tag{2.1}$$

In another words, determining the space of pointwise multipliers between two Banach ideal spaces, we may determine it on A and on B separately.

A function $\varphi : [0,\infty) \to [0,\infty]$ will be called the *Young function* if it satisfies $\varphi(0) = 0$, $\lim_{u\to\infty} \varphi(u) = \infty$ and is convex on $[0, b_\varphi)$ (or on $[0, b_\varphi]$ when $\varphi(b_\varphi) < \infty$), where

$$b_\varphi = \sup\{u \geq 0 : \varphi(u) < \infty\}.$$

We point out here that we allow $\varphi(u) = \infty$ for each $u > 0$. In such a case the corresponding Orlicz space L^φ contains only the zero function.

A function $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ is called the *Musielak–Orlicz function* if the following conditions hold:

(i) $\varphi(t,\cdot)$ is a Young function for a.e. $t \in \Omega$,
(ii) $\varphi(\cdot,u)$ is a measurable function for each $u \in [0,\infty)$.

Let φ be a Musielak–Orlicz function. We define the convex modular I_φ as

$$I_\varphi(x) = \int_\Omega \varphi(t, |x(t)|) d\mu(t).$$

The *Musielak–Orlicz space* L^φ is defined as

$$L^\varphi = \{x \in L^0 : I_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

and is equipped with the Luxemburg-Nakano norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(\frac{x}{\lambda}) \leq 1\}.$$

It is known that Musielak–Orlicz spaces have the Fatou property. Moreover, it follows immediately from the definition, that $\operatorname{supp} L^\varphi = \{t \in \Omega : b_\varphi(t) > 0\}$ (up to a set of measure zero).

For a given Musielak–Orlicz function φ we define two useful (functions) parameters

$$a_\varphi(t) = \sup\{u \geq 0 : \varphi(t,u) = 0\},$$

$$b_\varphi(t) = \inf\{u \geq 0 : \varphi(t,u) = \infty\}.$$

It is known, that both a_φ and b_φ are measurable [4, Proposition 5.1].

The following basic relation between the norm and the modular will be used frequently through the paper

$$\|x\|_\varphi \leq 1 \Rightarrow I_\varphi(x) \leq \|x\|_\varphi, \tag{2.2}$$

for $x \in L^\varphi$ (see [13, Theorem 1.1]). More information on Musielak–Orlicz and Orlicz spaces can be found for example in [17, 6, 7, 8].

3. Auxiliary results

Recall that our goal is to describe the space of pointwise multipliers $M(L^{\varphi_1}, L^\varphi)$ between two Musielak–Orlicz spaces and thus we will operate on two Musielak–Orlicz functions φ, φ_1, both defined over the same measure space Ω. The result will be given in terms of the third Musielak–Orlicz function $\varphi \ominus \varphi_1$ - the Young conjugate of φ_1 with respect to φ. In order to define it we need to introduce the following decomposition of the continuous part of the domain

Ω depending on behaviour of both φ, φ_1. Let φ, φ_1 be two Musielak–Orlicz functions. We define the following sets:

$$\begin{aligned}
\Omega_{0,0} &:= \{t \in \Omega^c : b_{\varphi_1}(t) = b_\varphi(t) = \infty\},\\
\Omega_{0,\infty} &:= \{t \in \Omega^c : b_{\varphi_1}(t) = \infty,\ b_\varphi(t) < \infty\},\\
\Omega_{\infty,0} &:= \{t \in \Omega^c : 0 < b_{\varphi_1}(t) < \infty,\ b_\varphi(t) = \infty\},\\
\Omega_{\infty,\infty} &:= \{t \in \Omega^c : 0 < b_{\varphi_1}(t) < \infty,\ b_\varphi(t) < \infty\},\\
\Omega_\infty &:= \Omega_{\infty,\infty} \cup \Omega_{\infty,0}.
\end{aligned}$$

Given two Musielak–Orlicz functions φ, φ_1, the *Young conjugate of φ_1 with respect to φ* is defined as

$$\varphi \ominus \varphi_1(t,u) := \begin{cases} \sup\{\varphi(t,su) - \varphi_1(t,s) : 0 \le s < b_{\varphi_1}(t)\} & \text{if } t \in \Omega^c,\\ \sup\{\varphi(t,su) - \varphi_1(t,s) : 0 \le s \le \min\{1/\varphi^{-1}(\frac{1}{\mu(t)}), \frac{b_{\varphi_1}(t)}{2}\}\} & \text{if } t \in \Omega^a. \end{cases}$$

Observe firstly that such defined function $\varphi \ominus \varphi_1$ satisfies $b_{\varphi\ominus\varphi_1}(\omega) > 0$ for each $\omega \in \Omega^a$. Moreover, it is easy to see, that for $t \in \Omega^c_{0,\infty}$

$$\varphi \ominus \varphi_1(t,u) = \begin{cases} 0 & \text{if } u = 0,\\ \infty & \text{if } u > 0. \end{cases}$$

In consequence,

$$\operatorname{supp}(L^{\varphi\ominus\varphi_1}) = (\Omega^c_{0,0} \cup \Omega^c_\infty \cup \Omega^a) \cap \operatorname{supp}(L^\varphi) \tag{3.1}$$

It may be instructive to realize what is $\varphi \ominus \varphi_1$, when φ, φ_1 are Nakano functions.

Example 1. Let $p, q : \Omega \to [1,\infty)$ be two measurable functions and define $\varphi(t,u) = \frac{1}{q(t)} u^{q(t)}$, $\varphi_1(t,u) = \frac{1}{p(t)} u^{p(t)}$ for $t \in \Omega, u \ge 0$. Assume that $q(t) \le p(t)$ for a.e. $t \in \Omega$. One can easily calculate that

$$\varphi \ominus \varphi_1(t,u) = \frac{1}{r(t)} u^{r(t)}$$

where $\frac{1}{p(t)} + \frac{1}{r(t)} = \frac{1}{q(t)}$ for a.e. $t \in \Omega$.

In the proof of the main theorem, we are going to imitate inductive argument used in [5] and in [12]. In order to do it we need a kind of decomposition of the measure space Ω. The following two lemmas provide it.

Lemma 2. Let Ω be a non-atomic measure space. Furthermore, let φ be a Musielak–Orlicz function such that $b_\varphi(t) = \infty$ for a.e. $t \in \Omega$. For each $a > 0$ there exists a sequence of pairwise disjoint measurable sets (A_n) such that $\bigcup_{n\in\mathbb{N}} A_n = \Omega$ and

$$\|\chi_{A_n}\|_\varphi \le \frac{1}{a}$$

for every $n \in \mathbb{N}$.

Proof. Fix $a > 0$. Define the sets

$$B_n := \{t \in \Omega : n - 1 \le \varphi(t,a) < n\}$$

for $n \in \mathbb{N}$. Evidently, each B_n is measurable, since the function $\varphi(\cdot, a)$ is measurable. Moreover $\bigcup_{n\in\mathbb{N}} B_n = \Omega$ and (B_n) is a sequence of pairwise disjoint sets. Since we operate on a non-atomic measure space, each B_n may be divided further into a sequence (finite or not) of pairwise disjoint sets $(C_j^n)_{j\in I_n}$ such that $\bigcup_{j\in I_n} C_j^n = B_n$ and $\mu(C_j^n) \le \frac{1}{n}$ for each $j \in I_n$. In consequence, we have for $n \in \mathbb{N}$ and $j \in I_n$

$$\begin{aligned} I_\varphi\left(a\chi_{C_j^n}\right) &= \int_{C_j^n} \varphi(t,a)d\mu(t) \\ &\le \mu(C_j^n) \sup_{t\in C_j^n} \varphi(t,a) \le 1. \end{aligned}$$

It follows that

$$\left\|\chi_{C_j^n}\right\|_\varphi \le \frac{1}{a},$$

for every $n \in \mathbb{N}$ and $j \in I_n$. Finally, we get the desired sequence (A_n) just by rearranging the (doubly indexed) sequence (C_j^n). □

Lemma 3. Let Ω be a non-atomic measure space and φ be a Musielak–Orlicz function such that $0 < b_\varphi(t) < \infty$ for a.e. $t \in \Omega$. There exists a sequence of pairwise disjoint measurable sets (A_n) such that $\bigcup_{n\in\mathbb{N}} A_n = \Omega$ and for each $n \in \mathbb{N}$

$$\|\chi_{A_n}\|_\varphi \le \frac{2}{\operatorname*{ess\,sup}_{t\in A_n}\{b_\varphi(t)\}}.$$

Proof. For each $k \in \mathbb{Z}$ define

$$B_k := \{t \in \Omega : 2^{k-1} < b_\varphi(t) \le 2^k\}.$$

Evidently, sets B_k are measurable, since b_φ is a measurable function. Next, for each $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ we define

$$B_{k,n} := \{t \in B_k : n-1 \le \varphi(t, 2^{k-1}) < n\}.$$

Then the doubly indexed sequence $(B_{k,n})$ consists of pairwise disjoint measurable sets such that $\bigcup_{n\in\mathbb{N},k\in\mathbb{Z}} B_{k,n} = \Omega$. Denote

$$I := \{(k,n) \in \mathbb{Z}^2 : B_{k,n} \ne \emptyset\}.$$

For each $(k,n) \in I$ we can further decompose $B_{k,n}$ into a (finite or not) sequence $(C_j^{k,n})_{j\in I_{k,n}}$ of pairwise disjoint measurable sets in such a way that $\bigcup_{j\in I_{k,n}} C_j^{k,n} = B_{k,n}$ and $\mu(C_j^{k,n}) \le \frac{1}{n}$ for each $j \in I_{k,n}$. Finally, for every $(k,n) \in I$ and $j \in I_{k,n}$ we have

$$\begin{aligned} I_\varphi\left(2^{k-1}\chi_{C_j^{k,n}}\right) &= \int_{C_j^{k,n}} \varphi(t, 2^{k-1})d\mu(t) \\ &\le \mu(C_j^{k,n}) \sup_{t\in C_j^{k,n}} \varphi(t, 2^{k-1}) \le 1. \end{aligned}$$

In consequence,

$$\left\|\chi_{C_j^{k,n}}\right\|_\varphi \le \frac{1}{2^{k-1}} \le \frac{2}{\operatorname*{ess\,sup}_{t\in B_{k,n}}\{b_\varphi(t)\}} \le \frac{2}{\operatorname*{ess\,sup}_{t\in C_j^{k,n}}\{b_\varphi(t)\}}.$$

Similarly as before, the desired sequence is obtained after rearranging the (triple indexed) sequence $(C_j^{k,n})$. □

Fact 4. If a Musielak–Orlicz function φ is such that $b_\varphi(t) < \infty$ for a.e. $t \in \Omega$, then

$$L^\varphi \subset L^\infty(1/b_\varphi).$$

Proof. Let $0 \leq y \notin L^\infty(1/b_\varphi)$. For each $n \in \mathbb{N}$ we define sets

$$A_n = \{t \in \Omega : n \leq \frac{y(t)}{b_\varphi(t)}\}.$$

Then there is $N \in \mathbb{N}$ such that $\mu(A_n) > 0$ for $n \geq N$. Fix $a > 0$ and choose $n \geq N$ satisfying $an > 2$. We can see that

$$nb_\varphi \chi_{A_n} \leq y.$$

In consequence,

$$I_\varphi(ay) \geq I_\varphi(anb_\varphi \chi_{A_n}) \geq I_\varphi(2b_\varphi \chi_{A_n}) = \infty.$$

Since $a > 0$ was arbitrary we conclude that $y \notin L^\varphi$. $\square$

Lemma 5. Let Ω be a non-atomic measure space and let φ, φ_1 be two Musielak–Orlicz functions such that $0 < b_{\varphi_1}(t) < \infty$ and $0 < b_\varphi(t) < \infty$ for a.e. $t \in \Omega$. Then

$$M(L^{\varphi_1}, L^\varphi) \subset L^\infty(b_{\varphi_1}/b_\varphi).$$

Proof. Let $0 \leq y \notin L^\infty(v)$, where $v(t) := \frac{b_{\varphi_1}(t)}{b_\varphi(t)}$. For each $n \in \mathbb{N}$ we define

$$A_n = \{t \in \Omega : n \leq y(t)v(t) < n+1\}.$$

Then there exist infinitely many $n \in \mathbb{N}$ for which $\mu(A_n) > 0$. Denote the set of such n's by I. Next, since Ω is non-atomic, for each $n \in I$ there is $B_n \subset A_n$ such that $\mu(B_n) > 0$ and

$$\int_{B_n} \varphi_1(t, \frac{b_{\varphi_1}(t)}{2}) d\mu(t) \leq \frac{1}{2^n}.$$

We define

$$f(t) := \sum_{n=1}^{\infty} \frac{b_{\varphi_1}(t)}{2} \chi_{B_n}.$$

Then

$$I_{\varphi_1}(f) = \int \varphi_1(t, f(t)) d\mu(t) = \sum_{n=1}^{\infty} \int_{B_n} \varphi_1(t, \frac{b_{\varphi_1}(t)}{2}) d\mu(t) \leq 1.$$

It means that $f \in L^{\varphi_1}$ and $\|f\|_{\varphi_1} \leq 1$. However,

$$y(t)f(t) \geq \frac{1}{2} y(t) b_{\varphi_1}(t) \geq \frac{n}{2} b_\varphi(t) \quad \text{ for a.e. } t \in B_n,$$

which implies that $yf \notin L^\varphi$, since $L^\varphi \subset L^\infty(\frac{1}{b_\varphi})$ by Fact 4. Consequently, $y \notin M(L^{\varphi_1}, L^\varphi)$ and the proof is finished. $\square$

Lemma 6. Suppose Ω is a non-atomic measure space and let φ, φ_1 be Musielak–Orlicz functions such that $\operatorname{supp} L^{\varphi_1} = \Omega$. Then

$$\operatorname{supp} M(L^{\varphi_1}, L^\varphi) \subset \Omega_{0,0} \cup \Omega_\infty.$$

Proof. We need only to show that $\mu(\Omega_{0,\infty} \cap \operatorname{supp} M(L^{\varphi_1}, L^\varphi)) = 0$. Suppose, for a contrary, there exists $A \subset \Omega_{0,\infty}$ such that $\mu(A) > 0$ and $\chi_A \in M(L^{\varphi_1}, L^\varphi)$. Let $C \subset A$ be chosen in such a way that $\mu(C) > 0$ and $\inf_{t \in C} b_\varphi(t) = \delta > 0$. From Lemma 2 it follows that for each $n \in \mathbb{N}$ there exists $A_n \subset C$ such that $\mu(A_n) > 0$ and

$$\|\chi_{A_n}\|_{\varphi_1} \leq \frac{1}{n}.$$

Moreover, by Fact 4, we know that $L^\varphi[\Omega_{0,\infty}] \subset L^\infty(\frac{1}{b_\varphi})[\Omega_{0,\infty}]$ with some inclusion constant $c > 0$. It means

$$\begin{aligned}\|\chi_{A_n}\|_\varphi &\geq c^{-1}\|\chi_{A_n}\|_{L^\infty(\frac{1}{b_\varphi})} \\ &\geq c^{-1}\sup_{t\in A_n}\frac{1}{b_\varphi(t)} \\ &= \frac{1}{c\inf\limits_{t\in A_n} b_\varphi(t)} \\ &\geq \frac{1}{c\inf\limits_{t\in C} b_\varphi(t)} = \frac{1}{c\delta}.\end{aligned}$$

Finally, for each $n \in \mathbb{N}$ define $x_n := n\chi_{A_n}$. Then $x_n \in B(L^{\varphi_1})$ and it follows

$$\|\chi_A\|_M \geq \|x_n\chi_A\|_\varphi = \|n\chi_{A_n}\|_\varphi \geq \frac{n}{c\delta},$$

for each $n \in \mathbb{N}$. In consequence, $\chi_A \notin M(L^{\varphi_1}, L^\varphi)$ which contradicts our assumption. □

Of course, the supremum in definition of function $\varphi \ominus \varphi_1$ need not be attained. To avoid such a situation, we introduce a truncated version of $\varphi \ominus \varphi_1$ (cf. [12, Definition 1]). Namely, for $a > 0$ we define the function $\varphi \ominus_a \varphi_1$ in the following way

$$\varphi\ominus_a\varphi_1(t,u) := \begin{cases}\sup\{\varphi(t,su) - \varphi_1(t,s) : 0 \leq s \leq a\} & \text{if } t \in \Omega_{0,0}\cup\Omega_{0,\infty}\\ \sup\{\varphi(t,su) - \varphi_1(t,s) : 0 \leq s \leq \frac{a}{a+1}b_{\varphi_1}(t)\} & \text{if } t \in \Omega_\infty \\ \sup\{\varphi(t,su) - \varphi_1(t,s) : 0 \leq s \leq \min\{1/\varphi^{-1}(\frac{1}{\mu(t)}), \frac{1}{b_{\varphi_1}(t)}2\}\} & \text{if } t \in \Omega^a\end{cases}$$

It is easy to see that

$$b_{\varphi\ominus_a\varphi_1}(t) = \frac{(a+1)b_\varphi(t)}{ab_{\varphi_1}(t)} \tag{3.2}$$

for $t \in \Omega_\infty$.

Lemma 7. Let Ω be a non-atomic measure space and φ, φ_1 be Musielak–Orlicz functions such that $\operatorname{supp} L^{\varphi_1} = \Omega$. If $A \subset \operatorname{supp} L^\varphi \setminus \Omega_{\infty,0}$ is a set of positive measure and numbers $a > 1, u > 0$ satisfy $\varphi \ominus_a \varphi_1(t, \frac{3}{2}u) < \infty$ for a.e. $t \in A$, then the function $x : A \to \mathbb{R}_+$, defined by

$$x(t) := \max\{0 \leq v \leq \min\{a, \frac{a}{a+1}b_{\varphi_1}(t)\} : \varphi_1(t,v) + \varphi\ominus_a\varphi_1(t,u) = \varphi(t,uv)\},$$

is measurable.

Proof. Without loss of generality we may assume that $\varphi_1(t,\cdot), \varphi(t,\cdot)$ are Orlicz functions for each $t \in A$. Fix $u > 0$ and $a > 1$ satisfying

$$\varphi\ominus_a\varphi_1(t, \frac{3}{2}u) < \infty \text{ for a.e. } t \in A \tag{3.3}$$

and let x be like in the statement. Let (r_k) be a dense sequence in $[0, a]$. For each $k, n \in \mathbb{N}$ define

$$B_k^n := \{t \in A : r_k \leq \frac{a}{a+1}b_{\varphi_1}(t),\ \varphi_1(t,r_k) + \varphi\ominus_a\varphi_1(t,u) - \varphi(t,ur_k) < 1/n\}$$

and

$$q_k^n := r_k\chi_{B_k^n}.$$

Just notice that by the definition of $\varphi \ominus_a \varphi_1$

$$0 \leq \varphi_1(t,v) + \varphi\ominus_a\varphi_1(t,w) - \varphi(t,wv)$$

for a.e. $t \in \Omega$ and $w, v \geq 0$. Therefore,

$$\varphi(t, ur_k) < \infty,$$

because for every $k \in \mathbb{N}$ we have $\varphi_1(t, r_k) < \infty$ and $\varphi \ominus_a \varphi_1(t, u) < \infty$. Of course, functions q_k^n are measurable, since sets B_k^n are measurable. We will show that

$$x = \limsup_{k,n\to\infty} q_k^n. \tag{3.4}$$

Firstly we will explain the inequality $\limsup_{k,n\to\infty} q_k^n \leq x$. Suppose, for a contradiction, that for some $t_0 \in A$ and some $\delta > 0$ there holds

$$\limsup_{k,n\to\infty} q_k^n(t_0) > x(t_0) + \delta.$$

This implies that there is a (singly-indexed) sequence $(q_{k_i}^{n_i})$ such that $\min\{a, \frac{a}{a+1} b_{\varphi_1}(t_0)\} \geq q_{k_i}^{n_i}(t_0) > x(t_0) + \delta$, $n_i, k_i \to \infty$ and

$$\varphi_1(t_0, u_{q_{k_i}^{n_i}}(t_0)) + \varphi \ominus_a \varphi_1(t_0, u) - \varphi(t_0, u_{q_{k_i}^{n_i}}(t_0)) < 1/n_i \tag{3.5}$$

for each $i = 1, 2, 3, \ldots$. On the other hand, there is a subsequence $(q_j) := (q_{k_{i_j}}^{n_{i_j}})$ of $(q_{k_i}^{n_i})$ and $q_0 > x(t_0)$ such that $q_j(t_0) \to q_0$. However, by (3.5) and continuity of respective functions, we get

$$\varphi_1(t_0, q_0) + \varphi \ominus_a \varphi_1(t_0, u) - \varphi(t_0, uq_0) = 0,$$

which contradicts maximality of $x(t_0)$ and proves inequality $\limsup_{k,n\to\infty} q_k^n \leq x$.

To see the opposite inequality fix $t \in A$ and denote

$$C_n := \{0 \leq v \leq \min\{a, \frac{a}{a+1} b_{\varphi_1}(t)\} : \varphi_1(t, v) + \varphi \ominus_a \varphi_1(t, u) - \varphi(t, uv) < 1/n\}.$$

We see that sets C_n are open and non-empty, since $x(t) \in C_n$ for each n. Therefore, one can select a sequence (r_{n_i}) such that $r_{n_i} \in C_i$ and $r_{n_i} \to x(t)$. Then $t \in B_{n_i}^i$ for each $i = 1, 2, 3, \ldots$ and, consequently,

$$x(t) \leq \limsup_{k,n\to\infty} q_k^n(t),$$

which finally proves measurability of x. □

4. Pointwise multipliers

Theorem 8. Let φ, φ_1 be Musielak–Orlicz functions over a measure space (Ω, Σ, μ) and assume that $\operatorname{supp} L^{\varphi_1} = \Omega$. Then

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}.$$

Proof. Without loss of generality we can assume that $\operatorname{supp}(L^{\varphi}) = \Omega$, since

$$M(L^{\varphi_1}, L^{\varphi})[\Omega \setminus \operatorname{supp}(L^{\varphi})] = \{0\} = L^{\varphi \ominus \varphi_1}[\Omega \setminus \operatorname{supp}(L^{\varphi})],$$

where the second equality follows from (3.1). The proof of inclusion

$$L^{\varphi \ominus \varphi_1} \subset M(L^{\varphi_1}, L^{\varphi})$$

is the same as in the case of Orlicz spaces and we omit it (see for example [12, Lemma 6]).

We only need to prove the remaining inclusion

$$M(L^{\varphi_1}, L^{\varphi}) \subset L^{\varphi \ominus \varphi_1}$$

Let $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ be a simple function such that $\|y\|_M \leq \frac{1}{4c}$, where $c \geq 1$ is the constant of inclusion

$$M(L^{\varphi_1}, L^{\varphi})[\Omega_{\infty,\infty}] \subset L^{\infty}(b_{\varphi_1}/b_{\varphi})[\Omega_{\infty,\infty}]$$

(cf. Lemma 5). We will show that

$$I_{\varphi\ominus_a\varphi_1}(y) \le 1 \tag{4.1}$$

for every $a > 1$. To prove this inequality, for each $a > 1$ we will construct a function $x(t)$ on Ω and a family of pairwise disjoint sets (A_n) satisfying:

(i) $\varphi \ominus_a \varphi_1(t, y(t)) = \varphi(t, x(t)y(t)) - \varphi_1(t, x(t))$ for a.e $t \in \Omega$,

(ii) $\|xy\chi_{A_n}\|_\varphi \le \frac{1}{2}$ for each $n \in \mathbb{N}$,

(iii) $\operatorname{supp}(M(L^{\varphi_1}, L^\varphi)) \subset \bigcup\limits_{n\in\mathbb{N}} A_n$,

(iv) $x \in L^{\varphi_1}$ and $\|x\|_{\varphi_1} \le 1$.

Let $a > 1$. Since y is a simple function we can write it in the form

$$y = \sum_{k=0}^{n} b_k \chi_{B_k} + \sum_{k=0}^{m} d_k \chi_{\omega_k},$$

where for every k we have $b_k, d_k > 0$, $B_k \subset \Omega_\infty \cup \Omega_{0,0}$, $\mu(B_k) < \infty$ and ω_k's are atoms. In order to construct the desired function x, we will apply Lemma 7 for each b_k and B_k. First of all we need to show that assumptions of Lemma 7 are fulfilled, i.e. for each $0 \le k \le n$ we have $\varphi \ominus_a \varphi_1(t, \frac{3}{2}b_k) < \infty$ for a.e. $t \in B_k$. Let $0 \le k \le n$. Then for a.e. $t \in B_k$ we have

$$b_k = y(t) \le \frac{b_{\varphi\ominus\varphi_1}(t)}{4} \le \frac{b_{\varphi\ominus_a\varphi_1}(t)}{2},$$

since, by Lemma 5,

$$\left\|y b^{-1}_{\varphi\ominus\varphi_1} \chi_{\Omega_{\infty,\infty}}\right\|_\infty \le c \left\|y\chi_{\Omega_{\infty,\infty}}\right\|_M \le \frac{1}{4}$$

and

$$b_{\varphi\ominus_a\varphi_1}(t) = \infty$$

for a.e. $t \in \Omega_{0,0} \cup \Omega_{\infty,0}$. Consequently $\varphi \ominus_a \varphi_1(t, \frac{3}{2}b_k) < \infty$ for a.e. $t \in B_k$. Thus using Lemma 7 for the set B_k and the number b_k we obtain measurable function $x_k(t)$ on B_k such that

$$\varphi \ominus_a \varphi_1(t, y(t)) = \varphi(t, x_k(t)y(t)) - \varphi_1(t, x_k(t))$$

and $0 \le x_k(t) \le \min\{a, \frac{a}{a+1} b_{\varphi_1}(t)\}$ for a.e. $t \in B_k$.

Now we will consider the atomic part of Ω. For every $0 \le k \le m$ let $c_k > 0$ satisfy

$$0 \le c_k \le \min\{1/\varphi^{-1}(\frac{1}{\mu(\omega_k)}), \frac{b_{\varphi_1}(\omega_k)}{2}\}\}$$

and

$$\varphi \ominus_a \varphi_1(\omega_k, y(\omega_k)) = \varphi(\omega_k, c_k y(\omega_k)) - \varphi_1(\omega_k, c_k).$$

Such numbers exist, since the supremum in definition of $\varphi \ominus_a \varphi_1$ is taken over compact set.

The function satisfying (i) is defined as

$$x(t) := \begin{cases} x_k(t) & \text{if } t \in B_k,\ 0 \le k \le n, \\ c_k & \text{if } t = \omega_k,\ 0 \le k \le m, \\ 0 & \text{if } t \notin \operatorname{supp}(y). \end{cases}$$

In the next step we will determine sets (A_n) satisfying (ii) and (iii).

We start with Ω_∞. By Lemma 3 there exists a sequence of pairwise disjoint measurable sets (A_n^1) such that $\bigcup\limits_{n\in\mathbb{N}} A_n^1 = \Omega_\infty$ and

$$\left\|\chi_{A_n^1}\right\|_{\varphi_1} \le \frac{2}{\sup\limits_{t\in A_n^1}\{b_{\varphi_1}(t)\}},$$

for every $n \in \mathbb{N}$. Since $0 \leq x(t) < b_{\varphi_1}(t)$, we have

$$\left\|xy\chi_{A_n^1}\right\|_\varphi \leq \sup_{t\in A_n^1}\{b_{\varphi_1}(t)\}\left\|y\right\|_M \left\|\chi_{A_n^1}\right\|_{\varphi_1} \leq \frac{1}{2}, \tag{4.2}$$

and therefore sets (A_n^1) satisfy (ii).
Secondly, by Lemma 2, there exists sequence (A_n^2) of pairwise disjoint measurable sets such that $\bigcup\limits_{n\in\mathbb{N}} A_n^2 = \Omega_{0,0}$ and

$$\left\|\chi_{A_n^2}\right\|_{\varphi_1} \leq \frac{1}{a}.$$

Moreover, we have

$$\left\|xy\chi_{A_n^2}\right\|_\varphi \leq \frac{a}{2}\left\|\chi_{A_n^2}\right\|_{\varphi_1} \leq \frac{1}{2}, \tag{4.3}$$

because $x(t) \leq a$.

Considering the atomic part, let's observe that for each atom ω

$$\|xy\chi_\omega\|_\varphi \leq \frac{1}{2\varphi^{-1}(\frac{1}{\mu(\omega)})}\|\chi_\omega\|_{\varphi_1} = \frac{1}{2}, \tag{4.4}$$

where the last equality follows by $\|\chi_\omega\|_{\varphi_1} = \varphi^{-1}(\frac{1}{\mu(\omega)})$. Therefore, we can take atoms as desired sets.

Finally, it is enough to renumerate the sequences $(A_n^1), (A_n^2), (\omega)_{\omega\in\mathrm{supp}(M(L^{\varphi_1},L^\varphi))^a}$ into one sequence (A_n). By Lemma 6,

$$\mathrm{supp}(M(L^{\varphi_1}, L^\varphi)) \subset \bigcup_{n\in\mathbb{N}} A_n,$$

thus the construction of desired sets (A_n) is finished.

It just left to show that (iv) is fulfilled, i.e.

$$\|x\|_{\varphi_1} \leq 1.$$

In order to prove it, we define functions $x_n := \sum\limits_{k=1}^{n} x\chi_{A_k}$ and we will inductively show that

$$I_{\varphi_1}(x_n) \leq \frac{1}{2}.$$

Since $x_n \uparrow x$ a.e., from the Fatou property, it will follow that $x \in L^{\varphi_1}$ and

$$\|x\|_{\varphi_1} \leq \sup_n \|x_n\|_{\varphi_1} \leq 1.$$

Firstly we need to show that for every $k \in \mathbb{N}$ there holds

$$\|x\chi_{A_k}\|_{\varphi_1} \leq \frac{1}{2}. \tag{4.5}$$

From the equality

$$\varphi \ominus_a \varphi_1(t, y(t)) = \varphi(t, x(t)y(t)) - \varphi_1(t, x(t))$$

we obtain two inequalities

$$\varphi_1(t, x(t)) \leq \varphi(t, x(t)y(t)) \text{ for a.e. } t \in \Omega, \tag{4.6}$$

$$\varphi \ominus_a \varphi_1(t, y(t)) \leq \varphi(t, x(t)y(t)) \text{ for a.e. } t \in \Omega. \tag{4.7}$$

From (4.6) and by inequality $\|xy\chi_{A_k}\|_\varphi \leq \frac{1}{2}$ we have

$$I_{\varphi_1}(x\chi_{A_k}) = \int_{A_k}\varphi_1(t, x(t))d\mu(t) \leq \int_{A_k}\varphi(t, y(t)x(t))d\mu(t) = I_\varphi(yx\chi_{A_k}) \leq \frac{1}{2} \tag{4.8}$$

for every $k \in \mathbb{N}$, where the last inequality follows from (2.2).
In particular, $I_{\varphi_1}(x_1) \leq \frac{1}{2}$, and we can proceed with the induction. Let $n \geq 1$ and suppose that

$$I_{\varphi_1}(x_n) \leq \frac{1}{2}.$$

We have

$$I_{\varphi_1}(x_{n+1}) = I_{\varphi_1}(x_n) + I_{\varphi_1}\left(x\chi_{A_{n+1}}\right) \leq 1$$

and thus $\|x_{n+1}\|_{\varphi_1} \leq 1$. Similarly, as in inequality (4.8), we obtain

$$I_{\varphi_1}(x_{n+1}) \leq I_{\varphi}(yx_{n+1}) \leq \frac{1}{2},$$

by $\|yx_{n+1}\|_{\varphi} \leq \frac{1}{2}\|x_{n+1}\|_{\varphi_1} \leq \frac{1}{2}$. It means that (4.5) is proved.

Finally, we are ready to show that $I_{\varphi\ominus_a\varphi_1}(y) \leq 1$. We have

$$\|yx\|_{\varphi} \leq \|y\|_M \|x\|_{\varphi_1} \leq \frac{1}{2}$$

and from inequality (4.7) we obtain

$$I_{\varphi\ominus_a\varphi_1}(y) \int \varphi \ominus_a \varphi_1(t, y(t))d\mu(t) \leq \int \varphi(t, y(t)x(t))d\mu(t) = I_{\varphi}(yx) \leq 1.$$

Clearly, $\varphi \ominus_a \varphi_1(t, y(t)) \uparrow \varphi \ominus \varphi_1(t, y(t))$ for a.e. $t \in \Omega$ when $a \uparrow \infty$. Applying the Fatou lemma we have

$$I_{\varphi\ominus\varphi_1}(y) = \int_{\Omega} \varphi \ominus \varphi_1(t, y(t))d\mu(t) \leq \liminf_{a\to\infty} \int_{\Omega} \varphi \ominus_a \varphi_1(t, y(t))d\mu(t) \leq 1,$$

which proves the inequality (4.1). It means that $y \in L^{\varphi\ominus\varphi_1}$ and

$$\|y\|_{\varphi\ominus\varphi_1} \leq 1.$$

Concluding, if $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ is a simple function, then $y \in L^{\varphi\ominus\varphi_1}$ and

$$\|y\|_{\varphi\ominus\varphi_1} \leq 4c\|y\|_M.$$

Thus the theorem is proved for positive simple functions. We will once again use the Fatou property to complete the argument for an arbitrary function.

Let $y \in M(L^{\varphi_1}, L^{\varphi})$. There exists a sequence of simple functions (y_n) such that $0 \leq y_n \uparrow |y|$ a.e. on Ω. Since $M(L^{\varphi_1}, L^{\varphi})$ is a Banach lattice, $\|y_n\|_M \leq \|y\|_M$ for every $n \in \mathbb{N}$. From the Fatou property of $L^{\varphi\ominus\varphi_1}$ we have $y \in L^{\varphi\ominus\varphi_1}$ and

$$\|y\|_{\varphi\ominus\varphi_1} \leq \sup_{n\in\mathbb{N}} \|y_n\|_{\varphi\ominus\varphi_1} \leq 4c\|y_n\|_M \leq 4c\|y\|_M,$$

which finishes the proof. $\square$

In the special case of variable exponent spaces we have the following corollary. It has been recently proved in [9] using elementary methods. Recall that the Nakano space (or variable exponent space) is defined as $L^{p(\cdot)} := L^{\varphi}$, where $\varphi(t, u) = u^{p(t)}$, for a measurable function $p : \Omega \to [1, \infty)$.

Corollary 9. Let Ω be non-atomic and let $p, q : \Omega \to [1, \infty)$ be two measurable functions satisfying $q(t) \leq p(t)$ for a.e. $t \in \Omega$. Then

$$M(L^{p(\cdot)}, L^{q(\cdot)}) = L^{r(\cdot)},$$

where $\frac{1}{p(t)} + \frac{1}{r(t)} = \frac{1}{q(t)}$ for a.e. $t \in \Omega$.

Proof. First of all, observe that each Nakano space $L^{p(\cdot)}$ may be equivalently defined by the Musielak–Orlicz function $\varphi_p(t,u) = \frac{1}{p(t)}u^{p(t)}$. In fact, we see that for $\varphi(t,u) = u^{p(t)}$ there holds

$$\varphi(t, \frac{u}{2}) = (\frac{u}{2})^{p(t)} \leq \frac{1}{p(t)}u^{p(t)} = \varphi_p(t,u) \leq \varphi(t,u),$$

for each $t \in \Omega$ and $u > 0$, which means that $L^{p(\cdot)} = L^{\varphi_p}$. Now the proof follows directly from Example 1 and the above theorem. $\square$

5. Pointwise products

For a given Musielak–Orlicz function φ on Ω we define the right-continuous inverse at point $t \in \Omega$

$$\varphi^{-1}(t,u) := \inf\{v \geq 0 : \varphi(t,v) > u\}.$$

If $\varphi, \varphi_1, \varphi_2$ are Musielak–Orlicz functions we write $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ if there exists a constant $C > 0$ such that

$$C\varphi_1^{-1}(t,u)\varphi_2^{-1}(t,u) \leq \varphi^{-1}(t,u)$$

for a.e. $t \in \Omega$ and $u \geq 0$. Similarly, we write $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ if there exists a constant $C > 0$ such that for a.e. $t \in \Omega$ and $u \geq 0$

$$C\varphi_1^{-1}(t,u)\varphi_2^{-1}(t,u) \geq \varphi^{-1}(t,u).$$

Moreover, $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ means that $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ and $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$.

Recall the classical Lozanovskii factorization theorem which says that each Banach ideal space E factorizes L^1, this is

$$E \odot M(E, L^1) = L^1.$$

Generalizing this idea, for a couple of Banach ideal spaces E, F we say that E factorizes F if

$$E \odot M(E,F) = F.$$

Recently the authors proved in [12, Theorem 2] that for a pair of Orlicz functions φ, φ_1, the function space L^{φ_1} may be factorized by L^{φ} if and only if

$$\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}. \tag{5.1}$$

That result is based on Theorem 5 in [11], which states that in the case of non-atomic and finite measure space, given three Orlicz functions $\varphi, \varphi_0, \varphi_1$, there holds

$$L^{\varphi} \odot L^{\varphi_1} = L^{\varphi_0}$$

if and only if

$$\varphi_1^{-1}\varphi_0^{-1} \approx \varphi^{-1}.$$

In this section we will show that, in the case of Musielak–Orlicz spaces, the condition (5.1) is sufficient, but not necessary to have the factorization

$$L^{\varphi} \odot L^{\varphi_1} = L^{\varphi_0}.$$

An immediate consequence of Theorem 8 is the following inclusion.

Lemma 10. Let (Ω, Σ, μ) be a measure space and let φ, φ_1 be Musielak–Orlicz functions. If $\mathrm{supp}(L^{\varphi_1}) = \Omega$ then

$$L^{\varphi} \subset L^{\varphi \ominus \varphi_1} \odot L^{\varphi_1}$$

Proof. Let $x \in L^{\varphi \ominus \varphi_1}$ and $y \in L^{\varphi_1}$. Then, since $M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}$, we see that

$$xy \in L^{\varphi}$$

and

$$\|xy\|_{\varphi} \leq \|x\|_M \|y\|_{\varphi_1} \leq c\|x\|_{\varphi \ominus \varphi_1} \|y\|_{\varphi_1}.$$

$\square$

Lemma 11. Let (Ω, Σ, μ) be a measure space and $\varphi, \varphi_0, \varphi_1$ be Musielak–Orlicz functions. Assume that $\varphi_1^{-1}\varphi_0^{-1} \succ \varphi^{-1}$ and $\operatorname{supp} L^{\varphi_1} = \Omega$. Then

$$L^{\varphi} \subset L^{\varphi_0} \odot L^{\varphi_1}.$$

Proof. Denote by $c \geq 1$ the constant of inclusion

$$L^{\varphi}[\Omega_{\infty,\infty}] \subset L^{\infty}(b_{\varphi}^{-1})[\Omega_{\infty,\infty}].$$

Let $0 \leq z \in L^{\varphi}$ be such that $\|z\|_{\varphi} = \frac{2}{3c}$. Put $y(t) := \varphi(t, z(t))$. We have $y(t) < \infty$ a.e., since $z(t) \leq \frac{2}{3} b_{\varphi}(t)$. For $i = 0, 1$, define

$$z_i(t) := \begin{cases} \varphi_i^{-1}(t, y(t))\sqrt{\frac{z(t)}{\varphi_0^{-1}(t,y(t))\varphi_1^{-1}(t,y(t))}} & \text{if } t \in \operatorname{supp}(z) \\ 0 & \text{if } t \notin \operatorname{supp}(z) \end{cases}$$

Note that $z = z_0 z_1$. We will show that $z_i \in L^{\varphi_i}$ for $i = 0, 1$. Let $D > 0$ be such that

$$D\varphi_1^{-1}(t, u)\varphi_0^{-1}(t, u) \geq \varphi^{-1}(t, u).$$

We claim that

$$\varphi_i(t, \frac{z_i(t)}{\sqrt{D}}) \leq y(t). \tag{5.2}$$

If $y(t) = 0$ then

$$z_i(t) = a_{\varphi}(t)\sqrt{\frac{z(t)}{a_{\varphi_0}(t)a_{\varphi_1}(t)}} \leq a_{\varphi}(t)\sqrt{\frac{a_{\varphi}(t)}{a_{\varphi_0}(t)a_{\varphi_1}(t)}} \leq a_{\varphi}(t)\sqrt{D}$$

thus

$$\varphi_i(t, \frac{z_i(t)}{\sqrt{D}}) = 0.$$

If $y(t) > 0$ then

$$\begin{aligned} z_i(t) &= \varphi_i^{-1}(t, y(t))\sqrt{\frac{z(t)}{\varphi_0^{-1}(t, y(t))\varphi_1^{-1}(t, y(t))}} \\ &\leq \varphi_i^{-1}(t, y(t))\sqrt{\frac{Dz(t)}{\varphi^{-1}(t, y(t))}} \\ &= \varphi_i^{-1}(t, y(t))\sqrt{\frac{Dz(t)}{z(t)}} = \varphi_i^{-1}(t, y(t))\sqrt{D}. \end{aligned}$$

Therefore,

$$\varphi_i(t, \frac{z_i(t)}{\sqrt{D}}) \leq \varphi_i(t, \varphi_i^{-1}(t, y(t))) = y(t)$$

and the claim is proved. Integrating both sides in (5.2) we obtain

$$I_{\varphi_i}\left(\frac{z_i}{\sqrt{D}}\right) \leq I_{\varphi}(z) \leq 1,$$

for $i = 0, 1$. It follows, that

$$\|z_i\|_{\varphi_i} \leq \sqrt{D} \leq \sqrt{2Dc\,\|z\|_{\varphi}}.$$

This means that $z \in L^{\varphi_0} \odot L^{\varphi_1}$ and

$$\|z\|_{L^{\varphi_0} \odot L^{\varphi_1}} \leq 2Dc\,\|z\|_{\varphi}.$$

$\square$

Recall that for Musielak–Orlicz functions φ, φ_1, the generalized Young inequality implies that

$$\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \prec \varphi^{-1}$$

(see for example [10]).

Corollary 12. Let φ, φ_1 be Musielak–Orlicz functions on a σ-finite measure space (Ω, Σ, μ). If $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}$ then L^{φ_1} factorizes L^{φ}.

We finish the paper providing an example, which shows that the opposite implication does not hold. In particular, Theorem 2 in [12] cannot be directly generalized to Musielak–Orlicz spaces.

Example 13. Let $\Omega = [0, 1/2)$. Consider the following Musielak–Orlicz functions

$$\varphi(t, u) = \max\{u - t, 0\},$$
$$\varphi_1(t, u) = u.$$

Moreover, $L^{\varphi} = L^{\varphi_1} = L^1$. We have

$$\varphi \ominus \varphi_1(t, u) = \begin{cases} 0 & \text{if } 0 \le u \le 1 \\ \infty & \text{if } u > 1, \end{cases}$$

thus $L^{\varphi \ominus \varphi_1} = L^{\infty}$. In consequence, the factorization

$$L^{\varphi_1} \odot L^{\varphi \ominus \varphi_1} = L^{\varphi}$$

holds. On the other hand an easy computations show that

$$(\varphi \ominus \varphi_1)^{-1}(t, u) = 1, \ \varphi^{-1}(t, u) = u + t, \ \varphi_1^{-1}(t, u) = u.$$

We have $\varphi_1^{-1}(t, u)(\varphi \ominus \varphi_1)^{-1}(t, u) = u$, thus there is no constant D such that

$$D\varphi_1^{-1}(t, u)(\varphi \ominus \varphi_1)^{-1}(t, u) \ge \varphi^{-1}(t, u)$$

for every t and u (take for example $u = 0$ and $t > 0$). Hence

$$\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \not\succ \varphi^{-1}.$$

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Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3a, 60-965 Poznań, Poland

E-mail address: klesnik@vp.pl

E-mail address: jakub.tomaszewski@put.poznan.pl