ON MAXIMALLY OSCILLATING PERFECT SPLINES AND SOME OF THEIR EXTREMAL PROPERTIES

OLEG KOVALENKO

Abstract. In this paper we study analogues of the perfect splines for weighted Sobolev classes of functions defined on half-line. These splines play important role in the solution of certain extremal problems.

1. Introduction

The classical result by Chebyshev [2] states that for a given continuous on a segment function f, a polynomial P_n of degree at most n minimizes the uniform norm of the difference $f - P_n$, if and only if the function $f - P_n$ has $n+2$ points of oscillation.

The notion of oscillation and the functions that oscillate maximally play important role in Approximation Theory. The maximally oscillating polynomials and splines together with their analogues and generalizations solve many extremal problems. See for example [5], [7], [8], and references therein.

Let I be a finite interval or the positive half-line $\mathbb{R}_+$. Denote by $C^m(I)$, $m \in \mathbb{Z}_+$, the set of all m – times continuously differentiable (continuous in the case $m = 0$) functions $x : I \to \mathbb{R}$; by $L_\infty(I)$ we denote the space of all measurable functions $x : I \to \mathbb{R}$ with finite norm

$$\|x\| = \|x\|_{L_\infty(I)} := \operatorname*{ess\,sup}_{t\in I} |x(t)|.$$

Let X be $C(I)$ or $L_\infty(I)$, $f_\pm \in C(I)$ be positive functions. For $x \in X$ set

$$\|x\|_{X,f_-,f_+} := \left\| \frac{\max\{x(\cdot), 0\}}{f_+(\cdot)} + \frac{\max\{-x(\cdot), 0\}}{f_-(\cdot)} \right\|_X .$$

If $f_- = f_+ =: f$, we write $\|x\|_{X,f}$ instead of $\|x\|_{X,f_-,f_+}$.

For brevity we denote $\lim\limits_{t\to+\infty} x(t)$ by $x(\infty)$.

For positive functions $f, g \in C(I)$ and natural r set

$$L^r_{f,g}(I) := \left\{ x \in C(I) : \|x\|_{C(I),f} < \infty,\, x^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}, \right.$$
$$\left. \|x^{(r)}\|_{L_\infty(I),g} < \infty \right\},$$

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$$W^r_{f,g}(I) := \left\{x \in L^r_{f,g}(I) : \|x^{(r)}\|_{L_\infty(I),g} \le 1\right\}.$$

In the case when $f \equiv 1$ we write $L^r_{\infty,g}(I)$ instead of $L^r_{f,g}(I)$.

For $k = 1, \dots, r-1$, we call the function

$$\omega(W^r_{f,g}(I), D^k, \delta) := \sup_{x \in W^r_{f,g},\, \|x\|_{C(I),f} \le \delta} \|x^{(k)}\|_{C(I)},\ \delta \ge 0 \tag{1}$$

a modulus of continuity of k-th order differentiation operator on the class $W^r_{f,g}(I)$.

Study of the function ω is closely connected to the sharp Kolmogorov type inequalities. The first results in this topic were obtained in the 1910s by Landau [6], Hadamard [3], Hardy and Littlewood [4]. Since then the topic was intensively studied. For a more detailed overview of the history of the question see [1] and references therein.

The goal of the article is to define analogues of maximally oscillating perfect splines for the class $W^r_{f,g}(0,\infty)$ and characterize the values of the modulus of continuity (1) under several assumptions on the functions f and g.

The article is organized as follows. In Section 2 we define perfect g-splines and prove the existence of maximally oscillating g-splines on the finite segments and the half-line. In Section 3 we study the non-symmetrical weighted norms of the maximally oscillating g-splines. Section 4 is devoted to the study of the modulus of continuity (1).

2. Perfect g-splines

2.1. **Definitions.** Let $I = (a,b)$, where $a \in \mathbb{R}$ and b denotes either a real number or the positive infinity. Let a positive function $g \in C(I)$ be given.

Definition 1. *The function $G \in C^{r-1}(I)$ is called a perfect g-spline of the order r with $n \in \mathbb{N}$ knots $a < t_1 < \ldots < t_n < b$, if on each of the intervals $(t_i, t_{i+1}), i = 0, 1, \ldots, n$, $t_0 := a, t_{n+1} := b$, there exists derivative $G^{(r)}$ and $\frac{G^{(r)}(t)}{g(t)} \equiv \epsilon \cdot (-1)^i$ on the intervals $(t_i, t_{i+1}), i = 0, 1, \ldots, n$, where $\epsilon \in \{1, -1\}$.*

Definition 2. *A primitive G of the order r of the function g on I we will call a perfect g-spline of the order r with* 0 *knots.*

Definition 3. *Denote by $\Gamma^r_{n,g}(I)$ the set of all perfect g-splines G defined on I of the order r with not more than $n \in \mathbb{Z}_+$ knots.*

2.2. **Perfect g-splines with given zeros.**

Lemma 1. *Let $A > 0$, $n \in \mathbb{N}$ and $0 \le s_1 < s_2 < \ldots < s_n < A$. There exists a perfect g-spline $G \in \Gamma^r_{n,g}[0,A]$ such that*

$$G^{(k)}(A) = 0,\ k = 0, 1, \ldots, r-1. \tag{2}$$

and

$$G(s_k) = 0, k = 1, \dots, n. \tag{3}$$

Moreover, conditions (2) *and* (3) *imply that* G *changes sign in each of the points* s_k, $k = 1, \dots, n$, *and has exactly* n *knots.*

Consider the sphere $S^n := \left\{(\xi_1, \dots, \xi_{n+1}) \in \mathbb{R}^{n+1} \colon \sum\limits_{k=1}^{n+1} |\xi_k| = A\right\}$. For arbitrary $\xi \in S^n$ consider the partition of the segment $[0, A]$ by the points $t_m = \sum\limits_{k=1}^{m} |\xi_k|$, $m = 1, \dots, n$. Let $G(\xi) \in C^{r-1}[0, A]$ be the function that satisfies boundary conditions (2) and such that $G^{(r)}(\xi, t) = \operatorname{sgn} \xi_k \cdot g(t)$ on the interval (t_{k-1}, t_k), $k = 1, \dots, n+1$, $t_0 := 0$, $t_{n+1} = A$; such function is uniquely determined by the imposed conditions. Moreover, $G(\xi) = -G(-\xi)$ for arbitrary $\xi \in S^n$ and $G(\xi)$ uniformly converges to $G(\xi_0)$ provided by $\xi \to \xi_0$.

Consider a map $\phi \colon S^n \to \mathbb{R}^n$, $\phi(\xi) = (G(\xi; s_1), \dots, G(\xi; s_n))$. Is is continuous and odd. By the Borsuk theorem, there exists $\xi^* \in S^n$ such that $\phi(\xi^*) = 0$. The function $G(\xi^*)$ satisfies (2) and (3), $G^{(r)}(\xi^*)$ is non-zero almost everywhere. Hence by the Rolle theorem the function $G^{(r)}(\xi^*)$ has at least n sign changes. From the construction it follows, that the function $G^{(r)}(\xi^*)$ has exactly n sign changes, hence $G(\xi^*) \in \Gamma^r_{n,g}[0, A] \setminus \Gamma^r_{n-1,g}[0, A]$ is a desired perfect g-spline. The lemma is proved.

Lemma 2. *Let* $A > 0$, $n \in \mathbb{N}$ *and* $0 < s_1 < s_2 < \ldots < s_n < A$. *If two perfect* g*-splines* $G_1, G_2 \in \Gamma^r_{n,g}[0, A]$ *satisfy conditions* (2) *and* (3), *then either* $G_1 \equiv G_2$, *or* $G_1 \equiv -G_2$.

We can assume that both G_1 and G_2 are positive on $[0, s_1)$. Hence G_1 and G_2 have same signs on each of the intervals (s_k, s_{k+1}), $k = 0, \dots, n$, $s_0 := 0$, $s_{n+1} := A$.

Let $k \in \{0, \dots, n\}$ and $s \in (s_k, s_{k+1})$ be fixed. We prove that

$$G_1(s) = G_2(s). \tag{4}$$

Assume the contrary, without loss of generality we may assume that $|G_1(s)| < |G_2(s)|$. There exists $\varepsilon \in (0, 1)$ such that $G_1(s) = (1 - \varepsilon) \cdot G_2(s)$. The difference $G := G_1 - (1 - \varepsilon) \cdot G_2$ satisfies conditions (2) and (3); moreover, it has additional zero in the point s. From the definition of the function G it follows that the function $G^{(r)}$ is non-zero almost everywhere and changes its sign only in the knots of the perfect g-spline G_1. Hence all zeros $s_1, \dots, s_n$, s and A of the function G are separated and thus the Rolle theorem implies that the function $G^{(r)}$ has at least $n + 1$ sign changes. This contradiction proves (4). Due to arbitrariness of the point s, this implies that $G_1 \equiv G_2$. The lemma is proved.

Lemma 3. *Let $A > 0$ and $n \in \mathbb{N}$ be given. Suppose that for each $m = 0, 1, \dots$ a point $\mathbb{R}^n \ni s^{(m)} = (s_{1,m}, \dots, s_{n,m})$, $0 \le s_{1,m} < \dots < s_{n,m} < A$ is given. Let $G_m \in \Gamma^r_{n,g}[0, A]$ be a perfect g-spline that satisfies (2) and vanishes in the points $s^{(m)}$, $m = 0, 1, \dots$. Denote by $t^{(m)} = (t_{1,m}, \dots, t_{n,m})$, $0 < t_{1,m} < \dots < t_{n,m} < A$ the knots of G_m. Then $s^{(m)} \to s^{(0)}$ as $m \to \infty$ implies $t^{(m)} \to t^{(0)}$ as $m \to \infty$.*

Assume the contrary, that the sequence $\left\{t^{(m)}\right\}_{m=1}^{\infty}$ has two different limit points $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$. Consider the prefect g splines G_u and G_v with knots in the point u and v that satisfy boundary conditions (2); each of the splines is determined up to the sign. Moreover, since the prefect spline continuously (in the sense of uniform convergence) depends on its knots, both G_u and G_v vanish at the points $s^{(0)}$. However this contradicts to Lemma 2. The lemma is proved.

2.3. **Maximally oscillating perfect g-splines on a segment.**

Definition 4. *Let I denote a segment or a half-line. Suppose that two functions $f_+, f_- \in C(I)$ such that $f_+(t) > f_-(t)$ for all $t \in I$ are given. We say that a function $h \in C(I)$ has $n \in \mathbb{N}$ points of oscillation between the functions f_- and f_+, if $f_-(t) \le h(t) \le f_+(t)$ for all $t \in I$ and there exist points $s_k \in I$, $k = 1, \dots, n$, $s_1 < s_2 < \dots < s_n$, such that for $k = 1, \dots, n$*

$$h(s_k) = \begin{cases} f_+(s_k), & k \text{ is odd,} \\ f_-(s_k), & k \text{ is even.} \end{cases}$$

Lemma 4. *Let $A > 0$ and two functions $f_+, f_- \in C[0, A]$ be given. Assume $f_\pm(t) > 0$ for all $t \in [0, A]$. Then for all $n \in \mathbb{Z}_+$ there exists $C > 0$ and a perfect g-spline $G \in \Gamma^r_{n,g}[0, A]$ that satisfies (2), has exactly n knots, and has $n+1$ points of oscillation between the functions $-C \cdot f_-$ and $C \cdot f_+$.*

For all $\varepsilon \in \left(0, \frac{A}{n+1}\right)$ consider the n-dimensional simplex

$$\Xi^n_\varepsilon := \left\{ (\xi_1, \dots, \xi_{n+1}) \in \mathbb{R}^{n+1} \colon \xi_1, \dots, \xi_{n+1} \ge \varepsilon, \sum_{k=1}^{n+1} \xi_k = A \right\}.$$

Set

$$\Xi^n_0 := \left\{ (\xi_1, \dots, \xi_{n+1}) \in \mathbb{R}^{n+1} \colon \xi_1, \dots, \xi_{n+1} > 0, \sum_{k=1}^{n+1} \xi_k = A \right\}.$$

For each point $\xi \in \Xi^n_\varepsilon$ consider the partition of the segment $[0, A]$ by the points $s_m = \sum\limits_{k=1}^{m} |\xi_k|$, $m = 1, \dots, n$. Due to the definition of the simplex, $0 < s_1 < \dots < s_n < A$. Hence, in virtue of Lemma 1, there exists a perfect g-spline $G(\xi)$ with exactly n knots that satisfies (2) and (3). The sign of $G(\xi)$ is chosen in such a way, that $G(\xi)$ is positive on $(0, s_1)$. For each $k = 1, \dots, n+1$, set

$$\delta_k(\xi) := \min\left\{\delta \geq 0\colon |G(\xi; t)| \leq \delta \cdot f_k(t), t \in (s_{k-1}, s_k)\right\},$$

where $s_0 := 0$, $s_{n+1} := A$, $f_k = f_+$ for odd k, and $f_k = f_-$ for even k.

Set $\Delta_k(\xi) := \delta_k(\xi) - \min\limits_{j=1,\dots,n+1} \delta_j(\xi)$, $k = 1, \dots, n+1$, and $\Delta(\xi) := \sum\limits_{k=1}^{n+1} \Delta_k(\xi)$.

If for some $\varepsilon > 0$ there exists $\xi \in \Xi^n_\varepsilon$ such that $\Delta(\xi) = 0$, then corresponding perfect g-spline $G(\xi)$ is a desired one.

Assume that for all $\varepsilon > 0$ and $\xi \in \Xi^n_\varepsilon$

$$\Delta(\xi) \neq 0. \tag{5}$$

In virtue of Lemma 3, δ_k and Δ_k are continuous functions of ξ, $k = 1, \dots, n+1$, hence Δ is also a continuous function of ξ.

Note, that since the functions f_+ and f_- are separated from zero, from

$$\sup_{G \in \Gamma^r_{n,g}[0,A]} \|G'\|_{C[0,A]} < \infty$$

it follows that there exists $\alpha > 0$ that does not depend on ξ and ε such that

$$\delta_k(\xi) \leq \alpha \cdot \xi_k,\ k = 1, \dots, n+1; \tag{6}$$

from $\inf\limits_{G \in \Gamma^r_{n,g}[0,A]} \|G\|_{C[0,A]} > 0$ it follows that

$$\inf_{\varepsilon > 0} \min_{\xi \in \Xi^n_\varepsilon} \max_{j=1,\dots,n+1} \delta_k(\xi) =: \beta > 0. \tag{7}$$

Then

$$\inf_{\varepsilon > 0} \min_{\xi \in \Xi^n_\varepsilon} \Delta(\xi) =: \gamma > 0. \tag{8}$$

Really, if (8) does not hold, then, due to (5), for arbitrary $\varepsilon > 0$ there exists $\xi_\varepsilon \in \Xi^n_0 \setminus \Xi^n_\varepsilon$ such that $\Delta(\xi_\varepsilon) < \frac{\beta}{2}$. Due to (7) this implies that $\min\limits_{k=1,\dots,n+1} \delta_k(\xi_\varepsilon) > \frac{\beta}{2}$. However, this contradicts to (6).

Define a map $\phi\colon \Xi^n_\varepsilon \to \Xi^n_\varepsilon$,

$$\phi(\xi) = \varepsilon + \frac{A - (n+1)\varepsilon}{\Delta(\xi)} \left(\Delta_2(\xi), \Delta_3(\xi), \dots, \Delta_{n+2}(\xi)\right),$$

where $\Delta_{n+2}(\xi) := \Delta_1(\xi)$. By the construction, ϕ is continuous, hence by the Brouwer fixed point theorem there exists $\xi^* \in \Xi^n_\varepsilon$ such that

$$\phi(\xi^*) = \xi^* = (\xi^*_1, \dots, \xi^*_{n+1}). \tag{9}$$

Hence for $k = 1, \dots, n+1$, due to (6) and (8), one has

$$\xi^*_k = \varepsilon + \frac{A - (n+1)\varepsilon}{\Delta(\xi^*)} \Delta_{k+1}(\xi^*) < \varepsilon + \frac{A}{\gamma} \delta_{k+1}(\xi^*) < \varepsilon + \frac{\alpha \cdot A}{\gamma} \xi^*_{k+1}. \tag{10}$$

From the construction of the functions Δ_k it follows that there exists $k \in \{1, \dots, n+1\}$ such that $\Delta_k(\xi^*) = 0$. Let for definiteness $\Delta_1(\xi^*) = 0$. Then, due to (9), $\xi^*_{n+1} = \varepsilon$. Consecutive application of estimate (10) implies that there exist positive numbers $\eta_1, \dots, \eta_n$ that are independent of ε and ξ^* and such that $\xi^*_n < \eta_n\varepsilon$, $\xi^*_{n-1} < \eta_{n-1}\varepsilon$ and so on, $\xi^*_1 < \eta_1\varepsilon$. However, if ε is chosen to be enough small, this contradicts to the fact that $\sum\limits_{k=1}^{n+1} \xi^*_k = A$. The lemma is proved.

2.4. **Maximally oscillating perfect g-splines on a half-line.** In what follows we assume that the following two conditions hold.

Assumption 1. *The functions $f, f_\pm \in C[0,\infty)$ are non-increasing positive and such that*

$$f_\pm(\infty) > 0,\ f(\infty) > 0. \tag{11}$$

Assumption 2. *The function $g \in C[0,\infty)$ is positive non-increasing and such that*

$$A_0 := \int\limits_0^\infty g(t)dt < \infty, \tag{12}$$

and for $k = 1, \dots, r-1$

$$A_k := \int\limits_0^\infty \left[\sum_{s=0}^{k-1} \frac{(-1)^{k-s-1}A_s}{(k-s-1)!} t^{k-s-1} + (-1)^k g_k(t)\right] dt < \infty, \tag{13}$$

where $g_k(t) := \int\limits_0^t g_{k-1}(s)ds$, $k = 1, 2, \dots, r$ and $g_0 := g$.

Definition 5. *Let conditions* (12) *and* (13) *hold. Set $P_0 := g$ and for $k = 1, \dots, r$, and $t \in [0,\infty)$ set*

$$P_k(t) := (-1)^k \sum_{s=0}^{k-1} \frac{(-1)^{k-s-1}A_s}{(k-s-1)!} t^{k-s-1} + g_k(t).$$

Note, that P_k is the k-th order primitive of the function g such that $P_k(\infty) = 0$, $k = 1, \dots, r$. Moreover, P_k is monotone and does not change sign on $[0,\infty)$.

Lemma 5. *Let $r \in \mathbb{N}$, $x \in L^r_{\infty,g}(0,\infty)$ and $\lim\limits_{t\to\infty} x(t) = A \in \mathbb{R}$. Then for all $t \in \mathbb{R}$, $|x(t)| \le |A| + |P_r(t)|$.*

It is enough to prove the lemma in the case when $A = 0$. This, in turn, can be easily done by induction on r. Really, using induction hypothesis for $r > 1$, or the definition of the class $L^r_{\infty,g}(0,\infty)$ for $r = 1$, we have

$$|x(t)| = |x(t) - x(\infty)| = \left| \int_t^\infty x'(s)ds \right| \le \int_t^\infty |x'(s)|\, ds \le \int_t^\infty |P_{r-1}(s)|\, ds$$

$$= \left| \int_t^\infty P_{r-1}(s)ds \right| = |P_r(t) - P_r(\infty)| = |P_r(t)|.$$

The lemma is proved.

Lemma 6. *Let $n \in \mathbb{Z}_+$ be given. For all $\alpha \in (0,\infty)$ there exists $C > 0$ and a perfect g-spline $G = G(n,\alpha) \in \Gamma^r_{n,g}[0,\infty)$ with exactly n knots that has $n+1$ points of oscillation between $-C \cdot f_-$ and $C \cdot f_+$ and such that*

$$\frac{Cf_+(\infty) - G(\infty)}{Cf_-(\infty) + G(\infty)} = \alpha. \tag{14}$$

Set $a := \frac{1}{1+\alpha} f_+(\infty) - \frac{\alpha}{1+\alpha} f_-(\infty)$. Then $a \in (-f_-(\infty), f_+(\infty))$ and hence both functions $f_+ - a$ and $f_- + a$ are positive continuous non-increasing on $[0,\infty)$ functions.

For arbitrary $A > 0$ consider the perfect g-spline $G_A \in \Gamma^r_{n,g}[0, A]$ that oscillates maximally between the functions $-C_A(f_- + a)$ and $C_A(f_+ - a)$, where $C_A > 0$ and the existence of such spline is guaranteed by Lemma 4. Let $0 \le s_1^A < \ldots < s_{n+1}^A \le A$ be the oscillation points of G_A and $0 < t_1^A < \ldots < t_n^A < A$ be its knots.

Letting $A \to \infty$ and switching to a subsequence, if needed, we may assume that each of the sequences s_k^A, $k = 1, \ldots, n+1$, t_k^A, $k = 1, \ldots, n$ and C_A, has a finite or infinite limit s_k, t_k and C respectively.

Since G_A satisfies (2), using arguments from the proof of Lemma 5, one can show that for all $A > 0$ and $t \in [0, A]$

$$|G_A(t)| \le |P_r(t)|. \tag{15}$$

The latter inequality implies that for all $A > 0$ one has $C_A \le \|P_r\|_{C[0,\infty),f_-,f_+}$; moreover, $\|P_r\|_{C[0,\infty),f_-,f_+}$ is finite due to (11). Hence C is finite.

Since (15) holds for $t = s_{n+1}^A$, from (11) and the fact that $P_r(\infty) = 0$, it follows that

$$s_{n+1} < \infty.$$

From (11) and the fact that the derivative G'_A can not become arbitrarily large, it follows that $s_k \ne s_j$, $k \ne j$.

Taking into account (15) and using standard compactness arguments, we can extract a limit function $G = \lim\limits_{A\to\infty} G_A$, where the convergence of the function and all derivatives of order $\le r-1$ is uniform on any bounded interval.

Then G has $n+1$ points of oscillation between $-C \cdot (f_- + a)$ and $C \cdot (f_+ - a)$. Therefore G has n knots, i. e. $t_k \ne t_j$, $k \ne j$. Really, G

has n zeros on $[0, s_{n+1}]$ and $G(\infty) = 0$. Hence G' has $n-1$ zeros on $(0, s_{n+1})$ and one on (s_{n+1}, ∞). Continuing the same way, we obtain that $G^{(r)}$ has n sign changes, hence G has n knots. This means that $G \in \Gamma^r_{n,g}[0,\infty)$.

The function $G + Ca$ is a desired one. Really, it is a perfect g-spline with exactly n knots that has $n+1$ points of oscillation between $-C \cdot f_-$ and $C \cdot f_+$. Moreover, condition (14) holds, due to the choice of a. The lemma is proved.

3. Maximally oscillating perfect g-spline with given norm

3.1. Definition of the functions C_n.

Lemma 7. *If there are two perfect g-splines $G_i \in \Gamma^r_{n,g}[0,\infty)$ that oscillate maximally between $-C_i \cdot f_-$ and $C_i \cdot f_+$, $i = 1, 2$, and*

$$G_1(\infty) = G_2(\infty), \tag{16}$$

then $C_1 = C_2$.

Assume the contrary, let for definiteness $C_1 > C_2$. Let $0 \le s_1 < \ldots < s_{n+1}$ be the oscillation points of G_1, $0 < t_1 < \ldots < t_n$ be the knots of G_1 and $0 < u_1 < \ldots < u_n$ be the knots of G_2. Set $\delta := G_1 - G_2$. Then $\mathrm{sgn}\delta(s_k) = \mathrm{sgn}G_1(s_k)$, $k = 1, \ldots, n+1$, and

$$\delta^{(k)}(\infty) = 0, k = 0, \ldots, r-1. \tag{17}$$

Hence δ has n separated zeros on $(0, s_{n+1})$. Due to (17), δ' has n separated zeros on $(0,\infty)$. Continuing the same way, we obtain that $\delta^{(r)}$ has n sign changes on $(0,\infty)$. This implies, that $\delta^{(r)}$ is not identical zero on each of the intervals (t_k, t_{k+1}), $k = 0, \ldots, n$, $t_0 := 0$, $t_{n+1} := \infty$. Hence $u_1 < t_1$, since otherwise $\delta^{(r)}(t) = 0$, $t \in (t_0, t_1)$; $u_2 < t_2$, since otherwise $(u_1, u_2) \supset (t_1, t_2)$, and hence $\delta^{(r)}(t) = 0$, $t \in (t_1, t_2)$, and so on, $u_n < t_n$. However, we obtain that $\delta^{(r)}(t) = 0$ for $t \in (t_n, t_{n+1})$ and hence $\delta^{(r)}$ has at most $n-1$ sign changes. Contradiction. The lemma is proved.

Lemma 8. *Statement of Lemma 7 remains true if condition* (16) *is substituted by* (14) *for both splines, i. e. if*

$$\frac{C_i f_+(\infty) - G_i(\infty)}{C_i f_-(\infty) + G_i(\infty)} = \alpha,$$

$i = 1, 2$, $\alpha > 0$.

Set $a := \frac{1}{1+\alpha} f_+(\infty) - \frac{\alpha}{1+\alpha} f_-(\infty)$. Then for $i = 1, 2$, $G_i(\infty) = C_i \cdot a$. The perfect g-splines $G_i - G_i(\infty)$ satisfy (16) and oscillate maximally between $-C_i(f_- + a)$ and $C_i(f_+ - a)$, $i = 1, 2$. Lemma 7 now implies that $C_1 = C_2$. The lemma is proved.

We do not prove the uniqueness of the maximally oscillating perfect g-spline. However, Lemma 8 implies the correctness of the following definition.

Definition 6. *For fixed $r, n \in \mathbb{N}$ and $\alpha > 0$ set*

$$C_n(\alpha) = C_{r,n,g,f_-,f_+}(\alpha) := \|G_\alpha\|_{C_{[0,\infty)},f_-,f_+},$$

where G_α is a spline from $\Gamma^r_{n,g}[0,\infty)$ built according to Lemma 6.

3.2. **Some properties of the functions C_n.**

Lemma 9. *For each $n \in \mathbb{N}$, C_n is continuous on $(0,\infty)$.*

Let $\beta > 0$ and $\beta_s \to \beta$, $s \to \infty$. Denote by G_β and G_s, $s \in \mathbb{N}$, the perfect g-splines built according to Lemma 6 with α in the boundary condition (14) substituted by β and β_s respectively.

Switching to a subsequence, if needed, we may assume that the sequences of the knots and the oscillation points of G_s, together with the sequence $\{C_n(\beta_s)\}_{s=1}^\infty$, have limits as $s \to \infty$. Moreover, analogously to the proof of Lemma 6, we can prove that all oscillation point limits are finite and different. Hence we obtain a perfect g-spline $G \in \Gamma^r_{n,g}[0,\infty)$ that oscillates maximally between $-Cf_-$ and Cf_+, $C = \lim\limits_{s\to\infty} C_n(\beta_s)$, and satisfies (14) with $\alpha = \beta$. Due to Lemma 8, we obtain $C_n(\beta) = C$, hence the lemma is proved.

Lemma 10. *The function C_n is non-decreasing in the case of odd n, and is non-increasing in the case of even n.*

We prove the lemma for the case of odd n; the case of even n can be proved using similar arguments.

For each $\alpha > 0$ denote by G_α a perfect g-spline built according to Lemma 6. Then, due to (14),

$$G_\alpha(\infty) = \frac{C_n(\alpha)}{\alpha+1}\left(f_+(\infty) - \alpha\cdot f_-(\infty)\right). \tag{18}$$

Since n is odd, G_α is increasing on the interval (M_α, ∞) for some $M_\alpha > 0$.

Assume the contrary, let n be odd and $0 < \gamma < \beta$ be such that $C_n(\gamma) > C_n(\beta)$. Then

$$G_\beta(\infty) > G_\gamma(\infty). \tag{19}$$

Really, otherwise we get an extra zero of $G_\gamma - G_\beta$ on the interval $(M_\gamma, \infty]$ and obtain a contradiction using arguments similar to the proof of Lemma 7.

For arbitrary $0 < \alpha < \gamma$ one has $G_\alpha(\infty) \neq G_\beta(\infty)$. Really, otherwise Lemma 7 implies $C_n(\alpha) = C_n(\beta)$, and hence, due to (18), $\alpha = \beta$, which is impossible.

From Lemma 9 and (19) it now follows, that

$$G_\beta(\infty) > G_\alpha(\infty) \tag{20}$$

for all $0 < \alpha < \gamma$. Hence for all small enough $\alpha > 0$ one has

$$C_n(\alpha) f_+(\infty) \le G_\beta(\infty) < C_n(\beta) f_+(\infty),$$

hence $C_n(\alpha) < C_n(\beta)$. However, the latter inequality implies that the difference $G_\beta - G_\alpha$ has n separated zeros between oscillation points of G_β and additional zero on the interval (M_β, ∞) due to (20). Using arguments similar to the proof of Lemma 7, we obtain a contradiction. The lemma is proved.

3.3. **Auxiliary results.**

Lemma 11. *Let $G \in \Gamma^r_{n,g}[0,\infty)$ have $n+1$ points of oscillation between f_- and f_+. Assume $s_1 < \ldots < s_n$ are all zeros of G' and $t_1 < \ldots < t_n$ are its knots. Then $t_k \geq s_k$, $k = 1, \ldots, n$.*

Assume the contrary, let $k \in \{1, \ldots, n\}$ be such that $t_k < s_k$. Then there exists $\varepsilon > 0$ such that G has at most $n-k$ knots on the interval $(s_k - \varepsilon, \infty)$. Since all zeros of G' are simple, on the interval $(s_k - \varepsilon, \infty)$ the function G' changes sign in each of the points s_l, $l = k, \ldots, n$, totally $n-k+1$ times. Since $G'(\infty) = 0$, G'' has at least $n-k+1$ sign changes on $(s_k - \varepsilon, \infty)$. Continuing in a similar way, we obtain that $G^{(r)}$ has at least $n-k+1$ sign changes on $(s_k - \varepsilon, \infty)$. However, this is impossible, since it has only at most $n-k$ knots on this interval. The lemma is proved.

Lemma 12. *If n is odd and*

$$\varliminf_{t\to\infty} \frac{f_-(t) - f_-(\infty)}{P_r(t)} = 0, \tag{21}$$

then there exists $C > 0$ and a perfect g-spline $G \in \Gamma^r_{n-1,g}[0,\infty)$ with exactly $n-1$ knots that has n points of oscillation between $-Cf_-$ and Cf_+ and such that

$$G(\infty) = -C \cdot f_-(\infty).$$

For each $\alpha > 0$ denote by G_α a perfect g-spline built according to Lemma 6. Let $s_1^\alpha < \ldots < s_{n+1}^\alpha$ be its oscillation points. Since n is odd, G_α is increasing on the interval (s^α_{n+1}, ∞). Let $\alpha \to \infty$ and $s_1 \leq \ldots \leq s_{n+1}$ be the limits of the sequences $s_1^\alpha, \ldots, s^\alpha_{n+1}$. Let G be a limiting g-spline in the sequence G_α.

Let z_n^α denote the last zero of G'_α and $z_n = \lim\limits_{\alpha\to\infty} z_n^\alpha$. We prove that

$$z_n = \infty. \tag{22}$$

Assume the contrary, let $z_n < \infty$. Then G increases on (z_n, ∞), and hence $G(t) = G(\infty) - |P_r(t)|$ for all big enough t, since its restriction to the interval (M, ∞) with enough big M is a primitive of the order r of either g or $-g$.

From (21) it follows, that there exists arbitrarily large t such that

$$\frac{f_-(t) - f_-(\infty)}{|P_r(t)|} < \frac{1}{C},$$

where $C := \|G\|_{C[0,\infty),f_-,f_+}$. Hence we obtain

$$\begin{aligned} G(t) &= G(\infty) - |P_r(t)| = -Cf_-(\infty) - |P_r(t)| \\ &< C(-f_-(\infty) - f_-(t) + f_-(\infty)) = -Cf_-(t), \end{aligned}$$

which is impossible. Thus (22) holds. Hence $s_{n+1} = \infty$.

Since by Lemma 5 for all $\alpha > 0$

$$\begin{aligned} &|G_\alpha(s_{n+1}^\alpha) - G_\alpha(s_n^\alpha)| = |G_\alpha(s_{n+1}^\alpha) - G_\alpha(\infty) - (G_\alpha(s_n^\alpha) - G_\alpha(\infty))| \\ &\le |G_\alpha(s_{n+1}^\alpha) - G_\alpha(\infty)| + |G_\alpha(s_n^\alpha) - G_\alpha(\infty)| \le |P_r(s_{n+1}^\alpha)| + |P_r(s_n^\alpha)|, \end{aligned}$$

then s_n^α can not become arbitrarily large, due to (11). Hence $s_n < \infty$.

From Lemma 11 and (22) it follows, that G can not have n knots. From the other hand, since G has n points of oscillation, it can not have less than $n-1$ knots. Hence G has exactly $n-1$ knots.

Thus G is a desired perfect g-spline and the lemma is proved.

Similarly to the previous lemma, one can prove the following result.

Lemma 13. *If n is even, $n \ge 2$ and*

$$\lim_{t\to\infty} \frac{f_+(t) - f_+(\infty)}{P_r(t)} = 0,$$

then there exists $C > 0$ and a perfect g-spline $G \in \Gamma^r_{n-1,g}[0,\infty)$ with exactly $n-1$ knots that has n points of oscillation between $-Cf_-$ and Cf_+ and such that

$$G(\infty) = C \cdot f_+(\infty).$$

3.4. **Relations between the functions C_n and C_{n-1}.** In Definition 6 we introduced the functions $C_n\colon (0,\infty) \to (0,\infty)$ for all $n \in \mathbb{N}$. Using the function P_r from Definition 5, for convenience we define a (constant) function C_0.

Definition 7. *For all $\alpha > 0$ set*

$$C_0 = C_{r,0,g,f_-,f_+}(\alpha) := \inf_{a\in\mathbb{R}} \|P_r(\cdot) + a\|_{C[0,\infty),f_-,f_+}.$$

Lemma 14. *For all $n \in \mathbb{N}$ and $\alpha, \beta > 0$ one has $C_n(\alpha) \le C_{n-1}(\beta)$.*

Assume the contrary, let $C_n(\alpha) > C_{n-1}(\beta)$. Let G_α and G_β be the maximally oscillating splines built according to Lemma 6 with n and $n-1$ knots respectively; if $n = 1$, then set $G_\beta := P_r + a$, where a is chosen in such a way that $\|G_\beta\|_{C[0,\infty),f_-,f_+} = C_0$. Set $\Delta := G_\alpha - G_\beta$.

Let $s_1 < \ldots < s_{n+1}$ be the oscillation points of G_α and let for definiteness $n = 2k-1$ be odd, $k \in \mathbb{N}$. Then for each $m = 1, \ldots, 2k-1$, there exists $s_m^1 \in (s_m, s_{m+1})$ such that $(-1)^m \Delta'(s_m^1) > 0$. Moreover, G_α increases on (s_{2k}, ∞) and G_β decreases on (M, ∞), where M is the last oscillation point of G_β. Hence there exists $s_{2k}^1 > s_{2k-1}^1$ such that $\Delta'(s_{2k}^1) > 0$. Thus Δ' has $2k-1$ sign changes and moreover, $\Delta'(\infty) = 0$. Hence Δ'' has $2k-1$ sign changes and moreover, $\Delta''(\infty) = 0$. Continuing the same way, we obtain that $\Delta^{(r)}$ has at least $2k-1 = n$

sign changes. However, $\Delta^{(r)}$ can change sign only in the knots of G_β, hence not more than $n-1$ times. Contradiction. The lemma is proved.

Lemma 15. *If*

$$\varliminf_{t\to\infty}\frac{f_\pm(t)-f_\pm(\infty)}{P_r(t)}=0, \tag{23}$$

then for all $n\in\mathbb{N}$,

$$\lim_{\alpha\to+0}C_{2n-1}(\alpha)=\lim_{\alpha\to+0}C_{2n}(\alpha)$$

and

$$\lim_{\alpha\to+\infty}C_{2n-1}(\alpha)=\lim_{\alpha\to+\infty}C_{2n-2}(\alpha).$$

The first equality follows from Lemmas 14 and 13; the second one follows from Lemmas 14 and 12. The lemma is proved.

3.5. **Perfect g-splines with given norms.**

Theorem 1. *Let Assumptions 1 and 2 and condition* (23) *hold. For all $r\in\mathbb{N}$ and $C\in\big(0,C_{r,0,g,f_-,f_+}\big]$ there exist $n\in\mathbb{Z}_+$ and a perfect g-spline $G_C\in\Gamma^r_{n,g}[0,\infty)$ that has exactly n knots and $n+1$ oscillation points between $-Cf_-$ and Cf_+.*

From Lemmas 9, 14 and 15 it follows that the splines $G(n,\alpha)$ from Lemma 6 can be numbered with a single real parameter, so that the norm $\|G(n,\alpha)\|_{C[0,\infty),f_-,f_+}$ continuously depends on it. Moreover, since

$$\bigvee_0^\infty G(n,\alpha)=\int_0^\infty|G'(n,\alpha,t)|dt,$$

then due to Lemma 5, all variations of the splines $G(n,\alpha)$ are bounded by some number independent of n and α. Assumption 1 now implies that

$$\|G(n,0)\|_{C[0,\infty),f_-,f_+}\to 0$$

as $n\to\infty$. Hence the functional $\|G(n,\alpha)\|_{C[0,\infty),f_-,f_+}$ takes all values from the interval $\big(0,C_{r,0,g,f_-,f_+}\big]$. The theorem is proved.

4. Extremal properties of maximally oscillating g-splines

4.1. **Discussion of known results.**

Let numbers $a>0$ and $r\in\mathbb{N}$ be fixed. Consider the infimum

$$\inf_{g_r}\|g_r\|_{C[0,a],f} \tag{24}$$

over all r-th order primitives on $[0,a]$ of the function g. Let the infimum in (24) be attained on the function $G_r(a,\cdot)$.

Definition 8. *Set $\varphi_{r,f}(a):=\|G_r(a)\|_{C[0,a],f}$, $a>0$.*

It is clear, that $\varphi_{r,f}$ is a non-decreasing function of a and in the case when $g \equiv 1$ one has $\varphi_{r,f}(\infty) := \lim\limits_{a\to+\infty} \varphi_{r,f}(a) = \infty$. However, when g is non-constant, it can happen that $\varphi_{r,f}(\infty)$ is finite. The following theorem was proved in [1].

Theorem 2. *For $r \in \mathbb{N}$, $\varphi_{r,f}(\infty) < \infty$ if and only if conditions* (12) *and* (13) *hold and*

$$\sup_{t\in[0,\infty)} \frac{|P_r(t)|}{f(t)} < \infty. \tag{25}$$

In [1] the following information about the modulus of continuity ω was found. In the case, when $\varphi_{r,f}(\infty) = \infty$, the values $\omega(\delta, D^k)$ were characterized for all $\delta > 0$. In the case, when $\varphi_{r,f}(\infty) < \infty$, the values $\omega(\delta, D^k)$ were characterized only for a non-increasing sequence of numbers $\{\delta_{r,n}\}_{n=0}^{\infty}$.

Note, that Assumptions 1 and 2 imply that $\varphi_{r,f}(\infty) < \infty$.

4.2. Characterization of the modulus of continuity.

Theorem 3. *Let $r \in \mathbb{N}$, $r \geq 2$ and $f, g \in C[0,\infty)$ be such that Assumptions 1 and 2 hold. Assume also that*

$$\varliminf_{t\to\infty} \frac{f(t) - f(\infty)}{P_r(t)} = 0.$$

Then for all $k = 1, \dots, r-1$,

$$\omega(W^r_{f,g}[0,\infty), D^k, \delta) = \begin{cases} |P_{r-k}(0)|, & \delta \geq C_{r,0,g,f,f}, \\ |G_\delta^{(k)}(0)|, & \delta \in (0, C_{r,0,g,f,f}), \end{cases}$$

where G_δ is a perfect g-spline from Theorem 1 with $f_- = f_+ =: f$; P_r is defined in Definition 5 and $C_{r,0,g,f,f}$ is defined in Definition 7.

Let $x \in W^r_{f,g}[0,\infty)$ and $\|x\|_{C[0,\infty);f} = \delta$.

Then $x^{(k)}(\infty) = 0$ and Lemma 5 implies that $\|x^{(k)}\| \leq |P_{r-k}(0)|$. This proves the theorem in the case when $\delta \geq C_{r,0,g,f,f}$.

Let now $\delta \in (0, C_{r,0,g,f,f})$ and assume that $\|x^{(k)}\| > |G_\delta^{(k)}(0)|$. Since f and g are non-increasing, for arbitrary $\alpha \geq 0$, $x(\cdot + \alpha) \in W^r_{f,g}[0,\infty)$, and $\|x\|_{C[0,\infty);f} \geq \|x(\cdot+\alpha)\|_{C[0,\infty);f}$. Hence without loss of generality we can assume that $\|x^{(k)}\| = |x^{(k)}(0)|$. There exists $\varepsilon > 0$ such that $(1-\varepsilon)|x^{(k)}(0)| > |G_\delta^{(k)}(0)|$. Set $\Delta = G_\delta - (1-\varepsilon)x$.

We prove the case $k = 1$ first. Multiplying the function x by -1, if needed, we can assume that $x'(0) = -\|x'\|$. Hence

$$\Delta'(0) > 0. \tag{26}$$

Assume G_δ has n knots and let $s_1 < \dots < s_{n+1}$ be its oscillation points. Then for all $k = 1, \dots, n+1$, $\operatorname{sgn}\Delta(s_k) = \operatorname{sgn} G_\delta(s_k)$, and

$$\Delta(s_1) > 0. \tag{27}$$

Hence Δ has at least n separated zeros on (s_1, s_{n+1}). Taking into account (26) and (27), we obtain that Δ' has at least n separated zeros on $(0, s_{n+1})$, moreover, $\Delta'(\infty) = 0$. Hence Δ'' has at least n sign changes on $(0,\infty)$. Moreover, $\Delta''(\infty) = 0$. Continuing the same way, we obtain that $\Delta^{(r)}$ has at least n sign changes. Since $(1-\varepsilon)|x^{(r)}(t)| < g(t)$ for almost all $t \geq 0$, the function $\Delta^{(r)}$ can change its sign only in the knots of the spline G_δ. Hence $\Delta^{(m)}$ has exactly n sign changes, $m = 1, \dots, r$. This implies that $\mathrm{sgn}\Delta^{(m)}(0) = -\mathrm{sgn}\Delta^{(m+1)}(0)$, $m = 1, \dots, r-1$, in particular $\mathrm{sgn}\Delta^{(r)}(0) = (-1)^{r+1}$, due to (26). From the other hand, $\mathrm{sgn}\Delta^{(r)}(0) = \mathrm{sgn}G_\delta^{(r)}(0) = (-1)^r$, since G_δ decreases on $[0, s_2]$. The obtained contradiction proves the theorem in the case $k = 1$.

The proof of the theorem in the case of arbitrary k is similar to the case $k = 1$. Extra sign change will be obtained on the level of k-th derivative.

The theorem is proved.

References

1. V. Babenko and O. Kovalenko, *On modulus of continuity of differentiation operator on weighted sobolev classes*, Journal of Inequalities and Applications **2015:295** (2015).
2. P. L. Chebyshev, *Théorie des mécanismes connus sous le nom de parallélogrammes*, Mémoires des Savants étrangers présentés á l'Académie de Saint-Pétersbourg **7** (1854), 539–586.
3. J. Hadamard, *Sur le module maximum d'une fonction et de ses derivees*, C. R. Soc. Math. France **41** (1914), 68–72.
4. G. H. Hardy and J. E. Littlewood, *Contribution to the arithmetic theory of series*, Proc. London Math. Soc. **11** (1913), no. 2, 411–478.
5. S. Karlin, S. A. Micchelli, A. Pinkus, and L. J. Schoenberg, *Spline functions and approximation theory*, Acad. Press., New York, 1976.
6. E. Landau, *Einige ungleichungen fur zweimal differenzierbare funktion*, Proc. London Math. Soc. **13** (1913), 43–49.
7. A. Pinkus, *Some extremal properties of perfect splines and the pointwise landau problem on the finite interval*, JAT **23** (1978), no. 1, 37–64.
8. V. M. Tihomirov, *Best methods of approximation and interpolation of differentiable functions in the space* $c_{[-1,1]}$, Mathematics of the USSR-Sbornik **9** (1969), no. 2, 275–289.

Oleg Kovalenko, Department of Mathematics and Mechanics, Oles Honchar Dnipro National University, Gagarina pr., 72, Dnipro, 49010, Ukraine

E-mail address: `olegkovalenko90@gmail.com`