Limits on the Universal Method for Matrix Multiplication

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December 21, 2018

Abstract

In a recent work, Alman and Vassilevska Williams [AW18b] proved limitations on designing matrix multiplication algorithms using the Galactic method applied to many tensors of interest, including the family of Coppersmith-Winograd tensors. In this note, we extend all their lower bounds to the more powerful Universal method, which can use any degeneration of a tensor instead of just monomial degenerations. Our main result is that the Universal method applied to any Coppersmith-Winograd tensor cannot yield a bound on ω better than 2.16805. We also study a slightly restricted form of Strassen's Laser method and prove that it is optimal: when it applies to a tensor T, it achieves $\omega = 2$ if and only if it is possible for the Universal method applied to T to achieve $\omega = 2$.

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1 Introduction

In [AW18b], it is shown that there is a universal constant $c > 2$ such that for all positive integers q, the Galactic method applied to the Coppersmith-Winograd tensor CW_q cannot yield a bound on ω better than c. In this note, we improve this result:

Theorem 1.1. $\omega_u(CW_q) \geq 2.16805$ *for all* q, *i.e. the Universal method applied to* CW_q *cannot yield a bound on* ω *better than* 2.16805.

In other words, we extend the main result of [AW18b] from the Galactic method to the *Universal* method, and we also further optimize our lower bound to show that it is at least 2.16805. In the Universal method, one may take a tensor T of known asymptotic rank $\tilde{R}(T)$, and degenerate any power $T^{\otimes n}$ in any way into a disjoint sum $\bigoplus_i \langle a_i, b_i, c_i \rangle$ of matrix multiplication tensors, yielding the bound $\sum_i (a_i b_i c_i)^{\omega/3} \leq \tilde{R}(T)^n$. The value $\omega_u(T)$ is the limsup, over all n and all such degenerations, of the resulting bound on ω. By comparison, the Galactic method only allowed for *monomial* degenerations rather than arbitrary degeneration.

The proof of [AW18b] works by introducing a suite of tools for giving upper bounds on $\tilde{I}(T)$, the *asymptotic independence number* (sometimes also called the 'galactic subrank' or the 'monomial degeneration subrank') for many tensors T with certain structure. Our main new idea is to show that those same tools also upper bound $\tilde{S}(T)$, the *asymptotic slice rank* of T. We then apply the known connection, that upper bounds on the slice rank of T yield lower bounds on $\omega_u(T)$. Since we show how to replace $\tilde{I}$ with $\tilde{S}$ in all the tools in [AW18b], it follows that all the lower bounds in [AW18b], including those for tensors other than CW_q, hold with ω_g replaced by the more powerful ω_u.

We briefly note that our lower bound of $2.16805 > 2 + \frac{1}{6}$ may be significant when compared to the recent algorithm of Cohen, Lee and Song [CLS18] which solves n-variable linear programs in time about $O(n^\omega + n^{2+1/6})$.

We will also give a wide class of tensors T for which our tools are tight, meaning they not only give an upper bound on $\tilde{S}(T)$, but they also give a matching lower bound. Hence, for these tensors, no better lower bound is possible by arguing only about $\tilde{S}(T)$. Interestingly, the class of tensors we prove this for is the set of tensors to which a slightly restricted version of the Laser method (as used by [Str86, CW90, Wil12, LG14]) applies, which includes the tensors CW_q, cw_q, and T_q^{lower}; see Definition 5.1 for the precise definition. For these tensors T, we show that the Laser method constructs a degeneration from $T^{\otimes n}$ to an independent tensor of size $\Lambda^{n-o(n)}$, where Λ is the upper bound on $\tilde{S}(T)$ implied by our Theorem 3.4. Combined, these imply that $\tilde{S}(T) = \Lambda$.

This has an intriguing consequence: If T is any tensor to which the restricted Laser method applies, and the Laser method does not yield $\omega = 2$ when applied to T, then in fact $\omega_u(T) > 2$, and even the substantially more general Universal method applied to T cannot yield $\omega = 2$. For example, the fact that Coppersmith-Winograd [CW90] applied the Laser method to the tensor CW_q and achieved an upper bound greater than 2 on ω *implies that* $\omega_u(CW_q) > 2$, and no *arbitrary degeneration* of powers of CW_q can yield $\omega = 2$. It is known [CW90, Wil12, LG14] that applying the Laser method to higher and higher powers of a tensor can improve the resulting upper bound on ω, but our result shows that if the Laser method applied to the first power of the tensor did not yield $\omega = 2$, then this sequence of Laser method applications (which is a special case of the Universal method) must converge to a value greater than 2 as well. Our result also generalizes the result of Kleinberg, Speyer and Sawin [KSS18], where it was shown that (what can be seen as) the Laser method achieves a tight lower bound on $\tilde{S}(T_q^{lower})$, matching the upper bound of Blasiak et al. [BCC+17].

In Section 2 we introduce the relevant notions related to slice rank. In Section 3 we present the proofs of our new strengthened lower bounding tools for asymptotic slice rank. In Section 4 we apply these tools to a number of tensors of interest including CW_q. Finally, in Section 5, we define and discuss the optimality of the restricted Laser method.

Concurrent Work On 12/17, Christandl, Vrana and Zuiddam posted an arXiv preprint [CVZ18a] in which they independently proved some of the same results as us, including Theorem 1.1. Our main results in this note were initially announced in a recorded talk at the Simons Institute for the Theory of Computing [AW18c] on 12/4.

Although we achieve the same upper bounds on $\tilde{S}(T)$ and hence $\omega_u(T)$ for a number of tensors, our techniques seem different: we use simple combinatorial tools generalizing those from our prior work [AW18b], while their bounds use the seemingly more complicated machinery of Strassen's asymptotic spectrum of tensors [Str86]. We also provide matching lower bounds for $\tilde{S}(CW_q)$, $\tilde{S}(cw_q)$, and $\tilde{S}(T_q)$ which they do not. Our techniques also extend to bounding the 'value' $V_\tau(T')$ of subtensors T' which may arise in the analysis of $\omega_u(T)$ for a tensor $T \supset T'$ (see Section 4.4 where we analyze the value of the infamous t_{112} tensor which appears as a block in $CW_q^{\otimes 2}$), whereas their 'irreversibility' approach only seems to bound $\omega_u(T')$ itself. Our results about the optimailty of the Laser method are also, as far as we know, entirely new.

2 Preliminaries

We assume familiarity with the notions introduced in [AW18b], and in particular, we will use the same notation introduced in [AW18b, Section 3]. The main new notation we will need relates to the slice rank of tensors.

We say a tensor T over X, Y, Z has *x-rank* 1 if it is of the form

$$T = \left(\sum_{x\in X} \alpha_x \cdot x\right) \otimes \left(\sum_{y\in Y}\sum_{z\in Z} \beta_{y,z} \cdot y \otimes z\right) = \sum_{x\in X, y\in Y, z\in Z} \alpha_x \beta_{y,z} \cdot xyz$$

for some choices of the α and β coefficients over the base field. More generally, the x-rank of T, denoted $\mathrm{S_x}(T)$, is the minimum number of tensors of x-rank 1 whose sum is T. We can similarly define the y-rank, $\mathrm{S_y}$, and the z-rank, $\mathrm{S_z}$. Then, the *slice rank* of T, denoted $\mathrm{S}(T)$, is the minimum k such that there are tensors T_X, T_Y and T_Z with $T = T_X + T_Y + T_Z$ and $\mathrm{S_x}(T_X) + \mathrm{S_y}(T_Y) + \mathrm{S_z}(T_Z) = k$.

Unlike tensor rank, the slice-rank is not submultiplicative in general, i.e. there are tensors A and B such that $\mathrm{S}(A \otimes B) > \mathrm{S}(A) \cdot \mathrm{S}(B)$. For instance, it is not hard to see that $\mathrm{S}(CW_5) = 3$, but since it is known [Wil12, LG14] that $\omega_u(CW_5) \leq 2.373$, it follows that $\mathrm{S}(CW_q^{\otimes n}) \geq 7^{n\cdot 2.373/3 - o(n)} \geq 4.66^{n-o(n)}$. We are thus interested in the *asymptotic slice rank*, $\tilde{S}(T)$, of tensors T, defined as

$$\tilde{S}(T) := \limsup_{n\in\mathbb{N}} [\mathrm{S}(T^{\otimes n})]^{1/n}.$$

We note a few simple properties of slice rank which will be helpful in our proofs:

Lemma 2.1. *For tensors A and B:*

(1) $\mathrm{S}(A) \leq \mathrm{S_x}(A) \leq \mathrm{R}(A)$,

(2) $\mathrm{S_x}(A \otimes B) \leq \mathrm{S_x}(A) \cdot \mathrm{S_x}(B)$,

(3) $\mathrm{S}(A+B) \le \mathrm{S}(A) + \mathrm{S}(B)$*, and* $\mathrm{S_x}(A+B) \le \mathrm{S_x}(A) + \mathrm{S_x}(B)$*, and*

(4) $\mathrm{S}(A \otimes B) \le \mathrm{S}(A) \cdot \max\{\mathrm{S_x}(B), \mathrm{S_y}(B), \mathrm{S_z}(B)\}$,

(5) If A is a tensor over X, Y, Z, then $\mathrm{S_x}(T) \le |X|$ *and hence* $\mathrm{S}(T) \le \min\{|X|, |Y|, |Z|\}$.

Proof. (1) and (2) are straightforward. (3) follows since the sum of the slice rank (resp. x-rank) expressions for A and for B gives a slice rank (resp. x-rank) expression for $A+B$. To prove (4), let $m = \max\{\mathrm{S_x}(B), \mathrm{S_y}(B), \mathrm{S_z}(B)\}$, and note that if $A = A_X + A_Y + A_Z$ such that $\mathrm{S_x}(A_X) + \mathrm{S_y}(A_Y) + \mathrm{S_z}(A_Z) = \mathrm{S}(A)$, then

$$A \otimes B = A_X \otimes B + A_Y \otimes B + A_Z \otimes B,$$

and so

$$\begin{aligned} \mathrm{S}(A \otimes B) &\le \mathrm{S}(A_X \otimes B) + \mathrm{S}(A_Y \otimes B) + \mathrm{S}(A_Z \otimes B) \\ &\le \mathrm{S_x}(A_X \otimes B) + \mathrm{S_y}(A_Y \otimes B) + \mathrm{S_z}(A_Z \otimes B) \\ &\le \mathrm{S_x}(A_X)\,\mathrm{S_x}(B) + \mathrm{S_y}(A_Y)\,\mathrm{S_y}(B) + \mathrm{S_z}(A_Z)\,\mathrm{S_z}(B) \\ &\le \mathrm{S_x}(A_X)m + \mathrm{S_y}(A_Y)m + \mathrm{S_z}(A_Z)m = \mathrm{S}(A) \cdot m. \end{aligned}$$

Finally, (5) follows since, for instance, any tensor with one only x-variable has x-rank 1. □

Asymptotic slice rank is interesting in the context of matrix multiplication algorithms because of the following facts.

Proposition 2.2 ([TS16] Corollary 2)**.** *If A and B are tensors such that A has a degeneration to B, then* $\mathrm{S}(B) \le \mathrm{S}(A)$*, and hence* $\tilde{S}(B) \le \tilde{S}(A)$.

Proposition 2.3 ([Tao16] Lemma 1; see also [BCC+17] Lemma 4.7)**.** *For any positive integer q, we have* $\mathrm{S}(\langle q \rangle) = \tilde{S}(\langle q \rangle) = q$*, where $\langle q \rangle$ is the independent tensor of size q.*

Proposition 2.4 ([Str86] Theorem 4; see also [AW18b] Lemma 4.2)**.** *For any positive integers a, b, c, the matrix multiplication tensor $\langle a, b, c \rangle$ has a (monomial) degeneration to an independent tensor of size at least* $0.75 \cdot abc/\max\{a, b, c\}$.

Corollary 2.5. *For any positive integers a, b, c, we have* $\tilde{S}(\langle a, b, c \rangle) = abc/\max\{a, b, c\}$.

Proof. Assume without loss of generality that $c \ge a, b$. For any positive integer n, we have that $\langle a, b, c \rangle^{\otimes n} \simeq \langle a^n, b^n, c^n \rangle$ has a degeneration to an independent tensor of size at least $0.75 \cdot a^n b^n$, meaning $S(\langle a, b, c \rangle^{\otimes n}) \ge 0.75 \cdot a^n b^n$ and hence $\tilde{S}(\langle a, b, c \rangle) \ge (0.75)^{1/n} ab$, which means $\tilde{S}(\langle a, b, c \rangle) \ge ab$. Meanwhile, $\langle a, b, c \rangle$ has ab different x-variables, so it must have $\mathrm{S_x}(\langle a, b, c \rangle) \le ab$ and more generally, $\mathrm{S}(\langle a, b, c \rangle^{\otimes n}) \le \mathrm{S_x}(\langle a, b, c \rangle^{\otimes n}) \le (ab)^n$, which means $\tilde{S}(\langle a, b, c \rangle) \le ab$. □

To summarize: we know that degenerations cannot increase asymptotic slice rank, and that matrix multiplication tensors have a high asymptotic slice rank. Hence, if T is a tensor such that $\omega_u(T)$ is 'small', meaning a power of T has a degeneration to a disjoint sum of many large matrix multiplication tensors, then T itself must have 'large' asymptotic slice rank. This can be formalized identically to [AW18b, Theorem 4.1 and Corollary 4.3] to show:

Theorem 2.6. *For any concise tensor T,*

$$\tilde{S}(T) \ge \tilde{R}(T)^{\frac{6}{\omega_g(T)} - 2}.$$

Corollary 2.7. *For any tensor T, if $\omega_g(T) = 2$, then $\tilde{S}(T) = \tilde{R}(T)$. Moreover, for every constant $s < 1$, there is a constant $w > 2$ such that every tensor T with $\tilde{I}(T) \leq \tilde{R}(T)^s$ must have $\omega_g(T) \geq w$.*

Almost all the tensors we consider in this note are *variable-symmetric* tensors, and for these tensors T we can get a better lower bound on $\omega_u(T)$ from an upper bound on $\tilde{S}(T)$. We say that a tensor T over X, Y, Z is variable-symmetric if T, as a tensor over X, Y, Z, is isomorphic to T as a tensor over Y, Z, X, or in other words, replacing the coefficient T_{ijk} with T_{jki} fr all i, j, k results in a tensor isomorphic to T.

Theorem 2.8. *For a variable-symmetric tensor T we have $\omega_u(T) \geq 2\log(\tilde{R}(T))/\log(\tilde{S}(T))$.*

Proof. As in the proof of [AW18b, Theorem 4.1], by definition of ω_u, we know that for every $\delta > 0$, there is a positive integer n such that $T^{\otimes n}$ has a degeneration to $F \odot \langle a, b, c\rangle$ for integers F, a, b, c such that $\omega_u(T)^{1+\delta} \geq 3\log(\tilde{R}(T)^n/F)/\log(abc)$. In fact, since T is symmetric, we know $T^{\otimes n}$ also has a degeneration to $F \odot \langle b, c, a\rangle$ and to $F \odot \langle c, a, b\rangle$, and so $T^{\otimes 3n}$ has a degeneration to $F^3 \odot \langle abc, abc, abc\rangle$. As above, it follows that $\tilde{S}(T^{\otimes 3n}) \geq \tilde{S}(F^3 \odot \langle abc, abc, abc\rangle) = F^3 \cdot (abc)^2$. Rearranging, we see

$$abc \leq \tilde{S}(T)^{3n/2}/F^{3/2}.$$

Hence,

$$\omega_u(T)^{1+\delta} \geq 3\frac{\log(\tilde{R}(T)^n/F)}{\log(abc)} \geq 3\frac{\log(\tilde{R}(T)^n/F)}{\log(\tilde{S}(T)^{3n/2}/F^{3/2})} = 2\frac{\log(\tilde{R}(T)) - \frac{1}{n}\log(F)}{\log(\tilde{S}(T)) - \frac{1}{n}\log(F)} \geq 2\frac{\log(\tilde{R}(T))}{\log(\tilde{S}(T))},$$

where the last step follows because $\tilde{R}(T) \geq \tilde{S}(T)$ and so adding the same quantity to both the numerator and denominator cannot increase the value of the fraction. This holds for all $\delta > 0$ and hence implies the desired result. □

3 Combinatorial Tools for Asymptotic Slice Rank Upper Bounds

We now move on to proving that the three main tools from [AW18b] for upper bounding $\tilde{I}$ also give the same bounds on $\tilde{S}$. Since $\tilde{I}(T) \leq \tilde{S}(T)$ for any tensor T, our new tools are more general. By using properties of slice rank, we are often able to generalize the tools to apply to a larger set of tensors as well.

3.1 Generalization of [AW18b, Theorem 5.3]

We begin by recalling two definitions. If T is a tensor over X, Y, Z, we say that X, Y, Z are *minimal* for T if each variable is used in at least one nonzero term of T. In other words, there is no variable in X, Y, or Z which, when zeroed out, leaves T unchanged. If X, Y, Z are minimal for T, then the *measure* of T, denoted $\mu(T)$, is given by $\mu(T) := |X| \cdot |Y| \cdot |Z|$. We state two simple facts about μ:

Fact 3.1. *For tensors A and B,*

- *$\mu(A \otimes B) \leq \mu(A) \cdot \mu(B)$, and*
- *if X, Y, Z are minimal for A then $\mathrm{S}(A) \leq \min\{|X|, |Y|, |Z|\} \leq \mu(A)^{1/3}$.*

Theorem 3.2. *Suppose T is a tensor, and $T_1, \ldots, T_k$ are tensors with $T = T_1 + \cdots + T_k$. Then, $\tilde{S}(T) \leq \sum_{i=1}^{k}(\mu(T_i))^{1/3}$.*

Proof. Note that

$$T^{\otimes n} = \sum_{(P_1,\dots,P_n)\in\{T_1,\dots,T_k\}^n} P_1 \otimes \cdots \otimes P_n.$$

It follows that

$$\begin{aligned}
\mathrm{S}(T^{\otimes n}) &\le \sum_{(P_1,\dots,P_n)\in\{T_1,\dots,T_k\}^n} \mathrm{S}(P_1 \otimes \cdots \otimes P_n) \\
&\le \sum_{(P_1,\dots,P_n)\in\{T_1,\dots,T_k\}^n} \mu(P_1 \otimes \cdots \otimes P_n)^{1/3} \\
&\le \sum_{(P_1,\dots,P_n)\in\{T_1,\dots,T_k\}^n} (\mu(P_1)\cdot\mu(P_2)\cdots\mu(P_n))^{1/3} \\
&= (\mu(T_1)^{1/3} + \cdots + \mu(T_k)^{1/3})^n,
\end{aligned}$$

which implies as desired that $\tilde{S}(T) \le (\mu(T_1)^{1/3} + \cdots + \mu(T_k)^{1/3})$. ∎

Remark 3.3. *[AW18b, Theorem 5.3], in addition to bounding $\tilde{I}$ instead of $\tilde{S}$, also required that $T = T_1 + \cdots + T_k$ be a* partition *of the terms of T; here we are allowed any tensor sum.*

3.2 Generalization of [AW18b, Theorem 5.2]

This tool will be the most important in upper bounding the asymptotic slice rank of many tensors of interest. We begin with a number of definitions regarding probability distributions and block partitions of tensors.

Suppose T is a tensor minimal over X, Y, Z, and let $X = X_1 \cup \cdots \cup X_{k_X}$, $Y = Y_1 \cup \cdots \cup Y_{k_Y}$, $Z = Z_1 \cup \cdots \cup Z_{k_Z}$ be partitions of the three variable sets. For $(i,j,k) \in [k_X] \times [k_Y] \times [k_Z]$, let T_{ijk} be T restricted to X_i, Y_j, Z_k, and let $L = \{T_{ijk} \mid (i,j,k) \in [k_X] \times [k_Y] \times [k_Z], T_{ijk} \neq 0\}$. For $i \in [k_X]$ let $L_{X_i} = L \cap (T_{ijk} \mid (j,k) \in [k_Y] \times [k_Z])$, and define similarly L_{Y_j} and L_{Z_k}.

For any probability distribution $p : L \to [0,1]$, we define a few things. For $i \in [k_X]$, let $p(X_i) := \sum_{T_{ijk}\in L_{X_i}} p(T_{ijk})$, and similarly $p(Y_j)$ and $p(Z_k)$. Define

$$p_X := \prod_{i\in[k_X]} \left(\frac{|X_i|}{p(X_i)}\right)^{p(X_i)},$$

and p_Y and p_Z similarly. Then,

Theorem 3.4. *For any tensor T whose variable sets are partitioned as described above,*

$$\tilde{S}(T) \le \limsup_p \min\{p_X, p_Y, p_Z\}.$$

Proof. For any positive integer n, we can write

$$T^{\otimes n} = \sum_{(P_1,\dots,P_n)\in L^n} P_1 \otimes \cdots \otimes P_n.$$

For a given $(P_1, \dots, P_n) \in L^n$, let $dist(P_1, \dots, P_n)$ be the probability distribution on L which results from picking a uniformly random $\alpha \in [n]$ and outputting P_α. For a probability distribution $p : L \to [0,1]$, define $L_{n,p} := \{(P_1, \dots, P_n) \in L^n \mid dist(P_1, \dots, P_n) = p\}$. Note that the number of p for which $L_{n,p}$ is nonempty is only poly(n), since they are the distributions which assign an integer multiple of $1/n$ to each element of L. Let D be the set of these probability distributions.

We can now rearrange:

$$T^{\otimes n} = \sum_{p \in D} \sum_{(P_1,\ldots,P_n) \in L_{n,p}} P_1 \otimes \cdots \otimes P_n.$$

Hence,

$$\begin{aligned}
\mathrm{S}(T^{\otimes n}) &\leq \sum_{p \in D} \mathrm{S}\left(\sum_{(P_1,\ldots,P_n) \in L_{n,p}} P_1 \otimes \cdots \otimes P_n \right) \\
&\leq \mathrm{poly}(n) \cdot \max_{p \in D} \mathrm{S}\left(\sum_{(P_1,\ldots,P_n) \in L_{n,p}} P_1 \otimes \cdots \otimes P_n \right).
\end{aligned}$$

For any probability distribution $p : L \to [0,1]$, let us count the number of x-variables used minimally in $\left(\sum_{(P_1,\ldots,P_n) \in L_{n,p}} P_1 \otimes \cdots \otimes P_n\right)$. These are the tuples of the form $(x_1, \ldots, x_n) \in X^n$ where, for each $i \in [k_X]$, there are exactly $n \cdot p(X_i)$ choices of j for which $x_j \in X_i$. The number of these is[1]

$$\binom{n}{n \cdot p(X_1), n \cdot p(X_2), \ldots, n \cdot p(X_{k_X})} \cdot \prod_{i \in [k_X]} |X_i|^{n \cdot p(X_i)}.$$

This is upper bounded by $p_X^{n+o(n)}$, where p_X is the quantity defined above the Theorem statement. It follows that $\mathrm{S_x}\left(\sum_{(P_1,\ldots,P_n) \in L_{n,p}} P_1 \otimes \cdots \otimes P_n\right) \leq p_X^{n+o(n)}$. We can similarly argue about $\mathrm{S_y}$ and $\mathrm{S_z}$. Hence,

$$\begin{aligned}
\mathrm{S}(T^{\otimes n}) &\leq \mathrm{poly}(n) \cdot \max_{p \in D} \mathrm{S}\left(\sum_{(P_1,\ldots,P_n) \in L_{n,p}} P_1 \otimes \cdots \otimes P_n \right) \\
&\leq \mathrm{poly}(n) \cdot \max_{p \in D} \min\{p_X, p_Y, p_Z\}^{n+o(n)} \\
&\leq \mathrm{poly}(n) \cdot \limsup_{p} \min\{p_X, p_Y, p_Z\}^{n+o(n)}.
\end{aligned}$$

Hence, $\mathrm{S}(T^{\otimes n})^{1/n} \leq \limsup_p \min\{p_X, p_Y, p_Z\}^{1+o_n(1)}$, and the desired result follows. □

Remark 3.5. *In the case when each X_i, Y_j, Z_k contains only one variable, Theorem 3.4 generalizes [AW18b, Theorem 5.2].*

Remark 3.6. *Suppose T is over X, Y, Z with $|Z| = |Y| = |Z| = q$. For any probability distribution p we always have $p_X, p_Y, p_Z \leq q$, and moreover we only have $p_X = q$ when $p(X_i) = |X_i|/q$ for each i. Similar to [AW18b, Corollary 5.1], we get that if no probability distribution p is δ-close (say, in ℓ_1 distance) to having $p(X_i) = |X_i|/q$ for all i, $P(Y_j) = |Y_j|/q$ for all j, and $P(Z_k) = |Z_k|/q$ for all k, simultaneously, then we get $\tilde{S}(T) \leq q^{1-f(\delta)}$ for some increasing function f with $f(\delta) > 0$ for all $\delta > 0$.*

[1] Here, $\binom{n}{p_1 n, p_2 n, \ldots, p_\ell n} = \frac{n!}{(p_1 n)!(p_2 n)! \cdots (p_\ell n)!}$, with each $p_i \in [0,1]$ and $p_1 + \cdots + p_\ell = 1$, is the multinomial coefficient, with the known bound from Stirling's approximation, for fixed p_is, that $\binom{n}{p_1 n, p_2 n, \ldots, p_\ell n} \leq \left(\prod_i p_i^{-p_i}\right)^{n+o(n)}$. Throughout this paper we use the convention that $p_i^{p_i} = 1$ when $p_i = 0$.

3.3 Generalization of [AW18b, Theorem 5.1]

The final remaining tool from [AW18b], their Theorem 5.1, turns out to be unnecessary for proving our tight lower bounds in the next section. Nonetheless, we sketch here how to extend it to give asymptotic slice rank upper bounds as well.

For a tensor T, let $m(T) := \max\{\mathrm{S_x}(T), \mathrm{S_y}(T), \mathrm{S_z}(T)\}$. Recall from Lemma 2.1 that for any two tensors A, B we have $\mathrm{S}(A \otimes B) \leq \mathrm{S}(A) \cdot m(B)$.

In general, for two tensors A and B, even if $\tilde{S}(A)$ and $\tilde{S}(B)$ are 'small', we might still have that $\tilde{S}(A+B)$ is 'large', much larger than $\tilde{S}(A) + \tilde{S}(B)$. Here we show that if, not only is $\tilde{S}(A)$ small, but even $\mathrm{S_x}(A)$ is small, then we can get a decent bound on $\tilde{S}(A+B)$.

Theorem 3.7. *Suppose T, A, B are tensors such that $A + B = T$. Then,*

$$\tilde{S}(T) \leq \left(\frac{m(A)}{(1-p) \cdot \mathrm{S_x}(A)}\right)^{1-p} \cdot \frac{1}{p^p},$$

where $p \in [0,1]$ is given by

$$p := \frac{\log\left(\frac{\mathrm{S_x}(B)}{\tilde{S}(B)}\right)}{\log\left(\frac{m(A)}{\mathrm{S_x}(A)}\right) + \log\left(\frac{\mathrm{S_x}(B)}{\tilde{S}(B)}\right)}.$$

Proof. We begin by, for any integers $n \geq k \geq 0$, giving bounds on $\mathrm{S}(A^{\otimes k} \otimes B^{\otimes(n-k)})$. First, since $\mathrm{S_x}$ is submultiplicative, we have

$$\mathrm{S}(A^{\otimes k} \otimes B^{\otimes(n-k)}) \leq \mathrm{S_x}(A^{\otimes k} \otimes B^{\otimes(n-k)}) \leq \mathrm{S_x}(A)^k \cdot \mathrm{S_x}(B)^{n-k}.$$

Second, from the definition of m, we have

$$\mathrm{S}(A^{\otimes k} \otimes B^{\otimes(n-k)}) \leq m(A^{\otimes k}) \cdot \mathrm{S}(B^{\otimes(n-k)}) \leq m(A)^k \cdot \tilde{S}(B)^{n-k}.$$

It follows that for any positive integer n we have

$$S(T^{\otimes n}) \leq \sum_{k=0}^{n} \binom{n}{k} \cdot \mathrm{S}(A^{\otimes k} \otimes B^{\otimes(n-k)}) \leq \sum_{k=0}^{n} \binom{n}{k} \cdot \min\{\mathrm{S_x}(A)^k \cdot \mathrm{S_x}(B)^{n-k}, m(A)^k \cdot \tilde{S}(B)^{n-k}\}.$$

As in the proof of [AW18b, Theorem 5.1], we can see that the quantity $\binom{n}{k} \cdot \min\{\mathrm{S_x}(A)^k \cdot \mathrm{S_x}(B)^{n-k}, m(A)^k \cdot \tilde{S}(B)^{n-k}\}$ is maximized at $k = pn$, and the result follows. □

Remark 3.8. *This result generalizes [AW18b, Theorem 5.1], no longer requiring that A be the tensor T restricted to a single x-variable. In [AW18b, Theorem 5.1], since A is T restricted to a single x-variable, and we required A to have at most q terms, we got the bounds $\mathrm{S_x}(A) = 1$ and $m(A) \leq q$. Similarly, B had at most $q-1$ different x-variables, so $\mathrm{S_x}(B) \leq q-1$. Substituting those values into Theorem 3.7 yields the original [AW18b, Theorem 5.1] with $\tilde{I}$ replaced by $\tilde{S}$.*

4 Computing the Slice Ranks for Tensors of Interest

In this section, we give slice rank upper bounds for a number of tensors of interest. It will follow from Section 5 that most of the bounds we prove here are tight.

4.1 Generalized Coppersmith-Winograd Tensors

We begin with the generalized CW tensors defined in [AW18b], which for a positive integer q and a permutation $\sigma : [q] \to [q]$ are given by

$$CW_{q,\sigma} := x_0 y_0 z_{q+1} + x_0 y_{q+1} z_0 + x_{q+1} y_0 z_0 + \sum_{i=1}^{q} (x_i y_{\sigma(i)} z_0 + x_i y_0 z_i + x_0 y_i z_i).$$

Just as in [AW18b, Section 7.1], we can see that Theorems 3.2 and 3.4 immediately apply to $CW_{q,\sigma}$ to show that there is a universal constant $\delta > 0$ such that for any q and σ we have $\tilde{S}(CW_{q,\sigma}) \le (q+2)^{1-\delta}$, and hence a universal constant $c > 2$ such that $\omega_u(CW_{q,\sigma}) \ge c$. Indeed, we get the exact same constants as in [AW18b].

That said, we will now prove that $c \ge 2.16805$. In fact, we will show that [AW18b, Theorem 5.2] was already sufficient to show that $\omega_g(CW_{q,\sigma}) \ge 2.16805$, and hence that our Theorem 3.4 shows that $\omega_u(CW_{q,\sigma}) \ge 2.16805$.

We begin by partitioning the variable sets of $CW_{q,\sigma}$, using the notation of Theorem 3.4. Let $X_0 = \{x_0\}$, $X_1 = \{x_1, \ldots, x_q\}$, and $X_2 = \{x_{q+1}\}$, so that $X_0 \cup X_1 \cup X_2$ is a partition of the x-variables of $CW_{q,\sigma}$.[2] Similarly, let $Y_0 = \{y_0\}$, $Y_1 = \{y_1, \ldots, y_q\}$, $Y_2 = \{y_{q+1}\}$, $Z_0 = \{z_0\}$, $Z_1 = \{z_1, \ldots, z_q\}$, and $Z_2 = \{z_{q+1}\}$. With this partition, we can see that $L = \{T_{002}, T_{020}, T_{200}, T_{011}, T_{101}, T_{110}\}$.

Consider any probability distribution p on L. By symmetry, we can assume without loss of generality that $p(T_{002}) = p(T_{020}) = p(T_{200}) = v$ and $p(T_{011}) = p(T_{101}) = p(T_{110}) = 1/3 - v$ for some value $v \in [0, 1/3]$ (see e.g. [CW90]). Applying Theorem 3.4 yields:

$$\tilde{S}(CW_q) \le \max_{v \in [0,1/3]} \frac{q^{2(1/3-v)}}{v^v (2/3 - 2v)^{2/3-2v} (1/3 + v)^{1/3+v}}.$$

In fact, we will see in the next section that this is tight (i.e. the value above is *equal* to $\tilde{S}(CW_q)$, not just an upper bound on it). The values for the first few q can be computed using optimization software as follows:

q	$\tilde{S}(CW_{q,\sigma})$
1	$2.7551\cdots$
2	$3.57165\cdots$
3	$4.34413\cdots$
4	$5.07744\cdots$
5	$5.77629\cdots$
6	$6.44493\cdots$
7	$7.08706\cdots$

Finally, using the lower bound $\tilde{R}(CW_{q,\sigma}) \ge q + 2$ (in fact, it is known that $\tilde{R}(CW_{q,\sigma}) = q + 2$), and the upper bound on $\tilde{S}(CW_{q,\sigma})$ we just proved, we can apply Theorem 2.8 to give lower bounds $\omega_u(CW_{q,\sigma}) \ge 2 \log(\tilde{R}(CW_{q,\sigma}))/\log(\tilde{S}(CW_{q,\sigma})) \ge 2 \log(q+2)/\log(\tilde{S}(CW_{q,\sigma}))$ as follows:

It is not hard to see that the resulting lower bound on $\omega_u(CW_{q,\sigma})$ is increasing with q and is always at least $2.16805\ldots$ (see Appendix A below for a proof), and hence that for any q and any σ we have $\omega_u(CW_{q,\sigma}) \ge 2.16805$ as desired.

[2]The sets of partitions were 1-indexed before, but we 0-index here for notational consistency with past work.

q	Lower Bound on $\omega_u(CW_{q,\sigma})$
1	$2.16805\cdots$
2	$2.17794\cdots$
3	$2.19146\cdots$
4	$2.20550\cdots$
5	$2.21912\cdots$
6	$2.23200\cdots$
7	$2.24404\cdots$

4.2 Generalized Simple Coppersmith-Winograd Tensors

Similar to $CW_{q,\sigma}$, we can define for a positive integer q and a permutation $\sigma : [q] \to [q]$ the simple Coppersmith-Winograd tensor $cw_{q,\sigma}$ given by:

$$cw_{q,\sigma} := \sum_{i=1}^{q} (x_i y_{\sigma(i)} z_0 + x_i y_0 z_i + x_0 y_i z_i).$$

These tensors, when σ is the identity permutation, are well-studied. For instance, Coppersmith and Winograd [CW90] showed that if $\tilde{R}(cw_{2,id}) = 2$ then $\omega = 2$.

We will again give a tight bound on $\tilde{S}(cw_{q,\sigma})$ using Theorem 3.4 combined with the next section. To apply Theorem 3.4, we again pick a partition of the variables. Let $X_0 = \{x_0\}$, $X_1 = \{x_1, \ldots, x_q\}$, $Y_0 = \{y_0\}$, $Y_1 = \{y_1, \ldots, y_q\}$, $Z_0 = \{z_0\}$, and $Z_1 = \{z_1, \ldots, z_q\}$. We can see that $L = \{T_{011}, T_{101}, T_{110}\}$. Similar to before, by symmetry, we need only consider the probability distribution p on L which assigns the same probability, 1/3, to each part. It follows that

$$\tilde{S}(cw_{q,\sigma}) \leq (1/3)^{-1/3}(2/3)^{-2/3} \cdot q^{2/3} = \frac{3}{2^{2/3}} \cdot q^{2/3}.$$

Again, we will see in the next section that this bound is tight. Using the lower bound $\tilde{R}(cw_{q,\sigma}) \geq q+1$, we get the lower bound

$$\omega_u(cw_{q,\sigma}) \geq 2\frac{\log(q+1)}{\log\left(\frac{3}{2^{2/3}} \cdot q^{2/3}\right)}.$$

The first few values are as follows; note that we cannot get a bound better than 2 when $q = 2$ because of Coppersmith and Winograd's remark.

q	Lower Bound on $\omega_u(cw_{q,\sigma})$
1	$2.17795\cdots$
2	2
3	$2.02538\cdots$
4	$2.06244\cdots$
5	$2.09627\cdots$
6	$2.12549\cdots$
7	$2.15064\cdots$

4.3 Cyclic Group Tensors

We next look at two tensors which were studied in [AW18b, Section 7.3]. Recall that, for each positive integer q, we defined the tensor T_q (the group tensor of the cyclic group C_q) as:

$$T_q = \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} x_i y_j z_{i+j \bmod q}.$$

We then defined the lower triangular version of T_q, called T_q^{lower}, as:

$$T_q^{lower} = \sum_{i=0}^{q-1} \sum_{j=0}^{q-1-i} x_i y_j z_{i+j}.$$

While Theorem 3.4 does not give any nontrivial upper bounds on $\tilde{S}(T_q)$, it does give nontrivial upper bounds on $\tilde{S}(T_q^{lower})$, as noted in [AW18b, Section 7.3]. Using computer optimization software, we can compute our lower bound on $\tilde{S}(T_q^{lower})$, using Theorem 3.4 where each partition contains eactly one variable, for the first few values of q:

q	Upper Bound on $\tilde{S}(T_q^{lower})$
2	$1.88988\cdots$
3	$2.75510\cdots$
4	$3.61071\cdots$
5	$4.46157\cdots$

We show in the next section that these numbers are also tight. It is known (see e.g. [AW18a]) that $\tilde{R}(T_q) = \tilde{R}(T_q^{lower}) = q$. Thus we get the following lower bounds on $\omega_u(T_q^{lower}) \geq 2\log(q)/\log(\tilde{S}(T_q^{lower}))$:

q	Lower Bound on $\omega_u(T_q^{lower})$
2	$2.17795\cdots$
3	$2.16805\cdots$
4	$2.15949\cdots$
5	$2.15237\cdots$

These numbers match the lower bounds obtained by [AW18a, BCC+17] in their study of T_q; our Theorem 3.4 can be viewed as an alternate tool to achieve those lower bounds. The bound approaches 2 as $q \to \infty$, as it is known that $\log(\mathrm{S}(T_q))/\log(q) = 1 - o(1)$ as $q \to \infty$. Interestingly, it is shown in [CVZ18b, Theorem 4.16] that T_q^{lower} degenerates to T_q over the field $\mathbb{F}_q$, which implies that our bounds above also hold for T_q over $\mathbb{F}_q$.

4.4 The Value of the Subtensor t_{112} of $CW_q^{\otimes 2}$

A key tensor which arises in applying the Laser method to increasing powers of CW_q, including [CW90, Wil12, LG14, LG12, GU18], is the tensor t_{112} which (for a given positive integer q) is given by

$$t_{112} := \sum_{i=1}^{q} x_{i,0} y_{i,0} z_{0,q+1} + \sum_{k=1}^{q} x_{0,k} y_{0,k} z_{q+1,0} + \sum_{i,k=1}^{q} x_{i,0} y_{0,k} z_{i,k} + \sum_{i,k=1}^{q} x_{0,k} y_{i,0} z_{i,k}.$$

For $\tau \in [2/3, 1]$, let $V_\tau(t_{112})$ denote the 'value' of t_{112} as defined by Coppersmith-Winograd [CW90]. In [CW90] it is shown that

$$V_\tau(t_{112}) \geq 2^{2/3} q^\tau (q^{3\tau} + 2)^{1/3}.$$

This bound has been used in all the subsequent work using CW_q, without improvement. Here we show it is tight and cannot be improved in the case $\tau = 2/3$:

Proposition 4.1. $V_{2/3}(t_{112}) = 2^{2/3} q^{2/3} (q^2 + 2)^{1/3}$.

Before proving Proposition 4.1, we begin with some definitions to help us tackle tensors like t_{112} which are not variable-symmetric. For a tensor T over X, Y, Z, let T^{sym} be the variable-symmetric tensor over $X \times Y \times Z, Y \times Z \times X, Z \times X \times Y$ which results from taking the tensor product of the three cyclic shifts of the variable sets of T. As shown by [CW90], we know that $V_\tau(T^{sym}) = V_\tau(T)^3$ for any tensor T and value $\tau \in [2/3, 1]$. Applying Theorem 3.4 to a symmetrized tensor is slightly simpler than the general case, as observed by [CW90], as follows:

Proposition 4.2 ([CW90])**.** *For any tensor T whose variable sets are partitioned as described above Theorem 3.4, we have $\tilde{S}(T^{sym}) \leq \limsup_p p_X \cdot p_Y \cdot p_Z$.*

Note that in Proposition 4.2, p is iterating over probability distributions on the L resulting from our blocking of T, and not over probability distributions on a blocking of T^{sym}. By comparison, Theorem 3.4 showed that $\tilde{S}(T) \leq \limsup_p \min\{p_X, p_Y, p_Z\}$.

Proof of Proposition 4.1. By definition of $V_{2/3}$, for every $\delta > 0$ there is a positive integer n such that $(t_{112}^{sym})^{\otimes n}$ has a degeneration to $\bigoplus_i \langle a_i a_i a_i \rangle$ for values such that $\sum_i a_i^2 \geq (V_{2/3}(T_{112}))^{3n(1-\delta)}$. In particular, this yields the bound $\tilde{S}((t_{112}^{sym})^{\otimes n}) \geq \sum_i a_i^2 \geq (V_{2/3}(t_{112}))^{3n(1-\delta)}$. Since this holds for all $\delta > 0$, it follows that $\tilde{S}(t_{112}^{sym}) \geq (V_{2/3}(t_{112}))^3 \geq 2^2 q^2 (q^2 + 2)$.

We now upper bound $\tilde{S}(t_{112}^{sym})$ using Proposition 4.2. We will use the following partition of the variables of t_{112}: $X_0 = \{x_{i,0} \mid i \in [q]\}$, $X_1 = \{x_{0,k} \mid k \in [q]\}$, $Y_0 = \{y_{i,0} \mid i \in [q]\}$, $Y_1 = \{y_{0,k} \mid k \in [q]\}$, $Z_0 = \{z_{i,k} \mid i, k \in [q]\}$, $Z_1 = \{z_{0,q+1}\}$, and $Z_1 = \{z_{q+1,0}\}$. Hence, $L = \{T_{001}, T_{112}, T_{010}, T_{100}\}$. We can assume by symmetry that any probability distribution p on L assigns the same value v to T_{010} and T_{100}, and the same value $1/2 - v$ to T_{001} and T_{112}. We finally get the bound:

$$\tilde{S}(t_{112}) \leq \limsup_{v \in [0,1/2]} (2q)^2 \cdot \frac{(q^2)^{2v}}{(2v)^{2v}(1/2 - v)^{1-2v}}.$$

This is maximized at $v = q^2/(2q^2+2)$, which yields exactly $\tilde{S}(t_{112}^{sym}) \leq 2^2 q^2 (q^2 + 2)$. The desired bound follows. □

The only upper bound we are able to prove on V_τ for $\tau > 2/3$ is the straightforward $V_\tau(t_{112}) \leq V_{2/3}(t_{112})^{3\tau/2} = 2^\tau q^\tau (q^2+2)^{\tau/2}$, which is slightly worse than the best known lower bound $V_\tau(t_{112}) \geq 2^{2/3} q^\tau (q^{3\tau} + 2)^{1/3}$. It is an interesting open problem to prove tight upper bounds on $V_\tau(T)$ for any nontrivial tensor T and value $\tau > 2/3$. $T = t_{112}$ may be a good candidate since the Laser method seems unable to improve $V_\tau(t_{112})$ for any τ, even when applied to any small tensor power $t_{112}^{\otimes n}$.

5 Slice Rank Lower Bounds via the Laser Method

In this section, we show that a slightly restricted version of the Laser method can be used to give matching upper and lower bounds on $\tilde{S}(T)$ for any tensor T to which it applies. We will build off of Theorem 3.4, which already looks similar to the expressions which arise in the Laser method.

Consider any tensor T which is minimal over X, Y, Z, and let $X = X_1 \cup \cdots \cup X_{k_X}$, $Y = Y_1 \cup \cdots \cup Y_{k_Y}$, $Z = Z_1 \cup \cdots \cup Z_{k_Z}$ be partitions of the three variable sets. Define T_{ijk}, L, and p_X for a probability distribution p on L, as in the top of subsection 3.2.

Definition 5.1. We say that T, along with the partitions of X, Y, Z, is a *laser-ready tensor partition* if the following conditions are satisfied:

(1) T is variable-symmetric, as are the partitions.[3]

(2) For every $(i, j, k) \in [k_X] \times [k_Y] \times [k_Z]$, either $T_{ijk} = 0$, or else $T_{ijk} = \langle a, b, c\rangle$ for integers a, b, c such that $ab = |X_i|$, $bc = |Y_j|$, and $ca = |Z_k|$.

(3) There is an integer ℓ such that $T_{ijk} \neq 0$ only if $i + j + k = \ell$.

(4) At least one of the following two conditions holds:

   (a) Each X_i, Y_j, and Z_k contains exactly one variable, or

   (b) Given any values $v(X_1), \ldots, v(X_{k_X}), v(Y_1), \ldots, v(Y_{k_Y}), v(Z_1), \ldots, v(Z_{k_Z}) \in [0, 1]$ with $\sum_{i\in[k_X]} v(X_i) = \sum_{j\in[k_Y]} v(Y_j) = \sum_{k\in[k_Z]} v(Z_k) = 1$, there is a unique probability distribution p on L such that $p(X_i) = v(X_i)$ for all $i \in [k_X]$, $p(Y_j) = v(Y_j)$ for all $j \in [k_Y]$, and $p(Z_k) = v(Z_k)$ for all $k \in [k_Z]$.

These conditions may seem strange at first, but in fact they are the conditions needed for the Laser method to apply in the most straightforward way to T. We refer to [Wil12, Section 3] for more details. More precisely, condition (1) can be assumed without loss of generality by symmetrizing the input tensor, then conditions (3) and (4) are necessary for the Laser method in the simplest setting to apply. (For the familiar reader: cases (4a) and (4b) in the definition of a laser-ready tensor partition correspond to the simplest cases in [Wil12, Section 3] when, in their notation, $\aleph = \aleph_{max}$.) Condition (2) is the only one which is not necessary for general use of the Laser method but which we will need here; note that it is also implied by condition (4b), but not (4a). It is not hard to see that CW_q, cw_q, and T_q, for any q, partitioned as they were in the previous section, are laser-ready tensor partitions.

The key remark about such tensor partitions is as follows:

Theorem 5.2. *Suppose tensor T, along with the partitions of X, Y, Z, is a laser-ready tensor partition. Then, $\tilde{S}(T) = \limsup_p \min\{p_X, p_Y, p_Z\}$, where the limsup is over probability distributions p on L.*

Proof. (Sketch, as we omit details from [Wil12, Section 3])

The upper bound, $\tilde{S}(T) \leq \limsup_p \min\{p_X, p_Y, p_Z\}$, follows immediately from Theorem 3.4. For the lower bound, let us assume we are in case (4a) of the laser-ready partition definition; the case (4b) is similar. From condition (1) in the definition of a laser-ready tensor partition, we can assume without loss of generality that the limsup is only over probability distributions p on L such that $p(T_{ijk}) = p(T_{jki})$ for all i, j, k, as in [CW90]. In this case, the Laser method as described in [Wil12, Section 3] implies that for any such p and for any positive integer n, the tensor $T^{\otimes n}$ has a degeneration into

$$\left(\prod_{i\in[k_X]} p(X_i)^{-p(X_i)}\right)^{n-o(n)} \odot \bigotimes_{T_{ijk}\in L} T_{ijk}^{n\cdot p(T_{ijk})},$$

[3] To be more precise, we assume that replacing $x_i \leftarrow y_i$, $y_i \leftarrow z_i$, and $z_i \leftarrow x_i$ for all i leaves both T and the block structure isomorphic.

where, because of condition (2) in the definition of a laser-ready tensor partition, the tensor $\bigotimes_{T_{ijk} \in L} T_{ijk}^{n \cdot p(T_{ijk})}$ is isomorphic to $\langle a, a, a \rangle$ for

$$a = \left(\prod_{T_{ijk} \in L} (|X_i| \cdot |Y_j| \cdot |Z_k|)^{n \cdot p(T_{ijk})/6} \right) = \left(\prod_{T_{ijk} \in L} (|X_i|)^{n \cdot p(T_{ijk})/2} \right),$$

where the second equality holds because of the symmetry we assumed on p. By Proposition 2.4, this means $T^{\otimes n}$ has a degeneration to an independent tensor of size

$$\left(\prod_{i \in [k_X]} p(X_i)^{-p(X_i)} \right)^{n - o(n)} \cdot a^2 = p_X^{n-o(n)}.$$

Applying Propositions 2.2 and 2.3 implies that $\tilde{S}(T) \geq p_X$, as desired. □

Corollary 5.3. *The upper bounds on $\tilde{S}(CW_{q,\sigma})$, $\tilde{S}(cw_{q,\sigma})$, and $\tilde{S}(T_q)$ from Section 4 are tight.*

Proof. CW_q, cw_q, and T_q, for any q, partitioned as they were in the previous section, are laser-ready tensor partitions. □

Corollary 5.4. *If T is a tensor with a laser-ready tensor partition, and applying the Laser method to T yields an upper bound on ω of $\omega \leq c$ for some $c > 2$, then $\omega_u(T) > 2$.*

Proof. When the Laser method shows, as in the proof of Theorem 5.2, that $T^{\otimes n}$ has a degeneration into

$$\left(\prod_{i \in [k_X]} p(X_i)^{-p(X_i)} \right)^{n - o(n)} \odot \langle a, a, a \rangle,$$

the resulting upper bound on ω is that

$$\left(\prod_{i \in [k_X]} p(X_i)^{-p(X_i)} \right)^{n - o(n)} \cdot a^{\omega} \geq \tilde{R}(T)^n.$$

In particular, this yields $\omega = 2$ if and only if $p_X = \tilde{R}(T)$, so if it yields $\omega \leq c$, then $\tilde{S}(T) = p_X < \tilde{R}(T)^{1-\delta}$ for some $\delta > 0$. Combined with Theorem 2.6 or Theorem 2.8, this means that $\omega_u(T) > 2$. □

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A Proof that $\omega_u(CW_{q,\sigma}) \geq 2.16805$ for all q

Define the function $f : [0, 1/3] \to \mathbb{R}$ by

$$f(v) := \frac{1}{v^v (2/3 - 2v)^{2/3-2v} (1/3 + v)^{1/3+v}}.$$

In Section 4.1, we showed that

$$\omega_u(CW_{q,\sigma}) \geq \liminf_{v \in [0,1/3]} 2 \frac{\log(q+2)}{\log(q^{2/3-2v} \cdot f(v))}.$$

This is a relatively simple optimization problem, which any computer optimization software can solve quickly for fixed q. In particular, the bound we got for $q = 1$ was computed to be $\ell := 2.16805\cdots$. Here we prove that, in fact, the bound is at least ℓ for all $q \geq 1$.

For $1 \leq q \leq 100$, we can simply compute the value using optimization software and confirm that it is true. For instance, the bound for $q = 100$ can be computed to be $2.49543\ldots$ by inputting the following into Wolfram Alpha:

```
minimize 2 * log(102) / log(100^(2/3 - 2*v) / ( v^v * (2/3 - 2*v)^(2/3 - 2*v)
* (1/3 + v)^(1/3 + v) )) for 0.000001 <= v <= 0.33333
```

Now consider when $q > 100$. Since

$$\max_{\substack{a,b,c\in[0,1]\\a+b+c=1}} \frac{1}{a^a b^b c^c} \leq \max_{\substack{a,b,c\in[0,1]\\a+b+c=1}} \lim_{n\to\infty} \binom{n}{an, bn, cn}^{1/n} \leq \max_{\substack{a,b,c\in[0,1]\\a+b+c=1}} \lim_{n\to\infty} (3^n)^{1/n} = 3,$$

achieved at $a = b = c = 1/3$, we see that $f(v) \leq 3$ for all $v \in [0, 1/3]$. It follows that our bound on $\omega_u(CW_{q,\sigma})$ is at least

$$2\frac{\log(q+2)}{\log(3q^{2/3})}. \tag{1}$$

The expression (1) evaluated at $q = 101$ is $2.220038\ldots > \ell$, and (1) is easily seen to be increasing with q when $q > 100$, as desired.