Minimax theorems in a fully non-convex setting

Dedicated to Professor Wataru Takahashi, with esteem and friendship, on his 75th birthday

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Abstract. In this paper, we establish two minimax theorems for functions $f : X \times I \to \mathbf{R}$, where I is a real interval, without assuming that $f(x, \cdot)$ is quasi-concave. Also, some related applications are presented.

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The most known minimax theorem ([7]) ensures the occurrence of the equality

$$\sup_Y \inf_X f = \inf_X \sup_Y f$$

for a function $f : X \times Y \to \mathbf{R}$ under the following assumptions: X, Y are convex sets in Hausdorff topological vector spaces, one of them is compact, f is lower semicontinuous and quasi-convex in X, and upper semicontinuous and quasi-concave in Y.

In the past years, we provided some contributions to the subject where, keeping the assumption of quasi-concavity on $f(x, \cdot)$, we proposed alternative hypotheses on $f(\cdot, y)$. Precisely, in [2], we assumed the inf-connectedness of $f(\cdot, y)$ and, the same time, that Y is a real interval, while, in [5], we assumed the inf-compactness and uniqueness of the global minimum of $f(\cdot, y)$.

In the present paper, we offer a new contribution where the hypothesis that $f(x, \cdot)$ is quasi-concave is no longer assumed.

Let T be a topological space. A function $g : T \to [-\infty, +\infty[$ is said to be relatively inf-compact if, for each $r \in \mathbf{R}$, there exists a compact set $K \subseteq T$ such that $g^{-1}(]-\infty, r[) \subseteq K$. Moreover, g is said to be inf-connected if, for each $r \in \mathbf{R}$, the set $g^{-1}(]-\infty, r[)$ is connected. For the basic notions on multifunctions, we refer to [1].

Our main results are as follows:

THEOREM 1. - *Let X be a topological space, let I be a real interval and let $f : X \times I \to \mathbf{R}$ be a continuous function such that, for each $\lambda \in I$, the set of all global minima of the function $f(\cdot, \lambda)$ is connected. Moreover, assume that there exists a non-decreasing sequence of compact intervals, $\{I_n\}$, with $I = \cup_{n \in \mathbf{N}} I_n$, such that, for each $n \in \mathbf{N}$, the following conditions are satisfied:*

(a_1) *the function $\inf_{\lambda \in I_n} f(\cdot, \lambda)$ is relatively inf-compact ;*

(b_1) *for each $x \in X$, the set of all global maxima of the restriction of the function $f(x, \cdot)$ to I_n is connected.*

Then, one has

$$\sup_Y \inf_X = \inf_X \sup_Y f \ .$$

THEOREM 2. - *Let X be a topological space, let I be a compact real interval and let $f : X \times I \to \mathbf{R}$ be an upper semicontinuous function such that $f(\cdot, \lambda)$ is continuous for all $\lambda \in I$. Assume that:*

(a_2) *there exists a set $D \subseteq I$, dense in I, such that the function $f(\cdot, \lambda)$ is inf-connected for all $\lambda \in D$;*

(b_2) *for each $x \in X$, the set of all global maxima of the function $f(x, \cdot)$ is connected.*

Then, one has

$$\sup_Y \inf_X = \inf_X \sup_Y f \ .$$

REMARK 1. - We want to remark that, in both Theorems 1 and 2, it is essential that I be a real interval. To see this, consider the following example. Take

$$X = I = \{(t, s) \in \mathbf{R}^2 : t^2 + s^2 = 1\}$$

and define $f : X \times I \to \mathbf{R}$ by

$$f(t, s, u, v) = tu + sv$$

for all $(t, s), (u, v) \in X$. Clearly, f is continuous, $f(\cdot, \cdot, u, v)$ is inf-connected and has a unique global minimum, and $f(t, s, \cdot, \cdot)$ has a unique global maximum. However, we have

$$\sup_X \inf_I f = -1 < 1 = \inf_X \sup_I f \ .$$

The common key tool in our proofs of Theorems 1 and 2 is provided by the following general principle:

THEOREM A ([2], Theorem 2.2). - *Let X be a topological space, let I be a compact real interval and let $S \subseteq X \times I$ be a connected set whose projection on I is the whole of I.*

Then, for every upper semicontinuous multifunction $\Phi : X \to 2^I$, with non-empty, closed and connected values, the graph of Φ intersects S.

Another known proposition which is used in the proof of Theorem 1 is as follows:

PROPOSITION A ([5], Proposition 2.1). - *Let X be a topological space, Y a non-empty set, $y_0 \in Y$ and $f : X \times Y \to \mathbf{R}$ a function such that $f(\cdot, y)$ is lower semicontinuous for all $y \in Y$ and relatively inf-compact for $y = y_0$. Assume also that there is a non-decreasing sequence of sets $\{Y_n\}$, with $Y = \cup_{n\in\mathbf{N}} Y_n$, such that*

$$\sup_{Y_n} \inf_X f = \inf_X \sup_{Y_n} f$$

for all $n \in \mathbf{N}$.

Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f \ .$$

A further result which is used in the proofs of Theorems 1 and 2 is provided by the following proposition which, in the given generality, is new:

PROPOSITION 1. - *Let X, Y be two topological spaces and let $f : X \times Y \to \mathbf{R}$ be a lower semicontinuous function such that $f(x, \cdot)$ is continuous for all $x \in X$. Moreover, assume that, for each $y \in Y$, there exists a neighbourhood V of y such that the function $\inf_{v\in V} f(\cdot, v)$ is relatively inf-compact. For each $y \in Y$, set*

$$F(y) = \left\{ u \in X : f(u, y) = \inf_{x\in X} f(x, y) \right\} \ .$$

Then, the multifunction F is upper semicontinuous.

PROOF. Let $C \subseteq X$ be a closed set. We have to prove that $F^-(C)$ is closed. So, let $\{y_\alpha\}_{\alpha\in D}$ be a net in $F^-(C)$ converging to some $\tilde{y} \in Y$. For each $\alpha \in D$, pick $u_\alpha \in F(y_\alpha) \cap C$. By assumption, there is a neighbourhood V of $\tilde{y}$ such that the function $\inf_{v\in V} f(\cdot, v)$ is relatively inf-compact. Since the function $\inf_{x\in X} f(x, \cdot)$ is upper semicontinuous, we can assume that it is bounded above on V. Fix $\rho > \sup_V \inf_X f$. Then, there is a compact set $K \subseteq X$ such that

$$\left\{ x \in X : \inf_{v\in V} f(x, v) < \rho \right\} \subseteq K \ .$$

But

$$\left\{ x \in X : \inf_{v\in V} f(x, v) < \rho \right\} = \bigcup_{v\in V} \{x \in X : f(x, v) < \rho\}$$

and so

$$\bigcup_{v\in V} \{x \in X : f(x,v) < \rho\} \subseteq K \ . \tag{1}$$

Let $\alpha_1 \in D$ be such that $y_\alpha \in V$ for all $\alpha \geq \alpha_1$. Consequently, by (1), $u_\alpha \in K$ for all $\alpha \geq \alpha_1$. By compactness, the net $\{u_\alpha\}_{\alpha\in D}$ has a cluster point $\tilde{u} \in K$. Clearly, $(\tilde{u}, \tilde{y})$ is a cluster point in $X \times Y$ of the net $\{(u_\alpha, y_\alpha)\}_{\alpha\in D}$. We claim that

$$f(\tilde{u}, \tilde{y}) \leq \limsup_\alpha f(u_\alpha, y_\alpha) \ .$$

Arguing by contradiction, assume the contrary and fix r so that

$$\limsup_\alpha f(u_\alpha, y_\alpha) < r < f(\tilde{u}, \tilde{y}) \ .$$

Then, there would be $\alpha_2 \in D$ such that

$$f(u_\alpha, y_\alpha) < r$$

for all $\alpha \geq \alpha_2$. On the other hand, since, by assumption, the set $f^{-1}(]r, +\infty[)$ is open, there would be $\alpha_3 \geq \alpha_2$ such that

$$r < f(u_{\alpha_3}, y_{\alpha_3})$$

which gives a contradiction. Now, fix $x \in X$. Then, since $u_\alpha \in F(y_\alpha)$, we have

$$f(\tilde{u}, \tilde{y}) \leq \limsup_\alpha f(u_\alpha, y_\alpha) \leq \lim_\alpha f(x, y_\alpha) = f(x, \tilde{y}) \ .$$

That is, $\tilde{u} \in F(\tilde{y})$. Since C is closed, $\tilde{u} \in C$. Hence, $\tilde{y} \in F^-(C)$ and this ends the proof. △

We now can prove Theorems 1 and 2.

Proof of Theorem 1. Fix $n \in \mathbf{N}$. Let us prove that

$$\sup_{I_n} \inf_X f = \inf_X \sup_{I_n} f \ . \tag{2}$$

Consider the multifunction $F : I_n \to 2^X$ defined by

$$F(\lambda) = \left\{ u \in X : f(u, \lambda) = \inf_{x\in X} f(x, \lambda) \right\}$$

for all $\lambda \in I_n$. Thanks to Proposition 1, F is upper semicontinuous and, by assumption, its values are non-empty, compact and connected. As a consequence, by Theorem 7.4.4 of [1], the graph of F is connected. Let S denote the graph of the inverse of F. So, S is connected as it is homeomorphic to the graph of F. Now, consider the multifunction $\Phi : X \to 2^{I_n}$ defined by

$$\Phi(x) = \left\{ \mu \in I_n : f(x, \mu) = \sup_{\lambda\in I_n} f(x, \lambda) \right\}$$

for all $x \in X$. By Proposition 1 again, the multifunction Φ is upper semicontinuous and, by assumption, its values are non-empty, closed and connected. After noticing that the projection of S on I_n is the whole of I_n, we can apply Theorem A. Therefore, there exists $(\tilde{x}, \tilde{\lambda}) \in S$ such that $\tilde{\lambda} \in \Phi(\tilde{x})$. That is

$$f(\tilde{x}, \tilde{\lambda}) = \inf_{x\in X} f(x, \tilde{\lambda}) = \sup_{\lambda\in I_n} f(\tilde{x}, \lambda) \ . \tag{3}$$

Clearly, (2) follows from (3). Now, the conclusion is a direct consequence of Proposition A. △

Proof of Theorem 2. Arguing by contradiction, assume the contrary and fix a constant r so that

$$\sup_I \inf_X f < r < \inf_X \sup_I f \ .$$

Let $G : I \to 2^X$ be the multifunction defined by

$$G(\lambda) = \{x \in X : f(x, \lambda) < r\}$$

for all $\lambda \in I$. Notice that $G(\lambda)$ is non-empty for all $\lambda \in I$ and connected for all $\lambda \in D$. Moreover, the graph of G is open in $X \times I$ and so G is lower semicontinuous. Then, by Proposition 5.7 of [3], the graph of G is connected and so the graph of the inverse of G, say S, is connected too. Consider the multifunction $\Phi : X \to 2^I$ defined by

$$\Phi(x) = \left\{\mu \in I : f(x, \mu) = \sup_{\lambda \in I} f(x, \lambda)\right\}$$

for all $x \in X$. Notice that $\Phi(x)$ is non-empty, closed and connected, in view of (b_2). By Proposition 1, the multifunction Φ is upper semicontinuous. Now, we can apply Theorem A. So, there exists $(\hat{x}, \hat{\lambda}) \in S$ such that $\hat{\lambda} \in \Phi(\hat{x})$. This implies that

$$f(\hat{x}, \hat{\lambda}) < r < \inf_X \sup_I f \leq \sup_{\lambda \in I} f(\hat{x}, \lambda) = f(\hat{x}, \hat{\lambda})$$

which is absurd. $\triangle$

Here is an application of Theorem 1.

THEOREM 3. - *Let $(H, \langle \cdot, \cdot \rangle)$ be a real inner product space, let $K \subset H$ be a compact and convex set, with $0 \notin K$, and let $f : X \to K$ be a continuous function, where*

$$X = \bigcup_{\lambda \in \mathbf{R}} \lambda K \ .$$

Assume that there are two numbers α, c, with

$$\inf_{x \in X} \|f(x)\| < c < \|f(0)\| \ ,$$

such that:
(a) $\{x \in X : \langle x, f(x) \rangle = \alpha\} \subset \{x \in X : \|f(x)\| < c\}$;
(b) $\{x \in X : c^2 \langle x, f(x) \rangle = \alpha \|f(x)\|^2\} \subset \{x \in X : \|f(x)\| \geq c\}$.
Then, there exists $\tilde{\lambda} \in \mathbf{R}$ such that the set

$$\{x \in X : x = \tilde{\lambda} f(x)\}$$

is disconnected.

PROOF. Consider the function $\varphi : X \times \mathbf{R} \to \mathbf{R}$ defined by

$$\varphi(x, \lambda) = \|x - \lambda f(x)\|^2 - c^2 \lambda^2 + 2\alpha\lambda$$

for all $(x, \lambda) \in X \times \mathbf{R}$. Notice that

$$\varphi(x, \lambda) = \|x\|^2 + (\|f(x)\|^2 - c^2)\lambda^2 - 2(\langle x, f(x) \rangle - \alpha)\lambda \ . \tag{4}$$

Further, observe that, when $\|f(x)\| \geq c$, in view of (a), we have

$$\sup_{\lambda \in \mathbf{R}} \varphi(x, \lambda) = +\infty \tag{5}$$

as well as

$$\varphi(x, -\lambda) \neq \varphi(x, \lambda) \tag{6}$$

for all $\lambda > 0$. When $\|f(x)\| \geq c$ again, the function $\varphi(x, \cdot)$ is convex and so, by (6), for each $\lambda > 0$, its restriction to $[-\lambda, \lambda]$ it has a unique global maximum. Clearly, $\varphi(x, \cdot)$ has the same uniqueness property also when $\|f(x)\| < c$. Now, observe that, for each $\lambda \in \mathbf{R}$, the function λf has a fixed point in X, in view of the Schauder theorem. Hence, we have

$$\sup_{\lambda\in\mathbf{R}} \inf_{x\in X} \varphi(x,\lambda) = \sup_{\lambda\in\mathbf{R}}(-c^2\lambda^2 + 2\alpha\lambda) = \frac{\alpha^2}{c^2} \ . \tag{7}$$

We claim that

$$\frac{\alpha^2}{c^2} < \inf_{x\in X} \sup_{\lambda\in\mathbf{R}} \varphi(x,\lambda) \ . \tag{8}$$

First, observe that, since $0 \notin K$, every closed and bounded subset of X is compact. This easily implies that, for each $\mu > 0$, the function $x \to \inf_{|\lambda|\leq\mu} \varphi(x,\lambda)$ is relatively inf-compact. Consequently, the sublevel sets of the function $x \to \sup_{\lambda\in\mathbf{R}} \varphi(x,\lambda)$ (which is finite if $\|f(x)\| < c$) are compact. Therefore, there exists $\tilde{x} \in X$ such that

$$\sup_{\lambda\in\mathbf{R}} \varphi(\tilde{x},\lambda) = \inf_{x\in X} \sup_{\lambda\in\mathbf{R}} \varphi(x,\lambda) \ . \tag{9}$$

So, by (5), one has $\|f(\tilde{x})\| < c$. Clearly, we also have

$$\sup_{\lambda\in\mathbf{R}} \varphi(\tilde{x},\lambda) = \|\tilde{x}\|^2 + \frac{|\langle \tilde{x}, f(\tilde{x})\rangle - \alpha|^2}{c^2 - \|f(\tilde{x})\|^2} \ . \tag{10}$$

Let us prove that

$$\|\tilde{x}\|^2 + \frac{|\langle \tilde{x}, f(\tilde{x})\rangle - \alpha|^2}{c^2 - \|f(\tilde{x})\|^2} > \frac{\alpha^2}{c^2} \ . \tag{11}$$

After some manipulations, one realizes that (11) is equivalent to

$$\frac{1}{c^2 - \|f(\tilde{x})\|^2}\left(2\alpha\langle \tilde{x}, f(\tilde{x})\rangle - |\langle \tilde{x}, f(\tilde{x})\rangle|^2 - \frac{\alpha^2}{c^2}\|f(\tilde{x})\|^2\right) < \|\tilde{x}\|^2 \ . \tag{12}$$

Now, for each $y \in X \setminus \{0\}$, $t \in \mathbf{R}$, set

$$I(y,t) = \{x \in H : \langle x, y\rangle = t\} \ .$$

Consider the inequality

$$\frac{1}{c^2 - \|y\|^2}\left(2\alpha t - t^2 - \frac{\alpha^2}{c^2}\|y\|^2\right) < \frac{t^2}{\|y\|^2} \ . \tag{13}$$

After some manipulations, one realizes that (13) is equivalent to

$$(\alpha\|y\|^2 - tc^2)^2 > 0 \ .$$

So, (13) is satisfied if and only if

$$\alpha\|y\|^2 \neq tc^2 \ . \tag{14}$$

Observe that

$$\frac{|t|}{\|y\|} = \mathrm{dist}(0, I(y,t)) \leq \mathrm{dist}(0, I(y,t)\cap X) \ . \tag{15}$$

Therefore, if (14) is satisfied, for each $x \in I(y,t)\cap X$, in view of (13) and (15), we have

$$\frac{1}{c^2 - \|y\|^2}\left(2\alpha\langle x, y\rangle - |\langle x, y\rangle|^2 - \frac{\alpha^2}{c^2}\|y\|^2\right) < \|x\|^2 \ . \tag{16}$$

At this point, taking into account that $c^2\langle \tilde{x}, f(\tilde{x})\rangle \neq \alpha\|f(\tilde{x})\|^2$ (by (b)), we draw (12) from (16) since $\tilde{x} \in I(f(\tilde{x}), \langle \tilde{x}, f(\tilde{x})\rangle)$. Summarizing: taking $I = \mathbf{R}$ and $I_n = [-n, n]$ $(n \in \mathbf{N})$, the continuous function φ satisfies (a_1) and (b_1) of Theorem 1, but, in view of $(7) - (11)$, it does not satisfy the conclusion of that theorem. As a consequence, there exists $\tilde{\lambda} \in \mathbf{R}$ such that the set of all global minima of $\varphi(\cdot, \tilde{\lambda})$ is disconnected. But such a set agrees with the set of all solutions of the equation $x = \tilde{\lambda} f(x)$, and the proof is complete. $\triangle$

REMARK 2. - We do not know whether Theorem 3 is still true when $0 \in K$ and (b) is (necessarily) changed in

$$\{x \in X : f(x) \neq 0,\ c^2\langle x, f(x)\rangle = \alpha\|f(x)\|^2\} \subset \{x \in X : \|f(x)\| \geq c\} .$$

However, the proof of Theorem 3 shows that the following is true:

THEOREM 4. - *Let $(X, \langle\cdot,\cdot\rangle)$ be a finite-dimensional real Hilbert space and let $f : X \to X$ be a continuous function with bounded range. Assume that there are two numbers α, c, with*

$$\inf_{x \in X} \|f(x)\| < c < \|f(0)\| ,$$

such that:
(a') $\{x \in X : \langle x, f(x)\rangle = \alpha\} \subset \{x \in X : \|f(x)\| < c\}$;
(b') $\{x \in X : f(x) \neq 0,\ c^2\langle x, f(x)\rangle = \alpha\|f(x)\|^2\} \subset \{x \in X : \|f(x)\| \geq c\}$.

Then, there exists $\tilde{\lambda} \in \mathbf{R}$ such that the set

$$\{x \in X : x = \tilde{\lambda} f(x)\}$$

is disconnected.

Finally, we present two applications of Theorem 2.

THEOREM 5. - *Let X be a Banach space, let $\varphi \in X^* \setminus \{0\}$ and let $\psi : X \to \mathbf{R}$ be a Lipschitzian functional whose Lipschitz constant is equal to $\|\varphi\|_{X^*}$. Moreover, let $[a, b]$ be a compact real interval, $\gamma : [a, b] \to [-1, 1]$ a convex (resp. concave) and continuous function, with $\mathrm{int}(\gamma^{-1}(\{-1, 1\})) = \emptyset$, and $c \in \mathbf{R} \setminus \{0\}$. Assume that*

$$\gamma(a)\psi(x) + ca \neq \gamma(b)\psi(x) + cb$$

for all $x \in X$ such that $\psi(x) > 0$ (resp. $\psi(x) < 0$).

Then (with the convention $\sup \emptyset = -\infty$), one has

$$\sup_{\lambda \in \gamma^{-1}(\{-1,1\})} \inf_{x \in X} (\varphi(x) + \gamma(\lambda)\psi(x) + c\lambda) = \inf_{x \in X} \sup_{\lambda \in [a,b]} (\varphi(x) + \gamma(\lambda)\psi(x) + c\lambda) .$$

PROOF. Consider the continuous function $f : X \times [a, b] \to \mathbf{R}$ defined by

$$f(x, \lambda) = \varphi(x) + \gamma(\lambda)\psi(x) + c\lambda$$

for all $(x, \lambda) \in X \times [a, b]$. By Theorem 2 of [4], for each $\lambda \in \gamma^{-1}(]-1, 1[)$, the function $f(\cdot, \lambda)$ is inf-connected and unbounded below. Also, notice that $\gamma^{-1}(]-1, 1[)$, by assumption, is dense in $[a, b]$. Now fix $x \in X$. If $\psi(x) > 0$ (resp. $\psi(x) < 0$) the function $f(x, \cdot)$ is convex and, by assumption, $f(x, a) \neq f(x, b)$. As a consequence, the unique global maximum of this function is either a or b. If $\psi(x) \leq 0$, the function is concave and so, obviously, the set of all its global maxima is connected. Now, the conclusion follows directly from Theorem 2. $\triangle$

Let $(T, \mathcal{F}, \mu)$ be a σ-finite measure space, E a real Banach space and $p \geq 1$.

As usual, $L^p(T, E)$ denotes the space of all (equivalence classes of) strongly μ-measurable functions $u : T \to E$ such that $\int_T \| u(t) \|^p \, d\mu < +\infty$, equipped with the norm

$$\| u \|_{L^p(T,E)} = \left(\int_T \| u(t) \|^p \, d\mu \right)^{\frac{1}{p}} .$$

A set $D \subseteq L^p(T, E)$ is said to be decomposable if, for every $u, v \in D$ and every $A \in \mathcal{F}$, the function

$$t \to \chi_A(t)u(t) + (1 - \chi_A(t))v(t)$$

belongs to D, where χ_A denotes the characteristic function of A.

A real-valued function on $T \times E$ is said to be a Caratéodory function if it is measurable in T and continuous in E.

THEOREM 6. - *Let $(T, \mathcal{F}, \mu)$ be a σ-finite non-atomic measure space, E a real Banach space, $p \in [1, +\infty[$, $X \subseteq L^p(T, E)$ a decomposable set, $[a, b]$ a compact real interval, $\gamma : [a, b] \to \mathbf{R}$ a convex (resp. concave) and continuous function, $c \in \mathbf{R} \setminus \{0\}$. Moreover, let $\varphi, \psi : T \times E \to \mathbf{R}$ be two Carathéodory functions such that, for some $M \in L^1(T)$, $k \in \mathbf{R}$, one has*

$$\max\{|\varphi(t, x)|, |\psi(t, x)|\} \leq M(t) + k\|x\|^p$$

for all $(t, x) \in T \times E$ and

$$\gamma(a) \int_T \psi(t, u(t))d\mu + ca \neq \gamma(b) \int_T \psi(t, u(t))d\mu + cb$$

for all $u \in X$ such that $\int_T \psi(t, u(t))d\mu > 0$ (resp. $\int_T \psi(t, u(t))d\mu < 0$).

Then, one has

$$\sup_{\lambda \in [a,b]} \inf_{u \in X} \left(\int_T (\varphi(t, u(t)) + \gamma(\lambda)\psi(t, u(t)))d\mu + c\lambda \right) = \inf_{u \in X} \sup_{\lambda \in [a,b]} \left(\int_T (\varphi(t, u(t)) + \gamma(\lambda)\psi(t, u(t)))d\mu + c\lambda \right) .$$

PROOF. The proof goes on exactly as that of Theorem 5. So, one considers the function $f : X \times [a, b] \to \mathbf{R}$ defined by

$$f(u, \lambda) = \int_T (\varphi(t, u(t)) + \gamma(\lambda)\psi(t, u(t)))d\mu + c\lambda$$

for all $(u, \lambda) \in X \times [a, b]$, and realizes that it satisfies the hypotheses of Theorem 2. In particular, for each $\lambda \in [a, b]$, the inf-connectedness of the function $f(\cdot, \lambda)$ is due to [6], Théorème 7. △

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