A TOEPLITZ-LIKE OPERATOR WITH RATIONAL SYMBOL HAVING POLES ON THE UNIT CIRCLE III: THE ADJOINT

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Abstract. This paper contains a further analysis of the Toeplitz-like operators T_ω on H^p with rational symbol ω having poles on the unit circle that were previously studied in [5, 6]. Here the adjoint operator T_ω^* is described. In the case where $p=2$ and ω has poles only on the unit circle $\mathbb{T}$, a description is given for when T_ω^* is symmetric and when T_ω^* admits a selfadjoint extension. Also in the case where $p=2$, ω has only poles on $\mathbb{T}$ and in addition ω is proper, it is shown that T_ω^* coincides with the unbounded Toeplitz operator defined by Sarason in [12].

1. Introduction

In this paper we proceed with our study of unbounded Toeplitz-like operators on H^p with rational symbols that have poles on the unit circle $\mathbb{T}$ which was initiated in [5]. Our previous work on such Toeplitz-like operators focused on their Fredholm properties (in [5]) and the various parts of their spectra (in [6]). Here we determine properties of the adjoint operator and conditions under which the operator is symmetric and when it has a selfadjoint extension.

Before we can define our Toeplitz-like operators, some notation has to be introduced. We write Rat for the space of rational complex functions, $\mathrm{Rat}(\mathbb{T})$ for the subspace of Rat consisting of rational complex functions with poles only on the unit circle $\mathbb{T}$, and $\mathrm{Rat}_0(\mathbb{T})$ for the subspace of strictly proper functions in $\mathrm{Rat}(\mathbb{T})$. Now let $\omega \in \mathrm{Rat}$, possibly with poles on $\mathbb{T}$. As in [5], we define the Toeplitz-like operator T_ω $(H^p \to H^p)$, for $1<p<\infty$, via

$$\mathrm{Dom}(T_\omega)=\{g\in H^p \mid \omega g = f+\rho, \text{ with } f\in L^p,\ \rho\in \mathrm{Rat}_0(\mathbb{T})\},\quad T_\omega g=\mathbb{P}f. \tag{1.1}$$

Here $\mathbb{P}$ is the Riesz projection of L^p onto H^p. The operator T_ω is densely defined and closed. In case $\omega \in \mathrm{Rat}(\mathbb{T})$, explicit formulas for the domain, kernel, range, and a complement of the range were obtained in [6], as an extension of a result in [5] for the case where T_ω is Fredholm. We briefly recall these results in Section 2, as they will be frequently used throughout the paper.

In case ω has no poles on $\mathbb{T}$, in fact for any $\omega\in L^\infty$, the adjoint of the Toeplitz operator T_ω on H^p can be identified with the Toeplitz operator T_{ω^*} on $H^{p'}$, with $1<p'<\infty$ such that $1/p+1/p'=1$ and with ω^* defined as $\omega^*(z)=\overline{\omega(z)}$ on $\mathbb{T}$.

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The identification of $(H^p)'$ and $H^{p'}$ goes via the usual pairing

$$\langle f, g\rangle_{p,p'} = \frac{1}{2\pi}\int_{\mathbb{T}} \overline{g(z)} f(z)\, dz \quad (f \in H^p, g \in H^{p'}).$$

In the sequel we use the same notation for the similarly defined pairing between L^p and $L^{p'}$ to identify $(L^p)'$ and $L^{p'}$, and in both cases the indices will often be omitted.

For the Toeplitz-like operators studied in this paper the situation is more complicated than for Toeplitz operators with L^∞ symbols. However, we do obtain that T_ω^* can be identified with the restriction of the Toeplitz-like operator T_{ω^*} on $H^{p'}$ to a dense subspace of its domain. Like for the operator T_ω, in case ω is in $\mathrm{Rat}(\mathbb{T})$ we obtain a more explicit description of T_ω^*, which we present after introducing some further notation.

Throughout the paper $\mathcal{P}$ denotes the space of complex polynomials and $\mathcal{P}_k$, for any non-negative integer k, denotes the subspace of $\mathcal{P}$ of polynomials of degree at most k. The degree of a polynomial $r \in \mathcal{P}$ is denoted as $\deg(r)$. Given $r \in \mathcal{P}$ with $\deg(r) = k$, say $r(z) = r_0 + zr_1 + \cdots + z^k r_k$, we define the polynomial $r^\sharp$ by

$$r^\sharp(z) = z^k \overline{r(1/\overline{z})} = \overline{r_0} z^k + \overline{r_1} z^{k-1} + \cdots + \overline{r_k}.$$

The following theorem is our first main result.

Theorem 1.1. *Let $\omega = s/q \in \mathrm{Rat}$ with $s, q \in \mathcal{P}$ co-prime and $1 < p < \infty$. Factor $s = s_- s_0 s_+$ and $q = q_- q_0 q_+$ with s_-, q_- having roots only inside $\mathbb{T}$, s_0, q_0 having roots only on $\mathbb{T}$, and s_+, q_+ having roots only outside $\mathbb{T}$. Set $m = \deg(q)$, $n = \deg(s)$, $m_\pm = \deg(q_\pm)$, $n_\pm = \deg(s_\pm)$ $m_0 = \deg(q_0)$, $n_0 = \deg(s_0)$ and let $1 < p' < \infty$ with $1/p + 1/p' = 1$. Then*

$$\mathrm{Dom}(T_\omega^*) = (q_0)^\sharp H^{p'} \subset \mathrm{Dom}(T_{\omega^*}) \quad \textit{and} \quad T_\omega^* = T_{\omega^*}|_{(q_0)^\sharp H^{p'}}. \tag{1.2}$$

Furthermore, we have

$$\begin{aligned} \mathrm{Ran}(T_\omega^*) &= T_{z^{m-n}(s_+)^\sharp/(q_+)^\sharp} Q_{n_0+n_--m_0-m_-}(s_0)^\sharp H^{p'}, \\ \mathrm{Ker}(T_\omega^*) &= \left\{ \frac{(q_-)^\sharp (q_0)^\sharp r}{(s_-)^\sharp} \mid \deg(r) < n_- - m_- - m_0 \right\}. \end{aligned} \tag{1.3}$$

Here $Q_k = I_{H^{p'}} - P_{\mathcal{P}_{k-1}}$, with $P_{\mathcal{P}_{k-1}}$ the standard projection in $H^{p'}$ onto $\mathcal{P}_{k-1} \subset H^{p'}$ to be interpreted as 0 if $k \leq 0$, i.e., $Q_k = I_{H^{p'}}$ if $k \leq 0$. Thus, for $n_0 + n_- \leq m_0 + m_-$ we have $\mathrm{Ran}(T_\omega^*) = T_{z^{m-n}/(q_+)^\sharp}(s_+ s_0)^\sharp H^{p'}$. Moreover,

$$\dim \mathrm{Ker}(T_\omega^*) = \max\left\{0, \#\{\textit{zeroes of } \omega \textit{ inside } \mathbb{D}\} - \#\{\textit{poles of } \omega \textit{ in } \overline{\mathbb{D}}\}\right\},$$

where the multiplicities of the zeroes and poles are taken into account. Hence, $\dim \mathrm{Ker}(T_\omega^*)$ is the maximum of 0 and $n_- - m_- - m_0$. In particular, T_ω^* is injective if and only if the number of poles of ω inside $\overline{\mathbb{D}}$ is greater than or equal to the number of zeroes of ω inside $\mathbb{D}$, multiplicities taken into account.

Before giving a proof of Theorem 1.1 in Section 4, we prove the specialization of this result for the case $\omega \in \mathrm{Rat}(\mathbb{T})$ in Section 3. For this purpose we first provide a description of T_{ω^*} in Section 2.

The injectivity result, but not the description of $\mathrm{Ker}(T_\omega^*)$, can also be derived from general theory and results on T_ω. Indeed, according to Theorem II.3.7 in [4], T_ω^* is injective if and only if T_ω has dense range, so that the claim follows from Proposition 2.4 in [6]. More can be obtained in this way, since H^p, $1 < p < \infty$, is reflexive. By Theorem II.2.14 of [4] it follows that $T_\omega^{**} = T_\omega$, with the usual identifications of the dual spaces. Hence, applying the above to T_ω^* we find that T_ω^* has dense range if and only if T_ω is injective; see also Theorem II.4.10 in [4]. By Banach's Closed Range Theorem, cf., [14], T_ω^* has closed range if and only if T_ω has closed range. Again applying results from [6] now gives the following result.

Corollary 1.2. *Let $\omega \in \mathrm{Rat}$ and $1 < p < \infty$. Then T_ω^* has closed range if and only if ω has no zeroes on $\mathbb{T}$, or equivalently, ω^* has no zeroes on $\mathbb{T}$. Moreover, T_ω^* has dense range if and only if*

$$\#\left\{\begin{matrix}\textit{poles of } \omega \textit{ inside } \overline{\mathbb{D}}\\ \textit{multi. taken into account}\end{matrix}\right\} \leq \#\left\{\begin{matrix}\textit{zeroes of } \omega \textit{ inside } \overline{\mathbb{D}}\\ \textit{multi. taken into account}\end{matrix}\right\}.$$

Beyond Section 4, and in the remainder of this introduction, we only consider the case $p = 2$ and $\omega \in \mathrm{Rat}(\mathbb{T})$. By comparing the results on T_ω and T_ω^* it is obvious T_ω cannot be selfadjoint, except when ω has no poles on $\mathbb{T}$. In Section 5 we describe in terms of ω when T_ω^* is symmetric, in which case $T_\omega^* \subset T_\omega$, and whenever T_ω^* is symmetric we describe when T_{ω^*} admits a selfadjoint extension. The following theorem collects some of the main results of Section 5; it follows directly from Theorem 5.1, Corollaries 5.2 and 5.7, Propositions 5.4 and 5.9.

Theorem 1.3. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime. Consider T_ω on H^2. Then*

$$T_\omega^* \textit{ is symmetric} \quad \Longleftrightarrow \quad \omega(\mathbb{T}) \subset \mathbb{R}.$$

In particular, if T_ω^* is symmetric, then $\deg(s) \leq \deg(q) \leq 2\deg(s)$. Furthermore, if T_ω^* is symmetric, then T_ω^* admits a selfadjoint extension if and only if the number of roots of $s - iq$ and $s + iq$ in $\mathbb{D}$, counting multiplicities, coincide. This happens in particular if $\omega(\mathbb{T}) \neq \mathbb{R}$, but cannot happen in case $\deg(q)$ is odd.

Several other conditions for T_ω^* to be symmetric and/or have a selfadjoint extension are derived in Section 5.

In [12] Sarason introduced and studied an unbounded Toeplitz-like operator with symbol in the Smirnov class. In Section 6 we show that if $\omega \in \mathrm{Rat}(\mathbb{T})$ is proper, then the adjoint operator T_ω^* is precisely a Toeplitz-like operator of the type studied by Sarason. Hence in this case our Toeplitz-like operator $T_\omega = T_\omega^{**}$ coincides with the adjoint of the Toeplitz-like operator considered in [12]. Based on ideas in [12], we also show that $H(\overline{\mathbb{D}})$, the space of functions analytic on a neighborhood of $\overline{\mathbb{D}}$, is contained in $\mathrm{Dom}(T_\omega)$ and in fact is a core of T_ω.

In the last section of [12], Sarason introduces a class of closed, densely defined Toeplitz-like operators on H^2 determined by algebraic properties, which was further investigated by Rosenfeld in [10, 11]. In particular, this class of Toeplitz-like operators contains the unbounded Toeplitz-like operator studied by Sarason and is closed under taking adjoints, and hence contains our Toeplitz-like operators with proper symbols in $\mathrm{Rat}(\mathbb{T})$. In fact, we will show in Section 6 that T_ω is contained in the class of Toeplitz-like operators for any ω in Rat.

2. The operator T_{ω^*} for $\omega \in \mathrm{Rat}(\mathbb{T})$

In this section we recall some results from [5, 6] on the operator T_ω for $\omega \in \mathrm{Rat}(\mathbb{T})$ that we will use in the sequel, and apply them to the operator T_{ω^*}. Hence, throughout this section let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime. We set $m = \deg(q)$ and $n = \deg(s)$. Furthermore, factor $s = s_- s_0 s_+$ with s_-, s_0 and s_+

polynomials having roots only inside, on, or outside $\mathbb{T}$, respectively. We then recall from Theorem 2.2 in [6] that

$$\begin{aligned}\operatorname{Ker}(T_\omega) &= \{r/s_+ \mid \deg(r) < m - \deg(s_-s_0)\}\,;\\ \operatorname{Dom}(T_\omega) = qH^p + \mathcal{P}_{m-1}; &\quad \operatorname{Ran}(T_\omega) = sH^p + \widetilde{\mathcal{P}},\end{aligned} \tag{2.1}$$

where $\widetilde{\mathcal{P}}$ is the subspace of $\mathcal{P}$ given by

$$\widetilde{\mathcal{P}} = \{r \in \mathcal{P} \mid rq = r_1 s + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{m-1}\} \subset \mathcal{P}_{n-1}. \tag{2.2}$$

Furthermore, $H^p = \overline{\operatorname{Ran}(T_\omega)} + \widetilde{\mathcal{Q}}$ forms a direct sum decomposition of H^p, where

$$\widetilde{\mathcal{Q}} = \mathcal{P}_{k-1} \quad \text{with} \quad k = \max\{\deg(s_-) - m, 0\}, \tag{2.3}$$

following the convention $\mathcal{P}_{-1} := \{0\}$. Furthermore, the action of T_ω is as follows.

$$\begin{aligned}&T_\omega g = sh + \widetilde{r} \quad (g = qh + r \in qH^p + \mathcal{P}_{m-1} = \operatorname{Dom}(T_\omega)),\\ &\text{where } \widetilde{r} \in \mathcal{P}_{n-1} \text{ is such that } rs = \widetilde{r}q + r_2 \text{ for some } r_2 \in \mathcal{P}_{m-1}.\end{aligned}$$

We also recall from Lemma 5.3 in [5] that

$$T_{z^\kappa \omega} = T_{z^\kappa} T_\omega \quad \text{for any integer } \kappa \leq 0. \tag{2.4}$$

Recall that ω^* is defined as $\omega^*(z) = \overline{\omega(z)}$ on $\mathbb{T}$, i.e., $\omega^*(z) = \overline{s(z)}/\overline{q(z)}$. For $z \in \mathbb{T}$

$$\overline{q(z)} = \overline{q_0 + zq_1 + \cdots + z^m q_m} = \overline{q_0} + \overline{q_1}\frac{1}{z} + \cdots + \overline{q_m}\frac{1}{z^m} = \frac{1}{z^m} q^\sharp(z).$$

Hence $q^\sharp(z) = z^m \overline{q(z)}$, and likewise $s^\sharp(z) = z^n \overline{s(z)}$. Thus we have

$$\omega^*(z) = \frac{z^{m-n} s^\sharp(z)}{q^\sharp(z)} \text{ if } m \geq n \quad \text{and} \quad \omega^*(z) = \frac{s^\sharp(z)}{z^{n-m} q^\sharp(z)} \text{ if } m < n. \tag{2.5}$$

In fact, the formula $\omega^*(z) = z^{m-n} s^\sharp(z)/q^\sharp(z)$ holds in both cases, but is not always a representation as the ratio of two polynomials. Note in particular that $\omega^* \in \operatorname{Rat}(\mathbb{T})$ in case ω is proper, while this need not be the case if ω is not proper. Thus, if ω is proper, the above formulas apply directly, while for the non-proper case, using (2.4) we can reduce certain questions to questions concerning the Toeplitz operator $T_{s^\sharp/q^\sharp}$ with symbol $s^\sharp/q^\sharp$ which is in $\operatorname{Rat}(\mathbb{T})$.

A polynomial $r \neq 0$ is called self-inversive in case $r = \gamma r^\sharp$ for a constant $\gamma \in \mathbb{C}$, which necessarily is unimodular. In fact, γ is the ratio $r_0/\overline{r_n}$ with $r_0 = r(0)$ and r_n the leading coefficient of r. By a theorem of Cohn [2], a polynomial r has all its roots on $\mathbb{T}$ if and only if r is self-inversive and its derivative has all its roots in the closed unit disc $\overline{\mathbb{D}}$. Hence, any polynomial with roots only on $\mathbb{T}$ is self-inversive. In particular, $q = \gamma q^\sharp$ and $s_0 = \rho(s_0)^\sharp$ for unimodular constants γ and ρ.

More generally, in the transformation $r \to r^\sharp$, the nonzero roots of r (including multiplicity) transfer along the unit circle via the map $\alpha \mapsto 1/\overline{\alpha} = |\alpha|^{-2}\alpha$, while the degree decreases by the multiplicity of 0 as a root of r. Consequently, in the factorization $s^\sharp = (s_+)^\sharp (s_0)^\sharp (s_-)^\sharp$, the polynomials $(s_+)^\sharp$, $(s_0)^\sharp$ and $(s_-)^\sharp$ contain the roots of $s^\sharp$ inside, on and outside $\mathbb{T}$, respectively, taking multiplicities into account. We write $(s_+)^\sharp$ rather than $s_+^\sharp$, etc., to avoid confusion with what one may interpret as $(s^\sharp)_+$.

We now apply the above to T_{ω^*} acting on $H^{p'}$, $1 < p' < \infty$, to fit better with the remainder of the paper.

Proposition 2.1. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime, $m = \deg(q)$ and $n = \deg(s)$. Factor $s = s_- s_0 s_+$ with s_-, s_0 and s_+ polynomials having roots only inside, on, or outside $\mathbb{T}$, respectively. Then for T_{ω^*} on $H^{p'}$, $1 < p' < \infty$, we have*

$$\mathrm{Ker}(T_{\omega^*}) = \left\{ r_0/(s_-)^\sharp \mid \deg(r_0) < \deg(s_-) \right\}, \quad \mathrm{Dom}(T_{\omega^*}) = q^\sharp H^{p'} + \mathcal{P}_{m-1}. \tag{2.6}$$

Moreover, we have

$$\begin{aligned} \mathrm{Ran}(T_{\omega^*}) &= z^{m-n} s^\sharp H^{p'} + \widetilde{\mathcal{P}}_* \quad \text{if } m \geq n, \\ \mathrm{Ran}(T_{\omega^*}) &= T_{z^{m-n}}(s^\sharp H^{p'} + \widetilde{\mathcal{P}}_*) \quad \text{if } m < n, \end{aligned} \tag{2.7}$$

where for $m \geq n$ the subspace $\widetilde{\mathcal{P}}_*$ is given by

$$\widetilde{\mathcal{P}}_* = \{ r \in \mathcal{P} \mid r q^\sharp = z^{m-n} r_1 s^\sharp + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{m-1} \} \subset \mathcal{P}_{m-n+\deg(s^\sharp)-1}, \tag{2.8}$$

while for $m < n$ we have

$$\widetilde{\mathcal{P}}_* = \{ r \in \mathcal{P} \mid r q^\sharp = r_1 s^\sharp + r_2 \text{ for } r_1, r_2 \in \mathcal{P}_{m-1} \} \subset \mathcal{P}_{\deg(s^\sharp)-1}. \tag{2.9}$$

Furthermore, $\mathrm{Ran}(T_{\omega^*})$ *is dense in* $H^{p'}$.

Proof. We separate the cases $m \geq n$ and $m < n$.

For $m \geq n$, we have $\omega^* = \widetilde{s}/\widetilde{q} \in \mathrm{Rat}(\mathbb{T})$ with $\widetilde{s} = z^{m-n} s^\sharp$ and $\widetilde{q} = q^\sharp$. Hence $\widetilde{s}$ factors as $\widetilde{s} = (z^{m-n}(s_+)^\sharp)(s_0)^\sharp(s_-)^\sharp$, where the factors have all their roots inside, on, or outside $\mathbb{T}$, respectively. Also, $\deg(q^\sharp) = \deg(q)$ and $\deg((s_+)^\sharp) = \deg(s_+)$. So the formulas for $\mathrm{Dom}(T_{\omega^*})$ and $\mathrm{Ran}(T_{\omega^*})$ follow directly from (2.1), while the formula for $\mathrm{Ker}(T_{\omega^*})$ follows because the bound on the degree of r_0 can be computed as

$$m - \deg(z^{m-n}(s_+)^\sharp(s_0)^\sharp) = n - \deg((s_+)^\sharp(s_0)^\sharp) = n - \deg(s_+ s_0) = \deg(s_-).$$

Finally, a complement of the closure of $\mathrm{Ran}(T_{\omega^*})$ is given by $\mathcal{P}_{k-1}$ with k the maximum of 0 and $\deg(z^{m-n}(s_+)^\sharp) - m = \deg((s_+)^\sharp) - n \leq 0$. Hence $\mathcal{P}_{-1} = \{0\}$. Thus T_{ω^*} has dense range, as claimed.

In case $m < n$, we have $T_{\omega^*} = T_{z^{m-n}} T_{s^\sharp/q^\sharp}$ and $s^\sharp/q^\sharp$ is in $\mathrm{Rat}(\mathbb{T})$. Applying the above results for T_ω to $T_{s^\sharp/q^\sharp}$ directly gives the formulas for $\mathrm{Dom}(T_{\omega^*})$ and $\mathrm{Ran}(T_{\omega^*})$.

To see that the formula for $\mathrm{Ker}(T_{\omega^*})$ holds, we follow the argumentation of the proof of Lemma 4.1 in [5]. For $g \in \mathrm{Dom}(T_{\omega^*}) = \mathrm{Dom}(T_{s^\sharp/q^\sharp})$ to be in $\mathrm{Ker}(T_{\omega^*})$ is equivalent to $T_{s^\sharp/q^\sharp} g \in \mathcal{P}_{n-m-1}$. In other words, by Lemma 3.2 in [5], to $s^\sharp g = q^\sharp \widetilde{r} + r_1$ with $r_1 \in \mathcal{P}_{m-1}$ and $\widetilde{r} \in \mathcal{P}_{n-m-1}$, since then $T_{s^\sharp/q^\sharp} g = \widetilde{r}$. The latter happens precisely when $g = r/(s_-)^\sharp$ with $r \in \mathcal{P}_{\deg(s_-)-1}$. Indeed, in that case $\deg((s_+)^\sharp(s_0)^\sharp r) < n$ which in the equation $(s_+)^\sharp(s_0)^\sharp r = s^\sharp g = q^\sharp \widetilde{r} + r_1$ corresponds to $\deg(\widetilde{r}) < m-1$, as required. Finally, we note that a complement of $\overline{\mathrm{Ran}(T_{s^\sharp/q^\sharp})}$ in $H^{p'}$ is given by $\mathcal{P}_{k-1}$ with $k = \max\{0, \deg s_+{}^\sharp - m\} \leq n-m$. Let $f \in H^{p'}$ and write $z^{n-m} f = h + r \in \overline{\mathrm{Ran}(T_{s^\sharp/q^\sharp})} + \mathcal{P}_{k-1}$. Then $f = T_{z^{m-n}} z^{n-m} f = T_{z^{m-n}}(h + r) = T_{z^{m-n}} h \in T_{z^{m-n}} \overline{\mathrm{Ran}(T_{s^\sharp/q^\sharp})} \subset \overline{\mathrm{Ran}(T_{z^{m-n}} T_{s^\sharp/q^\sharp})} = \overline{\mathrm{Ran}(T_{\omega^*})}$. Thus also in this case $\mathrm{Ran}(T_{\omega^*})$ is dense in $H^{p'}$. $\square$

We conclude this section with a lemma will be of use in the sequel.

Lemma 2.2. *Let $r_1, r_2 \in \mathcal{P}$. Set $n_i = \deg(r_i)$, for $i = 1, 2$, and $n = \deg(r_1 + r_2)$. Then*

$$(r_1 + r_2)^\sharp = z^{n-n_1} r_1^\sharp + z^{n-n_2} r_2^\sharp.$$

In case $n < \max\{n_1, n_2\}$, then $n_1 = n_2$ and 0 is a root of $r_1^\sharp + r_2^\sharp$ with multiplicity $n - n_1$, so that the left hand side in the above identity still is a polynomial without a root at 0.

Proof. By definition, for $z \in \mathbb{T}$ we have

$$\begin{aligned}(r_1+r_2)^\sharp(z) &= z^n(\overline{r_1(1/\overline{z})} + \overline{r_2(1/\overline{z})}) = z^{n-n_1}z^{n_1}\overline{r_1(1/\overline{z})} + z^{n-n_2}z^{n_2}\overline{r_2(1/\overline{z})}\\ &= z^{n-n_1}r_1^\sharp(z) + z^{n-n_2}r_2^\sharp(z). \qquad \square\end{aligned}$$

3. The adjoint of T_ω for $\omega \in \mathrm{Rat}(\mathbb{T})$

In this section we prove the first main result, Theorem 1.1, for the special case that $\omega \in \mathrm{Rat}(\mathbb{T})$. In this case, the result specializes to the following theorem, which we prove in this section.

Theorem 3.1. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime and $1 < p < \infty$. Set $m = \deg(q)$ and $n = \deg(s)$ and let $1 < p' < \infty$ with $1/p + 1/p' = 1$. Then*

$$\mathrm{Dom}(T_\omega^*) = q^\sharp H^{p'} \subset \mathrm{Dom}(T_{\omega^*}) \quad \textit{and} \quad T_\omega^* = T_{\omega^*}|_{q^\sharp H^{p'}}. \tag{3.1}$$

In fact, for $g = q^\sharp v \in q^\sharp H^{p'}$ we have $T_\omega^* g = T_{z^{m-n}} s^\sharp v$. Moreover, factorize $s = s_- s_0 s_+$ with s_-, s_0 and s_+ polynomials having roots only inside, on, or outside $\mathbb{T}$, respectively. Then

$$\mathrm{Ran}(T_\omega^*) = T_{z^{m-n}} s^\sharp H^{p'} \ \textit{and}\ \mathrm{Ker}(T_\omega^*) = \left\{ \frac{q^\sharp r}{(s_-)^\sharp} \mid \deg(r) < \deg(s_-) - m \right\}. \tag{3.2}$$

In particular, we have

$$\dim \mathrm{Ker}(T_\omega^*) = \max\{0, \#\{\textit{zeroes of } \omega^* \textit{ outside } \mathbb{T}\} - \#\{\textit{poles of } \omega^* \textit{ on } \mathbb{T}\}\},$$

where the multiplicities of the zeroes and poles are taken into account. Thus T_ω^* is injective if and only if ω has at least as many poles inside $\mathbb{T}$ as zeroes inside $\mathbb{T}$ unequal to 0, multiplicities taken into account.

We first present some auxiliary lemmas. Throughout, let $1 < p, p' < \infty$ such that $1/p + 1/p' = 1$. We will consider T_ω as an operator with domain in H^p and T_{ω^*} as an operator with domain in $H^{p'}$.

Lemma 3.2. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime, $m = \deg(q)$ and $n = \deg(s)$. Then*

$$q^\sharp H^{p'} \subset \mathrm{Dom}(T_\omega^*) \cap \mathrm{Dom}(T_{\omega^*}) \quad \textit{and} \quad T_\omega^*|_{q^\sharp H^{p'}} = T_{\omega^*}|_{q^\sharp H^{p'}}.$$

Moreover, for $g = q^\sharp v \in q^\sharp H^{p'}$, with $v \in H^{p'}$, we have $T_\omega^* g = T_{z^{m-n}} s^\sharp v$, and thus $T_\omega^*(q^\sharp H^{p'}) = T_{z^{m-n}} s^\sharp H^{p'}$.

Proof. The inclusion $q^\sharp H^{p'} \subset \mathrm{Dom}(T_{\omega^*})$ follows from Proposition 2.1. Let g be in $q^\sharp H^{p'}$, say $g(z) = q^\sharp(z) v(z)$ for $v \in H^{p'}$. We show that for $f \in \mathrm{Dom}(T_\omega)$ we have $\langle T_w f, g\rangle_{p,p'} = \langle f, T_{\omega^*} g\rangle_{p,p'}$. Let $f \in \mathrm{Dom}(T_\omega)$ and $h = T_\omega f \in H^p$, i.e., $sf = qh + r$ for some $r \in \mathcal{P}_{m-1}$, by [5, Lemma 2.3]. Then

$$\begin{aligned}\langle T_\omega f, g\rangle_{p,p'} &= \langle h, q^\sharp v\rangle_{p,p'} = \langle h, z^m \overline{q} v\rangle_{p,p'} = \langle qh, z^m v\rangle_{p,p'}\\ &= \langle sf - r, z^m v\rangle_{p,p'} = \langle sf, z^m v\rangle_{p,p'} \quad (\text{because } \deg(r) < m,\ v \in H^{p'})\\ &= \langle f, z^m \overline{s} v\rangle_{p,p'} = \langle f, z^{m-n} s^\sharp v\rangle_{p,p'} = \langle f, T_{z^{m-n}} s^\sharp v\rangle_{p,p'} \ (\text{because } f \in H^p).\end{aligned}$$

It remains to show that $T_{\omega^*}g = T_{z^{m-n}}s^\sharp v$. If $m \geq n$, then $\omega^* = z^{m-n}s^\sharp/q^\sharp$ is in $\mathrm{Rat}(\mathbb{T})$ and $\omega^* g = z^{m-n}s^\sharp v \in H^{p'}$, so that, $T_{\omega^*}g = z^{m-n}s^\sharp v = T_{z^{m-n}}s^\sharp v$, by Lemma 2.3 in [5]. In case $m < n$, we have $T_{\omega^*}g = T_{z^{m-n}}T_{s^\sharp/q^\sharp}g = T_{z^{m-n}}s^\sharp v$. □

Lemma 3.3. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime, $m = \deg(q)$ and $n = \deg(s)$. Let $g \in \mathrm{Dom}(T_\omega^*)$ and $k = T_\omega^* g \in H^{p'}$. Then for any $r \in \mathcal{P}_{n-1}$ and $r_1 \in \mathcal{P}_{m-1}$ so that*

$$sr_1 = qr + r_2 \text{ for some } r_2 \in \mathcal{P}_{m-1} \tag{3.3}$$

we have

$$\langle r_1, k\rangle_{p,p'} = \langle r, g\rangle_{p,p'}.$$

Moreover, we have

$$z^{m-n}s^\sharp g - q^\sharp k \in \mathcal{P}_{m-1} \text{ if } m \geq n \quad \text{and} \quad s^\sharp g - z^{n-m}q^\sharp k \in \mathcal{P}_{n-1} \text{ if } m < n. \tag{3.4}$$

In particular, $\mathrm{Dom}(T_\omega^*) \subset \mathrm{Dom}(T_{\omega^*})$ and $T_\omega^* = T_{\omega^*}|_{\mathrm{Dom}(T_\omega^*)}$.

Proof. Let $g \in \mathrm{Dom}(T_\omega^*)$ and $k = T_\omega^* g$. Hence $\langle T_\omega f, g\rangle_{p,p'} = \langle f, k\rangle_{p,p'}$ for each $f \in \mathrm{Dom}(T_\omega)$. Since $\omega \in \mathrm{Rat}(\mathbb{T})$, we have $\mathrm{Dom}(T_\omega) = qH^p + \mathcal{P}_{m-1}$. Let $f = qh + r_1 \in \mathrm{Dom}(T_\omega)$, with $h \in H^p$ and $r_1 \in \mathcal{P}_{m-1}$. Then $T_\omega f = sh + r$ where $r \in \mathcal{P}_{n-1}$ is uniquely determined by (3.3). Thus

$$\langle sh, g\rangle + \langle r, g\rangle = \langle sh + r, g\rangle = \langle T_\omega f, g\rangle = \langle f, k\rangle = \langle qh + r_1, k\rangle = \langle qh, k\rangle + \langle r_1, k\rangle.$$

We obtain that

$$\langle sh, g\rangle - \langle qh, k\rangle = \langle r_1, k\rangle - \langle r, g\rangle.$$

However, in choosing $f \in \mathrm{Dom}(T_\omega)$ we can choose $h \in H^p$ and $r_1 \in \mathcal{P}_{m-1}$ independently, and in particular set one or the other equal to zero, resulting in

$$\langle sh, g\rangle = \langle qh, k\rangle \quad (h \in H^p), \quad \langle r_1, k\rangle = \langle r, g\rangle \quad (r \in \mathcal{P}_{n-1}, r_1 \in \mathcal{P}_{m-1} \text{ as in (3.3)}).$$

The second identity proves the first claim of the lemma. From the first identity we obtain that

$$0 = \langle h, \overline{s}g - \overline{q}k\rangle_{p,p'} = \langle h, z^{-n}s^\sharp g - z^{-m}q^\sharp k\rangle_{p,p'} \quad (h \in H^p).$$

Thus $\mathbb{P}(z^{-n}s^\sharp g - z^{-m}q^\sharp k) = 0$. On the other hand, for $l = \max\{m, n\}$ we have

$$z^l(z^{-n}s^\sharp g - z^{-m}q^\sharp k) = z^{l-n}s^\sharp g - z^{l-m}q^\sharp k \in H^{p'}.$$

This can only occur if $z^{l-n}s^\sharp g - z^{l-m}q^\sharp k \in \mathcal{P}_{l-1}$, which proves the second claim.

To complete the proof, we show that $g \in \mathrm{Dom}(T_{\omega^*})$ and $T_{\omega^*}g = k$. For $m \geq n$ we have $\omega^* \in \mathrm{Rat}(\mathbb{T})$ and the first inclusion of (3.4) can be rewritten as

$$\omega^* g = \left(\frac{z^{m-n}s^\sharp}{q^\sharp}\right) g = k + \widetilde{r}/q^\sharp, \quad \text{for some } \ \widetilde{r} \in \mathcal{P}_{m-1}.$$

Since $\deg(q^\sharp) = \deg(q) = m$, it now follows that $g \in \mathrm{Dom}(T_{\omega^*})$ and $T_{\omega^*}g = k$. In case $m < n$ we have $T_{\omega^*} = T_{z^{m-n}}T_{s^\sharp/q^\sharp}$ and $s^\sharp/q^\sharp \in \mathrm{Rat}(\mathbb{T})$. Now the second inclusion of (3.4) gives

$$\left(\frac{s^\sharp}{q^\sharp}\right) g = z^{n-m}k + \widetilde{r}/q^\sharp, \quad \text{for some } \ \widetilde{r} \in \mathcal{P}_{n-1}.$$

Write $\widetilde{r} = \widetilde{r}_1 q^\sharp + \widetilde{r}_2$ with $\widetilde{r}_2 \in \mathcal{P}_{m-1}$. Then $\widetilde{r}/q^\sharp = \widetilde{r}_1 + \widetilde{r}_2/q^\sharp$ and $\deg(\widetilde{r}_1) < m - n$. Since $\widetilde{r}_2/q^\sharp \in \mathrm{Rat}_0(\mathbb{T})$ it follows that $g \in \mathrm{Dom}(T_{s^\sharp/q^\sharp}) = \mathrm{Dom}(T_{\omega^*})$ and $T_{s^\sharp/q^\sharp}g = z^{n-m}k + \widetilde{r}_1$. But then $T_{\omega^*}g = T_{z^{m-n}}T_{s^\sharp/q^\sharp}g = T_{z^{m-n}}(z^{n-m}k + \widetilde{r}_1) = k$. □

A special case of the following result was proven as part of the proof of Theorem 2.2 in [6].

Lemma 3.4. *Let $r, \widetilde{r} \in \mathcal{P}$ be co-prime. Then $rH^p \cap \widetilde{r}H^p = r\widetilde{r}H^p$.*

Proof. Let $\widetilde{r}f = rg$ with $f, g \in H^p$. Then $f = r \cdot g/\widetilde{r} \in H^p$, so we should show $\widetilde{f} := g/\widetilde{r} \in H^p$, i.e., $\widetilde{f}$ analytic on $\mathbb{D}$ and $\int_{\mathbb{T}} |\widetilde{f}(z)|^p\, dz < \infty$.

Since $g \in H^p$, the function $\widetilde{f}$ can only fail to be analytic at the roots of $\widetilde{r}$ inside $\mathbb{D}$. However, if this were the case, then $f = r\widetilde{f}$ would also fail to be analytic in $\mathbb{D}$, since r and $\widetilde{r}$ are co-prime. Thus $\widetilde{f}$ is analytic on $\mathbb{D}$.

Divide $\mathbb{T}$ as $\mathbb{T}_1 \cup \mathbb{T}_2$ with $\mathbb{T}_1 \cap \mathbb{T}_2 = \emptyset$ in such a way that $\mathbb{T}_1$ and $\mathbb{T}_2$ are both nonempty finite unions of line segments of $\mathbb{T}$ so that the interior of $\mathbb{T}_1$ contains the roots of r and the interior of $\mathbb{T}_2$ the roots of $\widetilde{r}$. Then $|\widetilde{r}(z)| > N_1$ on $\mathbb{T}_1$ and $|r(z)| > N_2$ on $\mathbb{T}_2$ for some $N_1, N_2 > 0$. Note that $f = r\widetilde{f}$ and $g = \widetilde{r}\widetilde{f}$. We then obtain

$$\int_{\mathbb{T}_2} |\widetilde{f}(z)|^p\, dz = \int_{\mathbb{T}_2} |f(z)/r(z)|^p\, dz \leq N_2^{-p} \int_{\mathbb{T}_2} |f(z)|^p\, dz \leq (2\pi N_2^p)^{-1} \|f\|^p_{H^p}.$$

Using $g = \widetilde{r}\widetilde{f}$, one obtains similarly that $\int_{\mathbb{T}_1} |\widetilde{f}(z)|^p\, dz \leq (2\pi N_1^p)^{-1} \|g\|^p_{H^p}$. Thus $\int_{\mathbb{T}} |\widetilde{f}(z)|^p\, dz < \infty$. $\square$

Proof of Theorem 3.1. By Lemma 3.2, in order to prove (3.1), the formula for the action of T_ω^* on $q^\sharp H^{p'}$ and for the range of T_ω^* in (3.2), it remains to show that $\mathrm{Dom}(T_\omega^*) \subset q^\sharp H^{p'}$.

View $\mathcal{P}$ and $\mathcal{P}_k$, $k = 1, 2, \ldots$, as subspaces of H^p or $H^{p'}$, write P_k for the projection onto $\mathcal{P}_{k-1}$ and set $Q_k = I - P_k$. Also, the standard $k \times k$ compression of a Toeplitz operator T_ϕ on H^p (or $H^{p'}$) is denoted by $T_{\phi,k}$, i.e., $T_{\phi,k} = P_k T_\phi|_{\mathcal{P}_{k-1}}$. Now, the relation (3.3) between $r \in \mathcal{P}_{n-1}$ and $r_1 \in \mathcal{P}_{m-1}$ can be rewritten as

$$T_s r_1 - T_q r \in \mathcal{P}_{m-1},$$

or, equivalently, as

$$Q_m T_s P_m r_1 = Q_m T_s r_1 = Q_m T_q r = Q_m T_q P_n r. \tag{3.5}$$

We now consider the cases $m \geq n$ and $m < n$ separately.

First assume $m \geq n$. We can then decompose $Q_m T_s P_m$ and $Q_m T_q P_n$ as

$$Q_m T_s P_m = \begin{bmatrix} 0 & T^*_{s^\sharp,n} T^*_{z^{m-n}} \\ 0 & 0 \end{bmatrix} : \mathcal{P}_{m-1} = \begin{bmatrix} \mathcal{P}_{m-n} \\ T_{z^{m-n}}\mathcal{P}_{n-1} \end{bmatrix} \to \begin{bmatrix} \mathcal{P}_{n-1} \\ T_z^n H^p \end{bmatrix},$$
$$Q_m T_q P_n = \begin{bmatrix} T^*_{q^\sharp,n} \\ 0 \end{bmatrix} : \mathcal{P}_{n-1} \to \begin{bmatrix} \mathcal{P}_{n-1} \\ T_{z^n} H^p \end{bmatrix}.$$

Hence, in this case the identity in (3.5) can be write as

$$T^*_{s^\sharp,n}(T^*_{z^{m-n}} r_1) = T^*_{q^\sharp,n} r.$$

Since all Toeplitz matrices are upper triangular, we in fact have

$$T^*_{s^\sharp,m} T^*_{z^{m-n},m} r_1 = T^*_{q^\sharp,m} r.$$

Note that $T^*_{q^\sharp,n}$ is invertible, because q has only roots on $\mathbb{T}$ so that $q(0) \neq 0$. We obtain that for given $r_1 \in \mathcal{P}_{m-1}$, the polynomial $r \in \mathcal{P}_{n-1}$ that satisfies (3.3) is uniquely determined by

$$r = (T^*_{q^\sharp,m})^{-1} T^*_{s^\sharp,m} T^*_{z^{m-n},m} r_1 = T^*_{s^\sharp,m} T^{*m-n}_{z,m} (T^*_{q^\sharp,m})^{-1} r_1,$$

where the commutation of Toeplitz matrices can occur since they all have analytic symbols. Now take $r_1 \in \mathcal{P}_{m-1}$ arbitrary, and define r as above, so that (3.3) holds. Then, by Lemma 3.3, we have

$$\begin{aligned}
\langle r_1, P_m k\rangle_{\mathcal{P}_{m-1}} &= \langle r_1, k\rangle_{p,p'} = \langle r, g\rangle_{p,p'} = \langle r, P_m g\rangle_{\mathcal{P}_{m-1}} \\
&= \langle T^*_{s^\sharp,m} T^{*m-n}_{z,m} (T^*_{q^\sharp,m})^{-1} r_1, P_m g\rangle_{\mathcal{P}_{m-1}} \\
&= \langle r_1, (T_{q^\sharp,m})^{-1} T^{m-n}_{z,m} T_{s^\sharp,m} P_m g\rangle_{\mathcal{P}_{m-1}}.
\end{aligned}$$

Since $r_1 \in \mathcal{P}_{m-1}$ is arbitrary, we obtain that $P_m k = (T_{q^\sharp,m})^{-1} T^{m-n}_{z,m} T_{s^\sharp,m} P_m g$, and thus

$$P_m T_{q^\sharp} k = T_{q^\sharp,m} P_m k = T^{m-n}_{z,m} T_{s^\sharp,m} P_m g = P_m T^{m-n}_z T_{s^\sharp} g.$$

This shows that $P_m q^\sharp k = P_m z^{m-n} s^\sharp g$. Together with the first inclusion in (3.4) we obtain that

$$q^\sharp k = z^{m-n} s^\sharp g.$$

Since $q^\sharp$ and $z^{m-n}s^\sharp$ are co-prime, we can apply Lemma 3.4 to conclude $g \in q^\sharp H^{p'}$.

Next assume $m < n$. We can then write $\omega = \omega_0 + \omega_1$ uniquely with $\omega_0 \in \mathrm{Rat}_0(\mathbb{T})$ and $\omega_1 \in \mathrm{Rat}$ with no poles on $\mathbb{T}$, i.e, $\omega_1 \in L^\infty(\mathbb{T})$, see [5, Lemma 2.4]. In fact $\omega_1 \in \mathcal{P}$, since all poles of ω are on $\mathbb{T}$, and $\omega_0 = \widetilde{s}/q$ with $\widetilde{s} \in \mathcal{P}_{m-1}$. It now follows that $\mathrm{Dom}(T^*_{\omega_0}) = q^\sharp H^{p'}$, and since T_{ω_1} is bounded, $\mathrm{Dom}(T^*_\omega) = \mathrm{Dom}(T^*_{\omega_0}) = q^\sharp H^{p'}$. Furthermore, $T^*_\omega = T^*_{\omega_0} + T^*_{\omega_1}|_{q^\sharp H^{p'}} = T_{\omega_0^*}|_{q^\sharp H^{p'}} + T_{\omega_1^*}|_{q^\sharp H^{p'}} = T_{\omega^*}|_{q^\sharp H^{p'}}$.

In the next part of the proof we prove the formula for $\mathrm{Ker}(T_{\omega^*})$, without distinguishing between the proper and non-proper case. Let $g = q^\sharp v \in \mathrm{Dom}(T^*_\omega)$ with $v \in H^{p'}$. Then $g \in \mathrm{Ker}(T^*_\omega)$ if and only if $g \in \mathrm{Ker}(T_{\omega^*})$, i.e., $g = q^\sharp v = r_1/(s_-)^\sharp$ for $r_1 \in \mathcal{P}_{\deg(s_-)-1}$, see Proposition 2.1. Thus $v = r_1/((s_-)^\sharp q^\sharp) \in \mathrm{Rat} \cap H^{p'}$. Then $v \in H^{p'}$ implies $r_1 = q^\sharp r$, and $\deg(r) = \deg(r_1) - m < \deg(s_-) - m$. Hence $g = q^\sharp r/(s_-)^\sharp$ with $\deg(r) < \deg(s_-) - m$. That all such functions are in $\mathrm{Ker}(T^*_\omega) = \mathrm{Ker}(T_{\omega^*}) \cap q^\sharp H^{p'}$ follows directly from the formula for $\mathrm{Ker}(T_{\omega^*})$ obtained in Proposition 2.1. The formula for the dimension of $\mathrm{Ker}(T^*_\omega)$ follows directly and the condition for injectivity follows since $\deg(s_-)^\sharp$ is equal to the number on nonzero roots of s_-, counting multiplicity. □

4. The adjoint of T_ω: General case

In the section we prove Theorem 1.1 in full generality. Hence let $\omega = s/q \in \mathrm{Rat}$ with $s, q \in \mathcal{P}$ co-prime. As in Theorem 1.1, factor $s = s_- s_0 s_+$ and $q = q_- q_0 q_+$ with s_-, q_- having roots only inside $\mathbb{T}$, s_0, q_0 having roots only on $\mathbb{T}$, and s_+, q_+ having roots only outside $\mathbb{T}$. Set $m = \deg(q)$, $n = \deg(s)$, $m_\pm = \deg(q_\pm)$, $n_\pm = \deg(s_\pm)$, and $m_0 = \deg(q_0)$, $n_0 = \deg(s_0)$. By Lemma 5.1 in [5], and its proof, we can factor ω as $\omega = \omega_-(z^\kappa \omega_0)\omega_+$ with $\kappa = n_- - m_-$, $\omega_- = s_-/(z^\kappa q_-)$ having only poles and zeroes inside $\mathbb{T}$, $\omega_0 = s_0/q_0$ having only poles and zeroes on $\mathbb{T}$, and $\omega_+ = s_+/q_+$ having only poles and zeroes outside $\mathbb{T}$, and we have $T_\omega = T_{\omega_-} T_{z^\kappa \omega_0} T_{\omega_+}$. Moreover, T_{ω_-} and T_{ω_+} are bounded and boundedly invertible.

Note that $T_{\omega_-} T_{z^\kappa \omega_0}$ is closed and densely defined and $\mathrm{Ran}(T_{\omega_+}) = H^p$, and thus by Corollary 1 in [13]

$$T^*_\omega = T^*_{\omega_+} \left(T_{\omega_-} T_{z^\kappa \omega_0}\right)^*.$$

Furthermore, T_{ω_-} is bounded and $T_{z^\kappa \omega_0}$ is closed and densely defined. By Theorem 4 in [1] one has

$$\left(T_{\omega_-} T_{z^\kappa \omega_0}\right)^* = T^*_{z^\kappa \omega_0} T^*_{\omega_-}.$$

Combining this and using that $T^*_{\omega_+} = T_{\omega_+^*}$ and $T^*_{\omega_-} = T_{\omega_-^*}$ we see that

$$T^*_\omega = T^*_{\omega_+} T^*_{z^\kappa \omega_0} T^*_{\omega_-} = T_{\omega_+^*} T^*_{z^\kappa \omega_0} T_{\omega_-^*} \quad \text{on } \mathrm{Dom}(T^*_\omega).$$

Note that

$$\omega_-^* = \frac{(s_-)^\sharp}{(q_-)^\sharp},\ \ \omega_0^* = z^{m_0 - n_0} \frac{(s_0)^\sharp}{(q_0)^\sharp},\ \ (z^\kappa \omega_0)^* = z^{m_0 - n_0 - \kappa} \frac{(s_0)^\sharp}{(q_0)^\sharp},\ \ \omega_+^* = z^{m_+ - n_+} \frac{(s_+)^\sharp}{(q_+)^\sharp}.$$

By construction, ω_- and $1/\omega_-$ are both anti-analytic. Consequently, ω_-^* and $1/\omega_-^*$ are both analytic functions. This implies $T^{\pm}_{\omega_-^*}(q_0)^\sharp H^{p'} \subset (q_0)^\sharp H^{p'}$, and thus $T_{\omega_-^*}(q_0)^\sharp H^{p'} = (q_0)^\sharp H^{p'}$. Since $T_{\omega_+^*}$ is invertible, to see that $\mathrm{Dom}(T^*_\omega) = (q_0)^\sharp H^{p'}$ it suffices to show $\mathrm{Dom}(T^*_{z^\kappa \omega_0}) = (q_0)^\sharp H^{p'}$. For the case where $\kappa \geq 0$, so that $z^\kappa \omega_0 \in \mathrm{Rat}(\mathbb{T})$, this follows directly from Theorem 3.1. For $\kappa < 0$, note that $T_{z^\kappa \omega_0} = T_{z^\kappa} T_{\omega_0}$, so that $T^*_{z^\kappa \omega_0} = T^*_{\omega_0} T^*_{z^\kappa} = T^*_{\omega_0} T_{z^{-\kappa}}$, again using Theorem 4 of [1]. Then $g \in \mathrm{Dom}(T^*_{z^\kappa \omega_0})$ holds if and only if $z^{-\kappa} g \in \mathrm{Dom}(T^*_{\omega_0}) = (q_0)^\sharp H^{p'}$. By Lemma 3.4 this is the same as $g \in (q_0)^\sharp H^{p'}$, since $z^{-\kappa}$ and $q_0^\sharp$ are co-prime. Thus in both cases we arrive at $\mathrm{Dom}(T^*_\omega) = (q_0)^\sharp H^{p'}$. Moreover, we also find that $T^*_{z^\kappa \omega_0} = T_{(z^\kappa \omega_0)^*}|_{(q_0)^\sharp H^{p'}}$, so that

$$T^*_\omega = T_{\omega_+^*} T^*_{z^\kappa \omega_0} T_{\omega_-^*} = T_{\omega_+^*} T_{(z^\kappa \omega_0)^*} T_{\omega_-^*}|_{(q_0)^\sharp H^{p'}} = T_{\omega^*}|_{(q_0)^\sharp H^{p'}}.$$

Hence (1.2) holds.

Next we derive the formula for $\mathrm{Ker}(T^*_\omega)$. For $\kappa \geq 0$ we have $g \in \mathrm{Ker}(T^*_\omega)$ if and only if $T_{\omega_-^*} g \in \mathrm{Ker}(T^*_{z^\kappa \omega_0}) = (q_0)^\sharp \mathcal{P}_{\kappa - m_0 - 1}$, where the last identity follows by applying Theorem 3.1 to $z^\kappa \omega_0$. Thus $g \in \mathrm{Ker}(T^*_\omega)$ if and only if $((s_-)^\sharp/(q_-)^\sharp) g = (q_0)^\sharp r$, i.e., $g = (q_-)^\sharp (q_0)^\sharp r/(s_-)^\sharp$, for some $r \in \mathcal{P}_{\kappa - m_0 - 1}$, as claimed. For $\kappa < 0$ we have $g \in \mathrm{Ker}(T^*_\omega)$ if and only if $z^{-\kappa} \omega_-^* g \in \mathrm{Ker}(T^*_{\omega_0})$. However, $\mathrm{Ker}(T^*_{\omega_0}) = \{0\}$, by Theorem 3.1, so that $\mathrm{Ker}(T^*_\omega) = \{0\}$, in line with the formula in (1.3). The formula for the dimension of $\mathrm{Ker}(T^*_\omega)$ follows directly.

Now we turn to the formula for $\mathrm{Ran}(T^*_\omega)$. Note that

$$\mathrm{Ran}(T^*_\omega) = T_{\omega_+^*} \mathrm{Ran}(T^*_{z^\kappa \omega_0} T_{\omega_-^*}) = T_{\omega_+^*} \mathrm{Ran}(T^*_{z^\kappa \omega_0}). \tag{4.1}$$

We first show that $\mathrm{Ran}(T^*_{z^\kappa \omega_0}) = T_{z^{m_0 - n_0 - \kappa}} (s_0)^\sharp H^{p'}$. Again, for the case $\kappa \geq 0$ this follows directly from Theorem 3.1. Assume $\kappa < 0$. Then $T^*_{z^\kappa \omega_0} = T^*_{\omega_0} T_{z^{-\kappa}}$. Hence,

$$\begin{aligned}\mathrm{Ran}(T^*_{z^\kappa \omega_0}) &= T^*_{\omega_0}(z^{-\kappa} H^{p'} \cap \mathrm{Dom}(T_{\omega_0})) = T^*_{\omega_0}(z^{-\kappa} H^{p'} \cap (q_0)^\sharp H^{p'}) \\ &= T^*_{\omega_0} z^{-\kappa} (q_0)^\sharp H^{p'}.\end{aligned}$$

The last identity follows by Lemma 3.4. Now the action of $T^*_{\omega_0}$, as described in Theorem 3.1, shows that $\mathrm{Ran}(T^*_{z^\kappa \omega_0}) = T_{z^{m_0 - n_0}} z^{-\kappa} (s_0)^\sharp H^{p'} = T_{z^{m_0 - n_0 - \kappa}} (s_0)^\sharp H^{p'}$. Since $1/q_+$ is analytic, $1/(q_+)^\sharp$ is anti-analytic, and therefore, independent of the sign of $m_+ - n_+$, we have

$$T_{\omega_+^*} = T_{1/(q_+)^\sharp} T_{z^{m_+ - n_+}} T_{(s_+)^\sharp}.$$

Thus

$$\mathrm{Ran}(T^*_\omega) = T_{1/(q_+)^\sharp} T_{z^{m_+ - n_+}} T_{(s_+)^\sharp} T_{z^{m_0 - n_0 - \kappa}} (s_0)^\sharp H^{p'}.$$

Note that $T_{(s_+)^\sharp}$ and $T_{z^{m_0-n_0-\kappa}}$ need not commute, in case $m_0 - n_0 - \kappa < 0$. However, we do have $T_{(s_+)^\sharp} T_{z^{m_0-n_0-\kappa}} = T_{z^{m_0-n_0-\kappa}} T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0}$. Moreover, since $(s_+)^\sharp$ is analytic, $T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0} = Q_{\kappa+n_0-m_0} T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0}$ and we have

$$T_{z^{m_+-n_+}} T_{z^{m_0-n_0-\kappa}} Q_{\kappa+n_0-m_0} = T_{z^{m_+-n_++m_0-n_0-\kappa}} Q_{\kappa+n_0-m_0} = T_{z^{m-n}} Q_{\kappa+n_0-m_0}.$$

Therefore, we have

$$\begin{aligned}\mathrm{Ran}(T_\omega^*) &= T_{1/(q_+)^\sharp} T_{z^{m-n}} T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0} (s_0)^\sharp H^{p'} \\ &= T_{z^{m-n}(s_+)^\sharp/(q_+)^\sharp} Q_{\kappa+n_0-m_0} (s_0)^\sharp H^{p'},\end{aligned}$$

again using that $1/(q_+)^\sharp$ is anti-analytic and $(s_+)^\sharp$ is analytic. This gives the general formula for $\mathrm{Ran}(T_\omega^*)$. In case $\kappa + n_0 - m_0 \leq 0$, we have $Q_{\kappa+n_0-m_0} = I$ and $T_{(s_+)^\sharp} Q_{\kappa+n_0-m_0} (s_0)^\sharp = (s_+ s_0)^\sharp$, as claimed.

5. Symmetric operators and selfadjoint extensions

For $\omega \in \mathrm{Rat}$, the second adjoint T_ω^{**} is well-defined and $T_\omega^{**} = T_\omega$, since T_ω is a closed, densely defined operator on a reflexive Banach space [8, Theorem III.5.24]. Now consider $\omega \in \mathrm{Rat}(\mathbb{T})$ and $p = 2$. From Theorem 1.1 it is obvious that $T_\omega \neq T_\omega^*$, except in the degenerate case where q is constant, since $\mathrm{Dom}(T_\omega) = qH^2 + \mathcal{P}_{\deg(q)-1}$ contains all polynomials while $\mathrm{Dom}(T_\omega^*) = q^\sharp H^2$ only contains the polynomials that contain $q^\sharp$ as a factor. Consequently, T_ω cannot be selfadjoint. In this section we consider the question when T_ω^* is symmetric, and, if this is the case, when does T_ω^* have a selfadjoint extension L. The first topic is addressed in the following theorem.

Theorem 5.1. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime. Set $n = \deg(s)$ and $m = \deg(q)$. Then the following are equivalent.*

1. *T_ω^* is symmetric;*
2. *$\omega(\mathbb{T}) \subset \mathbb{R}$;*
3. *$\omega(z) = \widetilde{\omega}(-i\frac{z+1}{z-1})$ with $\widetilde{\omega}$ a real rational function with poles only on $\mathbb{R}$;*
4. *the essential spectrum $\sigma_{\mathrm{ess}}(T_\omega)$ of T_ω is contained in $\mathbb{R}$;*
5. *ω is proper, $s = z^{m-n}\widetilde{s}$ with $\widetilde{s}$ self-inversive and $q_0\overline{s_n} = \overline{q_m} s_{m-n}$ holds, where $s(z) = \sum_{k=0}^n s_k z^k$ and $q(z) = \sum_{k=0}^m q_k z^k$.*

Moreover, if T_ω^* is symmetric, then $T_\omega^* \subset T_\omega$.

Proof. We first prove the equivalence of (1) and (2), and that (1) implies $T_\omega^* \subset T_\omega$. Assume (2). Then, for $z \in \mathbb{T}$, not a root of q, we have $\omega^*(z) = \overline{\omega(z)} = \omega(z)$. Hence $\omega^* = \omega$. Since q has only roots on $\mathbb{T}$, we have $q = \gamma q^\sharp$ for a unimodular constant γ. Hence $qH^2 = q^\sharp H^2$. This shows $T_\omega^* = T_{\omega^*}|_{q^\sharp H^2} = T_\omega|_{qH^2} \subset T_\omega$. Since $(T_\omega^*)^* = T_\omega$, it follows that T_ω^* is symmetric and $T_\omega^* \subset T_\omega$. Conversely, assume (1). Then we still have $qH^2 = q^\sharp H^2$ and $T_\omega^* \subset (T_\omega^*)^* = T_\omega$. Hence $T_\omega^* = T_\omega|_{qH^2}$. In particular, we have $\omega^* q = T_{\omega^*} q = T_\omega^* q = T_\omega q = \omega q$. This implies $\omega = \omega^*$. Hence $\omega(z) = \overline{\omega(z)}$ for $z \in \mathbb{T}$, not a root of q. Thus $\omega(\mathbb{T}) \subset \mathbb{R}$.

That (2) and (3) are equivalent follows simply because in (3) ω is the composition of $\widetilde{\omega}$ and the inverse Cayley transform, which maps the circle $\mathbb{T}$ bijectively onto $\mathbb{R}$. The fact that $\widetilde{\omega}$ is real rational, i.e., $\widetilde{\omega} = \widetilde{s}/\widetilde{q}$ with $\widetilde{s}$ and $\widetilde{q}$ real polynomials, is equivalent to $\widetilde{\omega}(\mathbb{R}) := \{\widetilde{\omega}(t) \colon t \in \mathbb{R},\ \widetilde{q}(t) \neq 0\} \subset \mathbb{R}$. Also, the equivalence of (2) and (4) is a direct consequence of the fact that $\sigma_{\mathrm{ess}}(T_\omega) = \omega(\mathbb{T})$, by [6, Theorem 1.1].

Finally, we prove (2) $\Leftrightarrow$ (5). Since $q = \gamma q^\sharp$, we have

$$\omega^* = z^{m-n}\frac{s^\sharp}{q^\sharp} = z^{m-n}\gamma\frac{s^\sharp}{q}.$$

Thus, we have $\omega = \omega^*$ if and only if $z^{m-n}\gamma s^\sharp = s$. Hence (2) is equivalent to $z^{m-n}\gamma s^\sharp = s$. Now assume (2). Since $\deg(s^\sharp) \le \deg(s)$, the identity $z^{m-n}\gamma s^\sharp = s$ can only occur if $m \ge n$, i.e., if ω is proper. The identity also shows that $s = z^{m-n}\widetilde{s}$ for $\widetilde{s} = \gamma s^\sharp$. On the other hand, $s^\sharp = (z^{m-n}\widetilde{s})^\sharp = \widetilde{s}^\sharp$. Thus $\widetilde{s} = \gamma s^\sharp = \gamma \widetilde{s}^\sharp$, which shows $\widetilde{s}$ is self-inversive, with constant γ. Note that $\gamma = q_0/\overline{q_m}$. Also, we have $s_0 = \cdots = s_{m-n-1} = 0$ and $\widetilde{s}(z) = \sum_{k=0}^{2n-m} s_{m-n+k}z^k$. Since $\widetilde{s}$ is self-inversive, $\widetilde{s} = \delta\widetilde{s}^\sharp$ with $\delta = s_{m-n}/\overline{s_n}$. But also $\delta = \gamma$, so $s_{m-n}/\overline{s_n} = q_0/\overline{q_m}$. Thus $q_0\overline{s_n} = \overline{q_m}s_{m-n}$. Hence (5) holds. Conversely, assume (5). Reversing the above argument, it follows that $q_0\overline{s_n} = \overline{q_m}s_{m-n}$ implies $\widetilde{s} = \delta\widetilde{s}^\sharp$ with $\delta = \gamma$. Thus $\gamma s^\sharp = \gamma\widetilde{s}^\sharp = \widetilde{s}$. This implies $s = z^{m-n}\widetilde{s} = z^{m-n}\gamma s^\sharp$, and hence (2). □

Corollary 5.2. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime.Assume T_ω^* is symmetric. Then $\deg(s) \le \deg(q) \le 2\deg(s)$.*

Proof. By Theorem 5.1 condition (5) holds with $m = \deg(q)$ and $n = \deg(s)$. Since $\widetilde{s}$ is self-inversive, we have $\widetilde{s}(0) \ne 0$. Consequently, 0 would be a non-removable singularity of $s = z^{m-n}\widetilde{s}$ in case $m < n$, which gives a contradiction. Hence $m \ge n$. Furthermore, comparing the degrees on both sides of $s = z^{m-n}\widetilde{s}$ yields, $n = m - n + \deg(\widetilde{s}) \ge m - n$. Hence $m \le 2n$. □

When T_ω^* is symmetric, it need not be the case that T_ω^* has a selfadjoint extension. In Proposition 5.4 below we characterize when T_ω^* does have a selfadjoint extension. However, we first give a concrete example that shows this does not always happen.

Example 5.3. In [7] Helson considered the functions $\omega_k(z) = \left(-i\frac{z+1}{z-1}\right)^k$ for $k \in \mathbb{N}$. For all k we have $\omega_k(\mathbb{T}) \subset \mathbb{R}$, see Theorem 5.1 (3) above, hence $T_{\omega_k}^*$ is symmetric by Theorem 5.1. In fact, for k even $\omega_k(\mathbb{T}) = \mathbb{R}_+$, while for k odd we have $\omega_k(\mathbb{T}) = \mathbb{R}$. We show that $T_{\omega_k}^*$ does not have a selfadjoint extension for $k = 1$. In Example 5.8 we return to this example for general k.

For $k = 1$ we have $\omega(z) = \omega_1(z) = -i\frac{z+1}{z-1}$. Hence $\mathrm{Dom}(T_\omega) = (z-1)H^2 + \mathbb{C}$ and $\mathrm{Dom}(T_\omega^*) = (z-1)H^2$. Suppose T_ω^* has a selfadjoint extension L. Then $L = L^*$ and thus $T_\omega^* \subset L = L^* \subset T_\omega^{**} = T_\omega$. Since T_ω is not selfadjoint, the inclusions are strict. Hence $\mathrm{Dom}(T_\omega^*) \subset \mathrm{Dom}(L) \subset \mathrm{Dom}(T_\omega)$, with strict inclusions. However, the complement of $\mathrm{Dom}(T_\omega^*)$ in $\mathrm{Dom}(T_\omega)$ is one-dimensional, hence not both inclusions can be strict. Thus T_ω does not admit a selfadjoint extension.

Proposition 5.4. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Then T_ω^* admits a selfadjoint extension if and only if the number of roots of $s - iq$ and $s + iq$ in $\mathbb{D}$, counting multiplicities, coincide.*

Proof. The operator T_ω^* is an adjoint, and hence closed, and by assumption symmetric. Following definition X.2.12 from [3] we define the deficiency subspaces of T_ω^* as the spaces

$$\mathcal{L}_+ = \mathrm{Ker}\,(T_\omega^{**} - i) = (\mathrm{Ran}\,(T_\omega^* + i))^\perp, \quad \mathcal{L}_- = \mathrm{Ker}\,(T_\omega^{**} + i) = (\mathrm{Ran}\,(T_\omega^* - i))^\perp,$$

and the deficiency indices as the integers $n_\pm = \dim \mathcal{L}_\pm$. Since $T_\omega^{**} = T_\omega$, we have

$$n_+ = \dim \mathrm{Ker}\,(T_\omega - i) \quad \text{and} \quad n_- = \dim \mathrm{Ker}\,(T_\omega + i).$$

Also, we have $T_\omega \pm i = T_{\omega \pm i}$. By item (b) of Theorem X.2.20 in [3], T_ω has a selfadjoint extension if and only if $n_+ = n_-$. Note that $\omega \pm i = (s \pm iq)/q$. We now apply Corollary 4.2 from [5] to $T_{\omega\pm i}$, to obtain that $n_\pm$ is equal to the maximum of 0 and the difference of m and the number of roots of $s \pm iq$ in $\overline{\mathbb{D}}$, counting multiplicities. However, since T_ω^* is symmetric, ω is proper so the number of roots cannot exceed m. Note also that $\omega(\mathbb{T}) \subset \mathbb{R}$, so $s \pm iq$ cannot have roots on $\mathbb{T}$. It thus follows that T_ω^* has a selfadjoint extension if and only if the number of roots in $\mathbb{D}$ of $s - iq$ and $s + iq$, counting multiplicities, coincide, as claimed. $\square$

Since T_ω^* is never selfadjoint for $\omega \in \mathrm{Rat}(\mathbb{T})$ having at least one pole on $\mathbb{T}$, the formulas for $n_\pm$ in the above proof along with item (a) of Theorem X.2.20 in [3] directly give the following corollary.

Corollary 5.5. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Then $s + iq$ or $s - iq$ must have a root in $\mathbb{D}$.*

Proposition 5.4 can be rephrased in terms of the index of the operators $T_{\omega\pm i}$.

Proposition 5.6. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Then $T_{\omega+i}$ and $T_{\omega-i}$ are both Fredholm and T_ω^* admits a selfadjoint extension if and only if the Fredholm indices of $T_{\omega+i}$ and $T_{\omega-i}$ coincide.*

Proof. This follows directly from Proposition 5.4 and Theorem 1.1 of [5] applied to $\omega + i$ and $\omega - i$, using that $\omega \pm i = (s \pm iq)/q$. $\square$

Corollary 5.7. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Assume $\omega(\mathbb{T}) \neq \mathbb{R}$. Then T_ω^* admits a selfadjoint extension.*

Proof. The Fredholm index of $T_{\omega-\lambda}$ is constant with respect to $\lambda \in \mathbb{C}$ on the connected components of $\mathbb{C}$ separated by the essential spectrum of T_ω, which is equal to $\omega(\mathbb{T})$; see [6, Theorem 1.1]. Hence if $\omega(\mathbb{T}) \neq \mathbb{R}$, but $\omega(\mathbb{T}) \subset \mathbb{R}$ since T_ω^* is symmetric, then i and $-i$ are in the same connected component and thus $T_{\omega+i}$ and $T_{\omega-i}$ have the same index. The conclusion now follows from Proposition 5.6. $\square$

Example 5.8. We return to the functions $\omega_k(z) = \left(-i\frac{z+1}{z-1}\right)^k$ considered in Example 5.3. Since $\omega_k(\mathbb{T}) = \mathbb{R}_+$ for k even, we obtain directly from Corollary 5.7 that $T_{\omega_k}^*$ admits a selfadjoint extension in case k is even.

For odd values of k we have $\omega_k(\mathbb{T}) = \mathbb{R}$, and thus no conclusion can be drawn from Corollary 5.7. To deal with the odd case we resort to Proposition 5.4. Take $s(z) = (-i)^k(z+1)^k$ and $q = (z-1)^k$ and write k as $k = 2l+1$. The polynomials $s \pm iq$ are given by

$$\begin{aligned}
s(z) \pm iq(z) &= i\left((-1)^{l+1}(z+1)^{2l+1} \pm (z-1)^{2l+1}\right) \\
&= i\left((-1)^{l+1}\sum_{j=0}^{2l+1}\binom{2l+1}{j}z^j \pm \sum_{j=0}^{2l+1}\binom{2l+1}{j}z^j(-1)^{2l+1-j}\right) \\
&= i\sum_{j=0}^{2l+1}\binom{2l+1}{j}z^j\left((-1)^{l+1} \pm (-1)^{2l+1-j}\right) \\
&= i\sum_{j=0}^{2l+1}\binom{2l+1}{j}z^j\left((-1)^{l+1} \pm (-1)^{j-1}\right).
\end{aligned}$$

For odd values of l one obtains:

$$
\begin{aligned}
s(z) - iq(z) &= -2i \left(\binom{2l+1}{0} + \cdots + \binom{2l+1}{2l-2} z^{2l-2} + \binom{2l+1}{2l} z^{2l} \right), \\
s(z) + iq(z) &= 2i \left(\binom{2l+1}{1} z + \cdots + \binom{2l+1}{2l-1} z^{2l-1} + \binom{2l+1}{2l+1} z^{2l+1} \right) \\
&= 2iz \left(\binom{2l+1}{2l} + \cdots + \binom{2l+1}{2} z^{2-2} + \binom{2l+1}{0} z^{2l} \right)
\end{aligned}
$$

Observe that $s+iq$ is of the form $izp_+(z^2)$ where p_+ is a real polynomial of degree $2l$ and that $s-ig$ is of the form $ip_-(z^2)$ where p_- is a real polynomial of degree $2l$. Because p_+ and p_- are real polynomials and the fact that z^2 is the variable rather than z itself, the nonzero roots of $zp_+(z^2)$ come either in pairs (z and $-z$) for real nonzero roots or in quadruples $(z, \bar{z}, -z, -\bar{z})$ for nonreal roots, while zero appears as a simple root. Similarly, the roots of $p_-(z^2)$ come in pairs (z and $-z$) or quadruples $(z, \bar{z}, -z, -\bar{z})$ and there is no root at zero. Hence $s+iq$ has an odd number of roots inside the unit disc, and $s-iq$ has an even number of roots inside the unit disc, so that the indices n_+ and n_- can never coincide. One further observes that $p_- = p_+^\sharp$. In a similar way, for even values of l the polynomial $s+iq$ will have an even number of roots inside the unit disc and $s-iq$ will have an odd number of roots inside the unit disc. Hence, in all cases where k is odd, T_ω^* does not have a selfadjoint extension.

We now present a proposition that rephrases the criteria of Proposition 5.4 in terms of the roots of $s+iq$ (or $s-iq$) only. The observation that $T_{\omega_k}^*$ in Example 5.8 has no selfadjoint extension follows as a special case. In general, T_ω^* cannot have a selfadjoint extension whenever $\deg(q)$ is odd for any $\omega \in \mathrm{Rat}(\mathbb{T})$.

Proposition 5.9. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Set $l_\pm = m - \deg(s \pm iq)$ and define*

$$
k_{\pm,1} = \# \left\{ \begin{matrix} \text{zeroes of } \omega \pm i \text{ inside } \mathbb{T} \\ \text{multi. taken into account} \end{matrix} \right\}, \quad k_{\pm,2} = \# \left\{ \begin{matrix} \text{zeroes of } \omega \pm i \text{ outside } \mathbb{T} \\ \text{multi. taken into account} \end{matrix} \right\},
$$

Then

$$
T_\omega^* \text{ has a selfadjoint extension} \iff l_+ + k_{+,2} = k_{+,1} \iff l_- + k_{-,2} = k_{-,1}.
$$

In particular, if T_ω^* has a selfadjoint extension, then $\deg(q)$ must be even.

The basis for the proof of Proposition 5.9 lies in the following lemma, which clarifies the relation between $s+iq$ and $s-iq$ under the assumption that T_ω^* is symmetric.

Lemma 5.10. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ coprime, be such that T_ω^* is symmetric. Set $l_\pm = \deg(q) - \deg(s \pm iq)$ and let γ be the unimodular constant such that $q = \gamma q^\sharp$. Then*

$$
s \pm iq = \gamma z^{l_\mp} (s \mp iq)^\sharp. \tag{5.1}
$$

Moreover, we have $l_\pm = 0$ if and only if $\omega(0) = \pm i$. In particular, only one of l_+ and l_- can be nonzero.

Proof. Since T_ω^* is symmetric, by assumption, ω has the properties listed in Theorem 5.1. In particular, ω is proper, $m := \deg(q) \geq \deg(s) =: n$, and $s = z^{m-n}\widetilde{s}$ with

$\widetilde{s}$ self-inversive and the unimodular constants that establish the self-inversiveness of $\widetilde{s}$ and q coincide (equivalently, $q_0\overline{s_n} = \overline{q_m}s_{m-n}$).

Note that $\deg(s\pm iq) \neq m$ occurs precisely when $\deg(s) = \deg(q)$ and the leading coefficients s_m and q_m of s and q, respectively, satisfy $s_m \pm iq_m = 0$, i.e., $s_m/q_m = \mp i$. Since $m = n$, the identity $q_0\overline{s_n} = \overline{q_m}s_{m-n}$ shows $\omega(0) = s_0/q_0 = \overline{s_m}/\overline{q_m}$. Hence $\deg(s \pm iq) \neq m$ holds if and only if $\omega(0) = \overline{\mp i} = \pm i$, as claimed.

We first prove (5.1) for the case $\omega(0) = 0$. So assume $\omega(0) = 0$, or equivalently, $s(0) = 0$. In this case $l_+ = l_- = 0$. Since $s = z^{m-n}\widetilde{s}$ and $\widetilde{s}(0) \neq 0$ (because $\widetilde{s}$ is self-inversive), we have $m > n$. Also note that $m - n$ is equal to the multiplicity of 0 as a root of s. We now employ Lemma 2.2, using that $\deg(s+iq) = m = \deg(iq)$, to obtain

$$\gamma(s \mp iq)^\sharp = z^{\deg(s+iq)-\deg(s)}\gamma s^\sharp \mp (-i)\gamma q^\sharp = z^{m-n}\gamma\widetilde{s}^\sharp \pm iq = z^{m-n}\widetilde{s} \pm iq = s \pm iq.$$

Hence (5.1) holds.

Now assume $\omega(0) \neq 0$, i.e., $s(0) \neq 0$. In that case $s = \widetilde{s}$. Hence s is self-inversive with the same constant γ that establishes the self-inversiveness of q. This also yields $m = n$. Since s and q are self-inversive with the same constant γ, we have

$$\overline{s_{m-k}}q_k = \overline{q_{m-k}s_{m-k}}\gamma = \overline{q_{m-k}}s_k \quad \text{for } k = 0, \dots, m.$$

Hence for all k we have

$$\overline{s_{m-k}}(s_k + iq_k) = s_k(\overline{s_{m-k}} + i\overline{q_{m-k}}) \quad \text{and} \quad \overline{q_{m-k}}(s_k + iq_k) = q_k(\overline{s_{m-k}} + i\overline{q_{m-k}}).$$

In case $s_{m-k} = 0$ and $q_{m-k} = 0$, also $s_k = 0$ and $q_k = 0$, since $s_k = \gamma\overline{s_{m-k}}$ and $q_k = \gamma\overline{q_{m-k}}$, and thus $s_k + iq_k = 0 = \gamma(\overline{s_{m-k}} + i\overline{q_{m-k}})$. If either $s_{m-k} \neq 0$ or $q_{m-k} \neq 0$, divide the first identity by $\overline{s_{m-k}}$ or the second identity by $\overline{q_{m-k}}$ to arrive at $s_k + iq_k = \gamma(\overline{s_{m-k}} + i\overline{q_{m-k}})$. Hence

$$s_k + iq_k = \gamma(\overline{s_{m-k} - iq_{m-k}}) \quad \text{for } k = 0, \dots, m. \tag{5.2}$$

In particular, $s_k + iq_k = 0$ if and only if $s_{m-k} - iq_{m-k} = 0$. It follows that 0 is a root of $s \pm iq$ with multiplicity $l_\mp$. Comparing coefficients, it follows that the identities in (5.1) correspond to the identities in (5.2). Hence (5.1) holds. □

Proof of Proposition 5.9. Since T_ω^* is assumed to be symmetric, (5.1) holds. Together with the fact that the $\sharp$ operator reflects roots over $\mathbb{T}$, this implies that the number of roots of $s \pm iq$ inside $\mathbb{T}$ are equal to $l_\pm$ plus the number of roots of $s \mp iq$ outside $\mathbb{T}$, counting multiplicities. In other words, we have

$$k_{+,1} = l_- + k_{-,2} \quad \text{and} \quad k_{-,1} = l_+ + k_{+,2}. \tag{5.3}$$

By Proposition 5.6, T_ω^* has a selfadjoint extension if and only if $s + iq$ and $s - iq$ have an equal number of roots inside $\mathbb{T}$, again counting multiplicities, equivalently, $k_{+,1} = k_{-,1}$. Given (5.3), it follows that $k_{+,1} = k_{-,1}$ is equivalent to $k_{+,1} = l_+ + k_{+,2}$, and likewise to $k_{-,1} = l_- + k_{-,2}$. This proves the two criteria for T_ω^* to have a selfadjoint extension.

By Lemma 5.10, either $l_+ = 0$ or $l_- = 0$. Say $l_+ = 0$. Since $s + iq$ cannot have roots on $\mathbb{T}$, we have $\deg(q) = \deg(s + iq) = k_{+,1} + k_{+,2}$. If T_ω^* admits a selfadjoint extension, then we have $k_{+,1} = l_+ + k_{+,2} = k_{+,2}$. Hence $\deg(q) = 2k_{+,1}$ is even. For $l_- = 0$ the arguments goes similarly. □

Combining the fact that T_ω^* cannot have a selfadjoint extension in case $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, s, q co-prime, and $\deg(q)$ odd with Corollary 5.7 immediately yields the following result.

Corollary 5.11. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime, be such that T_ω^* is symmetric and $\deg(q)$ is odd. Then $\omega(\mathbb{T}) = \mathbb{R}$.*

The next example shows that also with $\deg(q)$ even it can occur that T_ω^* does not admit a selfadjoint extension.

Example 5.12. Let $\omega = s/q$ with

$$s(z) = i(1 + az + z^2), \quad \text{for some } 0 \neq a \in \mathbb{R}\ , \quad \text{and} \quad q(z) = 1 - z^2.$$

Then $m = n$ and

$$s^\sharp = -s, \quad q^\sharp = -q.$$

So T_ω^* is symmetric by Theorem 5.1 (5). Also, we have

$$(s + iq)(z) = i(2 + az) \quad \text{and} \quad (s - iq)(z) = iz(a + 2z).$$

Hence the number of roots of $s - iq$ inside $\mathbb{D}$ is 1 if $|a| \geq 2$ and 2 if $0 \neq |a| < 2$, while the number of roots of $s + iq$ inside $\mathbb{D}$ is 1 if $|a| > 2$ and 0 if $0 \neq |a| \leq 2$. Thus T_ω^* admits a selfadjoint extension if and only if $|a| > 2$.

6. Comparison with the unbounded Toeplitz operator defined by Sarason

The Smirnov class N^+ consists of quotients $\frac{b}{a}$ with a and b H^∞-functions such that the denominator a is an outer function. The function $\varphi = \frac{b}{a} \in N^+$ is said to be in *canonical form* if $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$. By Proposition 3.1 of [12], every function $\varphi \in N^+$ can be uniquely written in canonical form.

In [12], Sarason investigated an unbounded Toeplitz operator T_φ^{Sa} with symbol φ in N^+, which is defined by

$$\mathrm{Dom}(T_\varphi^{\mathrm{Sa}}) = \{f \in H^2 : \varphi f \in H^2\}, \quad T_\varphi^{\mathrm{Sa}} f = \varphi f \ (f \in \mathrm{Dom}(T_\varphi^{Sa})).$$

More generally, T_φ^{Sa} can be defined in this way for any holomorphic function φ on $\mathbb{D}$, but for T_φ^{Sa} to be densely defined, φ must be in N^+; see [12, Lemma 5.2].

Let $\varphi = \frac{b}{a} \in N^+$ be the canonical representation of φ. Then it is shown in Proposition 5.3 of [12] that $\mathrm{Dom}(T_\varphi^{\mathrm{Sa}}) = aH^2$. The adjoint of the operator T_φ^{Sa} is motivated by the action of the conjugate transpose of the matrix representation of T_φ^{Sa}, which is lower triangular. The domain of the adjoint operator is shown to contain the space $H(\overline{\mathbb{D}})$ of functions that are analytic on some neighborhood of the closed unit disc $\overline{\mathbb{D}}$, and the adjoint is equal to the closure of the operator on $H(\overline{\mathbb{D}})$; see [12, Lemmas 6.1 and 6.4].

Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$ with $s, q \in \mathcal{P}$ co-prime. Set $n = \deg(s)$ and $m = \deg(q)$. Assume ω is proper, i.e., $n \leq m$. Then $\omega^*(z) = z^{m-n} s^\sharp / q^\sharp \in \mathrm{Rat}(\mathbb{T})$. Since $q^\sharp$ has zeroes only on $\mathbb{T}$ it is outer and thus $\omega^* \in N^+$. While in general T_ω and T_ω^{Sa} are different, the following proposition shows that T_ω coincides with $T_{\omega^*}^{\mathrm{Sa}}$, and hence $T_\omega = T_\omega^{**} = T_{\omega^*}^{\mathrm{Sa}}$. Without the properness assumption, ω^* is not in N^+, because ω^* has a pole at 0, and hence $T_{\omega^*}^{\mathrm{Sa}}$ is not defined.

Proposition 6.1. *Let $\widetilde{\omega} = \widetilde{s}/\widetilde{q} \in \mathrm{Rat}(\mathbb{T})$ with $\widetilde{s}, \widetilde{q} \in \mathcal{P}$ co-prime. Then $\mathrm{Dom}(T_{\widetilde{\omega}}^{\mathrm{Sa}}) = \widetilde{q}H^2$ and $T_{\widetilde{\omega}}^{\mathrm{Sa}} = T_{\widetilde{\omega}}|_{\widetilde{q}H^2}$. In particular, if $\omega \in \mathrm{Rat}(\mathbb{T})$ is proper, then $T_\omega^* = T_{\omega^*}^{\mathrm{Sa}}$.*

Proof. We first show $\mathrm{Dom}(T_{\widetilde{\omega}}^{\mathrm{Sa}}) = \widetilde{q}H^2$. Let $\widetilde{\omega} = a/b$ be the canonical form of $\widetilde{\omega}$. As noted above, $\mathrm{Dom}(T_{\widetilde{\omega}}^{\mathrm{Sa}}) = aH^2$. By the Fejér-Riesz Theorem there is a polynomial r such that on $\mathbb{T}$ we have $|r|^2 = |\widetilde{s}|^2 + |\widetilde{q}|^2$, r has no roots in

$\mathbb{D}$ and $\arg(r(0)) = \arg(\widetilde{q}(0))$. The latter is possible since $\widetilde{q}(0) \neq 0$ and implies $\widetilde{q}(0)/r(0) > 0$. Note that r also has no roots on $\mathbb{T}$, since $\widetilde{s}$ and $\widetilde{q}$ are co-prime. It follows that $\widetilde{q}/r$ and $\widetilde{s}/r$ are both H^∞-functions, $\widetilde{q}/r$ is outer and $\widetilde{q}(0)/r(0) > 0$. Hence $a = \widetilde{q}/r$ and $b = \widetilde{s}/r$, by the uniqueness of the canonical form. Also, since all the roots of r are outside $\mathbb{T}$, $r^{-1}H^2 = H^2$, so that $aH^2 = \widetilde{q}H^2$.

Now let $f \in \mathrm{Dom}(T^{\mathrm{Sa}}_{\widetilde{\omega}})$, say $f = \widetilde{q}h$ with $h \in H^2$. Then $T^{\mathrm{Sa}}_{\widetilde{\omega}} f = \widetilde{\omega} f = \widetilde{s}h$. On the other hand, the fact that $\widetilde{\omega} f = \widetilde{s}h$ and $\widetilde{s}h \in H^2$ shows $T_{\widetilde{\omega}} f = \mathbb{P}\widetilde{s}h = \widetilde{s}h$. Hence $T^{\mathrm{Sa}}_{\widetilde{\omega}} = T_{\widetilde{\omega}}|_{\widetilde{q}H^2}$. □

Next we employ some of the ideas from [12] to derive the following result. Recall that for a Hilbert space operator $T : \mathrm{Dom}(T) \to \mathcal{H}$ a linear submanifold $\mathcal{D} \subset \mathrm{Dom}(T)$ is called a *core* in case the graph $G(T|_\mathcal{D})$ of $T|_\mathcal{D}$ is dense in the graph $G(T)$ of T; cf., page 166 in [8].

Theorem 6.2. *Let $\omega \in \mathrm{Rat}(\mathbb{T})$. Then $H(\overline{\mathbb{D}})$ is contained in $\mathrm{Dom}(T_\omega)$. If ω is proper, then $H(\overline{\mathbb{D}})$ is a core of T_ω.*

Proof of $H(\overline{\mathbb{D}}) \subset \mathrm{Dom}(T_\omega)$. Write $\omega = \frac{s}{q} \in \mathrm{Rat}_0(\mathbb{T})$ with $s, q \in \mathcal{P}$ coprime. Let $f \in H(\overline{\mathbb{D}})$. Then there exists a $R > 1$ such that f is still analytic on an open neighborhood of the closed disc with radius R. Set $\widetilde{f}(z) = f(Rz)$, $\widetilde{q}(z) = q(Rz)$ and $\widetilde{s}(z) = s(Rz)$. Then $\widetilde{f} \in H^2$ and $\widetilde{q}$ is a polynomial with no roots on $\mathbb{T}$ and $\deg(q) = \deg(\widetilde{q})$. By Theorem 3.1 in [5], $H^2 = \widetilde{q}H^2 + \mathcal{P}_{\deg(q)-1}$. Thus $\widetilde{s}\widetilde{f} = \widetilde{q}\widetilde{h} + \widetilde{r}$ for some $\widetilde{h} \in H^2$ and $\widetilde{r} \in \mathcal{P}$ with $\deg(\widetilde{r}) < \deg(q)$. Now set $r(z) = \widetilde{r}(z/R)$ and $h(z) = \widetilde{h}(z/R)$. Then $r \in \mathcal{P}$ with $\deg(r) = \deg(\widetilde{r}) < \deg(q)$ and $h \in H^2$, even $h \in H(\overline{\mathbb{D}})$. Also, we have $sf = qh + r$. Thus $f \in \mathrm{Dom}(T_\omega)$. □

Before proving the second claim of Theorem 6.2 it is useful to consider the value of T_ω when applied to the evaluation functional or reproducing kernel element $k_\lambda(z) = (1 - \overline{\lambda}z)^{-1}$, where $\lambda \in \mathbb{D}$. Note that $k_\lambda \in H(\overline{\mathbb{D}})$, hence $k_\lambda \in H^2$, and k_λ has the reproducing kernel property for H^2:

$$\mathrm{span}\{k_\lambda \colon \lambda \in \mathbb{D}\} \text{ dense in } H^2 \quad \text{and} \quad \langle h, k_\lambda \rangle = h(\lambda) \quad (h \in H^2,\ \lambda \in \mathbb{D}).$$

See [9] for a recent account of the theory of reproducing kernel Hilbert spaces and further references.

Lemma 6.3. *Let $\omega = s/q \in \mathrm{Rat}(\mathbb{T})$, with $s, q \in \mathcal{P}$ co-prime, be proper. Then*

$$T_\omega k_\lambda = \overline{\omega^*(\lambda)} k_\lambda \quad (\lambda \in \mathbb{D}).$$

Proof. Suppose $g = T_\omega k_\lambda$ then $s(z)(1-\overline{\lambda}z)^{-1} = q(z)g(z) + r(z)$, where $r \in \mathcal{P}_{m-1}$. Here $m = \deg(q)$. Hence $(1 - \overline{\lambda}z)g = (s + (1 - \overline{\lambda}z)r)/q$ is in $\mathrm{Rat}(\mathbb{T})$ as well as in H^2. This can only occur if $(1 - \overline{\lambda}z)g$ is a polynomial, i.e., $g = k_\lambda \widetilde{r}$ for some $\widetilde{r} \in \mathcal{P}$. Thus $s + (1 - \overline{\lambda}z)r = q\widetilde{r}$. Since ω is proper, the degree of the left hand side is at most m. But then $\widetilde{r}$ is constant, say with value $\widetilde{c}$. This shows $T_\omega k_\lambda = \widetilde{c}k_\lambda$.

To determine $\widetilde{c}$ we evaluate the identity $s + (1 - \overline{\lambda}z)r = q\widetilde{c}$ at $1/\overline{\lambda}$. This gives $s(1/\overline{\lambda}) = q(1/\overline{\lambda})\widetilde{c}$. Note that

$$s^\sharp(\lambda) = \lambda^n \overline{s(1/\overline{\lambda})} \quad \text{and} \quad q^\sharp(\lambda) = \lambda^m \overline{q(1/\overline{\lambda})},$$

where $n = \deg(s)$. Hence

$$s(1/\overline{\lambda}) = \overline{\lambda}^{-n} \overline{s^\sharp(\lambda)} \quad \text{and} \quad q(1/\overline{\lambda}) = \overline{\lambda}^{-m} \overline{q^\sharp(\lambda)}.$$

This gives

$$\widetilde{c} = \frac{\overline{\lambda}^{-n}\overline{s^\sharp(\lambda)}}{\overline{\lambda}^{-m}\overline{q^\sharp(\lambda)}} = \overline{\left(\frac{\lambda^{m-n}s^\sharp(\lambda)}{q^\sharp(\lambda)}\right)} = \overline{\omega^*(\lambda)}. \qquad \square$$

Proof of Theorem 6.2. It remains to prove that $H(\overline{\mathbb{D}})$ is a core for T_ω in case ω is proper. So, assume ω is proper. We need to show that the graph of $T_\omega|_{H(\overline{\mathbb{D}})}$ is dense in the graph of T_ω. In other words, let $f, g \in H^2$ with (f, g) perpendicular to $G(T_\omega|_{H(\overline{\mathbb{D}})})$, then we need to show (f, g) is perpendicular to $G(T_\omega)$. Since $k_\lambda \in H(\overline{\mathbb{D}})$, for $\lambda \in \mathbb{D}$, we have

$$0 = \langle (f, g), (k_\lambda, T_\omega k_\lambda)\rangle = \langle f, k_\lambda\rangle + \langle g, \overline{\omega^*(\lambda)}k_\lambda\rangle = f(\lambda) + \omega^*(\lambda)g(\lambda) \quad (\lambda \in \mathbb{D}).$$

Hence $\omega^* g = -f$. In particular, $\omega^* g \in H^2$. Thus $g \in \mathrm{Dom}(T^{\mathrm{Sa}}_{\omega^*}) = \mathrm{Dom}(T^*_\omega)$ and $T^*_\omega g = -f$, by Proposition 6.1. For any $h \in \mathrm{Dom}(T_\omega)$ we have

$$\langle (f, g), (h, T_\omega h)\rangle = \langle (-T^*_\omega g, g), (h, T_\omega h)\rangle = -\langle T^*_\omega g, h\rangle + \langle g, T_\omega h\rangle = 0.$$

This proves our claim. $\square$

In Section 8 of [12], Sarason introduced the class of closed, densely defined operators T on H^2 which satisfy

(1) $T_z \,\mathrm{Dom}(T) \subset \mathrm{Dom}(T)$;
(2) $T^*_z T T_z = T$;
(3) $f \in \mathrm{Dom}(T)$, $f(0) = 0 \Rightarrow T^*_z f \in \mathrm{Dom}(T)$.

This class of operators was further studied by Rosenfeld in [11], see also [10], in which he referred to such operators as Sarason-Toeplitz operators. The operators T^{Sa}_φ, for $\varphi \in N^+$, are Sarason-Toeplitz operators, and the class of operators is closed under taking adjoints, by Proposition 2.1 in [11]. Hence, by Proposition 6.1, T_ω is a Sarason-Toeplitz operator whenever $\omega \in \mathrm{Rat}(\mathbb{T})$ is proper. We show that in fact T_ω is a Sarason-Toeplitz operator for any $\omega \in \mathrm{Rat}$.

Proposition 6.4. *Let $\omega \in \mathrm{Rat}$. Then T_ω on H^2 is a Sarason-Toeplitz operator.*

Proof. First consider $\omega \in \mathrm{Rat}(\mathbb{T})$. That T_ω satisfies (1) and (2) was proved in [5, Lemma 2.3]. We claim that $T^*_z \,\mathrm{Dom}(T_\omega) \subset \mathrm{Dom}(T_\omega)$. Write $\omega = s/q$ with $s, q \in \mathcal{P}$ co-prime. Then $\mathrm{Dom}(T_\omega) = qH^2 + \mathcal{P}_{\deg(q)-1}$. Let $f = qh + r \in \mathrm{Dom}(T_\omega)$ with $h \in H^2$ and $r \in \mathcal{P}$, $\deg(r) < \deg(q)$. Then $T^*_z f = qT^*_z h + h(0)T^*_z q + T^*_z r$, which is in $qH^2 + \mathcal{P}_{\deg(q)-1} = \mathrm{Dom}(T_\omega)$. Hence T_ω is a Sarason-Toeplitz operator in case $\omega \in \mathrm{Rat}(\mathbb{T})$.

Now take $\omega \in \mathrm{Rat}$ arbitrarily. By Lemma 5.1 in [5], see also Section 4 above, $\omega = \omega_- z^\kappa \omega_0 \omega_+$ with $\kappa \in \mathbb{Z}$, and ω_-, ω_0 and ω_+ in Rat with zeroes and poles only inside, on or outside $\mathbb{T}$, respectively. In particular, $\omega_0 \in \mathrm{Rat}(\mathbb{T})$, ω_- and ω_-^{-1} are both anti-analytic, and ω_+ and ω_+^{-1} are both analytic. Also, $T_\omega = T_{\omega_-} T_{z^\kappa \omega_0} T_{\omega_+}$. Note that $z^\kappa \omega_0 \in \mathrm{Rat}(\mathbb{T})$ in case $\kappa \geq 0$ and $T_{z^\kappa \omega_0} = T_{z^\kappa} T_{\omega_0}$ in case $\kappa < 0$ (by [5, Lemma 5.3]). In both cases it now easily follows that $T_{z^\kappa \omega_0}$ is a Sarason-Toeplitz operator. The claim for T_ω follows since $T^{\pm 1}_{\omega_+} T_z = T_z T^{\pm 1}_{\omega_+}$ and $T^{\pm 1}_{\omega_-} T^*_z = T^*_z T^{\pm 1}_{\omega_-}$. $\square$

In fact, by the same arguments one can show that T_ω on H^p, $1 < p < \infty$, satisfied (1)-(3) in case T^*_z is replaced by $T_{z^{-1}}$.

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