# Five not-so-easy pieces: open problems about vertex rings

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ABSTRACT. We present five open problems in the theory of vertex rings. They cover a variety of different areas of research where vertex rings have been, or are threatening to be, relevant. They have also been chosen because I personally find them interesting, and because I think each of them has a chance (the title of the paper notwithstanding!) of being solved. In each case we give some explanatory background and motivation, sometimes including proofs of special cases. Beyond vertex rings per se, the topics covered include connections to real Lie theory, formal group laws, modular linear differential equations, Pierce bundles, and genus 2 Siegel modular forms and the Moonshine Module.

#### 1. Introduction

The organizers of this Conference kindly asked if I might be interested in writing a paper about some open problems in the theory of vertex operator algebras. I liked the sound of the idea but they provided no further guidance, and I struggled with the question of an appropriate format and likely topics. Here's what I have not included: readvertising well-known current problems, for example the question of Mathieu Moonshine and its umbral variants [10], [4], [16], though this is indeed a fascinating area. Similarly, I've skipped the vast question of explaining the parallels between VOA theory and the theory of subfactors. I thought it could be worthwhile to discuss the following question: why is it so hard to construct VOAs? The long struggle to rigorously construct the VOAs on Schellekens list (i.e., the holomorphic, c=24 VOAs), now complete thanks to the efforts of many, is an illustration of this question. The putative Haggerup VOA of Terry Gannon would have made a worthy centerpiece of such a discussion. However, Gannon has recently circulated a preprint on this subject [17] and in doing so he unwittingly vitiated my idea.

In the end I've included questions according to the following criteria: (i) I've thought about them and find them interesting; (ii) they collectively exhibit some diversity of topic; (iii) I believe they're all doable. The titles of each Section are as follows:

- Real forms of vertex operator algebras.
- Vertex rings and formal group laws.
- Vertex operator algebras and modular linear differential equations.

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- Pierce bundles of local vertex rings.
- Genus 2 Monstrous Moonshine.

#### 2. Real forms of vertex operator algebras

The following is a fundamental fact in Lie theory: let  $\mathfrak{g}$  be a *semisimple* complex Lie algebra. Then  $\mathfrak{g}$  has a *compact real form*. (Compact means that the restriction of the Killing form to a Cartan subalgebra of a real form of  $\mathfrak{g}$  is *negative-definite*.)

**Problem 1.** Describe the class of complex VOAs V which have an analog of this theorem. Do strongly regular VOAs always have a real form?

Some discussion is appropriate to provide context and meaning to this problem.

In the following, by a VOA we always mean a *complex* VOA. For the sake of clarity it is sometimes convenient to include a subscript on vertex operators and modes to indicate which Fock space is intended: thus vertex operators for V are  $Y_V(u,z) = \sum_n u_V(n)z^{-n-1}$ , etc.

Let V be a complex linear space. We follow what is fairly standard notation and let  $V^{\mathbf{R}}$  denote the same set V regarded as a real linear space. If V is also a Lie algebra then  $V^{\mathbf{R}}$  is naturally a real Lie algebra. What about VOAs? If V is a VOA with  $u \in V^{\mathbf{R}}$  we can still define vertex operators

$$Y_{VR}(u,z) := \sum_{n} u_{V}(n)z^{-n-1}.$$

This means that  $u_{VR}(n)$  is the mode  $u_V(n)$  regarded as a real operator on  $V^R$ .

Let  $\omega \in V$  be the Virasoro element of V whose modes L(n) satisfy the standard identity

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12} \delta_{m,-n} c I d_V,$$

where c is the central charge of V. The statement that  $\omega$  is a Virasoro element of  $V^{\mathbf{R}}$  makes no sense unless c is a real number.

We shall say that  $\omega$  is real if the central charge c that it determines is also real. (We do not want to say that V is real in this situation; it could be misleading.)

DEFINITION 2.1. Let  $(V, Y, \mathbf{1}, \omega)$  be a (complex) VOA with vertex operator Y, vacuum element  $\mathbf{1}$ , and a real Virasoro element  $\omega$ . A real form of V is a conformal  $sub\,VOA\,U\subseteq V^{\mathbf{R}}$  such that

$$(2.1) V^{\mathbf{R}} = U \oplus iU.$$

A conformal subVOA of  $V^{\mathbf{R}}$  is a real subVOA  $(U,Y,\mathbf{1},\omega)$  whose vacuum and Virasoro elements are the same as that of V. Note that multiplication by i defines an isomorphism of U-modules  $i:U \stackrel{\cong}{\longrightarrow} iU$ . With a decomposition such as (2.1) in hand we can say, without invoking any ambiguity, that the states in U are the real states and those in iU are the imaginary states. For example,  $\mathbf{1}$  and  $\omega$  are both real states. If a and b are real states, and if  $u:=a+ib\in V^{\mathbf{R}}$ , then

(2.2) 
$$Y_{VR}(u,z) = Y_{U}(a-b,z) \oplus i Y_{U}(a+b,z).$$

Let us now assume that V is of strong CFT-type. In particular V is a simple VOA with a conformal decomposition of the shape

$$V = \mathbf{C} \mathbf{1} \oplus V_1 \oplus \dots$$

furthermore  $V_1$  consists of quasiprimary states, i.e., if  $u \in V_1$  then L(1)u=0. With this set-up, Haisheng Li has shown [28] that V has, up to scalars, a unique invariant bilinear form

$$b:V\times V\to \mathbf{C}.$$

For the definition of an invariant bilinear form for a VOA, cf. [15], [28]. In particular, reference [15] establishes that such a form is necessarily symmetric. Furthermore because V is simple then b is nondegenerate, in particular its restriction to each homogeneous space  $V_n$  is also nondegenerate. We may, and shall, canonically  $normalize\ b$  so that

$$b(1,1)=-1.$$

The resulting bilinear form is the best VOA analog of the Killing form for a semisimple Lie algebra.

Next we observe that Li's theory applies equally well to real VOAs as well as complex VOAs. In particular, a real VOA of *CFT*-type has a unique normalized, *real-valued*, invariant, bilinear form.

Let us now assume that V is not only of strong CFT-type, but in addition it has a real Virasoro element  $\omega$  and a real form U. Because  $\omega \in U$  it follows from (2.2) that

$$L_V(n)=L_U(n)+iL_U(n)$$
.

Thus  $V_n=U_n+iU_n$ . It follows that  $U_0=\mathbf{R1}$  and  $U_1$  is annihilated by  $L_U(1)$ , so that U is a real VOA of CFT-type. As a result, U also has unique normalized, invariant, real-valued, bilinear form.

Lemma 2.1. Suppose that V is a (complex) VOA that has strong CFT-type, a real Virasoro element and a real form U. Let b be the normalized invariant bilinear form on V. Then the restriction of b to  $U \times U$  is the normalized real-valued, invariant, bilinear form on U.

PROOF. The restriction  $b_U$  of b to U is an invariant bilinear form on U, and we have to show that it is real-valued.

The real and imaginary parts of b,  $\Re(b)$  and  $\Im(b)$  both define real-valued, invariant, bilinear forms on U, so there is a real scalar y such that  $y\Re(b_U)=\Im(b_U)$ . Hence restriction to U satisfies  $b_U=\Re(b_U)+i\Im(b_U)=(1+iy)\Re(b_U)$ . We now have

$$-1=b_U(\mathbf{1},\mathbf{1})=(1+iy)(-1),$$

so y=0. This proves the Lemma.

Problem 1 may now make sense. Given a (complex) VOA satisfying the conditions of Lemma 2.1 we have two related and canonically defined invariant bilinear forms, namely the  ${\bf C}$ -valued normalized, invariant, bilinear form b on V and its restriction to the  ${\bf R}$ -valued normalized, invariant, bilinear form on U. This is a VOA analog of the Lie algebra scenario in which we have a semisimple Lie algebra  ${\mathfrak g}$  and the restriction of the Killing form to a Cartan subalgebra of a real form for  ${\mathfrak g}$ .

Problem 1 has two parts. Unlike the case of semisimple Lie algebras, there is currently no available result guaranteeing that a suitable class of VOAs (satisfying the hypotheses of Lemma 2.1, say) actually has a real form. One part of Problem 1 asks for some resolution of this problem, and specifically asks if VOAs that are strongly regular might fit the bill. Strongly regular VOAs are the nicest class of VOAs of all [31]: by definition they are of strong CFT-type and they are also rational and  $C_2$ -cofinite. If V is strongly regular then the central charge is rational [8], so that the Virasoro element is certainly real. However it is generally unknown if V has a real form. Many strongly regular VOAs are known to have a  $\mathbf{Z}$ -form [6], [7], so they certainly have a real form. A strongly regular VOA without a real form would itself be interesting.

The other part of Problem 1 deals with *compactness*, i.e., the (negative-) definiteness of some real bilinear forms. It is somewhat vague in detail, and deliberately so. Signatures of real bilinear forms have not played a big part in VOA theory, although the fact that the Griess algebra  $V_2^{\natural}$  of the Moonshine module supports a real and compact bilinear form is important for Norton identities in monstrous moonshine [44]. Beyond this I am unaware of even so much as a passing reference to such questions, save for the comment of Kac and Raina at the end of Lecture 8 in [26].

One analog of the Lie theory set-up goes as follows: with earlier notation, and assuming V has a real form U, can we choose U so that the restriction of b to each  $U_n$  is compact? This seems like too much to expect in general, but what about lattice theories  $V_L$ ? In the Lie theory set-up we only consider restrictions to a Cartan subalgebra, so for VOAs we might hope to find some real subVOA  $U' \subseteq U$ , not too small, such that b is compact on each  $U'_n$ , or at the very least, b has a computable signature on each  $U'_n$ .

Suppose that V is strongly regular. Then the Lie algebra on  $V_1$  is reductive, and in some sense a Cartan subalgebra of  $V_1$  plays the rôle of a Cartan subalgebra of V. (For more on this perspective, see [31]). By the Lie theory we can find a Cartan subalgeba of  $V_1$  having a compact real form. Perhaps this is all that one can hope for? On the other hand by Theorem 1 of [31], for any Cartan subalgebra  $C \subseteq V_1$  we can find a lattice theory subVOA  $V_L \cong W \subseteq V$  such that  $L \subseteq C$  is cocompact. This suggests taking C to have a compact real form and looking for U' in W. Can we take U' to be a real form of W?

## 3. Vertex Rings and Formal Group Laws

**Problem 2**. Let k be a commutative ring, F a formal group law over k and V a vertex k-algebra. What are the axioms for an F-vertex k-algebra?

Haisheng Li made some substantial (and surprising, to me) contributions towards fostering the connections between VOAs and formal group laws (FGLs) in [29]. Li showed how to modify the weak associativity axiom for a VOA using a FGL F. In this way he produced what he called a  $vertex\ F$ -algebra, which is a variant of a VOA that satisfies axioms resembling the locality axioms (i.e., the Jacobi identity, locality, and weak commutativity and associativity) but are modified by F.

Problem 2 asks for a generalization of Li's results to  $vertex\ rings$  rather than VOAs. (In our nomenclature it was convenient to replace vertex F-algebra with

F-vertex algebra, but be assured they mean the same thing.) Vertex rings are just like VOAs, but the scalars lie in an arbitrary commutative ring k rather than a field of characteristic 0 such as  $\mathbf{C}$ . An axiomatic approach to vertex rings is given in [33]. Although the axioms for a vertex ring are virtually identical to those for a VOA, the lack of denominators in a vertex ring hampers attempts to prove analogs of results for VOAs, which become more complicated, or plain wrong, or meaningless.

So it is with FGLs. Working with a FGL F over a commutative ring k which is a  $\mathbf{Q}$ -algebra is facilitated by the existence of a logarithm for F, implying that F is isomorphic to the additive group law. Sure enough, in his applications to VOAs Li makes heavy use of the existence of a logarithm. Thus the main point of Problem 2 is to reproduce Li's results without using logarithms! (Well, you may use them, but they should not figure in the final answer!)

If one is going to meld FGLs with vertex algebra theory, it seems most natural to do it for vertex rings without denominators, for that is the natural domain in which the theory of FGLs resides. And by working at this level of generality one gains access to what could be some of the most interesting FGL's for vertex algebra theory. We have in mind Euler's FGL associated to an elliptic integral, the Lazard universal FGL, and the FGLs of generalized cohomology theories.

In the rest of this Section I will review some of the relevant background about FGLs designed to make the preceding paragraphs intelligible. Then I will answer Problem 2 in the special case of vertex rings of type  $(k, \underline{D})$ , where k is, as before, an arbitrary commutative ring and  $\underline{D}$  is an *iterative Hasse-Schmidt derivation* of k. These are the easiest vertex rings that are *not* VOAs (cf. [33], Theorem 5.5), and they provide a lifeline to the classical theory of rings with derivation [42]. This special case may suggest what to expect when attacking Problem 2 in full generality.

For somewhat different approaches to formal groups and FGLs, we may refer the reader to the encyclopedic text of Hazewinkel [22], and a more scheme-theoretic approach in a course of Strickland [48] which I found on the internet.

Fix a unital commutative ring k. A 1-dimensional commutative formal group law (FGL) over k is a power series  $F(X,Y) \in k[[X,Y]]$  satisfying the following properties:

(a) 
$$F(X,0)=X, F(0,Y)=Y,$$
  
(b)  $F(F(X,Y),Z)=F(X,F(Y,Z)),$   
(c)  $F(X,Y)=F(Y,X).$ 

These look like the right- and left-identity, associativity and commutativity axioms for an abelian group They should be supplemented by an axiom for inverses, but that's not necessary because it is readily proved (formal implicit function theorem) that

there is a power series  $\iota(X) \in k[[X]]$  such that  $F(X, \iota(X)) = 0$ .

We have

$$F(X,Y)=X+Y+\sum_{i,j\geq 0}c_{ij}X^{i}Y^{j} \ (c_{ij}\in k).$$

For 'nice' rings k it is true that (c) is a *consequence* of (a) and (b) (cf. [22], Theorem 6.1 for a precise statement).

There are the following basic examples of FGLs, defined for every k:

$$F_a(X,Y):=X+Y,$$
  
 $F_m(X,Y):=X+Y+XY.$ 

These are the additive and multiplicative FGLs respectively.

Let us introduce the notation

$$X+_F Y=F(X,Y).$$

It suggests how to produce a new abelian group structure on k[[X]] as follows:

$$a(X)+_{F}b(X):=F(a(X),b(X)).$$

This idea goes to the heart of how and why FGLs may be used to modify a vertex ring.

A Hasse-Schmidt derivation (HS)  $\underline{D} := (D_0, D_1, D_2, ...)$  of k is a sequence of endomorphisms  $D_m: k \to k$  satisfying

$$D_0 = Id_k, \quad D_m(1) = 0 \ (m \ge 1),$$
 
$$D_m(uv) = \sum_{i+j=m} D_i(u)D_j(v) \quad (m \ge 0, u, v \in k)$$

An HS derivation D is called *iterative* if it also satisfies the identity

$$(3.1) D_i \circ D_j = \binom{i+j}{i} D_{i+j}.$$

There is a very interesting generalization of the idea of an iterative HS derivation that uses a FGL, cf. [42], [21]. This goes as follows: Let F(X,Y) be a FGL over k and let  $\underline{D}$  be an HS derivation of k. We call  $\underline{D}$  an HS F-derivation if it satisfies the following identity:

(3.2) 
$$\sum_{i,j>0} D_j \circ D_i(u) X^i Y^j = \sum_{n>0} D_n(u) (X +_F Y)^n \quad (u \in k).$$

It can be checked [42] that an HS derivation  $\underline{D}$  is iterative if, and only if, it is an  $F_a$ -derivation (additive FGL). So this really is a generalization of iterativity. This concept has been well-studied in the literature. For further details in the case when F is the multiplicative FGL  $F_m$ , cf. [5], where one gets a glimpse of the rather complicated identities satisfied by the operators  $D_m$  that must replace (3.1).

Remark 3.1. It may not be quite clear why one needs an FGL in (3.2) as opposed to some other power series. This point is discussed in [21].

Now let us turn to the vertex rings  $(k, \underline{D})$  where  $\underline{D}$  is an HS-derivation of k. They are defined as follows ([33], Section 5.2): the vacuum element is  $1 \in k$  and vertex operators are defined by

(3.3) 
$$Y(u,z)v := \sum_{n>0} D_n(u)vz^n \quad (u,v \in k).$$

The relevant result is

Theorem 3.2. ([33], Theorems 3.5+5.5)  $(k,\underline{D})$  is a vertex ring if, and only if,  $\underline{D}$  is iterative.  $\Box$ 

In the spirit of Li's work and Problem 3, we now ask:

what happens if, in the above construction, we replace  $\underline{D}$  by a HS F-derivation?

We will get what can only be described as a vertex ring modified by an HS F-derivation. So we call this an F-vertex ring. It is certainly not a vertex ring in the usual sense unless  $F=F_a$  is the additive FGL, because only in this case will  $\underline{D}$  be iterative. So it must be that one of the locality axioms is changed, and indeed this is the case. We have the following generalization of Theorem 3.2.

THEOREM 3.3. Let  $(k, \underline{D}, F)$  consist of a unital commutative ring k, a HS derivation  $\underline{D}$  of k, and a FGL F over k. Let vertex operators be defined by (3.3). Then the following are equivalent:

(ii) 
$$Y(Y(a,z)b), w)c = Y(a,z+F)w)Y(b,w)c (a,b,c \in k)$$

PROOF. (ii) $\Rightarrow$ (i). Take b=c=1 in (ii) to obtain

$$Y(Y(a,z)1,w)1 = Y(a,z+_F w)Y(1,w)1$$

$$\Rightarrow Y\left(\sum_{m\geq 0} D_m(a)z^m,w\right) = \sum_{m\geq 0} D_m(a)(z+_F w)^m$$

$$\Rightarrow \sum_{m\geq 0} \sum_{\ell\geq 0} D_\ell(D_m(a))w^\ell z^m = \sum_{m\geq 0} D_m(a)(z+_F w)^m.$$

This is (3.2), so (i) holds.

 $(i) \Rightarrow (ii)$ . The left-hand-side of (ii) is equal to

$$\sum_{n\geq 0} D_n \left( \sum_{m\geq 0} D_m(a)bz^m \right) cw^n$$

$$= \sum_{i,j\geq 0} \sum_m D_i \circ D_m(a)D_j(b)cz^m w^{i+j} \quad \text{(using the HS property)}$$

$$= \sum_{k\geq 0} D_k(a)(z+_F w)^k \sum_{j\geq 0} D_j(b)cw^j \quad \text{(using the } F\text{-derivation property)}$$

$$= Y(a,z+_F w)Y(b,w)c,$$

and this is the right-hand-side of (ii). The Theorem is proved.

The thrust of Problem 2 should now be evident. It is asking for the generalization of Theorem 3.3 to *arbitrary* vertex k-algebras V. How do we modify V using a FGL F, and how are the vertex ring axioms affected?

The general case is a good deal more complicated than the special case that we just handled. Although HS derivations figure prominently in all vertex rings (cf. [33], esp. Sections 3-5) it is by no means true in general that  $\underline{D}$  determines the vertex operators as it does for  $(k,\underline{D})$ . Furthermore in the general case  $\underline{D}$  intervenes in the translation-covariance axiom (loc. cit.), something we did not have to consider in the special case. What seems clear is that the HS-derivation will have to be an HS

F-derivation just as before. But the F-weak associativity axiom (this is Li's name for (ii) in Theorem 3.3) will almost certainly get generalized to read

(3.4) 
$$(z+_F w)^N Y(Y(a,z)b), w)c = (z+_F w)^N Y(a,z+_F w)Y(b,w)c$$
  
 $(N \gg 0).$ 

To recapitulate, we do not want to suggest that Theorem 3.3 remains true if we simply replace (ii) by (3.4). We are suggesting that (3.4) together with HS F-derivations will be key ingredients in the solution of Problem 2. One thing is sure: a close reading of [29] will be required!

## 4. Vertex operator algebras and modular linear differential equations

**Problem 4**. State and prove the 3-dimensional Mathur-Mukhi-Sen theorem.

If I had to name a single paper that has most influenced my thinking about vertex operator algebras, then the answer would undoubtedly be Zhu's work on modular-invariance [52], which was essentially his Phd thesis. Zhu deals with a class of nice VOAs that are simple, of CFT-type, rational, and  $C_2$ -cofinite. Thus V is nearly strongly regular, but without any assumptions about existence of an invariant bilinear form, which is not relevant to Zhu's analysis. We will simply paraphrase these conditions by saying (somewhat inaccurately) that V is rational. Thus V has only finitely many irreducible modules. Let's name them  $M^1, \ldots, M^p$  and let the conformal weight of  $M^i$  be  $h^i$ . If V has central charge c then the formal graded dimension, or q-character, of  $M^i$  is defined to be

(4.1) 
$$Z_i(q) := \operatorname{Tr}_{M^i} q^{L(0) - \frac{c}{24}} = q^{h^i - \frac{c}{24}} \sum_{n \ge 0} \dim M_n^i,$$

where at first we are obliged to treat q as a formal variable.

Zhu's paper is concerned with these q-characters. He pioneered the use of differential equations in VOA theory by proving that the q-characters are convergent. The method is well worth studying. One uses the Virasoro operators to show that  $Z_i(q)$  satisfies a differential equation which has a regular singularity at q=0. The Frobenius method then proves convergence in a deleted neighborhood of q=0. (There are many references for background on the theory of differential equations that we need here, ranging from the old-fashioned [23] and the similar but easier-to-read [20], to the more modern [47].)

From the perspective of a dyed-in-the-wool algebraist, the beauty of this approach is that it is purely formal, and can be used to great effect in VOA theory. All of the estimates and analysis are taken care of by Frobenius! At the same time, Zhu proves that if we set  $q:=e^{2\pi i\tau}$  with  $\tau$  in the complex upper half-plane  $\mathbf{H}$ , then  $Z_i(q)$  is just the q-expansion of a periodic holomorphic function  $Z_i(\tau)$  in  $\mathbf{H}$ . Thus each  $Z_i(\tau)$  has a certain translation-invariance property. Zhu's main theorem is a related and more trenchant invariance result. To describe it, introduce the complex linear space

$$\mathfrak{ch}_V := \langle Z_1(\tau), \dots, Z_p(\tau) \rangle \subseteq F.$$

Here, we consider  $\mathfrak{ch}_V$  as a subspace of the space F of all holomorphic functions in  $\mathbf{H}$ . Zhu proved that  $\mathfrak{ch}_V$  furnishes a representation of the inhomogeneous modular group  $\Gamma:=SL_2(\mathbf{Z})$ . This means that if  $\gamma\in\Gamma$  then there are scalars  $c_{ij}(\gamma)\in\mathbf{C}$ 

(independent of  $\tau$ ) such that

$$Z_i(\gamma \tau) = \sum_j c_{ij}(\gamma) Z_j(\tau),$$

where we are using standard notation for the fractional linear action  $\Gamma \times \mathbf{H} \rightarrow \mathbf{H}$  given by

$$(\gamma,\tau) := \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \tau \right) \mapsto \gamma \tau := \frac{a\tau + b}{c\tau + d}.$$

For example, if V is holomorphic in the sense that it has a unique irreducible module  $(p=1 \text{ and } M^1=V)$  then Zhu's theorem says that  $Z_1(\tau)$  satisfies

$$Z_1(\gamma\tau)=c_{11}(\gamma)Z_1(\tau) \quad (\gamma \in \Gamma)$$

The assignment  $\chi:\gamma\mapsto c_{11}(\gamma)$  is the representation (a character of  $\Gamma$  in this case) furnished by  $\mathfrak{ch}_V$ . Thus  $Z_1(\tau)$  is a modular function of weight 0 and level 1 with character  $\chi$ . The theory of modular functions provides a complete description of such functions. For example, if  $\chi=1$  is the trivial character then the modular functions in question constitute the field  $\mathbf{C}(j)$  of rational functions in the absolute modular invariant

$$j(\tau) = q^{-1} + 744 + 196884q + \dots$$

The character  $\chi$  does not provide any real difficulty. If it is nontrivial then it has order 3, and the corresponding class of modular functions is readily described. We will not need it here.

Attempting to usefully organize all holomorphic VOAs V into some type of classification scheme seems problematic. There are infinitely many self-dual, positive-definite, integral lattices L, all of which realize holomorphic lattice theories  $V_L$ , moreover the class of such VOAs is closed with respect to tensor products. The q-character of V is probably as good an invariant as we can expect, although it does not always distinguish between VOAs. For example, the two self-dual, rank 16 lattices  $L_1 := E_8 + E_8$  and  $L_2 := \Gamma_{16}$  (a spin lattice with root system  $D_{16}$ ) have the same theta-function and therefore they define holomorphic lattice theories  $V_{L_1}, V_{L_2}$  that are not isomorphic but have the same q-characters.

This brings us to the Problem stated at the beginning of this Section which concerns a case when p>1, so that V has more than one<sup>1</sup> irreducible module. Unlike the holomorphic case, there is some hope of classifying such VOAs based on the theory of modular linear differential equations (MLDEs). For an introduction to this subject (but not its applications in VOA theory) cf. [13]; we will need some details here.

Fix an integer k. There is a right action of  $\Gamma$  on F defined by

$$(f(\tau), \gamma) \mapsto f|_{k}\gamma(\tau) := (c\tau + d)^{-k} f(\gamma \tau).$$

Here, and below, we take  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

<sup>&</sup>lt;sup>1</sup>Of course, this means more than one isomorphism class of irreducible modules

Let  $F_k$  denote this  $\Gamma$ -module. The modular derivative in weight k is the differential operator

$$D_k: F \to F, \quad f \mapsto D_k f := \frac{1}{2\pi i} \frac{df}{d\tau} - \frac{k}{12} E_2(\tau) f,$$

where  $E_2(\tau)$  is the weight 2 Eisenstein series

$$E_2(\tau) := 1 - 24 \sum_{n>1} \sum_{d|n} dq^n.$$

The strange-looking operator  $D_k$  is important because

$$D_k:F_k\to F_{k+2}$$

intertwines the  $\Gamma$ -modules  $F_k$  and  $F_{k+2}$ . This amounts to the standard formula [27], Chapter X:

$$(4.2) (D_k f)|_{k+2} \gamma(\tau) = D_k(f|_k \gamma)(\tau).$$

By iteration we obtain differential operators

$$D_k^n := D_{k+2n-2} \circ \cdots \circ D_{k+2} \circ D_k : F_k \to F_{k+2n}.$$

Let  $\widetilde{F}_k$  be the subspace of  $F_k$  consisting of holomorphic functions that are 'meromorphic at q=0'.  $\widetilde{F}_k$  is not a  $\Gamma$ -submodule, however we have

$$D_k:\widetilde{F}_k\to\widetilde{F}_{k+2}.$$

For example,  $E \subseteq F_0$  is a  $\Gamma$ -submodule by Zhu's theorem and  $E \subseteq \widetilde{F}_0$  because the condition of meromorphy holds thanks to the q-expansions (4.1). Hence  $D_k^n(E)$  is a  $\Gamma$ -module contained in  $\widetilde{F}_{2n}$ 

The graded algebra of classical holomorphic modular forms is

$$M := \bigoplus_{k>0} M_{2k}$$

where  $M_{2k}$  is the subspace of  $\widetilde{F}_{2k}^{\Gamma}$  (the  $\Gamma$ -invariants with a q-expansion) that are 'holomorphic at  $\infty$ '. These are the weight 2k modular forms. Now we are prepared to define an MLDE of order n and weight (2k,0). This is a DE of the type

(4.3) 
$$\left(\sum_{j=0}^{n} P_{2k+2n-2j}(\tau) D_0^j\right) u = 0 \quad (P_{2\ell}(\tau) \in M_{2\ell}).$$

When written using the usual derivative  $\frac{d}{d\tau}$ , (4.3) is a standard DE of order n with coefficients in M. Consequently, the space of solutions is an n-dimensional linear space. Solutions may develop singularities at the zeros of the leading coefficient  $P_{2k}(\tau)$  but are holomorphic elsewhere.

Where is all of this leading? We have purposely avoided going into the theory of vector-valued modular forms, which is actually intimately related to MLDEs. But by combining these two strands, the following result can be proved (cf. [32], [30]):

(4.4) There is an MLDE whose solution space is exactly  $\mathfrak{ch}_V$ .

Given Zhu's theory as we have outlined it, no further VOA input is needed for this result. It is purely a matter of vector-valued modular form theory and MLDEs. And in fact the theory says more than just (4.4), for in some important cases (discussed further below) it says that the MLDE can be chosen to take a specific form.

The reason we use MLDE's rather than ordinary DEs is that there is an evident natural action of  $\Gamma$  on the solution space. Indeed, if  $u(\tau)$  is a solution then  $\gamma \in \Gamma$  acts via the stroke operator  $u|_0\gamma(\tau)$ . This is obvious if the solution space is  $\mathfrak{ch}_V$  for some V, indeed the action is just the one we described before that was found by Zhu, but it is true in general thanks to (4.2). Let's denote this representation of  $\Gamma$  by  $\rho$ . It is in fact the *monodromy representation* and one of the main features of the MLDE. Thus

the monodromy  $\rho$  of an MLDE is a representation of the modular group.

 $\mathfrak{ch}_V$  and its associated MLDE are beautiful invariants of the VOA V, and may be compared to the usual S- and T-matrices of RCFT, or the modular tensor category that is V-Mod.  $\mathfrak{ch}_V$  neatly packages S- and T-matrices coming from the monodromy representation, as well as the q-characters of irreducible V-modules that span the solution space of the MLDE. This circumstance permits us to combine techniques from the theory of modular forms and the theory of DEs in order to study  $\mathfrak{ch}_V$ .

The Problem stated at the beginning of this Section is part of the program Classify rational VOAs according to  $\mathfrak{ch}_V$  and their associated MLDEs.

In fact this effort was initiated in 1987 by the physicists Mathur, Mukhi and Sen [43]. They obtained a partial result in the 2-dimensional case that was recently completed in [34]. These papers deal with the case of *monic* MLDEs of order 2. Monic here means that the leading coefficient  $P_{2k}(\tau)$  in (4.3) is equal to 1, and in particular k=0. Thus the DE in question is the simplest possible order 2 MLDE, which is

$$(4.5) (D_0^2 + \kappa E_4(\tau)) u = 0 (\kappa \in \mathbb{C})$$

where  $E_4$  is the usual normalized Eisenstein series of weight 4. (This comes about after perusal of (4.3) just because  $M_2=0$  and  $M_4=\mathbb{C}E_4$ .)

Here are the known rational VOAs with the property that  $\dim \mathfrak{ch}_V = 2$  and which have *irreducible monodromy*. All but one are affine algebras of level 1, the other is the Yang-Lee minimal model:

$$L(A_1,1), L(A_2,1), L(D_4,1), L(E_6,1), L(E_7,1), L(G_2,1), L(F_4,1), Vir_{c_{2,5}}$$

These rational VOAs are almost characterized by this property in [34], Theorem 1. But it is convenient to distinguish two cases:

- (a) V has exactly two irreducible modules.
- (b) V has more than two irreducible modules, but dim  $\mathfrak{ch}_V=2$ .

Now we can state the 2-dimensional Mathur-Mukhi-Sen Theorem:

THEOREM 4.1. ([43], [34] Theorem 2) Suppose that V is a strongly regular  $^2$  VOA satisfying (a). Assume that the associated MLDE has the form (4.5) and that it has irreducible monodromy. Then V is isomorphic to one of the following:

$$L(A_1,1), L(E_7,1), L(G_2,1), L(F_4,1), Vir_{c_{2.5}}$$

<sup>&</sup>lt;sup>2</sup>Here we need the full force of strong regularity. That's because this condition is assumed in some of the Theorems in the literature needed to prove Theorem 4.1.

The gist of Problem 4 is to prove a 3-dimensional analog of Theorem 4.1, preferably assuming only dim  $\mathfrak{ch}_V=3$  but perhaps using an analog of (a). In any case the MLDE looks like

$$(4.6) (D_0^3 + \kappa E_4(\tau) D_0 + \lambda E_6(\tau)) u = 0 (\kappa, \lambda \in \mathbb{C}).$$

Typically the monodromy of such an MLDE will be irreducible, but that will not always be the case. An important point that will certainly figure in the solution of Problem 4 is that (4.6) is actually a disguised version of a hypergeometric equation solved by certain hypergeometric functions  $_3F_2$  ([13], [14]).

The list of rational VOAs with dim  $\mathfrak{ch}_V{=}3$  is be wilderingly diverse, and includes the following:

$$(4.7) L(A_{3},1), L(A_{4},1), L(C_{2},1), L(A_{2},2), L(E_{8},2), L(D_{\ell},1) (\ell \geq 5), L(B_{\ell},1) (\ell \geq 3), Vir_{c_{3,4}}, Vir_{c_{2,7}}, Vir_{c_{2,5}} \times Vir_{c_{2,5}} V_{\sqrt{2}E_{8}}, V_{\Lambda}, VB^{\natural}. V_{\sqrt{2}E_{c}}^{+}, V_{\Lambda}^{+}.$$

Not all of these examples have irreducible monodromy. Among the last five examples are lattice theories  $V_L$  with L either a rescaled  $E_8$  root lattice or the Barnes-Wall lattice  $\Lambda$  of rank 16 together with their  $\mathbf{Z}_2$ -orbifolds. The other one is the Baby Monster VOA  $VB^{\sharp}$  [22].

I compiled this list from various interesting papers done in support of Problem 4 (and related problems) by Arike, Kaneko, Nagatomo and Sakai, including characterizations of some of the VOAs [1], [2], [3]. However we are still far from a complete solution.

Although Problem 4 is an obvious extension of Theorem 4.1, fresh ideas are needed for a complete solution of Problem 4. This is because the list (4.7) contains infinitely many different VOAs, and in particular there is no bound on the possible central charge c. In the proof of Theorem 4.1 in [34] one first shows that the assumptions of the Theorem imply that there are only finitely many possible values of c, then a detailed analysis of each possibility yields the final answer. So this approach must be modified. A modest start is made in [35].

I would have liked to discuss some additional questions about the MLDEs attached to rational VOAs, a subject which I find fascinating. For example are these MLDEs necessarily Fuchsian on the 3-punctured sphere? However an adequate discussion would require more space than is available here.

### 5. Pierce bundles of local vertex rings

**Problem 5**. Characterize *exchange* vertex rings.

In order to explain the meaning of this Problem we will first make an incursion into the theory of *Pierce bundles* of a commutative ring. This may seem like something of detour to the reader, but it is vital to explain and motivate Problem 5. Pierce's original paper [46] is, of course, a good reference for his theory, and the intriguing text of Johnson [25] also deals with this set of ideas in a broad framework. The extension to vertex rings was first given in Part II of [33].

The structure sheaf of a unital, commutative ring k is a fundamental geometric object [11] that has now infiltrated into graduate algebra texts (e.g., [9], Chapter 15). The same cannot be said of the Pierce sheaf of k, though they are closely related.

Let  $X:=Spec\ k$  be the prime ideal spectrum of k equipped with the Zariski topology. One considers the disjoint union  $E:=\bigcup_P k_P$  of the localizations  $k_P$  of k at a prime ideal  $P\subseteq k$ . Once E is topologized, we get a bundle of commutative rings  $\pi:E\to X$  and the fiber  $\pi^{-1}(P)$  over P is just  $k_P$ . Such a bundle gives rise to the structure sheaf  $\mathcal O$  of k; it is a sheaf of rings over X which arises from the local sections of  $\pi$  over the open sets of X. There is a categorical equivalence between such bundles and sheaves over X, so that they carry the same data.

In the Pierce construction, one begins with the same k but a very different X. Namely associated to k is the set of all  $idempotents\ e{\in}k$ . This set carries the structure of a  $Boolean\ algebra$  and it may also be regarded as a second commutative ring in which multiplication of idempotents e, f is just their product ef in k, whereas addition is defined by  $e \oplus f{:=}e{+}f{-}2ef\ (e{+}f$  is addition in k). Denote this ring by  $B{=}B(k)$ . It is a  $Boolean\ ring$  in the sense that every element is idempotent. For further details about this construction, cf. [46] and [24], Chapter 8.

For the purposes of the Pierce construction we take  $X:=Spec\ B(k)$ . This is an example of a *Boolean space*, namely the Zariski topology on X is both *Hausdorff* and *totally disconnected* (connected components are single points) and the topology has a *basis of clopen sets*.

The Pierce bundle  $\pi:E \to X$  of k is defined as follows: as in the case of the structure sheaf  $\mathcal{O}$ , E is defined as the disjoint union of what will eventually be the stalks, and if  $P \in X$  is a prime ideal of B then  $\pi^{-1}(P):=k/\overline{P}$  where  $\overline{P}:=\cup_{e\in P} ek$ .  $(\overline{P} \text{ really } is \text{ an ideal in } k$ , though the union may suggest otherwise).

Pierce showed [46] that k can be recovered as the global sections of  $\pi$ . Furthermore, the stalks  $\pi^{-1}(P)$  are indecomposable rings. This sets up an equivalence between the category Ring of commutative rings and the category of reduced bundles, which are ètale bundles of indecomposable rings over a Boolean base space.

Pierce's work gave impetus to a cottage industry focused on the question of what commutative rings can arise as global sections of suitably conditioned bundles of rings? This is sometimes referred to as representation theory of rings.

For example, Pierce obtained [46] a beautiful characterization of those commutative rings k having the additional property that the stalks of the Pierce bundle are not just indecomposable, but in fact simple. Of course, a simple commutative ring is a field, so the question amounts to this: which commutative rings have Pierce bundles whose stalks are fields? Remarkably, these are precisely the commutative  $von\ Neumann\ regular\ rings$ . This class of commutative rings may be characterized in several ways, one of which is that every principal ideal is generated by an idempotent. For the general theory of (not necessarily commutative) von Neumann regular rings, see [18].

As a second illustration, and one which almost brings us to the meaning of Problem 5, one may pose the following question: what commutative rings have Pierce bundles whose stalks are *local rings*? This is meaningful inasmuch as a local ring is necessarily indecomposable. On the other hand, local rings include fields,

so that after Pierce's theorem the class of rings that we are after here includes all commutative von Neumann regular rings. The question was answered by Monk [45] following work of Warfield [50]. See also [25], Chapter V. Monk showed that *exchange rings* are precisely the rings whose Pierce bundles have stalks which are local rings. Here, a commutative exchange ring k is defined by the following property:

(5.1) every element of k is the sum of an idempotent and a unit.

At last we turn to vertex rings. In [33], Part II, I showed that Pierce's theory as we have sketched it out, extends naturally to the category Ver of vertex rings in place of Ring. In particular, every vertex ring V has a Pierce bundle, which is an ètale bundle of indecomposable vertex rings  $\pi:E\to X$  over a Boolean base space X. V may be recovered as the vertex ring of global sections of  $\pi$ . An important point is the origin of the base space X. Indeed, define the  $center\ C(V)$  of V to consist of states  $v\in V$  whose vertex operator Y(v,z)=v(-1) is constant. Then C(V) is a unital commutative ring with respect to the  $-1^{th}$  product in V and a (highly degenerate) vertex subring of V. Then we take

$$X := B(C(V)).$$

I also obtained ([33], Theorem 11.1) the analog of Pierce's characterization of von Neumann regular rings. Surprisingly, the result is a more-or-less *verbatim* restatement of Pierce's theorem, but with 'vertex ring' in place of 'commutative ring'.

To be explicit, let V be a vertex ring. The 2-sided principal ideal generated by an element  $a \in V$  is the intersection of all 2-sided ideals of V that contain a, and an idempotent of V is just an idempotent of C(V) regarded as a commutative ring. Then we call V a von Neumann regular vertex ring if every 2-sided principal ideal of V is of the form e(-1)V for some idempotent  $e \in C(V)$ . Let  $E \to X$  be the Pierce bundle of V. Then

(5.2) V is a von Neumann regular vertex ring if, and only if, the stalks of the Pierce bundle are simple vertex rings.

Problem 5 asks for the vertex ring analog of Monk's theorem about exchange rings.

To be clear, define a vertex ring V to be a local vertex ring if V has a unique maximal ideal J (J is maximal if, and only if, V/J is a simple vertex ring). Such vertex rings are very familiar. If V is a VOA over  ${\bf C}$  of CFT-type, then its conformal grading takes the shape

$$V = \mathbf{C} \mathbf{1} \oplus V_1 \oplus V_2 \dots$$

Every such V that is self-dual is a local VOA, where J is the radical of any nonzero invariant bilinear form on V.

Cue Problem 5:

which vertex rings V have the property that their Pierce bundles have stalks all of which are local vertex rings?

This simultaneously generalizes (5.2) and Monk's theorem. We may call such a vertex ring an *exchange vertex ring*. So how should we generalize (5.1) so that it also applies to all vertex rings and not just commutative rings?

#### 6. Genus 2 Monstrous Moonshine

**Problem 5**. Construct the genus 2 Moonshine Module  $V^{\natural}$ .

We have already referred to the influence of Zhu's paper [52], in which he treated the arithmetic properties of the characters of modules of a rational VOA. In fact Zhu did much more than this. He defined n-point functions, and implicitly reduced their study to that of 1-point functions by establishing a recursive formula that they satisfy. These functions all reside on a complex torus, i.e., a compact Riemann surface of genus g=1. On the other hand, and in marked contrast to the physics literature where the use of 'higher loop' calculations is routine, there have been relatively few applications of n-point functions on a higher genus Riemann surface in the mathematical literature on VOAs.

Following his initial paper [49], Michael Tuite and I set out to do something about this [36]-[41]. Because the 'easiest' higher loop functions to study ought to be 0-point functions on a g=2 Riemann surface, i.e., the genus 2 characters of a VOA V, we introduced some functions that we expect will serve as the desired genus 2 characters. With the idea of modular-invariance in mind, we anticipated that these functions would have some g=2 modular-invariance properties, and for rational V would in fact be Siegel modular forms on a congruence subgroup of  $Sp_4(\mathbf{Z})$ .

Problem 5 asks for the determination of the genus 2 character of the Moonshine Module  $V = V^{\natural}$ . A precise Conjecture is stated at the end of this Section. Although Tuite and I managed to understand the case when V is a lattice theory  $V_L$  (more on this below) the case of  $V^{\natural}$  eluded us. I've always found this circumstance to be rather disappointing, and would be delighted to see the solution! In the rest of this Section I will give an account of the genus 2 theory for  $V_L$ . In effect, problem 5 asks a reproduction of the calculations that follow for related VOAs such as  $V_L^+$  and  $\mathbf{Z}_2$ -orbifolds of  $V_L$ .

A distinctive feature of the higher genus theory for VOAs V is that it emphasizes, and depends on, the geometric theory of Riemann surfaces of all genera, not just tori. In keeping with Zhu's perspective, we define the higher genus n-point functions of V in a sort of recursive manner. Thus we assume we have at our disposal the full gamut of n-point functions for V at g=1, and then define the higher genus n-point functions in terms of them.

When g=2, this leads to the circumstance that we obtain not just one, but two, different definitions of the genus 2 character of V. This is because there are two rather different ways to construct a g=2 compact Riemann surface by doing some plumbing (sewing) with g=1 surfaces (complex tori). In the first approach (the so-called  $\epsilon$ -formalism) we sew together a pair of complex tori  $X_1, X_2$  by excising a parameterized disk of radius  $\epsilon$  from each  $X_i$  and identifying the resulting boundaries. Here we are plumbing  $X_1$  and  $X_2$  by connecting them, if you will, using a cylinder running between the punctures. The resulting surface is often denoted by  $X_1 \# X_2$ . In the second approach (the  $\rho$ -formalism) we start out with a single

complex torus X, excise a disjoint pair of disks, and attach a handle running from one to the other.

Let V be a VOA. For  $v \in V$  the vertex operator for v is denoted by

$$Y(v,z) := \sum_{n \in \mathbf{Z}} v(n)z^{-n-1}$$

The zero mode of a homogeneous state  $v \in V_k$  is o(v) := v(k-1). This operator preserves all homogeneous spaces  $V_\ell$ . For general  $v \in V$  written as a sum of homogeneous states we define o(v) by linear extension. The 1-point functions at genus 1 for V are defined as follows:

$$\begin{split} Z^{(1)}(v,\tau) &:= \operatorname{Tr}_{V} Y(q^{L(0)}v,q) q^{L(0)-c/24} \\ &= \operatorname{Tr}_{V} q^{L(0)-c/24} o(v) = q^{-c/24} \sum_{\ell} \operatorname{Tr}_{V_{\ell}} o(v) q^{\ell} \end{split}$$

where, as usual, we have set  $q:=q_{\tau}:=e^{2\pi i\tau}$  for  $\tau$  in the complex upper half-plane **H**. The 2-point functions at g=1 may be defined as follows:

$$Z^{(1)}((v_1,q_1),(v_2,q_2)) := \operatorname{Tr}_V Y(q_1^{L(0)}v_1,q_1) Y(q_2^{L(0)}v_2,q_2) q^{L(0)-c/24},$$

where  $q_j := q_{\tau_i}$ . For additional background, cf. [52], [36], [39], [40].

We come to the genus 2 character in the  $\epsilon$ -formalism, which may be defined for nice enough V as follows:

(6.1) 
$$Z_{V,\epsilon}^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{k \ge 0} \epsilon^k \sum_{u \in V_{[k]}} Z_V^{(1)}(u, \tau_1) Z_V^{(1)}(\overline{u}, \tau_2).$$

We must explain the notation. We are assuming that  $V=\mathbf{C}\mathbf{1}\oplus V_1\oplus \ldots$  is a *self-dual* VOA of CFT-type, so that V comes equipped with a symmetric, nondegenerate, invariant, bilinear form. Associated to V is an *isomorphic* copy  $V[\ ]$  that arises from the *same* underlying linear space and a change-of-variables vertex operator (loc. cit.)

$$Y[v,z] := Y(q_z^{L(0)}v, q_z - 1),$$

and  $V[\ ]:=\mathbf{C}\mathbf{1}\oplus V_{[1]}\oplus\ldots$  is the (isomorphic) conformal grading. Let b be the normalized invariant bilinear form on  $V[\ ]$ . For each  $k,\ b$  sets-up an identification of  $V_{[k]}$  with its dual space, and  $\overline{u}\in V_{[k]}$  is the metric dual of u with respect to b, i.e.,  $\overline{u}:=\sum_i(b,u_i)u_i$  for a basis  $\{u_i\}$  of  $V_{[k]}$  satisfying  $b(u_i,u_j)=\delta_{ij}$ . This change of variables corresponds to a pivot from the original VOA  $(V,Y,\mathbf{1})$  on a punctured sphere to an isomorphic copy on  $(V[\ ],Y[\ ],\mathbf{1})$  on a cylinder. Then (6.1) relates 1-point functions on the cylinder to the plumbed g=2 Riemann surface  $X_1\# X_2$  with parameter  $\epsilon$ .

We think of  $\tau_i$  as the *modulus* of the elliptic curve  $X_i$ , i.e., the point in  $\mathbf{H}$  corresponding to  $X_i$ . Then there is a natural complex domain denoted (somewhat confusingly) by  $D^{\epsilon}$  consisting of the triples  $(\tau_1, \tau_2, \epsilon)$  for which the sewing procedure producing  $X_1 \# X_2$  is defined, and there is an evident diagram

$$D^{\epsilon} \xrightarrow{\Omega^{(2)}} \mathbf{H}^{2}$$

$$Z_{V,\epsilon}^{(2)} \qquad \mathbf{C}$$

where  $\Omega^{(2)}$  is the *period matrix* of  $X_1 \# X_2$ , which lies in the *genus 2 Siegel upper half space*  $\mathbf{H}^{(2)}$ . (For background, cf. [12]). It is known [37], [51] that  $\Omega^{(2)}$ , which describes the period matrix in terms of sewing variables, is a holomorphic map. Furthermore  $Z^{(2)}$  is holomorphic [39]. We may then ask:

(6.2) does 
$$Z^{(2)}$$
 factor through  $\Omega^{(2)}$ ?

There is a parallel story in the  $\rho$ -formalism, where the genus 2 character is defined by

(6.3) 
$$Z_{V,\rho}^{(2)}(\tau,\rho) = \sum_{k\geq 0} \rho^k \sum_{u\in V_{[k]}} Z_V^{(1)}(\overline{u},u,w,\tau).$$

Recall that in this formalism we are attaching a handle to an elliptic curve X of modulus  $\tau$ . This procedure defines the complex domain  $D^{\rho}$  consisting of the data  $(\tau, \rho, w)$  needed to implement the plumbing [41]. And there is a second diagram of holomorphic maps

$$D^{\rho} \xrightarrow{\Omega^{(2)}} \mathbf{H}^{2}$$

$$Z_{V,\rho}^{(2)} \downarrow \mathbf{C}$$

Of course we still want to know the answer to (6.2) in the  $\rho$ -formalism.

Let us now turn to a description of the genus 2 characters for lattice theories  $V=V_L$ . If we want to compare these functions with, say, Siegel modular forms, there is an evident difficulty, namely the *mismatch of variables*. For example, the genus 2 theta-function for L is defined in terms of a symmetric matrix  $\Omega$  with positive-definite real part, that is  $\Omega \in \mathbf{H}^{(2)}$ , which we may think of as the period matrix of a g=2 Riemann surface:

$$\Theta_L^{(2)}(\Omega) := \sum_{\alpha,\beta \in L} exp \left\{ \pi i \left( (\alpha,\alpha) \Omega_{11} + 2(\alpha,\beta) \Omega_{12} + (\beta,\beta) \Omega_{22} \right) \right\}.$$

This theta-function has the entries  $\Omega_{ij}$  of  $\Omega$  as variables, and these are vastly different to the variables arising from the sewing procedures.

Now the answers to (6.2) and its  $\rho$ -analog is no, but morally yes. To explain what this means, recall that the genus 1 character for a rank n lattice theory  $V_L$  is

$$Z_{V_L}^{(1)}(\tau) = \frac{\theta_L(\tau)}{\eta(\tau)^n}.$$

This arises from the containment of a rank n Heisenberg subVOA  $M^n \subseteq V_L$ , indeed we may rewrite the last display as

$$\frac{Z_{V_L}^{(1)}(\tau)}{Z_{M^n}^{(1)}(\tau)} = \theta_L(\tau).$$

Surprisingly, precisely the same formula holds at genus 2 in both formalisms [37], [41]. To be somewhat misleading, we have

$$\frac{Z_{V_L,\epsilon}^{(2)}(\tau_1,\tau_2,\epsilon)}{Z_{M_n}^{(2)}(\tau_1,\tau_2,\epsilon)} = \theta_L^{(2)}(\Omega) = \frac{Z_{V_L,\rho}^{(2)}(\tau,\rho,w)}{Z_{M_n}^{(2)}(\tau,\rho,w)}.$$

(It is misleading because  $\Omega$  really depends on the moduli, depending on which formalism we are in.) A more accurate statement is that the following diagrams commute:



We can now prescribe exactly what is needed in Problem 5. Regarding any of the VOAs  $V=V_L^+$  or its  $\mathbb{Z}_2$ -orbifold, what is required are analogs of the last display. The conjecture is that there is a Siegel modular form F (depending on V) such that the following commute:



In case  $L=\Lambda$  is the Leech lattice and  $V=V^{\natural}$  is the  $\mathbb{Z}_2$ -orbifold, alias the Moonshine Module, F will be long sought genus 2 Moonshine character.

None of this involves the Monster simple group  $per\ se$ , however if the existence of F can be established, one expects that similar results can be proved for 1-point functions at g=2, and also including twisted sectors and group elements along the lines of [8].

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