# NETWORK EFFECTS AND DEFAULT CLUSTERING FOR LARGE PORTFOLIOS

# KONSTANTINOS SPILIOPOULOS AND JIA YANG

ABSTRACT. We consider a large collection of dynamically interacting components defined on a weighted directed graph determining the impact of default of one component to another one. We prove a law of large numbers for the empirical measure capturing the evolution of the different components in the pool and from this we extract important information for quantities such as the loss rate in the overall pool as well as the mean impact on a given component from system wide defaults. A singular value decomposition of the adjacency matrix of the graph allows to coarse-grain the system by focusing on the highest eigenvalues which also correspond to the components with the highest contagion impact on the pool. Numerical simulations demonstrate the theoretical findings.

#### 1. Introduction

The financial crisis of 2007-2009 made clear to the mathematical finance community that connectedness and network effects in financial systems need to be better understood and modelled. Risk can propagate through the system and network topology can affect its propagation.

Exogenous risks acting as initial shocks, such as devaluation of mortgage-baked securities, changes in interest rates or commodity prices cannot fully explain crisis events, but can lead to contagion effects, [23, 2]. In particular, shocks can lead to spiral events within the system and the topology and connectedness of the system can then affect how these spiral events unfold and propagate. This can then lead to systemic risk events, see for example [24], which has been by now widely accepted to be a dynamic event, [2, 4].

In the past ten years researchers have tried to understand and model such behavior in different ways. A significant body of literature has emerged that is aiming in understanding and modeling complex financial systems. Before describing the main contributions of this paper, let us first briefly describe the three main different lines of research that have emerged in the study of systemic risk. Firstly, there is the network models for clustering and contagion that follow the earlier work of [1, 13], see also [19] for a review. Secondly, there is the dynamic mean field type of models literature, see for example [5, 6, 10, 14, 18, 7, 20, 15]. Thirdly, there is the reduced form credit and portfolio risk literature that is using intensity models of correlated default, [8, 16, 17, 25, 27, 26]. Despite this significant progress, many questions are still wide open.

Date: December 20, 2018.

The present research was partially supported by the National Science Foundation (DMS 1412529 and DMS 1550918). We would like to thank Kay Giesecke and Paolo Guasoni for discussions on this project.

Our work falls in the last category, i.e., in the reduced form credit risk literature. Motivated by the empirical work of [2] and following [16, 17], the intensity to default process for each individual name in the pool is characterized by three terms: an idiosyncratic term, which is specific to each name, a contagion term, which is responsible for clustering of defaults, and an exogenous risk term common to all names in the pool. As it has been established in [2, 16, 17], see also [25] for a review, these terms give important insights on how risk propagates and on how defaults cluster. Due to the interconnectedness of the system, the failure of a single component increases the likelihood of failure of other components in the system. Uncertainty becomes an issue which then leads market participants to fear even more losses in asset prices disproportional to the magnitude of the crisis. Reduceform point process models of correlated default are many times used to assess portfolio credit risk and are based on counting processes. We use dynamic portfolio credit risk models to understand large portfolio asymptotics and default clustering.

Our contribution in this paper is twofold. Firstly, we consider general stochastic intensity-to-default processes where the drift coefficient of the idiosyncratic component is only required to satisfy appropriate dissipative properties instead of requiring it to be affine. This generalization complicates the analysis as we cannot use anymore the properties of affine models, resulting in more delicate analysis and estimates. We prove well-posedness of the related stochastic intensity models and rigorously characterize the limit of the empirical survival distribution of the names in the pool as their number grows to infinity.

Secondly, we consider network effects, a feature missing from the earlier work of [8, 16] and its follow ups. To be more precise, we specify the interaction of names by a weighted, directed graph  $G(\Gamma, \mathcal{E}, \omega)$  where  $\Gamma$  is the set of vertices (i.e., names),  $\mathcal{E}$  is the set of (directed) edges and  $\omega: \mathcal{E} \to (0, \infty)$  is a function assigning weights to edges (as a convention we could define  $\omega(i,j)=0$  whenever  $(i,j)\notin\mathcal{E}$ ). An edge  $(i,j) \in \mathcal{E}$  implies a directed interaction, the impact that the default of name i has on name j. The weight  $\omega(i,j)$  measures the strength of the interaction. For example,  $\omega(i,j)$  could represent the loss of name j at the default of counterparty i (the loss is usually the positive part of the mark-to-market value of the contract at default). Let A be the matrix with elements  $\omega(i,j)$ . As it turns out, a singular value decomposition (SVD) of A allows us not only to identify the number of different clusters in the pool (this is the number of non-zero eigenvalues of A), but it also gives a physical meaning to the term responsible for contagion effects and spiral events. In addition, the SVD allows us to also quantify which of the clusters have the most significant impact on the whole system. It also allows us to reduce the dimensionality of the system via appropriate low-rank approximations. In this paper, we theoretically analyze the limit of the empirical measure of surviving names as  $N \to \infty$  and we also showcase the different cases by numerical studies. We demonstrate numerically that if there is sufficient spectral gap in the eigenvalues of A from the SVD, then the distribution of stochastic processes of interest is very well approximated by appropriate low rank approximations. This becomes practically useful, since without the low-rank approximation, as we will see, the computation of the quantities of interest can become prohibitively expensive.

We study the typical behavior of the loss rate both in the overall pool and within names of the same type. In addition, we study the mean impact to default on a given name from system wide defaults as the number of components  $N \to \infty$ . We allow

the pool to be heterogeneous with stochastic intensity that evolves dynamically in time and with different weights  $\omega(i,j)$  for different i,j. In addition, the loss rate (either overall in the pool or for names of specific types) and the mean default impact on a given from system wide defaults are dynamic quantities and their computation can be numerically cumbersome. We show numerically that low-rank approximations motivated through the SVD can be very effective in accurately reducing the dimension of the system and thus making their evaluation numerically tractable.

At this point, we want to mention that even though our primary motivation comes from interacting particle systems in financial mathematics, our results are broader applicable. In a given system with many different components, not all components are equally connected to other components or equally affected by the default of other components. The failure of one component due to external forcing giving rise to failure of other components of a given system is of broader interest.

The rest of the paper is organized as follows. In Section 2 we describe our model in detail. In Section 3 we lay down our assumptions that are assumed to hold throughout the paper. Section 4 contains the main results of this paper. The proof of the main theorem is in the subsequent sections. In particular, tightness of the empirical measure is discussed in Section 6, its limit characterization is in Section 7 followed by uniqueness of the limiting point in Section 8. Section 5 contains our simulation studies and numerical results on low-rank approximations. Technical results and their proofs have been gathered in Appendix A. Section 9 is about our conclusions and outlook for future work.

# 2. Model description

The model considered in this paper models the evolution of a system consisting of N names which are subject to default risk. The model for the default risk takes into account three terms: an idiosyncratic risk (specific to a given name), a systematic risk (common to all names) and a term modeling default contagion and spiral events. The last term takes into account the network topology.

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  where all random variables are defined. Let  $\{W^n\}_{n\in\mathbb{N}}$  be a collection of i.i.d. standard Brownian motions which are used to model the idiosyncratic risk for each component of the pool. Let V be a standard Brownian motion independent from  $W^n$ 's, driving the randomness of the systematic risk factor process X.  $\mathcal{V} = \sigma(V_s, 0 \leq s \leq t) \vee \mathcal{N}$ , where  $\mathcal{N}$  is the set of null sets. Let  $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$  be a collection of independent standard exponential random variables. For  $N \in \mathbb{N}$  and  $n \in \{1, 2, \ldots, N\}$ , denote by  $\tau^{N,n}$  the stopping time at which the

For  $N \in \mathbb{N}$  and  $n \in \{1, 2, ..., N\}$ , denote by  $\tau^{N,n}$  the stopping time at which the n-th component of the system fails. The failure time  $\tau^{N,n}$  has stochastic intensity process  $\lambda^{N,n}$  to be described below. The default time  $\tau^{N,n}$  is

$$\tau^{N,n} \stackrel{\text{def}}{=} \inf \Big\{ t \geq 0 : \int_0^t \lambda_s^{N,n} ds \geq \mathfrak{e}_n \Big\},$$

and, we can also write

$$\chi_{\{\tau^{N,n} \leq t\}} = \chi_{[\mathfrak{e}_n,\infty)} \Big( \int_0^t \lambda_s^{N,n} ds \Big),$$

where  $\chi_{\{B\}}$  is the indicator function for a set B.

Recall the network structure of the system, which is described by a directed graph  $G(\Gamma, \mathcal{E}, \omega)$  where  $\Gamma$  is the set of components in the system,  $\mathcal{E}$  is the set of

directed edges and  $\omega: \mathcal{E} \to (0, \infty)$  is the function assigning weights to edges.  $\omega(i, j)$  represents the default impact the *i*-th name has on the *j*-th firm.

Then, the total loss experienced by name j due to system wide defaults by time t is

(1) 
$$\sum_{i=1}^{N} \omega(i,j) 1_{\{\tau^{N,i} \le t\}}$$

Let A be the adjacency matrix of G, i.e. the (i,j)th entry of A is given by  $\omega(i,j)$  for  $(i,j) \in \mathcal{E}$  and 0 if  $(i,j) \notin \mathcal{E}$ .

Then, the classical singular value decomposition (SVD in short) yields

(2) 
$$A = \sum_{j=1}^{r} \xi_j^2 \, \ell_j u_j^{\mathsf{T}}$$

where  $\{\ell_1,\ldots,\ell_r\}$  are orthonormal vectors (spanning columns of A),  $\{u_1,\ldots,u_r\}$  are orthonormal vectors (spanning rows of A) and  $\xi_1^2 > \xi_2^2 > \cdots > \xi_r^2 > 0$  are real numbers known as the singular values.

Here,  $r \leq N$  is called the rank of A. In a sense r represents the complexity of the system. The larger r is, the more complex the structure of the interaction becomes. One can interpret r as the number of clusters in the system.

Let  $\ell_{i,j}$  be the *i*-th entry of  $\ell_j$  in (2) and similarly  $u_{i,j}$  be the *i*-th entry of the vector  $u_j$ . The mean default impact on n-th name from system wide defaults up to time t is

$$(3) \quad Q_t^{N,n} = \frac{1}{N} \sum_{i=1}^N \omega(i,n) \chi_{\{\tau^{N,i} \le t\}} = \sum_{i=1}^r \xi_j^2 u_{n,j} \frac{1}{N} \sum_{i=1}^N \ell_{i,j} \chi_{\{\tau^{N,i} \le t\}} = \beta_n^C \cdot L_t^N,$$

where  $\beta_n^C=(\xi_1^2u_{n,1},\xi_2^2u_{n,2},\ldots,\xi_r^2u_{n,r})^T$  and the vector-valued process  $L_t^N=(L_t^{N,1},L_t^{N,2},\ldots,L_t^{N,r})^T$  has elements

$$L_t^{N,j} = \frac{1}{N} \sum_{i=1}^{N} \ell_{i,j} \chi_{\{\tau^{N,i} \le t\}}.$$

The j-th entry of  $L_t^N$  can be loosely interpreted as the stochastic loss rate of the j-th cluster of the network.

An intensity is driven by an idiosyncratic risk represented by a Brownian motion  $W^n$ , a systematic risk represented by the process X, and spillover risk represented by the process  $\beta_n^C \cdot L^N$ . In particular, we consider the following interacting system

$$d\lambda_{t}^{N,n} = b(\lambda_{t}^{N,n}, a_{n})dt + \sigma_{n} \cdot (\lambda_{t}^{N,n})^{\rho}dW_{t}^{n} + \beta_{n}^{C} \cdot dL_{t}^{N} + \beta_{n}^{S}\lambda_{t}^{N,n}dX_{t}$$

$$\lambda_{0}^{N,n} = \lambda_{0,N,n}$$

$$dX_{t} = b_{0}(X_{t})dt + \sigma_{0}(X_{t})dV_{t}$$

$$X_{0} = x_{0}$$

$$L_{t}^{N,j} = \frac{1}{N} \sum_{n=1}^{N} \ell_{n,j}\chi_{\{\tau^{N,n} \leq t\}}. \qquad j = 1, 2, ..., r$$

Notice that we allow for a heterogeneous pool, which means that the intensity dynamics of each name can be different. In the model  $\sigma_n \in \mathbb{R}_+, a_n \in \mathbb{R}^k$  for some  $k > 0, \ \beta_n^S \in \mathbb{R}$  are constants and  $1/2 \le \rho < 1$ . Let us set  $\mathcal{P} = \mathbb{R}_+ \times \mathbb{R}^{k+2r+1}$  and

 $\hat{\mathcal{P}} = \mathcal{P} \times \mathbb{R}_+$ . For all  $n \in \{1, 2, ..., N\}$ , we capture these different dynamics by defining the "types"

$$p^n = (\sigma_n, a_n, \beta_{n,1}^C, \cdots, \beta_{n,r}^C, \beta_n^S, \ell_{n,1}, \cdots, \ell_{n,r}) \in \mathcal{P}$$

and

$$\hat{p}^n = (p^n, \lambda_{0,N,n}) \in \hat{\mathcal{P}}.$$

In addition, we let  $\hat{p}_t^n = (p^n, \lambda_t^{N,n}) \in \hat{\mathcal{P}}$ .

Our paper, extends significantly the result of [17]. Firstly, the drift term  $b(\lambda, a)$  only needs to have certain dissipative properties with respect to  $\lambda$ . Secondly, we now have a network structure described through the adjacency matrix A. As we shall see, the analysis of this model is not only more challenging, but it also requires new arguments and ideas. The introduction of the network structure through the adjacency A, allows for a far richer set of questions to be asked.

# 3. Notation and Assumptions

In this section, we go over our assumptions that are assumed to hold throughout the paper.

First, let us define

$$\pi^N = \frac{1}{N} \sum_{n=1}^N \delta_{p^n}$$

$$\Lambda_0^N = \frac{1}{N} \sum_{n=1}^N \delta_{\lambda_{0,N,n}}.$$

The measures  $\pi^N$  and  $\Lambda_0^N$  belong to the space of Borel probability measures on  $\mathcal{P}$  and  $\mathbb{R}$  respectively. These spaces will be denoted by  $\mathfrak{P}(\mathcal{P})$  and  $\mathfrak{P}(\mathbb{R})$  respectively.

**Assumption 3.1.** Assume that the limits

$$\pi = \lim_{N \to \infty} \pi^N$$

$$\Lambda = \lim_{N \to \infty} \Lambda_0^N$$

exist on  $\mathfrak{P}(\mathbb{P})$  and  $\mathfrak{P}(\mathbb{R})$  respectively.

**Assumption 3.2.** Assume that there is a constant  $K_{3.2} > 0$  such that all the coefficients  $\sigma_n$ ,  $a_n$ ,  $||\beta_n^C||$ ,  $|\beta_n^S|$  and  $|\ell_{n,j}|$  j = 1, 2, ..., r are bounded by  $K_{3.2}$  and there exists a  $\bar{\sigma} > 0$  that  $\inf_n \sigma_n^2 \geq \bar{\sigma}^2 > 0$ .

For the drift coefficient function  $b(\lambda, a)$  we assume the following growth and regularity conditions.

**Assumption 3.3.** Function  $b(\lambda, a)$  are locally Lipschitz and there exists finite constants d > 1, q > 1, K > 0 and positive bounded functions  $\gamma$  and k with  $\gamma(a) > 0$  and k(a) > 0 such that

$$\lambda b(\lambda, a) < -\gamma(a)|\lambda|^d$$
, for  $|\lambda| \ge K$   
 $|b(\lambda, a)| \le k(a)(1 + |\lambda|^q)$ ,

and

Furthermore we assume that for any  $\lambda \in \mathbb{R}_+$ ,  $a \mapsto b(\lambda, a)$  is a continuous function.

A remark in regards to Assumption 3.3 follows.

**Remark 3.4.** If we take  $a=(\bar{\alpha},\bar{\lambda})\in\mathbb{R}^2_+$  and  $b(\lambda,a)=-\bar{a}(\lambda-\bar{\lambda})$ , then the idiosyncratic part of the intensity process becomes the classical CEV model. Notice that in this case  $b(\lambda,a)=-\partial_{\lambda}V(\lambda,a)$  with  $V(\lambda,a)=\frac{\bar{a}}{2}(\lambda-\bar{\lambda})^2$  and the function  $V(\lambda,a)=\frac{\bar{a}}{2}(\lambda-\bar{\lambda})^2$  and  $V(\lambda,a)=\frac$ 

However, Assumption 3.3 relaxes the affine structure to a requirement about appropriate dissipativity of the drift coefficient  $b(\lambda, a)$ . This enlarges the class of drifts  $b(\lambda, a)$  that one can consider. For example, one could consider situations where  $b(\lambda, a) = -\partial_{\lambda}V(\lambda, a)$  with  $V(\lambda, a)$  being a bistable potential. Such situations could correspond to situations where the creditworthiness of certain names might have two equilibria, corresponding to two different parts of the business cycle.

The goal of this paper is to rigorously establish that such choices lead to well defined intensity-to-default processes, subsequently well defined mean field limits of the empirical survival distribution (Section 4), and, of course, to numerically explore (Section 5) the potential effects of the network structure and of low-rank approximations on the distribution of dynamically evolving stochastic processes of interest (see also Section 9 for a more elaborate related discussion).

The rest of the assumptions are related to the exogenous risk process X.

**Assumption 3.5.** Assume that function  $\sigma_0(\cdot)$  is bounded, that is there exists a constant  $K_{3.5}$  such that  $|\sigma_0(x)| < K_{3.5}$ . For  $b_0$  assume  $\sup_{t < \infty} \mathbb{E}|b_0(X_t)|^{4p} < \infty$  for some p > 1.

Let us define

$$\Gamma_t = -\beta^S \int_0^t b_0(X_s) ds.$$

**Assumption 3.6.** Assume that for some  $p \geq 1$ ,  $\sup_{t < \infty} \mathbb{E}[X_t^{2p}]$  and  $\sup_{t < \infty} \mathbb{E}[e^{4p|\Gamma_t|}]$  are bounded.

The last Assumption 3.7 makes sure that we can extend some technical lemmas from bounded drifts  $b_0(x)$  to potentially unbounded ones.

**Assumption 3.7.** Assume there is a function u(x) such that  $\sigma_0(x)u(x) = -b_0(x)$  and for any T > 0 we have

$$\mathbb{E}\left[e^{1/2\int_0^T |u(X_s)|^2 ds}\right] < \infty,$$

and that for any T there is a p>1 such that

$$\mathbb{E}\left[\left|e^{-\int_0^T u(X_s)dV_s-1/2\int_0^T |u(X_s)|^2ds}\right|^p\right]<\infty.$$

#### 4. Well-posedness of the model and main results

In this section we prove that the model is well-possed and we present our main results. Let us begin with well-posedness of the model, Lemma 4.1. For this purpose, let  $\xi$  be a vector of processes having r components, predictable, right-continuous, monotone and bounded with  $\xi_0 = 0$ .

**Lemma 4.1.** Let Assumptions 3.2-3.7 hold. There exists an unique nonnegative solution  $\lambda$  of the following SDE:

$$d\lambda_t = b(\lambda_t, a)dt + \sigma(\lambda_t \vee 0)^{\rho}dW_t + \beta^C \cdot d\xi_t + \beta^S \lambda_t dX_t$$
$$\lambda_0 = \lambda_o$$
$$dX_t = b_0(X_t)dt + \sigma_0(X_t)dV_t.$$

Lemma 4.2 is about an essential a-priori bound that will be used in many places of the subsequent proofs.

**Lemma 4.2.** Let  $p \ge 1$  be such that Assumptions 3.5 and 3.6 hold. Then, for such  $p \ge 1$  and for every  $T \ge 0$ ,

$$K_{4.2} \stackrel{\text{def}}{=} \sup_{0 \le t \le T, n \in \mathbb{N}} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[|\lambda_t^{N,n}|^p]$$

is finite.

Proofs of Lemmas 4.1 and 4.2 are in Appendix A. Let us denote the survival indicator process for a given name in the pool by

$$M_t^{N,n} = \chi_{\{\tau^{N,n} > t\}}$$

and define the empirical distribution of the  $\hat{p}^n$ 's corresponding to the names that have survived up to time t as follows:

$$\mu_t^N = \frac{1}{N} \sum_{n=1}^N \delta_{\hat{p}_t^n} M_t^{N,n}.$$

Notice that  $\mu_t^N$  captures the entire dynamics of the model (including the effect of the heterogeneities and network topology).

In order to study the convergence of  $\mu^N$ , we need to set up the appropriate topological framework. That is, let E be the collection of sub-probability measures on  $\hat{\mathcal{P}}$ , i.e., E consists of those Borel measures  $\nu$  on  $\hat{\mathcal{P}}$  such that  $\nu(\hat{\mathcal{P}}) \leq 1$ . Then fix a point  $\star$  which is not in  $\hat{\mathcal{P}}$  and let  $\hat{\mathcal{P}}^+ = \hat{\mathcal{P}} \cup \{\star\}$  (the so-called one-point compactification of  $\hat{\mathcal{P}}$ ). Open sets are those which are open subsets of  $\hat{\mathcal{P}}$  (endowed with the original topology) or complements in  $\hat{\mathcal{P}}^+$  of closed subsets of  $\hat{\mathcal{P}}$  (again, in the original topology of  $\hat{\mathcal{P}}$ ).

Define a bijection  $\zeta$  from E to the Borel probability measures on  $\hat{\mathcal{P}}^+$  as

$$(\zeta \nu)(A) = \nu(A \cap \mathcal{P}) + (1 - \nu(\mathcal{P}))\delta_{\star}(A),$$

for any  $A \in \mathcal{B}(\hat{\mathcal{P}}^+)$ . Then we can make E a Polish space.

We define the Skorokhod topology on  $\mathfrak{P}(\hat{\mathcal{P}}^+)$ , and define a corresponding metric on E by requiring  $\zeta$  to be an isometry. Then, the space E will be Polish.

Thus,  $\mu^N$  is an element of  $D_E[0,\infty)$ , i.e., is a map from  $[0,\infty)$  into E which is right-continuous and has left-hand limits. The space  $D_E[0,\infty)$  will be endowed with the Skorohod metric, which we denote by  $d_E$ , see [12].

Next, for each  $f \in C^{\infty}(\hat{\mathcal{P}})$  define

$$\langle f, \mu \rangle_E = \int_{\hat{p} \in \hat{\mathcal{P}}} f(\hat{p}) \mu(d\hat{p}).$$

In addition, define the generators

$$\mathcal{L}_{1}f(\hat{p}) = \frac{1}{2}\sigma^{2}\lambda^{2\rho}\frac{\partial^{2}f}{\partial\lambda^{2}}(\hat{p}) + b(\lambda, a)\frac{\partial f}{\partial\lambda}(\hat{p}) - \lambda f(\hat{p})$$

$$\mathcal{L}_{2}^{x}f(\hat{p}) = \frac{1}{2}(\beta^{S})^{2}\lambda^{2}\sigma_{0}^{2}(x)\frac{\partial^{2}f}{\partial\lambda^{2}}(\hat{p}) + \beta^{S}\lambda b_{0}(x)\frac{\partial f}{\partial\lambda}(\hat{p})$$

$$\mathcal{L}_{3}^{x}f(\hat{p}) = \beta^{S}\lambda\sigma_{0}(x)\frac{\partial f}{\partial\lambda}(\hat{p})$$

$$\mathcal{L}_{4}f = \beta^{C}\frac{\partial f}{\partial\lambda}(\hat{p})$$

$$\iota(\hat{p}) = \lambda$$

$$\nu(\hat{p}) = \ell$$

with  $\beta^C$ ,  $\ell$  are vector valued, of the form  $\beta^C = (\beta_1^C, \beta_2^C, \dots, \beta_r^C)$  and  $\ell = (l_1, l_2, \dots, l_r)$  respectively. We write  $\nu_j(\hat{p}) = l_j$  for  $j = 1, 2, \dots, r$ .

We will also use the notation

$$\mathbb{E}_{\mathcal{V}}[\cdot] = \mathbb{E}[\cdot|\mathcal{V}].$$

Now, we are in position to state the main result of the paper.

**Theorem 4.3.** Let Assumptions 3.1-3.7 hold. We have that  $\mu_{\cdot}^{N}$  converges in distribution to  $\bar{\mu}$  in  $D_{E}[0,T]$ . The evolution of  $\bar{\mu}$  is given by the measure evolution equation

$$d\langle f, \bar{\mu}_t \rangle_E = \left\{ \langle \mathcal{L}_1 f, \bar{\mu}_t \rangle_E + \langle \mathcal{L}_2^{X_t} f, \bar{\mu}_t \rangle_E + \langle \iota \nu, \bar{\mu}_t \rangle_E \cdot \langle \mathcal{L}_4 f, \bar{\mu}_t \rangle_E \right\} dt$$

$$+ \langle \mathcal{L}_3^{X_t} f, \bar{\mu}_t \rangle_E dV_t, \quad \forall f \in C^{\infty}(\hat{\mathcal{P}}) \ a.s.$$

In addition, if  $(Q_i(t), \lambda_t(\hat{p}), i = 1, ..., r)$  is the unique pair satisfying

$$Q_i(t) = \int_{\hat{p} \in \hat{\mathcal{P}}} l_i \mathbb{E}_{\mathcal{V}_t} \left\{ \lambda_t^*(\hat{p}) exp \left[ -\int_0^t \lambda_s^*(\hat{p}) ds \right] \right\} \pi(dp) \Lambda_0(d\lambda).$$

$$\lambda_t^*(\hat{p}) = \lambda_0 + \int_0^t b(\lambda_s, a) ds + \sigma \cdot (\lambda_s)^{\rho} dW_s^* + \int_0^t \sum_{i=1}^r \beta_i^C Q_i(s) ds + \beta^S \int_0^t \lambda_s^*(\hat{p}) dX_s.$$

then for any  $A \in \mathfrak{B}(\mathcal{P})$  and  $B \in \mathfrak{B}(\mathbb{R}_+)$ ,  $\bar{\mu}$  is given by

$$(5) \quad \bar{\mu_t}(A \times B) = \int_{\hat{p} \in \mathcal{P}} \chi_A(p) \mathbb{E}_{\mathcal{V}_t} \left[ \chi_B(\lambda_t^*(\hat{p})) exp \left[ -\int_0^t \lambda_s^*(\hat{p}) ds \right] \right] \pi(dp) \Lambda_0(d\lambda).$$

Proof of Theorem 4.3. The ingredients of the proof are in Sections 6, 7 and 8. In Section 6 we prove that the family  $\{\mu^N\}_{N\in\mathbb{N}}$  is relatively compact (as a  $D_E[0,\infty)$ -valued random variable). Therefore  $\{X,\mu^N\}_{N\in\mathbb{N}}$  is also relatively compact. If we denote by  $(X,\bar{\mu})$  an accumulation point of one of its convergent subsequences, then the computations of Section 7 show that it will satisfy (4). The results of Section 8 show that  $\bar{\mu}$  is actually unique and is actually given by (5). These results complete the proof of the theorem.

#### 5. Numerical studies and simulation results

In this section we demonstrate numerically the theoretical results of the paper. Before presenting the numerical studies, we first describe the numerical method that we follow and we also comment on general aspects and issues that are common in all examples.

One of the quantities that we are interested in is the overall loss rate in the pool, defined by

$$D_t^N = \frac{1}{N} \sum_{n=1}^N \chi_{\{\tau^{N,n} \le t\}} = \frac{1}{N} \sum_{n=1}^N (1 - M_t^{N,n}) = 1 - \mu_t^N(\hat{\mathcal{P}}).$$

Related to this quantity is also the loss rate for names of the same type, say type A, denoted by  $p_A$ :

$$D_t^N(p_A) = \frac{1}{N_A} \sum_{n=1}^N \chi_{\{\tau^{N,n} \le t\}} \chi_{\{p^{N,n} = p_A\}} = \frac{1}{N_A} \sum_{n=1}^N (1 - M_t^{N,n}) \chi_{\{p^{N,n} = p_A\}}$$
$$= 1 - \frac{N}{N_A} \mu_t^N(\{\hat{p} : p = p_A\}),$$

where  $N_A$  is the total number of names of type A in the pool.

We are also interested in the mean impact on name  $n \in \{1, \dots, N\}$  from system wide defaults by time t, which is  $Q_t^{N,n}$  defined by (3) as

$$Q_t^{N,n} = \beta_n^C \cdot L_t^N$$

with the contagion coefficient vector being

$$\beta_n^C = (\xi_1^2 u_{n1}, \xi_2^2 u_{n2}, \dots, \xi_r^2 u_{nr})$$

and the r-dimensional vector  $L_t^N = (L_t^{N,1}, L_t^{N,2}, \dots, L_t^{N,r})$  where

$$L_t^{N,j} = \frac{1}{N} \sum_{n=1}^N \ell_{n,j} \chi_{\{\tau^n \le t\}} \qquad j = 1, 2, \dots, r.$$

In order to be able to compute  $Q_t^{N,n}$  we need to be able to compute  $L_t^{N,j}$ , which is associated to the j-th cluster of the network:

$$L_t^{N,j} = \frac{1}{N} \sum_{n=1}^{N} \ell_{n,j} \chi_{\{\tau^n \le t\}} = \frac{1}{N} \sum_{n=1}^{N} \ell_{n,j} - \frac{1}{N} \sum_{n=1}^{N} \ell_{n,j} M_t^n$$
$$= \frac{1}{N} \sum_{n=1}^{N} \ell_{n,j} - \langle \nu_j, \mu_t^N \rangle,$$

where we recall that  $\nu_j(\hat{p}) = l_j$ .

In the numerical simulations, quantities like  $\mu_t^N(\hat{\mathcal{P}})$ ,  $\mu_t^N(\{\hat{p}:p=p_A\})$  and  $\langle \nu_j, \mu_t^N \rangle$  are approximated by  $\bar{\mu}_t(\hat{\mathcal{P}})$ ,  $\bar{\mu}_t(\{\hat{p}:p=p_A\})$  and  $\langle \nu_j, \bar{\mu}_t \rangle$  respectively; made possible via Theorem 4.3. In order to be able to compute the latter quantities, we first assume that  $\bar{\mu}_t(d\hat{p})$  has a density  $v(t,\hat{p})$ , i.e., we assume that we can write  $\bar{\mu}_t(d\hat{p}) = v(t,\hat{p})d\hat{p}$  with  $\hat{p} = (p,\lambda)$ . An integration by parts on the stochastic

evolution equation that  $\bar{\mu}_t(d\hat{p})$  satisfies, gives

$$dv(t,\hat{p}) = \left\{ \mathcal{L}_{1}^{*}v(t,\hat{p}) + \mathcal{L}_{2}^{*,X_{t}}v(t,\hat{p}) + \sum_{j=1}^{r} \left( \int_{\hat{p}' \in \hat{\mathcal{P}}} \nu_{j}(\hat{p}')\iota(\hat{p}')v(t,\hat{p}')d\hat{p}' \right) (\mathcal{L}_{4}^{*,j})v(t,\hat{p}) \right\} dt + \mathcal{L}_{3}^{*,X_{t}}v(t,\hat{p})dV_{t}$$

$$v(0,\hat{p}) = (\pi \times \Lambda_{0})(\hat{p})$$

$$v(t,p,\lambda=0) = \lim_{\lambda \to \infty} v(t,p,\lambda) = 0,$$

where the adjoint operators are given by:

$$\mathcal{L}_{1}^{*}v(t,\hat{p}) = \frac{\partial^{2}}{\partial\lambda^{2}}(\frac{1}{2}\sigma^{2}\lambda^{2\rho}v(t,\hat{p})) - \frac{\partial}{\partial\lambda}(b(\lambda,a)v(t,\hat{p})) - \lambda v(t,\hat{p}),$$

$$\mathcal{L}_{2}^{*,x}v(t,\hat{p}) = \frac{\partial^{2}}{\partial\lambda^{2}}(\frac{1}{2}(\beta^{S})^{2}\lambda^{2}\sigma_{0}^{2}(x)v(t,\hat{p})) - \frac{\partial}{\partial\lambda}(\beta^{S}\lambda b_{0}(x)v(t,\hat{p})),$$

$$\mathcal{L}_{3}^{*,x}v(t,\hat{p}) = -\frac{\partial}{\partial\lambda}(\beta^{S}\lambda\sigma_{0}(x)v(t,\hat{p})),$$

$$\mathcal{L}_{4}^{*,j}v(t,\hat{p}) = -\beta_{j}^{C}\frac{\partial v(t,\hat{p})}{\partial\lambda}, j = 1, 2, \dots, r,$$

$$\iota(\hat{p}) = \lambda,$$

$$\nu(\hat{p}) = (\nu_{1}(\hat{p}), \dots, \nu_{r}(\hat{p})) = \ell = (l_{1}, \dots, l_{r}).$$

Now, by applying Theorem 4.3 we approximate,

$$D_{t}^{N} \approx D_{t} = 1 - \bar{\mu}_{t}(\hat{\mathcal{P}}) = 1 - \int_{\hat{p} \in \hat{\mathcal{P}}} \bar{\mu}_{t}(d\hat{p}) = 1 - \int_{p \in \mathcal{P}} \int_{\lambda=0}^{\infty} v(t, p, \lambda) d\lambda \quad \pi(dp)$$

$$D_{t}^{N}(p_{A}) \approx D_{t}(p_{A}) = 1 - \kappa_{A} \bar{\mu}_{t}(\{\hat{p} : p = p_{A}\}) = 1 - \kappa_{A} \int_{\hat{p}: p = p_{A}} \bar{\mu}_{t}(d\hat{p})$$

$$= 1 - \int_{\lambda=0}^{\infty} v(t, p_{A}, \lambda) d\lambda$$

where  $\kappa_A = \lim_{N \to \infty} \frac{N}{N_A} = [\pi(\{p_A\})]^{-1}$  if the limit exists.

$$L_t^{N,j} \approx \frac{1}{N} \sum_{n=1}^N \ell_{n,j} - \langle \nu_j, \bar{\mu}_t \rangle = \frac{1}{N} \sum_{n=1}^N \ell_{n,j} - \int_{\hat{p} \in \hat{\mathcal{P}}} \nu_j(\hat{p}) \bar{\mu}_t(d\hat{p})$$
$$= \frac{1}{N} \sum_{n=1}^N \ell_{n,j} - \int_{p \in \mathcal{P}} l_j \int_{\lambda=0}^\infty v(t, p, \lambda) d\lambda \quad \pi(dp)$$

Hence, it is enough to be able to compute  $u_0(t,p) = \int_0^\infty v(t,\hat{p})d\lambda$ . In order to do so, we first define the k-th moment to be, see also [17],

(6) 
$$u_k(t,p) = \int_0^\infty \lambda^k v(t,\hat{p}) d\lambda.$$

The moment  $u_k(t,p)$  can be calculated from the evolution function of  $dv(t,\hat{p})$ , by multiplying it with  $\lambda^k$  and integrating by parts over  $[0,\infty)$ . As it will become clearer in the examples that follow,  $u_k(t,p)$  will satisfy a system of equations. However, this system is not a closed system in that for any  $k \in \mathbb{N}$ ,  $u_k$  depends on  $u_{k+1}$ . To

resolve this, we follow the method of truncation and in particular for a large enough K, we set  $u_{K+1} = u_K$  and then we solve backwards.

Now, if the number of clusters r is large or if the pool has a large degree of heterogeneity, then the number of equations  $u_k(t,p)$  in the system can be prohibitively large. To resolve this and make the computation numerically feasible one can result in appropriate low-rank approximations as dictated by the SVD. The SVD facilitates the decomposition of the network interaction into r mean-field type clusters. Given an adjacency matrix A of rank r corresponding to a network of size N one could also find an approximation to A which has a low rank, say  $\theta$  such that  $\theta < r$  and in general one would like to have  $\theta/N \to 0$  in the regime  $N \to \infty$ .

This approach would extract the most significant mean-field clusters from the originally specified network reducing the computation complexity but maintaining the main features of the network. To do so, we recall a classical result from matrix algebra stating if  $0 < \theta < r$  is a positive integer, then the minimal value of the  $L^2$  distance  $||A - B||_2$  (the standard Frobenius norm) over all matrices with rank less or equal to  $\theta$  is achieved at

$$A_{\theta} = \sum_{j=1}^{\theta} \xi_j^2 \, \ell_j u_j^{\top}$$

with  $\xi_j^2$  in decreasing order. In addition, we actually have

$$||A - A_{\theta}||_2 = \sum_{i=\theta+1}^{r} \xi_i^2.$$

In particular, the best low-rank approximation when rank is 1 is at  $A_1$ . This singles out the contribution of the most important cluster. To support this claim further note that the orthonormality of the vectors  $\{u_j, j=1, \cdots r\}$  and the definition  $\beta_{nj}^C = \xi_j^2 u_{n,j}$  gives that for every  $j=1\cdots r$ 

$$\|\beta_{\cdot,j}^C\|_2 = \xi_j^2 \|u_{\cdot,j}\|_2 = \xi_j^2,$$

which immediately gives a ranking of  $\|\beta_{\cdot,j}^C\|_2$  based on the eigenvalues  $\xi_j$ .

We will see the power of the low-rank approximation in the examples that follow. In particular, if there is enough of spectral gap in the eigenvalues given by the SVD, then the limiting loss rate  $D_t$  as well as the limiting mean impact on a given name n,  $Q_t^n$ , are very well approximated by only considering the clusters associated to the first few large eigenvalues and ignoring the rest.

In all the numerical examples that follow, we consider for simplicity a specific form of function  $b(\lambda, \alpha) = -\bar{\alpha}(\lambda - \bar{\lambda})$  and  $\rho = 1/2$ , and take the systematic risk process to be a CIR process  $dX_t = \kappa(\theta - X_t)dt + \epsilon\sqrt{X_t}dV_t$ . For the numerical purposes of this paper, we have restricted attention to the aforementioned choices as we want to be able to compare and draw intuition from the existing literature.

Before presenting the numerical studies, let us collect here their main findings:

- The ranking of the corresponding contagion parameter,  $\beta_{n,j}^C$ , gives a clear ranking of the mean impact on names belonging to the same cluster from system wide defaults. It also ranks appropriately the loss rate for names of different types within the same cluster.
- The ranking of the eigenvalues of A,  $\xi_j$ ,  $j = 1, \dots, r$  gives a clear ranking of the importance of the different clusters in explaining the behavior of the pool.

• In complicated networks with many different clusters or high degree of heterogeneity, the numerical computation of quantities like  $D_t$  or  $Q_t$  can be prohibitively large. The singular value decomposition allows us to reduce the dimension of the system making such computations feasible, while maintaining accuracy.

With these choices, we consider below three different numerical studies. The first example has one cluster, i.e. r=1 in the SVD, and the second example has two clusters, i.e. r=2 in the SVD. The third example is motivated by the well documented core-periphery network structure for financial models, see for example [11, 21]. Notice that names of different types may belong to the same cluster. Namely each cluster does not need to be homogenous. This becomes clear in the specific examples below.

5.1. One cluster case. In this example, we consider a situation where the adjacency matrix A has only one positive eigenvalue. This corresponds to having one cluster, r = 1, but of course the pool can still be heterogenous.

Let us start by fixing some values for the parameters  $\kappa = 4$ ,  $\theta = 0.5$ ,  $\epsilon = 0.5$ ,  $X_0 = 0.2$ ,  $\sigma = 0.9$ ,  $\bar{\alpha} = 4$ ,  $\bar{\lambda} = 0.2$ ,  $\lambda_0 = 0.2$  and  $\beta^S = 2$ . Also, let us consider a pool of N = 1000 names.

In addition, assume that 50% of the  $\beta_{n,1}^C$ 's are taking the value  $\beta_1^{C,1} = 1.2361$  and the rest 50% of the  $\beta_{n,1}^C$ 's are taking the value  $\beta_1^{C,2} = 0.6362$ , while all  $\ell_{n,1}$ 's take value  $\ell_1^1 = 0.0316$ . To describe this more effectively, we slightly abuse notation and consider discrete random variables  $\tilde{\beta}_1^C$  and  $\tilde{\ell}_1$  defined by

$$\mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C,1}) = 0.5, \quad \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C,2}) = 0.5 \text{ and } \mathbb{P}(\tilde{\ell}_1 = l_1^1) = 1.$$

The corresponding adjacency matrix A has a singular value decomposition with only one nonnegative eigenvalue 10. The first column of the left matrix takes one value 0.0316. The first column of the right matrix takes two values 0.12361 and 0.06362 with same frequencies. Notice that we indeed have  $\beta_1^{C,1} = 0.12361 \cdot 10 =$ 1.2361 and  $\beta_1^{C,2} = 0.06362 \cdot 10 = 0.6362$ , as expected.

Hence, we have a heterogeneous pool with two different types, where however both of them belong to the same cluster.

In this case, the moments, as defined by (6) satisfy the following pair of coupled

$$du_k(t, p_1) = \left\{ u_k(t, p_1)(-\alpha k + \beta^S \kappa(\theta - X_t)k + 0.5(\beta^S)^2 \epsilon^2 X_t k(k-1)) - u_{k+1}(t, p_1) \right\} dt$$

$$+ \left\{ u_{k-1}(t, p_1)(0.5\sigma^2 k(k-1) + \alpha \bar{\lambda}k + l_1^1 \beta_1^{C,1} k(1/2u_1(t, p_1) + 1/2u_1(t, p_2))) \right\} dt$$

$$+ \beta^S \epsilon \sqrt{X_t} k u_k(t, p_1) dV_t$$

$$du_k(t, p_2) = \left\{ u_k(t, p_2)(-\alpha k + \beta^S \kappa(\theta - X_t)k + 0.5(\beta^S)^2 \epsilon^2 X_t k(k-1)) - u_{k+1}(t, p_2) \right\} dt$$

$$+ \left\{ u_{k-1}(t, p_2)(0.5\sigma^2 k(k-1) + \alpha \bar{\lambda}k + l_1^1 \beta_1^{C,2} k(1/2u_1(t, p_1) + 1/2u_1(t, p_2))) \right\} dt$$

$$+ \beta^S \epsilon \sqrt{X_t} k u_k(t, p_2) dV_t$$

with 
$$u_k(0,p) = \int_0^\infty \lambda^k(\pi \times \Lambda_0)(\hat{p})d\lambda$$
.  
Then, we have that the overall loss rate is

$$D_t^N \approx D_t = 1 - (1/2u_0(t, p_1) + 1/2u_0(t, p_2))$$

The loss rate for type  $p_i$ , i = 1 or 2 is

$$D_t^N(p_i) \approx D_t(p_i) = 1 - u_0(t, p_i), \quad i = 1, 2$$

The mean impact, from the system wide defaults by time t, on name n, which comes from type  $p_i$ , i = 1 or 2 is

$$Q_t^{N,n} \approx Q_t(p_i) = \beta_1^{C,i} L_t, \quad i = 1, 2$$

where

$$L_t = l_1^1 - l_1^1 \left( 1/2u_0(t, p_1) + 1/2u_0(t, p_2) \right).$$

Now notice that the system that the moments satisfy is a non-closed system, since the equation for the k-th moment depends on the (k+1) moment. In order to solve this we truncate the system at a certain level, in particular at the level K=20, by setting  $u_{20}(t,p)=u_{21}(t,p)$  and solve backwards. This will then give us  $u_1(t,p)$  and  $u_0(t,p)$  for any time t. Here we choose the time endpoint to be T=1. We do the numerical iteration with time step being 0.01. We run 50,000 Monte Carlo trials and plot the overall limiting loss  $D_t$  and limiting loss for Type  $p_i$ ,  $D_t(p_i)$ , i=1,2 in Figure 1 left plot. We also plot the empirical mean of overall limiting loss rate  $D_T$  and the empirical mean of limiting loss rate for two types  $D_T(p_i)$ , i=1,2, up to time T=1 in Figure 1 right plot.

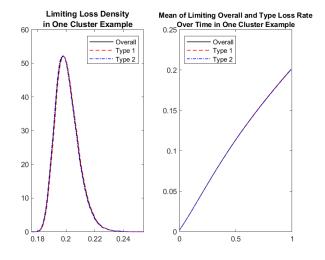


FIGURE 1. Left: Density for overall limiting loss  $D_T$  and Type i limiting loss for  $D_T(p_i)$ , i = 1, 2 at T = 1; Right: Empirical mean of overall limiting loss  $D_T$  and empirical mean of limiting loss for types  $D_T(p_i)$ , i = 1, 2 up to time T = 1.

In Figure 2, we plot the mean impact on a name n, i.e.,  $Q_t(p^n)$ , from system wide default as a function of time t for the two different types of names. Here the name n, can be one of two types, type 1 or type 2, as indicated by the parameters  $\beta_1^{C,1}, \beta_1^{C,2}$ . It is instructive to notice from the plots that  $Q_t(p_1) \geq Q_t(p_2)$ , which is to be expected due to the relation  $\beta_1^{C,1} > \beta_1^{C,2}$  of the contagion coefficients.

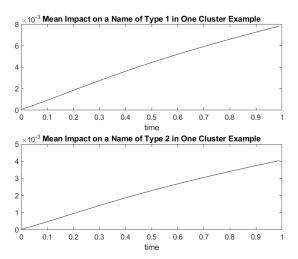


FIGURE 2. Mean impact on names of type 1  $Q_t(p_1)$  and type 2  $Q_t(p_2)$  from system wide default up to time T = 1.

5.2. **Two clusters case.** In this example now we consider the case where A has two positive eigenvalues. This corresponds to having a heterogeneous pool with two clusters, r=2. In this example, we will also test numerically the effect of the low-rank approximation on the limiting loss and on the mean impact on given names by system wide defaults.

Let us choose the following values for the parameters  $\kappa=4, \theta=0.5, \epsilon=0.5, X_0=0.2, \sigma=0.9, \bar{\alpha}=4, \bar{\lambda}=0.2, \lambda_0=0.2$  and  $\beta^S=2$ . Also, let us consider a pool of N=1000 names.

Furthermore, we assume that 50% of the  $\beta_{n,1}^C$ 's (first cluster) are taking the value  $\beta_1^{C,1} = 0.2050$  and the rest 50% of the  $\beta_{n,1}^C$ 's are taking  $\beta_1^{C,2} = 0.3980$ . All the  $l_{n,1}$ 's take the value  $l_1^1 = 0.0316$ .

In addition, 2/3 of the  $\beta_{n,2}^C$ 's (second cluster) are taking  $\beta_2^{C,1} = 0.0009$  and the rest 1/3 of the  $\beta_{n,2}^C$ 's are taking the value  $\beta_2^{C,2} = 0.0022$ . Finally, 50% of the  $l_{n,2}$ 's are taking the value  $l_2^1 = 0.0043$  whereas the rest 50% of the  $l_{n,2}$ 's are taking the value  $l_2^2 = -0.0022$ .

As with the previous example, we slightly abuse notation and define discrete random variables  $\tilde{\beta}_1^C$ ,  $\tilde{\beta}_2^C$ ,  $\tilde{\ell}_1$  and  $\tilde{\ell}_2$  such that

$$\begin{split} &\mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C,1}) = 1/2, \quad \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C,2}) = 1/2, \\ &\mathbb{P}(\tilde{\ell}_1 = l_1^1) = 1, \\ &\mathbb{P}(\tilde{\beta}_2^C = \beta_2^{C,1}) = 2/3, \quad \mathbb{P}(\tilde{\beta}_2^C = \beta_2^{C,2}) = 1/3 \\ &\mathbb{P}(\tilde{\ell}_2 = l_2^1) = 1/2, \quad \mathbb{P}(\tilde{\ell}_2 = l_2^2) = 1/2. \end{split}$$

For the corresponding adjacency matrix A, the SVD has two nonnegative eigenvalues 10 and 1. The first column of the right matrix takes two values 0.0205 and 0.0398 with same frequencies. This indeed corresponds to the two values  $\beta_1^{C,1} = 0.0205 \cdot 10 = 0.2050$  and  $\beta_1^{C,2} = 0.0398 \cdot 10 = 0.3980$ . The second column of

the right matrix takes two values 0.0009 and 0.0022 with ratio of frequencies being 2:1. This indeed corresponds to the two values  $\beta_2^{C,1}=0.0009\cdot 1=0.0009$  and  $\beta_2^{C,2}=0.0022\cdot 1=0.0022$ . The first column of the left matrix takes only one value 0.0316. The second column of the left matrix takes two values 0.0043 and -0.0022 with equal frequencies.

Let us now denote by  $u_k(t; k_1, k_2, k_3)$  to be the kth moment at time t with  $k_1=1,2,\,k_2=1,2,\,k_3=1,2$  being the choice index for  $\tilde{\beta}_1^C,\,\tilde{\beta}_2^C$  and  $\tilde{\ell}_2$  respectively. For example,  $k_1=1,\,k_2=1,\,k_3=2$  corresponds to the choice  $\tilde{\beta}_1^C=\beta_1^{C,1},\,\tilde{\beta}_2^C=\beta_2^{C,1}$  and  $\tilde{\ell}_2=l_2^2$ . Then there will be totally  $2^3=8$  equations in the coupled system. However because of the special structure we end up with only 4 different equations. In particular, for  $k_1,k_2,k_3\in\{1,2\}$  we have

$$du_k(t; k_1, k_2, k_3) = \left\{ u_k(t; k_1, k_2, k_3) (-\alpha k + \beta^S \kappa (\theta - X_t) k + 0.5 (\beta^S)^2 \epsilon^2 X_t k(k-1) \right\} - u_{k+1}(t; k_1, k_2, k_3) dt + u_{k-1}(t; k_1, k_2, k_3) \left\{ (0.5\sigma^2 k(k-1) + \alpha \bar{\lambda} k) + G_k(t; k_1, k_2) \right\} dt + \beta^S \epsilon \sqrt{X_t} k u_k(t; k_1, k_2, k_3) dV_t$$

Notice that  $u_k(t; k_1, k_2, 1) = u_k(t; k_1, k_2, 2)$  for  $k_1, k_2 = 1, 2$ . We supplement  $u_k(t; k_1, k_2, k_3)$  with initial conditions together with  $u_k(0; k_1, k_2, k_3) = \int_0^\infty \lambda^k(\pi \times \Lambda_0)(\hat{p})d\lambda$  and we define

$$G_{k}(t; k_{1}, k_{2}) = k l_{1}^{1} \beta_{1}^{C, k_{1}} \sum_{i_{1}, i_{2}, i_{3} = 1}^{2} u_{1}(t; i_{1}, i_{2}, i_{3}) \mathbb{P}(\tilde{\beta}_{1}^{C} = \beta_{1}^{C, i_{1}}) \mathbb{P}(\tilde{\beta}_{2}^{C} = \beta_{2}^{C, i_{2}}) \mathbb{P}(\tilde{\ell}_{2} = l_{2}^{i_{3}})$$

$$+ k \beta_{2}^{C, k_{2}} \sum_{i_{3} = 1}^{2} l_{2}^{i_{3}} \sum_{i_{1}, i_{2} = 1}^{2} u_{1}(t; i_{1}, i_{2}, k_{3}) \mathbb{P}(\tilde{\beta}_{1}^{C} = \beta_{1}^{C, i_{1}}) \mathbb{P}(\tilde{\beta}_{2}^{C} = \beta_{2}^{C, i_{2}}) \mathbb{P}(\tilde{\ell}_{2} = l_{2}^{i_{3}}),$$

where  $k_1, k_2 = 1, 2$ . Then we have that the overall loss rate is

$$D_t^N \approx D_t = 1 - \sum_{k_1, k_2, k_3 = 1}^2 u_0(t; k_1, k_2, k_3) \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C, k_1}) \mathbb{P}(\tilde{\beta}_2^C = \beta_2^{C, k_2}) \mathbb{P}(\tilde{\ell}_2 = l_2^{k_3}).$$

The loss rate for type  $(k_1, k_2, k_3)$ , where  $k_1, k_2, k_3 = 1, 2$  essentially changes only with  $k_1$  and  $k_2$  and takes the form,

$$D_t^N(k_1, k_2, 1) \approx D_t(k_1, k_2, 1) = 1 - u_0(t; k_1, k_2, 1), \quad D_t(k_1, k_2, 1) = D_t(k_1, k_2, 2).$$

The mean impact on name n from system wide defaults up to time t is determined only via the choices for  $k_1$  and  $k_2$  through  $\tilde{\beta}_1^C$  and  $\tilde{\beta}_2^C$  respectively. In particular, we have

$$Q_t^{N,n} \approx Q_t(k_1, k_2) = \beta_1^{C, k_1} L_t^1 + \beta_2^{C, k_2} L_t^2,$$

where for the j-th cluster, j = 1, 2, we have

$$L_t^1 = l_1^1 - l_1^1 \sum_{k_1, k_2, k_3 = 1}^2 u_0(t; k_1, k_2, k_3) \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C, k_1}) \mathbb{P}(\tilde{\beta}_2^C = \beta_2^{C, k_2}) \mathbb{P}(\tilde{\ell}_2 = l_2^{k_3})$$

$$\begin{split} L_t^2 &= \sum_{k_3=1}^2 l_2^{k_3} \mathbb{P}(\tilde{\ell}_2 = l_2^{k_3}) \\ &+ \sum_{k_2=1}^2 l_2^{k_3} \sum_{k_3=1}^2 u_0(t; k_1, k_2, k_3) \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C, k_1}) \mathbb{P}(\tilde{\beta}_2^C = \beta_2^{C, k_2}) \mathbb{P}(\tilde{\ell}_2 = l_2^{k_3}) \end{split}$$

As with the previous example, we truncate at the level K = 20, and choose the time endpoint to be T = 1. We do the numerical iteration with time step being 0.01. We run 50,000 Monte Carlo trials and plot overall limiting loss rate  $D_t$  and the limiting loss rate for different types  $D_t(k_1, k_2)$ ,  $k_1, k_1 = 1, 2$  in the left plot of Figure 3. We also plot the empirical mean of the overall limiting loss rate and the empirical mean of the loss rate  $D_T$  for different types over time  $D_T(k_1, k_2)$ ,  $k_1, k_1 = 1, 2$  in the right plot of Figure 3.

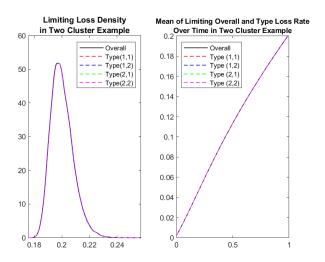


FIGURE 3. Left: Density for overall limiting loss  $D_T$  and limiting loss for types  $D_T(k_1, k_2)$  at T = 1; Right: Empirical mean of overall limiting loss  $D_T$  and empirical mean of limiting loss for types  $D_T(k_1, k_2)$  up to time T = 1.

In Figure 4 we plot the mean impact on a name from type  $(k_1, k_2)$ ,  $k_1, k_2 = 1, 2$  due to system wide defaults up to time T = 1.

As we discussed in the beginning of this section, the SVD facilitates the decomposition of the network interaction into r mean-field type clusters.

We test the effect of the low-rank approximation by only keeping the first cluster. This singles out the contribution of the most important cluster.

In other words, we replace A by

$$A_{\text{aprrox}} = \xi_1^2 \ell_1 u_1^T$$

which reduces the problem to a one cluster problem. Comparing the overall limiting loss that we get from the two cluster case  $D_t$  and its first cluster approximation  $D_{\mathrm{approx},t}$ , see left plot of Figure 5, we get that the distribution of the limiting loss processes are practically indistinguishable. Similar conclusion can be made from the right plot of Figure 5, where we plot the empirical mean of overall limiting loss rate over time in the two cluster example  $D_T$  and it first-cluster approximation  $D_{\mathrm{approx},T}$ . These in turn imply that the second cluster can be neglected for the purposes of these computations.

Lastly, we investigate how the mean impact on a name from system wide defaults for the two cluster case and its one cluster approximated version compare. In Figure

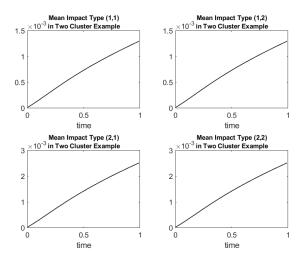


FIGURE 4. Mean impact on names of type  $(k_1, k_2)$ ,  $Q_t(k_1, k_2)$ , from system wide default by up to time T = 1.

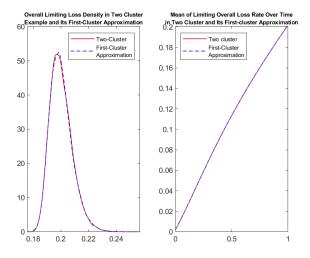


FIGURE 5. Left: Density for overall limiting loss  $D_T$  and overall limiting loss  $D_{\mathrm{approx},T}$  from its first cluster approximation at T=1; Right: Empirical mean of overall limiting loss  $D_T$  in the two cluster example and empirical mean of limiting loss in its first-cluster approximation  $D_{\mathrm{approx},T}$  up to time T=1.

4, we see that the mean impact on given names depends mainly on  $\tilde{\beta}_1^C$ , and not so much on  $\tilde{\beta}_2^C$ . This will be further verified in the one cluster approximation case, where we calculate the approximated mean impacts on these two types by using the information only from first entries of  $\beta^C$  and  $L_t^N$ , i.e., by using only the information from the first cluster, shown in Figure 6.

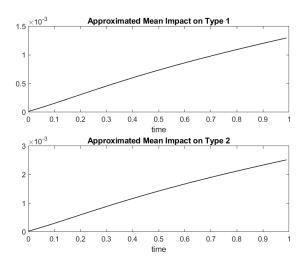


FIGURE 6. Approximated mean impact on names of different types from system wide default by time t in the coarse-grained case.

$$Q_{\mathrm{approx},t}^{N,n}(p_i) \approx \beta_1^{C,i} L_{\mathrm{approx},t}^1, \text{ for } i = 1, 2.$$

Comparing Figures 4 and 6 we see that the first cluster, which has the largest eigenvalue, indeed captures the behavior on the mean impact on a given name of type defined by  $\tilde{\beta}_1^C$ . In addition, notice that the mean default impact on names of type 2 is larger than the mean default impact on names of type 1 for all  $t \in [0,1]$ . This is to be expected due to the relation  $\beta_1^{C,2} > \beta_1^{C,1}$ .

5.3. Core-Periphery example. A reasonably realistic model for financial related applications is the core-periphery case, see for example [11, 21]. In a core-periphery model, one has a few names that constitute the core of the network and considerably depend on each other, in a sense forming the most influential part of the network, and the periphery which is composed by the rest of the names in the pool which depend less on each other. Motivated by this structure, let us consider the case of N=1000 names and an appropriate adjacency matrix A. For illustration purposes the first  $10\times 10$  block of A is given by:

The SVD for such a matrix gives 5 eigenvalues 105.1800, 34.8857, 28.5160, 11.7761 and 11.0924 significantly larger than the rest, with the first one being

dominantly big. Therefore, motivated by the low rank approximation, we can use the first few clusters to approximate the behavior of the network.

5.3.1. One cluster approximation for core-periphery. Let us choose the first eigenvalue to do the low rank approximation. Similarly to what was done for the previous examples, we define discrete random variables  $\tilde{\beta}_1^C$  and  $\tilde{\ell}_1$  taking values from the SVD with corresponding relative frequencies. It turns out that the SVD composition yields eight different values for  $\tilde{\beta}_1^C$  and four different values for  $\tilde{\ell}_1$ . We record the values and their corresponding empirical probability distribution in Tables 1 and 2 respectively.

$ ilde{eta}_1^C$	$\beta_1^{C,1}$	$\beta_1^{C,2}$	$\beta_1^{C,3}$	$\beta_1^{C,4}$	$\beta_1^{C,5}$	$\beta_1^{C,6}$			
<b>value</b> = $u_{n,1} \cdot 105.180$	9.6707	11.8406	11.9426	14.1198	15.6449	15.7501			
probability	0.70	0.13	0.14	0.01	0.01	0.01			
Table 1. Distribution for $\tilde{\beta}_1^C$ .									

$ ilde{\ell}_1$	$l_1^1$	$l_1^2$	$l_1^3$	$l_1^4$	
$\mathbf{value}$	0.0934	0.0989	0.2424	0.2533	
probability	0.78	0.20	0.01	0.01	

Table 2. Distribution for  $\tilde{\ell}_1$ .

Let us choose the following values for the parameters  $\kappa=4, \ \theta=0.5, \ \epsilon=0.5, \ X_0=0.2, \ \sigma=0.9, \ \bar{\alpha}=4, \ \bar{\lambda}=0.2, \ \lambda_0=0.2 \ \text{and} \ \beta^S=2.$ 

Let us denote by  $u_k(t;k_1,k_2)$  to be the k-th moment by time t with  $k_1 \in \{1,2,\ldots,6\}$  and  $k_2 \in \{1,2,3,4\}$  being the choice index for  $\tilde{\beta}_1^C$ , and  $\tilde{\ell}_1$  respectively. For example,  $k_1=1,k_2=2$  corresponds to the choice  $\tilde{\beta}_1^C=\beta_1^{C,1}$  and  $\tilde{\ell}_1=l_1^2$ . Then, there will be in total  $6\times 4=24$  equations in the coupled system. However, because of the special structure we end up with only 6 different equations as follows:

$$du_k(t; k_1, k_2) = \left\{ u_k(t; k_1, k_2)(-\alpha k + \beta^S \kappa(\theta - X_t)k + 0.5(\beta^S)^2 \epsilon^2 X_t k(k-1)) - u_{k+1}(t; k_1, k_2) \right\} dt + u_{k-1}(t; k_1, k_2) \left\{ (0.5\sigma^2 k(k-1) + \alpha \bar{\lambda}k) + G_k(t; k_1) \right\} dt + \beta^S \epsilon \sqrt{X_t} k u_k(t; k_1, k_2) dV_t,$$

together with  $u_k(0;k_1,k_2)=\int_0^\infty \lambda^k(\pi\times\Lambda_0)(\hat{p})d\lambda$  and where we define

$$G_k(t;k_1) = \left(\sum_{i_2=1}^4 l_1^{i_2} \sum_{i_1=1}^6 u_1(t;i_1,i_2) \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C,i_1}) \mathbb{P}(\tilde{\ell}_1 = l_1^{i_2})\right) k \beta_1^{C,k_1}$$

Therefore we have

$$u_k(t; k_1, k_2) = u_k(t; k_1, 1)$$

for any k, t and  $k_2 = 1, 2, 3, 4$ .

The overall loss rate in the one-cluster approximation is

$$D_{1\text{approx},t}^{N} \approx D_{1\text{approx},t} = 1 - \sum_{k_1=1}^{6} \sum_{k_2=1}^{4} u_0(t; k_1, k_2) \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C, k_1}) \mathbb{P}(\tilde{\ell}_1 = l_1^{k_2}).$$

The loss rate for type  $(k_1, k_2)$  where  $k_1 = 1, 2, ..., 6$  and  $k_2 = 1, 2, 3, 4$  in the one-cluster approximation are actually falling into 6 distinct categories indexed by  $k_1$ , the choice of  $\tilde{\beta}_1^C$ .

$$D_{\text{lapprox }t}^{N}(k_1, k_2) \approx D_{\text{lapprox},t}(k_1, k_2) = 1 - u_0(t; k_1, k_2).$$

The mean impact, from system wide defaults up to time t, on name n, which comes from type  $(k_1, k_2)$ , where  $k_1 = 1, 2, \ldots, 6$  and  $k_2 = 1, 2, 3, 4$ , turns out to be characterized by the first index  $k_1$ 

$$Q_{1\text{approx},t}^{N,n}(k_1,k_2) \approx Q_{1\text{approx},t}(k_1) = \beta_1^{C,k_1} L_{1\text{approx},t},$$

for any  $k_2 = 1, 2, 3, 4$  with

$$L_{1\text{approx},t} = \sum_{k_2=1}^{4} l_1^{k_2} \mathbb{P}(\tilde{\ell}_1 = l_1^{k_2})$$
$$- \sum_{k_2=1}^{4} l_1^{k_2} \sum_{k_1=1}^{6} u_0(t; k_1, k_2) \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C, k_1}) \mathbb{P}(\tilde{\ell}_1 = l_1^{k_2})$$

As with the previous two examples, we truncate at the level K=20, and choose the time endpoint to be T=1. We do the numerical iteration with time step being 0.01. We run 50,000 Monte Carlo trials and plot overall limiting loss rate  $D_{1\mathrm{approx},t}$  and the limiting loss rate for different types  $D_{1\mathrm{approx},t}^{k_1}$ ,  $k_1=1,2,\ldots,6$  in Figure 7. Notice how the mean of the distribution shifts to the right as the value for  $k_1$  increases, indicating an increase to the value that the random variable  $\tilde{\beta}_1^C$  takes. We plot the mean of the loss rate over time for the whole pool and for individual types in Figure 8. We observe that the plot indicates larger losses as the value for  $k_1$  increases.

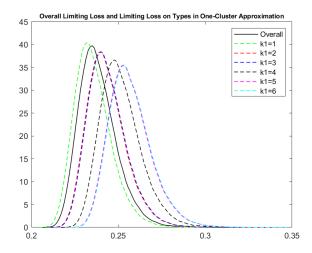


FIGURE 7. Density for overall limiting loss  $D_{1\text{approx},T}$  and limiting loss for types  $D_{1\text{approx},T}(k_1)$  at T=1.

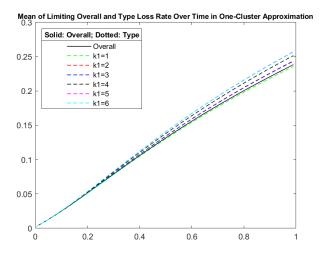


FIGURE 8. Empirical mean of overall limiting loss  $D_{1\text{approx},T}$  and empirical mean of limiting loss for types  $D_{1\text{approx},T}(k_1)$  up to time T=1

In Figure 9, we plot the mean impact on a name from system wide defaults up to time t, there are totally 6 different categories indexed by  $k_1$  the choice of  $\tilde{\beta}_1^C$  as we discussed before.

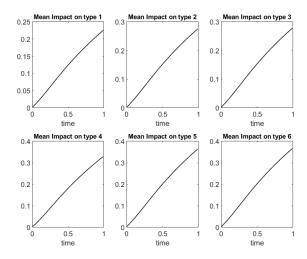


FIGURE 9. Mean impact on names of different types from system wide default by time t for the Core-Periphery case approximated by the first cluster.

5.3.2. Two clusters approximation for core-periphery. Let us now investigate the core-periphery case by doing a low rank approximation based on the first two clusters. From the SVD decomposition, the second largest eigenvalue is 34.8857. Below,

we summarize the empirical distributions of coefficients from the second columns of the matrices from the SVD decomposition. Table 3 is for coefficient  $\tilde{\beta}_2^C$  and Table 4 is for coefficient  $\tilde{\ell}_2$ .

Table 4. Distribution for  $\tilde{\ell}_2$ .

Let us now denote by  $u_k(t; k_1, k_2, k_3, k_4)$  to be the k-th moment by time t with  $k_1 \in \{1, 2, \dots, 6\}$ ,  $k_2 \in \{1, 2, \dots, 8\}$ ,  $k_3 \in \{1, 2, 3, 4\}$ , and  $k_4 \in \{1, 2, \dots, 8\}$  being the choice index for  $\tilde{\beta}_1^C$ ,  $\tilde{\beta}_2^C$ ,  $\tilde{\ell}_1$  and  $\tilde{\ell}_2$  respectively. For example,  $k_1 = 1, k_2 = 1, k_3 = 2, k_4 = 1$  corresponds to the choice  $\tilde{\beta}_1^C = \beta_1^{C,1}$ ,  $\tilde{\beta}_2^C = \beta_2^{C,1}$ ,  $\tilde{\ell}_1 = l_1^2$  and  $\tilde{\ell}_2 = l_2^1$ . Then there will be totally  $6 \times 8 \times 4 \times 8 = 1536$  equations in the coupled system. However because of the special structure we end up with only  $6 \times 8 = 48$  different equations.

We have

$$du_k(t; k_1, k_2, k_3, k_4) = \left\{ u_k(t; k_1, k_2, k_3, k_4) (-\alpha k + \beta^S \kappa (\theta - X_t) k + 0.5 (\beta^S)^2 \epsilon^2 X_t k (k - 1)) - u_{k+1}(t; k_1, k_2, k_3, k_4) \right\} dt + u_{k-1}(t; k_1, k_2, k_3, k_4) \left\{ (0.5\sigma^2 k (k - 1) + \alpha \bar{\lambda} k) + G_k(t; k_1, k_2) \right\} dt + \beta^S \epsilon \sqrt{X_t} k u_k(t; k_1, k_2, k_3, k_4) dV_t$$

together with  $u_k(0; k_1, k_2, k_3, k_4) = \int_0^\infty \lambda^k(\pi \times \Lambda_0)(\hat{p})d\lambda$  where we have defined

$$G_k(t;k_1,k_2)$$

$$\begin{split} &=k\beta_1^{C,k_1}\left[\sum_{i_3=1}^4 l_1^{i_3}\sum_{i_1=1}^6 \sum_{i_2=1}^8 \sum_{i_4=1}^8 u_1(t;i_1,i_2,i_3,i_4) \mathbb{P}(\tilde{\beta}_1^C=\beta_1^{C,i_1}) \mathbb{P}(\tilde{\beta}_2^C=\beta_2^{C,i_2}) \mathbb{P}(\tilde{\ell}_1=l_1^{i_3}) \mathbb{P}(\tilde{\ell}_2=l_2^{i_4})\right] \\ &+k\beta_2^{C,k_2}\left[\sum_{i_4=1}^8 l_2^{i_4}\sum_{i_1=1}^6 \sum_{i_2=1}^8 \sum_{i_3=1}^4 u_1(t;i_1,i_2,i_3,i_4) \mathbb{P}(\tilde{\beta}_1^C=\beta_1^{C,i_1}) \mathbb{P}(\tilde{\beta}_2^C=\beta_2^{C,i_2}) \mathbb{P}(\tilde{\ell}_1=l_1^{i_3}) \mathbb{P}(\tilde{\ell}_2=l_2^{i_4})\right] \end{split}$$

Therefore  $u_k(t; k_1, k_2, k_3, k_4) = u_k(t; k_1, k_2, 1, 1)$  for any  $k_3 = 1, 2, 3, 4$  and  $k_4 = 1, 2, ..., 8$ . The overall loss rate is

$$D_{\text{2approx},t}^{N} \approx D_{\text{2approx},t} = 1 - \sum_{k_{1}=1}^{6} \sum_{k_{2}=1}^{8} \sum_{k_{3}=1}^{4} \sum_{k_{4}=1}^{8} u_{0}(t; k_{1}, k_{2}, k_{3}, k_{4})$$
$$\cdot \mathbb{P}(\tilde{\beta}_{1}^{C} = \beta_{1}^{C, k_{1}}) \mathbb{P}(\tilde{\beta}_{2}^{C} = \beta_{2}^{C, k_{2}}) \mathbb{P}(\tilde{\ell}_{1} = l_{1}^{k_{3}}) \mathbb{P}(\tilde{\ell}_{2} = l_{2}^{k_{4}})$$

The loss rates for the type  $(k_1, k_2, k_3, k_4)$ , where  $k_1 = 1, 2, \dots, 6, k_2 = 1, 2, \dots, 8, k_3 = 1, 2, 3, 4$  and  $k_4 = 1, 2, \dots, 8$ , turn out to fall into one of the  $6 \times 8 = 48$  different categories indexed by  $(k_1, k_2)$ 

$$D_{\text{2approx},t}^{N}(k_1, k_2, k_3, k_4) \approx D_{\text{2approx},t}(k_1, k_2, 1, 1) = 1 - u_0(k_1, k_2, 1, 1).$$

The mean impact on name n from type  $(k_1, k_2, k_3, k_4)$ , where  $k_1 = 1, 2, \ldots, 6$ ,  $k_2 = 1, 2, \ldots, 8$ ,  $k_3 = 1, 2, 3, 4$  and  $k_4 = 1, 2, \ldots, 8$ , is again determined by the choice  $k_1$  and  $k_2$  for  $\tilde{\beta}_1^C$  and  $\tilde{\beta}_2^C$  respectively

$$Q_{\mathrm{2approx},t}^{N,n}(k_1,k_2,k_3,k_4) \approx Q_{\mathrm{2approx},t}(k_1,k_2) = \beta_1^{C,k_1} L_{\mathrm{2approx},t}^1 + \beta_2^{C,k_2} L_{\mathrm{2approx},t}^2,$$
 where for the j-th cluster,  $j=1,2$ , in the two-cluster approximation we have

$$\begin{split} L_{\mathrm{2approx},t}^{1} &= \sum_{k_{3}=1}^{4} l_{1}^{k_{3}} \mathbb{P}(\tilde{\ell}_{1} = l_{1}^{k_{3}}) - \sum_{k_{3}=1}^{4} l_{1}^{k_{3}} \sum_{k_{1}=1}^{6} \sum_{k_{2}=1}^{8} \sum_{k_{4}=1}^{8} u_{0}(t; k_{1}, k_{2}, k_{3}, k_{4}) \\ &\cdot \mathbb{P}(\tilde{\beta}_{1}^{C} = \beta_{1}^{C, k_{1}}) \mathbb{P}(\tilde{\beta}_{2}^{C} = \beta_{2}^{C, k_{2}}) \mathbb{P}(\tilde{\ell}_{1} = l_{1}^{k_{3}}) \mathbb{P}(\tilde{\ell}_{2} = l_{2}^{k_{4}}) \end{split}$$

$$\begin{split} L_{\text{2approx},t}^2 &= \sum_{k_4=1}^8 l_2^{k_4} \mathbb{P}(\tilde{\ell}_2 = l_2^{k_4}) - \sum_{k_4=1}^8 l_2^{k_4} \sum_{k_1=1}^6 \sum_{k_2=1}^8 \sum_{k_3=1}^4 u_0(t;k_1,k_2,k_3,k_4) \\ &\cdot \mathbb{P}(\tilde{\beta}_1^C = \beta_1^{C,k_1}) \mathbb{P}(\tilde{\beta}_2^C = \beta_2^{C,k_2}) \mathbb{P}(\tilde{\ell}_1 = l_1^{k_3}) \mathbb{P}(\tilde{\ell}_2 = l_2^{k_4}). \end{split}$$

As with the previous example, we truncate at the level K=20, and choose the time endpoint to be T=1. We do the numerical iteration with time step being 0.01. We run 50,000 Monte Carlo trials and plot the overall limiting loss  $D_{2\mathrm{approx},t}$  in the two cluster approximation. In the left plot of Figure 10, we see that the two approximations perform similarly in estimating the overall loss rate. This can be also verified via the plot of the mean of overall loss rate over time for each one of the two approximations in the right plot of Figure 10.

We can also investigate the mean impact on a name in the two-cluster approximation case. Since  $k_1 = 1, 2, ..., 6$  and  $k_2 = 1, 2, ..., 8$  we will have in total  $6 \times 8 = 48$  different types of mean impacts in the two-cluster approximation case. These are demonstrated in Figure 11.

It is instructive to compare the low rank approximation based on just the first cluster with the low rank approximation based on the first two clusters. The dotted lines are very well approximated by the solid line in Figure 11. In fact, we computed numerically the percent error of the mean impact on a name from the two different approximations, that is,

$$PE_t(k_1, k_2) = |Q_{2approx,t}(k_1, k_2) - Q_{1approx,t}(k_1)|/Q_{2approx,t}(k_1, k_2),$$

and in all cases the percent error made by using the one-cluster approximation versus the two-cluster approximation was not greater than 1.7% for all times  $t \in [0,1]$ . For comparison purposes we also mention that the computation of  $D_t$  and  $Q_t$  based on the two-cluster approximation took about twelve times more than the their computation based on the one-cluster approximation, indicating considerable savings in computational time while maintaining accuracy. Lastly, notice that the mean default impact on names of type  $k_1 = 1, \dots, 6$  from system wide defaults is ordered according to the order of the corresponding contagion coefficients  $\beta_1^{C,k_1}$  via Table 1.

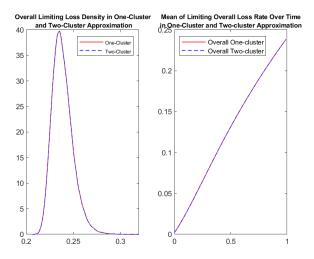


FIGURE 10. Left: Overall limiting loss from one-cluster approximation  $D_{1\text{approx},T}$  and two-cluster approximation  $D_{2\text{approx},T}$  at T=1; Right: Empirical mean of overall limiting loss for one-cluster approximation  $D_{1\text{approx},T}$  and two-cluster approximation  $D_{2\text{approx},T}$  up to time T=1.

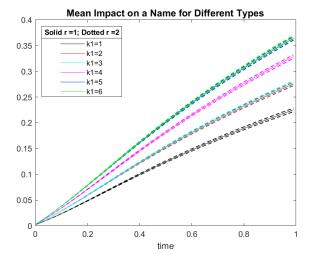


FIGURE 11. Mean impact on names of different types from system wide default by time t for the Core Periphery case approximated by the one-cluster (solid line) and two-cluster (dotted line) with colors to distinguish the choice of  $k_1$  in  $\beta_1^{C,k_1}$ 

#### 6. Tightness

In this section we prove that the family  $\{\mu^N\}_{N\in\mathbb{N}}$  is relatively compact as a  $D_E[0,\infty)$ -valued random variable. To do so, we first need to do some preliminary computations.

First we need to look at what happens to  $\langle f, \mu^N \rangle_E$  when one of the firms defaults. Assume that the n-th firm defaults at time t and the other firms do not default at time t (simultaneous defaults happen with probability zero). Then, at time t-,  $\langle f, \mu_{t-}^N \rangle_E$  is

$$\begin{split} \langle f, \mu_{t-}^N \rangle_E &= \frac{1}{N} \sum_{n'=1}^N \left\langle f, \ \delta_{\hat{p}_{t-}^{N,n'}} M_{t-}^{N,n'} \right\rangle_E = \frac{1}{N} \sum_{n'=1}^N \int_{\hat{p} \in \hat{\mathcal{P}}} f(\hat{p}) M_{t-}^{N,n'} \delta_{\hat{p}_{t-}^{N,n'}} (d\hat{p}) \\ &= \frac{1}{N} \sum_{n'=1}^N f(p^{N,n'}, \ \lambda_{t-}^{N,n'}) M_{t-}^{N,n'} \\ &= \frac{1}{N} \sum_{\substack{1 \leqslant n' \leqslant N \\ n' \neq n}} f(p^{N,n'}, \ \lambda_{t-}^{N,n'}) M_{t-}^{N,n'} + \frac{1}{N} f(p^{N,n}, \ \lambda_{t-}^{N,n}) \\ &= \frac{1}{N} \sum_{\substack{1 \leqslant n' \leqslant N \\ n' \neq n}} f(p^{N,n'}, \ \lambda_{t-}^{N,n'}) M_{t}^{N,n'} + \frac{1}{N} f(p^{N,n}, \ \lambda_{t-}^{N,n}). \end{split}$$

At time t,  $\langle f, \mu_t^N \rangle_E$  is

$$\begin{split} \langle f, \mu_t^N \rangle_E &= \frac{1}{N} \sum_{n'=1}^N \left\langle f, \ \delta_{\hat{p}_t^{N,n'}} M_t^{N,n'} \right\rangle_E = \frac{1}{N} \sum_{n'=1}^N \int_{\hat{p} \in \hat{\mathcal{P}}} f(\hat{p}) M_t^{N,n'} \delta_{\hat{p}_t^{N,n'}} (d\hat{p}) \\ &= \frac{1}{N} \sum_{n'=1}^N f(p^{N,n'}, \ \lambda_t^{N,n'}) M_t^{N,n'} \\ &= \frac{1}{N} \sum_{\substack{1 \leqslant n' \leqslant N \\ n' \neq n}} f(p^{N,n'}, \ \lambda_t^{N,n'}) M_t^{N,n'} + \frac{1}{N} f(p^{N,n}, \ \lambda_t^{N,n}) M_t^{N,n} \\ &= \frac{1}{N} \sum_{\substack{1 \leqslant n' \leqslant N \\ n' \neq n}} f(p^{N,n'}, \ \lambda_t^{N,n'}) M_t^{N,n'} \\ &= \frac{1}{N} \sum_{\substack{1 \leqslant n' \leqslant N \\ n' \neq n}} f(p^{N,n'}, \ \lambda_{t-}^{N,n'}) + \frac{1}{N} \sum_{j=1}^r \xi_j^2 \ u_{n',j} \ l_{n,j} \right) M_t^{N,n'}, \end{split}$$

where the last step comes from the jump in  $\lambda^{N,n'}$  at time t when there is a default in the n-th firm. The jump size is  $\frac{1}{N}\sum_{j=1}^r \xi_j^2 \ u_{n',j} \ l_{n,j}.$ Also note that  $M_t^{N,n}=0$  since n-th firm defaults at time t means  $\int_0^t \lambda_s^{N,n} ds=\mathfrak{e}_n.$ 

Therefore, we have that

$$\langle f, \mu_t^N \rangle_E - \langle f, \mu_{t-}^N \rangle_E = \mathcal{J}_{N,n}^f(t)$$

where

$$\mathcal{J}_{N,n}^{f}(t) = \frac{1}{N} \sum_{1 \leq n' \leq N} \left( f(p^{N,n'}, \lambda_{t-}^{N,n'} + \frac{1}{N} \sum_{j=1}^{r} \xi_{j}^{2} u_{n',j} \ell_{n,j} \right) - f(p^{N,n'}, \lambda_{t-}^{N,n'}) \right) M_{t}^{N,n'} - \frac{1}{N} f(p^{N,n}, \lambda_{t-}^{N,n}).$$

Then, the Itô formula for  $\langle f, \mu_t^N \rangle_E$  yields

$$\langle f, \mu_{t}^{N} \rangle_{E} = \langle f, \mu_{0}^{N} \rangle_{E} + \frac{1}{N} \int_{0}^{t} \sum_{n=1}^{N} \left[ \frac{1}{2} \sigma_{n}^{2} (\lambda_{s}^{N,n})^{2\rho} \frac{\partial^{2} f}{\partial \lambda^{2}} (\hat{p}_{s}^{N,n}) + b(\lambda_{s}^{N,n}, a_{n}) \frac{\partial f}{\partial \lambda} (\hat{p}_{s}^{N,n}) \right] M_{s}^{N,n} ds$$

$$+ \frac{1}{N} \int_{0}^{t} \sum_{n=1}^{N} \left[ \frac{1}{2} (\beta_{N,n}^{S})^{2} (\lambda_{s}^{N,n})^{2} \sigma_{0}^{2} (X_{s}) \frac{\partial^{2} f}{\partial \lambda^{2}} (\hat{p}_{s}^{N,n}) + \beta_{N,n}^{S} \lambda_{s}^{N,n} b_{0}(X_{s}) \frac{\partial f}{\partial \lambda} (\hat{p}_{s}^{N,n}) \right] M_{s}^{N,n} ds$$

$$+ \frac{1}{N} \int_{0}^{t} \sum_{n=1}^{N} \sigma_{n} (\lambda_{s}^{N,n})^{\rho} \frac{\partial f}{\partial \lambda} (\hat{p}_{s}^{N,n}) M_{s}^{N,n} dW_{s}^{N}$$

$$+ \frac{1}{N} \int_{0}^{t} \sum_{n=1}^{N} \beta_{N,n}^{S} \lambda_{s}^{N,n} \sigma_{0}(X_{s}) \frac{\partial f}{\partial \lambda} (\hat{p}_{s}^{N,n}) M_{s}^{N,n} dV_{s}$$

$$(7)$$

$$+ \sum_{n=1}^{N} \int_{0}^{t} \mathcal{J}_{N,n}^{f}(s) d [1 - M_{s}^{N,n}].$$

Next we prove that the sequence of measures  $\{\mu_{\cdot}^{N}\}_{N\in\mathbb{N}}$  is relatively compact as a  $D_{E}[0,\infty)$ -valued random variable. By Theorem 8.6 in [29], relative compactness is a consequence of Theorems 6.1 and 6.2 below.

**Theorem 6.1.** For any  $\eta > 0$  and T > 0, there exist a compact set K such that

$$\sup_{0 \le t < T, n \in \mathbb{N}} \mathbb{P}\{\mu_t^N \notin K\} < \eta.$$

Due to Lemma 4.2 proven in Appendix A, the proof of Theorem 6.1 is as that of Lemma 6.1 in [16]. Hence, the details are omitted. The following theorem gives regularity of  $\mu_t^N$ .

**Theorem 6.2.** There is a random variable  $H_N$  with  $\sup_{N \in \mathbb{N}} \mathbb{E}[H_N] < \infty$  such that for any  $0 \le t \le T$ ,  $0 \le u \le \delta$  and  $0 \le v \le \delta \wedge t$ ,

$$\mathbb{E}[q^2(\langle f, \mu^N_{t+u} \rangle_E, \langle f, \mu^N_{t} \rangle_E)q^2(\langle f, \mu^N_{t} \rangle_E, \langle f, \mu^N_{t-v} \rangle_E)|\mathcal{F}^N_t] \leq \delta^{\frac{1}{2}}\mathbb{E}[H_N|\mathcal{F}^N_t]$$

where

$$q(x, y) = min\{|x - y|, 1\}.$$

*Proof.* Notice that we can write

$$\langle f, \mu_t^N \rangle_E \ = \ \langle f, \mu_0^N \rangle_E + A_t^{1,N} + A_t^{2,N} + B_t^{1,N} + B_t^{2,N},$$

where

$$A_t^{1,N} = \frac{1}{N} \sum_{n=1}^{N} \int_0^t v_s^{N,n} ds,$$

$$\begin{split} v_s^{N,n} &= \left[ \frac{1}{2} \sigma_n^2 \; (\lambda_s^{N,n})^{2\rho} \; \frac{\partial^2 f}{\partial \lambda^2} (\hat{p}_s^{N,n}) \; + \; b(\lambda_s^{N,n}, a_n) \; \frac{\partial f}{\partial \lambda} (\hat{p}_s^{N,n}) \right. \\ &+ \left. \frac{1}{2} (\beta_{N,n}^S)^2 \; (\lambda_s^{N,n})^2 \; \sigma_0^2(X_s) \; \frac{\partial^2 f}{\partial \lambda^2} (\hat{p}_s^{N,n}) \; + \beta_{N,n}^S \lambda_s^{N,n} \; b_0(X_s) \; \frac{\partial f}{\partial \lambda} (\hat{p}_s^{N,n}) \right] \; M_s^{N,n}. \\ &A_t^{2,N} = \sum_{n=1}^N \int_0^t \; \mathcal{J}_{N,n}^f(s) \; d \; [1 - M_s^{N,n}]. \\ &B_t^{1,N} = \frac{1}{N} \int_0^t \sum_{n=1}^N \sigma_n \; (\lambda_s^{N,n})^\rho \; \frac{\partial f}{\partial \lambda} (\hat{p}_s^{N,n}) \; M_s^{N,n} \; dW_s^n. \\ &B_t^{2,N} = \frac{1}{N} \int_0^t \sum_{n=1}^N \beta_{N,n}^S \; \lambda_s^{N,n} \; \sigma_0(X_s) \; \frac{\partial f}{\partial \lambda} (\hat{p}_s^{N,n}) \; M_s^{N,n} \; dV_s. \end{split}$$

For T > 0, define

$$\Xi_N^{(1)} \stackrel{\text{def}}{=} \frac{1}{2} \left\{ 1 + \frac{1}{N} \sum_{n=1}^N \int_0^T (v_r^{N,n})^2 dr \right\}.$$

$$\begin{split} (v_s^{N,n})^2 &\leq 2 \Big[ \frac{1}{2} \sigma_n^2 \ (\lambda_s^{N,n})^{2\rho} \ \frac{\partial^2 f}{\partial \lambda^2} (\hat{p}_s^{N,n}) \ + \ b(\lambda_s^{N,n}, a_n) \ \frac{\partial f}{\partial \lambda} (\hat{p}_s^{N,n}) \\ &+ \frac{1}{2} (\beta_{N,n}^S)^2 \ (\lambda_s^{N,n})^2 \ \sigma_0^2(X_s) \ \frac{\partial^2 f}{\partial \lambda^2} (\hat{p}_s^{N,n}) \Big]^2 + 2 \Big[ \beta_{N,n}^S \lambda_s^{N,n} \ b_0(X_s) \ \frac{\partial f}{\partial \lambda} (\hat{p}_s^{N,n}) \Big]^2. \end{split}$$

By Assumption 3.3

$$\begin{split} (v_s^{N,n})^2 & \leq 2 \Big[ \frac{1}{2} \sigma_n^2 \ (\lambda_s^{N,n})^{2\rho} \ \frac{\partial^2 f}{\partial \lambda^2} (\hat{p}_s^{N,n}) \ + k(a) (1 + |\lambda_s^{N,n}|^q) \frac{\partial f}{\partial \lambda} (\hat{p}_s^{N,n}) \\ & + \frac{1}{2} (\beta_{N,n}^S)^2 \ (\lambda_s^{N,n})^2 \ \sigma_0^2(X_s) \ \frac{\partial^2 f}{\partial \lambda^2} (\hat{p}_s^{N,n}) \Big]^2 + 2 \Big[ \beta_{N,n}^S \lambda_s^{N,n} \ b_0(X_s) \ \frac{\partial f}{\partial \lambda} (\hat{p}_s^{N,n}) \Big]^2. \end{split}$$

Then together with Assumption 3.2, Assumption 3.5 and Lemma 4.2 we know

that  $\sup_{N\in\mathbb{N}}\mathbb{E}[\Xi_N^{(1)}]<\infty$ . Notice that for any  $0\leq s\leq t\leq T$  and h square-integrable function on [0,T], we have that

$$\begin{split} \int_{s}^{t} h(r)dr & \leq & \sqrt{t-s}\sqrt{\int_{0}^{T} h^{2}(r)dr} \\ & = & \frac{\sqrt{t-s}}{(t-s)^{1/4}}(t-s)^{1/4}\sqrt{\int_{0}^{T} h^{2}(r)dr} \\ & \leq & \frac{1}{2}\left\{\frac{t-s}{(t-s)^{1/2}} + (t-s)^{1/2}\int_{0}^{T} h^{2}(r)dr\right\} \\ & = & \frac{1}{2}(t-s)^{1/2}\left\{1 + \int_{0}^{T} h^{2}(r)dr\right\}. \end{split}$$

Therefore, we get that

$$\mathbb{E}[|A_t^{1,N} - A_s^{1,N}||\mathcal{F}_s^N] \le (t-s)^{1/2} \mathbb{E}[\Xi_N^{(1)}|\mathcal{F}_s^N]$$

Since

$$\begin{split} |\mathcal{J}_{N,n}^{f}(t)| &= |\frac{1}{N} \sum_{1 \leqslant n' \leqslant N} \left( f\left(p^{N,n'}, \ \lambda_{t-}^{N,n'} + \frac{1}{N} \sum_{j=1}^{r} \xi_{j}^{2} \ u_{n',j} \ell_{n,j} \right) - f(p^{N,n'}, \ \lambda_{t-}^{N,n'}) \right) \ M_{t}^{N,n'} \\ &- \frac{1}{N} f(p^{N,n}, \ \lambda_{t-}^{N,n})| \\ &\leq \frac{1}{N} \sum_{1 \leqslant n' \leqslant N} \left\{ \frac{\partial f}{\partial \lambda} (p^{N,n'}, \ \lambda_{t-}^{N,n'}) \frac{1}{N} \sum_{j=1}^{r} \xi_{j}^{2} \ u_{n',j} \ell_{n,j} + O(1/N^{2}) \right\} + \frac{1}{N} |f(p^{N,n}, \ \lambda_{t-}^{N,n})| \\ &\leq \frac{1}{N} \{C_{3.2} || \frac{\partial f}{\partial \lambda} || + ||f|| \}. \end{split}$$

Therefore, we have that

$$\mathbb{E}[|A_t^{2,N} - A_s^{2,N}||\mathcal{F}_s^N] \le (t-s)^{1/2} \{C_{3,2}||\frac{\partial f}{\partial \lambda}|| + ||f||\} \mathbb{E}[\Xi_N^{(2)}|\mathcal{F}_s^N],$$

where  $\Xi_N^{(2)}$  is defined as

$$\Xi_N^{(2)} \stackrel{\text{def}}{=} \frac{1}{2} \{ 1 + \frac{1}{N} \sum_{n=1}^N \int_0^T (\lambda_r^{N,n})^2 dr \},$$

and, by Lemma 4.2, we have  $\sup_{N\in\mathbb{N}} \mathbb{E}[\Xi_N^{(2)}] < \infty$ .

$$\begin{split} \mathbb{E}[|B_t^{1,N} - B_s^{1,N}|^2 | \mathcal{F}_s^N] &= \mathbb{E}[|\frac{1}{N} \int_s^t \sum_{n=1}^N \sigma_n \; (\lambda_r^{N,n})^\rho \; \frac{\partial f}{\partial \lambda}(\hat{p}_r^{N,n}) \; M_r^{N,n} \; dW_r^n|^2 | \mathcal{F}_s^N] \\ &= \mathbb{E}[\frac{1}{N^2} \sum_{n=1}^N \int_s^t \left( \sigma_n \; (\lambda_r^{N,n})^\rho \; \frac{\partial f}{\partial \lambda}(\hat{p}_r^{N,n}) \; M_r^{N,n} \right)^2 \; dr | \mathcal{F}_s^N] \\ &\leq \mathbb{E}[\frac{1}{N} \sum_{n=1}^N \int_s^t \left( \sigma_n \; (\lambda_r^{N,n})^\rho \; \frac{\partial f}{\partial \lambda}(\hat{p}_r^{N,n}) \right)^2 \; dr | \mathcal{F}_s^N] \\ &\leq (t-s)^{1/2} \mathbb{E}[\Xi_N^{(3)} | \mathcal{F}_s^N], \end{split}$$

where

$$\Xi_N^{(3)} \stackrel{\text{def}}{=} \frac{1}{2} \left\{ 1 + \frac{1}{N} \sum_{n=1}^N \int_0^T \left( \sigma_n \ (\lambda_r^{N,n})^\rho \ \frac{\partial f}{\partial \lambda} (\hat{p}_r^{N,n}) \right)^4 dr \right\}.$$

Together with Assumption 3.2, Assumption 3.5 and Lemma 4.2 we know that  $\sup_{N\in\mathbb{N}} \mathbb{E}[\Xi_N^{(3)}] < \infty$ .

$$\mathbb{E}[|B_{t}^{2,N} - B_{s}^{2,N}|^{2}|\mathcal{F}_{s}^{N}] = \mathbb{E}[|\frac{1}{N} \int_{s}^{t} \sum_{n=1}^{N} \beta_{N,n}^{S} \lambda_{r}^{N,n} \sigma_{0}(X_{r}) \frac{\partial f}{\partial \lambda}(\hat{p}_{r}^{N,n}) M_{r}^{N,n} dV_{r}|^{2}|\mathcal{F}_{s}^{N}] \\
= \mathbb{E}[\int_{s}^{t} (\frac{1}{N} \sum_{n=1}^{N} \beta_{N,n}^{S} \lambda_{r}^{N,n} \sigma_{0}(X_{r}) \frac{\partial f}{\partial \lambda}(\hat{p}_{r}^{N,n}) M_{r}^{N,n})^{2} dr|\mathcal{F}_{s}^{N}] \\
\leq \mathbb{E}[\frac{1}{N} \sum_{n=1}^{N} \int_{s}^{t} (\beta_{N,n}^{S} \lambda_{r}^{N,n} \sigma_{0}(X_{r}) \frac{\partial f}{\partial \lambda}(\hat{p}_{r}^{N,n}))^{2} dr|\mathcal{F}_{s}^{N}] \\
\leq (t-s)^{1/2} \mathbb{E}[\Xi_{N}^{(4)}|\mathcal{F}_{s}^{N}],$$

where

$$\Xi_N^{(4)} \stackrel{\text{def}}{=} \frac{1}{2} \left\{ 1 + \frac{1}{N} \sum_{n=1}^N \int_0^T \left( \beta_{N,n}^S \ \lambda_r^{N,n} \ \sigma_0(X_r) \ \frac{\partial f}{\partial \lambda}(\hat{p}_r^{N,n}) \ \right)^4 dr \right\}.$$

Again with Assumption 3.2, Assumption 3.5 and Lemma 4.2 we know that  $\sup_{N\in\mathbb{N}}\mathbb{E}[\Xi_N^{(4)}]<\infty.$  For any  $0\leq t-v\leq t\leq t+u\leq T,$ 

$$\begin{split} & \mathbb{E}[q^2(\langle f, \mu^N_{t+u} \rangle_E, \langle f, \mu^N_{t} \rangle_E)q^2(\langle f, \mu^N_{t} \rangle_E, \langle f, \mu^N_{t-v} \rangle_E)|\mathcal{F}^N_t] \\ \leq & \mathbb{E}[q^2(\langle f, \mu^N_{t+u} \rangle_E, \langle f, \mu^N_{t} \rangle_E)|\mathcal{F}^N_t]q^2(\langle f, \mu^N_{t} \rangle_E, \langle f, \mu^N_{t-v} \rangle_E) \\ \leq & \mathbb{E}[q^2(\langle f, \mu^N_{t+u} \rangle_E, \langle f, \mu^N_{t} \rangle_E)|\mathcal{F}^N_t] \\ \leq & 4 \big\{ \mathbb{E}[q^2(A^{1,N}_{t+u}, A^{1,N}_{t})|\mathcal{F}^N_{t}] + \mathbb{E}[q^2(A^{2,N}_{t+u}, A^{2,N}_{t})|\mathcal{F}^N_{t}] \\ & + \mathbb{E}[q^2(B^{1,N}_{t+u}, B^{1,N}_{t})|\mathcal{F}^N_{t}] + \mathbb{E}[q^2(B^{2,N}_{t+u}, B^{2,N}_{t})|\mathcal{F}^N_{t}] \big\} \\ \leq & 4 \big\{ \mathbb{E}[|A^{1,N}_{t+u} - A^{1,N}_{t}||\mathcal{F}^N_{t}] + \mathbb{E}[|A^{2,N}_{t+u} - A^{2,N}_{t}||\mathcal{F}^N_{t}] \\ & + \mathbb{E}[|B^{1,N}_{t+u} - B^{1,N}_{t}|^2|\mathcal{F}^N_{t}] + \mathbb{E}[|B^{2,N}_{t+u} - B^{2,N}_{t}|^2|\mathcal{F}^N_{t}] \big\}. \end{split}$$

The definition  $q^2(x,y)=\min\{|x-y|^2,1\}$  yields the desired bound for the last display, with  $H_N=\Xi_N^{(1)}+(C_{3.2}||\frac{\partial f}{\partial \lambda}||+||f||)\Xi_N^{(2)}+\Xi_N^{(3)}+\Xi_N^{(4)}$ . The previous computations show that  $\sup_{N\in\mathbb{N}}\mathbb{E} H_N<\infty$ . This concludes the proof of the theorem.  $\square$ 

### 7. Characterization of the limit

In Section 6 we proved relative compactness of the family  $\{\mu^N\}_{N\in\mathbb{N}}$ . Therefore, the laws of  $\mu^N$ 's will have at least one limit point. In this section, we identify the possible limit points.

Let S be the collection of elements  $\Phi$  in  $B(\mathbb{R} \times \mathscr{P}(\hat{\mathcal{P}}))$  of the form

$$\Phi(x,\mu) = \varphi_1(x)\varphi_2(\langle f_1,\mu\rangle_E, \langle f_2,\mu\rangle_E, \dots, \langle f_M,\mu\rangle_E)$$

for some  $M \in \mathbb{N}$ , some  $\varphi_1 \in C^{\infty}(\mathbb{R})$ ,  $\varphi_2 \in C^{\infty}(\mathbb{R}^M)$  and some  $\{f_m\}_{m=1}^M$ . Then  $\mathcal{S}$ separates the probability measure space  $\mathcal{P}(\hat{\mathcal{P}})$ . Then it is enough to consider the martingale convergence problem on S.

For  $f \in C^2(\mathbb{R})$  define the operator

$$\mathfrak{G}(f)(x) = b_0(x)\frac{\partial f}{\partial x}(x) + \frac{1}{2}\sigma_0^2(x)\frac{\partial^2 f}{\partial x^2}(x).$$

In addition, define the operators

$$(\mathcal{A}\Phi)(x,\mu) = \mathcal{G}(\varphi_1)(x) \ \varphi_2\big(\langle f_1,\mu\rangle_E, \ \langle f_2,\mu\rangle_E,\dots,\langle f_M,\mu\rangle_E\big)$$

$$+ \sum_{m=1}^M \varphi_1(x) \frac{\partial \varphi_2}{\partial x_m} \big(\langle f_1,\mu\rangle_E, \ \langle f_2,\mu\rangle_E,\dots,\langle f_M,\mu\rangle_E\big)$$

$$\times \big[\langle \mathcal{L}_1 f_m,\mu\rangle_E + \langle \mathcal{L}_2^x f_m,\mu\rangle_E + \langle \iota\nu,\mu\rangle_E \cdot \langle \mathcal{L}_4^x f_m,\mu\rangle_E\big]$$

$$+ \frac{1}{2} \sum_{m=1}^M \frac{\partial \varphi_1}{\partial x}(x) \ \frac{\partial \varphi_2}{\partial x_m} \big(\langle f_1,\mu\rangle_E, \ \langle f_2,\mu\rangle_E,\dots,\langle f_M,\mu\rangle_E\big) \langle \sigma_0(x)\mathcal{L}_3^x f_m,\mu\rangle_E$$

$$+ \frac{1}{2} \sum_{p,q=1}^M \varphi_1(x) \frac{\partial^2 \varphi}{\partial x_p \partial x_q} \big(\langle f_1,\mu\rangle_E, \ \langle f_2,\mu\rangle_E,\dots,\langle f_M,\mu\rangle_E\big)$$

$$\times \big(\langle \mathcal{L}_3 f_p,\mu\rangle_E \langle \mathcal{L}_3 f_q,\mu\rangle_E\big).$$

and

$$(\mathcal{B}\Phi)(x,\mu) = \sigma_0(x) \frac{\partial \varphi_1}{\partial x} (x) \varphi_2 (\langle f_1, \mu \rangle_E, \langle f_2, \mu \rangle_E, \dots, \langle f_M, \mu \rangle_E)$$

$$+ \varphi_1(x) \sum_{m=1}^M \frac{\partial \varphi_2}{\partial x_m} (\langle f_1, \mu \rangle_E, \langle f_2, \mu \rangle_E, \dots, \langle f_M, \mu \rangle_E) \langle \mathcal{L}_3^x f_m, \mu \rangle_E.$$

Then, Theorem 7.1 characterizes the possible limit points.

Theorem 7.1. We have that

$$\lim_{N \to \infty} \mathbb{E} \left[ \left\{ \Phi(X_{t_2}, \mu_{t_2}^N) - \Phi(X_{t_1}, \mu_{t_1}^N) - \int_{t_1}^{t_2} (\mathcal{A}\Phi)(X_s, \mu_s^N) ds - \int_{t_1}^{t_2} (\mathcal{B}\Phi)(X_s, \mu_s^N) dV_s \right\} \prod_{j=1}^J \psi_j(x_{r_j}, \mu_{r_j}^N) \right] = 0$$

for any  $\Phi \in \mathcal{S}$  and  $0 \le r_1 \le r_2 \le \cdots \le r_J = t_1 < t_2 < T$  and  $\{\psi_j\}_{j=1}^J \in B(E)$ . Proof. First, we notice that,

$$\mathcal{M}_{t}^{N,n} = 1 - M_{t}^{N,n} - \int_{0}^{t} \lambda_{s}^{N,n} M_{s}^{N,n} ds$$

is a martingale. This means that we can write

$$d(1 - M_t^{N,n}) = d\mathcal{M}_t^{N,n} + \lambda_t^{N,n} M_t^{N,n} dt$$

Going back to (7) we then get that

$$\begin{split} d\langle f, \mu_t^N \rangle_E &= \frac{1}{N} \sum_{n=1}^N \left[ \mathcal{L}_1 f(\hat{p}_t^{N,n}) + \lambda_t^{N,n} f(\hat{p}_t^{N,n}) \right] \, M_t^{N,n} \, dt + \frac{1}{N} \sum_{n=1}^N \left[ \mathcal{L}_2^{X_t} f(\hat{p}_t^{N,n}) \right] \, M_t^{N,n} \, dt \\ &+ \frac{1}{N} \sum_{n=1}^N \, \sigma_n \, (\lambda_t^{N,n})^\rho \, \frac{\partial f}{\partial \lambda} (\hat{p}_t^{N,n}) \, M_t^{N,n} \, dW_t^n + \frac{1}{N} \sum_{n=1}^N \, \mathcal{L}_3^{X_t} f(\hat{p}_t^{N,n}) \, M_t^{N,n} \, dV_t \\ &+ \sum_{n=1}^N \mathcal{J}_{N,n}^f(t) \, d \, [1 - M_t^{N,n}]. \end{split}$$

Then,

$$\begin{split} d\langle f, \mu_t^N \rangle_E &= \left[ \langle \mathcal{L}_1 f, \mu_t^N \rangle_E + \langle \iota f, \mu_t^N \rangle \right] dt + \langle \mathcal{L}_2^{X_t} f, \mu_t^N \rangle_E \ dt \\ &+ \frac{1}{N} \sum_{n=1}^N \ \sigma_n \ (\lambda_t^{N,n})^\rho \ \frac{\partial f}{\partial \lambda} (\hat{p}_t^{N,n}) \ M_t^{N,n} \ dW_t^n \\ &+ \langle \mathcal{L}_3^{X_t} f, \mu_t^N \rangle_E \ dV_t \\ &+ \sum_{n=1}^N \mathcal{J}_{N,n}^f(t) \ d \ [1 - M_t^{N,n}]. \end{split}$$

By Itô formula for  $\Phi(X_t, \mu_t^N)$  we obtain that

$$d \Phi(X_{t}, \mu_{t}^{N}) = \frac{\partial \varphi_{1}}{\partial x}(X_{t}) \varphi_{2}(\langle f_{1}, \mu \rangle_{E}, \langle f_{2}, \mu \rangle_{E}, \dots, \langle f_{M}, \mu \rangle_{E}) dX_{t}$$

$$+ \varphi_{1}(X_{t}) \sum_{m=1}^{M} \frac{\partial \varphi_{2}}{\partial x_{m}}(\langle f_{1}, \mu \rangle_{E}, \langle f_{2}, \mu \rangle_{E}, \dots, \langle f_{M}, \mu \rangle_{E}) [d\langle f_{m}, \mu_{t}^{N} \rangle_{E}]$$

$$+ \frac{1}{2} \frac{\partial^{2} \varphi_{1}}{\partial x^{2}}(X_{t}) \varphi_{2}(\langle f_{1}, \mu \rangle_{E}, \langle f_{2}, \mu \rangle_{E}, \dots, \langle f_{M}, \mu \rangle_{E}) [(dX_{t})^{2}]$$

$$+ \frac{1}{2} \sum_{m=1}^{M} \frac{\partial \varphi_{1}}{\partial x}(X_{t}) \frac{\partial \varphi_{2}}{\partial x_{m}}(\langle f_{1}, \mu \rangle_{E}, \langle f_{2}, \mu \rangle_{E}, \dots, \langle f_{M}, \mu \rangle_{E}) [dX_{t} d\langle f_{m}, \mu_{t}^{N} \rangle_{E}]$$

$$+ \frac{1}{2} \sum_{p=1}^{M} \sum_{q=1}^{M} \varphi_{1}(X_{t}) \frac{\partial^{2} \varphi_{2}}{\partial x_{p} \partial x_{q}}(\langle f_{1}, \mu \rangle_{E}, \langle f_{2}, \mu \rangle_{E}, \dots, \langle f_{M}, \mu \rangle_{E}) [d\langle f_{p}, \mu_{t}^{N} \rangle_{E} d\langle f_{q}, \mu_{t}^{N} \rangle_{E}]$$

$$+ \varphi_{1}(X_{t}) \sum_{n=1}^{N} \left\{ \varphi_{2}(\langle f_{1}, \mu_{t}^{N} \rangle_{E} + \mathcal{J}_{N,n}^{f_{1}}(t), \langle f_{2}, \mu_{t}^{N} \rangle_{E} + \mathcal{J}_{N,n}^{f_{2}}(t), \dots, \langle f_{M}, \mu_{t}^{N} \rangle_{E} + \mathcal{J}_{N,n}^{f_{M}}(t))$$

$$- \varphi_{2}(\langle f_{1}, \mu_{t}^{N} \rangle_{E}, \langle f_{2}, \mu_{t}^{N} \rangle_{E}, \dots, \langle f_{M}, \mu_{t}^{N} \rangle_{E}) \right\} d [1 - M_{t}^{N,n}],$$

where we have used the following notations for the terms  $[d\langle f_m, \mu_t^N \rangle_E]$ ,  $[(dX_t)^2]$ ,  $[dX_t \ d\langle f_m, \mu_t^N \rangle_E]$  and  $[d\langle f_p, \mu_t^N \rangle_E d\langle f_q, \mu_t^N \rangle_E]$ 

$$\begin{split} [d\langle f_m,\mu_t^N\rangle_E] &= \left[\langle \mathcal{L}_1 f_m,\mu_t^N\rangle_E + \left\langle \iota f_m,\mu_t^N\right\rangle\right] dt + \langle \mathcal{L}_2^{X_t} f_m,\mu_t^N\rangle_E \ dt \\ &+ \frac{1}{N} \sum_{n=1}^N \ \sigma_n \ (\lambda_t^{N,n})^\rho \ \frac{\partial f_m}{\partial \lambda} (\hat{p}_t^{N,n}) \ M_t^{N,n} \ dW_t^n + \langle \mathcal{L}_3^{X_t} f_m,\mu_t^N\rangle_E \ dV_t \\ [(dX_t)^2] &= \sigma_0^2(X_t) \ dt \\ [dX_t \ d\langle f_m,\mu_t^N\rangle_E] &= \sigma_0(X_t) \langle \mathcal{L}_3^{X_t} f_m,\mu_t^N\rangle_E \ dt \\ [d\langle f_p,\mu_t^N\rangle_E d\langle f_q,\mu_t^N\rangle_E] &= \frac{1}{N^2} \Big( \sum_{n=1}^N \sigma_n^2 \ (\lambda_t^{N,n})^{2\rho} \ \frac{\partial f_p}{\partial \lambda} (\hat{p}_t^{N,n}) \ \frac{\partial f_q}{\partial \lambda} (\hat{p}_t^{N,n}) \ M_t^{N,n} \Big) \ dt \\ &+ \left( \langle \mathcal{L}_3^{X_t} f_p,\mu_t^N\rangle_E \langle \mathcal{L}_3^{X_t} f_q,\mu_t^N\rangle_E \right) \ dt. \end{split}$$

Recall that

$$\mathfrak{G}(f)(x) = b_0(x)\frac{\partial f}{\partial x}(x) + \frac{1}{2}\sigma_0^2(x)\frac{\partial^2 f}{\partial x^2}(x).$$

and that

$$d(1 - M_t^{N,n}) = d\mathcal{M}_t^{N,n} + \lambda_t^{N,n} M_t^{N,n} dt$$

is a martingale. Rearrange terms and write  $\Phi(X_t, \mu_t^N)$  in the integral form,

$$\begin{split} \Phi(X_t,\mu_t^N) &= & \Phi(X_0,\mu_0^N) + \int_0^t \Im(\varphi_1)(X_s) \ \varphi_2\big(\langle f_1,\mu_s^N \rangle_E, \ \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E\big) \ ds \\ &+ \int_0^t \varphi_1(X_s) \sum_{m=1}^M \frac{\partial \varphi_2}{\partial x_m} \big(\langle f_1,\mu_s^N \rangle_E, \ \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E\big) \\ &\times \left\{ \langle \mathcal{L}_1 f_m, \mu_s^N \rangle_E + \langle t f_m, \mu_t^N \rangle + \langle \mathcal{L}_2^{X_s} f_m, \mu_s^N \rangle_E \right\} \ ds \\ &+ \frac{1}{2} \int_0^t \sum_{m=1}^M \frac{\partial \varphi_1}{\partial x} (X_s) \frac{\partial \varphi_2}{\partial x_m} \big(\langle f_1,\mu_s^N \rangle_E, \ \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E\big) \\ &\times \sigma_0(X_s) \langle \mathcal{L}_3^{X_s} f_m, \mu_s^N \rangle_E \ ds \\ &+ \int_0^t \varphi_1(X_s) \sum_{n=1}^N \lambda_s^{N,n} \left\{ \varphi_2 \big(\langle f_1,\mu_s^N \rangle_E, + \partial_{N,n}^f(s), \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \right) \right\} M_s^{N,n} \ ds \\ &+ \int_0^t \varphi_1(X_s) \sum_{n=1}^N \lambda_s^{N,n} \left\{ \varphi_2 \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \right) \right\} M_s^{N,n} \ ds \\ &+ \int_0^t \varphi_1(X_s) \sum_{m=1}^M \partial_{N,n}^{\varphi_2} \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \ dV_s \\ &+ \int_0^t \varphi_1(X_s) \sum_{m=1}^M \frac{\partial \varphi_2}{\partial x_m} \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \ dV_s \\ &+ \frac{1}{2N^2} \sum_{p,q=1}^M \int_0^t \varphi_1(X_s) \frac{\partial^2 \varphi}{\partial x_p \partial x_q} \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \ ds \\ &+ \frac{1}{2} \sum_{p,q=1}^M \int_0^t \varphi_1(X_s) \frac{\partial^2 \varphi}{\partial x_p \partial x_q} \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \ ds \\ &+ \frac{1}{N} \sum_{n=1}^N \sum_{m=1}^M \int_0^t \varphi_1(X_s) \frac{\partial^2 \varphi}{\partial x_p \partial x_q} \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \ \times \big(\langle \mathcal{L}_3^{X_s} f_p,\mu_s^N \rangle_E \mathcal{L}_3^{X_s} f_m \mu_s^N \rangle_E \big) \ ds \\ &+ \frac{1}{N} \sum_{n=1}^N \sum_{m=1}^M \int_0^t \varphi_1(X_s) \frac{\partial^2 \varphi}{\partial x_p \partial x_q} \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \ \times \big(\langle \mathcal{L}_3^{X_s} f_p,\mu_s^N \rangle_E \mathcal{L}_3^{X_s} f_m \mu_s^N \rangle_E \big) \ ds \\ &+ \frac{1}{N} \sum_{n=1}^N \sum_{m=1}^M \int_0^t \varphi_1(X_s) \frac{\partial \varphi_2}{\partial x_m} \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \ ds \\ &+ \frac{1}{N} \sum_{n=1}^N \sum_{m=1}^M \int_0^t \varphi_1(X_s) \frac{\partial \varphi_2}{\partial x_m} \big(\langle f_1,\mu_s^N \rangle_E, \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \ dM_s^{N,n} \\ &+ \int_0^t \varphi_1(X_s) \sum_{n=1}^M \left\{ \varphi_2(\langle f_1,\mu_s^N \rangle_E + \beta_{N,n}^f(s), \langle f_2,\mu_s^N \rangle_E, \ldots, \langle f_M,\mu_s^N \rangle_E \big) \right\} dM_s^{N,n} \\ &= \sum_{n=1}^M \int_0^1 \varphi_1(X_s) \frac{\partial \varphi_2}$$

where, for  $i=1,\cdots,11,\,J_i^N$  represents the  $i^{\rm th}$  term in the right hand side of the last display. Notice that,

$$J_6^N + J_7^N = \int_0^t (\mathcal{B}\Phi)(X_s, \mu_s^N) dV_s,$$

and

$$J_1^N + J_2^N + J_3^N + J_4^N + J_9^N = \Phi(X_0, \mu_0^N) + \int_0^t (\mathcal{A}\Phi)(X_s, \mu_s^N) - \tilde{A}_s^N ds$$

where the  $\tilde{A}_t^N$  is defined as

$$\tilde{A}_{t}^{N} = \sum_{m=1}^{M} \varphi_{1}(X_{t}) \frac{\partial \varphi_{2}}{\partial x_{m}} \left( \langle f_{1}, \mu_{t}^{N} \rangle_{E}, \langle f_{2}, \mu_{t}^{N} \rangle_{E}, \dots, \langle f_{M}, \mu_{t}^{N} \rangle_{E} \right)$$

$$\times \frac{1}{N} \sum_{n=1}^{N} \lambda_{t}^{N,n} \, \tilde{\mathcal{J}}_{N,n}^{f_{m}}(t) M_{t}^{N,n}.$$

and  $\tilde{\mathcal{J}}_{N,n}^f(t)$  is defined as

$$\tilde{\mathcal{J}}_{N,n}^{f}(t) = \frac{1}{N} \sum_{1 \leq n' \leq N} \left( \sum_{j=1}^{r} \xi_{j}^{2} \ u_{n',j} \ell_{n,j} \frac{\partial f}{\partial \lambda} (\hat{p}_{t}^{N,n'}) \right) \ M_{t}^{N,n'} - f(\hat{p}_{t}^{N,n}).$$

Notice that we have

$$\sum_{j=1}^{r} \xi_j^2 \ u_{n',j} \ell_{n,j} = \beta_{N,n'}^C \cdot l^n,$$

where  $\beta_{N,n'}^C = (\xi_1^2 u_{n',1}, \xi_2^2 u_{n',2}, \dots, \xi_r^2 u_{n',r})$  and  $l^n = (l_{n,1}, l_{n,2}, \dots, l_{n,r})$ . Recalling that

$$\mathcal{L}_4 f = \beta^C \frac{\partial f}{\partial \lambda}(\hat{p}).$$

where  $\beta^C = (\xi_1^2 u_1, \xi_2^2 u_2, \dots \xi_r^2 u_r)$ , we get that

$$\tilde{\mathcal{J}}_{N,n}^f(t) = l^n \cdot \left\langle \mathcal{L}_4 f, \mu_t^N \right\rangle_E - f(\hat{p}_t^{N,n}).$$

Therefore we obtain that

$$\begin{split} \tilde{A}_t^N &= \sum_{m=1}^M \varphi_1(X_t) \frac{\partial \varphi_2}{\partial x_m} \left( \langle f_1, \mu_t^N \rangle_E, \ \langle f_2, \mu_t^N \rangle_E, \dots, \langle f_M, \mu_t^N \rangle_E \right) \\ &\qquad \times \frac{1}{N} \sum_{n=1}^N \lambda_t^{N,n} \Big[ l^n \cdot \left\langle \mathcal{L}_4 f_m, \mu_t^N \right\rangle_E \ - \ f_m(\hat{p}_t^{N,n}) \Big] M_t^{N,n} \\ &= \sum_{m=1}^M \varphi_1(X_t) \frac{\partial \varphi_2}{\partial x_m} \left( \langle f_1, \mu_t^N \rangle_E, \ \langle f_2, \mu_t^N \rangle_E, \dots, \langle f_M, \mu_t^N \rangle_E \right) \\ &\qquad \times \Big[ \frac{1}{N} \sum_{n=1}^N \lambda_t^{N,n} l^n \cdot \left\langle \mathcal{L}_4 f_m, \mu_t^N \right\rangle_E M_t^{N,n} \ - \ \frac{1}{N} \sum_{n=1}^N \lambda_t^{N,n} f_m(\hat{p}_t^{N,n}) M_t^{N,n} \Big] \\ &= \sum_{m=1}^M \varphi_1(X_t) \frac{\partial \varphi_2}{\partial x_m} \left( \langle f_1, \mu_t^N \rangle_E, \ \langle f_2, \mu_t^N \rangle_E, \dots, \langle f_M, \mu_t^N \rangle_E \right) \\ &\qquad \times \left[ \langle \iota \nu, \mu_t^N \rangle_E \cdot \left\langle \mathcal{L}_4 f_m, \mu_t^N \right\rangle_E \ - \ \langle \iota f_m, \mu_t^N \rangle_E \Big]. \end{split}$$

Now we prove that  $\left|J_5^N - \int_0^t \tilde{A}_s^N ds\right| \to 0$  as  $N \to \infty$ . Denote the operator

$$\mathcal{L}_5 f(\hat{p}) = \sigma \lambda^{\rho} \frac{\partial f}{\partial \lambda}$$

Denote the jump term  $J_5^N$  in the expression  $\Phi(X_t, \mu_t^N)$  as  $\int_0^t A_s^N ds$ . Now we look at the limit of this term as  $N \to \infty$ .

Hence there exists a constant K which depends on the uppper bound of the coefficients such that

$$\left|\mathcal{J}_{N,n}^f(t)\right| - \left|\frac{1}{N}\tilde{\mathcal{J}}_{N,n}^f(t)\right| \leqslant \frac{K^2}{N^2} \|\frac{\partial^2 f}{\partial \lambda^2}\|.$$

Hence, we get that

$$\lim_{N \to \infty} \mathbb{E} \left[ \int_0^t |A_s^N - \tilde{A}_s^N| \ ds \right] = 0.$$

Let us next show that  $J_8^N \to 0$ . The term  $J_8^N$  above can be written as,

$$\frac{1}{2N} \sum_{p,q=1}^{M} \int_{0}^{t} \varphi_{1}(X_{s}) \frac{\partial^{2} \varphi}{\partial x_{p} \partial x_{q}} \left( \langle f_{1}, \mu_{s}^{N} \rangle_{E}, \langle f_{2}, \mu_{s}^{N} \rangle_{E}, \dots, \langle f_{M}, \mu_{s}^{N} \rangle_{E} \right) \\
\times \left\{ \frac{1}{N} \sum_{n=1}^{N} (\mathcal{L}_{5} f_{p}) \left( \hat{p}_{s}^{N,n} \right) (\mathcal{L}_{5} f_{q}) \left( \hat{p}_{s}^{N,n} \right) M_{s}^{N,n} \right\} ds \\
= \frac{1}{2N} \sum_{p,q=1}^{M} \int_{0}^{t} \varphi_{1}(X_{s}) \frac{\partial^{2} \varphi}{\partial x_{p} \partial x_{q}} \left( \langle f_{1}, \mu_{s}^{N} \rangle_{E}, \langle f_{2}, \mu_{s}^{N} \rangle_{E}, \dots, \langle f_{M}, \mu_{s}^{N} \rangle_{E} \right) \\
\times \left\langle (\mathcal{L}_{5} f_{p} \mathcal{L}_{5} f_{q}), \mu_{s}^{N} \rangle_{E} ds.$$

This term goes to zero as N goes to infinity. Indeed, for the given M and  $\{f_m\}_{m=1}^M$  and t there exists a constant C depending on  $\max_{\{p,q=1,\dots,M\}} \|\frac{\partial^2 \varphi}{\partial x_p \partial x_q}\|$  and  $\max_{\{m=1\dots M\}} \|f_m\|$  and the upper bound of the coefficients such that,

$$\left| \frac{1}{2N} \sum_{p,q=1}^{M} \int_{0}^{t} \varphi_{1}(X_{s}) \frac{\partial^{2} \varphi}{\partial x_{p} \partial x_{q}} \left( \langle f_{1}, \mu_{s}^{N} \rangle_{E}, \langle f_{2}, \mu_{s}^{N} \rangle_{E}, \dots, \langle f_{M}, \mu_{s}^{N} \rangle_{E} \right) \times \left\langle (\mathcal{L}_{5} f_{p} \mathcal{L}_{5} f_{q}), \mu_{s}^{N} \rangle_{E} ds \right| \leqslant \frac{C}{N} \longrightarrow 0.$$

Lastly, we treat the terms  $J_{10}^N$  and  $J_{11}^N$ . Notice that the second to the last term  $J_{10}^N$  is a Brownian martingale and the term  $J_{11}^N$  is also a martingale. Denote their sum as a martingale  $\mathcal{M}_t^N$ . Calculations similar to the ones done above yield that

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}|\mathcal{M}_t^N|^2=0,$$

and the proof of the theorem is complete.

#### 8. Identification of the unique limit point

The uniqueness of the solution to the limiting martingale problem implied by Theorem 7.1 is analogous to the duality argument of Lemma 7.1 of [16] and the proof will not be repeated here.

Let us now identify this unique solution in the following two lemmas. Lemma 8.1 will give us the existence of a unique solution to a certain stochastic differential equation which will then be used in identifying the unique limiting solution in Lemma 8.2.

Let us now define the  $\sigma$ -algeba  $\mathcal{V}_t = \sigma(V_s, s \leq t)$  and let us set  $\mathcal{V} = \bigcup_{t \geq 0} \mathcal{V}_t$ . Let us set

$$\mathbb{E}_{\mathcal{V}}[\cdot] = \mathbb{E}[\cdot|\mathcal{V}].$$

**Lemma 8.1.** Let  $W^*$  be a reference Brownian motion and  $T < \infty$ . For each  $\hat{p} \in \hat{\mathcal{P}}$ , each  $t \leq T$  there is a unique pair of  $(Q_i(t), \lambda_t(\hat{p}), i = 1, \dots, r)$ 

$$Q_i(t) = \int_{\hat{p} \in \hat{\mathcal{P}}} l_i \mathbb{E}_{\mathcal{V}_t} \left\{ \lambda_t^*(\hat{p}) \exp\left[ -\int_0^t \lambda_s^*(\hat{p}) ds \right] \right\} \pi(dp) \Lambda_0(d\lambda).$$

$$\lambda_t^*(\hat{p}) = \lambda_0 + \int_0^t b(\lambda_s, a) ds + \sigma \cdot (\lambda_s)^{\rho} dW_s^* + \int_0^t \sum_{i=1}^r \beta_i^C Q_i(s) ds + \beta^S \int_0^t \lambda_s^*(\hat{p}) dX_s.$$

Lemma 8.1 is proven in the Appendix.

**Lemma 8.2.** For any  $A \in \mathfrak{B}(\mathbb{P})$  and  $B \in \mathfrak{B}(\mathbb{R}_+)$ ,  $\bar{\mu}$  is given by

$$\bar{\mu_t}(A \times B) = \int_{\hat{p} \in \hat{\mathcal{P}}} \chi_A(p) \mathbb{E}_{\mathcal{V}_t} \left[ \chi_B(\lambda_t^*(\hat{p})) \exp \left[ -\int_0^t \lambda_s^*(\hat{p}) ds \right] \right] \pi(dp) \Lambda_0(d\lambda).$$

*Proof.* For any  $f \in C^{\infty}(\hat{\mathcal{P}})$ ,

$$\langle f, \bar{\mu}_t \rangle_E = \int_{\hat{p} \in \hat{\mathcal{P}}} \mathbb{E}_{\mathcal{V}_t} \left[ f(p, \lambda_t^*(\hat{p})) \exp \left[ - \int_0^t \lambda_s^*(\hat{p}) ds \right] \right] \pi(dp) \Lambda_0(d\lambda)$$

and by Itô formula, we obtain, using Lemmas B.1 and B.2 in [17], that  $d\langle f, \bar{\mu}_t \rangle_E =$ 

$$\begin{split} &= \left\{ \int_{\hat{p} \in \hat{\mathcal{P}}} \mathbb{E}_{\mathcal{V}_t} \left[ \left[ (\mathcal{L}_1 f)(p, \lambda_t^*(\hat{p})) + (\mathcal{L}_2^{X_t} f)(p, \lambda_t^*(\hat{p})) \right] \exp \left[ - \int_0^t \lambda_s^*(\hat{p}) ds \right] \right] \\ &\quad + \left\{ \int_{\hat{p} \in \hat{\mathcal{P}}} \mathbb{E}_{\mathcal{V}_t} \left[ (\mathcal{L}_3^{X_t} f)(p, \lambda_t^*(\hat{p})) \exp \left[ - \int_0^t \lambda_s^*(\hat{p}) ds \right] \right] \pi(dp) \Lambda_0(d\lambda) \right\} dV_t \\ &\quad + \left\{ \int_{\hat{p} \in \hat{\mathcal{P}}} \mathbb{E}_{\mathcal{V}_t} \left[ \sum_{i=1}^r Q_i(t) (\mathcal{L}_4 f)_i(p, \lambda_t^*(\hat{p})) \exp \left[ - \int_0^t \lambda_s^*(\hat{p}) ds \right] \right] \pi(dp) \Lambda_0(d\lambda) \right\} dt \\ &\quad = \left\{ \langle \mathcal{L}_1 f, \bar{\mu_t} \rangle_E + \langle \mathcal{L}_2^{X_t} f, \bar{\mu_t} \rangle_E + \sum_{i=1}^r Q_i(t) \langle (\mathcal{L}_4 f)_i, \bar{\mu_t} \rangle_E \right\} dt + \left\{ \langle \mathcal{L}_3^{X_t} f, \bar{\mu_t} \rangle_E \right\} dV_t. \end{split}$$

where  $\iota(\lambda, p) = \lambda$ . Define now

$$G_i(t) = \int_{\hat{\boldsymbol{n}} \in \hat{\mathcal{P}}} \mathbb{E}_{\mathcal{V}_t} l_i \left[ \exp \left[ - \int_0^t \lambda_s^*(\hat{\boldsymbol{p}}) ds \right] \right] \pi(d\boldsymbol{p}) \Lambda_0(d\lambda).$$

Then, we have that

$$G_i'(t) = -\int_{\hat{p} \in \hat{\mathcal{P}}} \mathbb{E}_{\mathcal{V}_t} l_i \left[ \lambda_t^*(\hat{p}) \exp \left[ -\int_0^t \lambda_s^*(\hat{p}) ds \right] \right] \pi(dp) \Lambda_0(d\lambda) = -\langle l_i \iota, \bar{\mu_t} \rangle_E.$$

On the other hand by Lemma 8.1, we have

$$G_i'(t) = -Q_i(t),$$

concluding the proof of the lemma due to uniqueness.

#### 9. Conclusions and further research work

We consider a general point process model of correlated default timing in a pool of components (e.g. firms or names) interacting via a weighted directed graph which determines the impact of default among the different components. The model is empirically motivated and incorporates contagion effects, common systematic risk factors as well as idiosyncratic effects.

We prove a law of large numbers for the empirical survival distribution. This is then used to study the behavior of dynamic quantities of interest, such as mean loss rate in the pool or mean impact on given names from system wide defaults. The presence of the network structure enlarges the set of interesting questions that we can ask and at the same time allows via singular value decomposition arguments to reduce the computational burden via low rank approximations.

One of the interesting questions that we did not address here is that of the effect of choices such as bistability in the idiosyncratic component of the intensity-to-default process. Questions motivated by such choices, as well as others including the study of most likely paths to default, are more suitable for large deviations analysis in the spirit of [26], which will be done in a follow up work. In the present work we focus on establishing mathematical well-posedness of such models and on numerically exploring the effects of the network structure and low rank approximations on the typical behavior of quantities of interest.

#### Appendix A. Appendix

In this appendix we prove lemmas used throughout the paper. We remark here that most of the technical difficulties arising from dropping the affine structure in the idiosyncratic part of the intensity process are encountered in the proofs of the results in this Appendix.

Let  $\xi$  be a vector of processes having r components, predictable, bounded, right continuous, monotone with  $\xi_0 = 0$ . Define the process

$$Z_t = \lambda_0 + \beta^C \cdot \int_0^t e^{\Gamma_s} d\xi_s.$$

**Lemma A.1.** Let  $p \geq 1$  be such that Assumptions 3.5 and 3.6 hold. Then we have that

$$\begin{split} & \mathbb{E}[Z_t^{2p}]^{1/(2p)} \leq \lambda_0 + ||\beta^C||_1 \mathbb{E}[e^{2p\Gamma_t}]^{1/(2p)} \\ & + t^{1-1/2p} \left[ \left( \int_0^t \mathbb{E}\left[e^{4p\Gamma_s}\right] ds \right) \right]^{1/4p} \left\{ \left( \int_0^t ||\beta^C||_1^{1/4p} (\beta^S)^{1/4p} \mathbb{E}\left([b_0(X_s)]^{4p}\right) ds \right) \right\}^{1/4p}. \end{split}$$

In particular, we have that there is a finite constant  $0 < K < \infty$  such that  $\mathbb{E}[Z_t^{2p}] \le K$ .

Proof of Lemma A.1. Notice that  $Z_t$  can be written as

$$Z_t = \lambda_0 + \beta^C \cdot \{e^{\Gamma_t} \xi_t + \int_0^t e^{\Gamma_s} \xi_s \beta^S b_0(X_s) ds\}$$
$$= \lambda_0 + \int_0^t e^{\Gamma_s} \beta^C \cdot \xi_s \beta^S b_0(X_s) ds + \beta^C \cdot \xi_t e^{\Gamma_t}$$

Next given that  $\beta^C \cdot \xi_t \leq \sum_{j=1}^r |\beta_j^C| = ||\beta^C||_1$ , we obtain

$$\left\{ \mathbb{E}[Z_t^{2p}] \right\}^{1/(2p)} \leq \lambda_0 + ||\beta^C||_1 \mathbb{E}[e^{2p\Gamma_t}]^{1/(2p)} + \left\{ \mathbb{E}\left[ \left( \int_0^t \beta^C \cdot \xi_s e^{\Gamma_s} \beta^S b_0(X_s) ds \right)^{2p} \right] \right\}^{1/(2p)}.$$

By Cauchy-Schwartz inequality, we have

$$\left(\int_0^t \beta^C \cdot \xi_s e^{\Gamma_s} \beta^S b_0(X_s) ds\right)^2 \le \int_0^t e^{2\Gamma_s} ds \int_0^t \left[\beta^C \cdot \xi_s \beta^S b_0(X_s)\right]^2 ds.$$

Therefore

$$\left\{ \mathbb{E} \left[ \left( \int_{0}^{t} \beta^{C} \cdot \xi_{s} e^{\Gamma_{s}} \beta^{S} b_{0}(X_{s}) ds \right)^{2p} \right] \right\}^{1/(2p)}$$

$$\leq \left\{ \mathbb{E} \left[ \left( \int_{0}^{t} e^{2\Gamma_{s}} ds \right)^{p} \left( \int_{0}^{t} \left[ \beta^{C} \cdot \xi_{s} \beta^{S} b_{0}(X_{s}) \right]^{2} ds \right)^{p} \right] \right\}^{1/(2p)}$$

$$\leq \left\{ \left[ \mathbb{E} \left( \int_{0}^{t} e^{2\Gamma_{s}} ds \right)^{2p} \right]^{1/2} \left[ \mathbb{E} \left( \int_{0}^{t} \left[ \beta^{C} \cdot \xi_{s} \beta^{S} b_{0}(X_{s}) \right]^{2} ds \right)^{2p} \right]^{1/2} \right\}^{1/(2p)}$$

$$= \left[ \mathbb{E} \left( \int_{0}^{t} e^{2\Gamma_{s}} ds \right)^{2p} \right]^{1/4p} \left[ \mathbb{E} \left( \int_{0}^{t} \left[ \beta^{C} \cdot \xi_{s} \beta^{S} b_{0}(X_{s}) \right]^{2} ds \right)^{2p} \right]^{1/4p} .$$

By Holder inequality,

$$\int_0^t e^{2\Gamma_s} ds \le \left[ \int_0^t \left( e^{2\Gamma_s} \right)^{2p} \right]^{1/(2p)} \left[ \int_0^t 1 ds \right]^{1-1/2p} = t^{1-1/2p} \left( \int_0^t e^{4p\Gamma_s} ds \right)^{1/2p}.$$

So

$$\left[\mathbb{E}\left(\int_0^t e^{2\Gamma_s}ds\right)^{2p}\right]^{1/4p} = \left[t^{2p-1}\mathbb{E}(\int_0^t e^{4p\Gamma_s}ds)\right]^{1/4p}.$$

Similarly, we get

$$\int_{0}^{t} [\beta^{C} \cdot \xi_{s} \beta^{S} b_{0}(X_{s})]^{2} ds \leq \left[ \int_{0}^{t} [\beta^{C} \cdot \xi_{s} \beta^{S} b_{0}(X_{s})]^{4p} ds \right]^{1/2p} \left[ \int_{0}^{t} 1 ds \right]^{1-1/2p} 
= t^{1-1/2p} \left[ \int_{0}^{t} [||\beta^{C}||_{1} \beta^{S} b_{0}(X_{s})]^{4p} ds \right]^{1/2p}.$$

So, we have that

$$\left[\mathbb{E}\left(\int_0^t \left[\beta^C \cdot \xi_s \beta^S b_0(X_s)\right]^2 ds\right)^{2p}\right]^{1/4p} \leq \left[t^{2p-1}\mathbb{E}\left(\int_0^t \left[||\beta^C||_1 \beta^S b_0(X_s)\right]^{4p} ds\right)\right]^{1/4p}.$$

Therefore, we have

$$\left\{ \mathbb{E} \left[ \left( \int_0^t \beta^C \cdot \xi_s e^{\Gamma_s} \beta^S b_0(X_s) ds \right)^{2p} \right] \right\}^{1/(2p)} \\
\leq t^{1-1/2p} \left[ \left( \int_0^t \mathbb{E} \left[ e^{4p\Gamma_s} \right] ds \right) \right]^{1/4p} \left\{ \mathbb{E} \left( \int_0^t \left[ ||\beta^C||_1 \beta^S b_0(X_s) \right]^{4p} ds \right) \right\}^{1/4p} \\
= t^{1-1/2p} \left[ \left( \int_0^t \mathbb{E} \left[ e^{4p\Gamma_s} \right] ds \right) \right]^{1/4p} \left\{ \left( \int_0^t ||\beta^C||_1^{4p} (\beta^S)^{4p} \mathbb{E} \left( \left[ b_0(X_s) \right]^{4p} \right) ds \right) \right\}^{1/4p},$$
concluding the proof of the lemma.

*Proof of Lemma 4.1.* The proof of this lemma will be given in several steps. Let us first discuss existence and uniqueness of the equation for  $\lambda_t$  assuming that  $b(\lambda, \alpha)$  is uniformly bounded.

The existence and uniqueness of the solution  $\lambda_t$  follows along similar lines as in chapter V.11 in [28]. However, due to the peculiarities of the model considered here, the derivation of the bounds for the necessary norms are more complicated here. Below we mention the adjustments needed for the proof of uniqueness as the adjustments needed for the proof of existence are basically the same.

For any M > 0, let us set

$$b_M(\lambda, a) = b(\lambda, a)$$
, for all  $|\lambda| \leq M$ .

Let  $Y^M$  satisfy the equation

$$Y_t^M \stackrel{\text{def}}{=} \int_0^{t \wedge \tau_M} e^{\Gamma_s} [b_M(e^{-\Gamma_s}((Y_s^M + Z_s) \vee 0), a)] ds + \sigma \int_0^{t \wedge \tau_M} e^{\Gamma_s(1-\rho)} ((Y_s^M + Z_s) \vee 0)^{\rho} dW s + \beta^S \int_0^{t \wedge \tau_M} \sigma_0(X_s) ((Y_s^M + Z_s) \vee 0) dV_s,$$

where  $\tau_M$  is the random time defined via

(8) 
$$\tau_M = \inf \{ t \ge 0 : |e^{-\Gamma_t} ((Y_t^M + Z_t) \lor 0)| > M \} \land M.$$

It is clear that up to time  $\tau_M$ , the process  $Y_t^M$  will be the same as the process  $Y_t$ , which has b in place of  $b_M$  as its corresponding drift coefficient.

Now, we assume that the equation for  $Y_t^M$  has one more solution, potentially different than  $Y_t^M$ , denoted by  $Y_t'^M$ , and we denote by  $\tau_M'$  the corresponding random time.

Let us consider  $0 < \eta \ll 1$  and define the function

(9) 
$$\psi_{\eta}(x) = \frac{2}{\ln \eta^{-1}} \int_{0}^{|x|} \left\{ \int_{0}^{y} \frac{1}{z} \chi_{[\eta, \eta^{1/2}]}(z) dz \right\} dy.$$

Notice that  $\psi_{\eta}$  is an even function. In addition, its first and second derivatives satisfy

$$\psi_{\eta}'(x) = \frac{2}{\ln \eta^{-1}} \int_{z=0}^{x} \frac{1}{z} \chi_{[\eta, \eta^{1/2}]}(z) dz \quad \text{and} \quad \psi_{\eta}''(x) = \frac{2}{\ln \eta^{-1}} \frac{1}{x} \chi_{[\eta, \eta^{1/2}]}(x)$$

for all x > 0. Monotonicity arguments then show that for all  $x \in \mathbb{R}$  and  $\eta > 0$ ,  $|\psi'_n(x)| \leq 1$ , and

$$|x| \le \psi_{\eta}(x) + \sqrt{\eta}.$$

Additionally, we note that

$$\left|\psi_{\eta}''(x)\right| \leq \frac{2}{\ln \eta^{-1}} \frac{1}{|x|} \chi_{[\eta,\sqrt{\eta})}(|x|) \leq \frac{2}{\ln \eta^{-1}} \min \left\{ \frac{1}{|x|}, \frac{1}{\eta} \right\},$$

and that  $x\psi'_{\eta}(x) \geq 0$  for all  $x \in \mathbb{R}$ . We have

$$|Y_t^M - Y_t^{'M}| \leq \psi_{\eta}(Y_t^M - Y_t^{'M}) + \sqrt{\eta} \leq D_t^{1,M} + \sigma^2 D_t^{2,M} + (\beta^S)^2 D_t^{3,M} + \mathcal{M}_t + \sqrt{\eta} dt + (\beta^S)^2 D_t^{3,M} + \mathcal{M}_t + \sqrt{\eta} dt + (\beta^S)^2 D_t^{3,M} + \mathcal{M}_t + \sqrt{\eta} dt + (\beta^S)^2 D_t^{3,M} + (\beta^S)$$

where  $\mathcal{M}_t$  is a martingale, and

$$\begin{split} D_t^{1,M} &= \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi_\eta'(Y_s^M - Y_s'^M) e^{\Gamma_s} \\ & \left| b_M(e^{-\Gamma_s} ((Y_s^M + Z_s) \vee 0), a) - b_M(e^{-\Gamma_s} ((Y_s'^M + Z_s) \vee 0), a) \right| ds \\ & \leq \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi_\eta'(Y_s^M - Y_s'^M) C_{M,1} |Y_s^M - Y_s'^M| ds, \end{split}$$

where  $C_{M,1}$  is the Lipschitz constant for the truncated function  $b_M(\cdot,a)$ . Also,

$$\begin{split} D_t^{2,M} &= 1/2 \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi''_{\eta} (Y_s^M - Y_s'^M) e^{2\Gamma_s (1-\rho)} \\ & \times \left[ ((Y_s^M + Z_s) \vee 0)^{\rho} - ((Y_s'^M + Z_s) \vee 0)^{\rho} \right]^2 ds \\ & \leq 1/2 \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi''_{\eta} (Y_s^M - Y_s'^M) e^{2\Gamma_s (1-\rho)} \\ & \times \left[ ((Y_s^M + Z_s) \vee 0)^{2\rho} - ((Y_s'^M + Z_s) \vee 0)^{2\rho} \right] ds \\ & \leq 1/2 \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi''_{\eta} (Y_s^M - Y_s'^M) e^{2\Gamma_s} \\ & \times \left[ \left( e^{-\Gamma_s} ((Y_s^M + Z_s) \vee 0) \right)^{2\rho} - \left( e^{-\Gamma_s} ((Y_s'^M + Z_s) \vee 0) \right)^{2\rho} \right] ds \\ & \leq 1/2 \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi''_{\eta} (Y_s^M - Y_s'^M) e^{\Gamma_s} C_{M,2} |Y_s^M - Y_s'^M| ds \\ & \leq \frac{C_{M,2}}{\ln \eta^{-1}} \int_0^{t \wedge \tau_M \wedge \tau_M'} e^{\Gamma_s} ds \end{split}$$

for some constant  $K_2$ , where (10) was used. Here  $C_{M,2}$  is the Lipschitz coefficient of the locally Lipschitz function  $f(x) = x^{2\rho}$  for  $|x| \leq M$ . Similarly, using (10) and

Assumption 3.5 we can show

$$\begin{split} D_t^{3,M} &= 1/2 \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi''_{\eta} (Y_s^M - Y_s^{'M}) \sigma_0^2(X_s) \left[ ((Y_s^M + Z_s) \vee 0) - ((Y_s^{'M} + Z_s) \vee 0) \right]^2 ds \\ &\leq 1/2 \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi''_{\eta} (Y_s^M - Y_s^{'M}) \sigma_0^2(X_s) \left[ ((Y_s^M + Z_s) \vee 0)^2 - ((Y_s^{'M} + Z_s) \vee 0)^2 \right] ds \\ &\leq K_3 \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi''_{\eta} (Y_s^M - Y_s^{'M}) \left| Y_s^M - Y_s^{'M} \right| e^{\Gamma_s} \left| e^{-\Gamma_s} (Y_s^M + Z_s) + e^{-\Gamma_s} (Y_s^{'M} + Z_s) \right| ds \\ &\leq K_3 C_{M,3} \int_0^{t \wedge \tau_M \wedge \tau_M'} \psi''_{\eta} (Y_s^M - Y_s^{'M}) \left| Y_s^M - Y_s^{'M} \right| e^{\Gamma_s} ds \\ &\leq \frac{K_3 C_{M,3}}{\ln \eta^{-1}} \int_0^{t \wedge \tau_M \wedge \tau_M'} e^{\Gamma_s} ds \end{split}$$

Therefore, we get that

$$\sup_{t \le T \wedge \tau_M \wedge \tau_M'} \mathbb{E}|Y_t^M - Y_t^{'M}| \le \sqrt{\eta} + (\sigma^2 + \beta^S)^2 T \frac{K_2 C_{M,2} + K_3 C_{M,3}}{\ln \eta^{-1}}$$

$$+ C_{M,1} \int_0^T \sup_{s \le t \wedge \tau_M \wedge \tau_M'} \mathbb{E}|Y_s^M - Y_s^{'M}| dt.$$
(11)

By Gronwall's lemma, we obtain that

$$\sup_{t \leq T \wedge \tau_{M} \wedge \tau'_{M}} \mathbb{E}|Y_{t}^{M} - Y_{t}^{'M}| \leq \left(\sqrt{\eta} + (\sigma^{2} + \beta^{S})^{2} T \frac{K_{2}C_{M,2} + K_{3}C_{M,3}}{\ln \eta^{-1}}\right) \exp\{C_{M,1}T\}.$$

Let  $\eta \downarrow 0$ , we have for any T > 0.

$$\sup_{t \leq T \wedge \tau_{M} \wedge \tau_{M}^{\prime}} \mathbb{E} |Y_{t}^{M} - Y_{t}^{'M}| = 0.$$

That is  $Y_t^M = Y_t^{'M}$  for any  $M \in \mathbb{N}$  and  $t \leq T \wedge \tau_M \wedge \tau_M'$ . Then let  $M \to \infty$  and together with the observation that  $\tau_M, \tau_M^{'}$  increase to infinity almost surely, which follows by Lemma A.2, we obtain uniqueness of the solution  $Y_t$  to the following SDE

$$Y_{t} = \int_{0}^{t} e^{\Gamma_{s}} [b(e^{-\Gamma_{s}}((Y_{s} + Z_{s}) \vee 0), a)] ds + \sigma \int_{0}^{t} e^{\Gamma_{s}(1-\rho)} ((Y_{s} + Z_{s}) \vee 0)^{\rho} dW s$$
$$+ \beta^{S} \int_{0}^{t} \sigma_{0}(X_{s}) ((Y_{s} + Z_{s}) \vee 0) dV_{s}$$

Let us set now  $\bar{Y}_t \stackrel{\text{def}}{=} Y_t + Z_t$ . Then,  $\bar{Y}_t$  satisfies

$$\bar{Y}_t = Z_t + \int_0^t e^{\Gamma_s} [b(e^{-\Gamma_s}(\bar{Y}_s \vee 0), a) + \sigma \int_0^t e^{\Gamma_s(1-\rho)}(\bar{Y}_s \vee 0)^{\rho} dWs + \beta^S \int_0^t \sigma_0(X_s)(\bar{Y}_s \vee 0) dV_s.$$

It is easy to see now that  $\lambda_t \stackrel{\text{def}}{=} e^{-\Gamma_t} \bar{Y}_t$  is the unique solution defined in the lemma 4.1. Next we show that  $\lambda_t \geq 0$ . First, we notice that  $\bar{Y}_0 = Z_0 = \lambda_0 > 0$ .

By Itô formula for the function  $\psi_n(\cdot)$ 

$$\psi_{\eta}(\bar{Y}_{t})\chi_{\mathbb{R}_{-}}(\bar{Y}_{t}) = \psi_{\eta}(\bar{Y}_{0})\chi_{\mathbb{R}_{-}}(\bar{Y}_{0})$$

$$+ \int_{0}^{t} \psi'_{\eta}(\bar{Y}_{s})\chi_{\mathbb{R}_{-}}(\bar{Y}_{s})b(e^{-\Gamma_{s}}(\bar{Y}_{s}\vee 0), a)ds$$

$$+ (1/2)\sigma^{2} \int_{0}^{t} \psi''_{\eta}(\bar{Y}_{s})\chi_{\mathbb{R}_{-}}(\bar{Y}_{s})e^{2\Gamma_{s}(1-\rho)}(\bar{Y}_{s}\vee 0)^{2\rho}ds$$

$$+ 1/2(\beta^{S})^{2} \int_{0}^{t} \psi'_{\eta}(\bar{Y}_{s})\chi_{\mathbb{R}_{-}}(\bar{Y}_{s})\sigma_{0}(X_{s})(\bar{Y}_{s}\vee 0)^{2}ds + \mathcal{M}_{t}$$

where  $\mathcal{M}_t$  is a martingale. Notice that for s>0 at least one of  $\chi_{\mathbb{R}_-}(\bar{Y}_s)$  and  $(\bar{Y}_s \vee 0)$  have to be zero, then taking expectation for both sides:

$$\mathbb{E}[\psi_{\eta}(\bar{Y}_t)\chi_{\mathbb{R}_-}(\bar{Y}_t)] = \mathbb{E}[\int_0^t \psi_{\eta}'(\bar{Y}_s)\chi_{\mathbb{R}_-}(\bar{Y}_s)b(e^{-\Gamma_s}(\bar{Y}_s \vee 0), a)]ds$$

Notice that  $\chi_{\mathbb{R}_-}(\bar{Y}_s)b(e^{-\Gamma_s}(\bar{Y}_s\vee 0),a)$  can only take the nonzero value b(0,a)>0when  $\bar{Y}_s \leq 0$ . Also notice  $\psi'_{\eta}(x)$  takes non-positive values when  $x \leq 0$  and is 0 when  $|x| < \eta$ . Thus, if we let  $\eta \to 0$  the right hand side of the above equation is no greater than zero. On the left hand side, recall that as  $\eta \to 0$ ,  $\psi_{\eta}(x)$  goes to |x|. Therefore, letting  $\eta \to 0$ , we have

$$\mathbb{E}[\bar{Y}_t^-] = \mathbb{E}[|\bar{Y}_t|\chi_{\mathbb{R}_-}(\bar{Y}_t)] \le 0$$

Hence, we get that

$$\mathbb{E}[\bar{Y}_t^-] = 0,$$

i.e.,  $\bar{Y}_t$  is nonnegative and as a consequence  $\lambda_t = e^{-\Gamma_t}\bar{Y}_t$  is also nonnenative. This concludes the proof of the lemma.

Proof of Lemma 4.2. For each  $N \in \mathbb{N}$  and  $n \in \{1, 2, ..., N \}$  define

$$\begin{split} \Gamma_t^{N,n} &= -\beta_{N,n}^S \int_0^t b_0(X_s) ds \\ Z_t^{N,n} &= \lambda_{0,N,n} + \beta_{N,n}^C \cdot \int_0^t e^{\Gamma_s^{N,n}} dL_s^N \end{split}$$

$$Y_t^{N,n} = \int_0^t e^{\Gamma_s^{N,n}} [b(e^{-\Gamma_s^{N,n}} (Y_s^{N,n} + Z_s^{N,n}), a_n)] ds + \sigma^{N,n} \int_0^t e^{\Gamma_s^{N,n} (1-\rho)} (Y_s^{N,n} + Z_s^{N,n})^{\rho} dW_s^n + \beta_{N,n}^S \int_0^t \sigma_0(X_s) (Y_s^{N,n} + Z_s^{N,n}) dV_s.$$

Then  $\lambda_t^{N,n} = e^{-\Gamma_t^{N,n}} (Y_s^{N,n} + Z_t^{N,n})$ . So, we have

$$|\lambda_t^{N,n}|^p \leq \frac{1}{2} \left[ e^{-2p\Gamma_t^{N,n}} + (Y_t^{N,n} + Z_t^{N,n})^{2p} \right]$$

Hence, due to Assumption 3.6, it is enough to show that  $\sup_{t < T} \mathbb{E}|Y_t^{N,n}|$ 

 $Z_t^{N,n}|^{2p} \leq K$  for some appropriate finite constant K. Apply Itô formula to  $|Y_t^{N,n}+Z_t^{N,n}|^{2p}$ . We claim that without loss of generality the martingale terms that appear in the Itô formula can be considered to be true martingales and thus have zero expectation. With this in mind,  $1 - M_t^{N,n}$  $\int_0^t \lambda_s^{N,n} M_s^{N,n} ds \text{ is a martingale and we write } \lambda_t^{N,n} = e^{-\Gamma_t^{N,n}} (Y_s^{N,n} + Z_t^{N,n}).$ 

Then, we can write down

(12)

$$\begin{split} & \mathbb{E}|Y_t^{N,n} + Z_t^{N,n}|^{2p} \\ & = \mathbb{E}\int_0^t 2p|Y_s^{N,n} + Z_s^{N,n}|^{2p-1}e^{\Gamma_s^{N,n}}[b(e^{-\Gamma_s^{N,n}}((Y_s^{N,n} + Z_s^{N,n}) \vee 0), a_n)]ds \\ & + \mathbb{E}\frac{\sigma^{N,n^2}}{2}\int_0^t 2p(2p-1)|Y_s^N + Z_s|^{2p-2}e^{2\Gamma_s^{N,n}(1-\rho)}((Y_s^{N,n} + Z_s^{N,n}) \vee 0)^{2\rho}ds \\ & + \mathbb{E}\frac{(\beta_{N,n}^S)^2}{2}\int_0^t 2p(2p-1)|Y_s^{N,n} + Z_s^{N,n}|^{2p-2}(\sigma_0(X_s))^2(Y_s^{N,n} + Z_s^{N,n}) \vee 0)^2ds \\ & + \mathbb{E}\int_0^t 2p|Y_s^{N,n} + Z_s^{N,n}|^{2p-1}e^{\Gamma_s^{N,n}}\beta_{N,n}^C \cdot \frac{1}{N}\sum_{i=1}^N l^i\left(e^{-\Gamma_s^{N,i}}(Y_s^{N,i} + Z_s^{N,i})M_s^{N,i}\right)ds, \end{split}$$

By Assumption 3.3, we have that there is some K > 0 such that  $\lambda b(\lambda, a) \le -\gamma(a)|\lambda|^d$  for  $|\lambda| \ge K$ . Without loss of generality, we can assume that the dissipativity condition holds everywhere (if not we just consider separately the cases  $|\lambda| < K$  and  $|\lambda| \ge K$ ). Then, we have the estimate

$$(13) \qquad \mathbb{E} \int_{0}^{t} 2p|Y_{s}^{N,n} + Z_{s}^{N,n}|^{2p-1}e^{\Gamma_{s}^{N,n}} [b(e^{-\Gamma_{s}^{N,n}}((Y_{s}^{N,n} + Z_{s}^{N,n}) \vee 0), a_{n})]ds$$

$$\leq -\mathbb{E} \int_{0}^{t} 2p|Y_{s}^{N,n} + Z_{s}^{N,n}|^{2p-2}e^{2\Gamma_{s}^{N,n}}\gamma(a_{n})|e^{-\Gamma_{s}^{N,n}}(Y_{s}^{N,n} + Z_{s}^{N,n})|^{d}ds.$$

$$\leq 0$$

For the second term, we have

$$\begin{aligned} & (14) \qquad & \mathbb{E} \int_{0}^{t} e^{2\Gamma_{s}^{N,n}(1-\rho)} |Y_{s}^{N,n} + Z_{s}^{N,n}|^{2p-2} ((Y_{s}^{N,n} + Z_{s}^{N,n}) \vee 0)^{2\rho} ds \\ & \leq & \mathbb{E} \int_{0}^{t} e^{2\Gamma_{s}^{N,n}(1-\rho)} |Y_{s}^{N,n} + Z_{s}^{N,n}|^{2p-2} |Y_{s}^{N,n} + Z_{s}^{N,n}|^{2\rho} ds \\ & \leq & 2^{2(p-1+\rho)-1} \mathbb{E} \int_{0}^{t} e^{2\Gamma_{s}^{N,n}(1-\rho)} \left( |Y_{s}^{N,n}|^{2p-2+2\rho} + |Z_{s}^{N,n}|^{2p-2+2\rho} \right) ds \\ & \leq & 2^{2(p-1+\rho)-1} \mathbb{E} \int_{0}^{t} \left[ \frac{p-1+\rho}{p} |Y_{s}^{N,n}|^{2p} + \frac{1-\rho}{p} e^{2p\Gamma_{s}^{N,n}} \right] ds \\ & + 2^{2(p-1+\rho)-1} \mathbb{E} \int_{0}^{t} \left[ \frac{1-\rho}{p} e^{2p\Gamma_{s}^{N,n}} + \frac{p-1+\rho}{p} |Z_{s}^{N,n}|^{2p} \right] ds. \end{aligned}$$

The third term is similar with the second term with the help of Assumption 3.5 on the bound for  $\sigma_0$ .

(15) 
$$\mathbb{E} \int_{0}^{t} |Y_{s}^{N,n} + Z_{s}^{N,n}|^{2p-2} (\sigma(X_{s}))^{2} ((Y_{s}^{N,n} + Z_{s}^{N,n}) \vee 0)^{2} ds$$

$$\leq \mathbb{E} \int_{0}^{t} (\sigma(X_{s}))^{2} |Y_{s}^{N,n} + Z_{s}^{N,n}|^{2p} ds$$

$$\leq 2^{2p-1} K_{3.5}^{2} \mathbb{E} \int_{0}^{t} (|Y_{s}^{N,n}|^{2p} + |Z_{s}^{N,n}|^{2p}) ds$$

For the fourth term we apply subsequently Young's inequality, use Assumption 3.6 and we get

$$\mathbb{E} \int_{0}^{t} 2p |Y_{s}^{N,n} + Z_{s}^{N,n}|^{2p-1} e^{\Gamma_{s}^{N,n}} \beta_{N,n}^{C} \cdot \frac{1}{N} \sum_{i=1}^{N} l^{i} \left( e^{-\Gamma_{s}^{N,i}} (Y_{s}^{N,i} + Z_{s}^{N,i}) M_{s}^{N,i} \right) ds$$

$$\leq C_{0} K_{3.2} \mathbb{E} \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} |Y_{s}^{N,n} + Z_{s}^{N,n}|^{2p-1} e^{\Gamma_{s}^{N,n}} \left( e^{-\Gamma_{s}^{N,i}} |Y_{s}^{N,i} + Z_{s}^{N,i}| \right) ds$$

$$\leq C_{1} \left( 1 + \frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \int_{0}^{t} \left[ |Y_{s}^{N,n}|^{2p} + |Z_{s}^{N,n}|^{2p} \right] ds \right)$$

for appropriate constants  $C_0, C_1 < \infty$ .

Notice now that

$$|Y_t^{N,n}|^{2p} = |Y_t^{N,n} + Z_t^{N,n} - Z_t^{N,n}|^{2p}$$

$$\leq 2^{2p-1}|Y_t^{N,n} + Z_t^{N,n}|^{2p} + 2^{2p-1}|Z_t^{N,n}|^{2p}.$$

Next step is to bound (17) using (13), (14), (15), (16). First we average equations (13), (14), (15), (16) over  $n \in \{1, \dots, N\}$  and together with Assumption 3.2, Assumption 3.5, Assumption 3.6, Lemma A.1, we have that there is a constant K such that

$$\frac{1}{N} \sum_{n=1}^N \mathbb{E}[|Y^{N,n}_t|^{2p}] \leq K + K \int_0^t \frac{1}{N} \sum_{n=1}^N \mathbb{E}[|Y^{N,n}_s|^{2p}] ds.$$

By Gronwall lemma, we obtain that

(18) 
$$\sup_{0 \le t \le T} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[|Y_t^{N,n}|^{2p}] \le Ke^{KT}.$$

In addition, notice that using (18) now, (17) together with (13), (14), (15), (16), also gives that for any  $n \in \{1, \dots, N\}$ 

(19) 
$$\sup_{0 \le t \le T} \mathbb{E}[|Y_t^{N,n}|^{2p}] \le K,$$

for an appropriate constant  $K < \infty$  with the upper bounds being independent of N.

Together with Assumption 3.6 and Lemma A.1 we can finally get from (18) the bound advertised in the lemma.

It remains to address the claim on the martingale property of the stochastic integrals. Indeed, using the same truncation argument as in the proof of Lemma 4.1 we get that for each fixed M>0 the terms in question are true martingales. Then, because the corresponding upper bound in (18) turns out to be uniform with respect to M>0 and due to Lemma A.2 the claim is proven, concluding the proof of the lemma.

Proof of Lemma 8.1. As in the proof of Lemma 4.1, if we can prove that the result holds for the truncated processes which has  $b_M$  in place of b, then, due to Lemma A.2, the result will be true for the limit as  $M \to \infty$  as well. Therefore, we can restrict attention to the case where  $b(\lambda, \alpha)$  is replaced by  $b_M(\lambda, \alpha)$  for an arbitrary constant  $M < \infty$ .

In addition, let  $S(\mathbb{R}_+)$  be the set of  $\mathbb{R}_+$  valued, adapted, continuous processes  $\{\lambda_t\}_{t\in[0,T]}$  such that

$$\|\lambda\|_{T,1} = \sup_{0 \le t \le T} \mathbb{E}|\lambda_t| < \infty.$$

The space  $S(\mathbb{R}_+)$  endowed with the norm  $\|\cdot\|_{T,1}$  is a Banach space.

Consider a nonnegative process  $U_t(\hat{p}) \in S(\mathbb{R}_+)$  and set  $\xi(U)_t = (\xi_1(U)_t, \dots, \xi_r(U)_t)$ 

$$\xi_i(U)_t = \int_{\hat{p} \in \mathcal{P}} \left( 1 - \mathbb{E}_{\mathcal{V}_t} \left\{ \exp \left[ - \int_0^t U_s(\hat{p}) ds \right] \right\} \right) \pi(dp) \Lambda_0(d\lambda).$$

So, for given  $U_t(\hat{p}), U_t'(\hat{p}) \in S(\mathbb{R}_+)$ , we have that  $\xi_t = \xi(U)_t = (\xi_1(U)_t, \dots, \xi_r(U)_t)$ ,  $\xi_t' = \xi(U')_t = (\xi_1(U')_t, \dots, \xi_r(U')_t)$ . Then the process  $R_{t \wedge \tau_M \wedge \tau_M'} = \lambda_{t \wedge \tau_M \wedge \tau_M'} - \lambda_{t \wedge \tau_M \wedge \tau_M'}'$  satisfies

$$R_{t \wedge \tau_{M} \wedge \tau'_{M}} = \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} \left( b_{M}(\lambda_{s}, a) - b_{M}(\lambda'_{s}, a) \right) ds + \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} \sigma \left( \lambda_{s}^{\rho} - \lambda'_{s}^{\rho} \right) dW_{s}$$
$$+ \sum_{i=1}^{r} \beta_{i}^{C} \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} \left( d\xi_{s} - d\xi'_{s} \right) + \beta^{S} \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} R_{s} dX_{s}.$$

Apply Itô formula to  $\psi_{\eta}(R_t)$  where  $\psi$  is defined in Equation (9), and get

$$\psi_{\eta}(R_{t \wedge \tau_{M} \wedge \tau'_{M}}) = \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} (b_{M}(\lambda_{s}, a) - b_{M}(\lambda'_{s}, a)) \psi'_{\eta}(R_{s}) ds$$

$$+ \sum_{i=1}^{r} \beta_{i}^{C} \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} \psi'_{\eta}(R_{s}) (d\xi_{s} - d\xi'_{s})$$

$$+ \frac{\sigma^{2}}{2} \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} (\lambda_{s}^{\rho} - \lambda'_{s}^{\rho})^{2} \psi''_{\eta}(R_{s}) ds + \sigma \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} (\lambda_{s}^{\rho} - \lambda'_{s}^{\rho}) \psi'_{\eta}(R_{s}) dW_{s}$$

$$+ \beta^{S} \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} b_{0}(X_{s}) R_{s} \psi'_{\eta}(R_{s}) ds + \beta^{S} \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} \sigma_{0}(X_{s}) R_{s} \psi'_{\eta}(R_{s}) dV s$$

$$+ \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} \frac{1}{2} (\beta^{S} \sigma_{0}(X_{s}) R_{s})^{2} \psi''_{\eta}(R_{s}) ds.$$

Taking expectation of  $\psi_{\eta}(R_t)$  we get

$$\mathbb{E}\psi_{\eta}(R_{t\wedge\tau_{M}\wedge\tau'_{M}}) = \mathbb{E}\int_{0}^{t\wedge\tau_{M}\wedge\tau'_{M}} \left(b_{M}(\lambda_{s},a) - b_{M}(\lambda'_{s},a)\right)\psi'_{\eta}(R_{s})ds$$

$$+ \sum_{i=1}^{r} \beta_{i}^{C} \mathbb{E}\int_{0}^{t\wedge\tau_{M}\wedge\tau'_{M}} \psi'_{\eta}(R_{s})(d\xi_{s} - d\xi'_{s})$$

$$+ \frac{\sigma^{2}}{2} \mathbb{E}\int_{0}^{t\wedge\tau_{M}\wedge\tau'_{M}} \left(\lambda_{s}^{\rho} - {\lambda'_{s}}^{\rho}\right)^{2} \psi''_{\eta}(R_{s})ds + \beta^{S} \mathbb{E}\int_{0}^{t\wedge\tau_{M}\wedge\tau'_{M}} b_{0}(X_{s})R_{s}\psi'_{\eta}(R_{s})ds$$

$$+ \mathbb{E}\int_{0}^{t\wedge\tau_{M}\wedge\tau'_{M}} \frac{1}{2} \left(\beta^{S}\sigma_{0}(X_{s})R_{s}\right)^{2} \psi''_{\eta}(R_{s})ds.$$

As in Lemma A.2. in [17], the latter expression yields

$$\mathbb{E}|\xi_t - \xi_t'| \le K_{3.2}t \int_{\hat{p} \in \mathcal{P}} ||U_{\cdot}(\hat{p}) - U_{\cdot}'(\hat{p})||_t \pi(dp) \Lambda_0(d\lambda).$$

Therefore, we have

$$\left| \sum_{i=1}^{r} \beta_{i}^{C} \mathbb{E} \int_{0}^{t \wedge \tau_{M} \wedge \tau'_{M}} \psi'_{\eta}(R_{s}) (d\xi_{s} - d\xi'_{s}) \right| \leq t C_{0} K_{3.2} \int_{\hat{p} \in \mathcal{P}} ||U_{\cdot}(\hat{p}) - U'_{\cdot}(\hat{p})||_{t} \pi(dp) \Lambda_{0}(d\lambda).$$

At the same time, we have

$$\left| \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} \left( b_M(\lambda_s, a) - b_M(\lambda_s', a) \right) \psi_\eta'(R_s) ds \right| \leq C_{1,M} \int_0^{t \wedge \tau_M \wedge \tau_M'} \mathbb{E} |R_s| ds.$$

For the third term we obtain

$$\left| \frac{\sigma^2}{2} \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} \left( \lambda_s^{\rho} - {\lambda_s'}^{\rho} \right)^2 \psi_{\eta}^{"}(R_s) ds \right| \leq \left| \frac{\sigma^2}{2} \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} \left| \lambda_s^{2\rho} - {\lambda_s'}^{2\rho} \right| \psi_{\eta}^{"}(R_s) ds \right|$$

$$\leq \left| \frac{\sigma^2}{2} \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} C_{2,M} |R_s| \psi_{\eta}^{"}(R_s) ds \right|$$

$$\leq C_{2,M} K_{3,2}^2 \frac{2t}{\ln \eta^{-1}}$$

$$= C_2(\eta, t, M).$$

Now, let us assume  $b_0$  is bounded. Then we have

$$\left| \beta^S \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} b_0(X_s) R_s \psi_\eta'(R_s) ds \right| \le K_{3.2} K \int_0^{t \wedge \tau_M \wedge \tau_M'} \mathbb{E} |R_s| ds.$$

For the last term

$$\begin{split} & \left| \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} \frac{1}{2} \left( \beta^S \sigma_0(X_s) R_s \right)^2 \psi_\eta''(R_s) ds \right| \\ & \leq \frac{1}{2} (\beta^S)^2 \left| \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} (\sigma_0(X_s))^2 (\lambda_s^2 - {\lambda_s'}^2) \psi_\eta''(R_s) ds \right| \\ & \leq \frac{1}{2} (\beta^S)^2 \left| \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} (\sigma_0(X_s))^2 C_{3,M} |R_s| \psi_\eta''(R_s) ds \right| \\ & \leq \frac{1}{2} C_{3,M} K_{3.2}^2 \frac{1}{\ln \eta^{-1}} \mathbb{E} \int_0^{t \wedge \tau_M \wedge \tau_M'} (\sigma_0(X_s))^2 ds \\ & = C_3(\eta, t, M). \end{split}$$

For any  $0 < M < \infty$ , we have that the both terms  $C_2(\eta, t, M)$  and  $C_3(\eta, t, M)$  go to zero as  $\eta \downarrow 0$  or  $t \downarrow 0$ .

Thus, for any  $M < \infty$ , we have

$$\mathbb{E}\psi_{\eta}(R_{t\wedge\tau_{M}\wedge\tau'_{M}}) \leq (C_{1} + KK_{3.2}) \int_{0}^{t\wedge\tau_{M}\wedge\tau'_{M}} \mathbb{E}|R_{s}|ds + tC_{0}K_{3.2} \int_{\hat{p}\in\hat{\mathcal{P}}} ||U_{\cdot}(\hat{p}) - U'_{\cdot}(\hat{p})||_{t}\pi(dp)\Lambda_{0}(d\lambda) + C_{2}(\eta, t, M) + C_{3}(\eta, t, M).$$

Then applying  $|x| \leq \psi_{\eta}(x) + \sqrt{\eta}$  and using Gronwall's Lemma we have

$$\mathbb{E}|R_{t \wedge \tau_{M} \wedge \tau'_{M}}| \leq \left[ tC_{0}K_{3.2} \int_{\hat{p} \in \hat{\mathcal{P}}} ||U_{\cdot}(\hat{p}) - U'_{\cdot}(\hat{p})||_{t} \pi(dp) \Lambda_{0}(d\lambda) + C_{2}(\eta, t, M) + C_{3}(\eta, t, M) + \sqrt{\eta} \right] \cdot e^{(C_{1} + KK_{3.2})t}.$$

Send  $\eta \downarrow 0$  and notice that we can pick t small enough such that  $C(t) = tC_0K_{3,2}e^{(C_1+KK_{3,2})t} < 1$ . Hence, we obtain

$$\mathbb{E}|R_{t\wedge\tau_M\wedge\tau_M'}| \le C(t) \int_{\hat{p}\in\hat{\mathcal{P}}} ||U(\hat{p}) - U'(\hat{p})||_{t,1} \pi(dp) \Lambda_0(d\lambda),$$

where C(t) < 1.

Hence, we have obtained that the map  $\Phi$  defined by  $\lambda = \Phi(U)$  with  $U \in S(\mathbb{R}_+)$  is a contraction on  $S(\mathbb{R}_+)$  equipped with the  $L^1$  norm. Standard Picard iteration shows that there is a fixed point  $\lambda^*$  such that  $\lambda_t^* = \Phi_t(\lambda^*)$  for  $0 \le t \le t_1 \wedge \tau_M \wedge \tau_M'$  with  $C(t_1) < 1$ . This fixed point is unique, since

$$\sup_{t \leq t_1 \wedge \tau_M \wedge \tau_M'} \left| \lambda_t^*(\hat{p}) - \lambda_t^{'*}(\hat{p}) \right| \leq C(t) \int_{\hat{p}} \sup_{t \leq t_1 \wedge \tau_M \wedge \tau_M'} \left| \lambda_t^*(\hat{p}) - \lambda_t^{'*}(\hat{p}) \right| \pi(dp) \Gamma_0(d\lambda)$$

So, we have that

$$\sup_{t \le t_1 \wedge \tau_M \wedge \tau_M'} \left| \lambda_t^*(\hat{p}) - \lambda_t'^*(\hat{p}) \right| = 0$$

Thus, we have proven uniqueness of  $\lambda_t^*$  on  $[0, t_1 \wedge \tau_M \wedge \tau_M']$ . Then, starting from  $t_1$  we obtain uniqueness on  $[t_1 \wedge \tau_M \wedge \tau_M', (2t_1) \wedge \tau_M \wedge \tau_M']$  in the same way and we conclude by filling in the whole interval  $[0, T \wedge \tau_M \wedge \tau_M']$ .

Next, letting  $M \to \infty$ , and using Lemma A.2 which implies that  $\tau_M, \tau_M'$  converge to infinity almost surely, we have the proof of the lemma for bounded  $b_0$ .

For the case of general  $b_0$ , Assumption 3.7 guarantees that

$$M_T = e^{-\int_0^T u(X_s)dV_s - 1/2\int_0^T |u(X_s)|^2 ds},$$

is a martingale by Novikov's condition. Assumption 3.7 also assumes  $\mathbb{E}|M_T|^p < \infty$ . Then the result follows from the proof of Lemma A.6. in [17].

**Lemma A.2.** For any T > 0 and for  $\tau_M$  defined via (8), we have that

$$\lim_{M \to \infty} \mathbb{P}[\tau_M < T] = 0.$$

Proof of Lemma A.2. For any T > 0,

$$\mathbb{P}[\tau_M < T] \le \frac{1}{M} \mathbb{E}\left[ \sup_{t \wedge \tau_M < T} |e^{-\Gamma_t} ((Y_t^M + Z_t) \vee 0)| \right].$$

Due to Assumption 3.6 and Lemma A.1, it is enough to prove that

$$\mathbb{E} \sup_{t \wedge \tau_M \le T} |Y_t^M + Z_t|^2 \le \tilde{K}$$

where  $\tilde{K}$  is independent of M. Now  $\sup_{t \leq T} |Y_t^M + Z_t|^2$  can be estimated similarly as before. Indeed, applying Itô formula to  $|Y_t^M + Z_t|^2$ , we get

$$\begin{split} |Y_t^M + Z_t|^2 &= \lambda_0 + \int_0^t 2|Y_s^M + Z_s| \ e^{\Gamma_s} [b(e^{-\Gamma_s}((Y_s^M + Z_s) \vee 0), a)] ds \\ &+ \frac{1}{2}\sigma^2 \int_0^t 2e^{2\Gamma_s(1-\rho)}((Y_s^M + Z_s) \vee 0)^{2\rho} ds \\ &+ \frac{1}{2}(\beta^S)^2 \int_0^t 2(\sigma_0(X_s))^2((Y_s^M + Z_s) \vee 0)^2 ds \\ &+ \sigma \int_0^t 2|Y_s^M + Z_s| \ e^{\Gamma_s(1-\rho)}((Y_s^M + Z_s) \vee 0)^\rho dW_s \\ &+ \beta^S \int_0^t 2|Y_s^M + Z_s| \ \sigma_0(X_s)((Y_s^M + Z_s) \vee 0) dV_s \\ &+ \int_0^t 2|Y_s^M + Z_s| \ e^{\Gamma_s}\beta^C \cdot d\xi_s \end{split}$$

The first line in the right hand side of the expression above is bounded due to Assumption 3.3 (similarly to (13) we can assume without loss of generality that the dissipativity condition holds everywhere) and we have

$$\lambda_{0} + \int_{0}^{t} 2|Y_{s}^{M} + Z_{s}| e^{\Gamma_{s}} [b(e^{-\Gamma_{s}}((Y_{s}^{M} + Z_{s}) \vee 0), a)] ds$$

$$\leq \lambda_{0} - \int_{0}^{t} 2 e^{2\Gamma_{s}} \gamma(a) |e^{-\Gamma_{s}}(Y_{s}^{M} + Z_{s})|^{d} ds \leq \lambda_{0}$$

Therefore, if we square both sides in the Itô's formula expression, we will get

$$\begin{split} &|Y_t^M + Z_t|^4 \\ & \leq 6\lambda_0^2 + 6\sigma^4 \left[ \int_0^t e^{2\Gamma_s(1-\rho)} |Y_s^M + Z_s|^{2\rho} ds \right]^2 \\ & + 6(\beta^S)^4 \left[ \int_0^t (\sigma_0(X_s))^2 |Y_s^M + Z_s|^2 ds \right]^2 \\ & + 24\sigma^2 \left[ \int_0^t |Y_s^M + Z_s| \ e^{\Gamma_s(1-\rho)} \ |Y_s^M + Z_s|^\rho dW_s \right]^2 \\ & + 24(\beta^S)^2 \left[ \int_0^t |Y_s^M + Z_s| \ \sigma_0(X_s) \ |Y_s^M + Z_s| dV_s \right]^2 \\ & + 24 \left[ \int_0^t |Y_s^M + Z_s| \ e^{\Gamma_s} \beta^C \cdot d\xi_s \right]^2 \end{split}$$

Taking expectation of the supremum of the second term and using Hölder inequality, together with the fact that  $\rho < 1$  and Assumption 3.6 we have

$$\mathbb{E} \sup_{t \wedge \tau_M \le T} \left[ \int_0^t e^{2\Gamma_s(1-\rho)} |(Y_s^M + Z_s)|^{2\rho} ds \right]^2$$

$$\le \mathbb{E} \left[ \int_0^T e^{2\Gamma_s(1-\rho)} |Y_s^M + Z_s|^{2\rho} ds \right]^2$$

$$\le \mathbb{E} \left[ \int_0^T |Y_s^M + Z_s|^{4\rho} ds \int_0^T e^{4\Gamma_s(1-\rho)} ds \right]$$

$$\le c_{p_1} \mathbb{E} \int_0^T \sup_{u \wedge \tau_M \le s} |Y_u^M + Z_u|^4 ds,$$

for some constant  $c_{p_1} > 0$ . Similar calculations together with Assumption 3.5, gives a similar bound for the third term as well. Using Burkholder-Davis-Gundy inequality for the fourth term, together with Young's inequality, the fact that  $\rho < 1$  and Assumption 3.6 we get

$$\mathbb{E}\left\{ \sup_{t \wedge \tau_{M} \leq T} \left[ \int_{0}^{t} |Y_{s}^{M} + Z_{s}| \ e^{\Gamma_{s}(1-\rho)} \ |Y_{s}^{M} + Z_{s}|^{\rho} dW_{s} \right]^{2} \right\} \\
\leq c_{p_{2}} \mathbb{E} \int_{0}^{T} |Y_{s}^{M} + Z_{s}|^{2(1+\rho)} \ e^{2\Gamma_{s}(1-\rho)} ds \\
\leq c_{p_{3}} + c_{p_{4}} \mathbb{E} \int_{0}^{T} \sup_{u \wedge \tau_{M} \leq s} |Y_{u}^{M} + Z_{u}|^{4} ds$$

where  $c_{p_2}$ ,  $c_{p_3}$  and  $c_{p_4}$  are some positive constants. The stochastic integral term with respect to the V- Brownian motion is treated analogously using Assumption 3.5. For the last term, we use Young's inequality and Assumptions 3.2 and 3.5. We obtain

$$\mathbb{E} \sup_{t \wedge \tau_{M} \leq T} \left[ 24 \int_{0}^{t} |Y_{s}^{M} + Z_{s}| \ e^{\Gamma_{s}} \beta^{C} \cdot d\xi_{s} \right]^{2}$$

$$\leq c_{p_{5}} \mathbb{E} \left[ \sup_{t \wedge \tau_{M} \leq T} \int_{0}^{t} |Y_{s}^{M} + Z_{s}| \ e^{\Gamma_{s}} \beta^{C} \cdot d\xi_{s} \right]^{2}$$

$$\leq c_{p_{5}} \mathbb{E} \left[ \sup_{t \wedge \tau_{M} \leq T} |Y_{t}^{M} + Z_{t}| \sup_{t \wedge \tau_{M} < T} \int_{0}^{t} e^{\Gamma_{s}} \beta^{C} \cdot d\xi_{s} \right]^{2}$$

$$\leq c_{p_{5}} \mathbb{E} \left[ \frac{\epsilon}{2} \sup_{t \wedge \tau_{M} \leq T} |Y_{t}^{M} + Z_{t}|^{4} + \frac{1}{2\epsilon} \left[ \sup_{t \wedge \tau_{M} \leq T} \int_{0}^{t} e^{\Gamma_{s}} \beta^{C} \cdot d\xi_{s} \right]^{4} \right]$$

$$\leq c_{p_{5}} \epsilon \mathbb{E} \sup_{t \wedge \tau_{M} \leq T} |Y_{t}^{M} + Z_{t}|^{4} + c_{p_{\epsilon}}$$

for any  $\epsilon>0$  and correspondent constant  $c_{p_{\epsilon}}>0$ . Therefore we can choose  $\epsilon$  small enough so  $c_{p_5}\epsilon<1$  and we can move this term to the left hand side.

Thus, combining all the terms together with Assumption 3.2 leads to the estimate

$$\mathbb{E} \sup_{t \wedge \tau_M \le T} |Y_t^M + Z_t|^4 \le K \left( 1 + \int_0^T \mathbb{E} \sup_{u \wedge \tau_M \le s} |Y_u^M + Z_u|^4 ds \right).$$

Then by Gronwall lemma, the term  $\mathbb{E}\sup_{t\wedge \tau_M\leq T}|Y_t^M+Z_t|^4$  is bounded by a constant which is independent of M and we conclude by Fatou's lemma followed Jensen's inequality.

## References

- [1] F. Allen and D. Gale. Financial Contagion. Journal Political Economy 108, (2000), pp. 133.
- [2] Shahriar Azizpour, Kay Giesecke, and Gustavo Schwenkler. Exploring the sources of default clustering. Journal of Financial Economics, 129(1), (2018), pp. 154-183.
- [3] Y. Ait-Sahalia, J. Cacho-Diaz, and R. Laeven. Modeling financial contagion using mutually exciting jump processes. Journal of Financial Economics Vol. 117, Issue 3, (2015), pp. 585-606.
- [4] M.K. Brunnermeier, G. Gorton, and A. Krishnamurthy. Risk Topography. NBER Macroeconomics Annual, Vol. 26, (2012), pp. 149176.
- [5] Lijun Bo and Agostino Capponi. Bilateral credit valuation adjustment for large credit derivatives portfolios. Finance and Stochastics, Vol. 18, Issue 2, (2014), pp 431482.
- [6] Lijun Bo and Agostino Capponi. Systemic Risk in Interbanking Networks. SIAM Journal of Financial Mathematics, 6, (2015), pp. 386424.
- [7] Nick Bush, Ben Hambly, Helen Haworth, Lei Jin, and Christoph Reisinger. Stochastic evolution equations in portfolio credit modelling. SIAM Journal on Financial Mathematics, 2, (2011), pp. 627–664.
- [8] Jakša Cvitanić, Jin Ma, and Jianfeng Zhang. The law of large numbers for self-exciting correlated defaults. Stochastic Processes and Their Applications, 122(8), (2012), pp. 2781– 2810
- [9] Paolo Dai Pra, Wolfgang Runggaldier, Elena Sartori, and Marco Tolotti. Large portfolio losses: A dynamic contagion model. The Annals of Applied Probability, 19, (2009), pp. 347– 394.
- [10] Paolo Dai Pra and Marco Tolotti. Heterogeneous credit portfolios and the dynamics of the aggregate losses. Stochastic Processes and their Applications, 119, (2009), pp. 2913–2944.
- [11] Ben Vraig and Goetz von Peter Interbank tiering and money center banks. Journal of Financial Intermediation, Vol. 23, issue 3, (2014), pp. 322-347.
- [12] Stewart N. Ethier and Thomas G. Kurtz. Markov processes: Characterization and Convergence. John Wiley & Sons Inc., New York. 1986.
- [13] L. Eisenberg and T.H. Noe. Systemic risk in nancial systems. Management Science 47, (2001), pp. 236249.
- [14] Jean-Pierre Fouque and Tomoyuki Ichiba. Stability in a model of inter-bank lending. SIAM Journal on Financial Mathematics, 4, (2013), pp. 784–803.
- [15] Josselin Garnier, George Papanicolaou, and Tzu-Wei Yang. Large deviations for a mean field model of systemic risk. SIAM Journal on Financial Mathematics, 4, (2012), pp. 151–184.
- [16] Kay Giesecke, Konstantinos Spiliopoulos, and Richard Sowers. Default clustering in large portfolios: Typical events. The Annals of Applied Probability, 23(1), (2013), pp. 348–385.
- [17] Kay Giesecke, Konstantinos Spiliopoulos, Richard Sowers, and Justin A. Sirignano. Large portfolio asymptotics for loss from default. *Mathematical Finance*, Vol. 25, No. 1, (2015), pp. 77114.
- [18] Kay Giesecke and Stefan Weber. Credit contagion and aggregate losses. Journal of Economic Dynamics and Control, 30:741–767, 2006.
- [19] Paul Glasserman and H.P. Young. Contagion in Financial Networks. *Journal of Economic Literature*, 54, (2016), pp. 779831.
- [20] B.M. Hambly and Andreas Sojmark. An SPDE model for systemic risk with endogenous contagion. arXiv: 1801.10088, 2018.
- [21] Daniel A. Hojman and Adam Szeidl. Core and periphery in networks. Journal of Economic Theory, 139, (2008), pp. 295-309.
- [22] Andre Lucas, Pieter Klaassen, Peter Spreij, and Stefan Straetmans. An analytic approach to credit risk of large corporate bond and loan portfolios. *Journal of Banking and Finance*, 25, (2001), 1635–1664.
- [23] C. Meinerding. Asset allocation and asset pricing in the face of systemic risk: a literature overview and assessment. *International Journal of Theoretical and Applied Finance (IJTAF)*, 15(03), (2012), 1250023–1–1250023–27.

- [24] L.H. Pedersen. When everyone runs for the exit. International Journal of Central Banking, Vol. 5, (2009), pp. 177199.
- [25] Konstantinos Spiliopoulos Systemic Risk and Default Clustering for Large Financial Systems. In: Friz, P., Gatheral, J., Gulisashvili, A., Jacqier, A., Teichmann, J. (eds.) Large Deviations and Asymptotic Methods in Finance, Springer, (2015), pp. 529557.
- [26] Konstantinos Spiliopoulos and Richard Sowers. Default clustering in large pools: Large deviations. SIAM Journal on Financial Mathematics, Vol. 6, (2015), pp. 86116.
- [27] Konstantinos Spiliopoulos, Justin A. Sirignano, and Kay Giesecke. Fluctuation analysis for the loss from default. Stochastic Processes and their Applications, 124(7), (2014), pp. 2322-2362.
- [28] L.C.G.Rogers, D Williams. Diffusions, Markov Processes and Martingales (2000). Cambridge University Press.
- [29] Ioannis Karatzas, Steven E. Shreve, Brownian Motion and Stochastic Calculus(1991), 2nd ed. Graduate Texts in Mathematics. Springer, New York.

DEPARTMENT OF MATHEMATICS AND STATISTICS, BOSTON UNIVERSITY, BOSTON, MA 02215 E-mail address, Konstantinos Spiliopoulos: kspiliop@math.bu.edu E-mail address, Jia Yang: jiayang@bu.edu