

# NORMS, KERNELS AND EIGENVALUES OF SOME INFINITE GRAPHS

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## 1. INTRODUCTION

The study of infinite graphs, and in particular of their spectral properties, is relatively new. Perhaps one of the first studies on the subject was the 1982 paper by Mohar [13], which examined the basic properties of the spectrum of the adjacency matrix of an infinite graph. It gave norm estimates and described other functional analytic properties. In [15], Mohar and Woess give a survey of the results known up to 1998. It seems that many results about the spectrum of infinite graphs were proven in the context of harmonic analysis on graphs and on discrete groups, and can be obtained as corollaries of those results, especially in the case of regular graphs and regular trees.

Nevertheless, the study of the spectra of infinite graphs has only recently been studied in greater depth, perhaps motivated by the many results that have been obtained for the spectrum of the Laplacian of an infinite graph. For example in [7] Golinskii computes the spectra in  $L^2$  of several graphs which are formed by attaching an “infinite ray” to a finite graph. Further results on the spectrum of infinite graphs can be found in [1, 5, 8, 9, 10], to cite just a few.

In the present paper, we deal with the adjacency matrix of an infinite, locally finite graph. We consider this matrix as an operator, which we call the *shift operator*, defined on the set of  $p$ -summable functions themselves defined on the vertices of the graph.

We first obtain some results about the norm of the shift operator: we show that the shift on the  $L^p$  space of a graph  $G$  is bounded if and only if  $G$  has bounded degree. We show the norm is bounded by the maximum degree (Theorem 3.1); this result is well-known, but we include a proof since it seems that only the case  $p = 2$  is available in the literature. Most importantly, we show that for a class of graphs, the norm is exactly the maximum degree (Theorem 3.3). This improves an (unmentioned) corollary of Theorem 2.1 in [1], where it is shown that, under stronger hypotheses than ours, the spectral radius in  $L^2$  is the maximum degree (from this, it follows that the norm equals the maximum degree). We also obtain a norm estimate for trees in Theorem 3.4; a special case of this theorem appears to have been noticed in the literature but the proof for the case  $p \neq 2$  does not seem to be available elsewhere. We show (Theorem 3.5) that this estimate is sharp if the tree is “almost regular”, which does not seem to have been observed elsewhere.

We then move to study the triviality of the kernel of the shift operator on the  $L^p$  space of a tree. This question looks to have been overlooked in the literature (a related result can be found in [3]). We show (Theorem 4.3) that the kernel of the shift is trivial if some conditions on  $p$ , on the “essential” maximum degree, and on the “essential minimum” degree hold; and we show that these conditions cannot be improved (Example 4.4). We also show, in Theorem 4.5, that if some conditions hold, then the kernel of the shift is nontrivial, and we give an example (Example 4.6) which shows that, in some cases, these conditions cannot be improved. We leave open the question of whether these conditions can be improved in the rest of the cases.

Lastly, in Section 5 we describe an elementary method to find the eigenvalues of some graphs obtained by attaching a ray to a finite graph. Results similar to ours have been obtained in [7], but we believe our method is more elementary. We give several examples of this method: the kite with an infinite tail, the fly-swatter with an infinite tail, and the comb with an infinite tail. We also show that finding these eigenvalues actually gives the full spectrum of the shift, and finish by finding the spectrum of the infinite comb.

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## 2. PRELIMINARIES

Let us set the notation that we will use throughout this article. Recall that a *graph*  $G$  is a pair  $(V, E)$ , where  $V$  is a nonempty set, called the set of *vertices* of the graph, and  $E$  is a collection of subsets of  $V$  of cardinality two. The set  $E$  is called the set of *edges* of the graph. If  $\{u, v\} \in E$  we will say that  $u$  and  $v$  are *adjacent* and denote this relation by  $u \sim v$ . A *path of length*  $n$  between two distinct vertices  $u$  and  $v$  is a finite sequence  $\{u = u_0, u_1, u_2, \dots, u_{n-1}, u_n = v\}$  of distinct vertices such that  $u_{j-1} \sim u_j$  for  $j = 1, 2, \dots, n$ . We say that the graph is *connected* if there exists a path between any pair of distinct vertices. All of the graphs we will consider here are assumed to be connected. Clearly there exists a path of minimum length between any two vertices  $u$  and  $v$  and we denote the length of such path by  $d(u, v)$ , which we define to be zero if  $u = v$ . It is clear that  $d$  defines a metric on the set of vertices of the graph.

Here, we will deal mostly with infinite graphs. A graph is *infinite* if the set  $V$  is countably infinite. All of the graphs here will be *locally finite*, which means that the set  $\{v \in V : u \sim v\}$  is finite for every  $u \in V$ . The *degree* of  $u$  is the cardinality of said set, which we denote by  $\deg(u)$ . We will say that a locally finite graph has *bounded degree* if the set  $\{\deg(v) : v \in V\}$  is bounded.

All of our graphs will have a distinguished vertex, which we will usually denote by the letter  $o$ . We define  $|v| := d(o, v)$ . We denote by  $\gamma(n)$  the cardinality of the set  $\{v \in V : |v| = n\}$ .

We say that a graph  $T$  is a *tree* if for every pair of distinct vertices there is only one path between them. The distinguished vertex of a tree is called the *root* of the tree and we refer to  $T$  as a *rooted tree*. For every vertex  $v \neq o$  in a tree, there exists a unique vertex  $w$  such that  $d(v, w) = 1$  and  $w$  is in the path from  $o$  to  $v$ . We call such a vertex the *parent* of  $v$  and denote it by  $\text{par}(v)$ . We define  $\text{par}^n(v)$  inductively: obviously  $\text{par}^1(v) := \text{par}(v)$  for every vertex  $|v| \geq 1$  and for  $n \in \mathbb{N}$ , with  $n \geq 2$  we define  $\text{par}^n(v) := \text{par}(\text{par}^{n-1}(v))$  for every vertex  $|v| \geq n$ . For every  $v \in V$ , the set  $\{u \in V : u \sim v, u \neq \text{par}(v)\}$  is called the set of *children* of  $v$  and is denoted by  $\text{Chi}(v)$ . Also, for every  $v \in V$  and  $n \in \mathbb{N}$  with  $n \geq 2$ , we set

$$\text{Chi}^n(v) := \{u \in V : \text{par}^n(u) = v\}.$$

If  $\text{Chi}(v)$  is empty for some  $v \in V$ , we say that  $v$  is a *leaf* of  $T$ . If  $T$  has no leaves, we say the tree is leafless.

We denote the vector space of all functions  $f : V \rightarrow \mathbb{C}$  as  $\mathcal{F}$ . As is customary, we denote by  $\mathbf{L}^p(G)$  the set

$$\{f \in \mathcal{F} : \|f\|_p := \left( \sum_{v \in V} |f(v)|^p \right)^{1/p} < \infty\}$$

if  $1 \leq p < \infty$ , and by  $\mathbf{L}^\infty(G)$  the set

$$\{f \in \mathcal{F} : \|f\|_\infty := \sup_{v \in V} |f(v)| < \infty\}.$$

Clearly  $\mathbf{L}^p(G)$  is a Banach space, since it is isomorphic to  $\ell^p(V)$ .

Our main object of study is the *shift operator*  $S$ . This is defined on  $\mathcal{F}$  as

$$(Sf)(u) = \sum_{v \sim u} f(v).$$

If the graph is finite, the matrix of  $S$  with respect to the canonical basis of the vector space  $\mathcal{F}$  is the well-known and much-studied *adjacency* matrix of the graph.

We denote by  $\chi_A$  the characteristic function of the set  $A \subset V$ . Clearly  $\chi_A$  is in  $\mathbf{L}^\infty(G)$  but, for  $1 \leq p < \infty$ , the function  $\chi_A$  is in  $\mathbf{L}^p(G)$  if and only if  $A$  is finite. If  $A$  is a singleton  $\{v\}$  we write  $\chi_v := \chi_A$ .

Lastly, we will use the following version of Jensen's inequality. If  $\{a_1, a_2, \dots, a_n\}$  are nonnegative numbers, then

$$\left( \frac{a_1 + a_2 + \dots + a_n}{n} \right)^p \leq \frac{a_1^p + a_2^p + \dots + a_n^p}{n}$$

for  $1 \leq p < \infty$ . We will also use Young's inequality: if  $a, b \geq 0$  then  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$  for  $1 < p < \infty$  and  $q = \frac{p}{p-1}$ . Lastly, we will use the following inequality, which follows from the convexity of the function  $t \mapsto t^p$  (for  $1 \leq p < \infty$ ) and a trivial induction argument: if  $\{a_1, a_2, \dots, a_n\}$  are nonnegative numbers, then

$$(a_1 + a_2 + \dots + a_n)^p \geq a_1^p + a_2^p + \dots + a_n^p$$

for  $1 \leq p < \infty$ .

From now on every graph is assumed to be locally finite and connected.

## 3. BOUNDEDNESS

In this section we study the norm of the shift. This has been studied for the case  $p = 2$  in several places (see for example, Theorem 3.2 in [13] or Theorem 3.1 in [15]). The following result, for arbitrary  $p \geq 1$ , is probably well-known, but we include here the proof for completeness since we have not found it elsewhere.

**Theorem 3.1.** *Let  $G = (V, E)$  be a graph and let  $1 \leq p \leq \infty$ . Then  $S$  is bounded in  $\mathbf{L}^p(G)$  if and only if  $G$  has bounded degree. In this case,*

$$\|S\| \leq \max\{\deg(v) : v \in V\}.$$

*Proof.* First, assume that  $G$  has bounded degree and let  $M := \max\{\deg(v) : v \in V\}$ .

- (1) Suppose  $1 \leq p < \infty$ . Then, for every  $f \in \mathbf{L}^p(G)$

$$\begin{aligned} \|Sf\|_p^p &= \sum_{v \in V} \left| \sum_{u \sim v} f(u) \right|^p \leq \sum_{v \in V} \left( \sum_{u \sim v} |f(u)| \right)^p \\ &\leq \sum_{v \in V} \deg(v)^{p-1} \sum_{u \sim v} |f(u)|^p \quad (\text{by Jensen's inequality}) \\ &\leq M^{p-1} \sum_{v \in V} \sum_{u \sim v} |f(u)|^p \\ &\leq M^p \|f\|_p^p. \end{aligned}$$

Hence  $\|Sf\|_p^p \leq M^p \|f\|_p^p$  for all  $f \in \mathbf{L}^p(G)$  which implies that  $S$  is a bounded operator from  $\mathbf{L}^p(G)$  into  $\mathbf{L}^p(G)$  and  $\|S\| \leq M$ .

- (2) Suppose  $p = \infty$ . Let  $f \in \mathbf{L}^\infty(G)$ . For every  $v \in V$  we have

$$|Sf(v)| = \sum_{u \sim v} |f(u)| \leq M \|f\|_\infty.$$

Taking the supremum over all  $v \in V$  we obtain  $\|Sf\|_\infty \leq M \|f\|_\infty$  and hence  $S$  is a bounded operator from  $\mathbf{L}^p(G)$  into  $\mathbf{L}^p(G)$  and  $\|S\| \leq M$ .

Now assume  $G$  has unbounded degree.

- (1) Suppose  $1 \leq p < \infty$ . Since  $G$  has unbounded degree, there exists a sequence of vertices  $\{v_n\}$  such that  $\deg(v_n) \geq 2^{np}$ .

Let  $f \in \mathcal{F}$  be a function such that for all  $v \in V$ ,

$$f(v) = \begin{cases} 2^{-n}, & \text{if } v = v_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|f\|_p^p = \sum_{v \in V} |f(v)|^p = \sum_{n=1}^{\infty} |2^{-n}|^p < \infty$$

so  $f \in \mathbf{L}^p(G)$ . But

$$\|Sf\|_p^p = \sum_{v \in V} \left( \sum_{u \sim v} f(u) \right)^p \geq \sum_{v \in V} \sum_{u \sim v} (f(u))^p = \sum_{n=1}^{\infty} \deg(v_n) (f(v_n))^p \geq \sum_{n=1}^{\infty} 2^{np} \left( \frac{1}{2^n} \right)^p = \infty.$$

Hence  $S$  is unbounded.

- (2) Now suppose  $p = \infty$ . Let  $f \in \mathcal{F}$  such that  $f(v) = 1$  for all  $v \in V$ . Then  $\|f\|_\infty = 1$ , so  $f \in \mathbf{L}^\infty(G)$ . But

$$\|Sf\| = \sup_{v \in V} \left| \sum_{u \sim v} f(u) \right| = \sup_{v \in V} |\deg(v)| = \infty,$$

since the degree of  $G$  is unbounded. Hence  $S$  is unbounded.  $\square$

In addition to the last theorem, observe that in the case  $p = 1$  and  $p = \infty$ , we have  $\|S\| = \max\{\deg(v) : v \in V\}$ . Indeed, let  $v$  be a vertex with  $k := \deg(v) = \max\{\deg(u) : u \in V\}$ . If  $p = 1$ , then  $\|\chi_v\|_1 = 1$  and  $\|S\chi_v\|_1 = k$ , which shows that  $\|S\| = k$ . For  $p = \infty$ , let  $A = \{u \in V : u \sim v\}$ . Then  $\|\chi_A\|_\infty = 1$  and  $\|S\chi_A\|_\infty = k$ , which shows that  $\|S\| = k$ .

The natural question is whether the bound for the norm of the shift found above is optimal also for the cases  $1 < p < \infty$ . We now introduce several definitions which allow us to introduce a class of graphs for

which the bound will be optimal. The name on the second definition is based on the stronger definition in [1].

**Definition 3.2.** Let  $G = (V, E)$  be an infinite graph.

- We say  $G$  is *almost  $k$ -regular* if  $\deg(v) \leq k$  for all  $v \in V$  and there exists a finite set of vertices  $V_0$  such that  $\deg(v) = k$  for all  $v \in V \setminus V_0$ .
- For the following definition, recall that, for  $n \in \mathbb{N}$ ,  $\gamma(n)$  is defined to be the number of vertices of distance  $n$  from the distinguished vertex. We say  $G$  is  *$k$ -almost-euclidean* if the graph is almost  $k$ -regular, if  $\gamma$  is a nondecreasing function of  $n$  and if

$$\lim_{n \rightarrow \infty} \frac{\gamma(n+1)}{\gamma(1) + \gamma(2) + \cdots + \gamma(n)} = 0.$$

If  $G$  is  $k$ -almost-euclidean, observe that, since  $\gamma$  is nondecreasing we have that for fixed  $m \in \mathbb{N}_0$  with  $m \leq n$  that

$$\lim_{n \rightarrow \infty} \frac{\gamma(m)}{\gamma(0) + \gamma(1) + \gamma(2) + \cdots + \gamma(n)} = 0,$$

and for  $m < n$  that

$$\lim_{n \rightarrow \infty} \frac{\gamma(m) + \gamma(m+1) + \cdots + \gamma(n-1)}{\gamma(0) + \gamma(1) + \gamma(2) + \cdots + \gamma(n)} = 1.$$

We can now prove that for a certain class of graphs, the norm of the shift equals the maximum degree of the graph.

**Theorem 3.3.** Let  $G = (V, E)$  be a graph, let  $1 \leq p < \infty$  and let  $S : \mathbf{L}^p(G) \rightarrow \mathbf{L}^p(G)$  be the shift operator. If  $G$  is  $k$ -almost-euclidean, then  $\|S\| = k$ .

*Proof.* Let  $o$  be the distinguished vertex of  $G$ . For every  $n \in \mathbb{N}$ , define  $f_n : V \rightarrow \mathbb{C}$  as  $f_n(v) = 1$  if  $|v| \leq n$  and  $f_n(v) = 0$  otherwise. Choose  $m \in \mathbb{N}$  large enough such that if  $|v| \geq m-1$ , then  $\deg(v) = k$ .

First, observe that if  $m \leq |v| < n$  then  $(Sf_n)(v) = k$ , and if  $|v| > n+1$  then  $(Sf_n)(v) = 0$ .

It then follows that, for  $n > m$ ,

$$\|Sf_n\|_p^p = \sum_{v \in V} |(Sf_n)(v)|^p = \sum_{|v| < m} |(Sf_n)(v)|^p + \sum_{m \leq |v| < n} |(Sf_n)(v)|^p + \sum_{|v|=n} |(Sf_n)(v)|^p + \sum_{|v|=n+1} |(Sf_n)(v)|^p.$$

We have

$$\sum_{m \leq |v| < n} |(Sf_n)(v)|^p = \sum_{m \leq |v| < n} |k|^p = k^p(\gamma(m) + \gamma(m+1) + \cdots + \gamma(n-1)).$$

We also obtain the inequalities

$$\sum_{|v|=n} |(Sf_n)(v)|^p + \sum_{|v|=n+1} |(Sf_n)(v)|^p \leq \sum_{|v|=n} |k|^p + \sum_{|v|=n+1} |k|^p = k^p(\gamma(n) + \gamma(n+1)).$$

and

$$\sum_{|v| < m} |(Sf_n)(v)|^p \leq \sum_{|v| < m} |k|^p = k^p(\gamma(0) + \gamma(1) + \gamma(2) + \cdots + \gamma(m-1)).$$

Since  $\|f_n\|_p^p = \gamma(0) + \gamma(1) + \gamma(2) + \cdots + \gamma(n)$ , we have that

$$\lim_{n \rightarrow \infty} \frac{\sum_{m \leq |v| < n} |(Sf_n)(v)|^p}{\|f_n\|_p^p} = \lim_{n \rightarrow \infty} \frac{k^p(\gamma(m) + \gamma(m+1) + \cdots + \gamma(n-1))}{\gamma(0) + \gamma(1) + \gamma(2) + \cdots + \gamma(n)} = k^p.$$

Also, since the graph is  $k$ -almost-euclidean we have

$$0 \leq \lim_{n \rightarrow \infty} \frac{\sum_{|v| \leq m} |(Sf_n)(v)|^p}{\|f_n\|_p^p} \leq \lim_{n \rightarrow \infty} \frac{k^p(\gamma(0) + \gamma(1) + \gamma(2) + \cdots + \gamma(m))}{\gamma(0) + \gamma(1) + \gamma(2) + \cdots + \gamma(n)} = 0,$$

Using the same argument, we have

$$\lim_{n \rightarrow \infty} \frac{\sum_{|v|=n} |(Sf_n)(v)|^p + \sum_{|v|=n+1} |(Sf_n)(v)|^p}{\|f_n\|_p^p} = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{\|Sf_n\|_p^p}{\|f_n\|_p^p} = k^p,$$

and therefore  $\|S\| \geq k$ . By the previous theorem, we obtain the equality.  $\square$

Examples of  $k$ -almost-euclidean graphs include the lattice  $\mathbb{Z}^d$  (which is  $2d$ -almost-euclidean), a triangular tessellation of the plane (which is 6-almost-euclidean), the hexagonal tessellation of the plane (which is 3-almost-euclidean), and the graphs obtained by removing finitely many edges from these graphs (while keeping them connected).

There is a class of graphs for which there is a smaller bound for the norm: the trees. The following theorem is well-known for  $p = 2$ , but the case  $p \neq 2$  seems to have gone unnoticed. First recall that if  $1 < p < \infty$  and  $q := \frac{p}{p-1}$ , the duality between  $\mathbf{L}^p(G)$  and  $\mathbf{L}^q(G)$  is realized by the pairing

$$\langle f, g \rangle = \sum_{v \in V} f(v)g(v),$$

where  $f \in \mathbf{L}^p(G)$  and  $g \in \mathbf{L}^q(G)$ .

**Theorem 3.4.** *Let  $T = (V, E)$  be a rooted tree with  $\deg(v) \leq k$  for every  $v \in V$ , with  $k \geq 2$ . Let  $1 < p < \infty$  and let  $q = \frac{p}{p-1}$ . Then  $\|S\| \leq (k-1)^{1/p} + (k-1)^{1/q}$ .*

*Proof.* Let  $f \in \mathbf{L}^p(T)$  and  $g \in \mathbf{L}^q(T)$  with  $\|f\|_p \leq 1$  and  $\|g\|_q \leq 1$ . Since  $T$  is a tree we can write

$$\begin{aligned} \langle Sf, g \rangle &= \sum_{v \in V} \left( \sum_{w \sim v} f(w) \right) g(v) \\ &= \sum_{v \in V} \left( \sum_{w \in \text{Chi}(v)} f(w) \right) g(v) + \sum_{v \in V \setminus \{o\}} f(\text{par}(v))g(v) \\ &= \sum_{v \in V} \left( \sum_{w \in \text{Chi}(v)} f(w) \right) g(v) + \sum_{v \in V} f(v) \left( \sum_{w \in \text{Chi}(v)} g(w) \right). \end{aligned}$$

Applying the triangle inequality and then Young's inequality to each summand we obtain

$$\begin{aligned} |\langle Sf, g \rangle| &\leq \sum_{v \in V} \sum_{w \in \text{Chi}(v)} |f(w)| |g(v)| + \sum_{v \in V} \sum_{w \in \text{Chi}(v)} |f(v)| |g(w)| \\ &\leq \sum_{v \in V} \sum_{w \in \text{Chi}(v)} \frac{1}{p} |f(w)|^p (k-1)^{p/(p+q)} + \frac{1}{q} |g(v)|^q (k-1)^{-q/(p+q)} \\ &\quad + \sum_{v \in V} \sum_{w \in \text{Chi}(v)} \frac{1}{p} |f(v)|^p (k-1)^{-p/(p+q)} + \frac{1}{q} |g(w)|^q (k-1)^{q/(p+q)}. \end{aligned}$$

We then have

$$\begin{aligned} |\langle Sf, g \rangle| &\leq \frac{(k-1)^{p/(p+q)}}{p} \sum_{v \in V} \sum_{w \in \text{Chi}(v)} |f(w)|^p + \frac{(k-1)^{-q/(p+q)}}{q} \sum_{v \in V} \sum_{w \in \text{Chi}(v)} |g(v)|^q \\ &\quad + \frac{(k-1)^{-p/(p+q)}}{p} \sum_{v \in V} \sum_{w \in \text{Chi}(v)} |f(v)|^p + \frac{(k-1)^{q/(p+q)}}{q} \sum_{v \in V} \sum_{w \in \text{Chi}(v)} |g(w)|^q \\ &\leq \frac{(k-1)^{p/(p+q)}}{p} \sum_{v \in V} \sum_{w \in \text{Chi}(v)} |f(w)|^p + (k-1) \frac{(k-1)^{-q/(p+q)}}{q} \sum_{v \in V} |g(v)|^q \\ &\quad + (k-1) \frac{(k-1)^{-p/(p+q)}}{p} \sum_{v \in V} |f(v)|^p + \frac{(k-1)^{q/(p+q)}}{q} \sum_{v \in V} \sum_{w \in \text{Chi}(v)} |g(w)|^q \\ &\leq \frac{(k-1)^{p/(p+q)}}{p} \|f\|_p^p + \frac{(k-1)^{p/(p+q)}}{q} \|g\|_q^q \\ &\quad + \frac{(k-1)^{q/(p+q)}}{p} \|f\|_p^p + \frac{(k-1)^{q/(p+q)}}{q} \|g\|_q^q, \end{aligned}$$

where we used the fact that  $-\frac{q}{p+q} + 1 = \frac{p}{p+q}$  and  $-\frac{p}{p+q} + 1 = \frac{q}{p+q}$ . Recalling that  $\frac{p}{p+q} = \frac{1}{q}$  and  $\frac{q}{p+q} = \frac{1}{p}$  we obtain

$$|\langle Sf, g \rangle| \leq \frac{(k-1)^{1/q}}{p} + \frac{(k-1)^{1/q}}{q} + \frac{(k-1)^{1/p}}{p} + \frac{(k-1)^{1/p}}{q} = (k-1)^{1/q} + (k-1)^{1/p}.$$

Since  $\|S\| = \sup\{|\langle Sf, g \rangle| : \|f\|_p \leq 1, \|g\|_q \leq 1\}$  (see, for example, Proposition 1.10.11 in [12]), we obtain the desired result.  $\square$

In some cases, the bound for the norm above is attained.

**Theorem 3.5.** *Let  $T = (V, E)$  be an almost  $k$ -regular rooted tree. Let  $1 < p < \infty$  and let  $q = \frac{p}{p-1}$ . Then  $\|S\| = (k-1)^{1/p} + (k-1)^{1/q}$ .*

*Proof.* Since  $T$  is almost  $k$ -regular, there exists  $N \in \mathbb{N}$  such that  $\deg(v) = k$  if  $|v| \geq N$ . For each  $n > N$  we define the function

$$f_n(v) = \begin{cases} (k-1)^{-\frac{|v|}{p}}, & \text{if } N < |v| \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly  $f_n \in \mathbf{L}^p(G)$  for each  $n \in \mathbb{N}$ .

Let  $C = \gamma(N)$ . It then follows that for each  $j \geq N$ , there are  $C(k-1)^{j-N}$  vertices  $v$  with  $|v| = j$ . With this in mind, it follows that

$$\|f_n\|_p^p = \sum_{j=N+1}^n C(k-1)^{j-N} (k-1)^{-j} = (n-N)C(k-1)^{-N}.$$

We also have that

$$\begin{aligned} \|Sf_n\|_p^p &= C \left| 1 \cdot 0 + (k-1)(k-1)^{-\frac{N+1}{p}} \right|^p + C(k-1) \left| 1 \cdot 0 + (k-1)(k-1)^{-\frac{N+2}{p}} \right|^p \\ &\quad + \sum_{j=N+2}^{n-1} C(k-1)^{j-N} \left| (k-1)^{-\frac{j-1}{p}} + (k-1)(k-1)^{-\frac{j+1}{p}} \right|^p \\ &\quad + C(k-1)^{n-N} \left| (k-1)^{-\frac{n-1}{p}} + (k-1)0 \right|^p + C(k-1)^{n+1-N} \left| (k-1)^{-\frac{n}{p}} + (k-1)0 \right|^p. \end{aligned}$$

Simplifying yields

$$\begin{aligned} \|Sf_n\|_p^p &= C \left( (k-1)^{p-(N+1)} + (k-1)^{1+p-(N+2)} \right. \\ &\quad \left. + \sum_{j=N+2}^{n-1} (k-1)^{-N} \left| (k-1)^{\frac{j}{p} - \frac{j-1}{p}} + (k-1)(k-1)^{\frac{j}{p} - \frac{j+1}{p}} \right|^p \right. \\ &\quad \left. + (k-1)^{-N+1} + (k-1)^{-N+1} \right) \\ &= C(k-1)^{-N} \left( 2(k-1)^{p-1} + \sum_{j=N+2}^{n-1} \left| (k-1)^{\frac{1}{p}} + (k-1)^{1-\frac{1}{p}} \right|^p + 2(k-1) \right) \\ &= C(k-1)^{-N} \left( 2(k-1)^{p-1} + ((n-1) - (N+1)) \left| (k-1)^{\frac{1}{p}} + (k-1)^{1-\frac{1}{p}} \right|^p + 2(k-1) \right). \end{aligned}$$

From this, it follows that

$$\frac{\|Sf_n\|_p^p}{\|f_n\|_p^p} = \frac{2(k-1)^{p-1} + (n-N-2) \left| (k-1)^{\frac{1}{p}} + (k-1)^{1-\frac{1}{p}} \right|^p + 2(k-1)}{n-N}.$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{\|Sf_n\|_p^p}{\|f_n\|_p^p} = \left| (k-1)^{\frac{1}{p}} + (k-1)^{1-\frac{1}{p}} \right|^p,$$

and hence

$$\|S\| \geq (k-1)^{\frac{1}{p}} + (k-1)^{\frac{1}{q}}.$$

By the previous theorem, it follows that

$$\|S\| = (k-1)^{\frac{1}{p}} + (k-1)^{\frac{1}{q}}$$

as desired.  $\square$

#### 4. KERNEL OF TREES

In this section, we try to answer the question of when the kernel of the shift is trivial or nontrivial in the case of rooted (and “almost leafless”) trees. This question seems to have been overlooked in the literature and it yields some interesting results. Some results for a modification of the shift can be found in [3].

First, we need some definitions.

**Definition 4.1.** Let  $T = (V, E)$  be a rooted tree and let  $v \in V$ . We define  $\beta(v)$  to be the number of children of  $v$ .

Observe that  $\beta(v) = \deg(v) - 1$  for every vertex  $v \neq o$  and  $\beta(o) = \deg(o)$ . Also,  $\beta(v) = 0$  if and only if  $v$  is a leaf.

**Definition 4.2.** Let  $T = (V, E)$  be a rooted leafless tree with bounded degree. Consider the set  $\mathcal{M}$  defined as

$$\mathcal{M} := \{v \in V : \beta(v) = \beta(w) \text{ for infinitely many } w \in V\}.$$

(Notice that only finitely many vertices do not belong to  $\mathcal{M}$ .) We define the numbers

$$M := \max_{v \in \mathcal{M}} \{\beta(v)\}, \quad \text{and} \quad m := \min_{v \in \mathcal{M}} \{\beta(v)\}.$$

We will need the following observation. If  $T = (V, E)$  is a leafless tree and if  $f \neq 0$  satisfies  $Sf = 0$ , then there must exist  $v^* \in V$  such that  $f(v^*) \neq 0$ . Let  $w^* \in \text{Chi}(v^*)$ . Since  $(Sf)(w^*) = 0$ , we must have

$$0 = (Sf)(w^*) = f(v^*) + \sum_{u \in \text{Chi}(w^*)} f(u),$$

and hence there exists  $u^* \in \text{Chi}^2(v^*)$  with  $f(u^*) \neq 0$ . That is, if  $f \in \ker(S)$  and  $f$  is not zero at a vertex, then there is a grandchild of the vertex in which  $f$  does not vanish. Applying this argument inductively, it is clear that for all  $n \in \mathbb{N}$  there exists  $u_n \in \text{Chi}^{2n}(v^*)$  with  $f(u_n) \neq 0$ .

We are now ready to prove the following theorem.

**Theorem 4.3.** Let  $T = (V, E)$  be a tree with bounded degree and a finite number of leaves (equivalently,  $m \geq 1$ ) and consider the shift  $S$  on  $\mathbf{L}^p(T)$ , with  $1 \leq p \leq \log_M(m) + 1$  if  $M > 1$ , or  $1 \leq p < \infty$  if  $M = 1$ . Then  $\ker(S)$  is trivial.

*Proof.* Assume there exists  $f \in \ker(S)$  such that  $f \neq 0$ . Then there exists  $v^* \in V$  such that  $f(v^*) \neq 0$ .

Since  $T$  has bounded degree, there exists  $N \in \mathbb{N}$  such that for all  $|v| \geq 2N$ , we have  $v \in \mathcal{M}$ . We claim that

$$(1) \quad C \left( \frac{m}{M^{p-1}} \right)^k \leq \sum_{u \in \text{Chi}^{2k}(v^*)} |f(u)|^p$$

for all  $k \in \mathbb{N}$  with  $k \geq N$ , where

$$C := \left( \frac{m}{M^{p-1}} \right)^{-N} \sum_{u \in \text{Chi}^{2N}(v^*)} |f(u)|^p.$$

Observe that, since  $f(v^*) \neq 0$ , the observation before this theorem implies that there exists  $u \in \text{Chi}^{2N}(v^*)$  such that  $f(u) \neq 0$ . Hence,  $C$  is positive.

We proceed by induction on  $k$ . Observe that for  $k = N$  the inequality (1) is trivially satisfied. Now, assume that for a fixed  $k \in \mathbb{N}$ , with  $k \geq N$  the inequality (1) is satisfied.

Let  $w^* \in \text{Chi}^{2k}(v^*)$  and let  $v_0$  be a child of  $w^*$ . Then, since  $(Sf)(v_0) = 0$  we have

$$0 = f(w^*) + \sum_{u \in \text{Chi}(v_0)} f(u).$$

It then follows that

$$|f(w^*)| \leq \sum_{u \in \text{Chi}(v_0)} |f(u)|,$$

and therefore, by Jensen's inequality and since  $\beta(v_0) \leq M$ , we have

$$|f(w^*)|^p \leq \beta(v_0)^{p-1} \sum_{u \in \text{Chi}(v_0)} |f(u)|^p \leq M^{p-1} \sum_{u \in \text{Chi}(v_0)} |f(u)|^p.$$

The above inequality holds for all  $v_0 \in \text{Chi}(w^*)$ . We know  $w^*$  has at least  $m$  children, so

$$m|f(w^*)|^p \leq \sum_{v_0 \in \text{Chi}(w^*)} |f(w^*)|^p \leq M^{p-1} \sum_{v_0 \in \text{Chi}(w^*)} \sum_{u \in \text{Chi}(v_0)} |f(u)|^p = M^{p-1} \sum_{u \in \text{Chi}^2(w^*)} |f(u)|^p,$$

and therefore

$$\frac{m}{M^{p-1}} |f(w^*)|^p \leq \sum_{u \in \text{Chi}^2(w^*)} |f(u)|^p.$$

The above holds for all  $w^* \in \text{Chi}^{2k}(v)$ . By the induction hypothesis we have

$$\begin{aligned} C \left( \frac{m}{M^{p-1}} \right)^{k+1} &\leq \frac{m}{M^{p-1}} \sum_{w^* \in \text{Chi}^{2k}(v^*)} |f(w^*)|^p \\ &= \sum_{w^* \in \text{Chi}^{2k}(v^*)} \frac{m}{M^{p-1}} |f(w^*)|^p \\ &\leq \sum_{w^* \in \text{Chi}^{2k}(v^*)} \sum_{u \in \text{Chi}^2(w^*)} |f(u)|^p \\ &= \sum_{u \in \text{Chi}^{2(k+1)}(v^*)} |f(u)|^p. \end{aligned}$$

Thus, our claim is true by induction.

To finish the proof, observe that

$$\begin{aligned} \sum_{w \in V} |f(w)|^p &\geq \sum_{k=N}^{\infty} \sum_{u \in \text{Chi}^{2k}(v^*)} |f(u)|^p \\ &\geq C \sum_{k=N}^{\infty} \left( \frac{m}{M^{p-1}} \right)^k. \end{aligned}$$

Since  $\frac{m}{M^{p-1}} \geq 1$  by hypothesis, the above series diverges, which is a contradiction, since  $f \in \mathbf{L}^p(T)$ . Hence  $\ker(S) = \{0\}$  for  $1 \leq p \leq 1 + \log_M(m)$ , if  $M > 1$  and for  $1 \leq p < \infty$ , if  $M = 1$ .  $\square$

The following example shows that the bound for  $p$  in the theorem above cannot be improved.

**Example 4.4.** Let  $M \geq m$  be natural numbers and construct a tree  $T$  as follows. Let  $o$  be the root and add  $m$  children to it. To each of these new vertices add  $M$  children. To each of the latter vertices add  $m$  children, and to each of those vertices add  $M$  children. Continue in this manner. Consider the shift  $S$  on  $\mathbf{L}^p(T)$ , with  $\log_M(m) + 1 < p \leq \infty$  if  $M > 1$  or  $p = \infty$  if  $M = 1$ . Then  $\ker(S)$  is nontrivial.

*Proof.* Define  $f : T \rightarrow \mathbb{C}$  as follows:

$$f(v) = \begin{cases} 0, & \text{if } |v| \text{ is odd,} \\ (-M)^{-|v|/2}, & \text{if } |v| \text{ is even.} \end{cases}$$

We first verify that  $Sf = 0$ . Indeed, if  $|v|$  is even, then if  $w$  is adjacent to  $v$ , we have  $f(w) = 0$  and hence  $(Sf)(v) = 0$ . Now, if  $|v| = 2k + 1$  is odd, then  $f(\text{par}(v)) = (-M)^{-k}$  and, if  $w \in \text{Chi}(v)$  then  $f(w) = (-M)^{-k-1}$ . Hence,

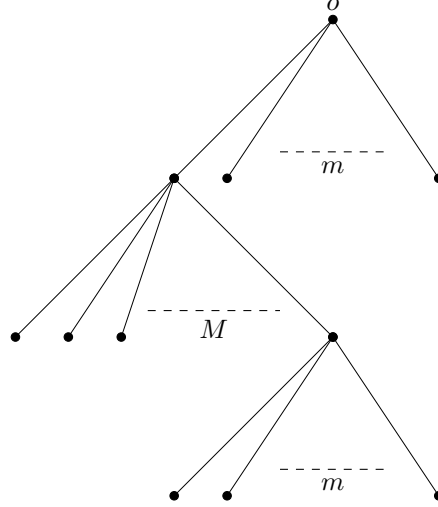
$$(Sf)(v) = f(\text{par}(v)) + \sum_{w \in \text{Chi}(v)} f(w) = (-M)^{-k} + \sum_{w \in \text{Chi}(v)} (-M)^{-k-1} = (-M)^{-k} + M(-M)^{-k-1} = 0.$$

It is clear that  $f$  is bounded, which settles the case  $p = \infty$ .

Now, if  $M > 1$ , let  $\log_M(m) + 1 < p < \infty$ . We will show that  $f \in \mathbf{L}^p(T)$ . Observe that, for each  $k \in \mathbb{N}_0$ , there are exactly  $(Mm)^k$  vertices  $v$  with  $|v| = 2k$ . We then have

$$\sum_{v \in V} |f(v)|^p = \sum_{|v| \text{ even}} |f(v)|^p = \sum_{k=0}^{\infty} \sum_{|v|=2k} |f(v)|^p = \sum_{k=0}^{\infty} \sum_{|v|=2k} |M|^{-kp} = \sum_{k=0}^{\infty} (Mm)^k |M|^{-kp} = \sum_{k=0}^{\infty} \left( \frac{m}{M^{p-1}} \right)^k,$$



FIGURE 1. Constructing  $T$  in Example 4.4.

which converges since  $p - 1 > \log_M(m)$ . Hence  $f \in \mathbf{L}^p(T)$  and  $\ker S$  is nontrivial.  $\square$

The idea in the construction of the nontrivial function in the example above is used below to give a more general theorem.

**Theorem 4.5.** *Let  $T = (V, E)$  be a rooted leafless tree with bounded degree and consider the shift  $S$  on  $\mathbf{L}^p(T)$ , with  $\log_m(M) + 1 < p \leq \infty$ , if  $m > 1$ ; or  $p = \infty$ , if  $m = 1$ . Then  $\ker(S)$  is nontrivial.*

*Proof.* First, we will construct a function  $f : V \rightarrow \mathbb{C}$  such that  $(Sf)(v) = 0$  for all  $v \in V$ . Later, we will show that  $f \in \mathbf{L}^p(T)$ .

We proceed inductively. Define  $f(o) = 1$ . For all  $v \in V$  with  $|v| = 1$ , define  $f(v) = 0$ . It is then clear that  $(Sf)(o) = 0$ . Now, assume that, for a fixed  $n \in \mathbb{N}$ , we have defined  $f(v)$  for all  $|v| \leq n$  and that  $Sf(v) = 0$  for all  $v$  with  $|v| < n$ .

For each  $v \in V$  with  $|v| = n + 1$ , we define  $f(v) = \frac{-f(\text{par}^2(v))}{\beta(\text{par}(v))}$ . Then for all  $v \in V$  with  $|v| = n$  we have

$$\begin{aligned}
 (Sf)(v) &= \sum_{u \sim v} f(u) = f(\text{par}(v)) + \sum_{u \in \text{Chi}(v)} f(u) \\
 &= f(\text{par}(v)) + \sum_{u \in \text{Chi}(v)} \frac{-f(\text{par}^2(u))}{\beta(\text{par}(u))} \\
 &= f(\text{par}(v)) + \sum_{u \in \text{Chi}(v)} \frac{-f(\text{par}(v))}{\beta(v)} \\
 &= f(\text{par}(v)) + \beta(v) \frac{-f(\text{par}(v))}{\beta(v)} \\
 &= 0.
 \end{aligned}$$

This completes the induction step; we conclude that  $Sf(v) = 0$  for all  $v \in V$ . Note that  $f$  is clearly not identically equal to 0, as  $f(o) = 1$ . Note also that for all odd  $n \in \mathbb{N}$ , if  $v \in V$  and  $|v| = n$ , then  $f(v) = 0$ .

It remains to be shown that  $f \in \mathbf{L}^p(T)$ . Observe that  $|f(v)| \leq 1$  for all  $v \in V$  and hence  $f \in \mathbf{L}^\infty(T)$ . Assume then that  $m > 1$  and  $\log_m(M) + 1 < p < \infty$ .

Since  $T$  has bounded degree, there exists  $N \in \mathbb{N}$  such that for all  $|v| \geq 2N$ , we have  $v \in \mathcal{M}$ . We now prove that for all  $n \in \mathbb{N}$  with  $n \geq N$ ,

$$(2) \quad \sum_{|u|=2n} |f(u)|^p \leq \frac{M^{n-N} C}{m^{(p-1)(n-N)}},$$

where

$$C = \sum_{|u|=2N} |f(u)|^p.$$

Clearly the desired inequality (2) is true for  $n = N$ . Now, assume (2) holds for some fixed  $n$  with  $n \geq N$ . By the definition of  $f$ , we obtain

$$\begin{aligned} \sum_{|u|=2(n+1)} |f(u)|^p &= \sum_{|w|=2n+1} \sum_{u \in \text{Chi}(w)} |f(u)|^p \\ &= \sum_{|w|=2n+1} \sum_{u \in \text{Chi}(w)} \left| \frac{-f(\text{par}^2(u))}{\beta(\text{par}(u))} \right|^p \\ &= \sum_{|w|=2n+1} \beta(w) \left| \frac{f(\text{par}(w))}{\beta(w)} \right|^p \\ &= \sum_{|w|=2n+1} \frac{1}{\beta(w)^{p-1}} |f(\text{par}(w))|^p \\ &\leq \frac{1}{m^{p-1}} \sum_{|w|=2n+1} |f(\text{par}(w))|^p, \end{aligned}$$

where in the last line we use the definition of  $m$ . We also obtain

$$\begin{aligned} \sum_{|w|=2n+1} |f(\text{par}(w))|^p &= \sum_{|v|=2n} \sum_{w \in \text{Chi}(v)} |f(\text{par}(w))|^p \\ &= \sum_{|v|=2n} \sum_{w \in \text{Chi}(v)} |f(v)|^p \\ &= \sum_{|v|=2n} \beta(v) |f(v)|^p \\ &\leq M \sum_{|v|=2n} |f(v)|^p. \end{aligned}$$

Combining the past two expressions, and using the induction hypothesis, we get

$$\sum_{|u|=2(n+1)} |f(u)|^p \leq \frac{M}{m^{p-1}} \sum_{|v|=2n} |f(v)|^p \leq \frac{M}{m^{p-1}} \frac{M^{n-N} C}{m^{(p-1)(n-N)}} = \frac{M^{n+1-N} C}{m^{(p-1)(n+1-N)}}.$$

Thus the desired inequality holds for  $n+1$  and the induction step is complete.

Observe that, since  $f(u) = 0$  for all  $v \in V$  such that  $|u|$  is odd, to show that  $f \in \mathbf{L}^p(T)$  we only need to prove that

$$\sum_{n=N}^{\infty} \sum_{|u|=2n} |f(u)|^p$$

converges. But

$$\sum_{n=N}^{\infty} \sum_{|u|=2n} |f(u)|^p \leq \sum_{n=N}^{\infty} \frac{M^{n-N} C}{m^{(p-1)(n-N)}} = C \sum_{n=N}^{\infty} \left( \frac{M}{m^{p-1}} \right)^{n-N},$$

which converges for  $p > 1 + \log_m M$ . Therefore  $f \in \mathbf{L}^p(T)$ . Having shown  $Sf = 0$  while  $f \neq 0$  and  $f \in \mathbf{L}^p(T)$ , we conclude that  $\ker(S)$  is nontrivial.  $\square$

If  $m = 1$ , the above theorem cannot be improved, as the following example shows.

**Example 4.6.** Let  $M \in \mathbb{N}$ , with  $M > 1$ . Choose a sequence of positive integers  $(t_j)$  such that

$$\sum_{j=1}^{\infty} \frac{t_j}{M^{(j-1)p}}$$

diverges for every  $p > 1$ . (For example, choose  $t_j = 2^{(j-1)^2}$  or  $t_j = j^{j-1}$ .)

Define a tree  $T$  as follows. Set the root  $o$  and assign  $M$  children to  $o$ . Now, to each of these children, assign a path of length  $2t_1 - 1$ . At each vertex at the end of each of these paths (we call each such vertex

a bifurcation node), assign  $M$  children. To each of these children, attach a path of length  $2t_2 - 1$ . At each vertex at the end of each of these paths, assign  $M$  children. To each of these children, attach a path of length  $2t_3 - 1$ . Keep going in this way.

Then,  $\ker S$  is trivial in  $\mathbf{L}^p(T)$  for  $1 < p < \infty$ .

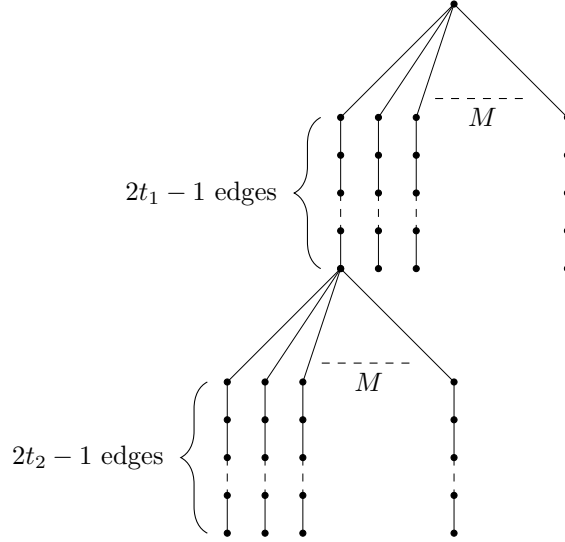


FIGURE 2. The tree  $T$  in Example 4.6.

*Proof.* Let  $f \in \mathbf{L}^p(T)$  with  $Sf = 0$ .

We first label a few nodes, to explain our example. Let  $v_0$  be a bifurcation node, and let  $v_1, v_2, \dots, v_{2s-1}$  be those vertices that comprise a path off of  $v_0$ ; that is,  $(v_0, v_1, \dots, v_{2s-1})$  is a path of length  $2s - 1$  that begins at  $v_0$ .

Since  $(Sf)(v_j) = 0$  for each vertex  $j = 1, 2, \dots, 2s - 2$ , we must have

$$f(v_0) = -f(v_2) = f(v_4) = \dots = (-1)^{s+1} f(v_{2s-2})$$

and

$$f(v_1) = -f(v_3) = f(v_5) = \dots = (-1)^{s+1} f(v_{2s-1}).$$

If  $f \neq 0$ , there must exist a bifurcation node  $w^*$  such that  $f(w^*) \neq 0$  or  $f(v^*) \neq 0$  for some  $v^* \in \text{Chi}(w^*)$ , otherwise  $f$  would be identically zero. Let us choose such a  $w^*$  closest to the root.

We have two cases:

- If  $f(w^*) \neq 0$ , then assume, without loss of generality, that  $f(w^*) = 1$ . But observe that, if  $v_1 \in \text{Chi}^2(w^*)$ , then  $f(v_1) = -1$ , since  $0 = (Sf)(\text{par}(v_1)) = f(w^*) + f(v_1)$ . Using the observation above, we obtain that, at the vertex  $v_{2t_1-1}$  at the end of the path, the absolute value of the function is also 1. But then, if  $w_1 \in \text{Chi}^2(v_{2t_1-1})$ , then  $|f(w_1)| = 1$ . Continuing in this manner, it is clear that there are infinitely many vertices  $u$  such that  $|f(u)| = 1$ , and hence  $f \notin \mathbf{L}^p(T)$ . Thus  $\ker S$  is trivial.
- If  $v_0 \in \text{Chi}(w^*)$  satisfies that  $f(v_0) \neq 0$ , again, we may assume that  $f(v_0) = 1$ . By the observation above, we must have that

$$|f(v_0)| = |f(v_2)| = |f(v_4)| = \dots = |f(v_{2t_1-2})| = 1,$$

where  $v_1, v_2, \dots, v_{2t_1-1}$  are the vertices in the path attached to  $v_0$ . Hence there are at least  $t_1$  vertices  $u$  where  $|f(u)| = 1$ .

Since  $(Sf)(v_{2t_1-1}) = 0$ , there must exist a vertex  $w_0 \in \text{Chi}(v_{2t_1-1})$  with  $|f(w_0)| \geq 1/M$ . Again, by the observation above, we must have

$$|f(w_0)| = |f(w_2)| = |f(w_4)| = \dots = |f(w_{2t_2-2})| \geq 1/M,$$

where  $w_1, w_2, \dots, w_{2t_2-1}$  are the vertices in the path attached to  $w_0$ . Hence there are at least  $t_2$  vertices  $u$  where  $|f(u)| \geq 1/M$ .

Since  $(Sf)(w_{2t_2-1}) = 0$ , there must exist a vertex  $x_0 \in \text{Chi}(v_{2t_2-1})$  with  $|f(x_0)| \geq 1/M^2$ . Again, by the observation above, we must have at least  $t_3$  vertices  $u$  where  $|f(u)| \geq 1/M^2$ .

Continuing in this manner, it should be clear that, for each  $j \in \mathbb{N}$ , we must have at least  $t_j$  vertices  $u$  with  $|f(u)| \geq 1/M^{j-1}$ . But this implies that

$$\sum_{v \in V} |f(v)|^p \geq \sum_{j=1}^{\infty} t_j \frac{1}{M^{(j-1)p}},$$

which diverges. This proves that  $f \notin \mathbf{L}^p(T)$  and hence  $\ker S$  is trivial.  $\square$

Can Theorem 4.5 be improved for the case  $m > 1$ ? We have not been able to answer the question and we leave it open for future research. As the above results show, the cases where  $\log_m(M) + 1 < p < \log_M(m) + 1$  seem to depend on the “distribution” of the numbers  $m$  and  $M$  along the tree, and not on the numbers themselves. A question for future research: is there a way to characterize the triviality of the kernel in terms of this “distribution”?

## 5. EIGENVALUES OF FINITE GRAPHS WITH INFINITE TAILS

The objective of this section is to present a method to find the spectrum of certain infinite graphs obtained by attaching to a finite graph an “infinite tail”. Some of the results we obtain here have been obtained by Golinskii in [7]: our method has the advantage of being elementary, while Golinskii’s method uses the theory of Jacobi matrices.

Concretely, say  $H$  is a graph with vertices  $\{v_0, v_1, v_2, \dots, v_n\}$  and  $L$  is the graph with vertices  $\{u_k : k \in \mathbb{N}\}$  and such that  $u_k$  is adjacent to  $u_j$  if and only if  $|k - j| = 1$  (what we call an “infinite tail”). We form the new graph  $G_H$  by taking the union of the vertices of  $H$  and  $L$ , with the same adjacency relations as before, but adding an edge between the vertex  $v_0$  and  $u_1$ .

We would like to solve the equation  $Sf = \lambda f$  for some complex-valued function  $f$  defined on the set  $V$  of vertices of  $G_H$ . Furthermore, we would like  $f$  to be in  $\mathbf{L}^p(G_H)$ , for  $1 \leq p < \infty$ . Let us look at the equations obtained from  $Sf = \lambda f$  when we evaluate  $f$  at the vertices  $u_j$ , for  $j \in \mathbb{N}$ . For simplicity, denote by  $f_j = f(u_j)$  for  $j \in \mathbb{N}$  and  $f_0 = f(v_0)$ . Then

$$f_{j-1} + f_{j+1} = \lambda f_j$$

for  $j \in \mathbb{N}$ . It is well known, and easy to see, that the solutions to this system are given by

$$f_j = C_1 b^j + C_2 c^j,$$

where  $b$  and  $c$  are the solutions to the equation  $t^2 - \lambda t + 1 = 0$  and  $C_1, C_2$  are constants to be determined. Since  $bc = 1$  we may assume that  $|b| \leq 1$  and  $c = 1/b$ . Since we want  $f \in \mathbf{L}^p(G_H)$ , we must have  $C_2 = 0$  and either  $|b| < 1$  or  $C_1 = 0$ . Observe that in the former case, since  $\lambda = b + c = b + 1/b$ , we must have  $|\lambda| > 2$ .

In either case, the problem of finding  $\lambda$  reduces to solving the finite system of equations

$$(3) \quad (Sf)(v_j) = \lambda f(v_j)$$

for  $j = 0, 1, 2, \dots, n$  and such that  $f(v_0) = C_1$  and  $f(u_1) = C_1 b$ .

If  $C_1 = 0$ , then  $f_j = 0$  for all  $j \in \mathbb{N}_0$  and hence the system of equations can be seen as finding an eigenvalue for the adjacency matrix of  $H$  such that the corresponding eigenvector has value 0 at  $v_0$ .

If  $C_1 \neq 0$ , dividing  $f$  by a constant, we may always assume that  $C_1 = 1$ . In this case, the system of equations does not correspond to the eigenvalue equations for the adjacency matrix of  $H$ , since the vertex  $u_1$ , which is not in  $H$ , is also adjacent to  $v_0$ . Nevertheless, in several cases the finite system of equations can be solved. We show a few examples where we can achieve this.

**5.1. The kite with infinite tail.** Let us consider a cycle graph with an infinite tail attached. That is,  $H$  is the  $(n+1)$ -cycle (with  $n \geq 2$ ), which consists of the vertices  $v_0, v_1, v_2, \dots, v_n$  and the edges given by the relations  $v_j \sim v_{j+1}$  for  $j = 0, 1, 2, \dots, n-1$  and  $v_n \sim v_0$ . The infinite graph  $G_H$  is obtained by attaching an infinite tail to the vertex  $v_0$ . We call this graph the *kite with an infinite tail*.

We will use the argument outlined above to obtain the set of eigenvalues of the kite.

Let us deal with the case  $C_1 = 0$  first. It is well-known (see, for example, [2, p. 8-9]) that the eigenvalues of the adjacency matrix of the  $(n+1)$ -cycle are the numbers  $\{2 \cos(2\pi j/(n+1)) : j = 0, 1, 2, \dots, [(n+1)/2]\}$  (here, as usual,  $[x]$  denotes the floor of  $x$ ).

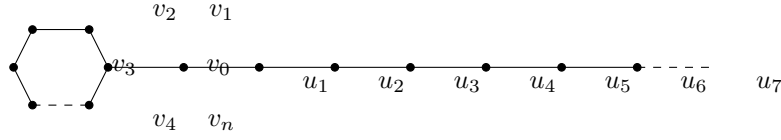


FIGURE 3. The kite with an infinite tail.

The corresponding eigenspaces are obtained in the following manner. Let  $\omega = \exp(i2\pi/(n+1))$ . Then the eigenspace corresponding to the eigenvalue  $2\cos(2\pi j/(n+1))$  is the linear span of the vectors

$$(1, \omega^j, \omega^{2j}, \omega^{3j}, \dots, \omega^{nj}) \quad \text{and} \quad (1, \omega^{-j}, \omega^{-2j}, \omega^{-3j}, \dots, \omega^{-nj}).$$

Observe that if  $n$  is even, then these eigenspaces are of dimension 2 if  $j = 1, 2, \dots, n/2$ , and of dimension 1 if  $j = 0$ . If  $n$  is odd, these eigenspaces are of dimension 2 if  $j = 1, 2, \dots, (n-1)/2$ , and of dimension 1 if  $j = 0$  or  $j = (n+1)/2$ .

Each nonzero vector in the eigenspaces of dimension 1 does not have a zero entry at the position corresponding to  $v_0$ . But clearly, we can choose an eigenvector in each of the eigenspaces of dimension 2 such that the value at  $v_0$  is zero. Hence, the eigenvalues for the kite with infinite tail, in this case, are the numbers

$$\{2\cos(2\pi j/(n+1)) : j = 1, 2, \dots, [n/2]\}.$$

Now, let us deal with the case  $C_1 \neq 0$ . As we mentioned above, we can always assume that  $C_1 = 1$ . If we set  $x_j := f(v_j)$ , the system of equations (3) becomes:

$$\begin{aligned} 1 + x_2 &= \lambda x_1 \\ x_2 + x_3 &= \lambda x_2 \\ x_3 + x_4 &= \lambda x_3 \\ &\vdots \\ x_{n-1} + 1 &= \lambda x_n \\ x_1 + x_n + b &= \lambda. \end{aligned}$$

The first  $n$  equations can be rewritten, in matrix form, as  $AX = C$ , where

$$A = \begin{pmatrix} \lambda & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & \lambda & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & \lambda & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & \lambda \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

It is well known (e.g., [2]) that the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

are all in the interval  $(-2, 2)$ . Since  $|\lambda| > 2$ , it follows that  $A$  is invertible and there is a unique solution to the equation  $AX = C$ .

If  $n$  is even, it is a tedious, but straightforward, computation to show that the solution of  $AX = C$  is given by the expressions

$$x_k = x_{n-k+1} = \frac{b^k(1 + b^{n-2k+1})}{1 + b^{n+1}},$$

for  $k = 1, 2, \dots, \frac{n}{2} - 1$  and

$$x_{\frac{n}{2}} = x_{\frac{n}{2}+1} = \frac{b^{\frac{n}{2}}}{1 + b^{n+1}}.$$

If  $n$  is odd, it can also be checked that the solution of  $AX = C$  is given by the expressions

$$x_k = x_{n-k+1} = \frac{b^k(1 + b^{n-2k+1})}{1 + b^{n+1}},$$

for  $k = 1, 2, \dots, \frac{n-1}{2}$  and

$$x_{\frac{n+1}{2}} = \frac{2b^{\frac{n+1}{2}}}{1 + b^{n+1}}.$$

In both cases ( $n$  odd and  $n$  even) the solutions also need to satisfy the equation

$$x_1 + x_n = \lambda - b.$$

Hence we obtain

$$\frac{b(1 + b^{n-1})}{1 + b^{n+1}} + \frac{b(1 + b^{n-1})}{1 + b^{n+1}} = \frac{1}{b},$$

which simplifies to

$$b^{n+1} + 2b^2 - 1 = 0.$$

Consider the polynomials  $p_n(x) = x^{n+1} + 2x^2 - 1$ . The case  $n = 2$  is special. In that case,  $p_2(x) = x^3 + 2x^2 - 1 = (x+1)(x^2 + x - 1)$  has roots

$$-1, \quad \frac{-1 - \sqrt{5}}{2} \approx -1.618, \quad \text{and} \quad \frac{-1 + \sqrt{5}}{2} \approx 0.618,$$

Since only one of the roots is of modulus less than 1, this choice of  $b$  gives the eigenvalue  $\lambda = \sqrt{5}$ .

If  $n$  is odd, it is easy to check that  $p_n$  is decreasing in the interval  $(-\infty, 0)$  and increasing in the interval  $(0, \infty)$ . Since  $p_n(0) = -1 < 0$  and  $p_n(1) = p_n(-1) = 2 > 0$ , it follows that there is exactly one root in the interval  $(-1, 0)$  and one in the interval  $(0, 1)$ . Hence, if  $n$  is odd, in this case there will be two eigenvalues (one the negative of the other) for the kite.

Now, if  $n$  is even,  $n \geq 4$ , define

$$\alpha_n = \left( \frac{-4}{n+1} \right)^{1/(n-1)}$$

It is clear that  $-1 < \alpha_n < 0$  and it is easy to check that  $p_n$  is increasing on the intervals  $(-\infty, \alpha_n)$  and  $(0, \infty)$  and decreasing on the interval  $(\alpha_n, 0)$ . Since  $p_n(-1) = 0$ ,  $p_n(0) = -1$  and  $p_n(1) = 2$ , it follows that there is exactly one root in the interval  $(-1, 0)$  and one in the interval  $(0, 1)$ .

In both cases, it turns out there are exactly two roots of  $p_n$  (for  $n \neq 2$ ) in the interval  $(-1, 1)$ , denoted by  $x_1(n)$  and  $x_2(n)$  and such that by  $-1 < x_2(n) < 0 < x_1(n) < 1$ .

Putting these results together with the computations for the case  $C_1 = 0$ , we obtain that the eigenvalues of the kite  $G_H$ , in  $\mathbf{L}^p(G_H)$ , are

$$\{-1, \sqrt{5}\}$$

if  $n = 2$ , and

$$\{2 \cos(2\pi j/(n+1)) : j = 1, 2, \dots, [n/2]\} \cup \{x_1(n) + \frac{1}{x_1(n)}, x_2(n) + \frac{1}{x_2(n)}\}$$

if  $n \geq 3$ .

**5.2. The fly-swatter with infinite tail.** Let us consider the fly-swatter. That is,  $H$  is the complete graph  $K_{n+1}$ , with  $n \geq 2$ , and  $G_H$  is obtained by attaching an infinite tail to one of the vertices (say the vertex  $v_0$ ). We call this graph the *fly-swatter*. We will use the argument outlined above to obtain the set of eigenvalues of the fly-swatter.

Again, let us deal with the case  $C_1 = 0$  first. It is well-known (see, for example, [2, p. 8]) that the set of eigenvalues of the adjacency matrix of the complete graph on  $(n+1)$  vertices is  $\{-1, n\}$ . The eigenspace corresponding to the eigenvalue  $n$  is spanned by the vector  $(1, 1, \dots, 1)$  and the eigenspace corresponding to the eigenvalue  $-1$  is the set

$$\{(v_0, v_1, v_2, \dots, v_n) \in \mathbb{C}^{n+1} : v_0 + v_1 + v_2 + \dots + v_n = 0\}.$$

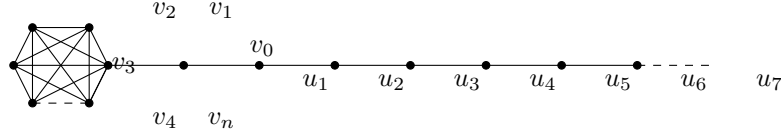


FIGURE 4. The fly-swatter.

Since an eigenvector with  $v_0 = 0$  can be chosen only in the case where the eigenvalue is  $-1$ , this shows that in the case  $C_1 = 0$ , the only eigenvalue of the fly-swatter is  $-1$ .

Now, if  $C_1 = 1$ , and we set  $x_j := f(v_j)$ , the system of equations (3) becomes:

$$\begin{aligned}
 1 + x_2 + x_3 + x_4 + \cdots + x_n &= \lambda x_1 \\
 1 + x_1 + x_3 + x_4 + \cdots + x_n &= \lambda x_2 \\
 1 + x_1 + x_2 + x_4 + \cdots + x_n &= \lambda x_3 \\
 &\vdots \\
 1 + x_1 + x_2 + x_3 + \cdots + x_{n-1} &= \lambda x_n \\
 x_1 + x_2 + x_4 + \cdots + x_n + b &= \lambda
 \end{aligned}$$

The first  $n$  equations can be rewritten, in matrix form, as  $AX = C$ , where

$$A = \begin{pmatrix} -\lambda & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & -\lambda & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & -\lambda & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & -\lambda & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & -\lambda \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

and the solution  $(x_1, x_2, x_3, \dots, x_n)$  must satisfy the last equation:

$$x_1 + x_2 + x_4 + \cdots + x_n = \lambda - b.$$

Now, the matrix  $A$  is invertible if  $\lambda \neq -1$  and  $\lambda \neq n$ , since if we set  $\lambda = 0$ , then the matrix  $A$  is the adjacency matrix for the complete graph on  $n + 1$  vertices. Since we are dealing with the case  $C_1 = 1$ , we have that  $\lambda > 2$  and hence we only need to discard the case  $\lambda = n$ . Clearly, however,  $C \in \ker A$  if  $\lambda = n$ , and  $A$  is selfadjoint, so we must have that the range of  $A$  is orthogonal to the kernel of  $A$ . Hence there is no solution in this case.

Now, for every other value of  $\lambda$  there is a unique solution to  $AX = C$ , and it can be easily checked that it is given by

$$x_1 = x_2 = \cdots = x_n = \frac{1}{\lambda - (n - 1)}.$$

Now, this solution should satisfy the equation

$$x_1 + x_2 + x_4 + \cdots + x_n + b = \lambda,$$

from which we obtain  $\frac{n}{\lambda - (n - 1)} + b = \lambda$ . Simplifying and using the fact that  $\lambda = b + 1/b$  we obtain

$$(n - 1)b^2 + (n - 1)b - 1 = 0.$$

The solutions to this equation are

$$b = -\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{n+3}{n-1}}.$$

It is easy to check that  $0 < -\frac{1}{2} + \frac{1}{2} \sqrt{\frac{n+3}{n-1}} < 1$  and  $-2 < -\frac{1}{2} - \frac{1}{2} \sqrt{\frac{n+3}{n-1}} < -1$ . Hence,  $b = -\frac{1}{2} + \frac{1}{2} \sqrt{\frac{n+3}{n-1}}$  and

$$\lambda = b + 1/b = \frac{1}{2} \left( -1 + \sqrt{\frac{n+3}{n-1}} \right) + \frac{2}{-1 + \sqrt{\frac{n+3}{n-1}}}$$

Putting these results together with the computations for the case  $C_1 = 0$ , we obtain that the eigenvalues of the fly-swatter  $G_H$ , in  $\mathbf{L}^P(G_H)$  are

$$\left\{ -1, \frac{1}{2} \left( -1 + \sqrt{\frac{n+3}{n-1}} \right) + \frac{2}{-1 + \sqrt{\frac{n+3}{n-1}}} \right\}.$$

**5.3. The comb with an infinite tail.** Let us consider the comb with an infinite tail. That is,  $H$  is the graph with vertices  $\{v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n\}$  and such that  $v_j \sim v_{j+1}$  for  $j = 1, 2, \dots, n-1$ , and  $v_j \sim w_j$  for  $j = 1, 2, \dots, n$ , where  $n \geq 2$ ; and  $G_H$  is obtained by attaching an infinite tail to the vertex  $v_n$ .

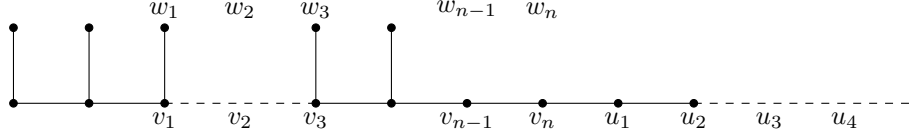


FIGURE 5. The comb with an infinite tail.

We call this graph  $G_H$  the *comb with an infinite tail* (or just the comb, for short). We will use the argument outlined above to obtain the set of eigenvalues of the comb.

Let us show that there are no eigenvalues in the case  $C_1 = 0$ . Assume  $f$  is an eigenvector with  $f(v_n) = 0$  and set  $y_j = f(v_j)$  and  $z_j = f(w_j)$  for  $j = 1, 2, \dots, n$ . The adjacency matrix of  $S$  for the graph  $H$  is the following  $2n \times 2n$  matrix:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ \hline 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

If we set  $T$  to be the upper-left  $n \times n$  corner of the matrix above,  $Y = (y_1, y_2, \dots, y_n)$  and  $Z = (z_1, z_2, \dots, z_n)$ , the eigenvalue problem  $Sf = \lambda f$  becomes  $TY + Z = \lambda Y$  and  $Y = \lambda Z$ . If  $\lambda = 0$  then  $Y = 0$  and hence  $Z = 0$ , hence  $\lambda \neq 0$ .

Now, since we want  $y_n = 0$ , it follows that  $z_n = 0$ . The last entry of the equation  $TY + Z = \lambda Y$  is  $y_{n-1} + z_n = \lambda y_n$  which implies that  $y_{n-1} = 0$  and hence that  $z_{n-1} = 0$ . The penultimate entry of  $TY + Z = \lambda Y$  is  $y_{n-2} + y_n + z_{n-1} = \lambda y_{n-1}$  which implies that  $y_{n-2} = 0$  and hence  $z_{n-2} = 0$ . Proceeding in this manner, we obtain that  $Y = 0$  and  $Z = 0$  and hence there are no eigenvalues in the case  $C_1 = 0$ .

Now let us deal with the case  $C_1 = 1$ . In this case, setting  $x_j = f(v_j)$  and  $a_j = f(w_j)$  for  $j = 1, 2, \dots, n$ , the system of equations (3) becomes

$$\begin{aligned} a_1 + x_2 &= \lambda x_1 \\ a_j + x_{j-1} + x_{j+1} &= \lambda x_j \quad \text{for all } j = 2, 3, \dots, n-2 \\ a_{n-1} + x_{n-2} + 1 &= \lambda x_{n-1} \\ a_n + x_{n-1} + b &= \lambda \\ x_j &= \lambda a_j \quad \text{for all } j = 1, 2, \dots, n, \end{aligned}$$

where, as mentioned before,  $x_n = 1$ .



First, observe that if  $\lambda = 0$ , this would contradict the fact that  $1 = x_n = \lambda a_n$ , and hence we may assume that  $\lambda \neq 0$ . Since we have  $a_j = \frac{x_j}{\lambda}$  for all  $j = 1, 2, \dots, n$  we can rewrite the above system as

$$\begin{aligned} \frac{1}{\lambda}x_1 + x_2 &= \lambda x_1 \\ \frac{1}{\lambda}x_j + x_{j-1} + x_{j+1} &= \lambda x_j \quad \text{for all } j = 2, 3, \dots, n-2 \\ \frac{1}{\lambda}x_{n-1} + x_{n-2} + 1 &= \lambda x_{n-1} \\ \frac{1}{\lambda} + x_{n-1} + b &= \lambda. \end{aligned}$$

Thus we need to find a solution to the system  $AX = C$  where

$$A = \begin{pmatrix} \lambda - 1/\lambda & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & \lambda - 1/\lambda & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \lambda - 1/\lambda & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda - 1/\lambda & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & \lambda - 1/\lambda \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \lambda - 1/\lambda - b \end{pmatrix},$$

where  $A$  is a  $n \times (n-1)$  matrix. Clearly, the columns of  $A$  are linearly independent, and hence the rank of  $A$  is  $n-1$ . Therefore there is a solution to  $AX = C$  if and only if the augmented matrix  $(A|C)$  has rank  $n-1$ . This occurs if and only if

$$\det \begin{pmatrix} \lambda - 1/\lambda & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & \lambda - 1/\lambda & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & \lambda - 1/\lambda & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda - 1/\lambda & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & \lambda - 1/\lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \lambda - 1/\lambda - b \end{pmatrix} = 0$$

Since  $\lambda = b + 1/b$ , we have  $\lambda - 1/\lambda = \frac{b^4 + b^2 + 1}{b(b^2 + 1)}$  and  $\lambda - 1/\lambda - b = \frac{1}{b(b^2 + 1)}$ , and the above determinant can be thought of as a function of  $b$ . Denote the determinant of the  $n \times n$  matrix by  $P_n(b)$ .

An explicit expression for the function  $P_n$  can be obtained recursively. Indeed, set  $P_1(b) := \lambda - 1/\lambda - b$  and compute  $P_2$  directly to obtain

$$P_2(b) = \frac{-b^6 - b^4 + 1}{(b(b^2 + 1))^2}.$$

Then, it easily follows that  $P_n(b) = (\lambda - 1/\lambda)P_{n-1}(b) - P_{n-2}(b)$ , for  $n \geq 3$ .

We would like to find out if there are solutions  $b$  of the equation  $P_n(b) = 0$  of modulus less than one. It can be shown by induction that, for  $n \geq 2$  we have that  $h_n(b) := (b(b^2 + 1))^n P_n(b)$  is an even polynomial of degree  $4n - 2$ , the coefficient of the term of degree  $4n - 2$  is  $-1$  and constant term is equal to 1. In fact, it can be easily seen that, if we set  $h_1(b) = 1$  and  $h_2(b) = -b^6 - b^4 + 1$ , the polynomial  $h_n$  is given by

$$h_n(b) = (b^4 + b^2 + 1)h_{n-1}(b) - b^2(b^2 + 1)^2 h_{n-2}(b),$$

for  $n \geq 3$ .

To obtain eigenvalues of the comb, it therefore suffices to compute the roots of  $h_n$  of modulus less than one. Since  $h_n(0) = 1$  and the coefficient of the term of degree  $4n - 2$  is  $-1$ , it follows that  $h_n$  has at least one positive root, for each  $n \geq 2$ .

It is straightforward to check that  $h_2(b) = -b^6 - b^4 + 1$  has exactly one positive root, which happens to be less than 1. Let us denote by  $\alpha_n$  the smallest positive root of  $h_n$ , for  $n \geq 2$ . We will show that the sequence  $(\alpha_n)$  is decreasing. First, observe that

$$h_3(\alpha_2) = (\alpha_2^4 + \alpha_2^2 + 1)h_2(\alpha_2) - \alpha_2^2(\alpha_2^2 + 1)^2 h_1(\alpha_2) = -\alpha_2^2(\alpha_2^2 + 1)^2.$$

Hence  $h_3(\alpha_2) < 0$  and since  $h_3(0) = 1 > 0$ , it follows that  $\alpha_3 < \alpha_2$ . Now, assume that  $\alpha_2 > \alpha_3 > \dots > \alpha_{k-1} > \alpha_k$ . Then,

$$h_{k+1}(\alpha_k) = (\alpha_k^4 + \alpha_k^2 + 1)h_k(\alpha_k) - \alpha_k^2(\alpha_k^2 + 1)^2 h_{k-1}(\alpha_k) = -\alpha_k^2(\alpha_k^2 + 1)^2 h_{k-1}(\alpha_k).$$

Since  $h_{k-1}(\alpha_k) > 0$  (otherwise, since  $h_{k-1}(0) = 1$ , it would follow that  $\alpha_{k-1} \leq \alpha_k$ ) we have that  $h_{k+1}(\alpha_k) < 0$  and hence  $\alpha_{k+1} < \alpha_k$ . Since the sequence  $(\alpha_n)$  is decreasing and  $\alpha_2 < 1$ , it follows that  $h_n$  has at least two roots in the interval  $(-1, 1)$  and hence the comb has at least two eigenvalues (one the negative of the other).

One can check using a computer algebra system or graphical software, that for  $2 \leq n \leq 6$  there are exactly two roots of  $h_n$  inside the interval  $(-1, 1)$ , for  $7 \leq n \leq 10$  there are exactly four roots of  $h_n$  inside the interval  $(-1, 1)$ , for  $11 \leq n \leq 14$  there are exactly six roots of  $h_n$  inside the interval  $(-1, 1)$ , for  $15 \leq n \leq 19$  there are exactly eight roots of  $h_n$  inside the interval  $(-1, 1)$ , for  $20 \leq n \leq 23$  there are exactly ten roots of  $h_n$  inside the interval  $(-1, 1)$ , and for  $n \geq 24$  there are at least twelve roots of  $h_n$  inside the interval  $(-1, 1)$ .

This will correspond to two eigenvalues of the comb for  $2 \leq n \leq 6$ , four eigenvalues of the comb for  $7 \leq n \leq 10$ , to six eigenvalues of the comb for  $11 \leq n \leq 14$ , to eight eigenvalues of the comb for  $15 \leq n \leq 19$ , to ten eigenvalues of the comb for  $20 \leq n \leq 23$ , and to at least twelve eigenvalues of the comb for  $n \geq 24$ . Can one compute the number of eigenvalues for any given  $n$ ? We leave that question open for future research.

One can also check, with the help of a computer, that for  $2 \leq n \leq 30$  all eigenvalues of the comb are contained in the interval  $[-1 - \sqrt{2}, -2] \cup [2, 1 + \sqrt{2}]$ . Does all of the above hold for all values of  $n$ ?

**5.4. Full spectrum of finite graphs with an infinite tail.** We should point out that the above results actually give the full spectrum of each of the graphs with an infinite tail attached. Indeed, in [4] Duren finds the spectrum of Toeplitz operators with finitely many nonzero diagonals. Applying those results to the case of the Toeplitz operator with matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we obtain that the spectrum of  $T$  in  $\ell^p$  ( $1 \leq p \leq \infty$ ) is the interval  $[-2, 2]$ . In fact, it can be deduced from Duren's result that the essential spectrum (see, for example, [16] for the definition of the essential spectrum) of  $T$  is the interval  $[-2, 2]$ .

Observe that  $T$  is the adjacency matrix of the infinite tail. Since all the graphs in this section can be seen as perturbations of the infinite tail by an operator of finite rank, it follows that the essential spectrum of each finite graph with an infinite tail attached (in  $\ell^p$ , with  $1 \leq p \leq \infty$ ) is the interval  $[-2, 2]$ . Hence, the spectrum (in  $\ell^p$ ) of each finite graph with an infinite tail attached is the union of the interval  $[-2, 2]$  and the set of eigenvalues (see [16] for details on the essential spectrum and point spectrum).

## 6. SPECTRUM OF THE INFINITE COMB

Let us consider the "infinite comb". That is,  $G$  is the graph formed with the vertices  $\{v_j, w_j : j \in \mathbb{N}\}$  and the edges given by the relations  $v_j \sim w_j$  for all  $j \in \mathbb{N}$ ,  $v_j \sim v_{j+1}$  for all  $j \in \mathbb{N}$ . First, let us observe that there are no eigenvalues for  $S$  in this case. Indeed, assume  $Sf = \lambda f$  and set  $x_j = f(v_j)$  and  $a_j = f(w_j)$  for  $j \in \mathbb{N}$ . We obtain the equations

$$\begin{aligned} a_1 + x_2 &= \lambda x_1, \\ a_j + x_{j-1} + x_{j+1} &= \lambda x_j, & \text{for all } j \geq 2, \\ x_j &= \lambda a_j, & \text{for all } j \in \mathbb{N}. \end{aligned}$$

Clearly, if  $\lambda = 0$ , then  $f = 0$  so we may assume  $\lambda \neq 0$ . Then, the above equations can be written as

$$(4) \quad \begin{aligned} x_2 &= \left( \lambda - \frac{1}{\lambda} \right) x_1, \\ x_{j-1} + x_{j+1} &= \left( \lambda - \frac{1}{\lambda} \right) x_j, & \text{for all } j \geq 2. \end{aligned}$$

But, as is well-known, the infinite matrix

$$T := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has no eigenvalues in  $\ell^p$ , for  $1 \leq p < \infty$ , and hence the system (4) has no  $p$ -summable solution.

Let us compute the spectrum of  $S$  in  $\mathbf{L}^p(G)$ . Let  $f_j := \chi_{v_j}$  and  $g_j := \chi_{w_j}$ . Then it is clear that the set  $\{f_j : j \in \mathbb{N}\} \cup \{g_j : j \in \mathbb{N}\}$  is an orthonormal basis for  $\mathbf{L}^p(G)$ . If we write the matrix of  $S$  with respect to this basis, it is clear that it can be written, in block form, as

$$S = \left( \begin{array}{c|c} T & I \\ \hline I & 0 \end{array} \right),$$

where  $T$  is as defined above. To find the spectrum of  $S$  we need to find for what values of  $\lambda$  the matrix

$$S - \lambda = \left( \begin{array}{c|c} T - \lambda & I \\ \hline I & -\lambda \end{array} \right),$$

is invertible. If  $\lambda = 0$ , then  $S$  is invertible and

$$S^{-1} = \left( \begin{array}{c|c} 0 & I \\ \hline I & -T \end{array} \right),$$

Now assume  $\lambda \neq 0$ . It is well known and easily verified that a  $2 \times 2$  block matrix of the form above is invertible if and only if  $T - \lambda + 1/\lambda$  is invertible. In that case, in the style of [6, p. 18], the inverse is

$$(S - \lambda)^{-1} = \left( \begin{array}{c|c} (T - \lambda + \frac{1}{\lambda})^{-1} & \frac{1}{\lambda}(T - \lambda + \frac{1}{\lambda})^{-1} \\ \hline \frac{1}{\lambda}(T - \lambda + \frac{1}{\lambda})^{-1} & -\frac{1}{\lambda} + \frac{1}{\lambda^2}(T - \lambda + \frac{1}{\lambda})^{-1} \end{array} \right).$$

Hence  $\lambda \in \sigma(S)$  if and only if  $\lambda - \frac{1}{\lambda} \in \sigma(T)$ . Recall  $\sigma(T) = [-2, 2]$  (see, for example, [11]). It can easily be checked that  $\lambda - \frac{1}{\lambda} \in [-2, 2]$  if and only if  $\lambda \in [-1 - \sqrt{2}, 1 - \sqrt{2}] \cup [-1 + \sqrt{2}, 1 + \sqrt{2}]$ . Hence

$$\sigma(S) = [-1 - \sqrt{2}, 1 - \sqrt{2}] \cup [-1 + \sqrt{2}, 1 + \sqrt{2}].$$

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