

# POSITIVE FIXED POINTS OF CUBIC OPERATORS ON $\mathbb{R}^2$ AND GIBBS MEASURES

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**ABSTRACT.** In this paper we consider one model with nearest-neighbor interactions and with the set  $[0, 1]$  of spin values on the Cayley tree of order three. Translation-invariant Gibbs measures for the model are studied. Results are proved by using properties of the positive fixed points of a cubic operator in the cone  $\mathbb{R}_+^2$ .

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## 1. INTRODUCTION

A lot of nonlinear operators are connected with problems in statistical physics, biology, thermodynamics, statistical mechanics and so on. One of the central problem in statistical physics is studying the existence of phase transitions. On the other hand side, phase transitions are connected with the theory of Gibbs measures [9]. In the theory of Gibbs measures there are a lot of papers which devoted to Gibbs measures on a Cayley tree [16]. As a usual, the problems of studying splitting Gibbs measure for models on a Cayley tree can be divided into three classes: 1) for models with a finite set of spin values; 2) for models with a countable set of spin values; 3) for models with a continuum set of spin values. Note that, the problems of studying Gibbs measures for models with a finite and a countable set of spin values on a Cayley tree are reduced to study the systems of algebraical or functional equations [1]- [8], [18]- [20]. One of the main factor is that studying translation-invariant Gibbs measures for models with a continuum set of spin values is reduced to find positive fixed points of nonlinear integral operator [10]- [15], [22].

For the case continuum set of spin values (i.e.,  $[0, 1]$ ): it's considered that models with the nearest-neighbor interactions on a Cayley tree [10]- [15], [22]. In [13], [22] is given that the existence of translation-invariant Gibbs measure for the models is equivalent to the existence of a positive fixed point of Hammerstein's nonlinear integral operator. It is proved that the existence of translation-invariant Gibbs measures for the models on a Cayley tree of an arbitrary order and shown that the uniqueness of translation-invariant Gibbs measures on the Cayley tree of order one (see [14], [12]).

It is found that a sufficient condition for the model has the unique translation-invariant splitting Gibbs measure and constructed models that each constructed model has at least two periodic Gibbs measures (see [12], [11]).

In [22] it is considered the models on the Cayley tree of order two and reduced to study translation-invariant Gibbs measures to the description of the positive fixed points of some quadratic operator on  $\mathbb{R}^2$ . Also, it's given sufficient conditions, which the model has one, two or three translation-invariant Gibbs measure by using quadratic operators.

In this paper we consider the translation-invariant Gibbs measures for models in [22] on the Cayley tree of order three.

## 2. PRELIMINARIES

A Cayley tree  $\Gamma^k = (V, L)$  of order  $k \geq 1$  is an infinite homogeneous tree, i.e., a graph without cycles, with exactly  $k + 1$  edges incident to each vertices. Here  $V$  is the set of vertices and  $L$  that of edges.

Consider models where the spin takes values in the set  $[0, 1]$ , and is assigned to the vertices of the tree. For  $A \subset V$  a configuration  $\sigma_A$  on  $A$  is an arbitrary function  $\sigma_A : A \rightarrow [0, 1]$ . Denote  $\Omega_A = [0, 1]^A$  is the set of all configurations on  $A$ . A configuration  $\sigma$  on  $V$  is defined as a function  $x \in V \mapsto \sigma(x) \in [0, 1]$ ; the set of all configurations is  $[0, 1]^V$ . The Hamiltonian of the model is :

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \xi_{\sigma(x), \sigma(y)}, \quad \sigma \in \Omega_V \quad (2.1)$$

where  $J \in \mathbb{R} \setminus \{0\}$  and  $\beta = \frac{1}{T}$ ,  $T > 0$  is temperature,  $\xi : (u, v) \in [0, 1]^2 \rightarrow \xi_{uv} \in \mathbb{R}$  is a given bounded, measurable function. As usually,  $\langle x, y \rangle$  stands for the nearest neighbor vertices.

Write  $x < y$  if the path from  $x^0$  to  $y$  goes through  $x$ . Call vertex  $y$  a direct successor of  $x$  if  $y > x$  and  $x, y$  are nearest neighbors. Denote by  $S(x)$  the set of direct successors of  $x$ . Observe that any vertex  $x \neq x^0$  has  $k$  direct successors and  $x^0$  has  $k + 1$ .

Let  $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0, 1]) \in R^{[0,1]}$  be mapping of  $x \in V \setminus \{x^0\}$ .

Now, we consider the following equation:

$$f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{tu}) f(u, y) du}{\int_0^1 \exp(J\beta\xi_{0u}) f(u, y) du}. \quad (2.2)$$

Here, and below  $f(t, x) = \exp(h_{t,x} - h_{0,x})$ ,  $t \in [0, 1]$  and  $du = \lambda(du)$  is the Lebesgue measure.

It is known that, for the splitting Gibbs measure for the model (2.1) to exist, the existence a solution of the equation (2.2) for any  $x \in V \setminus \{x^0\}$  is necessary and sufficient. Thus, we know that the splitting Gibbs measure  $\mu$  for the model (2.1) depends on the function  $f(t, x)$  and each a splitting Gibbs measure corresponds to a solution  $f(t, x)$  of the equation (2.2). Thus, note that, the number of the Gibbs measures for the model (2.1) is equal to the number of the positive solutions of the integral equation (2.2).

A detailed definition of the splitting Gibbs measure for models with nearest-neighbor interactions and a continuum set of spin values on the Cayley tree is given in the papers [10]- [15], [22]. In the future, instance of the term the splitting Gibbs measure, we will call the Gibbs measure.

Note, that the analysis of solutions to (2.2) is not easy. It's difficult to give a full description for the given potential function  $\xi_{t,u}$ . We study Gibbs measures of the model (2.2) in the case,  $f(t, x) = f(t)$  for all  $x \in S(x)$ . Such Gibbs measure is called translation-invariant.

We put

$$C_+[0, 1] = \{f \in C[0, 1] : f(x) \geq 0\}, \quad C_{>}[0, 1] = C_+[0, 1] \setminus \{\theta \equiv 0\}.$$

Let  $\xi_{tu}$  is a continuous function. For every  $k \in \mathbb{N}$  we consider an integral operator  $H_k$  acting in the cone  $C_+[0, 1]$  by the rule

$$(H_k f)(t) = \int_0^1 K(t, u) f^k(u) du, \quad k \in \mathbb{N}$$

where  $K(t, u) = \exp(J\beta\xi_{tu})$ .

The operator  $H_k$  is called Hammerstein's integral operator of order homogeneously  $k$ .

**Lemma 2.1.** [12] *Let  $k \geq 2$ . The Hamiltonian  $H$  (2.1) has a translation-invariant Gibbs measure iff the Hammerstein's integral operator  $H_k$  has a positive eigenvalue, i.e. the Hammerstein's integral equation*

$$H_k f = \lambda f, \quad f \in C_+[0, 1] \quad (2.3)$$

*has a nonzero positive solution for some  $\lambda > 0$ .*

Moreover, if the number  $\lambda_0 > 0$  is an eigenvalue of the operator  $H_k, k \geq 2$ , then an arbitrary positive number is an eigenvalue of the operator  $H_k$ . On the other hand, a number of positive eigenfunctions corresponding to the positive eigenvalues  $\lambda_1 > 0$  and  $\lambda_2 > 0$  of the operator  $H_k$  are equal (see [12]). Hence, we can conclude that the following lemma:

**Lemma 2.2.** *Let  $k \geq 2$ . A number  $N^{tigm}(H)$  of translation-invariant Gibbs measures for the model (2.1) the following equality holds:*

$$N^{tigm}(H) = N_+^{fix}(H_k),$$

*where  $N_+^{fix}(B)$  is a number of nontrivial positive fixed points of the operator  $B$ .*

### 3. MAIN RESULTS

Let  $\varphi_1(t)$ ,  $\varphi_2(t)$  and  $\phi_1(t)$ ,  $\phi_2(t)$  are strictly positive functions belong to  $C_+[0, 1]$ . We consider the Hamiltonian (2.1) on the Cayley tree  $\Gamma^3$  with the function of potential

$$\xi_{t,u} = \frac{\ln(\phi_1(t)\varphi_1(u) + \phi_2(t)\varphi_2(u))}{J\beta}. \quad (3.1)$$

We consider Hammerstein's integral operator  $H_3$  on  $C_+[0, 1]$  by the equality

$$(H_3 f)(t) = \int_0^1 (\phi_1(t)\varphi_1(u) + \phi_2(t)\varphi_2(u)) f^3(u) du.$$

Introduce the following denotes

$$\begin{aligned} \alpha_{11} &= \int_0^1 \varphi_1(u) \phi_1^3(u) du, \quad \alpha_{12} = \int_0^1 \varphi_1(u) \phi_1^2(u) \phi_2(u) du, \quad \alpha_{21} = \int_0^1 \varphi_1(u) \phi_1(u) \phi_2^2(u) du, \\ \alpha_{22} &= \int_0^1 \varphi_1(u) \phi_2^3(u) du, \quad \beta_{11} = \int_0^1 \varphi_2(u) \phi_1^3(u) du, \quad \beta_{12} = \int_0^1 \varphi_2(u) \phi_1^2(u) \phi_2(u) du, \\ \beta_{21} &= \int_0^1 \varphi_2(u) \phi_1(u) \phi_2^2(u) du, \quad \beta_{22} = \int_0^1 \varphi_2(u) \phi_2^3(u) du. \end{aligned}$$

It's easy to check that  $\alpha_{ij} > 0$  and  $\beta_{ij} > 0$  for all  $i, j \in \{1, 2\}$ .

We define a polynomial  $P_4(\xi)$  with fourth degree

$$P_4(\xi) = \mu_0 \xi^4 + \mu_1 \xi^3 + 3\mu_2 \xi^2 + \mu_3 \xi - \mu_4, \quad (3.2)$$

where

$$\mu_0 = \alpha_{22}, \quad \mu_1 = 3\alpha_{21} - \beta_{22}, \quad \mu_2 = \alpha_{12} - \beta_{21}, \quad \mu_3 = \alpha_{11} - 3\beta_{12}, \quad \mu_4 = \beta_{11}.$$

Denote

$$\begin{aligned} D &= \mu_1^2 - 8\mu_0\mu_2, \\ \theta_k &= 2\sqrt{-\frac{p}{3}} \cos\left(\frac{\alpha + 2\pi(k-2)}{3}\right), \quad k = \overline{1, 2, 3}, \end{aligned}$$

where

$$p = -\frac{3\mu_1^2}{16\mu_0^2} + \frac{3\mu_2}{2\mu_0}, \quad q = \frac{\mu_1^3}{32\mu_0^3} - \frac{3\mu_1\mu_2}{8\mu_0^2} + \frac{\mu_3}{4\mu_0}$$

and

$$\cos \alpha = \frac{q}{3} \left( -\frac{3}{p} \right)^{\frac{3}{2}}, \quad \alpha \in [0, \pi].$$

We put

$$\gamma_1 = \theta_3 - \frac{\mu_1}{4\mu_0}, \quad \gamma_2 = \theta_1 - \frac{\mu_1}{4\mu_0}, \quad \gamma_3 = \theta_2 - \frac{\mu_1}{4\mu_0}.$$

**Theorem 3.1.** *Let  $D \leq 0$ . Then the model (2.1) on the Cayley tree of order three has the unique translation-invariant Gibbs measure, i.e.,  $N^{tigm}(H) = 1$ .*

**Theorem 3.2.** *Let  $D > 0$ . If one of the following conditions*

(a)  $\gamma_2 \leq 0$ ,

(b)  $\gamma_2 > 0$ ,  $P_4(\gamma_2) < 0$ ,

(c)  $\gamma_2 > 0$ ,  $P_4(\gamma_3) > 0$ ,

*is satisfied, then the model (2.1) on the Cayley tree of order three has the unique translation-invariant Gibbs measure, i.e.,  $N^{tigm}(H) = 1$ .*

**Theorem 3.3.** *Let  $D > 0$ . If one of the following conditions*

(d)  $\gamma_2 > 0$ ,  $P_4(\gamma_2) = 0$ ,

(e)  $\gamma_2 > 0$ ,  $P_4(\gamma_3) = 0$ ,

*is satisfied, then the model (2.1) on the Cayley tree of order three has two translation-invariant Gibbs measures, i.e.,  $N^{tigm}(H) = 2$ .*

**Theorem 3.4.** *Let  $D > 0$ . If the following condition*

(f)  $\gamma_2 > 0$ ,  $P_4(\gamma_2) > 0$ ,  $P_4(\gamma_3) < 0$ ,

*is satisfied, then the model (2.1) on the Cayley tree of order three has three translation-invariant Gibbs measures, i.e.,  $N^{tigm}(H) = 3$ .*

#### 4. POSITIVE FIXED POINTS OF CUBIC OPERATORS ON $\mathbb{R}^2$

We put

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\},$$

$$\mathbb{R}_>^2 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

We consider the following cubic operator (CO)  $\mathcal{C}$  on the cone  $\mathbb{R}_+^2$ :

$$\mathcal{C}(x, y) = (a_{11}x^3 + 3a_{12}x^2y + 3a_{21}xy^2 + a_{22}y^3, b_{11}x^3 + 3b_{12}x^2y + 3b_{21}xy^2 + b_{22}y^3), \quad (4.1)$$

where  $a_{ij} > 0$  and  $b_{ij} > 0$  for all  $i, j \in \{1, 2\}$ .

Clearly, an arbitrary nontrivial positive fixed point  $(x_0, y_0) \in \mathbb{R}_+^2$  of the CO  $\mathcal{C}$  is a strictly positive, i.e.  $x_0 > 0$ ,  $y_0 > 0$ . We denote a number of fixed points of the CO  $\mathcal{C}$  belong to  $\mathbb{R}_>^2$  by  $N_{>}^{fix}(\mathcal{C})$ .

**Lemma 4.1.** *If  $\omega = (x_0, y_0) \in \mathbb{R}_{>}^2$  is a fixed point of the CO  $\mathcal{C}$ , then  $\omega \in \mathbb{R}_{>}^2$  and  $\xi_0 = \frac{y_0}{x_0}$  is root of the algebraic equation*

$$a_{22}\xi_0^4 + (3a_{21} - b_{22})\xi_0^3 + 3(a_{12} - b_{21})\xi_0^2 + (a_{11} - 3b_{12})\xi_0 - b_{11} = 0. \quad (4.2)$$

*Proof.* Let the point  $\omega = (x_0, y_0) \in \mathbb{R}_{>}^2$  be a fixed point of CO  $\mathcal{C}$ . Then

$$a_{11}x_0^3 + 3a_{12}x_0^2y_0 + 3a_{21}x_0y_0^2 + a_{22}y_0^3 = x_0, \quad b_{11}x_0^3 + 3b_{12}x_0^2y_0 + 3b_{21}x_0y_0^2 + b_{22}y_0^3 = y_0.$$

From the notation  $\frac{y_0}{x_0} = \xi_0$  we obtain

$$a_{11}x_0^3 + 3a_{12}x_0^3\xi_0 + 3a_{21}x_0^3\xi_0^2 + a_{22}x_0^3\xi_0^3 = x_0, \quad b_{11}x_0^3 + 3b_{12}x_0^3\xi_0 + 3b_{21}x_0^3\xi_0^2 + b_{22}x_0^3\xi_0^3 = x_0\xi_0.$$

Consequently,

$$x_0^3(a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3) = x_0, \quad x_0^3(b_{11} + 3b_{12}\xi_0 + 3b_{21}\xi_0^2 + b_{22}\xi_0^3) = \xi_0 x_0.$$

Hence, we have

$$\frac{1}{\xi_0} = \frac{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3}{b_{11} + 3b_{12}\xi_0 + 3b_{21}\xi_0^2 + b_{22}\xi_0^3}.$$

By the last equality we get

$$a_{22}\xi_0^4 + (3a_{21} - b_{22})\xi_0^3 + 3(a_{12} - b_{21})\xi_0^2 + (a_{11} - 3b_{12})\xi_0 - b_{11} = 0.$$

This completes the proof. ■

**Lemma 4.2.** *If  $\xi_0$  is a root of the algebraic equation (4.2), then the point  $\omega_0 = (x_0, \xi_0 x_0) \in \mathbb{R}_{>}^2$  is a fixed point of the CO  $\mathcal{C}$ , where*

$$x_0 = \frac{1}{\sqrt{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3}}. \quad (4.3)$$

*Proof.* Let  $\xi_0 > 0$  and  $\xi_0$  is a root of the equation (4.2). Put  $y_0 = \xi_0 x_0$ , where  $x_0$  is a given by the equality (4.3) and  $\omega_0 = (x_0, \xi_0 x_0)$ . From the equality  $y_0 = \xi_0 x_0$  we have

$$\begin{aligned} a_{11}x_0^3 + 3a_{12}x_0^2y_0 + 3a_{21}x_0y_0^2 + a_{22}y_0^3 &= a_{11}x_0^3 + 3a_{12}x_0^2(\xi_0 x_0) + 3a_{21}x_0(\xi_0 x_0)^2 + a_{22}(\xi_0 x_0)^3 = \\ &= x_0^3 \cdot (a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3) = \frac{1}{\sqrt{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3}}, \end{aligned}$$

i.e.

$$a_{11}x_0^3 + 3a_{12}x_0^2y_0 + 3a_{21}x_0y_0^2 + a_{22}y_0^3 = x_0.$$

On the other hand

$$a_{22}\xi_0^4 + (3a_{21} - b_{22})\xi_0^3 + 3(a_{12} - b_{21})\xi_0^2 + (a_{11} - 3b_{12})\xi_0 - b_{11} = 0.$$

Then we get

$$b_{11} + 3b_{12}\xi_0 + 3b_{21}\xi_0^2 + b_{22}\xi_0^3 = a_{11}\xi_0 + 3a_{12}\xi_0^2 + 3a_{21}\xi_0^3 + a_{22}\xi_0^4 = \xi_0(a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3).$$

From the last equality we have

$$\begin{aligned} \frac{\xi_0}{\sqrt{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3}} &= \frac{b_{11} + 3b_{12}\xi_0 + 3b_{21}\xi_0^2 + b_{22}\xi_0^3}{(\sqrt{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3})^3} = \\ &= b_{11}x_0^3 + 3b_{12}x_0^2y_0 + 3b_{21}x_0y_0^2 + b_{22}y_0^3 = y_0. \end{aligned}$$

This completes the proof. ■

We put

$$\mu_0 = a_{22}, \quad \mu_1 = 3a_{21} - b_{22}, \quad \mu_2 = a_{12} - b_{21}, \quad \mu_3 = a_{11} - 3b_{12}, \quad \mu_4 = b_{11}$$

and define a polynomial  $P_4(\xi)$  with fourth degree:

$$P_4(\xi) = \mu_0 \xi^4 + \mu_1 \xi^3 + 3\mu_2 \xi^2 + \mu_3 \xi - \mu_4. \quad (4.4)$$

**Lemma 4.3.** *The CO C has at least one positive fixed point in  $\mathbb{R}_>^2$ , i.e.  $N_{>}^{fix}(\mathcal{C}) \geq 1$ .*

*Proof.* Clearly, that  $P_4(0) = -b_{11}$  and  $P_4(+\infty) = +\infty$ . It means that there exists  $c > 0$  such that  $P_4(c) = 0$ . By the lemma 4.2  $(x_0, cx_0)$  is a fixed point of CO C, where

$$x_0 = \frac{1}{\sqrt{a_{11} + 3a_{12}c + 3a_{21}c^2 + a_{22}c^3}}.$$

■

**Lemma 4.4.** *A number of strictly positive fixed points of the CO C less than or equal to three, i.e.,  $1 \leq N_{>}^{fix}(\mathcal{C}) \leq 3$ .*

*Proof.* We have the following table for the number of sign changes of the coefficients of the polynomial  $P_4(\xi)$ :

| $P_4(\xi)$ | $\mu_0$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | the number of sign changes |
|------------|---------|---------|---------|---------|---------|----------------------------|
| 1.         | +       | +       | +       | +       | -       | 1                          |
| 2.         | +       | +       | +       | -       | -       | 1                          |
| 3.         | +       | +       | -       | -       | -       | 1                          |
| 4.         | +       | -       | -       | -       | -       | 1                          |
| 5.         | +       | -       | -       | +       | -       | 3                          |
| 6.         | +       | -       | +       | +       | -       | 3                          |
| 7.         | +       | +       | -       | +       | -       | 3                          |
| 8.         | +       | -       | +       | -       | -       | 3                          |

By using this table and the Descartes rule, we can conclude that a number of the positive solutions of the polynomial  $P_4(\xi)$  is not more than three (see [21] pp. 27-29), i.e.  $1 \leq N_{>}^{fix}(\mathcal{C}) \leq 3$ .

■

Denote

$$D = \mu_1^2 - 8\mu_0\mu_2, \\ \theta_k = 2\sqrt{-\frac{p}{3}} \cos\left(\frac{\alpha + 2\pi(k-2)}{3}\right), \quad k = \overline{1, 2, 3},$$

where

$$p = -\frac{3\mu_1^2}{16\mu_0^2} + \frac{3\mu_2}{2\mu_0}, \quad q = \frac{\mu_1^3}{32\mu_0^3} - \frac{3\mu_1\mu_2}{8\mu_0^2} + \frac{\mu_3}{4\mu_0}$$

and

$$\cos \alpha = \frac{q}{3} \left(-\frac{3}{p}\right)^{\frac{3}{2}}, \quad \alpha \in [0, \pi].$$

We put

$$\gamma_1 = \theta_3 - \frac{\mu_1}{4\mu_0}, \quad \gamma_2 = \theta_1 - \frac{\mu_1}{4\mu_0}, \quad \gamma_3 = \theta_2 - \frac{\mu_1}{4\mu_0}.$$

**Theorem 4.5.** *Let  $D \leq 0$ , then the CO C has the unique fixed point in  $\mathbb{R}_>^2$ , i.e.,  $N_{>}^{fix}(\mathcal{C}) = 1$ .*

*Proof.* Let  $D \leq 0$ . Then  $P_4''(\xi) \geq 0$  for all  $\xi \in \mathbb{R}$ . This shows that schedule of the function  $P_4(\xi)$  is convex on  $\mathbb{R}$  and  $P_4(0) = -b_{11} < 0$ ,  $P_4(\pm\infty) = +\infty$ . Consequently, there exists the unique number  $\xi_0 > 0$  such that  $P_4(\xi_0) = 0$ . ■

**Theorem 4.6.** *Let  $D > 0$ . If the CO  $\mathcal{C}$  satisfies one of the following conditions*

$$(a) \quad \gamma_2 \leq 0,$$

$$(b) \quad \gamma_2 > 0, \quad P_4(\gamma_2) < 0,$$

$$(c) \quad \gamma_2 > 0, \quad P_4(\gamma_3) > 0,$$

*then the CO  $\mathcal{C}$  has the unique fixed point in  $\mathbb{R}_{>}^2$ , i.e.  $N_{>}^{fix}(\mathcal{C}) = 1$ .*

*Proof.* Let  $D > 0$ . We have

$$P_4'(\xi) = 4\mu_0\xi^3 + 3\mu_1\xi^2 + 6\mu_2\xi + \mu_3. \quad (4.5)$$

We can find roots of the equation  $P_4'(\xi) = 0$  by the Vieta's method (see [23]). Put

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2.$$

From  $D > 0$  we have  $Q < 0$ . Then for the numbers  $\theta_1, \theta_2, \theta_3$  we obtain  $\theta_3 < \theta_1 < \theta_2$ . By the Viet's method the numbers  $\gamma_1, \gamma_2, \gamma_3$  are roots of the polynomial  $P_4'(\xi)$ . Then the polynomial  $P_4'(\xi)$  (4.5) has the following form

$$P_4'(\xi) = 4\mu_0(\xi - \gamma_1)(\xi - \gamma_2)(\xi - \gamma_3).$$

It follows the function  $P_4(\xi)$  is an increasing (decreasing) on the set  $(\gamma_1, \gamma_2) \cup (\gamma_3, +\infty)$   $((-\infty, \gamma_1) \cup (\gamma_2, \gamma_3))$ . The function  $P_4(\xi)$  has a local maximum value at the point  $\gamma_2$  and local minimum values at the points  $\gamma_1$  and  $\gamma_3$ .

(a) Let  $\gamma_2 < 0$ . Clearly,  $\min_{\xi \in (\gamma_2, +\infty)} P_4(\xi) = P_4(\gamma_3)$  and the function  $P_4(\xi)$  is an increasing on the interval  $(\gamma_3, +\infty)$ . On the other hand, we have  $P_4(0) < 0$ . Consequently, we get  $P_4(\gamma_3) < 0$ . It means the polynomial  $P_4(\xi)$  has the unique positive root.

(b) Let  $\gamma_2 > 0$  and  $P_4(\gamma_2) < 0$ . Then  $\max_{\xi \in (\gamma_1, \gamma_3)} P_4(\xi) = P_4(\gamma_2) < 0$ . Analogically, by an increasing property on the interval  $(\gamma_3, +\infty)$  of the function  $P_4(\xi)$  the polynomial  $P_4(\xi)$  has the unique positive root  $\xi_1$  and  $\xi_1 \in (\gamma_3, +\infty)$ .

(c) Let  $\gamma_2 > 0$  and  $P_4(\gamma_3) > 0$ . Then  $\max_{\xi \in (\gamma_1, \gamma_3)} P_4(\xi) = P_4(\gamma_2) > 0$  and  $\min_{\xi \in (\gamma_2, +\infty)} P_4(\xi) = P_4(\gamma_3) > 0$ . Consequently, by the inequality  $P_4(0) < 0$  the polynomial  $P_4(\xi)$  has the unique positive root  $\xi_1$  and  $\xi_1 \in (0, \gamma_2)$ . ■

**Theorem 4.7.** *Let  $D > 0$ . If the CO  $\mathcal{C}$  satisfies one of the following conditions*

$$(d) \quad \gamma_2 > 0, \quad P_4(\gamma_2) = 0,$$

$$(e) \quad \gamma_2 > 0, \quad P_4(\gamma_3) = 0,$$

*then the CO  $\mathcal{C}$  has two fixed points in  $\mathbb{R}_{>}^2$ , i.e.,  $N_{>}^{fix}(\mathcal{C}) = 2$ .*

*Proof.* (d) Let  $\gamma_2 > 0$  and  $P_4(\gamma_2) = 0$ . Then  $\max_{\xi \in (\gamma_1, \gamma_3)} P_4(\xi) = P_4(\gamma_2) = 0$  and the number  $\xi_1 = \gamma_2$  is the root of the polynomial  $P_4(\xi)$ . Since  $P_4(\xi)$  is an increasing on the interval  $(\gamma_3, \infty)$ , the polynomial  $P_4(\xi)$  has a root  $\xi_2 \in (\gamma_3, \infty)$ , because of  $\gamma_3 > 0$  and  $P_4(\gamma_3) < 0$ . Clearly, that the polynomial  $P_4(\xi)$  doesn't have any other root in the  $(\gamma_3, \infty)$ .

(e) Let  $\gamma_2 > 0$ ,  $P_4(\gamma_3) = 0$ . Then by an increasing property of the function  $P_4(\xi)$  on the  $(-\infty, \gamma_2)$  the polynomial  $P_4(\xi)$  has a positive root  $\xi_1 \in (0, \gamma_2)$ . By the other hand  $\min_{\xi \in (\gamma_2, \infty)} P_4(\xi) = P_4(\gamma_3) = 0$  and the number  $\xi_2 = \gamma_3$  is the second positive root of the polynomial  $P_4(\xi)$ . The polynomial  $P_4(\xi)$  doesn't have another root. ■

**Theorem 4.8.** *Let  $D > 0$ . If the CO  $\mathcal{C}$  is satisfied the following condition*

$$(f) \quad \gamma_2 > 0, \quad P_4(\gamma_2) > 0, \quad P_4(\gamma_3) < 0,$$

*then the CO  $\mathcal{C}$  has three fixed points in  $\mathbb{R}_{>}^2$ , i.e.  $N_{>}^{fix}(\mathcal{C}) = 3$ .*

*Proof.* Let  $\gamma_2 > 0$ ,  $P_4(\gamma_2) > 0$ ,  $P_4(\gamma_3) < 0$ . In this case since the function  $P_4(\xi)$  is an increasing on the set  $(\gamma_1, \gamma_2) \cup (\gamma_3, +\infty)$  and a decreasing property on the interval  $(\gamma_2, \gamma_3)$  of the function  $P_4(\xi)$  the polynomial  $P_4(\xi)$  has three positive roots  $\xi_1 \in (0, \gamma_2)$ ,  $\xi_2 \in (\gamma_2, \gamma_3)$  and  $\xi_3 \in (\gamma_3, \infty)$ , as  $P_4(0) = -b_{11} < 0$ ,  $P_4(\gamma_2) > 0$ ,  $P_4(\gamma_3) < 0$ ,  $P_4(+\infty) = +\infty$ . ■

## 5. PROOFS OF THE MAIN RESULTS

Let  $\varphi_1(t)$ ,  $\varphi_2(t)$  and  $\phi_1(t)$ ,  $\phi_2(t)$  are strictly positive functions belong to  $C_+[0, 1]$ . We consider Hammerstein's integral operator  $H_3$  on  $C_+[0, 1]$  by the equality

$$(H_3 f)(t) = \int_0^1 (\phi_1(t)\varphi_1(u) + \phi_2(t)\varphi_2(u))f^3(u)du$$

and cubic operator  $\mathcal{C}$  on  $\mathbb{R}^2$  by the rule

$$\mathcal{C}(x, y) = (\alpha_{11}x^3 + 3\alpha_{12}x^2y + 3\alpha_{21}xy^2 + \alpha_{22}y^3, \quad \beta_{11}x^3 + 3\beta_{12}x^2y + 3\beta_{21}xy^2 + \beta_{22}y^3).$$

Here

$$\begin{aligned} \alpha_{11} &= \int_0^1 \varphi_1(u)\phi_1^3(u)du, \quad \alpha_{12} = \int_0^1 \varphi_1(u)\phi_1^2(u)\phi_2(u)du, \quad \alpha_{21} = \int_0^1 \varphi_1(u)\phi_1(u)\phi_2^2(u)du, \\ \alpha_{22} &= \int_0^1 \varphi_1(u)\phi_2^3(u)du, \quad \beta_{11} = \int_0^1 \varphi_2(u)\phi_1^3(u)du, \quad \beta_{12} = \int_0^1 \varphi_2(u)\phi_1^2(u)\phi_2(u)du, \\ \beta_{21} &= \int_0^1 \varphi_2(u)\phi_1(u)\phi_2^2(u)du, \quad \beta_{22} = \int_0^1 \varphi_2(u)\phi_2^3(u)du. \end{aligned}$$

Clearly, that  $\alpha_{ij} > 0$  and  $\beta_{ij} > 0$  for all  $i, j \in \{1, 2\}$ .

**Lemma 5.1.** *The Hammerstein's integral operator  $H_3$  has a nontrivial positive fixed point iff the CO  $\mathcal{C}$  has a nontrivial positive fixed point and  $N_+^{fix}(H_3) = N_{>}^{fix}(\mathcal{C})$ .*

*Proof.* Let the Hammerstein's integral operator  $H_3$  has a nontrivial positive fixed point  $f(t) \in C_+[0, 1]$ . Put

$$c_1 = \int_0^1 \varphi_1(u)f^3(u)du \tag{5.1}$$

and

$$c_2 = \int_0^1 \varphi_2(u)f^3(u)du. \tag{5.2}$$

Clearly,  $c_1 > 0$ ,  $c_2 > 0$ , i.e.  $(c_1, c_2) \in \mathbb{R}_{>}^2$ . Then for the function  $f(t)$  the equality

$$f(t) = \phi_1(t)c_1 + \phi_2(t)c_2 \tag{5.3}$$

holds and  $f(t) \in C_{>}[0, 1]$ .



Consequently, for the parameters  $c_1, c_2$  from the equality (5.1) and (5.2) we have the following two identities:

$$\begin{aligned} c_1 &= \alpha_{11}c_1^3 + 3\alpha_{12}c_1^2c_2 + 3\alpha_{21}c_1c_2^2 + \alpha_{22}c_2^3, \\ c_2 &= \beta_{11}c_1^3 + 3\beta_{12}c_1^2c_2 + 3\beta_{21}c_1c_2^2 + \beta_{22}c_2^3. \end{aligned}$$

Therefore, the point  $(c_1, c_2)$  is the fixed point of the CO  $\mathcal{C}$ .

(b) Assume, that a point  $(x_0, y_0)$  be a nontrivial positive fixed point of the CO  $\mathcal{C}$  and the numbers  $x_0, y_0$  satisfies the following equalities

$$\begin{aligned} \alpha_{11}x_0^3 + 3\alpha_{12}x_0^2y_0 + 3\alpha_{21}x_0y_0^2 + \alpha_{22}y_0^3 &= x_0, \\ \beta_{11}x_0^3 + 3\beta_{12}x_0^2y_0 + 3\beta_{21}x_0y_0^2 + \beta_{22}y_0^3 &= y_0. \end{aligned}$$

It is easy to verify that the function  $f_0(t) = \phi_1(t)x_0 + \phi_2(t)y_0$  is the fixed point of the Hammerstein's integral operator  $H_3$  and  $f_0(t) \in C_{>}[0, 1]$  as  $(x_0, y_0) \in \mathbb{R}_+^2$ . This completes the proof. ■

For the model (2.1) on the  $\Gamma^3$  with the function of potential (3.1) from Lemma 2.2 and Lemma 5.1 the following equality holds:

$$N^{tigm}(H) = N_+^{fix}(H_3) = N_{>}^{fix}(\mathcal{C}).$$

By the last equality and the Theorems 4.5 - 4.8, we get the Theorems 3.1 - 3.4, respectively.

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