WEYL FAMILIES OF ESSENTIALLY UNITARY PAIRS

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ABSTRACT. It is known that the Weyl families corresponding to unitary boundary pairs $(\mathcal{H},\Gamma)$ belong to the class $\widetilde{\mathcal{R}}(\mathcal{H})$ of Nevanlinna families. Here we extends the theorem to the case of essentially unitary pairs by showing that the closures of members of the Weyl families belong to the class $\widetilde{\mathcal{R}}(\mathcal{H})$. Thus bounded Weyl functions of essentially unitary pairs are of class $\mathcal{R}[\mathcal{H}]$.

1. INTRODUCTION

Throughout $\mathfrak{H}$ and $\mathcal{H}$ denote Hilbert spaces. Let $\Gamma\subseteq\mathfrak{H}^2\times\mathcal{H}^2$ be a linear relation from a $J_{\mathfrak{H}}$-space to a $J_{\mathcal{H}}$-space [AI89, Section 1], where the canonical symmetry $J_{\mathfrak{H}}$ ($J_{\mathcal{H}}$) acts on $\mathfrak{H}^2$ ($\mathcal{H}^2$) as the multiplication operator by the second Pauli matrix $\left(\begin{smallmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{smallmatrix}\right)$. Let $\Gamma^{[*]}$ denote the Krein space adjoint of Γ [DHM17, Equation (2.6)], [DHMdS12, Section 7.2]. Then Γ is said to be $(J_{\mathfrak{H}},J_{\mathcal{H}})$-isometric if $\Gamma^{-1}\subseteq\Gamma^{[*]}$ and $(J_{\mathfrak{H}},J_{\mathcal{H}})$-unitary if $\Gamma^{-1}=\Gamma^{[*]}$ [DHM17, Definition 2.2], and essentially $(J_{\mathfrak{H}},J_{\mathcal{H}})$-unitary if $\overline{\Gamma}^{\,-1}=\Gamma^{[*]}$ [DHMdS06], where the overbar denotes the closure. For notational convenience, Γ is simply referred to as either isometric or (essentially) unitary.

Let A be a closed symmetric linear relation in $\mathfrak{H}$ and let Γ be an isometric linear relation. We assume that $A_*:=\operatorname{dom}\Gamma$ is dense in A^* with respect to the topology on $\mathfrak{H}^2$. We put $A:=A_*^*=\operatorname{mul}\Gamma^{[*]}$, so that the above assumptions are always satisfied by default. Likewise, putting $\widetilde{A}_*:=\operatorname{dom}\overline{\Gamma}$ and $\widetilde{A}:=\widetilde{A}_*^*\equiv(\widetilde{A}_*)^*$, and using that $\overline{\Gamma}$ is isometric, one finds that the linear relation $\widetilde{A}$ is closed and symmetric in $\mathfrak{H}$, and $\widetilde{A}_*$ is dense in $\widetilde{A}^*\equiv(\widetilde{A})^*$. Note that $A^*=\widetilde{A}^*$; if in addition Γ is essentially unitary, that is, if $\overline{\Gamma}$ is unitary, then also $A=\widetilde{A}$; if moreover Γ itself is unitary, then $A=\widetilde{A}=S$, where $S:=\ker\Gamma$.

By the above assumptions, the pair $(\mathcal{H},\Gamma)$ is an isometric/unitary boundary pair (for A^*) if Γ is isometric/unitary [DHM17, Definition 3.1]. In the terminology of [DHMdS06, Definition 3.1] Γ is a boundary relation (for S^*) iff it is unitary; see also [DHMdS06, Proposition 3.2]. If Γ is essentially unitary, we also say that the pair $(\mathcal{H},\Gamma)$ is an essentially unitary pair (for A^*).

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The first part of [DHMdS06, Theorem 3.9] states that the Weyl family $M_\Gamma(z)$, $z \in \mathbb{C}_* := \mathbb{C} \setminus \mathbb{R}$, corresponding to a boundary relation Γ is a Nevanlinna family, that is, it belongs to the class $\widetilde{\mathcal{R}}(\mathcal{H})$ (Definition 4.1). Here we prove an analogue of this statement for an essentially unitary Γ.

Theorem 1.1. *Let $M_\Gamma(z)$, $z \in \mathbb{C}_*$, be the Weyl family corresponding to an essentially unitary pair $(\mathcal{H}, \Gamma)$. Then the closure $\overline{M_\Gamma(z)} = M_{\overline{\Gamma}}(z)$ belongs to a Nevanlinna family.*

Here $\{M_{\overline{\Gamma}}(z)\}$ is the Weyl family corresponding to the unitary boundary pair $(\mathcal{H}, \overline{\Gamma})$. For Γ closed (that is, unitary), the theorem clearly reduces to the first part of [DHMdS06, Theorem 3.9]. By assuming additionally that the Weyl function $M_\Gamma(z)$ is bounded, and hence closed, one deduces another corollary.

Corollary 1.2. *Let $(\mathcal{H}, \Gamma)$ be an essentially unitary pair and $M_\Gamma(z) \in \mathrm{B}(\mathcal{H})$ $\forall z \in \mathbb{C}_*$. Then the Weyl function $M_\Gamma(z) = M_{\overline{\Gamma}}(z)$ belongs to the subclass $\mathcal{R}[\mathcal{H}]$ of Nevanlinna functions.* $\square$

According to [DHMdS06, Proposition 5.9] a Nevanlinna function of class $\mathcal{R}[\mathcal{H}]$ can be realized as the Weyl function of a B-generalized boundary pair $(\mathcal{H}, \Gamma)$ [DHM17, Definition 3.5]. Let us recall that ordinary, B-generalized, S-generalized, ES-generalized boundary pairs are all unitary boundary pairs; see [DHM17] for more details. Yet Corollary 1.2 shows that one can find a non-unitary boundary pair with the same Weyl function.

Assuming the hypotheses in Corollary 1.2 and $\operatorname{ran} \overline{\Gamma} = \mathcal{H}^2$, one concludes that the Weyl function $M_\Gamma(z) = M_{\overline{\Gamma}}(z)$ belongs to the subclass $\mathcal{R}^u[\mathcal{H}]$ of uniformly strict Nevanlinna functions. The single-valued linear relation Γ with such properties arises, for example, in the study of triplet extensions associated to the scales of Hilbert spaces of self-adjoint operators [Jur18, Section 7.5]; see also Section 5.

The proof of the main theorem is organized as follows: In Section 2 we list some preparatory results. In Section 3 we compute the adjoint $M_\Gamma(z)^*$ for an isometric pair $(\mathcal{H}, \Gamma)$; it follows that $M_\Gamma(z)^* = M_{\overline{\Gamma}}(\overline{z})$ for Γ essentially unitary. Since $\{M_{\overline{\Gamma}}(\overline{z})\}$ is a Nevanlinna family for $\overline{\Gamma}$ unitary—the fact that we actually show without referring to [DHMdS06, Theorem 3.9]—this implies Theorem 1.1; see Section 4.

Throughout we use the standard symbols dom, ran, mul, and ker to denote the domain, the range, the multivalued part, and the kernel of a linear relation. For more details related to the theory of linear relations and Nevanlinna families the reader may consult the papers in [BBM+18, DM17, BMN15, BHdS+13, dSWW11, DHMdS09, HdSS09, BHdS08, HSdSS07, HdS96, DM91] and also an extensive list of references therein.

2. Preliminaries

Here and elsewhere below a linear relation $\Gamma \subseteq \mathfrak{H}^2 \times \mathcal{H}^2$ from a $J_{\mathfrak{H}}$-space to a $J_{\mathcal{H}}$-space is assumed to be isometric, unless explicitly stated otherwise. Then the Green identity holds:

$$[\widehat{f}, \widehat{g}]_{\mathfrak{H}} = [\widehat{h}, \widehat{k}]_{\mathcal{H}} \tag{2.1}$$

for $(\widehat{f}, \widehat{h}) \in \Gamma$, $(\widehat{g}, \widehat{k}) \in \Gamma$. The $J_{\mathfrak{H}}$-metric $[\cdot,\cdot]_{\mathfrak{H}}$ is written in terms of the $\mathfrak{H}^2$-scalar product $\langle \cdot,\cdot \rangle_{\mathfrak{H}^2}$ according to

$$[\widehat{f}, \widehat{g}]_{\mathfrak{H}} := \langle \widehat{f}, J_{\mathfrak{H}} \widehat{g} \rangle_{\mathfrak{H}^2} = -\mathrm{i}(\langle f, g' \rangle_{\mathfrak{H}} - \langle f', g \rangle_{\mathfrak{H}}) \tag{2.2}$$

for $\widehat{f} = (f, f') \in \mathfrak{H}^2$, $\widehat{g} = (g, g') \in \mathfrak{H}^2$, provided that the $\mathfrak{H}$-scalar product $\langle \cdot,\cdot \rangle_{\mathfrak{H}}$ is conjugate-linear in the first argument. The same applies to the $J_{\mathcal{H}}$-metric $[\cdot,\cdot]_{\mathcal{H}}$.

The Krein space adjoint $\Gamma^{[*]}$ is defined by

$$\Gamma^{[*]} := \{(\widehat{k}, \widehat{g}) \in \mathcal{H}^2 \times \mathfrak{H}^2 \,|\, (\forall (\widehat{f}, \widehat{h}) \in \Gamma)\, [\widehat{f}, \widehat{g}]_{\mathfrak{H}} = [\widehat{h}, \widehat{k}]_{\mathcal{H}}\}\,.$$

Thus $\Gamma^{-1} \subseteq \Gamma^{[*]}$, and the equality holds iff Γ is unitary.

As usual, the eigenspaces of A^* are denoted by

$$\mathfrak{N}_z(A^*) := \ker(A^* - z)\,, \quad \widehat{\mathfrak{N}}_z(A^*) := \{\widehat{f}_z = (f_z, z f_z) \,|\, f_z \in \mathfrak{N}_z(A^*)\}$$

for $z \in \mathbb{C}$, and similarly for other linear relations. Since A^* is closed in $\mathfrak{H}$, its eigenspace is also closed, and one attains the orthogonal decomposition $\mathfrak{H} = \mathfrak{N}_{\overline{z}}(A^*) \oplus \mathfrak{N}_{\overline{z}}(A^*)^{\perp}$, where $\mathfrak{N}_{\overline{z}}(A^*)^{\perp} = \overline{\mathrm{ran}}(A - z)$. Then the $J_{\mathfrak{H}}$-orthogonal complement [AI89, Definition 1.11] $\widehat{\mathfrak{N}}_z(A_*)^{[\perp]}$ of $\widehat{\mathfrak{N}}_z(A_*)$ can be written as

$$\widehat{\mathfrak{N}}_z(A_*)^{[\perp]} = \widehat{\mathfrak{N}}_{\overline{z}}(A^*) \,\widehat{+}\, \mathfrak{D}_z\,, \quad \mathfrak{D}_z := \overline{z} I_{\mathfrak{N}_{\overline{z}}(A^*)^{\perp}} \,\widehat{+}\, (\{0\} \times \mathfrak{N}_z(A_*)^{\perp}) \tag{2.3}$$

where $\widehat{+}$ denotes the componentwise sum [HdSS09, Section 2.4] and $I_{\mathfrak{N}_{\overline{z}}(A^*)^{\perp}}$ denotes (the graph of) the identity operator restricted to $\mathfrak{N}_{\overline{z}}(A^*)^{\perp}$.

Remark 2.4. Let $\mathfrak{D} := \cap_{z \in \mathbb{C}_*} \mathfrak{D}_z$; then $\mathfrak{D} = \{0\} \times \mathfrak{M}$ where

$$\mathfrak{M} := \bigcap_{z \in \mathbb{C}_*} \mathfrak{N}_z(A_*)^{\perp}\,.$$

Clearly $\mathfrak{D} \subseteq S$ iff $\mathfrak{M} \subseteq \mathrm{mul}\, S$. Since $A_* \subseteq A^*$ densely, $\mathfrak{M} = \{0\}$ iff A is simple, in which case a closed symmetric linear relation A (and hence $S \subseteq A$) is an operator. The equality $\mathfrak{M} = \{0\}$ also shows that the closed linear span

$$\mathfrak{H}_s := \bigvee \{\mathfrak{N}_z(A_*) \,|\, z \in \mathbb{C}_*\}$$

coincides with $\mathfrak{H}$. Note that A is simple iff so is $\widetilde{A}$. When $A = S$ is simple, a unitary Γ is minimal [DHMdS06, Definition 3.4], and vice verse.

3. Weyl families

The Weyl family of A corresponding to an isometric pair $(\mathcal{H},\Gamma)$ is defined by [DHM17, Definition 3.2] $M_\Gamma(z):=\Gamma\widehat{\mathfrak{N}}_z(A_*)$ for $z\in\mathbb{C}_*$. Put

$$\Gamma_z:=\Gamma\,|_{\widehat{\mathfrak{N}}_z(A_*)}:=\Gamma\cap(\widehat{\mathfrak{N}}_z(A_*)\times\mathcal{H}^2)$$

then the linear relation $M_\Gamma(z)$ and its adjoint can be described by

$$M_\Gamma(z)=\operatorname{ran}\Gamma_z\,,\quad M_\Gamma(z)^*=\ker\Gamma_z^{[*]} \tag{3.1}$$

where $\Gamma_z^{[*]}$ denotes the Krein space adjoint of Γ_z. Since Γ is isometric and $\Gamma_z\subseteq\Gamma$, it is evident that

$$\Gamma_z^{-1}\subseteq\Gamma^{-1}\subseteq\Gamma^{[*]}\subseteq\Gamma_z^{[*]}\,,$$

that is, Γ_z is also isometric.

Lemma 3.2. *$\mathfrak{D}_z\subseteq\operatorname{mul}\Gamma_z^{[*]}$ for $z\in\mathbb{C}$.*

Proof. Let $\widehat{g}\in\mathfrak{D}_z$; then by (2.3) $\widehat{g}=(g,\overline{z}g+f)$, $g\in\mathfrak{N}_{\overline{z}}(A^*)^\perp$, $f\in\mathfrak{N}_z(A_*)^\perp$. Then by (2.2) $(\forall(\widehat{f}_z,\widehat{h})\in\Gamma_z)$

$$[\widehat{f}_z,\widehat{g}]_{\mathfrak{H}}=-\mathrm{i}\,\langle f_z,f\rangle_{\mathfrak{H}}=0=[\widehat{h},(0,0)]_{\mathcal{H}}$$

hence $((0,0),\widehat{g})\in\Gamma_z^{[*]}$. □

Theorem 3.3. *The adjoint is given by*

$$M_\Gamma(z)^*=(\Gamma^{[*]})^{-1}\widehat{\mathfrak{N}}_{\overline{z}}(A^*)$$

for $z\in\mathbb{C}_*$.

Proof. We split the proof into three steps.

Step 1. First we prove the next lemma.

Lemma 3.4. *The adjoint is given by*

$$M_\Gamma(z)^*=(\Gamma_z^{[*]})^{-1}\widehat{\mathfrak{N}}_z(A_*)^{[\perp]}$$

for $z\in\mathbb{C}_*$.

Proof. Consider $\widehat{k}\in M_\Gamma(z)^*=M_\Gamma(z)^{[\perp]}$; then $(\forall(\widehat{f}_z,\widehat{h})\in\Gamma_z\Leftrightarrow\forall\widehat{h}\in M_\Gamma(z))$ $(\forall\widehat{g}\in\widehat{\mathfrak{N}}_z(A_*)^{[\perp]})$

$$[\widehat{f}_z,\widehat{g}]_{\mathfrak{H}}=0=[\widehat{h},\widehat{k}]_{\mathcal{H}} \tag{3.5}$$

and so $\widehat{k}\in(\Gamma_z^{[*]})^{-1}\widehat{\mathfrak{N}}_z(A_*)^{[\perp]}$. Conversely, consider $\widehat{k}\in(\Gamma_z^{[*]})^{-1}\widehat{\mathfrak{N}}_z(A_*)^{[\perp]}$; then $(\exists\widehat{g}\in\widehat{\mathfrak{N}}_z(A_*)^{[\perp]})$ $(\widehat{k},\widehat{g})\in\Gamma_z^{[*]}$, and so $(\forall(\widehat{f}_z,\widehat{h})\in\Gamma_z)$ equation (3.5) holds; hence $\widehat{k}\in M_\Gamma(z)^{[\perp]}$. □

Step 2. By using (2.3) and Lemma 3.4, the adjoint $M_\Gamma(z)^*$ consists of $\widehat{h}\in\mathcal{H}^2$ such that $(\exists\widehat{f_{\overline{z}}}\in\widehat{\mathfrak{N}}_{\overline{z}}(A^*))$ $(\exists\widehat{g}\in\mathfrak{O}_z)$ $(\widehat{h},\widehat{f_{\overline{z}}}+\widehat{g})\in\Gamma_z^{[*]}$. By Lemma 3.2, on the other hand, $\mathfrak{O}_z\subseteq\operatorname{mul}\Gamma_z^{[*]}$. Therefore $((0,0),\widehat{g})\in\Gamma_z^{[*]}$, which shows $(\widehat{h},\widehat{f_{\overline{z}}})\in\Gamma_z^{[*]}$ by linearity of a subspace $\Gamma_z^{[*]}$. It follows that $M_\Gamma(z)^*=(\Gamma_z^{[*]})^{-1}\widehat{\mathfrak{N}}_{\overline{z}}(A^*)$.

Step 3. Let

$$X_{\overline{z}}:=(\Gamma_z^{[*]})^{-1}\,|_{\widehat{\mathfrak{N}}_{\overline{z}}(A^*)}\,,\quad Y_{\overline{z}}:=(\Gamma^{[*]})^{-1}\,|_{\widehat{\mathfrak{N}}_{\overline{z}}(A^*)}\,.$$

To accomplish the proof of the theorem, it remains to show that $X_{\overline{z}}=Y_{\overline{z}}$. Since the inclusion $X_{\overline{z}}\supseteq Y_{\overline{z}}$ is clear from $(\Gamma_z^{[*]})^{-1}\supseteq(\Gamma^{[*]})^{-1}$, we show the reverse inclusion.

Consider $\widehat{k}\in\mathcal{H}^2$ and $\widehat{g}_{\overline{z}}\in\widehat{\mathfrak{N}}_{\overline{z}}(A^*)$ such that $(\widehat{g}_{\overline{z}},\widehat{k})\in X_{\overline{z}}\smallsetminus Y_{\overline{z}}$. Then $(\forall(\widehat{f_z},\widehat{h})\in\Gamma_z)$ equation (3.5) holds, where $\widehat{g}\equiv\widehat{g}_{\overline{z}}$, but $(\forall(\widehat{f},\widehat{h})\in\Gamma)$ the Green identity (2.1), with $\widehat{g}\equiv\widehat{g}_{\overline{z}}$, fails. Now take $\widehat{f}=\widehat{f_z}\in\widehat{\mathfrak{N}}_z(A_*)$ and deduce a contradiction; hence $X_{\overline{z}}\subseteq Y_{\overline{z}}$. □

Remark 3.6. It follows from (3.1) that the intersection $M_\Gamma(z)\cap M_\Gamma(z)^*$ is a subset of the set of neutral vectors [AI89, Definition 1.3] of a $J_{\mathcal{H}}$-space. Thus, by applying the Green identity (2.1) for $(\widehat{f_z},\widehat{h})\in\Gamma_z$ such that $[\widehat{h},\widehat{h}]_{\mathcal{H}}=0$, one deduces that $M_\Gamma(z)\cap M_\Gamma(z)^*=\operatorname{mul}\Gamma_z=\operatorname{mul}\Gamma$. This result also follows from Theorem 3.3 by noting that

$$\Gamma\widehat{\mathfrak{N}}_z(A_*)\cap(\Gamma^{[*]})^{-1}\widehat{\mathfrak{N}}_{\overline{z}}(A^*)=\Gamma(\widehat{\mathfrak{N}}_z(A_*)\cap\widehat{\mathfrak{N}}_{\overline{z}}(A^*))$$

that $\widehat{\mathfrak{N}}_z(A_*)$ is non-degenerate [AI89, Definition 1.14] for $\Im z\neq0$, and that $\Gamma\{(0,0)\}=\operatorname{mul}\Gamma$. We therefore have yet another proof of the relation $M_\Gamma(z)\cap M_\Gamma(z)^*=\operatorname{mul}\Gamma$ $(z\in\mathbb{C}_*)$, which is stated without the proof in [DHM17, Lemma 3.6(i)], [DHMdS12, Lemma 7.52(i)], and which is shown in [DHMdS06, Lemma 4.1(i)] for a unitary pair $(\mathcal{H},\Gamma)$. By using Theorem 3.3 one finds the remaining invariance results for $M_\Gamma(\cdot)$.

Corollary 3.7. *Let $(\mathcal{H},\Gamma)$ be an essentially unitary pair. Then $M_\Gamma(z)^*=M_{\overline{\Gamma}}(\overline{z})$ for $z\in\mathbb{C}_*$.*

Proof. Since $\overline{\Gamma}^{\,-1}=\Gamma^{[*]}$, one has by Theorem 3.3 $M_\Gamma(z)^*=\overline{\Gamma}\widehat{\mathfrak{N}}_{\overline{z}}(A^*)$, that is, $M_\Gamma(z)^*$ is the set of $\widehat{h}\in\mathcal{H}^2$ such that $(\exists\widehat{f_{\overline{z}}}\in\widehat{\mathfrak{N}}_{\overline{z}}(A^*))$ $(\widehat{f_{\overline{z}}},\widehat{h})\in\overline{\Gamma}$. But also, it must hold $\widehat{f_{\overline{z}}}\in\operatorname{dom}\overline{\Gamma}=:\widetilde{A}_*$. Using that $A^*\supseteq\widetilde{A}_*$ one concludes that $\overline{\Gamma}\widehat{\mathfrak{N}}_{\overline{z}}(A^*)=\overline{\Gamma}\widehat{\mathfrak{N}}_{\overline{z}}(\widetilde{A}_*)$. □

If in addition Γ is unitary, Corollary 3.7 shows that $M_\Gamma(z)^*=M_\Gamma(\overline{z})$ $\forall z\in\mathbb{C}_*$.

4. Nevanlinna families

The following definition of a Nevanlinna family is due to [DHMdS12, Definition 9.12], [BHdS08, Definition 2.1], [DHMdS06, Section 2.6].

Definition 4.1. A family $M(z)$, $z\in\mathbb{C}_*$, of linear relations in $\mathcal{H}$ belongs to the class $\widetilde{\mathcal{R}}(\mathcal{H})$ of Nevanlinna families, or is said to be a Nevanlinna family, if:

(a) For $\Im z > 0/\Im z < 0$, the relation $M(z)$ is maximal dissipative/accretive, and the operator family $(M(z)+w)^{-1} \in \mathrm{B}(\mathcal{H})$, $w \in \mathbb{C}_+/\mathbb{C}_-$, is analytic;
(b) $M(\overline{z}) = M(z)^*$.

Here $\mathbb{C}_+/\mathbb{C}_-$ is the set of $z \in \mathbb{C}$ such that $\Im z > 0/\Im z < 0$; hence $\mathbb{C}_* = \mathbb{C}_+ \cup \mathbb{C}_-$. A linear relation $M(z)$ is dissipative (resp. accretive or accumulative) if $(\forall (h,h') \in M(z))$ $\Im \langle h, h' \rangle_{\mathcal{H}} \geq 0$ (resp. ≤ 0). We emphasize that the $\mathcal{H}$-scalar product is conjugate-linear in the first argument. A dissipative (resp. accretive) $M(z)$ is maximal dissipative (resp. maximal accretive or maximal accumulative) if $M(z)$ has no proper dissipative (resp. accretive) extensions.

The Weyl family $M_\Gamma(z)$, $z \in \mathbb{C}_*$, corresponding to an isometric pair $(\mathcal{H},\Gamma)$ is dissipative/accretive for $\Im z > 0/\Im z < 0$. Indeed, in view of (3.1), $\widehat{h} = (h,h') \in M_\Gamma(z)$ implies that $(\widehat{f}_z, \widehat{h}) \in \Gamma_z$ for some $\widehat{f}_z \in \widehat{\mathfrak{N}}_z(A_*)$. Then, by the Green identity (2.1), $\Im \langle h, h' \rangle_{\mathcal{H}} = (\Im z)\|f_z\|^2_{\mathfrak{H}}$; hence the claim. But then $(M_\Gamma(z)+w)^{-1}$ is an operator by [DdS74, Theorem 3.1(i)].

If in addition $M_\Gamma(\overline{z}) = M_\Gamma(z)^*$, then $M_\Gamma(z) = M_\Gamma(\overline{z})^*$, and therefore each member of the Weyl family is closed in this case: $M_\Gamma(z)^{**} = M_\Gamma(\overline{z})^* = M_\Gamma(z)$. But then the operator $(M_\Gamma(z)+w)^{-1}$ is bounded by [DdS74, Theorem 3.1(vi)], and $M_\Gamma(z)$ is maximal dissipative/accretive by [DdS74, Theorem 3.4(ii)].

It follows from the above that:

Proposition 4.2. *The Weyl family $M_\Gamma(z)$, $z \in \mathbb{C}_*$, is a Nevanlinna family iff $M_\Gamma(\overline{z}) = M_\Gamma(z)^*$.* □

By applying Corollary 3.7 and Proposition 4.2 one accomplishes the proof of Theorem 1.1.

Remark 4.3. Let us recall that the Weyl family of A and its simple part coincide. Indeed, let A_s be the simple part [LT77, Proposition 1.1] of A and let Γ_s be the restriction of Γ to $\mathfrak{H}^2_s$. Put $A_{s*} := \operatorname{dom}\Gamma_s = A_* \cap \mathfrak{H}^2_s$. Then $\widehat{\mathfrak{N}}_z(A_{s*}) = \widehat{\mathfrak{N}}_z(A_*) \cap \mathfrak{H}^2_s = \widehat{\mathfrak{N}}_z(A_*)$. Thus, since Γ is isometric, $\Gamma_s \subseteq \Gamma$ is also isometric, and the corresponding Weyl family of A_s is given by $M_{\Gamma_s}(z) = M_\Gamma(z)$, $z \in \mathbb{C}_*$, by noting that $\Gamma_s \cap \Gamma_z = \Gamma_z$. In addition, given an isometric Γ, assume that Γ_s is essentially unitary. Then Γ is also essentially unitary, whose closure $\overline{\Gamma} = \overline{\Gamma_s}$.

5. Example

Let $\widetilde{A}$ be a densely defined, closed, symmetric operator in a Hilbert space $\mathfrak{H}$ with defect numbers (d,d). Let L be a self-adjoint extension of $\widetilde{A}$ in $\mathfrak{H}$. Then by the von Neumann formula the adjoint $\widetilde{A}^* \supseteq L$ is described by $\operatorname{dom}\widetilde{A}^* = \operatorname{dom} L \dotplus \mathfrak{N}_z(\widetilde{A}^*)$, where the eigenspace is spanned by the deficiency elements $g_\sigma(z)$, $z \in \mathbb{C}_*$, with σ ranging over an index set $\mathcal{S}$ of cardinality d. That is, $\mathfrak{N}_z(\widetilde{A}^*) = g_z(\mathbb{C}^d)$, where $g_z(c) := \sum_{\sigma\in\mathcal{S}} c_\sigma g_\sigma(z)$ for $c = (c_\sigma) \in \mathbb{C}^d$.

Let $(\mathfrak{H}_n)_{n\in\mathbb{Z}}$ be the scale of Hilbert spaces associated with L; hence $\mathfrak{H}_2 = \operatorname{dom} L$ and $\mathfrak{H}_0 = \mathfrak{H}$. Then $g_\sigma(z)$ can be defined in the generalized sense as $g_\sigma(z) = (L-z)^{-1}\varphi_\sigma$ with some functional $\varphi_\sigma \in \mathfrak{H}_{-2} \setminus \mathfrak{H}_{-1}$. Thus, $\widetilde{A}$ is the symmetric restriction of L to the domain of $u \in \mathfrak{H}_2$ such that $\langle\varphi, u\rangle = 0$; here $\langle\varphi,\cdot\rangle = (\langle\varphi_\sigma,\cdot\rangle)$.

Let

$$\mathfrak{K} := \operatorname{span}\{g_\alpha := g_\sigma(z_j) \,|\, \alpha = (\sigma, j) \in \mathcal{S}\times J\}$$

where an index set $J := \{1, 2, \dots, m\}$, $m \in \mathbb{N}$, and the points $z_j \in \mathbb{C}_*$ such that $z_j \neq z_{j'}$ for $j \neq j'$. Then the system $\{g_\alpha\}$ is linearly independent, and so the associated Gram matrix $\mathcal{G} := (\langle g_\alpha, g_{\alpha'}\rangle_{\mathfrak{H}})$ is Hermitian and positive definite.

Let

$$\mathfrak{K}' := \mathfrak{H}_{2m+2} \dotplus \mathfrak{M}_z\,, \quad z \in \mathbb{C}_* \cap (\mathbb{C} \setminus \{z_j \,|\, j \in J\})$$

where $\mathfrak{M}_z \subseteq \mathfrak{H}_{2m}$ is defined by

$$\mathfrak{M}_z := \operatorname{span}\{\sum_{j\in J} b_j(z_j)^{-1}(L-z_j)^{-1} g_\sigma(z) \,|\, \sigma \in \mathcal{S}\}$$

where

$$b_j(z_j) := \prod_{j'\in J\setminus\{j\}} (z_j - z_{j'})\,.$$

Define the operator K in $\mathfrak{H}$ by

$$\operatorname{dom} K := \mathfrak{K}' \dotplus \mathfrak{K}\,, \quad K(u+k) := Lu + \sum_{\alpha,\alpha'\in\mathcal{S}\times J} C_{\alpha\alpha'}\,\langle g_{\alpha'}, k\rangle_{\mathfrak{H}}\, g_\alpha$$

for $u+k \in \mathfrak{K}' \dotplus \mathfrak{K}$, where $C_{\sigma j,\sigma' j'} := z_j[\mathcal{G}^{-1}]_{\sigma j,\sigma' j'}$. Define the linear relation $\Gamma \subseteq \mathfrak{H}^2 \times \mathcal{H}^2$, with $\mathcal{H} := \mathbb{C}^d$, by

$$\Gamma := \{\big((u+k, K(u+k)), (c(k), \langle\varphi, u\rangle + \mathcal{M}d(k))\big) \,|\, u+k \in \mathfrak{K}' \dotplus \mathfrak{K}\}$$

where the matrix $\mathcal{M} = (\mathcal{M}_{\sigma\alpha'}) \in \mathrm{B}(\mathbb{C}^{md}, \mathbb{C}^d)$ is defined by $\mathcal{M}_{\sigma,\sigma' j'} := R_{\sigma\sigma'}(z_{j'})$ for some matrix-valued Nevanlinna family $R(\cdot) = (R_{\sigma\sigma'}(\cdot))$ of class $\mathcal{R}^s[\mathcal{H}]$. In fact, if the functional $\langle\varphi_\sigma^{\mathrm{ex}}, \cdot\rangle$ extends $\langle\varphi_\sigma, \cdot\rangle : \mathfrak{H}_2 \to \mathbb{C}$ to $\operatorname{dom}\widetilde{A}^*$, then $R_{\sigma\sigma'}(z) := \langle\varphi_\sigma^{\mathrm{ex}}, g_{\sigma'}(z)\rangle$ for $z \in \mathbb{C}_*$. One verifies that $\ker \Im R(z) = \{0\}$: By definition $\Im R(z)$ is the matrix with entries $(\Im z)\,\langle g_\sigma(z), g_{\sigma'}(z)\rangle_{\mathfrak{H}}$ $\forall(\sigma,\sigma') \in \mathcal{S}\times\mathcal{S}$. Then $\ker \Im R(z)$ is the set of $c \in \mathbb{C}^d$ such that $g_z(c) \in g_z(\mathbb{C}^d)^\perp$; hence $c = 0$.

Let $A_* := \operatorname{dom}\Gamma = K$. Since the adjoint $K^* = \widetilde{A}$, one has $A := A_*^* = \widetilde{A}$. It also follows from the definition of K that the eigenspace $\mathfrak{N}_z(A_*) = \mathfrak{N}_z(\widetilde{A}^*)$ for $z \in \mathbb{C}_*$.

The Krein space adjoint of Γ is given by $\Gamma^{[*]} = \overline{\Gamma}^{\,-1}$, where the closure

$$\overline{\Gamma} = \{\big((u + g_z(c), \widetilde{A}^*(u + g_z(c))), (c, \langle\varphi, u\rangle + R(z)c)\big) \,|\, u \in \mathfrak{H}_2\,;\, c \in \mathbb{C}^d\}$$

and $z \in \mathbb{C}_*$. Then $\widetilde{A}_* := \operatorname{dom} \overline{\Gamma} = \widetilde{A}^*$ and $\mathfrak{N}_z(A_*) = \mathfrak{N}_z(\widetilde{A}_*)$. Since $\Gamma \subseteq (\Gamma^{[*]})^{-1} = \overline{\Gamma}$ is essentially unitary and $\overline{A_*} = A^*$, the pair $(\mathcal{H}, \Gamma)$ is an essentially unitary pair for A^*; the pair $(\mathcal{H}, \overline{\Gamma})$ is a unitary pair for $\widetilde{A}^*$, with the associated Weyl family given by $M_{\overline{\Gamma}}(z) = R(z)$, $z \in \mathbb{C}_*$. Note that $\ker \overline{\Gamma} = \widetilde{A}$.

Associate with Γ the linear relation

$$\Gamma_0 := \{(\widehat{f}, h) \,|\, (\exists h' \in \mathcal{H})\, (\widehat{f}, \widehat{h}) \in \Gamma\, ; \widehat{h} = (h, h')\}\,.$$

Since $\operatorname{dom} M_\Gamma(z) = \Gamma_0 \widehat{\mathfrak{N}}_z(A_*) = \mathbb{C}^d$, Corollary 1.2 shows that $M_\Gamma(z) = M_{\overline{\Gamma}}(z) = R(z)$, $z \in \mathbb{C}_*$, is a Nevanlinna family of class $\mathcal{R}^s[\mathcal{H}]$. One also verifies the equality $M_\Gamma(z) = R(z)$ by computing $\Gamma \widehat{\mathfrak{N}}_z(A_*)$ directly. Moreover, since $\operatorname{ran} \overline{\Gamma} = \mathcal{H}^2$, one concludes that actually $R(\cdot) \in \mathcal{R}^u[\mathcal{H}]$. For more details related to the analysis of operator K the reader may refer to [Jur18].

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