INVARIANT SUBSPACES FOR BISHOP OPERATORS AND BEYOND

FERNANDO CHAMIZO, EVA A. GALLARDO-GUTIÉRREZ, MIGUEL MONSALVE-LÓPEZ AND ADRIÁN UBIS

ABSTRACT. Bishop operators T_{α} acting on $L^2[0,1)$ were proposed by E. Bishop in the fifties as possible operators which might entail counterexamples for the Invariant Subspace Problem. We prove that all the Bishop operators are biquasitriangular and, derive as a consequence that they are norm limits of nilpotent operators. Moreover, by means of arithmetical techniques along with a theorem of Atzmon, the set of irrationals $\alpha \in (0,1)$ for which T_{α} is known to possess non-trivial closed invariant subspaces is considerably enlarged, extending previous results by Davie [11], MacDonald [21] and Flattot [14]. Furthermore, we essentially show that when our approach fails to produce invariant subspaces it is actually because Atzmon Theorem cannot be applied. Finally, upon applying arithmetical bounds obtained, we deduce local spectral properties of Bishop operators proving, in particular, that neither of them satisfy the *Dunford property* (C).

1. Introduction

Perhaps, one of the best-known unsolved problems in Functional Analysis is the *Invariant Subspace Problem*:

Does every bounded linear operator on a (separable, infinite-dimensional, complex) Hilbert space have a non-trivial closed invariant subspace?

In this regard, one of the earliest and most elegant invariant subspace theorems is the result of von Neumann in the Hilbert space setting (unpublished) and Aronszajn and Smith [3] in the context of Banach spaces which states, in particular, that compact operators have non-trivial closed invariant subspaces. In 1973 operator theorists were stunned by the generalization achieved by Lomonosov [20], who proved one of the most general positive results to provide invariant subspaces, namely: any linear bounded operator T acting on a Banach space commuting with a non-zero compact operator has a non-trivial closed invariant subspace. Moreover, T has a non-trivial hyperinvariant closed subspace, that is, a closed subspace which is invariant under every operator in the commutant of T. Accordingly, any linear bounded operator T has a non-trivial invariant closed subspace if it commutes with a non-scalar operator that commutes with a nonzero compact operator. But, it was not until 1980 that Hadwin, Nordgren, Radjavi, Rosenthal [16] showed the existence of an operator in the Hilbert space setting having non-trivial invariant subspaces to which Lomonosov's Theorem does not apply.

Date: November 27, 2018.

 $^{2010\} Mathematics\ Subject\ Classification.\ 47A15,\ 47B37,\ 47B38.$

Key words and phrases. Bishop operators, invariant subspace problem, Dunford property (C).

F. Chamizo and A. Ubis are partially supported by Plan Nacional I+D grant no. MTM2017-83496-P (Spain); E. A. Gallardo-Gutiérrez and M. Monsalve-López are partially supported by Plan Nacional I+D grant no. MTM2016-77710-P (Spain). The first three authors are also partially supported by "Severo Ochoa Programme for Centres of Excellence in R&D" (SEV-2015-0554). Finally, M. Monsalve-López also acknowledges support of the grant Ayudas de la Universidad Complutense de Madrid para contratos predoctorales de personal investigador en formación, ref. no. CT27/16.

In the meantime, two remarkable counterexamples came into scene. Firstly, in 1975 Enflo announced in the Séminaire Maurey-Schwarz at the École Polytechnique in Paris the existence of a separable Banach space and a linear bounded operator T without non-trivial closed invariant subspaces; though its publication was delayed for more than ten years [13]. Then, in 1985, Read [27] constructed a bounded linear operator without non-trivial closed invariant subspaces in the well-known sequence space ℓ^1 (see also [26] for a previous construction). Indeed, the construction carried over in [27] is the first known example of such an operator on any of the classical Banach spaces.

For decades a number of authors worked on extending these results to more general classes of operators, and significant progress has been made by developing deep tools in allied areas like Harmonic Analysis, Function Theory or finite dimension Linear Algebra in the framework of Operator Theory. Among different approaches, two have been specially fruitful in order to provide invariant subspaces for a given operator: one coming from the behavior of such operator acting on finite dimensional subspaces leading to the concept of quasitriangular operators. The other one, mostly based on function theory techniques, consist of developing an "appropriate" functional calculus which allows to produce hyperinvariant subspaces from the fact that two non-zero functions may have pointwise zero product.

Regarding the first approach, recall that a linear bounded operator T in a separable infinite dimensional Hilbert space H is said to be *quasitriangular* if there exists an increasing sequence $(P_n)_{n=1}^{\infty}$ of finite rank projections converging to the identity I strongly as $n \to \infty$ such that

$$||TP_n - P_n TP_n|| \to 0$$
, as $n \to \infty$.

Based on Aronszajn and Smith's Theorem, Halmos [17] introduced the concept of quasitriangular operators in the sixties to prove the existence of invariant subspaces. It is completely apparent that given a triangular operator in H, that is, a linear bounded operator which admits a representation as an upper triangular matrix with respect to a suitable orthonormal basis, there exists an increasing sequence $(P_n)_{n=1}^{\infty}$ of finite rank projections converging to the identity I strongly as $n \to \infty$ such that

$$TP_n - P_n TP_n = (I - P_n)TP_n = 0$$
, for all $n = 0, 1, 2, ...$

Hence, the definition of quasitriangularity says, roughly speaking, that T has a sequence of "approximately invariant" finite-dimensional subspaces. Compact operators, operators with finite spectrum, decomposable operators or compact perturbations of normal operators are examples of quasitriangular operators. On the other hand, the shift operator of index one is not quasitriangular; and remarkable results due to Douglas and Pearcy [12] and Apostol, Foias and Voiculescu [2] yield that the *Invariant Subspace Problem* is reduced to be proved for quasitriangular operators (see Herrero's book [18] for more on the subject).

In what the second approach refers, $Beurling\ algebras$ have played an important role in this context. The starting point was a theorem of Wermer [29] in 1952 which states that an invertible linear bounded operator T on H such that the series

$$\sum_{n=-\infty}^{\infty} \frac{\log \|T^n\|}{1+n^2}$$

converges and its spectrum is not a singleton is either a multiple of the identity or has a non-trivial hyperinvariant closed subspace. A stronger variant was proved by Atzmon (see [4] and [5], for instance). The common feature is the definition of a functional calculus, particularly in [5] mapping an algebra \mathcal{A}_{ρ} of functions defined on the unit circle \mathbb{T} into $\mathcal{L}(H)$, the Banach algebra of linear bounded operators acting on H. For more on the subject we refer to the classical monograph by Radjavi and Rosenthal [25] and the recent one by Chalendar and Partington [9].

The main goal of this work is addressing both approaches in the context of Bishop operators. Given an irrational number $\alpha \in (0,1)$, recall that the Bishop operator T_{α} is defined on $L^{p}[0,1)$, $1 \leq p \leq \infty$, by

$$T_{\alpha}f(t) = tf(\{t + \alpha\}), \qquad t \in [0, 1),$$

where $\{\cdot\}$ denotes the fractional part. As explained by Davie [11], these examples were suggested by Bishop as candidates for operators without non-trivial closed invariant subspaces. By means of a functional calculus approach, Davie proved the existence of non-trivial closed hyperinvariant subspaces in $L^2[0,1)$ for T_{α} whenever α is a non-Liouville irrational number in (0,1). Later, subsequent extensions strengthening it due to Blecher and Davie [7], MacDonald [21], [22] and Flattot [14] provided a large class of irrationals $\alpha \in (0,1)$ including some Liouville numbers.

Our main results in this context will be showing, on one hand, that every Bishop operator T_{α} as well as its adjoint T_{α}^* are quasitriangular operators in $L^2[0,1)$, having therefore a good approximation by approximately invariant finite-dimensional subspaces. On the other hand, in Theorem 3.7 we will extend the class of irrationals $\alpha \in (0,1)$ such that T_{α} has non-trivial closed hyperinvariant subspaces in $L^p[0,1)$ by considering arithmetical techniques which allow to strengthen the analysis of the behavior of certain functions associated to the functional calculus model. Indeed, those Liouville irrationals α escaping the condition set up in Theorem 3.7 are so extreme that Theorem 4.1 will show that, essentially, Atzmon Theorem cannot be applied for such irrationals. Roughly speaking, we prove that when our approach fails to produce invariant subspaces it is actually because Atzmon Theorem cannot be applied, what establishes, somehow, the threshold limit in the growth of the denominators of the convergents of those α . In some sense, this corroborates an approach to look for invariant subspaces for every T_{α} based on different functional analytic tools; which will be the goal in the final section.

On the other hand, observe that by Jarník-Besicovitch Theorem (see [8, Section 5.5], for instance), Liouville irrationals form a set of vanishing Hausdorff dimension. Nevertheless, it is possible to measure the difference between those cases covered by Davie and Flattot Theorems and Theorem 3.7, by considering the logarithmic Hausdorff dimension through the use of the family of functions $|\log x|^{-s}$ (instead of the usual x^s). With such a dimension, by means of [8, Theorem 6.8], one can easily deduce that the set of exceptions in Davie, Flattot and our case have dimension ∞ , 4 and 2, respectively.

The rest of the manuscript is organized as follows. In Section 2 we introduce some preliminaries and prove that every Bishop operator T_{α} is biquasitriangular in $L^{2}[0,1)$. In Section 3, we recall the functional calculus provided by Davie and its extension by Atzmon (a good reference for that is [9, Chapter 5]); and construct explicit functions in $L^{p}[0,1)$ which allow to extend the class of Liouville numbers $\alpha \in (0,1)$ such that T_{α} has non-trivial closed hyperinvariant subspaces. In Section 4, we show the limits of Atzmon's Theorem approach in the context of Bishop operators. Finally, in Section 5 we discuss some consequences regarding spectral subspaces, which constitute a class of invariant linear manifolds to look for non-trivial closed hyperinvariant subspaces. We will show, in particular, that T_{α} does not satisfy the *Dunford property* (C) in $L^{p}[0,1)$ by exhibiting that some spectral subspaces are not closed.

A word about notation. In this paper we employ a form of Vinogradov's notation. We write $A \ll B$ meaning $|A| \leq K|B|$ for some absolute constant K > 0. Note that in particular we have A = O(B).

2. Quasitriangular Bishop operators

As mentioned in the introduction, a linear bounded operator T in a separable infinite dimensional Hilbert space H is quasitriangular if there exists an increasing sequence $(P_n)_{n=1}^{\infty}$ of finite rank projections converging to the identity I strongly as $n \to \infty$ such that

$$||TP_n - P_n TP_n|| \to 0$$
, as $n \to \infty$.

In this Section, we show that every Bishop operator T_{α} is indeed biquasitriangular in $L^{2}[0,1)$, that is, both T_{α} and its adjoint T_{α}^{*} are quasitriangular operators. We will derive some consequences regarding the approximation of T_{α} .

In order to prove the result, we will consider semi-Fredholm operators. Let T be in $\mathcal{L}(H)$ and denote by Ker T and Ran T its kernel and its range, respectively. Recall that T is called *semi-Fredholm* if Ran T is closed and either the dimension of the kernel of T or the dimension of the kernel of the adjoint T^* is finite. In this case, the *index* of T is defined by

$$\operatorname{index} T = \dim(\operatorname{Ker} T) - \dim(\operatorname{Ker} T^*)$$

The following remarkable theorem by Douglas and Pearcy [12] and Apostol, Foias and Voiculescu [2] (see also [18, Chapter 6]) is the key fact relating semi-Fredholm operators to quasitriangular ones:

Theorem 2.1 (Douglas, Pearcy- Apostol, Foias, Voiculescu). An operator T is quasitriangular in H if and only if $\operatorname{index}(T - \lambda I) \geq 0$ for each complex number $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is semi-Fredholm.

We are in position now the prove the following result:

Theorem 2.2. For every irrational $\alpha \in (0,1)$, the Bishop operator T_{α} in $L^{2}[0,1)$ is biquasitriangular.

Proof. Let $\lambda \in \mathbb{C}$ such that $T_{\alpha} - \lambda I$ is semi-Fredholm. In particular, both $T_{\alpha} - \lambda I$ and its adjoint $T_{\alpha}^* - \overline{\lambda} I$ have closed range. Since the point spectrum of T_{α} is empty, one has that λ is in the resolvent of T_{α} , that is, $T_{\alpha} - \lambda I$ is invertible. Hence, index $(T_{\alpha} - \lambda I) = 0$. Since index $(T_{\alpha}^* - \overline{\lambda} I) = -\text{index}(T_{\alpha} - \lambda I)$, it follows that both T_{α} and its adjoint T_{α}^* are quasitriangular operators in $L^2[0,1)$, and the theorem is proved.

A few consequences may be derived from Theorem 2.2 in terms of approximation of T_{α} by linear bounded operators. For instance, by means of [18, Theorem 6.15], one has straightforwardly that for every irrational $\alpha \in (0,1)$, the operator T_{α} is the norm limit of algebraic operators. Recall that an operator is called algebraic if there exists a polynomial p such that p(T) is the zero operator. Clearly algebraic operators have non-trivial closed invariant subspaces. At this regard, it is worthy to point out that indeed, for every irrational $\alpha \in (0,1)$, T_{α} is norm limit of nilpotent operators in $L^p[0,1)$. Namely, for any positive integer n, let $\phi_n(t) = t \cdot 1_{[1/n,1)}(t)$ for $t \in [0,1)$ and consider the Bishop-type operator $T_{\phi_n,\alpha}$ defined by

$$T_{\phi_n,\alpha}f(t) = \phi_n(t)f(\{t+\alpha\}), \qquad t \in [0,1)$$

for $f \in L^p[0,1)$, $1 \le p < \infty$. Clearly, $(T_{\phi_n,\alpha})_{n \ge 1}$ are linear bounded operators in $L^p[0,1)$ converging in norm to T_{α} . Moreover, having in mind that $\tau_{\alpha}(t) = \{t + \alpha\}$ with α irrational in [0,1) is an ergodic transformation in $L^p[0,1)$, one deduces that $T_{\phi_n,\alpha}$ is nilpotent for every $n \ge 1$.

Remark. The fact that for every irrational $\alpha \in (0,1)$, both the Bishop operator T_{α} and its adjoint T_{α}^* in $L^2[0,1)$ are norm limit of nilpotent operators in $L^2[0,1)$ could be derived upon applying a theorem of Apostol, Foias and Voiculescu [2] which states that a linear bounded operator T is the norm limit of nilpotent operators if and only if it is biquasitriangular and both its spectrum and essential spectrum are connected and contain 0. The spectrum of T_{α} was first studied by Parrott

[24] in his Ph.D. thesis, who analyzed the different parts of the spectrum and proved, in particular, that

(2.1)
$$\sigma(T_{\alpha}) = \{ \lambda \in \mathbb{C} : |\lambda| \le e^{-1} \}$$

for any irrational $\alpha \in (0,1)$. Moreover, he also showed that the spectrum $\sigma(T_{\alpha})$ coincides with the essential spectrum $\sigma_e(T_{\alpha})$. Recall that if $\mathcal{K}(H)$ denotes the two-sided ideal of the compact operators in H, the essential spectrum of a linear bounded operator T consists of the set of complex numbers $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible modulo compact operators, that is, $T - \lambda I$ is not invertible in the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$ (see Conway's monograph [10], for instance, for more on the essential spectrum).

3. Bishop operators T_{α} with non-trivial invariant subspaces: enlarging the class of irrationals α

In this Section, we extend the set of known values of α for which the Bishop operator T_{α} acting on $L^p[0,1)$, $1 \leq p < \infty$, has non-trivial closed invariant subspaces (observe that for $p = \infty$, the existence follows since $L^{\infty}[0,1)$ is not separable).

The main goal of this section will be providing a careful approach to those irrationals in order to apply Atzmon's Theorem [5], by means of a functional calculus based on Beurling algebras, that is, algebras of continuous functions on the unit circle \mathbb{T} with a restricted growth of the Fourier sequences. In order to consider such approach, we will consider the operator

$$\widetilde{T}_{\alpha} = e \, T_{\alpha}$$

which, by means of the Spectral Theorem and equation (2.1), satisfies that the spectrum $\sigma(\widetilde{T}_{\alpha}) = \overline{\mathbb{D}}$. For the sake of completeness, we recall some results regarding Atzmon's Theorem and Flattot's result [14] to state the result in context. We refer to Chapter 5 in [9] for a complete account of it.

3.1. Beurling algebras and a theorem of Atzmon. Given $(\rho_n)_{n\in\mathbb{Z}}$ a sequence in $[1,+\infty)$, let \mathcal{A}_{ρ} consists of the Banach space of functions f continuous in \mathbb{T} such that the norm is given by

$$||f||_{\rho} = \sum_{n \in \mathbb{Z}} |\hat{f}(n)| \rho_n,$$

where $(\hat{f}(n))_{n\in\mathbb{Z}}$ denotes the sequence of Fourier coefficients of f. Observe that if $(\log \rho_n)_{n\in\mathbb{Z}}$ is sub-additive, that is, if $\rho_{m+n} \leq \rho_n \rho_m$ for all $n, m \in \mathbb{Z}$, then \mathcal{A}_{ρ} is a unital Banach algebra under pointwise multiplication. Note that the function algebra \mathcal{A}_{ρ} is isometrically isomorphic to the weighted convolution algebra $\ell^1(\mathbb{Z}, (\rho_n)_n)$; commonly known as Beurling algebra.

Definition 3.2. A sequence of real numbers $(\rho_n)_{n\in\mathbb{Z}}$ such that $\rho_0=1$ and $\rho_n\geq 1$ for all $n\in\mathbb{Z}$, is called a *Beurling sequence* if

$$\rho_{m+n} \le \rho_m \rho_n \quad \forall m, n \in \mathbb{Z} \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \frac{\log \rho_n}{1 + n^2} < \infty.$$

One of the key results regarding the Banach algebra \mathcal{A}_{ρ} when $\rho = (\rho_n)_{n \in \mathbb{Z}}$ is a Beurling sequence is that $f \in \mathcal{A}_{\rho}$ is invertible if and only if $f(e^{i\theta}) \neq 0$ for all $\theta \in [0, 2\pi]$. Moreover, the Banach algebra \mathcal{A}_{ρ} is regular [9, Theorem 5.1.7]. Recall that a function algebra \mathcal{A} on a compact space X is said to be regular if for all $p \in X$ and all compact subsets K of X with $p \notin K$, there exists $f \in \mathcal{A}$ such that f(p) = 1 and f = 0 in K. One advantage of regularity in a function algebra on \mathbb{T} is that it enables to construct two non-zero functions whose product is identically zero; and this, combined with a functional calculus argument, gives a strategy for obtaining invariant subspaces. This was pursued

by Davie [11] and refined by MacDonald [21] and Flattot [14] by means of Atzmon's theorem. In order to state it, let us recall the definition of ρ -regular numbers:

Definition 3.3. Let $\rho = (\rho_n)_{n \in \mathbb{Z}}$ be a Beurling sequence. An irrational α is said to be ρ -regular if there exists $m_0 \in \mathbb{N}$ and two functions h_1 , h_2 satisfying

$$\frac{h_1(n) \log \rho_n}{n \log n} \to \infty$$
 and $\frac{h_2(n) \log n}{\log \rho_n} \to 0$ as $n \to \infty$,

such that, for all $n > n_0$, there exists $p, q \in \mathbb{N}$, with (p, q) = 1, satisfying

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^2}$$
 and $h_1(n) \le q \le h_2(n)$.

Davie [11] made the choices $\log \rho_n = |n|^{\rho}$ with $\frac{1}{2} < \rho < 1$, $h_1(n) = n^{\rho'}$ with $0 < \rho' < \frac{1}{2}$ and $h_2(n) = n^{1/2}$, which characterized the non-Liouville numbers. Flattot [14, Theorem 4.6] extended it to a larger class including some Liouville numbers, by taking $h_1(n) = (\log n)^{2+\varepsilon}$, $h_2(n) = n/h_1(n)$ and $\log \rho_n = h_2(n)\sqrt{h_1(n)}$. In particular, using the language of continued fractions, if $(a_j/q_j)_{j=0}^{\infty}$ are the convergents of α , the limit of his result (see [14, Remark 5.4]) gives the existence of non-trivial invariant subspaces for T_{α} when

(3.2)
$$\log q_{j+1} = O(q_j^{1/2-\varepsilon}) \quad \text{for any } \varepsilon > 0.$$

Note that the condition (3.2) holds for instance for the classical Liouville number $\alpha = \sum_{j=0}^{\infty} 10^{-j!}$.

As mentioned, Atzmon's Theorem [5] was a key result in MacDonald and Flattot approaches. In order to state it, we say that a sequence $(a_n)_{n\in\mathbb{Z}}$ is dominated by another sequence, both nonnegative, $(b_n)_{n\in\mathbb{Z}}$ if $a_n \leq c \, b_n$ for all $n\in\mathbb{Z}$ and some constant c>0.

Theorem 3.4 (Atzmon [5]). Let \mathcal{X} be a Banach space and T a linear bounded operator in \mathcal{X} . Suppose that there exist sequences $(x_n)_{n\in\mathbb{Z}}$ in \mathcal{X} and $(y_n)_{n\in\mathbb{Z}}$ in \mathcal{X}^* with $x_0 \neq 0$, $y_0 \neq 0$ such that

$$Tx_n = x_{n+1}$$
 and $T^*y_n = y_{n+1}$

for all $n \in \mathbb{Z}$. Suppose further that both sequences $(\|x_n\|)_{n \in \mathbb{Z}}$ and $(\|y_n\|)_{n \in \mathbb{Z}}$ are dominated by Beurling sequences, and there are at least $\lambda \in \mathbb{T}$ at which the following vector-valued functions G_x and G_y defined on $\mathbb{C} \setminus \mathbb{T}$ do not both possess analytic continuation into a neighborhood of λ :

(3.3)
$$G_x(z) = \begin{cases} \sum_{n=1}^{\infty} x_{-n} z^{n-1} & \text{if } |z| < 1, \\ -\sum_{n=-\infty}^{0} x_{-n} z^{n-1} & \text{if } |z| > 1, \end{cases}$$

(3.4)
$$G_y(z) = \begin{cases} \sum_{n=1}^{\infty} y_{-n} z^{n-1} & \text{if } |z| < 1, \\ -\sum_{n=-\infty}^{0} y_{-n} z^{n-1} & \text{if } |z| > 1, \end{cases}$$

Then either T is a multiple of the identity or it has a non-trivial hyperinvariant subspace.

A few remarks are in order. Here $T^{-1}x = w$ means Tw = x even if T is not invertible, and in this way $(\|T^nx\|)_{n\in\mathbb{Z}}$ means a sequence $(\|x_n\|)_{n\in\mathbb{Z}}$ with $x_0 = x$ and $x_{n+1} = Tx_n$.

Observe also that the fact that $(\|x_n\|)_{n\in\mathbb{Z}}$ and $(\|y_n\|)_{n\in\mathbb{Z}}$ are dominated by Beurling sequences ensures that the Laurent series defining G_x and G_y converge absolutely in $\mathbb{C}\setminus\mathbb{T}$. In addition, both G_x and G_y are analytic functions in $\mathbb{C}\setminus\mathbb{T}$ and at ∞ , and hence, by Liouville's Theorem, each must have at least one singularity on the unit circle. At this regard, in order to apply Atzmon's Theorem to T_α , as observed by MacDonald (see [21, Claim pp. 307]) one has the following:

Proposition 3.5. Let $\alpha \in (0,1)$ be an irrational number and T_{α} the Bishop operator acting on $L^p[0,1)$. Let $x_0 \in L^p[0,1)$ and consider G_{x_0} and $G_{e^{2\pi it}x_0}$ the analytic functions in $\mathbb{C} \setminus \mathbb{T}$ given by equation (3.3) associated to x_0 and $e^{2\pi it}x_0$, respectively. Then $z_0 \in \mathbb{T}$ is a singularity of G_{x_0} if and only if $e^{-2\pi i\alpha}z_0$ is a singularity of $G_{e^{2\pi it}x_0}$.

The proof is just a consequence of the fact that T_{α} is similar to $e^{2\pi i\alpha}T_{\alpha}$ via the bilateral shift operator $W = M_{e^{2\pi it}}$ in $L^p[0,1)$ (and unitary equivalent in $L^2[0,1)$).

With Proposition 3.5 at hand, one deduces that T_{α} has non-trivial hyperinvariant subspaces in $L^p[0,1), \ 1 \leq p < \infty$, by means of Atzmon's Theorem as far as there exist $x,y \in L^p[0,1) \setminus \{0\}$ such that $\left(\|\widetilde{T}_{\alpha}^n x\|\right)_{n \in \mathbb{Z}}$ and $\left(\|(\widetilde{T}_{\alpha}^*)^n y\|\right)_{n \in \mathbb{Z}}$ are dominated by Beurling sequences. We state it for later reference.

Theorem 3.6. Given $\alpha \in (0,1)$ be an irrational number, if there exist $x, y \in L^p[0,1) \setminus \{0\}$, $1 \leq p < \infty$, such that $(\|\widetilde{T}_{\alpha}^n x\|)_{n \in \mathbb{Z}}$ and $(\|(\widetilde{T}_{\alpha}^*)^n y\|)_{n \in \mathbb{Z}}$ are dominated by Beurling sequences then T_{α} has a non-trivial hyperinvariant closed subspace.

We are now in position to state the main result of this section:

Theorem 3.7. Let $\alpha \in (0,1)$ an irrational number and $(a_j/q_j)_{j=0}^{\infty}$ the convergents in its continuous fraction. If

(3.5)
$$\log q_{j+1} = O\left(\frac{q_j}{(\log q_j)^3}\right)$$

then T_{α} has a non-trivial closed hyperinvariant subspace in $L^{p}[0,1)$, for $1 \leq p \leq \infty$.

Observe that Theorem 3.7 relaxes the condition provided by Flattot (3.2), allowing the exponent 1 instead of 1/2 and quantifying the role of ε . As we shall establish in Section 4, Theorem 3.7 is essentially the best possible result attainable from Theorem 3.4 and any improvement beyond the power of $\log q_j$ seems to require different functional analytical results.

Before proving Theorem 3.7, we consider a short derivation of the results of Davie and Flattot from Theorem 3.6 which highlights arithmetical considerations encapsulated in the Banach algebra arguments and may give some insight into the problem. In particular, it constitutes a simplification of the Theorem in [14].

We will see that the aforementioned results from [11] and [14] follow choosing in Theorem 3.6

$$x = y = 1_{\mathcal{B}_{\alpha}}$$
 with $\mathcal{B}_{\alpha} = \left\{ \frac{1}{20} < t < \frac{19}{20} : \langle t - n\alpha \rangle > \frac{1}{20n^2}, \ \forall n \in \mathbb{Z}^* \right\}$

where $\langle x \rangle = \min(\{x\}, 1 - \{x\})$ is the distance to the closest integer and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. As a matter of fact \mathcal{B}_{α} is none other than a variant of the sets E_t appearing in those papers. We point out that replacing in the definition of \mathcal{B}_{α} the condition by $\langle q_j t \rangle > Cq_j^{-1}$, with C a certain constant, would give a more manageable set but we prefer not to proceed in this way to keep the analogy with [11] and [14].

Observe also that \mathcal{B}_{α} has positive measure and hence x and y do not vanish identically as elements of $L^p[0,1)$. Note that $\langle t \rangle \leq \delta$ defines in [0,1) a set of measure 2δ for $\delta < 1/2$. Then the measure of the complement of \mathcal{B}_{α} in [0,1) is at most one twentieth of $2 + 2\sum_{n \in \mathbb{Z}^*} n^{-2} = 2 + 2\pi^2/3 < 20$ and consequently \mathcal{B}_{α} has positive measure.

In what follows, if α is an irrational number, and $(q_j)_{j=0}^{\infty}$ denotes the denominators of its convergents, an important fact we are going to use about continued fractions is that $(q_j)_{j=0}^{\infty}$ is an increasing sequence of positive integers such that [23, §7.5]

(3.6)
$$(2Q)^{-1} < \langle q\alpha \rangle < Q^{-1} \quad \text{for} \quad q = q_j, \quad Q = q_{j+1}.$$

This is more precise than Dirichlet's theorem, which assures $\langle q\alpha \rangle < q^{-1}$ for infinitely many values of q. From here on out, we use q and Q to indicate consecutive terms of $(q_j)_{j=0}^{\infty}$ as in (3.6).

3.8. The results of Davie and Flattot. In this subsection, we derive the results of Davie and Flattot, providing a simplification of the Theorem in [14].

In the sequel, the real function

(3.7)
$$L_n(t) = \sum_{j=0}^{n-1} (1 + \log\{t + j\alpha\})$$

plays a fundamental role because it is plain to check

(3.8)
$$\begin{cases} \widetilde{T}_{\alpha}^{n} f(t) = e^{L_{n}(t)} f(\{t + n\alpha\}), & \widetilde{T}_{\alpha}^{-n} f(\{t + n\alpha\}) = e^{-L_{n}(t)} f(t), \\ (\widetilde{T}_{\alpha}^{*})^{n} f(\{t + n\alpha\}) = e^{L_{n}(t)} f(t), & (\widetilde{T}_{\alpha}^{*})^{-n} f(t) = e^{-L_{n}(t)} f(\{t + n\alpha\}) \end{cases}$$

for $n \in \mathbb{Z}^+$ and also for n = 0 defining $L_0 = 0$.

For latter reference it is convenient to manipulate a little the definition of $L_n(t)$ when n=q.

Lemma 3.9. For q as in (3.6) fixed there exist $1/2 < |\delta| < 1$ and $|\delta_{\ell}| < 1$ with the same sign such that for any $k \in \mathbb{Z}$

$$L_q(t + kq\alpha) = \sum_{\ell=0}^{q-1} \left(1 + \log\left\{t + \frac{\ell}{q} + \frac{k\delta + \delta_{\ell}}{Q}\right\} \right).$$

Proof. By (3.6), we can write $\alpha = a/q + \delta/(qQ)$ where a/q is a convergent of α . Then the fractional part in (3.7) is $\{t + ja/q + k\delta/Q + j\delta/(qQ)\}$. The map $j \mapsto aj$ is invertible modulo q. If $\ell \mapsto j_{\ell}$ is its inverse with $0 \le j_{\ell} < q$, the result follows taking $\delta_{\ell} = j_{\ell}\delta/q$.

The following estimates for L_n are variations on those for F_m in [11].

Lemma 3.10. There exists an absolute constant C > 0 such that for $n \in \mathbb{Z}^+$

$$L_n(t) \le C\left(r + \frac{n}{q}\log(q+1)\right)$$
 for every $t \in \mathbb{R}$

where r is the remainder when n is divided by q. Moreover we have

$$L_n(t) \ge -C\left(r' + \frac{n+q}{q}\log(\mu^{-1} + q)\right) \qquad if \quad \min_{0 \le j < r' + n} \{t + j\alpha\} \ge \mu > 0$$

where r' = 0 if r = 0 and r' = q - r otherwise.

Proof. Separating the last r terms in L_n , we have

$$L_n(t) \le r + \sum_{j=0}^{n-r-1} (1 + \log\{t + j\alpha\}) = r + \sum_{k=0}^{\lfloor n/q \rfloor - 1} L_q(t + kq\alpha).$$

Applying Lemma 3.9, as δ_{ℓ} has constant sign and Q > q, on each interval $[\ell/q, (\ell+1)/q]$ there is exactly one value $\ell/q + \delta_{\ell}/Q$ and then we have

$$L_q(t + kq\alpha) \le q + \sum_{\ell=2}^{q-1} \log \frac{\ell}{q} \le C \log(q+1),$$

using Stirling's approximation, which proves the first inequality.

For the second, we expand the sum to the first multiple of q not less than n. Then

$$L_n(t) = -\sum_{j=n}^{n+r'-1} + \sum_{j=0}^{n+r'-1} \ge -r' + \sum_{k=0}^{\lceil n/q \rceil - 1} L_q(t + kq\alpha).$$

As the values of $\ell/q + \delta_{\ell}/Q$ are confined to disjoint intervals of length q^{-1} , at most two of the fractional parts in L_q could nearly coincide and the smallest fractional part appearing in $L_q(t+kq\alpha)$ is the minimum indicated in our hypothesis. Then

$$L_q(t + kq\alpha) \ge -2\log(\mu^{-1}) + q + \sum_{\ell=1}^{q-2}\log\frac{\ell}{q} \ge -C\log(\mu^{-1} + q)$$

and the result follows.

Corollary 3.11. Let $x = y = 1_{\mathcal{B}_{\alpha}}$. Then for $n \in \mathbb{Z}$

$$\log (1 + \|\widetilde{T}_{\alpha}^{n} x\| + \|(\widetilde{T}_{\alpha}^{*})^{n} y\|) \ll q + \frac{|n| + q}{q} \log(|n| + q + 1).$$

Proof. The result is trivial for n=0 and it follows immediately from the first part of Lemma 3.10 via (3.8) if n>0. On the other hand, the second part gives the expected bound for $\log\left(1+\|\widetilde{T}_{\alpha}^{-n}x\|\right)$ with $n\in\mathbb{Z}^+$ because for $t\in\mathcal{B}_{\alpha}$, we can take $\mu^{-1}=20n^2$. The same works for $\log\left(1+\|(\widetilde{T}_{\alpha}^*)^{-n}y\|\right)$ with $n\in\mathbb{Z}^+$ because $\{t+n\alpha\}\in\mathcal{B}_{\alpha}$ implies $\langle t+n\alpha-\ell\alpha\rangle>1/(20\ell)^2$ for $-q<\ell\leq n,\ \ell\neq 0$ and then $\mu^{-1}=20(n+q)^2$ is a valid choice.

With this bound we can easily derive the best known result from Theorem 3.6.

Corollary 3.12. Let $\alpha \in (0,1)$ be an irrational number such that the convergents $(a_j/q_j)_{j=0}^{\infty}$ in its continuous fraction satisfy (3.2). Then T_{α} has a non-trivial closed hyperinvariant subspace.

Proof. The sequence $(\rho_n)_{n\in\mathbb{Z}}$ given by $\log \rho_n = C_{\sigma}|n|\log^{-\sigma}(2+|n|)$ is clearly a Beurling sequence for any $\sigma > 1$ and $C_{\sigma} > 0$. By Corollary 3.11 and Theorem 3.6, it is enough to show that for |n| large we can always find q such that

$$q + \frac{|n| + q}{q} \log(|n| + q + 1) = O\left(\frac{|n|}{\log^{\sigma}(2 + |n|)}\right)$$
 for some $\sigma > 1$.

Take q such that $q \leq |n|^{2/3} < Q$. By (3.2) we have $\log Q = O(q^{1/2-\varepsilon})$, hence $q \gg (\log |n|)^{1+\sigma}$ for $1 + \sigma = (1/2 - \varepsilon)^{-1}$ and the expected bound follows.

3.13. **Proof of Theorem 3.7.** In this subsection, we address the proof of Theorem 3.7.

Firstly, for $t \in \mathcal{B}_{\alpha}$ we can take $\mu^{-1} = 20(n+q)^2$ in Lemma 3.10 and if n is very large in comparison with q there is an asymmetry in the bounds obtained in this lemma being the upper bound stronger. This is reasonable since in (3.7) a fractional part can be very small but not large since it is bounded by 1. Anyway, we shall see that it is possible to partially recover the symmetry getting a non biased bound for $|L_n(t)|$ by a more careful analysis than the one in §3.8. The improvement is achieved when n is very large in comparison to q, in such a way that $\log(|n| + q + 1)$ is not comparable to $\log(q+1)$ in Corollary 3.11, but it is controlled by Q (see Proposition 3.16 below).

Lemma 3.14. If $q \mid n, 1 \le n \le Q/(100q)$ and $t_0 \in \mathcal{B}_{\alpha}$ then

$$|L_n(t)| \ll \frac{n}{q} \log(q+1)$$
 for every $t \in [0,1)$ with $|t-t_0| \le \frac{1}{(10q)^2}$.

10 FERNANDO CHAMIZO, EVA A. GALLARDO-GUTIÉRREZ, MIGUEL MONSALVE-LÓPEZ, AND ADRIÁN UBIS

Proof. We can write $L_n(t) = \sum_{k=0}^{n/q-1} L_q(t+kq\alpha)$. It is enough to prove $\min_{0 \le j < q} \{t+kq\alpha+j\alpha\} > Cq^{-2}$ with C some constant because in this case Lemma 3.10 assures that each term in the sum contributes $O(\log(q+1))$.

By (3.6), $\alpha = a/q + \eta/(qQ)$ with $|\eta| < 1$. Then

$$\left\{t + kq\alpha + j\alpha\right\} \ge \left\{t_0 + j\alpha + \frac{\eta k}{Q}\right\} - \frac{1}{(10q)^2} \ge \left\langle t_0 + j\alpha + \frac{\eta k}{Q}\right\rangle - \frac{1}{(10q)^2} > \frac{1}{20j^2} - \frac{2}{100q^2}$$

where we have used $t_0 \in \mathcal{B}_{\alpha}$ and $k < Q/(100q^2)$ for the last inequality. This is greater than Cq^{-2} when $1 \le j < q$. A similar argument applies for j = 0 using that $\langle t_0 \rangle > 1/20$ for $t_0 \in \mathcal{B}_{\alpha}$.

Lemma 3.15. For $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, if $q \mid n_2 - n_1 \text{ and } Q/(100q) \leq n_2 - n_1 \leq Q - q \text{ then}$

$$L_{n_2}(t) - L_{n_1}(t) \ll \log n_2 + \frac{n_2 - n_1}{q} \log(q + 1)$$
 for every $t \in \mathcal{B}_{\alpha}$.

Proof. We start writing

$$L_{n_2}(t) - L_{n_1}(t) = \sum_{k=0}^{K-1} L_q(t + n_1\alpha + kq\alpha)$$
 with $K = \frac{n_2 - n_1}{q}$.

Let us call μ to the minimum of the fractional parts appearing in these terms. We have $\mu > 1/(20n_2^2)$ because $t \in \mathcal{B}_{\alpha}$.

By Lemma 3.9 and doing a translation $\ell \mapsto \ell + \ell_0$ modulo q if the minimum is reached for a certain $k = k_0$ and $\ell = \ell_0$, this can be expanded as

$$L_{n_2}(t) - L_{n_1}(t) = \sum_{k=0}^{K-1} \sum_{\ell=0}^{q-1} \left(1 + \log \left\{ \mu + \frac{\ell}{q} + \frac{(k-k_0)\delta + \delta_{\ell}}{Q} \right\} \right).$$

Note that we have employed $\delta_{\ell+\ell_0} - \delta_{\ell_0} = \delta_{\ell}$. We know that $K \leq Q/q - 1$ and recalling the properties of δ and δ_{ℓ} in Lemma 3.9, we have

$$\left|\frac{|k-k_0|}{2Q} < \left|\frac{(k-k_0)\delta + \delta_\ell}{Q}\right| < \frac{K}{Q} \le \frac{1}{q} - \frac{1}{Q}.$$

If $\ell \neq 0$ then the fractional part can be safely compared with that of ℓ/q to get $O(\log(q+1))$ for the sum on $\ell \neq 0$ and fixed each value of k. This gives $O(K\log(q+1))$. The contribution of $\ell = 0$ is comparable to

$$\log(\mu^{-1}) + K + \left| \sum_{k=1}^K \log\left(\frac{k}{Q}\right) \right| \ll \log n_2 + K + \left| \sum_{k=1}^K \log\frac{K}{Q} \right| + \left| \sum_{k=1}^K \log\frac{k}{K} \right|.$$

The last sum is O(K) by Stirling's approximation. This gives the expected bound noting $Q/K \ll q^2$.

With these lemmas we are ready to get an improvement of Lemma 3.10 for restricted values of t.

Proposition 3.16. Assume $Q \ge 4(10q)^4$, $1 \le n \le Q^{3/2}$ and let N be the closest multiple of Q to n. Then for $t \in \mathcal{B}_{\alpha}$ we have

$$L_n(t) \ll q + \frac{|n-N|}{q} \log(q+1) + \frac{n+Q}{Q} \log(n+1).$$

Proof. We introduce the decomposition

$$(3.9) L_n = L_N + (L_n - L_{n'}) + (L_{n'} - L_N)$$

where $n' = N \pm m$, $m \in \mathbb{Z}^+$, with $\pm m$ the closest multiple of q to n - N (here the \pm indicates the sign of n - N). Clearly we have $0 \le m \le |n - N| + q/2$.

Applying Lemma 3.10 with Q instead of q and $\mu^{-1} = 20N^2$, we have

$$L_N(t) \ll \frac{N}{Q} \log(N+1) \ll \frac{n}{Q} \log(n+1).$$

If n > n', $L_n(t) - L_{n'}(t)$ is $L_{n-n'}(t+n'\alpha)$ and if n < n' is $-L_{n'-n}(t+n\alpha)$. As |n-n'| < q in both cases Lemma 3.10 with $\mu^{-1} = 20(n+2q)^2$ assures

$$L_n(t) - L_{n'}(t) \ll q + \log(n+q) \ll q + \log n.$$

Finally, we have to deal with the last term in (3.9). If Q/(100q) < m then we are under the hypotheses of Lemma 3.15 that gives

$$L_{n'}(t) - L_N(t) \ll \log(N+m) + \frac{m}{q}\log(q+1).$$

Hence

$$L_{n'}(t) - L_N(t) \ll \log(n+1) + \frac{|n-N|}{q} \log(q+1).$$

If $m \leq Q/(100q)$, note firstly

$$\langle N\alpha \rangle \le \frac{N}{Q} \langle Q\alpha \rangle \le \frac{N}{Q^2} \le \frac{n + Q/2}{Q^2} \le \frac{1}{(10q)^2}.$$

If $n' \geq N$ we write $L_{n'}(t) - L_N(t) = L_m(t + N\alpha)$ and the previous bound proves that we can apply Lemma 3.14 to get $O(q^{-1}m\log(q+1))$. If n' < N then $L_N(t) - L_{n'}(t)$ coincides with $L_m(t + N\alpha)$ formally changing α by $-\alpha$ in the definition of L_m . As the denominators of the convergents of α and $-\alpha$ coincide except for a unit shift in the indexes, the same argument applies.

Adding the contribution of the three terms in (3.9) we get the result.

The analogue of Corollary 3.11 is:

Corollary 3.17. Let $x = y = 1_{\mathcal{B}_{\alpha}}$. For any $|n| \leq Q^{3/2}$ we have

$$\log \left(1 + \|\widetilde{T}_{\alpha}^{n} x\| + \|(\widetilde{T}_{\alpha}^{*})^{n} y\|\right) \ll q + \frac{|n|}{q} \log(q+1) + \frac{|n| + Q}{Q} \log(|n| + 2).$$

Proof. We are going to show that the bound holds for $A_n = \log (1 + \|\widetilde{T}_{\alpha}^n x\| + \|(\widetilde{T}_{\alpha}^*)^n y\|)$, $B_n = \log (1 + \|\widetilde{T}_{\alpha}^{-n} x\|)$ and $C_n = \log (1 + \|(\widetilde{T}_{\alpha}^*)^{-n} y\|)$ with $n \in \mathbb{Z}^+$.

For A_n , it follows substituting in (3.8) the first bound of Lemma 3.10.

If $Q < 4(10q)^4$ then $\log(|n| + q + 1) \ll \log(q + 1)$ and the bound for B_n and C_n follows from Corollary 3.11.

If $Q \ge 4(10q)^4$ Proposition 3.16 gives the bound for B_n .

It remains to bound C_n if $Q \ge 4(10q)^4$. With this purpose, we rewrite the last formula in (3.8) as $(T_{\alpha}^*)^{-n} f(\{t - n\alpha\}) = e^{-L_n(t - n\alpha)} f(t)$ and we note

$$L_n(t - n\alpha) = -\log\{t\} + \log\{t - n\alpha\} + \sum_{i=0}^{n-1} (1 + \log\{t - j\alpha\}).$$

The sum coincides with $L_n(t)$ replacing α by $-\alpha$. As we mentioned before, the convergents of α and $-\alpha$ have the same denominators and then Proposition 3.16 applies also for this sum. On the other hand, $\log\{t\}$ and $\log\{t-n\alpha\}$ are $O(\log(|n|+1))$ if $t \in \mathcal{B}_{\alpha}$.

Once we have got this bound, the proof of our main result parallels that of Corollary 3.12.

Proof of Theorem 3.7. Given $n \neq 0$, choose q such that $q \leq |n|^{2/3} < Q$. In this range

$$q + \frac{|n| + Q}{Q} \log(|n| + 2) \ll |n|^{2/3} + |n|^{1/3} \log(|n| + 2)$$

and by the condition (3.5),

$$\frac{|n|}{q}\log(q+1) \ll \frac{|n|}{\log Q(\log\log Q)^2} \ll \frac{|n|}{\log|n|(\log\log|n|)^2}.$$

Therefore by Corollary 3.17, there exists C > 0 such that for every $n \in \mathbb{Z}$

(3.10)
$$\max (\|\widetilde{T}_{\alpha}^{n}x\|, \|(\widetilde{T}_{\alpha}^{*})^{n}y\|) \leq \exp \left(\frac{C|n|}{\log(2+|n|)(\log\log(4+|n|))^{2}}\right)$$

and the result follows from Theorem 3.6 because the right hand side is a Beurling sequence.

Remark. Note that for Bishop-type operators of the form $T_{s,\alpha}f(t) = t^s f(\{t + \alpha\})$ where s > 0, all the bounds computed above remain true replacing $L_n(t)$ by

$$L_{s,n}(t) = \sum_{j=0}^{n-1} (s + s \log\{t + j\alpha\}),$$

and considering again $x = y = 1_{\mathcal{B}_{\alpha}}$. This clearly follows from the fact $L_{s,n}(t) = sL_n(t)$. Therefore, Theorem 3.7 is also valid for every $T_{s,\alpha}$ with s > 0; and, in particular, we obtain a generalization of [14, Theorem 4.7].

4. The limits of Atzmon Theorem

In this section we shall show that it is not possible to improve much on Theorem 3.7 by applying Atzmon's Theorem (Theorem 3.4) to \widetilde{T}_{α} . Before stating the main result of the section, observe that if $L^0[0,1)$ denotes the space of (classes of) measurable functions defined almost everywhere on [0,1), \widetilde{T}_{α} is a bijection in $L^0[0,1)$ with inverse:

$$\widetilde{T}_{\alpha}^{-1}f(t) = e^{-1}\frac{f(\{t-\alpha\})}{\{t-\alpha\}}, \quad t \in [0,1).$$

Nevertheless, in $L^p[0,1)$, $1 \leq p < \infty$, the operator \widetilde{T}_{α} is an injective, dense range operator. Hence, there exists a dense set of functions $g \in L^p[0,1)$ which have an infinite chain of backward iterates, that is, for all n > 0 there is $g_n \in L^p[0,1)$, unique, such that $\widetilde{T}_{\alpha}^n g_n = g$ (see [6, Corollary 1.B.3], for instance). As an abuse of notation in the next theorem, for $f \in L^p[0,1)$ and n > 0, we will denote by $\|\widetilde{T}_{\alpha}^{-n}f\|$ the norm of the n-th backward iterate $\widetilde{T}_{\alpha}^{-n}f$ whenever it belongs to $L^p[0,1)$ or ∞ , otherwise. Our main aim in this section is to prove the following:

Theorem 4.1. Let us define \mathcal{M} as the set of irrationals such that the convergents $(a_j/q_j)_{j=0}^{\infty}$ in its continuous fraction satisfy

$$\log q_{j+1} = O\left(\frac{q_j}{\log q_j}\right)$$

for every $j \geq 0$. Then, if α is an irrational not in \mathcal{M} we have

$$\sum_{n=-\infty}^{+\infty} \frac{\log(1+\|\widetilde{T}_{\alpha}^n f\|)}{1+n^2} = +\infty$$

for any non-zero $f \in L^p[0,1)$, for $1 \le p < \infty$.

Note that this result shows that there does not exist a sequence $(x_n)_{n\in\mathbb{Z}}$ satisfying the requirements in the statement of Theorem 3.4 whenever $\alpha\in\mathcal{M}$, and hence establishes a threshold limit in the growth of the denominators of the convergents of α for the application of Atzmon's Theorem to Bishop operators.

In order to prove Theorem 4.1, we will show that either $\|\widetilde{T}_{\alpha}^n f\|$ or $\|\widetilde{T}_{\alpha}^{-n} f\|$ is large for many values of n. To accomplish such a task, we consider the equation

(4.1)
$$\|\widetilde{T}_{\alpha}^{n}f\|^{p} + \|\widetilde{T}_{\alpha}^{-n}f\|^{p} = \int_{0}^{1} (e^{pL_{n}(t-n\alpha)} + e^{-pL_{n}(t)})|f(t)|^{p} dt$$

for any $n \geq 1$, which follows directly from (3.7), (3.8) and a change of variable. Now, $\alpha \notin \mathcal{M}$ means that it is very well approximable by some rationals a/q, which will imply that $L_n(t-n\alpha)$ is essentially identical to $L_n(t-n\frac{a}{q}) = L_n(t)$ for any n near q and divisible by it. In this situation, it appears that the integral in (4.1) must be large unless $|L_n(t)|$ is small, which should happen rarely. That is the basic idea behind the following result.

Lemma 4.2. Let a/q and A/Q be two consecutive convergents of α an irrational number, $q \geq 2$. For any $\varepsilon \in (0, 1/4)$ there exists a set $S_{q,\varepsilon} \subset [0, 1)$ of measure at most 20ε such that

$$\min(|L_n(t-n\alpha)|, |L_n(t)|) > \varepsilon \frac{n}{q} \log q$$

for every $t \not\in S_{q,\varepsilon}$ and every $n \in [\varepsilon^{-2}q^2 \log q, \varepsilon^2 Q/q]$.

Proof. Given $\varepsilon \in (0, 1/4)$, pick any $n \in [\varepsilon^{-2}q^2 \log q, \varepsilon^2 Q/q]$. By (3.6) we have $\alpha = a/q + \delta/(qQ)$ with $|\delta| < 1$ and our hypothesis assures $|j\delta/(qQ)| < \varepsilon^2/q^2$ for every $|j| \le n$. Hence for $\langle qt \rangle > 2\varepsilon$, we have $\langle t + ja/q \rangle > 2\varepsilon/q$ and

$$\left|\log\left\{t+j\alpha\right\}-\log\left\{t+\frac{ja}{q}\right\}\right| \leq \left|\log\left(\frac{2\varepsilon}{q}-\frac{\varepsilon^2}{q^2}\right)-\log\left(\frac{2\varepsilon}{q}\right)\right| \leq \frac{\varepsilon}{q}.$$

With this and the q-periodicity in j of $\log\{t + ja/q\}$ we deduce

$$\left| L_n(t) - \frac{n'}{q} L(\{qt\}) \right| \le \varepsilon \frac{n}{q} + \left| \sum_{j=n'}^{n-1} \left(1 + \log\{t + \frac{ja}{q}\} \right) \right|$$

where $L(x) = \sum_{\ell=0}^{q-1} (1 + \log((x+\ell)/q))$ and $n' = q \lfloor n/q \rfloor$. The trivial bound for the last term is $q(1 - \log(2\varepsilon/q))$ which is less than $2\varepsilon n/q$ in our range. A similar argument applies for $L_n(t - n\alpha)$. Hence

(4.2)
$$\min\left(|L_n(t-n\alpha)|,|L_n(t)|\right) > \frac{n-q+1}{q}|L(\{qt\})| - 3\varepsilon\frac{n}{q} \quad \text{for } \langle qt \rangle > 2\varepsilon.$$

The function L is increasing in (0,1) and $L'(x) \ge \log q$. Then the measure of $\{x : |L(x)| \le 8\varepsilon \log q\}$ is at most 16ε and (4.2) gives the expected bound except in the set

$$S_{q,\varepsilon} = \{ t \in [0,1) : \langle qt \rangle \le 2\varepsilon \} \cup \{ t \in [0,1) : |L(\{qt\})| \le 8\varepsilon \log q \}$$

which has measure at most 20ε .

With this lemma, we are ready to prove the theorem.

Proof of Theorem 4.1. Without loss of generality, assume that $f \in L^p[0,1)$ has an infinite chain of backward iterates $\widetilde{T}_{\alpha}^{-n}f \in L^p[0,1)$ and suppose ||f|| = 1. If α is an irrational not in \mathcal{M} , we have $\limsup_{j\to\infty} \frac{\log q_{j+1}}{q_j/\log q_j} = +\infty$, there exists a subsequence $(q_{j_m})_{m\in\mathbb{N}}$ such that

$$\frac{\log Q_{j_m}}{q_{j_m}/\log q_{j_m}} > m^2, \quad \text{with} \quad Q_{j_m} = q_{j_m+1}$$

for every m > 2. Now, consider the sets $S_{m_*} = \bigcup_{m \geq m_*} S_{q_{j_m},1/m^2}$, with $S_{q,\varepsilon}$ defined as in Lemma 4.2. Since $\sum_{m=1}^{\infty} m^{-2} < \infty$ we have that $\lim_{m_* \to \infty} \int_{S_{m_*}} |f(t)|^p dt = 0$, so there exists m_* such that $\int_{S_{m_*}} |f(t)|^p dt < 1/2$. This and (4.1) imply that

$$\|\widetilde{T}_{\alpha}^{n}f\|^{p} + \|\widetilde{T}_{\alpha}^{-n}f\|^{p} \ge \frac{1}{2} \inf_{t \notin S_{m_{n}}} \left(e^{pL_{n}(t-n\alpha)} + e^{-pL_{n}(t)} \right).$$

By Lemma 4.2 with $q = q_{j_m}$, $m \ge m_*$, and $\varepsilon = 1/m^2$ we have

$$\|\widetilde{T}_{\alpha}^{n}f\|^{p} + \|\widetilde{T}_{\alpha}^{-n}f\|^{p} \ge \frac{1}{2}e^{pn\log q_{j_{m}}/(m^{2}q_{j_{m}})}$$

for any $n \in [m^2 q_{j_m}^2 \log q_{j_m}, m^{-2} Q_{j_m}/q_{j_m}]$, so that

(4.3)
$$\sum_{m^2 q_{j_m}^3 < |n| < m^{-2} Q_{j_m}/q_{j_m}} \frac{\log(1 + \|\tilde{T}_{\alpha}^n f\|)}{1 + n^2} \gg \log\left(\frac{Q_{j_m}}{m^4 q_{j_m}^4}\right) \frac{\log q_{j_m}}{q_{j_m}} \frac{1}{m^2} \gg 1$$

for any m sufficiently large. As a consequence of

$$\frac{Q_{j_m}}{m^2 q_{j_m}} < Q_{j_m} = q_{j_m+1} \le q_{j_{m+1}} \le (m+1)^2 q_{j_{m+1}}^3,$$

we observe that the intervals defined by the indexes of the sum in (4.3) do not overlap for different values of m, hence the theorem follows.

5. Spectral subspaces of Bishop operators

In this section, we deal with $local\ spectral\ subspaces$ of Bishop operators, which are hyperinvariant subspaces (not necessarily closed) associated to closed subsets of the spectrum. While local spectral subspaces are closed for a large class of operators, those satisfying the so-called $Dunford\ property$ (C), as a consequence of the estimates obtained in the previous section, our main result in this section is that all Bishop operators do not belong to such a class; and therefore there exist local spectral subspaces which are not closed.

Before going further, we recall some preliminaries regarding local spectral theory, and refer to Laursen and Neumann monograph [19] for more on the subject.

5.1. Local spectral theory background. Let \mathcal{X} denote an arbitrary complex Banach space and $\mathcal{L}(\mathcal{X})$ the space of linear bounded operators on \mathcal{X} . For an open subset $U \subseteq \mathbb{C}$, let $\mathcal{H}(U,\mathcal{X})$ be the Fréchet space of analytic functions from U to \mathcal{X} endowed with the topology of uniform convergence on compact subsets.

Given any $T \in \mathcal{L}(\mathcal{X})$ and $x \in \mathcal{X}$, let $\rho_T(x)$ be the local resolvent of T at x, i.e. the set of $\lambda \in \mathbb{C}$ for which there exists an open neighborhood $U_{\lambda} \ni \lambda$ and an analytic function $f \in \mathcal{H}(U_{\lambda}, \mathcal{X})$ which fulfills the equation

(5.1)
$$(T - zI) f(z) = x, \text{ for every } z \in U_{\lambda}.$$

By $\sigma_T(x)$ we will denote the local spectrum of T at x, i.e. the complementary set of the local resolvent. Of course, bearing in mind that the function $f(z) = (T - zI)^{-1} x$ is analytic in the whole resolvent set, we have $\sigma_T(x) \subseteq \sigma(T)$. In the sequel, we shall use the following properties concerning the local spectra of an operator:

- (a) $\sigma_T(ax + by) \subseteq \sigma_T(x) \cup \sigma_T(y)$ for every $x, y \in \mathcal{X}$ and $a, b \in \mathbb{C}$.
- (b) $\sigma_T(x) \subseteq \sigma(T)$ for every $x \in \mathcal{X}$.
- (c) $\sigma_T(S x) \subseteq \sigma_T(x)$ for every $S \in \mathcal{L}(\mathcal{X})$ which commutes with T.

Whenever the solution of (5.1) is unique for every $\lambda \in \mathbb{C}$, we will say that T satisfies the single-valued extension property (abbrev. SVEP) and we will denote by $f_x(z)$ such local resolvent function. In such a case, the local spectral radius $r_T(x)$ fulfills the equality

$$r_T(x) = \max\{|\lambda| : \lambda \in \sigma_T(x)\}$$
 for every $x \in \mathcal{X}$.

Reminding the point spectrum and the compression spectrum of the Bishop operators, $\sigma_p(T_\alpha) = \emptyset$ and $\sigma_c(T_\alpha) = \emptyset$, it is somewhat direct to prove that both T_α and T_α^* have the SVEP, indeed, the same holds for every Bishop-type operator [15, Prop. 3.6].

Our first result regarding the local spectrum of Bishop operators by means of the estimates obtained in the previous section is the following:

Theorem 5.2. Let $\alpha \in (0,1)$ be any irrational number, T_{α} the Bishop operator acting on $L^p[0,1)$, $1 \leq p < \infty$, and

$$\mathcal{B}_{\alpha} = \left\{ \frac{1}{20} < t < \frac{19}{20} : \langle t - n\alpha \rangle > \frac{1}{20n^2}, \ \forall n \in \mathbb{Z}^* \right\}.$$

Then, both local spectra $\sigma_{T_{\alpha}}(1_{\mathcal{B}_{\alpha}})$ and $\sigma_{T_{\alpha}^*}(1_{\mathcal{B}_{\alpha}})$ are contained in the circle of radius e^{-1} , that is,

$$\sigma_{T_{\alpha}}(1_{\mathcal{B}_{\alpha}}) \subseteq \partial D(0, e^{-1}) \text{ and } \sigma_{T_{\alpha}^*}(1_{\mathcal{B}_{\alpha}}) \subseteq \partial D(0, e^{-1}).$$

Proof. We will just prove the theorem for T_{α} , an analogous argument works for T_{α}^* . Let us denote the convergents of α by $(a_j/q_j)_{j=0}^{\infty}$. Then, by Corollary 3.17, we know that for every $q_m \leq n^{2/3} \leq q_{m+1}$, we have

$$\log ||\widetilde{T}_{\alpha}^{-n} 1_{\mathcal{B}_{\alpha}}|| \le C \cdot \left(q_m + \frac{n}{q_m} \log(q_m + 1) + \frac{n + q_{m+1}}{q_{m+1}} \log(n + 2)\right),$$

where C > 0 is an absolute constant independent of m. Taking into account the range of n, this implies

$$||\widetilde{T}_{\alpha}^{-n} 1_{\mathcal{B}_{\alpha}}|| \le \exp\left(C \cdot \left(n^{-1/3} + \frac{1}{q_m} \log(q_m + 1) + n^{-2/3} \log(n + 2)\right)\right)^n.$$

Nevertheless, for every $\varepsilon > 0$, there exists m such that

$$C \cdot \left(n^{-1/3} + \frac{1}{q_m} \log(q_m + 1) + n^{-2/3} \log(n + 2)\right) \le \varepsilon$$

for every $q_m \leq n^{2/3} \leq q_{m+1}$. In particular, as a consequence of this bound, we have that the function

$$f_{1_{\mathcal{B}_{\alpha}}}(z) = \sum_{n=1}^{\infty} \left(\widetilde{T}_{\alpha}^{-n} \, 1_{\mathcal{B}_{\alpha}} \right) \cdot z^{n-1}$$

is analytic for $z \in D(0, e^{-\varepsilon})$. Since $f_{1_{\mathcal{B}_{\alpha}}}$ fulfills the equation $(\widetilde{T}_{\alpha} - zI) f_{1_{\mathcal{B}_{\alpha}}}(z) = 1_{\mathcal{B}_{\alpha}}$, this implies $D(0, e^{-\varepsilon}) \subseteq \rho_{\widetilde{T}_{\alpha}}(1_{\mathcal{B}_{\alpha}})$. Finally, making ε arbitrarily small, the theorem follows.

As it is pointed out in [4], since T_{α} has the SVEP, it may be seen that $\sigma_{T_{\alpha}}(1_{\mathcal{B}_{\alpha}})$ coincides with the singular points within $\partial D(0, e^{-1})$ of its local resolvent function. This allows us to identify easily some of the basic properties which satisfy $\sigma_{T_{\alpha}}(1_{\mathcal{B}_{\alpha}})$ (and therefore, $\sigma_{T_{\alpha}^*}(1_{\mathcal{B}_{\alpha}})$ as well).

Corollary 5.3. Let $\alpha \in (0,1)$ be any irrational number. Then, $\sigma_{T_{\alpha}}(1_{\mathcal{B}_{\alpha}})$ (resp. $\sigma_{T_{\alpha}^*}(1_{\mathcal{B}_{\alpha}})$) is symmetric with respect to the real axis and contains the point $\lambda = e^{-1}$.

Proof. We will just prove the result for T_{α} . The first claim is a consequence of $\overline{f_{1_{\mathcal{B}_{\alpha}}(z)}} = f_{1_{\mathcal{B}_{\alpha}}}(\overline{z})$, where $f_{1_{\mathcal{B}_{\alpha}}}$ is as above. Note that it may be deduced from the fact that $\widetilde{T}_{\alpha}^{-n} 1_{\mathcal{B}_{\alpha}}$ are all real-valued for every $n \in \mathbb{Z}^+$.

For the second claim, given any $z_0 \in \mathbb{D}$, the Taylor series of $f_{1_{\mathcal{B}_{\alpha}}}$ about z_0 is

$$f_{1_{\mathcal{B}_{\alpha}}}(z) = \sum_{m=0}^{\infty} \frac{\partial^m f_{1_{\mathcal{B}_{\alpha}}}(z_0)}{m!} \cdot (z - z_0)^m,$$

where

$$\partial^m f_{1_{\mathcal{B}_{\alpha}}}(z_0) = \sum_{n=m}^{\infty} n \cdot (n-1) \cdots (n-m+1) \cdot \left(\widetilde{T}_{\alpha}^{-n-1} \, 1_{\mathcal{B}_{\alpha}} \right) \cdot z_0^{n-m}.$$

Let $e^{i\vartheta}$ be a singular point on $\partial \mathbb{D}$ (there must exist at least one) for $f_{1_{\mathcal{B}_{\alpha}}}$ and choose any 0 < r < 1. By hypothesis, the series

$$f_{1_{\mathcal{B}_{\alpha}}}(z) = \sum_{m=0}^{\infty} \frac{\partial^{m} f_{1_{\mathcal{B}_{\alpha}}}(re^{\imath\vartheta})}{m!} \cdot (z - re^{\imath\vartheta})^{m}$$

has radius of convergence 1-r. Nevertheless, by the positivity of $\widetilde{T}_{\alpha}^{-n}$ and recalling that $1_{\mathcal{B}_{\alpha}}(t) \geq 0$ a.e., it may be seen that

$$\left|\left|\partial^m f_{1_{\mathcal{B}_{\alpha}}}(re^{i\vartheta})\right|\right| \leq \left|\left|\partial^m f_{1_{\mathcal{B}_{\alpha}}}(r)\right|\right| \ \text{ for every } m \geq 0,$$

what, in particular, implies that the radius of convergence of the Taylor series of $f_{1_{\mathcal{B}_{\alpha}}}$ about r cannot be greater than 1-r. This proves that 1 is a singular point for \widetilde{T}_{α} and the result follows.

Remark. The second part of the proof given for Corollary 5.3 is the vector-valued analogue of the classical result in Complex Analysis, known as Pringsheim Theorem; see, for example, [28, Sec. 7.21].

In general, given an arbitrary operator $T \in \mathcal{L}(\mathcal{X})$, determining the local spectrum at a non-zero $x \in \mathcal{X}$ is known to be a difficult problem. Actually, finding vectors $x \in \mathcal{X}$ with non-trivial local spectra may be a hopeful starting point in order to seek for (hyper-)invariant subspaces, since the subsets of \mathcal{X} defined as

$$X_T(F) = \{ x \in \mathcal{X} : \sigma_T(x) \subseteq F \},$$

turn out to be T-hyperinvariant linear manifolds by means of the properties (a), (b) and (c), and behave well via functional calculus tools. They are called *local spectral subspaces* though they are not closed a priori. Indeed, those operators $T \in \mathcal{L}(\mathcal{X})$ for which $X_T(F)$ is closed for every closed subset $F \subseteq \mathbb{C}$ are said to satisfy the Dunford property (C). Our next result states that Bishop operators do not satisfy the Dunford property (C).

Theorem 5.4. Let $\alpha \in (0,1)$ be any irrational. Then, the local spectral subspace $X_{T_{\alpha}}(\partial D(0,e^{-1}))$ (resp. $X_{T_{\alpha}^*}(\partial D(0,e^{-1}))$) is dense in $L^p[0,1)$ for $1 . In particular, neither <math>T_{\alpha}$ nor T_{α}^* have property (C) on $L^p[0,1)$.

Proof. Along the proof, let M_{ϕ} be the operator on $L^{p}[0,1)$ consisting on multiplying by ϕ . As a direct consequence of the following identity

$$M_{e^{2\pi\imath mt}} T_{\alpha} = e^{-2\pi\imath m\alpha} T_{\alpha} M_{e^{2\pi\imath mt}},$$

we deduce that T_{α} and $e^{2\pi i m \alpha} T_{\alpha}$ are similar for every $m \in \mathbb{Z}$. In particular, this implies that

$$\operatorname{span}\left\{e^{2\pi\imath mt}\cdot 1_{\mathcal{B}_{\alpha}}(t): m\in\mathbb{Z}\right\}\subseteq X_{T_{\alpha}}\left(\partial D(0,e^{-1})\right),$$

and so, reminding that the span of the set $\{e^{2\pi i m t}\}_{m \in \mathbb{Z}}$ is dense in $L^p[0,1)$ for 1 , we infer

$$\{x \in L^p[0,1) : \operatorname{supp} x \subseteq \mathcal{B}_\alpha\} \subseteq \overline{X_{T_\alpha}(\partial D(0,e^{-1}))}.$$

Moreover, since $T_{\alpha}(X_{T_{\alpha}}(\partial D(0,e^{-1}))) = X_{T_{\alpha}}(\partial D(0,e^{-1}))$, we can try to perform the same argument with the set

$$\operatorname{span}\left\{T_{\alpha} M_{e^{2\pi \imath m t}} 1_{\mathcal{B}_{\alpha}}(t) : m \in \mathbb{Z}\right\} = \operatorname{span}\left\{t e^{2\pi \imath m t} \cdot 1_{\mathcal{B}_{\alpha}}(\left\{t + \alpha\right\}) : m \in \mathbb{Z}\right\}.$$

But, since M_t is a dense range operator, we deduce that the set $\{t e^{2\pi i mt}\}_{m \in \mathbb{Z}}$ spans densely within $L^p[0,1)$ again; hence

$$\{x \in L^p[0,1) : \operatorname{supp} x \subseteq \tau_\alpha^{-1}(\mathcal{B}_\alpha)\} \subseteq \overline{X_{T_\alpha}(\partial D(0,e^{-1}))},$$

where $\tau_{\alpha}(t) = \{t + \alpha\}$. Now, as any operator of the form $M_{\{t+j\alpha\}}$ is of dense range and the finite product of dense range operators is again of this kind, we are in position to mimic our previous argument: for any $N \in \mathbb{N}$ we have

$$\overline{X_{T_{\alpha}}(\partial D(0, e^{-1}))} \supseteq \sup_{j=-N,\dots,N} \left\{ x \in L^{p}[0, 1) : \operatorname{supp} x \subseteq \tau_{\alpha}^{-j}(\mathcal{B}_{\alpha}) \right\}$$
$$= \left\{ x \in L^{p}[0, 1) : \operatorname{supp} x \subseteq \bigcup_{j=-N}^{N} \tau_{\alpha}^{-j}(\mathcal{B}_{\alpha}) \right\},$$

so it also contains the set W of x with supp $x \subset E = \bigcup_{j=-\infty}^{\infty} \tau_{\alpha}^{-j}(\mathcal{B}_{\alpha})$. Since \mathcal{B}_{α} has strictly positive measure and τ_{α}^{-1} is ergodic, we have that E has measure 1 and therefore $W = L^p[0,1)$.

Finally, for T_{α}^{*} an analogous argument works.

Remark. Observe that Theorem 5.4 applies not only for T_{α} or T_{α}^* , but also for every non-invertible Bishop-type operator $T_{\phi,\alpha} \in \mathcal{L}(L^p[0,1))$ which satisfies [21, Thm. 2.6]. In addition, this result somewhat complements the work begun in [15] consisting on identifying the local spectral properties fulfilled by those non-invertible Bishop-type operators.

Finally, as a consequence of Theorem 5.4, we show that the invariant linear manifolds consisting of the hyperrange, the analytical core and the algebraic core of T_{α} are dense in $L^{p}[0,1)$ for 1 .Recall, the hyperrange of an operator $T \in \mathcal{L}(\mathcal{X})$ is defined as

$$T^{\infty}(\mathcal{X}) = \bigcap_{n=1}^{\infty} T^n(\mathcal{X}).$$

In particular, by the injectivity of T_{α} , we have that its hyperrange matches with its algebraic core $C(T_{\alpha})$, which is defined as the greatest submanifold $M \subseteq L^{p}[0,1)$ such that $T_{\alpha}(M) = M$. On the other hand, the analytical core K(T) is defined as the set of $x \in \mathcal{X}$ for which there exists a sequence $(x_n)_{n\geq 0}$ and a constant $\delta>0$ such that

- $x = x_0$ and $Tx_{n+1} = x_n$ for every $n \ge 0$.
- $||x_n|| \le \delta^n ||x||$ for every $n \ge 0$.

In general, one has that $K(T) \subseteq C(T)$ and $K(T) = X_T(\mathbb{C} \setminus \{0\})$, see [1, Thms. 1.21 and 2.18].

Corollary 5.5. Let $\alpha \in (0,1)$ be any irrational number and $T_{\alpha} \in \mathcal{L}(L^p[0,1))$ for $1 . Then, all <math>K(T_{\alpha})$, $C(T_{\alpha})$ and $T_{\alpha}^{\infty}(L^p[0,1))$ are non-closed dense linear submanifolds of $L^p[0,1)$. The same holds for $T_{\alpha}^* \in \mathcal{L}(L^p[0,1))$ for 1 .

Proof. Firstly, observe that $C(T_{\alpha})$ is a dense linear submanifold of $L^{p}[0,1)$ since T_{α} is an injective dense range operator (see [6, Corollary 1.B.3]). In addition, since $K(T_{\alpha})$ trivially contains $X_{T_{\alpha}}(\partial D(0, e^{-1}))$, the result follows from Theorem 5.4. An analogous proof works for T_{α}^{*} .

References

- P. Aiena. Fredholm and local spectral theory, with applications to multipliers. Kluwer Academic Publishers, Dordrecht. 2004.
- [2] C. Apostol, C. Foiaş, and D. Voiculescu. Some results on non-quasitriangular operators. VI. Rev. Roumaine Math. Pures Appl., 18:1473–1494, 1973. Hommage au Professeur Miron Nicolescu pour son 70eme anniversaire, I.
- [3] N. Aronszajn and K. T. Smith. Invariant subspaces of completely continuous operators. Ann. of Math. (2), 60:345–350, 1954.
- [4] A. Atzmon. Operators which are annihilated by analytic functions and invariant subspaces. Acta Math., 144(1-2):27-63, 1980.
- [5] A. Atzmon. On the existence of hyperinvariant subspaces. J. Operator Theory, 11(1):3-40, 1984.
- [6] B. Beauzamy. Introduction to Operator Theory and Invariant Subspaces, volume 42 of North-Holland Mathematical Library. Elsevier Science Publisher, 1988.
- [7] D. P. Blecher and A. M. Davie. Invariant subspaces for an operator on $L^2(\Pi)$ composed of a multiplication and a translation. J. Operator Theory, 23(1):115–123, 1990.
- [8] Y. Bugeaud. Approximation by algebraic numbers, volume 160 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2004.
- [9] I. Chalendar and J. R. Partington. *Modern approaches to the invariant-subspace problem*, volume 188 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011.
- [10] J. B. Conway. A course in functional analysis, volume 96 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1990.
- [11] A. M. Davie. Invariant subspaces for Bishop's operators. Bull. London Math. Soc., 6:343-348, 1974.
- [12] R. G. Douglas and C. Pearcy. A note on quasitriangular operators. Duke Math. J., 37:177–188, 1970.
- [13] P. Enflo. On the invariant subspace problem for Banach spaces. Acta Math., 158(3-4):213-313, 1987.
- [14] A. Flattot. Hyperinvariant subspaces for Bishop-type operators. Acta Sci. Math. (Szeged), 74(3-4):689-718, 2008.
- [15] E. A. Gallardo-Gutiérrez and M. Monsalve-López. Power-regular bishop operators. Submitted, 2018.
- [16] D. W. Hadwin, E. A. Nordgren, H. Radjavi, and P. Rosenthal. An operator not satisfying Lomonosov's hypothesis. J. Funct. Anal., 38(3):410–415, 1980.
- [17] P. R. Halmos. Quasitriangular operators. Acta Sci. Math. (Szeged), 29:283–293, 1968.
- [18] D. A. Herrero. Approximation of Hilbert space operators. Vol. I, volume 72 of Research Notes in Mathematics. Pitman (Advanced Publishing Program), Boston, MA, 1982.
- [19] K. B. Laursen and M. M. Neumann. An introduction to local spectral theory, volume 20 of London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 2000.
- [20] V. I. Lomonosov. Invariant subspaces of the family of operators that commute with a completely continuous operator. Funkcional. Anal. i Priložen., 7(3):55–56, 1973.
- [21] G. W. MacDonald. Invariant subspaces for Bishop-type operators. J. Funct. Anal., 91(2):287–311, 1990.
- [22] G. W. MacDonald. Decomposable weighted rotations on the unit circle. J. Operator Theory, 35(2):205-221, 1996.
- [23] S. J. Miller and R. Takloo-Bighash. An invitation to modern number theory. Princeton University Press, Princeton, NJ, 2006. With a foreword by Peter Sarnak.
- [24] S. K. Parrott. Weighted translation operators. ProQuest LLC, Ann Arbor, MI, 1965. Thesis (Ph.D.)—University of Michigan.
- [25] H. Radjavi and P. Rosenthal. Invariant subspaces. Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77.
- [26] C. J. Read. A solution to the invariant subspace problem. Bull. London Math. Soc., 16(4):337-401, 1984.
- [27] C. J. Read. A solution to the invariant subspace problem on the space l₁. Bull. London Math. Soc., 17(4):305–317, 1985.
- [28] E. C. Titchmarsh. The theory of functions. Oxford University Press, Oxford, second edition, 1939.

[29] J. Wermer. The existence of invariant subspaces. Duke Math. J., 19:615–622, 1952.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, MADRID 28049, SPAIN AND INSTITUTO DE CIENCIAS MATEMÁTICAS ICMAT (CSIC-UAM-UC3M-UCM), MADRID, SPAIN

 $E ext{-}mail\ address, corresponding author: fernando.chamizo@uam.es}$

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO Y MATEMÁTICA APLICADA, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE CIENCIAS 3, 28040 MADRID, SPAIN AND INSTITUTO DE CIENCIAS MATEMÁTICAS ICMAT (CSIC-UAM-UC3M-UCM), MADRID, SPAIN

E-mail address: eva.gallardo@mat.ucm.es

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO Y MATEMÁTICA APLICADA, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE CIENCIAS 3, 28040 MADRID, SPAIN AND INSTITUTO DE CIENCIAS MATEMÁTICAS ICMAT (CSIC-UAM-UC3M-UCM), MADRID, SPAIN

 $E ext{-}mail\ address: migmonsa@ucm.es}$

Departamento de Matemáticas, Universidad Autónoma de Madrid, Madrid 28049, Spain

 $E ext{-}mail\ address: adrian.ubis@gmail.com}$