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A pair of commuting hypergeometric operators on the complex plane and bispectrality

Vladimir F. Molchanov, Yury A. Neretin[1]

We consider the standard hypergeometric differential operator $\mathfrak{D}$ regarded as an operator on the complex plane $\mathbb{C}$ and the complex conjugate operator $\overline{\mathfrak{D}}$. These operators formally commute and are formally adjoint one to another with respect to an appropriate weight. We find conditions when they commute in the Nelson sense and write explicitly their joint spectral decomposition. It is determined by a two-dimensional counterpart of the Jacobi transform (synonyms: generalized Mehler–Fock transform, Olevskii transform). We also show that the inverse transform is an operator of spectral decomposition for a pair of commuting difference operators defined in terms of shifts in imaginary direction.

1. Introduction

1.1. Spectral problem. Denote by $\dot{\mathbb{C}}$ the complex plane without points 0 and 1, by $\mathcal{D}(\dot{\mathbb{C}})$ the space of smooth compactly supported functions on $\dot{\mathbb{C}}$. Denote by $d\overline{\overline{z}}$ the standard Lebesgue measure on $\mathbb{C}$.

Fix real a and b. Consider the following measure on $\dot{\mathbb{C}}$

$$\mu_{a,b}(z)\, d\overline{\overline{z}} := |z|^{2a+2b-2}|1-z|^{2a-2b}\, d\overline{\overline{z}} \tag{1.1}$$

and the corresponding space $L^2(\mathbb{C}, \mu_{a,b})$,

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}\, \mu_{a,b}(z)\, d\overline{\overline{z}}.$$

Consider the following pair of differential operators in the space $L^2(\mathbb{C}, \mu_{a,b})$:

$$\mathfrak{D} := z(1-z)\frac{\partial^2}{\partial z^2} + \big(a+b-(2a+1)z\big)\frac{\partial}{\partial z} - a^2; \tag{1.2}$$

$$\overline{\mathfrak{D}} := \overline{z}(1-\overline{z})\frac{\partial^2}{\partial \overline{z}^2} + \big(a+b-(2a+1)\overline{z}\big)\frac{\partial}{\partial \overline{z}} - a^2. \tag{1.3}$$

These operators formally commute, i.e.,

$$\mathfrak{D}\overline{\mathfrak{D}}f = \overline{\mathfrak{D}}\mathfrak{D}f, \qquad \text{where } f \in \mathcal{D}(\dot{\mathbb{C}}).$$

A straightforward calculation shows that they are formally adjoint,

$$\langle \mathfrak{D}f, g\rangle = \langle f, \overline{\mathfrak{D}}g\rangle, \qquad \text{where } f,\, g \in \mathcal{D}(\dot{\mathbb{C}}).$$

Therefore the operators $\frac{1}{2}(\mathfrak{D}+\overline{\mathfrak{D}})$, $\frac{1}{2i}(\mathfrak{D}-\overline{\mathfrak{D}})$ are symmetric on the domain $\mathcal{D}(\dot{\mathbb{C}})$.

The purpose of this paper is to construct an explicit spectral decomposition of this pair, i.e., a unitary operator U, which diagonalizes both the operators $\mathfrak{D}$, $\overline{\mathfrak{D}}$.

As we know after the famous work of Edward Nelson [28], 1959, (see, also [37], Sect. VIII.5) a question about commutativity of two unbounded self-adjoint operators can be highly nontrivial. Recall that two self-adjoint operators A, B commute if they can be simultaneously realized as operators of multiplication by functions in some L^2. Equivalently, the corresponding one-parametric groups commute:

$$e^{isA}e^{itB} = e^{itB}e^{isA}, \qquad \text{where } s,\, t \text{ in } \mathbb{R}.$$

Equivalently, resolvents $(A-\lambda)^{-1}$ and $(B-\mu)^{-1}$, commute. However these properties do not follow from the identity $AB = BA$ and are difficult for a verification.

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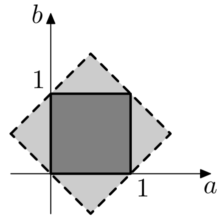

FIGURE 1. To Theorem 1.1. The domain Π of commutativity, and the domain $\Pi_{\mathrm{cont}} \subset \Pi$, where the spectrum is purely continuous.

There are some useful sufficient conditions for commutativity (in particular, in the Nelson's paper), but quite often a question remains to be heavy[2].

Define two domains $\Pi \supset \Pi_{\mathrm{cont}}$ of parameters (a, b):

$$\Pi: \quad 0 < a + b < 2, \qquad -1 < a - b < 1. \tag{1.4}$$

$$\Pi_{\mathrm{cont}}: \quad 0 \leqslant a \leqslant 1, \qquad 0 \leqslant b \leqslant 1, \quad \text{and } (a,b) \neq (\pm 1, \pm 1), (\pm 1, \mp 1). \tag{1.5}$$

Theorem 1.1. *The operators* $\frac{1}{2}(\mathfrak{D} + \overline{\mathfrak{D}})$, $\frac{1}{2i}(\mathfrak{D} - \overline{\mathfrak{D}})$ *admit extensions to a pair of commuting self-adjoint operators if and only if* $(a, b) \in \Pi$.

Next, we define a natural domain for our operators. Consider the subspace $\mathcal{R}_{a,b}(\dot{\mathbb{C}}) \subset L^2(\dot{\mathbb{C}}, \mu_{a,b})$ consisting of smooth functions f on $\dot{\mathbb{C}}$ satisfying the following conditions[3]:

$1°$. In a neighborhood of $z = 0$ a function f has an expansion of the form[4]

$$f(z) = \begin{cases} \alpha(z) + \beta(z)|z|^{2-2a-2b}, & \text{if } a + b \neq 1; \\ \alpha(z) + \beta(z) \ln |z|, & \text{if } a + b = 1, \end{cases} \tag{1.6}$$

where $\alpha(z)$, $\beta(z)$ are smooth functions.

$2°$. In a neighborhood of $z = 1$ a function f has an expansion of the form

$$f(z) = \begin{cases} \gamma(z) + \delta(z)|z - 1|^{2b-2a}, & \text{if } a - b \neq 0; \\ \gamma(z) + \delta(z) \ln |z - 1|, & \text{if } a - b = 0, \end{cases} \tag{1.7}$$

where $\gamma(z)$, $\delta(z)$ are smooth.

$3°$. For each p, q, N we have

$$\frac{\partial^{p+q} f}{\partial z^p \partial \overline{z}^q} = O\Big(|z|^{-2a-p-q} (\ln |z|)^{-N}\Big) \qquad \text{as } z \to \infty. \tag{1.8}$$

Theorem 1.2. a) *For* $(a, b) \in \Pi$ *the operators* $\frac{1}{2}(\mathfrak{D} + \overline{\mathfrak{D}})$, $\frac{1}{2i}(\mathfrak{D} - \overline{\mathfrak{D}})$ *are essentially self-adjoint on* $\mathcal{R}_{a,b}(\dot{\mathbb{C}})$ *and commute in the Nelson sense.*

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[2]A famous example is a problem, see [10], which was raised by Irving Segal in 1958 and which was discussed during almost 30 years: Let Ω be an open connected domain in $\mathbb{R}^n$. Assume that the operators $i\partial/\partial x_k$ in $\mathcal{D}(\Omega)$ admit commuting self-adjoint extensions. Is it correct that Ω is essentially a fundamental domain of $\mathbb{R}^n$ with respect to a certain discrete group? An answer is affirmative.

[3]If $(a, b) \notin \Pi$, then $\mathcal{R}_{a,b}(\dot{\mathbb{C}})$ is not contained in $L^2(\mathbb{C}, \mu_{a,b})$.

[4]Boundary conditions in this spirit sometimes arise in spectral theory of ordinary differential operators D for operators with deficiency indices $(1, 1)$ or $(2, 2)$, see, e.g., [31], Section 1.

b) *If* $(a,b)\in\Pi_{\mathrm{cont}}$*, then the spectrum of the problem*

$$\mathfrak{D}f=\zeta f,\qquad \overline{\mathfrak{D}}f=\overline{\zeta}f \tag{1.9}$$

is multiplicity free and consists of σ *having the form*

$$\zeta=\bigl(\tfrac{k+is}{2}\bigr)^2,\qquad \text{where } k\in\mathbb{Z},\ s\in\mathbb{R}. \tag{1.10}$$

If $(a,b)\in\Pi\setminus\Pi_{\mathrm{cont}}$*, then the spectrum consists on the same set plus one eigenvalue* $\zeta_0>0$*.*

Let us explain the obstacle for commutativity. Consider a second order differential operator D on an interval. For each $\zeta\in\mathbb{R}$ the differential equation $Df=\zeta f$ has two solutions, and we can select generalized eigenfunctions of D as solutions, which have L^2- or almost L^2-asymptotics at the ends of the interval. In our case the system (1.9) locally has 4 solutions. However $\dot{\mathbb{C}}$ is not simply connected, solutions are ramified at 0, 1, ∞. As a result there are few single valued solutions and we have no freedom for selection of asymptotics. Such considerations (see Section 4) allow to establish necessity of conditions of Theorem 1.1.

Unfortunately, we do not know a priori proof of sufficiency and obtain it as a byproduct of the explicit joint spectral decomposition of operators $\mathfrak{D}$, $\overline{\mathfrak{D}}$. Such detour makes our work long and requires numerous explicit calculations and estimates.

1.2. The index hypergeometric transform. Our work is a counterpart of the following classical topic. Consider a hypergeometric differential operator

$$D:=x(x+1)\frac{d^2}{dx^2}+\bigl((a+b)+(2a+1)x\bigr)\frac{d}{dx}+a^2$$

on the half-line $\mathbb{R}_+$, i.e., $x>0$. Consider the integral operator

$$I_{a,b}f(s):=\frac{1}{\Gamma(a+b)}\int_0^\infty f(x)\,{}_2F_1\begin{bmatrix}a+is,a-is\\ a+b\end{bmatrix};x\Bigr]\,x^{a+b-1}(1+x)^{a-b}\,dx. \tag{1.11}$$

Then $I_{a,b}$ is a unitary operator

$$L^2\Bigl(\mathbb{R}_+,x^{a+b+1}(1+x)^{a-b}\,dx\Bigr)\to L^2\Bigl(\mathbb{R}_+,\pi^{-1}\Bigl|\frac{\Gamma(a+is)\Gamma(b+is)}{\Gamma(2is)}\Bigr|^2\,ds\Bigr). \tag{1.12}$$

The operator $I_{a,b}$ sends D to the multiplication by s^2, see [40], [38], [35], [20], [19], [29], [32]. This transform[5] is known as *'the generalized Mehler-Fock transform', 'the Olevskii transform', 'the Jacobi transform'.*

Such operators arise in a natural way in the analysis on rank one Riemannian symmetric spaces, on the other hand they are special cases of multi-dimensional Harish-Chandra (or Heckman–Opdam, see [16]) transforms, which arise as spectral decompositions of certain families of commuting partial differential operators.

Next, consider the following difference operator in the space of even functions depending on the variable s:

$$\begin{split}Lg(s):=\frac{(a-is)(b-is)}{(-2is)(-2is+1)}\bigl(g(s+i)-g(s)\bigr)+\\ +\frac{(a+is)(b+is)}{(2is)(2is+1)}\bigl(g(s-i)-g(s)\bigr),\end{split} \tag{1.13}$$

[5]A special case $a=1/2$, $b=1$ of this transform was discovered by Gustav Mehler in 1881, the general transform was obtained by Weyl in 1910.

where $i^2=-1$. A domain of this operator is a space of even functions holomorphic in the strip $|\operatorname{Im} s|<1+\varepsilon$ with some condition of decreasing at infinity. It turns out that L is essentially self-adjoint in the weight L^2 and the operator $I_{a,b}^{-1}$ sends it to the operator of multiplication by x.

So we have a bispectrality in the spirit of Grünbaum [15]. Notice that simpler index integral transforms as Kontorovich–Lebedev transform and ${}_1F_1$-Wimp transforms also are bispectral, see [33].

Cherednik showed [4] that inverse Harish-Chandra transforms provide spectral decompositions of families of commuting difference operators, see also van Diejen, Emsiz [6].

1.3. Radial parts of Laplace operators. Recall one's more classical topic. Consider the usual sphere $S^2_{\mathbb{R}}$:

$$x^2+y^2+z^2=1,$$

the orthogonal group $\mathrm{SO}(3)$ acts in $L^2(S^2_{\mathbb{R}})$ by rotations. Recall one of possible ways to decompose this unitary representation into irreducible components. Consider the Beltrami–Laplace operator Δ on the sphere and restrict it to the space of functions depending on the height z. We get a differential operator

$$L_z:=(1-z^2)\frac{\partial^2}{\partial z^2}-2z\frac{\partial}{\partial z}$$

in $L^2[-1,1]$. Eigenfunctions of L_z are the Legendre polynomials. Simple arguments show that the spectral decomposition of Δ is a priori equivalent to the spectral decomposition of L_z (the reason of this equivalence is compactness of the group $\mathrm{SO}(2)$ of rotations of $S^2_{\mathbb{R}}$ about the vertical axis).

Now consider the complex manifold $S^2_{\mathbb{C}}\subset\mathbb{C}^3$ defined by the same equation $x^2+y^2+z^2=1$. The complex orthogonal group $\mathrm{SO}(3,\mathbb{C})$ (the Lorentz group) acts on the quadric $S^2_{\mathbb{C}}$, the action admits an $\mathrm{SO}(3,\mathbb{C})$-invariant measure, and again we come to a problem[6] of decomposition of the unitary representation of $\mathrm{SO}(3,\mathbb{C})$ in L^2 on $S^2_{\mathbb{C}}$. Now we have two Beltrami-Laplace operators, a holomorphic operator Δ and an antiholomorphic operator $\overline{\Delta}$. They commute in the Nelson sense. Restricting them to functions depending on the coordinate $z\in\mathbb{C}$ we get two operators[7]:

$$L_z:=(1-z^2)\frac{\partial^2}{\partial z^2}-2z\frac{\partial}{\partial z},\qquad L_{\overline{z}}:=(1-\overline{z}^2)\frac{\partial^2}{\partial \overline{z}^2}-2\overline{z}\frac{\partial}{\partial \overline{z}}.$$

However, now the stabilizer of the point $(x,y,z)=(0,0,1)$ is a *noncompact* subgroup $\mathrm{SO}(2,\mathbb{C})$, and this breaks a priori argumentation. A joint spectral decomposition of Δ, $\overline{\Delta}$ can be reformulated as a certain problem[8] for L_z, $\overline{L}_z$, but this is not precisely a problem of a joint spectral decomposition of L_z, $\overline{L}_z$.

Notice that a similar separation of variables can be done for L^2 on arbitrary rank one complex symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$ (and, more generally, for spaces of L^2-sections of line bundles on $G_{\mathbb{C}}/H_{\mathbb{C}}$). In all the cases we get pairs of hypergeometric operators of our type. We hope that our spectral decomposition allows to write

[6]This problem was solved by Naimark in [25] in a completely different way.

[7]This pair corresponds to $a=b=1/2$ in our parameters.

[8]Such reductions for families of Laplace operators were widely used by Harish-Chandra (in his famous works on the Plancherel formula for real semisimple Lie groups) and by his successors. The problem for L_z, $\overline{L}_z$ is more similar to decompositions of L^2 on real rank one pseudo-Riemannian symmetric spaces, which was solved by one of the authors [22]–[24].

the explicit Plancherel formula for such spaces and to give another proof of old Naimarks's results [25]–[27] on tensor products of representations of the Lorentz group. However, the present paper does not have such purposes.

1.4. Homographic transformations of the operators $\mathfrak{D}$, $\overline{\mathfrak{D}}$. Our next purpose is to present the explicit joint spectral decomposition of the pair $\mathfrak{D}$, $\overline{\mathfrak{D}}$. We need some preparations.

Consider the following 8 transformation of functions on $\dot{\mathbb{C}}$:

$$f(z) \mapsto \gamma_j(z) f(z), \qquad f(z) \mapsto \gamma_j(z) f(1-z),$$

where

$$\gamma_j(z) = 1, \quad |1-z|^{2(b-a)}, \quad |z|^{2(1-a-b)}, \quad |z|^{2(1-a-b)}|1-z|^{2(b-a)},$$

cf. Erdélyi etc., [7], Subsect. 2.6.1. It can be readily checked that these transformations send operators $\mathfrak{D}$, $\overline{\mathfrak{D}}$ to operators of the same type with other values of parameters (a,b), as

$$(a,b) \mapsto (b,a), \qquad (a,b) \mapsto (1-a,b), \qquad \text{etc.}$$

Thus we get all isometries of the square Π. In particular, such transformations send spectral problems to equivalent spectral problems.

1.5. Notation. Generalized powers. Denote by $\mathbb{C}^\times$ the multiplicative group of $\mathbb{C}$. We need a notation for characters of $\mathbb{C}^\times$. Let $z \in \mathbb{C}^\times$ and a, $a' \in \mathbb{C}$ satisfy $a - a' \in \mathbb{Z}$. We define a *generalized power* of z by

$$z^{\mathbf{a}} = z^{a|a'} := z^a \overline{z}^{a'} = e^{a \ln z + a' \overline{\ln z}} = |z|^{2a}\, \overline{z}^{a'-a},$$

Denote by $\Lambda^{\mathbb{C}}$ the set of all pairs $a|a'$ such that $a - a' \in \mathbb{Z}$. Denote by $\Lambda \subset \Lambda_{\mathbb{C}}$ the set of all pairs

$$a|a' = \tfrac{1}{2}(k+is)\Big|\tfrac{1}{2}(-k+is), \qquad \text{where } k \in \mathbb{Z},\ s \in \mathbb{R}. \tag{1.14}$$

Equivalently, $a|a' \in \Lambda$ if $a - a' \in \mathbb{Z}$, $a + a' \in i\mathbb{R}$. We also will use the notation

$$[\mathbf{a}] = [a|a'] := \tfrac{1}{2} \operatorname{Re}(a + a'). \tag{1.15}$$

We have

$$\left|z^{a|a'}\right| = |z|^{2[a|a']},$$

in particular, for $\mathbf{a} \in \Lambda$ we have $\left|z^{a|a'}\right| = 1$.

We fix the standard Lebesgue measure $\widetilde{d}\lambda$ on the set Λ:

$$\int_\Lambda \varphi(\lambda)\, \widetilde{d}\lambda := \sum_k \int_{\mathbb{R}} \varphi\bigl(\tfrac{k+is}{2}\bigr)\, ds.$$

1.6. Hypergeometric function of the complex field. Following Gelfand, Graev, and Retakh [11], we define the gamma function $\Gamma^{\mathbb{C}}$, the beta function $B^{\mathbb{C}}$, and the hypergeometric function ${}_2F_1^{\mathbb{C}}$ of the complex field. The *gamma function*

$\Gamma^{\mathbb{C}}$ is

$$\begin{aligned}(1.16)\quad \Gamma^{\mathbb{C}}(\mathbf{a}) = \Gamma^{\mathbb{C}}(a|a') &:= \frac{1}{\pi}\int_{\mathbb{C}} z^{\mathbf{a-1}} e^{2i\operatorname{Re} z}\, d\overline{\overline{z}} := \\ &:= \frac{1}{\pi}\int_{\mathbb{C}} z^{a-1|a'-1} e^{2i\operatorname{Re} z}\, d\overline{\overline{z}} = \\ = i^{a-a'}\frac{\Gamma(a)}{\Gamma(1-a')} &= i^{a'-a}\frac{\Gamma(a')}{\Gamma(1-a)} = \frac{i^{a'-a}}{\pi}\Gamma(a)\Gamma(a')\sin\pi a'.\end{aligned}$$

The *beta function* $B^{\mathbb{C}}$ is

(1.17)

$$\mathrm{B}^{\mathbb{C}}(\mathbf{a},\mathbf{b}) := \frac{1}{\pi}\int_{\mathbb{C}} t^{\mathbf{a}-1}(1-t)^{\mathbf{b}-1}\; d\overline{\overline{t}} = \frac{\Gamma^{\mathbb{C}}(\mathbf{a})\Gamma^{\mathbb{C}}(\mathbf{b})}{\Gamma^{\mathbb{C}}(\mathbf{a}+\mathbf{b})} = \frac{\Gamma(a)\Gamma(b)\Gamma(1-a'-b')}{\Gamma(a+b)\Gamma(a')\Gamma(b')}.$$

The *hypergeometric function of the complex field* is defined by

$$\begin{aligned}(1.18)\quad {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] = {}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{a};\mathbf{b}\\ \mathbf{c}\end{matrix};z\right] &= {}_2F_1^{\mathbb{C}}\left[\begin{matrix}a|a',b|b'\\ c|c'\end{matrix};z\right] := \\ &:= \frac{1}{\pi B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\int_{\mathbb{C}} t^{\mathbf{b-1}}(1-t)^{\mathbf{c-b-1}}(1-zt)^{-\mathbf{a}}\, d\overline{\overline{t}}.\end{aligned}$$

Recall that the Gauss hypergeometric functions are defined by

$${}_2F_1[a,b;c;z] := \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \sum_{p=0}^{\infty}\frac{(a)_p(b)_p}{p!\,(c)_p}\,z^p,$$

where $(c)_p := c(c+1)\dots(c+p-1)$ is the Pochhammer symbol. The functions ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ admit expressions in the terms of ${}_2F_1$, see Theorem 3.9.

1.7. Spectral decomposition. For $(a,b)\in\Pi$ we define the kernel $\mathcal{K}_{a,b}(z,\lambda)$ on $\mathbb{C}\times\Lambda$ by

$$(1.19)\qquad \mathcal{K}_{a,b}(z,\lambda) = \frac{1}{\Gamma^{\mathbb{C}}(a+b|a+b)}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}a+\lambda|a-\overline{\lambda},\ a-\lambda|a+\overline{\lambda}\\ a+b|a+b\end{matrix};z\right].$$

Theorem 1.3. *Let $(a,b)\in\Pi_{\mathrm{cont}}$. Then the operator*

$$J_{a,b}f(\lambda) := \int_{\mathbb{C}} \mathcal{K}_{a,b}(z,\lambda)\, f(z)\, \mu_{a,b}(z)\; d\overline{\overline{z}}$$

is a unitary operator from $L^2(\mathbb{C},\mu_{a,b})$ to $L^2_{\mathrm{even}}(\Lambda,\varkappa_{a,b})$ of even functions on Λ with respect to the Plancherel measure

$$(1.20)\qquad dK_{a,b}(\lambda) = \varkappa_{a,b}(\lambda)\,\widetilde{d}\lambda = \frac{1}{4\pi^2}\Bigl|\lambda\,\Gamma^{\mathbb{C}}(a-\lambda|a+\overline{\lambda})\,\Gamma^{\mathbb{C}}(b+\lambda|b-\overline{\lambda})\Bigr|^2\,\widetilde{d}\lambda.$$

Next, we modify the definition of the measure for $(a,b)\in\Pi\setminus\Pi_{\mathrm{cont}}$. Due to the homographic transformations[9] it is sufficient to consider the case $a<0$. We define the Plancherel measure $dK_{a,b}(\lambda)$ on $\Lambda_{\mathbb{C}}$ that is the sum of $\varkappa_{a,b}\,\widetilde{d}\lambda$ and two δ-measures located at points $\pm a|\pm a\in\Lambda_{\mathbb{C}}$,

$$(1.21)\qquad \Gamma^{\mathbb{C}}(a+b|a+b)\,\Gamma^{\mathbb{C}}(b-a|b-a)\,\Gamma^{\mathbb{C}}(2a|2a)\cdot\bigl(\delta_{a|a}+\delta_{-a|-a}\bigr).$$

[9]Changing of kernels $\mathcal{K}_{a,b}$ by the homographic transformation can be observed from Proposition 3.5.

Define a constant function $v(z)$ on $\dot{\mathbb{C}}$ by

$$v(z) = \Gamma^{\mathbb{C}}(a+b|a+b)^{-1}.$$

For $f \in \mathcal{D}(\dot{\mathbb{C}})$ we define an even function $J_{a,b}(\lambda)$ on the support of $dK_{a,b}(\lambda)$ given by the same formula (1.20) on Λ, its value at $(\pm a| \pm a)$ is

$$J_{a,b}f(\pm a| \pm a) := \langle f, v\rangle_{L^2(\mathbb{C},\mu_{a,b})}.$$

Theorem 1.4. *If $(a,b) \in \Pi$ and $a < 0$, then the operator $J_{a,b}$ is unitary as an operator $L^2(\mathbb{C}, \mu_{a,b})$ to $L^2_{\text{even}}(\Lambda_{\mathbb{C}}, dK_{a,b})$.*

Our operator really determines the spectral decomposition:

Theorem 1.5. *For each $(a,b) \in \Pi$ for any $f \in \mathcal{D}_{\text{even}}(\dot{\mathbb{C}})$ we have*

$$J_{a,b}\mathfrak{D}f(\lambda) = \lambda^2 J_{a,b}f(\lambda), \qquad J_{a,b}\overline{\mathfrak{D}}f(\lambda) = \overline{\lambda}^2 J_{a,b}f(\lambda).$$

This means that

$$\mathfrak{D}\mathcal{K}(z,\lambda) = \lambda^2\mathcal{K}(z,\lambda), \qquad \overline{\mathfrak{D}}\mathcal{K}(z,\lambda) = \overline{\lambda}^2\mathcal{K}(z,\lambda).$$

Next, we consider the space $\mathcal{D}_{\text{even}}(\dot{\Lambda})$, which consists of even smooth compactly supported functions on Λ that are zero at a neighborhood of the point $0|0$. The following statement explains the appearance of the space $\mathcal{R}_{a,b}$ and also is one of arguments for proofs of our main statements.

Theorem 1.6. *If $F \in \mathcal{D}_{\text{even}}(\dot{\Lambda})$, then $J^*_{a,b}F \in \mathcal{R}_{a,b}$.*

The images of δ-functions also are contained in $\mathcal{R}_{a,b}$.

1.8. The transformation $J_{a,b}$ in the complex domain. Let us extend our kernel $\mathcal{K}$ to the complex domain. For

$$\{\lambda|\lambda'\} = \left\{\tfrac{k+\sigma}{2}\middle|\tfrac{-k+\sigma}{2}\right\} \in \Lambda_{\mathbb{C}}$$

we set

$$\mathcal{K}_{a,b}(z;\lambda|\lambda') = \mathcal{K}_{a,b}(z;k,\sigma) := \tag{1.22}$$

$$:= \frac{1}{\Gamma^{\mathbb{C}}(a+b|a+b)}\, {}_2F_1^{\mathbb{C}}\left[\begin{matrix} a+\frac{k+\sigma}{2}\big|a+\frac{-k+\sigma}{2},\ a+\frac{-k-\sigma}{2}\big|a+\frac{k-\sigma}{2} \\ a+b|a+b\end{matrix};z\right],$$

where k ranges in $\mathbb{Z}$, σ ranges in $\mathbb{C}$. The previous expression (1.19) corresponds to a pure imaginary σ.

For $f \in \mathcal{D}(\dot{\mathbb{C}})$ we define a meromorphic function on $\Lambda_{\mathbb{C}}$ by

$$J_{a,b}f(k,\sigma) := \int_{\dot{\mathbb{C}}} f(z)\,\mathcal{K}(z;k,\sigma)\,d\mu_{a,b}\,d\overline{\overline{z}}.$$

Theorem 1.7. *For $f \in \mathcal{D}(\dot{\mathbb{C}})$ the function $J_{a,b}f$ is contained in the space $\mathcal{W}_{a,b}$ defined as follows.*

We define a space $\mathcal{W}_{a,b}$ as the space of all meromorphic functions[10] $F(k,\sigma)$ on $\Lambda_{\mathbb{C}}$ satisfying the conditions a)–d):

a) F is even, i.e., $F(-k,-\sigma) = F(k,\sigma)$.

b) Possible poles of $F(k,\sigma)$ are located at points

$$\sigma = \pm(-2a+|k|+2j), \qquad \pm(-2b+|k|+2j), \qquad \text{where } j = 1,\,2,\,3,\,\ldots \tag{1.23}$$

[10]We say that a function $F(k,\sigma)$ is meromorphic if it is meromorphic as a function in σ for any fixed k.

A maximal possible order of a pole at a point (l,c) is a multiplicity of (l,c) in the collection[11] (1.23)

c) For each $A>0$ for each $N>0$ in the union of strips $|\operatorname{Re}\sigma|<A$ we have an estimate

$$F(k,\sigma)=O\big(k^2+(\operatorname{Im}\sigma)^2\big)^{-N} \qquad \text{as } k^2+(\operatorname{Im}\sigma)^2\to\infty. \tag{1.24}$$

d) For each p, $q\in\mathbb{Z}$

$$F(p,q)=F(q,p). \tag{1.25}$$

Next, we extend the spectral density $\varkappa_{a,b}$ to the complex domain.

$$\varkappa_{a,b}(\lambda|\lambda')=\varkappa_{a,b}(k,\sigma):= \tag{1.26}$$
$$:=\frac{1}{4\pi^2}(k+\sigma)(k-\sigma)\,\Gamma^{\mathbb{C}}\big(a+\tfrac{k+\sigma}{2}\big|a+\tfrac{-k+\sigma}{2}\big)\,\Gamma^{\mathbb{C}}\big(a+\tfrac{-k-\sigma}{2}\big|a+\tfrac{-k-\sigma}{2}\big)\times$$
$$\Gamma^{\mathbb{C}}\big(b+\tfrac{k+\sigma}{2}\big|b+\tfrac{-k+\sigma}{2}\big)\,\Gamma^{\mathbb{C}}\big(b+\tfrac{-k-\sigma}{2}\big|b+\tfrac{-k-\sigma}{2}\big).$$

In the case $a<0$ discussed above, $\varkappa_{a,b}$ has a pole at $k=0$, $\sigma=a$ and the inner product in $L^2_{\mathrm{even}}(\Lambda,dK_{a,b})$ can be written as

$$\langle F,G\rangle=\frac{1}{i}\sum_k\int_{-i\infty}^{i\infty}F(k,\sigma)\,\overline{G(k,-\overline{\sigma})}\varkappa_{a,b}(k,\sigma)\,d\sigma+$$
$$+2\operatorname{res}_{s=a}\Big(F(k,\sigma)\,\overline{G(k,-\overline{\sigma})}\varkappa_{a,b}(0,\sigma)\Big).$$

If $a>1$, then the spectral density has a zero at $k=0$, $\sigma=a-1$ but both the functions $F(k,\sigma)$, $\overline{G(k,-\overline{\sigma})}$ admit simple poles at this point, and we have a similar formula.

1.9. Difference spectral problem. It turns out that our problem is bispectral, and the bispectrality is a crucial argument of our proof. We define analogs of the difference operator (1.13). Consider meromorphic functions Φ depending on

$$\lambda|\lambda'=\tfrac12(k+is)\Big|\tfrac12(-k+is)\in\Lambda_{\mathbb{C}}$$

and operators in the space of meromorphic functions defined by

$$T\Phi(k,s)=\Phi(k+1,s-i),\qquad \widetilde{T}\Phi(k,s)=\Phi(k+1,s+i),$$

or, equivalently,

$$T\Phi(\lambda|\lambda')=\Phi(\lambda+1|\lambda'),\quad \widetilde{T}\Phi(\lambda|\lambda')=\Phi(\lambda|\lambda'+1). \tag{1.27}$$

We define the following difference operators

$$\mathfrak{L}:=\frac{(a+\lambda)(b+\lambda)}{2\lambda(1+2\lambda)}\,(T-1)+\frac{(a-\lambda)(b-\lambda)}{-2\lambda(1-2\lambda)}\,(T^{-1}-1); \tag{1.28}$$

$$\overline{\mathfrak{L}}:=\frac{(a+\lambda')(b+\lambda')}{2\lambda'(1+2\lambda')}\,(\widetilde{T}^{-1}-1)+\frac{(a-\lambda')(b-\lambda')}{-2\lambda(1-2\lambda')}\,(\widetilde{T}-1). \tag{1.29}$$

Formally,

$$\mathfrak{L}\overline{\mathfrak{L}}=\overline{\mathfrak{L}}\mathfrak{L}.$$

[11] For $(a,b)\in\Pi$ orders of poles $\leqslant 2$. Poles of order 2 arise only if $a=b$, $a=1$, $b=1$.

Theorem 1.8. a) *The operators* $\frac{1}{2}(\mathfrak{L}+\overline{\mathfrak{L}})$, $\frac{1}{2i}(\mathfrak{L}-\overline{\mathfrak{L}})$ *defined on the space* $\mathcal{W}_{a,b}$ *are essentially self-adjoint and commute in the Nelson sense.*

b) *For* $\Phi \in J_{a,b}\mathcal{D}(\dot{\mathbb{C}})$ *we have*

$$J_{a,b}^{-1}\,\mathfrak{L}\,\Phi = z\,J_{a,b}^{-1}\,\Phi(z), \qquad J_{a,b}^{-1}\,\overline{\mathfrak{L}}\,\Phi(z) = \overline{z}\,J_{a,b}^{-1}\,\Phi(z). \tag{1.30}$$

Thus $J_{a,b}$ is an operator of joint spectral decomposition of $\mathfrak{L}$, $\overline{\mathfrak{L}}$.

1.10. Structure of proofs. We derive asymptotics of the kernel $\mathcal{K}(z,\lambda)$ as $z\to 0, 1, \infty$ for fixed λ (Theorem 3.9) and as $|\lambda|\to\infty$ for fixed z (Theorem 7.1). Next, we show inclusions

$$J_{a,b}^*\,\mathcal{D}_{\mathrm{even}}(\dot{\Lambda}) \subset \mathcal{R}_{a,b}, \qquad J_{a,b}\,\mathcal{D}(\dot{\mathbb{C}}) \subset \mathcal{W}_{a,b}$$

(Proposition 5.2 and Corollary 8.2) and symmetries

$$\langle \mathfrak{D} f, g\rangle_{L^2(\mathbb{C},\mu_{a,b})} = \langle f, \overline{\mathfrak{D}} g\rangle_{L^2(\mathbb{C},\mu_{a,b})}, \qquad \text{where } f,\, g\in \mathcal{R}_{a,b}; \tag{1.31}$$

$$\langle \mathfrak{L} F, G\rangle_{L^2(\dot{\Lambda},dK_{a,b})} = \langle F, \overline{\mathfrak{L}} G\rangle_{L^2(\Lambda,dK_{a,b})}, \quad \text{where } F,\, G\in \mathcal{W}_{a,b}, \tag{1.32}$$

see Proposition 5.5 and Theorem 8.4. This implies a generalized orthogonality, i.e.,

$$\langle J_{a,b}^*F, J_{a,b}^*G\rangle_{L^2(\mathbb{C},\mu_{a,b})} = 0 \quad \text{if supports of } F,\, G\in\mathcal{D}_{\mathrm{even}}(\dot{\Lambda}) \text{ are disjoint,}$$

and a similar statement for $J_{a,b}$, see Lemmas 9.2, 6.4. Next, we show that for any F, $G\in\mathcal{D}_{\mathrm{even}}(\dot{\Lambda})$ we have the following representation:

$$\langle J_{a,b}^*F, J_{a,b}^*G\rangle_{L^2(\mathbb{C},\mu_{a,b})} = \langle F, G\rangle_{L^2(\Lambda,dK_{a,b})} + \int_\Lambda\int_\Lambda H(\lambda_1,\lambda_2)F(\lambda_1)G(\lambda_2)\,\widetilde{d}\lambda_1\,\widetilde{d}\lambda_2,$$

where H is a locally integrable function, see Lemma 6.4. We also prove the similar statement for $J_{a,b}$, see Lemma 9.4. Then generalized orthogonality implies $H(\cdot,\cdot) = 0$. Thus we get

$$J_{a,b}^*J_{a,b} = 1, \qquad J_{a,b}J_{a,b}^* = 1, \tag{1.33}$$

and this is our main statement. Some steps of this double way are straightforward, some points require long calculations and estimates, and we meet some points of good luck (proofs of Theorem 8.4 and Lemma 9.4). We also need lot of information about functions ${}_2F_1^{\mathbb{C}}$ (in particular, to cover the cases $a+b\in\mathbb{Z}$ and $a-b\in\mathbb{Z}$ we need in a tedious examination of possible degenerations of functions ${}_2F_1^{\mathbb{C}}$).

The bispectrality allows to avoid a direct proof of completeness of the system of generalized eigenfunctions of $\mathfrak{D}$, $\overline{\mathfrak{D}}$.

To prove necessary conditions of self-adjointness in Theorem 1.1 we analyze common generalized eigenfunctions of $\mathfrak{D}$, $\overline{\mathfrak{D}}$ for $(a,b)\notin\Pi$ and after a natural selection we reduce a set of possible candidates to a finite family. This is done in Section 4.

This text is focused to a proof of unitarity of $J_{a,b}$. An introduction to functions ${}_2F_1^{\mathbb{C}}$ in Section 3 can be a point of an independent interest. Also, we get two relatively pleasant statements about asymptotic behavior of integrals

$$M(\varepsilon) = \int_{\mathbb{C}} t^{\alpha-1|\alpha'-1}(\varepsilon-t)^{\beta-1|\beta'-1}\psi(t)\;d\overline{\overline{t}} \qquad \text{as } \varepsilon\to 0$$

and

$$I(\lambda) = \int_{\mathbb{C}} |f(t)|^2 e^{i\,\mathrm{Re}\left(\lambda\varphi(t)\right)}\;d\overline{\overline{t}} \qquad \text{as } |\lambda|\to\infty,$$

where f, φ are holomorphic and $\lambda\in\mathbb{C}$ (Theorems 2.3 and 7.2).

1.11. Final remarks. The index hypergeometric transform (1.11) can be applied as a heavy tool of theory of special functions, see [20], [30], [32]. In [34] we use our operators $J_{a,b}$ to obtain a beta integral over Λ, which is a counterpart of the Dougall ${}_5H_5$-summation formula and of the de Branges–Wilson integral.

Also, we notice that functions, which can be regarded as higher hypergeometric functions ${}_4F_3^{\mathbb{C}}$ of the complex field, arise in a natural way in the work of Ismagilov [18] as analogs of the Racah coefficients for *unitary* representations of the Lorentz group $\mathrm{SL}(2,\mathbb{C})$ (see, also a continuation in [5]).

It seems that our problem can be a representative of some family of spectral problems, but now it is too early to claim something certainly.

2. Preliminaries. Gamma function, the Mellin transform, weak singularities

This section is a union of 3 disjoint topics:

— some properties of function $\Gamma^{\mathbb{C}}$, which are intensively used below:

— some properties of the Mellin transform on $\mathbb{C}$, they are used in a proof of Proposition 3.1 and in Sections 7–9:

— a lemma from asymptotic analysis, which is used only in a proof of Theorem 3.9 (the last statement can be independently established by a straightforward tiresome way):

2.1. Some properties of the gamma function. The usual functional equations for Γ-function can be easily rewritten for $\Gamma^{\mathbb{C}}$ (recall that $a - a' \in \mathbb{Z}$!):

$$\Gamma^{\mathbb{C}}(a|a') = \Gamma^{\mathbb{C}}(a'|a); \tag{2.1}$$

$$\Gamma^{\mathbb{C}}(a+1|a') = i\, a\, \Gamma^{\mathbb{C}}(a|a'); \tag{2.2}$$

$$\Gamma^{\mathbb{C}}(a|a')\,\Gamma^{\mathbb{C}}(1-a|1-a') = (-1)^{a-a'}; \tag{2.3}$$

$$\overline{\Gamma^{\mathbb{C}}(a|a')} = (-1)^{a-a'}\,\Gamma^{\mathbb{C}}(\overline{a}|\overline{a}'). \tag{2.4}$$

Also,

$$\prod_{p=0}^{m-1} \Gamma^{\mathbb{C}}\Bigl(a + \frac{p-1}{m}\Bigl| a' + \frac{p-1}{m}\Bigr) = m^{1-m(a+a')}\,\Gamma^{\mathbb{C}}(ma|ma').$$

The identity (2.4) implies

$$\overline{B^{\mathbb{C}}[a|a', b|b']} = B^{\mathbb{C}}[\overline{a}|\overline{a}', \overline{b}|\overline{b}'].\tag{2.5}$$

Let k_1, $k_2 \in \mathbb{Z}$. Then

$$\Gamma^{\mathbb{C}}(k_1|k_2) = \begin{cases} \infty, & \text{if } k_1,\ k_2 \in \mathbb{Z}_-,\\ 0, & \text{if } k_1,\ k_2 \in \mathbb{N},\\ i^{k_1-k_2}\frac{(k_1-1)!}{(-k_2)!}, & \text{if } k_1 \in \mathbb{N},\ k_2 \in \mathbb{Z}_-,\\ i^{k_2-k_1}\frac{(k_2-1)!}{(-k_1)!}, & \text{if } k_2 \in \mathbb{N},\ k_1 \in \mathbb{Z}_-,\end{cases}\tag{2.6}$$

where $\mathbb{Z}_-$ denotes the set of integers $\leqslant 0$.

The following lemma gives us the asymptotics of the Plancherel density (1.20).

Lemma 2.1. *We have the following asymptotics in $\lambda \in \Lambda$:*

$$\Gamma^{\mathbb{C}}(a-\lambda|a+\overline{\lambda})\,\Gamma^{\mathbb{C}}(b+\lambda|b-\overline{\lambda}) \sim \lambda^{a+b-1|a+b-1} \qquad \text{as } |\lambda|\to\infty.\tag{2.7}$$

The asymptotics is uniform in a, b if they range in a bounded domain.

PROOF. Denote $\operatorname{Re}\lambda = k/2$. Let $|\arg\lambda| < \pi - \varepsilon$. Then we represent our expression as

$$\frac{i^k\Gamma(a+\overline{\lambda})}{\Gamma(1-a+\lambda)}\cdot\frac{i^{-k}\Gamma(b+\lambda)}{\Gamma(1-b+\overline{\lambda})}$$

and apply the standard asymptotic formula $\Gamma(z+\alpha)/\Gamma(z+\beta) \sim z^{\alpha-\beta}$ in the sector $|\arg z| < \pi-\varepsilon$, see Erdélyi etc., [7], formula (1.18.4). If $|\arg(-\lambda)| < \pi-\varepsilon$, we write

$$\frac{i^{-k}\Gamma(a-\lambda)}{\Gamma(1-a-\overline{\lambda})}\cdot\frac{i^{k}\Gamma(b-\overline{\lambda})}{\Gamma(1-b-\lambda)}$$

and come to the same asymptotics. $\square$

2.2. The Mellin transform. Denote by $\mathbb{C}^\times := \mathbb{C}\setminus 0$ the multiplicative group of $\mathbb{C}$. The *Mellin transform* (see, e.g., [11]) on $\mathbb{C}^\times$ is defined by

$$g(\boldsymbol{\mu}) = \mathcal{M}f(\boldsymbol{\mu}) = \frac{1}{2\pi}\int_{\mathbb{C}} f(z) z^{\boldsymbol{\mu}-\mathbf{1}}\, d\overline{\overline{z}}, \tag{2.8}$$

where $\boldsymbol{\mu} = \{\mu|\mu'\} = \bigl\{\frac{k+is}{2}\big|\frac{-k+is}{2}\bigr\} \in \Lambda_{\mathbb{C}}$ (here we allow complex s). This operator is the Fourier transform on the group $\mathbb{C}^\times \simeq (\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R}$, so it is reduced to the Fourier transforms on $(\mathbb{R}/2\pi\mathbb{Z})$ and on $\mathbb{R}$. Indeed, changing variables

$$z = e^\rho e^{i\varphi}$$

we come to

$$g(k,s) = \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} f\bigl(e^\rho e^{i\varphi}\bigr)e^{ik\varphi+is\rho}\, d\rho\, d\varphi.$$

The inversion formula is given by

$$f(z) = \mathcal{M}^{-1}g(z) = \frac{1}{2\pi}\int_\Lambda g\bigl(\mu\big|-\overline{\mu}\bigr) z^{-\mu|\overline{\mu}}\, \widetilde{d}\mu.$$

Equivalently, $\mathcal{M}$ is a unitary operator $L^2(\mathbb{C}^\times, |z|^{-2}) \to L^2(\Lambda)$.

2.3. The Mellin transform of even functions. We say that a function f on $\mathbb{C}^\times$ is $\times$-*even* if $f(z^{-1}) = f(z)$. Denote by $L^2_+(\mathbb{C}^\times, |z|^{-2})$ the subspace of $L^2(\mathbb{C}^\times, |z|^{-2})$ consisting of $\times$-even functions. Obviously, the Mellin transform sends $\times$-even functions in z to even functions in μ. Also, for a $\times$-even function f we have

$$\mathcal{M}f(\boldsymbol{\mu}) = \frac{1}{2}\int_{\mathbb{C}} f(z)(z^{\boldsymbol{\mu}-\mathbf{1}} + z^{-\boldsymbol{\mu}-\mathbf{1}})\, d\overline{\overline{z}}, \qquad \text{where } f \text{ is } \times\text{-even.} \tag{2.9}$$

2.4. The Mellin transform of smooth compactly supported functions.

Theorem 2.2. a) *Let f be a compactly supported smooth function on $\mathbb{C}$. Then $\mathcal{M}f(\mu|\mu')$ extends to a meromorphic function in the variable μ with possible poles located at points $\mu|\mu' \in \mathbb{Z}_-\times\mathbb{Z}_-$. Moreover, for any p, $p' \in \mathbb{Z}_+$ for $\operatorname{Re}(\mu+\mu') > -p-p'$ we have*

$$I(\mu|\mu') = \frac{(-1)^{p+p'}}{2\pi(-\mu)_p\,(-\mu')_{p'}}\int_{\mathbb{C}} z^{\mu-p-1|\mu'-p'-1}\frac{\partial^{p+p'}}{\partial z^p\,\partial\overline{z}^{p'}}f(z)\ d\overline{\overline{z}}. \tag{2.10}$$

The residues at poles are

$$\operatorname{res}_{\mu|\mu'=-p|-p'} I(\mu|\mu') = \frac{1}{(p-1)!\,(p'-1)!}\frac{\partial^{p+p'}f(0,0)}{\partial z^p\,\partial\overline{z}^{p'}}. \tag{2.11}$$

b) *For each N for each A for all pairs (k,s) satisfying $|\operatorname{Im}s| < A$ we have*

$$\mathcal{M}f\bigl(\tfrac{k+is}{2}\big|\tfrac{-k+is}{2}\bigr) = O(k^2+|s|^2)^{-N} \qquad \text{as } |k^2|+|s^2|\to\infty.$$

For a proof of statement a), see Gelfand, Shilov [13], Sect. B.1.3, or equivalently Russian edition of Gelfand, Graev, Vilenkin [12], Addendum 1.3 (term 'Mellin transform' in that place is absent, but the statement is proved). The formula (2.10) is obtained from (2.8) with integration by parts. The statements about location of poles and about residues require more careful arguments.

Proof of statement b). We pass to polar coordinates, $z = e^{i\theta}$ and get

$$\mathcal{M}f\big(\tfrac{k+is}{2}\big|\tfrac{-k+is}{2}\big) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} H(\theta,r)\, r^{-1+is} e^{i\theta k}\, d\theta\, dr,$$

where $H(\theta,r) := \Phi(re^{i\theta})$ is a smooth function 2π-periodic in θ, the $H(\theta,0)$ does not depend on θ, also $H(\theta+\pi,-r) = H(\theta,r)$. Integrating by parts, we get

$$\mathcal{M}f\big(\tfrac{k+is}{2}\big|\tfrac{-k+is}{2}\big) = \frac{(-1)^l}{2\pi\,(is)_l}\int_0^{2\pi}\int_0^\infty \frac{\partial^l}{\partial r^l} H(\theta,r)\, r^{-1+is+l}\, dr\; e^{i\theta k}\, d\theta.$$

For $l > A$ the integral absolutely converges. Integrating by parts in θ, we get

$$\frac{(-1)^{k+l}}{2\pi\,(ik)^m (is)_l}\int_0^{2\pi}\int_0^\infty \frac{\partial^{l+m}}{\partial r^l\,\partial\theta^m} H(\theta,r) r^{-1+is+l} dr e^{i\theta k}\, d\theta,$$

and

$$|\mathcal{M}f\big(\tfrac{k+is}{2}\big|\tfrac{-k+is}{2}\big)| \leqslant \frac{\mathrm{const}}{|(2\pi i k)^m (is)_l|}.$$

If $|s| > |k|$ we take $m = 0$ and large l, if $|k| > |s|$, we take $l > |\operatorname{Im} s|$ and large m. $\square$

2.5. Weak singularities. Here we imitate one standard trick of asymptotic analysis, see, e.g., [8], Sect. I.4. Consider the integrals of the following type

$$M(\varepsilon) = M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\varepsilon) := \int_{|t|\leqslant R} t^{\alpha-1|\alpha'-1}(\varepsilon - t)^{\beta-1|\beta'-1}\psi(t)\; d\,\overline{\overline{t}},$$

where $\psi(t)$ is a smooth function on $\mathbb{C}$. Clearly, $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\varepsilon)$ is holomorphic in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ in the domain of convergence and admits a meromorphic continuation[12] to $(\alpha,\beta) \in Lambda^2$.

Theorem 2.3. *Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ satisfy the condition*

$$\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{1} \notin \mathbb{Z}_- \times \mathbb{Z}_-. \tag{2.12}$$

Then $M(\varepsilon)$ (defined in the sense of analytic continuation) admits the following asymptotic expansion at 0:

(2.13)

$$M(\varepsilon) \sim \sum_{i,i'\geqslant 0} B^{\mathbb{C}}(\alpha+i|\alpha'+i',\, \beta|\beta')\cdot\frac{1}{i!\,i'!}\frac{\partial^{i+i'}\psi(0,0)}{\partial t^i \partial \overline{t}^{i'}}\cdot \varepsilon^{\alpha+\beta+i-1|\alpha'+\beta'+i'-1} + \\ + \sum_{j,j'\geqslant 0} r_{j|j'}\varepsilon^{j|j'}.$$

The coefficients of the expansion are meromorphic in $\alpha|\alpha'$, $\beta|\beta'$.

If[13] $[\alpha|\alpha'] > 0$, $[\beta|\beta'] > 0$, $[\alpha|\alpha'] + [\beta|\beta'] > 1$, then

$$r_{00} = \int_{|t|<R} t^{\alpha+\beta-2|\alpha'+\beta'-2}\psi(t)\, d\,\overline{\overline{t}}. \tag{2.14}$$

---

[12]For instance, see proof of Proposition 3.1 below.

[13]Recall notation (1.15).

First, we prove the following lemma

Lemma 2.4. *Let α, α', β, β' satisfy the condition* (2.12). *Then the following integral* (*defined in the sense of analytic continuation*)

$$R(\varepsilon) = \int_{|t|<R} t^{\alpha-1|\alpha'-1}(\varepsilon - t)^{\beta-1|\beta'-1}\, d\,\overline{\overline{t}} \tag{2.15}$$

admits an asymptotic expansion of the form

$$R(\varepsilon) = B^{\mathbb{C}}(\alpha|\alpha',\beta|\beta')\cdot \varepsilon^{\alpha+\beta-1|\alpha'+\beta'-1} + \sum_{j\geqslant 0, j'\geqslant 0} p_{j|j'}\varepsilon^{j|j'}. \tag{2.16}$$

Moreover, the series

$$\sum_{j\geqslant 0, j'\geqslant 0} p_{j|j'}\varepsilon^{j|j'} \tag{2.17}$$

converges in the circle $|\varepsilon| < 1/R$, and coefficients $p_{j|j'}(\mathbf{a},\mathbf{b})$ are holomorphic in the domain (2.12).

Proof. Set

$$Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\varepsilon_1,\varepsilon_2) := (-1)^{\beta'-\beta} \int\limits_{|t|>R} t^{\alpha+\beta-2|\alpha'+\beta'-2}\Bigl(1-\frac{\varepsilon_1}{t}\Bigr)^{\beta-1}\Bigl(1-\frac{\varepsilon_2}{t}\Bigr)^{\beta'-1}\, d\,\overline{\overline{t}}.$$

This function is meromorphic in α, β, and in ε_1, ε_2 in the bidisk $|\varepsilon_1| < 1/R$, $|\varepsilon_2| < 1/R$. Let

$$[\alpha|\alpha'] > 0, \quad [\beta|\beta'] > 0, \quad [\alpha|\alpha'] + [\beta|\beta'] < 1. \tag{2.18}$$

Under these conditions the integral $R(\varepsilon)$ converges, and

$$R(\varepsilon) = \int_{\mathbb{C}} - \int_{|t|>R} = B^{\mathbb{C}}(\alpha|\alpha',\beta|\beta')\cdot\varepsilon^{\alpha+\beta-1|\alpha'+\beta'-1} - Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\varepsilon,\overline{\varepsilon}).$$

Expanding the integrand in $Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in a series in ε_1, ε_2 and integrating termwise we come to

$$\begin{gathered} Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\varepsilon_1,\varepsilon_2) = \\ = \sum_{j\geqslant 0, j'\geqslant 0:\, \alpha+\beta-j=\alpha'+\beta'-j'} \frac{(-\beta+1)_j\,(-\beta'+1)_j\, R^{\alpha+\alpha'+\beta+\beta'-j-j'}}{(j+j'-\alpha-\alpha'-\beta-\beta')\, j!\, j'!}\varepsilon_1^{j}\varepsilon_2^{j'}. \end{gathered} \tag{2.19}$$

Now we can omit restrictions (2.18). Indeed, under conditions (2.12) the series (2.19) converges in the bidisk $|\varepsilon_1| < 1$, $|\varepsilon_2| < 1$ and therefore its sum coincides with the meromorphic continuation. □

Proof of Theorem 2.3. We expand the function ψ as a sum

$$\psi(t,\overline{t}) = \sum_{j+j'\leqslant N} \frac{1}{j!\,j'!}\frac{\partial^{j+j'}\psi(0)}{\partial t^j\,\partial t^{j'}}\, t^{j|j'} + H_N(t),$$

where $H_N(t)$ is a smooth function and

$$H_N(t) = O(|t|^{N+1}) \qquad \text{as } t\to 0.$$

Substituting this to the initial integral we get a sum of integrals of the form (2.15), we apply Lemma 2.4 to each summand. Also we get a summand

$$I(\varepsilon) = \int_{|t|\leqslant R} t^{\alpha-1|\alpha'-1}(\varepsilon+t)^{\beta-1|\beta'-1}H_N(t)\, d\,\overline{\overline{t}}.$$

We wish to show that $T(\varepsilon)$ has partial derivatives at 0 up to order $N-k$, where k is constant depending only on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Consider a partition of unity, $1=\varphi_1+\varphi_2$ such that φ_2 is zero at some smaller circle $|t|<R'$. According this, we split $I=I_1+I_2$. Obviously, I_2 has an expansion of the form

$$I_2 \sim \sum_{j,j'\geqslant 0} c_{j|j'}\varepsilon^{j|j'}$$

with coefficient meromorphic in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$. Next, we integrate I_1 by parts many times,

$$I_1(\varepsilon)=\frac{1}{(\beta)_m(\beta')_m}\int_{|t|\leqslant R}(\varepsilon+t)^{\beta-1+m|\beta'+m-1}\frac{\partial t^{2m}}{\partial^m\partial\overline{t}^m}\Bigl(t^{\alpha-1|\alpha'-1}H_N(t)\varphi_1(t)\Bigr)\,d\,\overline{\overline{t}}.$$

Choosing m we can make $\beta+m-1$, $\beta'+m-1$ as large, as we want, say $>q$. Next, we choose a large N, such that $\frac{\partial^{2m}}{\partial t^m\partial\overline{t}^m}\bigl(\dots\bigr)$ is continuous at 0. Now we can q times differentiate $I_1(\varepsilon)$ by ε, $\overline{\varepsilon}$ at 0 and to take its Taylor expansion. This finishes a derivation of the asymptotic expansion for $R(\varepsilon)$.

If the integral $R(0)$ converges, we substitute $\varepsilon=0$ to the expansion and get the expression for r_{00}. □

3. The hypergeometric function of the complex field

Here we discuss basic properties of the functions ${}_2F_1^{\mathbb{C}}[\cdot]$.

3.1. Domain of convergence and analytic continuation. The hypergeometric function ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ of the complex field is defined by the Euler type integral (1.18):

$$(3.1)\qquad {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]=\frac{1}{\pi B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\int_{\mathbb{C}} t^{\mathbf{b-1}}(1-t)^{\mathbf{c-b-1}}(1-zt)^{-\mathbf{a}}\,d\,\overline{\overline{t}}.$$

For $z\neq 0$, 1, the integral absolutely converges (see notation (1.15) if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is contained in the following tube $\widehat{\Xi}$,

$$(3.2)\qquad \widehat{\Xi}:\quad [\mathbf{b}]>0,\quad [\mathbf{c}]-[\mathbf{b}]>0,\quad [\mathbf{a}]<1,\quad [\mathbf{c}]-[\mathbf{a}]<2.$$

In other words, the integral absolutely converges iff a point $\bigl([\mathbf{a}],[\mathbf{b}],[\mathbf{c}]\bigr)$ is contained in the simplex Ξ in $\mathbb{R}^3$ with vertices

$$(3.3)\qquad (1,0,0),\qquad (-1,0,0),\qquad (1,0,1),\qquad (1,2,2).$$

We have $\Lambda^{\mathbb{C}}\simeq\mathbb{C}\times\mathbb{Z}$, therefore triples ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$) depend on 3 integer and 3 complex parameters. Clearly, each component of the set $\mathbb{Z}^3\times\mathbb{C}^3$ has an open intersection with the domain of convergence[14].

Proposition 3.1. *For $z\in\dot{\mathbb{C}}$, the expression ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ as a function of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ admits a meromorphic extension to arbitrary values of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with poles at a countable union of surfaces*

$$(3.4)\qquad \mathbf{a}\in\mathbb{N}\times\mathbb{N},\ \mathbf{b}\in\mathbb{N}\times\mathbb{N},\ \mathbf{c}-\mathbf{a}\in\mathbb{N}\times\mathbb{N},\ \mathbf{c}-\mathbf{b}\in\mathbb{N}\times\mathbb{N},$$

$$(3.5)\qquad \mathbf{c}\in\mathbb{Z}_-\times\mathbb{Z}_-$$

and vanishes for all $z\in\dot{\mathbb{C}}$ at

$$(3.6)\qquad \mathbf{c}\in\mathbb{N}\times\mathbb{N}.$$

[14]The map $(\mathbf{a},\mathbf{b},\mathbf{c})\to\bigl([\mathbf{a}],[\mathbf{b}],[\mathbf{c}]\bigr)$ from $\Lambda^3\to\mathbb{R}^3$ is surjective on all components.

Proof. Consider a partition of unity $1 = \varphi_0(t) + \varphi_1(t) + \varphi_{1/z}(t) + \varphi_\infty(t) + \varphi_\varnothing(z)$, where all summands are smooth and nonnegative, φ_0, φ_1, $\varphi_{1/z}$, φ_∞ are zero outside small neighborhoods of of 0, 1, $1/z$, ∞ respectively, and $\varphi_\varnothing = 0$ in small neighborhoods of these points. Denote $P(t,\overline{t})$ the integrand in the integral representation of ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$. Then

$$ {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] = \int \varphi_0 P\, d\overline{\overline{t}} + \int \varphi_1 P\, d\overline{\overline{t}} + \int \varphi_{1/z} P\, d\overline{\overline{t}} + \int \varphi_\infty P\, d\overline{\overline{t}} + \int \varphi_\varnothing P\, d\overline{\overline{t}}. $$

The last summand is an entire function in $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. By Theorem 2.2 other summands are meromorphic and can have poles at

$$ \mathbf{b} \in \mathbb{Z}_- \times \mathbb{Z}_-,\ \ \mathbf{c}-\mathbf{b} \in \mathbb{Z}_- \times \mathbb{Z}_-,\ \ \mathbf{a} \in \mathbb{N}\times\mathbb{N},\ \ \mathbf{c}-\mathbf{a} \in \mathbb{N}\times\mathbb{N}. $$

However, $B^{\mathbb{C}}$-factor in the front of the integral (3.1) kills the first and the second families of poles and produces new poles and also zeros. This gives us (3.4)–(3.6), in particular the factor $\Gamma^{\mathbb{C}}(\mathbf{c})$ produces poles (3.5) and zeros (3.6).

All these possible poles really are poles, the simplest way to observe this is to look at formulas (3.26)–(3.34) derived below. Formulas (3.26)–(3.28) show that (3.4) are poles. To check a presence of poles (3.5) we apply (3.32)–(3.34). □

3.2. Kummer symmetries. This section contains a collection of elementary formulas, they partially depend on Theorem 3.9 proved below. However, proof of this theorem is based on differential equations and asymptotic analysis and is independent of our formulas.

First we notice two trivial identities

$$ {}_2F_1^{\mathbb{C}}\begin{bmatrix} a|a', b|b' \\ c|c' \end{bmatrix}; \overline{z} = {}_2F_1^{\mathbb{C}}\begin{bmatrix} a'|a, b'|b \\ c'|c \end{bmatrix}; z ; \tag{3.7} $$

$$ \overline{{}_2F_1^{\mathbb{C}}\left[\begin{matrix} a'|a, b'|b \\ c'|c \end{matrix}; z\right]} = {}_2F_1^{\mathbb{C}}\left[\begin{matrix} \overline{a}'|\overline{a}, \overline{b}'|\overline{b} \\ \overline{c}'|\overline{c} \end{matrix}; z\right]. \tag{3.8} $$

To verify (3.7) we substitute $t \mapsto \overline{t}$ to the integral (3.1).

Proposition 3.2. a) (Gauss identity) *Let* $[\mathbf{c}] - [\mathbf{a}] - [\mathbf{b}] > 0$. *Then*

$$ {}_2F_1^{\mathbb{C}}\left[\begin{matrix} \mathbf{a},\mathbf{b} \\ \mathbf{c} \end{matrix}; 1\right] := \lim_{z\to 1} {}_2F_1^{\mathbb{C}}\left[\begin{matrix} \mathbf{a},\mathbf{b} \\ \mathbf{c} \end{matrix}; z\right] = \frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a}-\mathbf{b})}{\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{b})}. \tag{3.9} $$

b) *Let*[15] $[\mathbf{c}] < 1$. *Then*

$$ {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};0] := \lim_{z\to 0} {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] = 1. $$

Proof. a) We substitute $z = 1$ to (3.1) and come to a beta function,

$$ \pi B^{\mathbb{C}}[\mathbf{b},\mathbf{c}-\mathbf{a}]/\pi B^{\mathbb{C}}[\mathbf{b},\mathbf{c}-\mathbf{c}]. $$

However, this argument is valid only if the beta integral $B^{\mathbb{C}}[\mathbf{b},\mathbf{c}-\mathbf{a}]$ converges. The general statement follows from Theorem 3.9.b proved below.

b) also is reduced to a beta-function with the same problem with the domain of convergence. The general statement follows from Theorem 3.9.a. □

Proposition 3.3.

$$ {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] = {}_2F_1^{\mathbb{C}}[\mathbf{b},\mathbf{a};\mathbf{c};z]. \tag{3.10} $$

[15] If $[\mathbf{c}] \geqslant 1$, then $\lim_{z\to 0}\big|{}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]\big| = \infty$.

This will became obvious after Theorem 3.9. We use this symmetry in the next two proofs.

Proposition 3.4. *(Euler and Pfaff transformations),*

$$ {}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{a},\mathbf{b}\\ \mathbf{c}\end{matrix};z\right]=(1-z)^{-\mathbf{a}}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{a},\mathbf{c}-\mathbf{b}\\ \mathbf{c}\end{matrix};\frac{z}{z-1}\right] \tag{3.11}$$

$$ =(1-z)^{-\mathbf{b}}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{c}-\mathbf{a},\mathbf{b}\\ \mathbf{c}\end{matrix};\frac{z}{z-1}\right] \tag{3.12}$$

$$ =(1-z)^{\mathbf{c}-\mathbf{a}-\mathbf{b}}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{c}-\mathbf{a},\mathbf{c}-\mathbf{b}\\ \mathbf{c}\end{matrix};z\right] \tag{3.13}$$

Proof. We substitute $t=1-s$ to (3.1) and get (3.11). Applying (3.10) we get (3.12). Applying (3.11) and (3.12), we get (3.13). $\square$

Proposition 3.5. (Kummer symmetries) *The following functions $u_j^{\mathbb{C}}(z)$ are equal*[16]:

$$u_1^{\mathbb{C}}(z) = {}_2F_1^{\mathbb{C}}\begin{bmatrix}\mathbf{a},\mathbf{b}\\ \mathbf{c}\end{bmatrix}; z\Bigr] \qquad \text{(compare with [7], (2.2.9.1));} \tag{3.14}$$

$$\begin{aligned}u_5^{\mathbb{C}}(z) &= \\ &= \frac{(-1)^{\mathbf{c}-\mathbf{a}-\mathbf{b}}\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{c}-1)}{\Gamma^{\mathbb{C}}(\mathbf{a})\,\Gamma^{\mathbb{C}}(\mathbf{b})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{b})} z^{1-c}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{b}-\mathbf{c}+\mathbf{1};\ \mathbf{a}-\mathbf{c}+\mathbf{1}\\ \mathbf{2}-\mathbf{c}\end{matrix}; z\right]\end{aligned} \tag{3.15}$$

(see [7], (2.2.9.17)) and ratio of coefficients at u_1, u_5 in (2.2.10.35));

$$u_3^{\mathbb{C}}(z) = \frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{b}-\mathbf{a})}{\Gamma^{\mathbb{C}}(\mathbf{b})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a})}\,(-z)^{-\mathbf{a}}\;{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{a},\mathbf{a}-\mathbf{c}+\mathbf{1}\\ \mathbf{a}-\mathbf{b}+\mathbf{1}\end{matrix}; z^{-1}\right] \tag{3.16}$$

(see [7], (2.2.9.9) and B_1 in (2.2.10.5));

$$u_4^{\mathbb{C}}(z) = \frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{a}-\mathbf{b})}{\Gamma^{\mathbb{C}}(\mathbf{a})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{b})}(-z)^{-\mathbf{b}}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{b},\mathbf{b}-\mathbf{c}+\mathbf{1}\\ \mathbf{b}-\mathbf{a}+\mathbf{1}\end{matrix}; z^{-1}\right] \tag{3.17}$$

(see [7], (2.2.9.10) and B_2 in (2.2.10.5));

$$u_2^{\mathbb{C}}(z) = \frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a}-\mathbf{b})}{\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a})\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{b})}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{a},\mathbf{b}\\ \mathbf{a}+\mathbf{b}+\mathbf{1}-\mathbf{c}\end{matrix}; 1-z\right] \tag{3.18}$$

(see [7], (2.2.9.5) and A_1 in (2.2.10.5));

$$u_6^{\mathbb{C}}(z) = \frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{a}+\mathbf{b}-\mathbf{c})}{\Gamma^{\mathbb{C}}(\mathbf{a})\Gamma^{\mathbb{C}}(\mathbf{b})}\,(1-z)^{\mathbf{c}-\mathbf{a}-\mathbf{b}}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{c}-\mathbf{a},\mathbf{c}-\mathbf{b}\\ \mathbf{c}-\mathbf{a}-\mathbf{b}+\mathbf{1}\end{matrix}; 1-z\right] \tag{3.19}$$

(see [7], (2.2.9.5) and A_1 in (2.2.10.5)).

Remark. For each expression (3.14)–(3.19) we can apply one of transformations (3.11)-(3.13). In this way we get 24 expressions of this type. ⊠

Proof. Formula for u_3. Changing a variable $t = 1/s$ in (3.1) we come to

$$\begin{aligned}&\frac{(-1)^{\mathbf{c}-\mathbf{a}-\mathbf{b}-\mathbf{1}}z^{-a}}{\pi B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\int_{\mathbb{C}} s^{\mathbf{a}-\mathbf{c}}(1-s)^{\mathbf{c}-\mathbf{b}-\mathbf{1}}(1-s/z)^{-\mathbf{a}}\,d\overline{\overline{s}} = \\ &\qquad \frac{(-1)^{\mathbf{c}-\mathbf{b}-\mathbf{1}}\pi B^{\mathbb{C}}(\mathbf{a}-\mathbf{c}+\mathbf{1},\mathbf{c}-\mathbf{b})}{\pi B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}(-z)^{-a}\;{}_2F_1^{\mathbb{C}}\left[\begin{matrix}\mathbf{a},\mathbf{a}-\mathbf{c}+\mathbf{1}\\ \mathbf{a}-\mathbf{b}+\mathbf{1}\end{matrix}; z^{-1}\right].\end{aligned}$$

We cancel $\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{b})$ and apply (2.3) two times.

Formula for u_4. We transpose $\mathbf{a}$ and $\mathbf{b}$ in the formula for u_3.

Formula for u_5. We combine transformations (3.16) and (3.17).

Formula for u_2. We combine transformations (3.16), (3.11), and again (3.16).

We combine transformations (3.16), (3.12), and again (3.16). □

Remark. Proposition 3.5 is a self-closed collection of identities. However, they are reflections of the Kummer table of solutions of the hypergeometric equation

$$\Bigl(z(1-z)\frac{\partial^2}{\partial z^2} + \bigl(c-(a+b+1)z\bigr)\frac{\partial}{\partial z} - ab\Bigr)\,u(z) = 0,$$

[16] The meaning of subscripts j in $u_j^{\mathbb{C}}$, references, and comments is explained in a remark after the proof.

see Erdélyi, et al., [7], Section 2.2.9, formulas (1)–(24). The Kummer table contains 6 solutions, each of them is defined in a neighborhood of one of singular points 0, 1, ∞.

$$\begin{aligned} u_1(z) &= \alpha_1(z), & u_5(z) &= z^{1-c}\alpha_5(z),\\ u_3(z) &= (-z)^{-a}\alpha_3(z^{-1}), & u_4(z) &= (-z)^{-b}\alpha_4(z^{-1}),\\ u_2(z) &= \alpha_2(1-z), & u_6(z) &= (1-z)^{c-a-b}\alpha_6(1-z), \end{aligned}$$

where $\alpha_j(x)$ are power series, $\alpha_j(0)=1$. Generally, these solutions are ramified at points 0, 1, ∞. Each solution is represented in 4 forms, which can be obtained one from another by Pfaff transformations, see Erdélyi, et al., [7], Sect. 2.1, (22)–(23). In the table above we present corresponding expressions for ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$, they correspond to Kummer's expressions with change $(a,b,c)\mapsto(\mathbf{a},\mathbf{b},\mathbf{c})$. The resulting functions $u_j^{\mathbb{C}}$ are non-ramified (by definition) and differ by factors independent of z, we normalize them to make them equal one to another. Counterparts of these factors (except one formula) are present in the Kummer formulas as coefficients of transfer-matrices (u_1,u_5) to (u_3,u_4) and (u_2,u_6), with the same replacement $(a,b,c)\mapsto(\mathbf{a},\mathbf{b},\mathbf{c})$, see Erdélyi, et al., [7], display (2.2.10.5) and coefficients A_1, A_2, B_1, B_2. So, in each line of Proposition 3.5 we give a reference to the corresponding formula in Erdélyi, et al., [7], (2.2.9.1)–(2.2.9.24) and to the corresponding coefficient in [7], display (2.2.10.5). ⊠

3.3. Differential equations.

Lemma 3.6.

$$\frac{\partial}{\partial z}\,{}_2F_1^{\mathbb{C}}\begin{bmatrix}a|a';b|b'\\ c|c'\end{bmatrix};z\Bigr] = \frac{ab}{c}\,{}_2F_1^{\mathbb{C}}\begin{bmatrix}a+1|a';b+1|b'\\ c+1|c'\end{bmatrix};z\Bigr];$$
$$\frac{\partial}{\partial \overline{z}}\,{}_2F_1^{\mathbb{C}}\begin{bmatrix}a|a';b|b'\\ c|c'\end{bmatrix};z\Bigr] = \frac{a'b'}{c'}\,{}_2F_1^{\mathbb{C}}\begin{bmatrix}a|a'+1;b|b'+1\\ c|c'+1\end{bmatrix};z\Bigr].$$

Proof. We differentiate the integral with respect to the parameter z, and get an integral of the same form. The calculation is valid if $\Xi\cap\bigl(\Xi+(\frac12,\frac12,\frac12)\bigr)\ne\varnothing$, where Ξ is the simplex defined by (3.2)–(3.3). This intersection is open and nonempty. It remains to refer to the meromorphic continuation. □

Denote

$$D = D[a,b,c] := z(1-z)\frac{\partial^2}{\partial z^2} + \bigl(c-(a+b+1)z\bigr)\frac{\partial}{\partial z} - ab; \tag{3.20}$$

$$D' = D'[a',b',c'] := \overline{z}(1-\overline{z})\frac{\partial^2}{\partial \overline{z}^2} + \bigl(c'-(a'+b'+1)\overline{z}\bigr)\frac{\partial}{\partial \overline{z}} - a'b'. \tag{3.21}$$

Proposition 3.7. *The complex hypergeometric function* $\mathcal{F}(z) = {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ *satisfies the following system of partial differential equations*

$$D[a,b,c]\ \mathcal{F}=0; \qquad D'[a',b',c']\ \mathcal{F}=0. \tag{3.22}$$

We call these equation by *complex hypergeometric system.*

Proof. This follows from the identity

$$D[a,b,c]\Bigl(t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\Bigr) = -a\frac{\partial}{\partial t}\Bigl(t^{b}(1-t)^{c-b}(1-tz)^{-a-1}\Bigr) \tag{3.23}$$

(cf. [7], (2.1.3.11)). Consider sufficiently small positive ε, δ, $\varkappa$ and take $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ such that

$$[b|b'] = \varepsilon, \quad [c|c'] = \varepsilon + \delta, \qquad [a|a'] = -\tfrac{1}{2} + \varepsilon + \delta + \varkappa.$$

We multiply both parts of (3.23) by $\overline{t}^{\,b'-1}(1-\overline{t})^{c'-b'-1}(1-\overline{t}\overline{z})^{-a'}$ and integrate over $\mathbb{C}$. In the left hand side for such values in parameter we can change the integration with differentiations in z. In the right hand side the integrand is an integrable derivative of an integrable function. Therefore the right hand side is zero. □

Proposition 3.8. a) *Any solution of system* (3.20)–(3.21) *is real analytic in* z.

b) *Let* $z_0 \neq 0, 1, \infty$. *Denote by* $\varphi_1(z)$, $\varphi_2(z)$ *a pair of independent holomorphic solutions of the ordinary differential equation* $D[a,b,c]\, f(z) = 0$ *at a neighborhood of* z_0. *Denote by* $\psi_1(\overline{z})$, $\psi_2(\overline{z})$ *a pair of antiholomorphic solutions of the ordinary differential equation* $D'[a',b',c']\, f(\overline{z}) = 0$. *Then any solution of the system* (3.20)–(3.21) *can be represented as*

$$\sum_{i,j=1,2} \tau_{ij}(\mathbf{a},\mathbf{b},\mathbf{c})\varphi_i(z)\psi_j(\overline{z}). \tag{3.24}$$

c) *If we choose* φ_i, ψ_j *meromorphic in parameters* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *in some domain in* $\Lambda^3_{\mathbb{C}}$, *then* τ_{ij} *also are meromorphic in parameters* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Proof. a) Indeed, $D[a,b,c]$ is an elliptic differential operator, therefore solutions of the equation $D\mathcal{F} = 0$ are analytic functions, see, e.g., [17],Theorem 9.5.1.

b) Consider a solution

$${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] = h_{00} + h_{10}(z-z_0) + h_{01}(\overline{z}-\overline{z}_0) + h_{11}(z-z_0)(\overline{z}-\overline{z}_0) + \dots$$

of the system of partial differential equations (3.22). These equations determine recurrence relations to Taylor coefficients h_{ij} of ${}_2F_1^{\mathbb{C}}[\dots]$ at z_0. It is can be easily checked that all the coefficients h_{ij} admit linear expressions in terms of h_{00}, h_{01}, h_{10}, h_{11}. On the other hand, for given h_{00}, h_{01}, h_{10}, h_{11}, we can find a local solution of the complex hypergeometric system (3.22) in the form $\sum C_{ij}\varphi_i(z)\psi_j(\overline{z})$.

c) By Lemma 3.6, coefficients h_{00}, h_{10}, h_{01}, h_{11} depend on $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ meromorphically. If $\varphi_i(z_0)$, $\varphi_i'(z_0)$, $\psi_j(z_0)$, $\psi_j'(z_0)$ are meromorphic in parameters, then C_{ij} also are meromorphic. □

3.4. Expressions for ${}_2F_1^{\mathbb{C}}$. Let us write expansions of ${}_2F_1^{\mathbb{C}}[\dots;z]$ near singular points $z = 0, 1, \infty$. Explicit formulas for fundamental systems of solutions of the hypergeometric differential equation are well-known, see Erdélyi, et al., [7], 2.9 (the Kummer series). To be definite, consider $z_0 = 0$. If $c \notin \mathbb{Z}$, then for generic values of parameters the hypergeometric equation $D[a,b,c]f(z) = 0$ has two holomorphic solutions at a neighborhood of 0,

$$\varphi_1(z) = {}_2F_1[a,b;c;z], \qquad \varphi_2(z) = z^{1-c}F[a+1-c, b+1-c; 2-c; z].$$

The equation $D'[a',b',c']f(\overline{z}) = 0$ has two antiholomorphic solutions

$$\psi_1(z) = {}_2F_1[a',b';c';\overline{z}], \qquad \psi_2(z) = \overline{z}^{1-c'}F[a'+1-c', b'+1-c'; 2-c'; \overline{z}].$$

Therefore near $z = 0$ we have solutions of system (3.20)–(3.21) of the same form (3.24) with new φ, ψ. We get a family of functions depending of 4 parameters τ_{ij}, therefore for generic $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ this formula gives all multivalued solutions near $z = 0$.

Solutions (3.24) that are single valued in a neighborhood of 0 have the form

$$A_1\varphi_1(z)\psi_1(\overline{z}) + A_2\varphi_2(z)\psi_2(\overline{z}). \tag{3.25}$$

Theorem 3.9. a) *In the disk $|z|<1$ we have the following expansion*

$$(3.26)\quad {}_2F_1^{\mathbb{C}}\begin{bmatrix}\mathbf{a};\mathbf{b}\\ \mathbf{c}\end{bmatrix} ;z\Bigr] = A_0\cdot {}_2F_1\begin{bmatrix}a,b\\ c\end{bmatrix};z\Bigr]\,{}_2F_1\begin{bmatrix}a',b'\\ c'\end{bmatrix};\overline{z}\Bigr]+$$

$$+A_1\cdot z^{1-c|1-c'}\,{}_2F_1\Bigl[\begin{matrix}a+1-c,b+1-c\\ 2-c\end{matrix};z\Bigr]\,{}_2F_1\Bigl[\begin{matrix}a'+1-c',b'+1-c'\\ 2-c'\end{matrix};\overline{z}\Bigr],$$

where

$$(3.27)\qquad A_0=1,$$

$$(3.28)\qquad A_1=(-1)^{\mathbf{c}-\mathbf{a}-\mathbf{b}}\frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{1})}{\Gamma^{\mathbb{C}}(\mathbf{a})\,\Gamma^{\mathbb{C}}(\mathbf{b})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{b})}.$$

b) *In the disk $|z-1|<1$ the following expansion take place*

$$(3.29)\quad {}_2F_1^{\mathbb{C}}\Bigl[\begin{matrix}\mathbf{a};\mathbf{b}\\ \mathbf{c}\end{matrix};z\Bigr] = B_0\cdot {}_2F_1\Bigl[\begin{matrix}a,b\\ a+b+1-c\end{matrix};1-z\Bigr]\,{}_2F_1\Bigl[\begin{matrix}a',b'\\ a'+b'+1-c'\end{matrix};1-\overline{z}\Bigr]+$$

$$+B_1\cdot(1-z)^{c-a-b|c'-a'-b'}\,{}_2F_1\Bigl[\begin{matrix}c-a,c-b\\ c+1-a-b\end{matrix};1-z\Bigr]\,{}_2F_1\Bigl[\begin{matrix}c'-a',c'-b'\\ c'+1-a'-b'\end{matrix};1-\overline{z}\Bigr],$$

where

$$(3.30)\qquad B_0=\frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a}-\mathbf{b})}{\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a})\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{b})},$$

$$(3.31)\qquad B_1=\frac{\Gamma^{\mathbb{C}}(\mathbf{c})\Gamma^{\mathbb{C}}(\mathbf{a}+\mathbf{b}-\mathbf{c})}{\Gamma^{\mathbb{C}}(\mathbf{a})\,\Gamma^{\mathbb{C}}(\mathbf{b})}.$$

c) *In the disk $|z|>1$ the following expansion holds*

(3.32)

$$ {}_2F_1^{\mathbb{C}}\Bigl[\begin{matrix}\mathbf{a};\mathbf{b}\\ \mathbf{c}\end{matrix};z\Bigr] = C_0\cdot(-z)^{-a|-a'}\,{}_2F_1\Bigl[\begin{matrix}a,a+1-c\\ a+1-b\end{matrix};z^{-1}\Bigr]\,{}_2F_1\Bigl[\begin{matrix}a',a'+1-c'\\ a'+1-b'\end{matrix};\overline{z}^{\,-1}\Bigr]+$$

$$+C_1\cdot(-z)^{-b|-b'}\,{}_2F_1\Bigl[\begin{matrix}b,b+1-c\\ b+1-a\end{matrix};z^{-1}\Bigr]\,{}_2F_1\Bigl[\begin{matrix}b',b'+1-c'\\ b'+1-a'\end{matrix};\overline{z}^{\,-1}\Bigr],$$

where

$$(3.33)\qquad C_0:=\frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{b}-\mathbf{a})}{\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{a})\,\Gamma^{\mathbb{C}}(\mathbf{b})},$$

$$(3.34)\qquad C_1:=\frac{\Gamma^{\mathbb{C}}(\mathbf{c})\,\Gamma^{\mathbb{C}}(\mathbf{a}-\mathbf{b})}{\Gamma^{\mathbb{C}}(\mathbf{c}-\mathbf{b})\,\Gamma^{\mathbb{C}}(\mathbf{a})}.$$

3.5. Proof of Theorem 3.9. Forms (3.26), (3.29), (3.32) for the desired expressions follow from the preceding considerations. Also we know that the coefficients A_0, A_1, B_0, B_1, C_0, C_1 are meromorphic in $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Now we apply asymptotic expansions from Theorem 2.3.

1. *Asymptotic of ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ as $z\to\infty$.* Assume that the defining integral for ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ converges, and also

$$(3.35)\qquad [\mathbf{b}]-[\mathbf{a}]>0$$

Then

$$\frac{1}{\pi B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\int_{\mathbb{C}} t^{\mathbf{b}-\mathbf{1}}(1-t)^{\mathbf{c}-\mathbf{b}-\mathbf{1}}(1-zt)^{-\mathbf{a}}\, d\overline{\overline{t}} =$$
$$= \frac{z^{-\mathbf{a}}}{\pi B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\int_{\mathbb{C}} t^{\mathbf{b}-\mathbf{1}}(1-t)^{\mathbf{c}-\mathbf{b}-\mathbf{1}}(z^{-1}-t)^{-\mathbf{a}}\, d\overline{\overline{t}} \sim$$
$$\sim \frac{B^{\mathbb{C}}(\mathbf{b}-\mathbf{a},\mathbf{c}-\mathbf{b})}{B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\cdot(-z)^{-\mathbf{a}}\cdot\Bigl(1+\sum_{(i,i')\neq(0,0)} p_{ii'} z^{-i|-i'}\Bigr)+$$
$$+\frac{B^{\mathbb{C}}(\mathbf{b},\mathbf{1}-\mathbf{a})}{B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\cdot z^{-\mathbf{b}}\cdot\Bigl(1+\sum_{(i,i')\neq(0,0)} q_{ii'} z^{-i|-i'}\Bigr).$$

Precisely, denote z^{-1} by ε, and denote the integrand in the last integral by $H(\cdot)$. Let $\varphi(t)\geqslant 0$, $\psi(t)\geqslant 0$ be smooth functions such that $\varphi(t)+\psi(t)=1$, $\varphi(t)=1$ near 0, and $\psi(t)=1$ near ∞. A straightforward differentiation by the parameter ε shows that

$$\int H(t;\varepsilon)\psi(t)\, d\overline{\overline{t}}$$

is smooth near $\varepsilon=0$. For

$$\int_{\mathbb{C}} H(t;z)\varphi(t;\varepsilon)\, d\overline{\overline{t}}$$

we apply Theorem 2.13, due to the restriction (3.35) we can also apply (2.14). Thus we get explicit coefficients C_0, C_1 in the expansion (3.32). To remove restrictions for parameters, we refer to the analytic continuation.

Finally, we transform $B^{\mathbb{C}}(\mathbf{b},\mathbf{1}-\mathbf{a})$ with formula (2.3),

$$B^{\mathbb{C}}(\mathbf{b},\mathbf{1}-\mathbf{a})=\frac{\Gamma^{\mathbb{C}}(\mathbf{b})\,\Gamma^{\mathbb{C}}(\mathbf{1}-\mathbf{a})}{\Gamma^{\mathbb{C}}(\mathbf{1}+\mathbf{b}-\mathbf{a})}=(-1)^{\mathbf{b}}\frac{\Gamma^{\mathbb{C}}(\mathbf{b})\Gamma^{\mathbb{C}}(\mathbf{a}-\mathbf{b})}{\Gamma^{\mathbb{C}}(\mathbf{a})}.$$

2. *Asymptotic of* ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ *as* $z\to 0$. Substituting $t=1/s$ to the definition (1.18) of ${}_2F_1^{\mathbb{C}}$, we get

$${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]=\frac{(-1)^{\mathbf{c}-\mathbf{a}-\mathbf{b}}}{\pi B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\int_{\mathbb{C}} s^{-\mathbf{c}+\mathbf{a}}(z-s)^{-\mathbf{a}}(1-s)^{\mathbf{c}-\mathbf{b}-\mathbf{1}}\, d\overline{\overline{s}}\sim$$
$$\sim\frac{(-1)^{\mathbf{c}-\mathbf{a}-\mathbf{b}}B^{\mathbb{C}}(\mathbf{a}-\mathbf{c}+\mathbf{1},\mathbf{1}-\mathbf{a})}{B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\cdot z^{\mathbf{1}-\mathbf{c}}\cdot\Bigl(1+\sum_{(i,i')\neq(0,0)} p_{ii'} z^{i|i'}\Bigr)+$$
$$+\frac{(-1)^{\mathbf{c}-\mathbf{b}}B^{\mathbb{C}}(\mathbf{1}-\mathbf{c},\mathbf{c}-\mathbf{b})}{B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\cdot\Bigl(1+\sum_{(i,i')\neq(0,0)} q_{ii'} z^{i|i'}\Bigr).$$

3. *Asymptotic of* ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ *as* $z\to 1$. We substitute $t=\frac{1}{1-s}$ to (1.18) and get

$$\begin{aligned}{}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] &= \frac{(-1)^{\mathbf{c}-\mathbf{b}}}{\pi B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\int_{\mathbb{C}} s^{\mathbf{c}-\mathbf{b}-\mathbf{1}}(1-s)^{\mathbf{a}-\mathbf{c}}(1-z-s)^{-\mathbf{a}}\,d\overline{\overline{s}}\sim\\ &\sim \frac{(-1)^{\mathbf{c}-\mathbf{b}}B^{\mathbb{C}}(\mathbf{c}-\mathbf{b},\mathbf{1}-\mathbf{a})}{B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\cdot(1-z)^{\mathbf{c}-\mathbf{b}-\mathbf{a}}\cdot\Bigl(1+\sum_{(i,i')\neq(0,0)}p_{ii'}(1-z)^{i|i'}\Bigr)+\\ &\qquad+\frac{(-1)^{\mathbf{c}-\mathbf{b}-\mathbf{a}}B^{\mathbb{C}}(\mathbf{c}-\mathbf{b}-\mathbf{a},\mathbf{1}+\mathbf{a}-\mathbf{c})}{B^{\mathbb{C}}(\mathbf{b},\mathbf{c}-\mathbf{b})}\cdot\Bigl(1+\sum_{(j,j')\neq(0,0)}q_{ii'}z^{j|j'}\Bigr).\end{aligned}$$

Remark. Another way of proof of Theorem 3.9. Applying the Kummer formulas, Erdélyi, et al., [7], Section 2.9, we can write analytic continuation of (3.25) to a neighborhood of this point. The resulting expression for ${}_2F_1^{\mathbb{C}}$ must be non-ramified at $z=1$. This gives us coefficients in (3.25) up to a common factor. In fact this calculation is done below in the proof of Proposition 3.11. . The scalar factor can be evaluated using (3.9). It remains to apply the Kummer formulas [7], Section 2.9, for analytic continuation again and to get an expansion at ∞. ⊠

3.6. Additional symmetry.

Proposition 3.10. *Let* $a-b\in\mathbb{Z}$. *Then*

$$(3.36)\qquad {}_2F_1^{\mathbb{C}}\begin{bmatrix} a|a',b|b' \\ c|c' \end{bmatrix};z\Bigr] = {}_2F_1^{\mathbb{C}}\begin{bmatrix} a|b',b|a' \\ c|c' \end{bmatrix};z\Bigr].$$

Proof. Expansions (3.26)–(3.28) at 0 for both the functions are identical. We only must verify coincidence of denominators in (3.28):

$$(3.37)\quad \begin{aligned}&\Gamma^{\mathbb{C}}(a|a')\,\Gamma^{\mathbb{C}}(b|b')\,\Gamma^{\mathbb{C}}(c-a|c'-a')\,\Gamma^{\mathbb{C}}(c-b|c'-b')=\\ &\qquad\qquad \Gamma^{\mathbb{C}}(a|b')\,\Gamma^{\mathbb{C}}(b|a')\,\Gamma^{\mathbb{C}}(c-a|c'-b')\,\Gamma^{\mathbb{C}}(c-b|c'-a').\end{aligned}$$

The both sides are equal to

$$\frac{(-1)^{c'-c}\pi^4\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')\Gamma(c-a)\Gamma(c-a')\Gamma(c-b)\Gamma(c-b')}{\sin\pi a'\sin\pi b'\sin\pi(c'-a')\sin\pi(c'-b')}.\qquad\square$$

3.7. Degenerations and logarithmic expressions.

a) Residues and zeros. Notice that poles and zeros of ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ as function of **a**, **b**, **c** depend on a normalizing factor in the front of the integral (3.1).

It is easy to see that residues at poles also are solutions of the complex hypergeometric system (3.22). The expressions for residues can be obtained from our expansions.

For obtaining residues at $\{a|a'\}\in\mathbb{N}\times\mathbb{N}$ we can use the expansion of ${}_2F_1^{\mathbb{C}}$ at $z=0$, see (3.26)–(3.28). We get

$$z^{1-c|1-c'}\,{}_2F_1\Bigl[\begin{matrix} a+1-c,b+1-c \\ 2-c \end{matrix};z\Bigr]\,{}_2F_1\Bigl[\begin{matrix} a'+1-c',b'+1-c' \\ 2-c' \end{matrix};\overline{z}\Bigr]$$

with an obvious $\Gamma^{\mathbb{C}}$-factor. Applying Pfaff transformations of ${}_2F_1$, we observe that these expressions are elementary functions. Formulas (3.26)–(3.28) allow to calculate residues at poles of all the types (3.4).

Next, consider another normalization[17] of functions ${}_2F_1^{\mathbb{C}}$:

$$
{}_2\widetilde{F}_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] := \frac{1}{\Gamma(\mathbf{c})}\, {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]. \tag{3.38}
$$

This operation cancels the factor $\Gamma^{\mathbb{C}}(\mathbf{c})$ in expansion of ${}_2F_1^{\mathbb{C}}[z]$ at ∞, see (3.32)–(3.34). So we get a finite expression at poles (3.5) and non-zero function at zeros (3.6).

Thus, at all exceptional planes (3.4)–(3.6) we get explicit nonzero expressions. Such expressions also depend on normalization of ${}_2F_1^{\mathbb{C}}[\dots,z]$, but for a point $(\mathbf{a}_0,\mathbf{b}_0,\mathbf{c}_0)$ being in a general position on an exceptional plane such nonzero expression is canonically defined up to a constant factor.

b) Further degenerations. Classical hypergeometric differential equation has a sophisticated list of degenerations, see [7], Sect. 2.2. In our case new difficulties arise if at least two of parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}-\mathbf{a}$, $\mathbf{c}-\mathbf{b}$ are contained in $\mathbb{Z}\times\mathbb{Z}$. We stop here further analysis and only notice that *for exceptional values of parameters a solution of the complex hypergeometric system* (3.22) *can be non-unique.*

For instance, if $\mathbf{a}\in\mathbb{Z}_-\times Z_-$, $\mathbf{c}-\mathbf{b}\in\mathbb{N}\times\mathbb{N}$, than both the summands in (3.25) are single-valued (since all hypergeometric series are terminating).

c) Logarithmic expressions. To be definite we discuss the case

$$
\mathbf{c}\in\mathbb{N}\times\mathbb{N}
$$

(which is interesting for our further purposes). Consider the function ${}_2\widetilde{F}_1^{\mathbb{C}}$ defined by (3.38). It has a removable singularity at our $\mathbf{c}$. Recall that for $c=n\in\mathbb{N}$ the usual hypergeometric differential equation $D[a,b,n]f=0$ has two solutions. The first is ${}_2F_1[a,b;c;z]$ and the second has the form

$$
\Psi[a,b;n;z] = \sum_{j=-n+1}^{\infty} p_j z^j + \ln z\cdot {}_2F_1[a,b;n;z], \tag{3.39}
$$

where p_j are explicit coefficients, $p_{-n+1}\neq 0$, and this form does not depend on further degenerations, see [1], Section 2.3. Passing around 0 gives

$$
\Psi[a,b;n;e^{i\varphi}z]\Big|_{\varphi=2\pi} - \Psi[a,b;n;z] = {}_2F_1[a,b;n;z].
$$

Thus the system

$$
D[a,b;n]\mathfrak{F}=0,\qquad D[a',b';n']\mathfrak{F}=0
$$

has two solutions that are single-value near zero, the first is obvious

$$
{}_2F_1[a,b,n;z]\,{}_2F_1[a',b';n';\overline{z}],
$$

and the second is

$$
{}_2F_1[a,b;n;z]\,\Psi[a',b';n';\overline{z}] + \Psi[a,b;n;z]\,{}_2F_1[a',b';n';\overline{z}]. \tag{3.40}
$$

Our function ${}_2\widetilde{F}_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{n};z]$ is certain linear combination of these solutions.

d) On uniqueness of a solution of the hypergeometric system.

[17] In fact, in the main part of our work we use this normalization of the kernel, see (1.19). Due to this we do not lose the case of L^2 on the complex quadric discussed in Subsect. 1.3.

Proposition 3.11. *Let*

$$a,\ b,\ c,\ c-a-b,\ c-a,\ c-b \notin \mathbb{Z},$$
$$a',\ b',\ c',\ c'-a'-b',\ c'-a',\ c'-b' \notin \mathbb{Z}.$$

Let the system $D[a,b,c]\mathcal{F}=0$, $D'[a',b',c']\mathcal{F}=0$ have a non-ramified non-zero solution. Then $c-c'\in\mathbb{Z}$ and

$$a-a',\ b-b'\in\mathbb{Z} \text{ or } a-b',\ b-a'\in\mathbb{Z} \tag{3.41}$$

Such solution is unique up to a scalar factor and therefore is ${}_2F_1^{\mathbb{C}}[a|a',b|b';c|c';z]$ or ${}_2F_1^{\mathbb{C}}[a|b',b|a';c|c';z]$

Proof. First, we examine a behavior of a solution near $z=0$. Let

$$\varphi(z) := {}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right], \qquad \psi(z) := {}_2F_1\left[\begin{matrix}a+1-c,b+1-c\\ 2-c\end{matrix};z\right],$$

i.e., φ, $z^{1-c}\psi$ are the Kummer solutions of the hypergeometric equation $D[a,b,c]f=0$ at 0, see [7], (2.9.1), (2.9.17). Denote by $\varphi(\overline{z})$, $\psi_1(\overline{z})$ the similar functions obtained be the change $a\mapsto a'$, $b\mapsto b'$, $c\mapsto c'$, $z\mapsto \overline{z}$. A solution of our system near 0 has the form

$$G(z)=\sigma\,\varphi(z)\varphi'(\overline{z})+\mu\,\overline{z}^{1-\overline{c}'}\varphi(z)\psi'(\overline{z})+\nu\, z^{1-c}\psi(z)\varphi'(\overline{z})+\tau\, z^{1-c|1-\overline{c}'}\psi(z)\psi'(\overline{z}).$$

Passing m times around 0 we come to

$$\begin{aligned}G(z)=\sigma\,\varphi(z)\varphi'(\overline{z})+\mu e^{2\pi m c' i}\,\overline{z}^{1-\overline{c}'}\varphi(z)\psi'(\overline{z})+\\ +\nu e^{-2\pi c i}\, z^{1-c}\psi(z)\varphi'(\overline{z})+\tau e^{2\pi m(c'-c)i}\, z^{1-c|1-\overline{c}'}\psi(z)\psi'(\overline{z}).\end{aligned}$$

Since c, $c'\notin\mathbb{Z}$, we have $e^{2\pi m c' i}$, $e^{-2\pi m c i}\ne 1$, on the other hand they are $\ne e^{2\pi m(c'-c)i}$. If $G(z)$ single-valued then $\mu=\nu=0$. Also, we need $\tau=0$ or $c-c'\in\mathbb{Z}$.

To examine behavior of G near $z=1$ we apply a formula for analytic continuation, see [7], Subsect. 2.10. Near $z=1$ we have

$$\begin{aligned}F\left[\begin{matrix}a,b\\ c\end{matrix};z\right]=A_1(a,b,c)\,{}_2F_1\left[\begin{matrix}a,b\\ a+b-c+1\end{matrix};1-z\right]+\\ +A_2(a,b,c)\,(1-z)^{c-a-b}{}_2F_1\left[\begin{matrix}c-a,c-b\\ c-a-b+1\end{matrix};1-z\right],\end{aligned} \tag{3.42}$$

where

$$A_1(a,b,c):=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad A_2(a,b,c):=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{3.43}$$

Since $c-a-b$, $c'-a'-b'\notin\mathbb{Z}$, the expression $\varphi(z)\varphi'(\overline{z})$ is not single valued. Thus $\tau\ne 0$, $c-c'\in\mathbb{Z}$, and

$$G(z)=\sigma\,\varphi(z)\varphi'(\overline{z})+\tau\, z^{1-c|1-\overline{c}'}\psi(z)\psi'(\overline{z}).$$

Applying for φ, φ', ψ, ψ' formula (3.42) and the identity

$${}_2F_1(\alpha,\beta;\gamma;z)=(1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z),$$

we come to

$$G(z) = \sigma A(a,b,c)A(a',b',c')\,{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};1-z\right]{}_2F_1\left[\begin{matrix}a',b'\\c'\end{matrix};1-\overline{z}\right]+$$

$$+\Big\{\sigma A_1(a,b,c)A_2(a',b',c')+\tau A_1(a+1-c,b+1-c,2-c)A_2(a'+1-c',b'+1-c',2-c')\Big\}\times$$

$$\times(1-\overline{z})^{c'-a'-b'}\,{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};1-z\right]{}_2F_1\left[\begin{matrix}c'-a',c'-b'\\c'-a'-b'+1\end{matrix};1-\overline{z}\right]+$$

$$+\Big\{\sigma A_2(a,b,c)A_1(a',b',c')+\tau A_2(a+1-c,b+1-c,2-c)A_1(a'+1-c',b'+1-c',2-c')\Big\}\times$$

$$\times(1-z)^{c-a-b}\,{}_2F_1\left[\begin{matrix}c-a,c-b\\c-a-b+1\end{matrix};1-z\right]{}_2F_1\left[\begin{matrix}a',b'\\c'\end{matrix};1-\overline{z}\right]+$$

$$+A_2(a+1-c,b+1-c,2-c)A_2(a'+1-c',b'+1-c',2-c')\times$$

$$\times(1-z)^{c-a-b|c'-a'-b'}\,{}_2F_1\left[\begin{matrix}c-a,c-b\\c-a-b+1\end{matrix};1-z\right]{}_2F_1\left[\begin{matrix}c'-a',c'-b'\\c'-a'-b'+1\end{matrix};1-\overline{z}\right]$$

The coefficients $A_1(\cdot)$, $A_2(\cdot)$ have no zeroes and no poles under our restrictions. The expression is single-valued iff two curly brackets are zero and $(c-a-b)-(c'-a'-b')\in\mathbb{Z}$. This implies

$$(a+b)-(a'+b')\in\mathbb{Z}.$$

Two curly brackets give a system of linear equations for σ, τ. It has a nonzero solution iff its determinant Δ is zero. Straightforward calculations give

$$\Delta=\pi^{-4}\Gamma(c)\Gamma(c')\Gamma(2-c)\Gamma(2-c')\Gamma(c-a-b)\Gamma(c'-a'-b')\Gamma(a+b-c)\Gamma(a'+b'-c')\cdot\Xi,$$

where

$$\Xi=\sin\pi(c-a)\sin\pi(c-b)\sin\pi a'\sin\pi b'-\sin\pi(c'-a')\sin\pi(c'-b')\sin\pi a\sin\pi b.$$

Clearly, the set $\Xi(a,b,c,a',b',c')=0$ is invariant with respect to shifts $a\mapsto a+1$, $b\mapsto b+1$, $c\mapsto c+1$. Therefore to examine the set of zeroes we can assume $c'=c$, $b'=a+b-a'$. Under these conditions Ξ can be reduced to the following form:

$$\Xi(a,b,c,a',b',c')=\sin\pi(a-a')\sin\pi(a'-b)\sin\pi c\sin\pi(c-a-b)$$

(this non-obvious identity can be verified by decompositions of both sides into sums if exponentials). This implies (3.41).

If $\Delta=0$ then σ, τ are defined up to a common scalar factor, this proves the uniqueness (and gives an expression for σ/τ). $\square$

e) Non-interesting solutions. However, we have seen that the complex hypergeometric system for some values of parameters has two single-valued solutions. Also, there are solutions that do not seem reasonable. For instance, we have

$$D[0,b_1,c_1]\cdot 1=0,\qquad D'[0,b_2,c_2]\cdot 1=0$$

for arbitrary b_1, c_1, b_2, $c_2\in\mathbb{C}$.

3.8. Differential-difference equations for ${}_2F_1^{\mathbb{C}}$. We can regard ${}_2F_1[a,b;c;z]$ as a family of functions of a complex variable z depending on 3 parameters a, b, c. But we also can regard ${}_2F_1[a,b;c;z]$ as *one function* of 4 complex variables a, b, c, z. Then ${}_2F_1[a,b;c;z]$ satisfy a non-obvious system of linear differential-difference equations, some examples of such equations are in Erdélyi, et al., [7], (2.8.20-45). Below we show that such equations can be automatically transformed to differential-difference equations for the function ${}_2F_1^{\mathbb{C}}[a|a',b|b';c|c';z]$ of 7 complex variables.

Consider a space of functions in variables a, b, c, z. Define operators

$$T_a f(a,b,c,z) = T_a f(a+1,b,c,z), \qquad T_b f(a,b,c,z) = f(a,b+1,c,z),$$
$$T_c f(a,b,c,z) = f(a,b,c+1,z).$$

Consider *finite* sums of the form

$$\mathcal{L} = \sum_{j\geqslant 0} \sum_{k,l,m\in\mathbb{Z}} U_{j,k,l,m}(a,b,c,z)\, T_a^k\, T_b^l\, T_c^m \frac{\partial^j}{\partial z^j}, \tag{3.44}$$

where $U_{j,k,l,m}(a,b,c,z)$ are polynomial expressions in z with coefficients rationally depending on a, b, c.

Assume that

$$\mathcal{L}\ {}_2F_1[a,b;c;z] = 0.$$

We can regard an operator (3.44) as an operator on functions $f(a|a',b|b',c|c',z)$ on $\Lambda^3 \times \dot{\mathbb{C}}$. We also define operators

$$T_{a'} f(a|a',\, b|b',\, c|c',\, z) = f(a|a'+1,\, b|b',\, c|c',\, z),$$
$$T_{b'} f(a|a',\, b|b',\, c|c',\, z) = f(a|a',\, b|b'+1,\, c|c',\, z),$$
$$T_{c'} f(a|a',\, b|b',\, c|c',\, z) = f(a|a',\, b|b',\, c|c'+1,\, z).$$

For any operator $\mathcal{L}$ we define operator $\mathcal{L}'$ by

$$\mathcal{L} = \sum_{j\geqslant 0} \sum_{k,l,m\in\mathbb{Z}} U_{j,k,l,m}(a',b',c',\overline{z}) T_{a'}^k\, T_{b'}^l\, T_{c'}^m \frac{\partial^j}{\partial \overline{z}^j}.$$

By the definition,

$$\mathcal{L}\mathcal{L}' = \mathcal{L}'\mathcal{L}.$$

Proposition 3.12. *Let the function $Q(a,b,c,z) = {}_2F_1[a,b;c;z]$ satisfy an equation $\mathcal{L}\, Q = 0$. Then the function*

$$R(a|a',\, b|b',\, c|c',\, z) := {}_2F_1^{\mathbb{C}}[a|a',\, b|b',\, c|c',\, z]$$

satisfies the system of equations

$$\mathcal{L}\, R(a|a',\, b|b',\, c|c',\, z) = 0, \qquad \mathcal{L}'\, R(a|a',\, b|b',\, c|c',\, z) = 0. \tag{3.45}$$

Lemma 3.13. *Let $Q = {}_2F_1[a,b;c;z]$ satisfy an equation $\mathcal{L}Q = 0$. Then*

$$\frac{e^{\pi i(c-a-b)}\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} z^{1-c}{}_2F_1[a+1-c, b+1-c; 2-c, z] \tag{3.46}$$

satisfies the same equation.

Remark. The same statement holds for functions

$$u_1 = \frac{\Gamma(c-a)\Gamma(c-a-b)}{\Gamma(c)\Gamma(c-b)} {}_2F_1\left[\begin{matrix} a, b \\ a+b-c+1 \end{matrix}; 1-z\right]; \tag{3.47}$$

$$u_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix}; 1-z\right]; \tag{3.48}$$

$$u_3 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-b)} z^{-a} {}_2F_1\left[\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}; z^{-1}\right]$$

and also for other summands in the right-hand sides of formulas Erdélyi, et al. [7], (2.10.1)–(2.10.4). □

Proof of Lemma 3.13. First, let a, b, c be in a general position. By Erdélyi, et al. [7], (2.10.1), (2.10.5),

$$F(a,b;c;z) = u_1 + u_2, \tag{3.49}$$

where u_1, u_2 are given by (3.47)–(3.48). The function u_2 is ramified at $z = 1$. Passing around this point we get a function

$$\widetilde{F} := u_1 + e^{2\pi i(c-a-b)} u_2.$$

By the analytic continuation, $\mathcal{L}\widetilde{F} = 0$. The factor $e^{2\pi i(c-a-b)}$ does not change under the shifts T_a, T_b, T_c. Therefore the summands u_1, u_2 satisfy the same equation, $\mathcal{L}u_1 = 0$, $\mathcal{L}u_2 = 0$. We apply the same transformation (3.49) to the summand u_1 and repeat the same reasoning. We observe that

$$\frac{\pi\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)\sin\pi(a+b-c)} z^{1-c} {}_2F_1[a+1-c, b+1-c; 2-c, z]$$

satisfies the same equation. This expression differs from (3.49) by the factor $e^{i\pi(a+b-c)}\sin\pi(a+b-c)$, which is invariant under the shifts T_a, T_b, T_c.

Passing to a limit we omit restrictions to a, b, c. $\square$

Proof of Proposition 3.12. We use the expression (3.26) for ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$. Obviously, the first summand satisfies the system (3.45). By Lemma 3.13, the expression

$$\frac{e^{\pi i(c-a-b)}\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} z^{1-c} {}_2F_1\begin{bmatrix} a+1-c, b+1-c \\ 2-c \end{bmatrix}; z\Bigg] \times$$
$$\times \frac{e^{\pi i(c'-a'-b')}\Gamma(c')\Gamma(c'-1)}{\Gamma(a')\Gamma(b')\Gamma(c'-a')\Gamma(c'-b')} \overline{z}^{1-c'} {}_2F_1\begin{bmatrix} a'+1-c', b'+1'-c \\ 2-c' \end{bmatrix}; \overline{z}\Bigg].$$

satisfies the system (3.45). It differs from the second summand in (3.26) by a factor

$$\frac{i^0 \sin\pi a' \sin\pi b' \sin\pi(c'-a')\sin\pi(c'-b')}{\sin\pi c' \sin\pi(c'-1)}.$$

This expression is invariant with respect to shifts T_a, $T_{a'}$, ... Therefore the second summand in (3.26) also satisfies the system. $\square$

3.9. One difference operator. By [29], formula (2.3), the Gauss hypergeometric function ${}_2F_1[p,q;r;z]$ satisfies the following difference equation

$$\begin{aligned} -\, y {}_2F_1(p,q;r;z) = \\ &= \frac{q(r-p)}{(q-p)(1+q-p)}\, {}_2F_1(p-1,q+1;r;z) - \\ &- \Big[\frac{q(r-p)}{(q-p)(1+q-p)} + \frac{p(r-q)}{(p-q)(1+p-q)}\Big] {}_2F_1(p,q;r;z) + \\ &+ \frac{p(r-q)}{(p-q)(1+p-q)}\, {}_2F_1(p+1,q-1;r;z). \end{aligned} \tag{3.50}$$

Define difference operators L, L' acting on functions in variables $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, z by

$$L = \frac{b(c-a)}{(b-a)(1+b-a)}\big(T_a^{-1}T_b - 1\big) - \frac{a(c-b)}{(a-b)(1+a-b)}\big(T_a T_b^{-1} - 1\big); \tag{3.51}$$

$$L' = \frac{b'(c'-a')}{(b'-a')(1+b'-a')}\big(T_{a'}^{-1}T_{b'} - 1\big) - \frac{a'(c'-b')}{(a'-b')(1+a'-b')}\big(T_{a'}T_{b'}^{-1} - 1\big). \tag{3.52}$$

Corollary 3.14. *The complex hypergeometric function ${}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]$ satisfies the following system of difference equations*

$$L\ {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] = z\ {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]; \tag{3.53}$$

$$L'\ {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z] = \overline{z}\ {}_2F_1^{\mathbb{C}}[\mathbf{a},\mathbf{b};\mathbf{c};z]. \tag{3.54}$$

3.10. Some properties of the kernel $\mathcal{K}$. We have the following corollaries from our previous considerations.

1) By (3.10) $\mathcal{K}_{a,b}$ *is even,*

$$\mathcal{K}_{a,b}(z;-k,-\sigma) = \mathcal{K}_{a,b}(z;k,\sigma). \tag{3.55}$$

2) By (3.8),

$$\overline{\mathcal{K}_{a,b}(z;k,-\overline{\sigma})} = \mathcal{K}_{a,b}(z;k,\sigma). \tag{3.56}$$

In particular, $\mathcal{K}_{a,b}(z;k,\sigma)$ *is real on* Λ.

3) By Proposition 3.7, $\mathcal{K}_{a,b}(z;k,\sigma)$ *satisfies the differential equations*

$$\mathfrak{D}\,\mathcal{K}_{a,b}(z;k,\sigma) = \tfrac{1}{4}(k+\sigma)^2\,\mathcal{K}_{a,b}(z;k,\sigma); \tag{3.57}$$

$$\overline{\mathfrak{D}}\,\mathcal{K}_{a,b}(z;k,\sigma) = \tfrac{1}{4}(k-\sigma)^2\,\mathcal{K}_{a,b}(z;k,\sigma). \tag{3.58}$$

4) By Corollary 3.14, $\mathcal{K}_{a,b}(z;k,\sigma)$ *satisfies the difference equations*

$$\mathfrak{L}\,\mathcal{K}_{a,b}(z;k,\sigma) = z\,\mathcal{K}_{a,b}(z;k,\sigma); \tag{3.59}$$

$$\overline{\mathfrak{L}}\,\mathcal{K}_{a,b}(z;k,\sigma) = \overline{z}\,\mathcal{K}_{a,b}(z;k,\sigma). \tag{3.60}$$

4. Nonexistence of commuting self-adjoint extensions

Here we prove that for $(a,b)\notin\Pi$ the operators $\frac12(\mathfrak{D}+\overline{\mathfrak{D}})$, $\frac{1}{2i}(\mathfrak{D}-\overline{\mathfrak{D}})$ defined on $\mathcal{D}(\dot{\mathbb{C}})$ do not admit commuting self-adjoint extensions. We analyze the set of possible generalized eigenfunctions and show that this set is too small.

4.1. Generalized eigenfunctions. Denote by $\mathcal{D}'(\dot{\mathbb{C}})$ the space of distributions on $\dot{\mathbb{C}}$. We have a nuclear rigging (see [2], Section 14.2)

$$\mathcal{D}(\dot{\mathbb{C}}) \subset L^2(\mathbb{C},\mu_{a,b}) \subset \mathcal{D}'(\dot{\mathbb{C}}),$$

and apply the usual formalism of generalized eigenfunctions, see [2], Chapter 15.

Recall that we have formally symmetric and formally commuting operators

$$D_+ := \tfrac12(\mathfrak{D}+\overline{\mathfrak{D}}), \qquad D_- := \tfrac{1}{2i}(\mathfrak{D}-\overline{\mathfrak{D}})$$

in $L^2(\mathbb{C},\mu)$ (defined on the domain $\mathcal{D}(\dot{\mathbb{C}})$) and the spectral problem

$$\mathfrak{D}\Phi = \zeta\Phi, \qquad \overline{\mathfrak{D}}\Phi = \overline{\zeta}\Phi. \tag{4.1}$$

Suppose that the operators D_+, D_- admit commuting self-adjoint extensions. Then the operator U of spectral decomposition can be written in terms of generalized eigenfunctions. Precisely, there exist a space R equipped with a measure ρ and an injective measurable map $r\mapsto\varphi_r$ from R to $\mathcal{D}'(\dot{\mathbb{C}})$ such that

$$D_+\varphi_r = a(r)\varphi_r, \qquad D_-\varphi_r = b(r)\varphi_r,$$

where $a(r)$, $b(r)$ are real-valued functions, and the pairing

$$Uf(s,t) = \{f,\varphi_{s,t}\}$$

of $f\in\mathcal{D}(\dot{\mathbb{C}})$ and φ_r determines a unitary operator $L^2(\mathbb{C},\mu)\to L^2(R,\rho)$, see [2], Subsect. 15.2.3).

Since the operator $\mathfrak{D}$ is elliptic, generalized eigenfunctions are smooth on $\dot{\mathbb{C}}$, see e.g., [2], Theorem 16.2.1. Therefore in our case generalized eigenfunctions φ_r are usual smooth solutions of the system of differential equations.

We also can identify the measure space R with its image, and so we can think that *the measure ρ is sitting on the space Ω of smooth solutions of systems* (4.1), *where ζ ranges in* $\mathbb{C}$. We intend to show that for any measure ρ on Ω the operator $J : L^2(\Omega, \rho) \to L^2(\mathbb{C}, \mu_{a,b})$ defined by

$$Uh(z) = \int_\Omega h(r)\varphi_r(z)\, d\rho(r)$$

is not unitary. Precisely:

Lemma 4.1. *Let $(a, b) \notin \Pi$. Let ρ be a measure on Ω, and let the corresponding operator U be bounded. Then ρ is an atomic measure supported by a finite set.*

The idea of a proof is simple, it is explained in the next subsection, a formal proof is completed in Subsect. 4.3.

Lemma 4.1 *implies that for $(a, b) \notin \Pi$ the operators D_+, D_- have no commuting self-adjoint extensions.*

4.2. Almost proof of Lemma 4.1. For ζ being in a general position, the system (4.1) has a unique solution, and it has a form ${}_2F_1^{\mathbb{C}}[\cdot; z]$. Denote by Ω_{hyp} the subset of Ω consisting of functions ${}_2F_1^{\mathbb{C}}[\cdot; z]$. We wish to prove the following statement:

Lemma 4.2. *Let $(a, b) \notin \Pi$ and $a + b$, $a - b$, a, $b \notin \mathbb{Z}$. Let ρ be a measure on Ω, and let the corresponding operator U be bounded. Then ρ is atomic on Ω_{hyp}.*

Proof. Set $\zeta = \lambda^2$. Then a hypergeometric solution of the system (4.1) has one of two forms:

$${}_2F_1^{\mathbb{C}}\begin{bmatrix} a+\lambda|a-\overline{\lambda}, a-\lambda|a+\overline{\lambda} \\ a+b|a+b \end{bmatrix}, z\Bigg], \qquad {}_2F_1^{\mathbb{C}}\begin{bmatrix} a+\lambda|a+\overline{\lambda}, a-\lambda|a-\overline{\lambda} \\ a+b|a+b \end{bmatrix}, z\Bigg].$$

In the first case we have $(a + \lambda) - (a - \overline{\lambda}) = 2 \operatorname{Re} \lambda \in \mathbb{Z}$, hence $\lambda = \frac{1}{2}(k + is)$, where $k \in \mathbb{Z}$, $s \in \mathbb{R}$. We come to functions $\mathcal{K}_{a,b}(z; k, is)$.

In the second case we have $\lambda - \overline{\lambda} \in \mathbb{Z}$, i.e., $\lambda = \tau \in \mathbb{R}$. We come to functions

$$\mathcal{K}(z; 0, \tau) = {}_2F_1^{\mathbb{C}}\left[\begin{matrix} a+\tau|a+\tau, a-\tau|a-\tau \\ a+b|a+b \end{matrix}, z\right]. \tag{4.2}$$

Next, we will show that

$$\text{the measure } \rho \text{ is zero on the set of all } \lambda = \tfrac{1}{2}(k + is) \text{ with } s \neq 0. \tag{4.3}$$

Our kernel has the following asymptotics at $z = 0$ and $z = 1$:

$$\mathcal{K}(z; k, is) = \big(1 + O(z)\big) + B(k, is)|z|^{2-2a-2b}\big(1 + O(z)\big) \qquad \text{as } z \to 0, \tag{4.4}$$

$$\begin{aligned} \mathcal{K}(z; k, is) = &C(k, is)\big(1 + O(1 - z)\big)+ \\ &+ D(k, is)|1 - z|^{2b-2a}\big(1 + O(1 - z)\big) \qquad \text{as } z \to 1, \end{aligned} \tag{4.5}$$

where the coefficients B, C, D are continuous non-vanishing functions on Λ and all $O(\cdot)$ are uniform on compact subsets of Λ (see formulas (3.26)–(3.31)).

To be definite, assume that $a+b>2$. Consider a point $(k_0, is_0)\in\Lambda$, $s_0\ne 0$ and a neighborhood $\mathcal{N}$ of (k_0, is_0). Assume that $\rho(\mathcal{N})>0$. Denote by $I_\mathcal{N}$ the indicator function of the set $\mathcal{N}$. The function $UI_\mathcal{N}$ has the following asymptotics at $z=0$:

$$UI_\mathcal{N}(z)=\alpha\,(1+O(z))+\beta\,|z|^{2-2a-2b}\big(1+O(z)\big)\qquad \text{as } z\to 0.$$

Due to uniformity $O(\cdot)$, for a sufficiently small neighborhood $\mathcal{N}$ we have $\alpha\ne 0$, $\beta\ne 0$. Since $a+b>2$, the actual asymptotics is

$$UI_\mathcal{N}(z)=\beta\,|z|^{2-2a-2b}(1+O(z)).$$

Therefore

$$UI_\mathcal{N}\notin L^2\big(\mathbb{C},|z|^{2a+2b-2}|1-z|^{2a-2b}\,d\overline{\overline{z}}\big).$$

This contradicts to boundedness of U. Thus any point has a neighborhood of zero measure, and this implies claim (4.3) in the case $a+b>1$.

In domains $a+b<0$, $a-b<-1$, $a-b>1$ we get the same effect.

Next, examine the complementary series $\mathcal{K}(z;0,\tau)$ of eigenfunctions, see (4.2). We have the same asymptotics (4.4)–(4.5), we only must write coefficients of form $A(0,\tau)$, $B(0,\tau)$, $C(0,\tau)$, $D(0,\tau)$ in (4.4)–(4.5). These functions have zeros and poles on the axis $\tau\in\mathbb{R}$. The same argument as above shows that if τ_0 is not a zero and not a pole of all our coefficients, then the measure ρ is zero on a sufficiently small neighborhood of τ_0. The set of zeros and poles is countable. This completes a proof of the lemma. $\square$

4.3. Proof of non-self-adjointness. However, our system of differential equations (4.1) has solutions that have not form ${}_2F_1^{\mathbb{C}}$, and enumeration of all possible degenerations is tedious. So we start proof of Lemma 4.1 again. Due to the homographic transformations, without loss of generality we can set

$$a+b>2.$$

First, we examine asymptotics at a neighborhood of $z=0$.

Asymptotics at $z=0$. Non-logarithmic case. If $a+b\ne 2, 3, \dots$, then the equation $\mathfrak{D}\Phi=\lambda^2\Phi$ has two holomorphic solutions,

$$\Psi_1(z):={}_2F_1\begin{bmatrix}a+\lambda, a-\lambda\\ a+b\end{bmatrix};z\Big],\qquad \Psi_2(z):=z^{1-a-b}{}_2F_1\begin{bmatrix}1-b+\lambda,1-b-\lambda\\ 2-a-b\end{bmatrix};z\Big].$$

The equation $\overline{\mathfrak{D}}\Phi=\overline{\lambda}^2\Phi$ has two antiholomorphic solutions

$$\widetilde\Psi_1(\overline z):={}_2F_1\left[\begin{matrix}a+\overline\lambda, a-\overline\lambda\\ a+b\end{matrix};\overline z\right],\qquad \widetilde\Psi_2(\overline z):=\overline z^{\,1-a-b}{}_2F_1\left[\begin{matrix}1-b+\overline\lambda,1-b-\overline\lambda\\ 2-a-b\end{matrix};\overline z\right].$$

Therefore a single-valued solution of the system must have the form

$$A\Psi_1(z)\widetilde\Psi_1(\overline z)+B\Psi_2(z)\widetilde\Psi_2(\overline z).$$

The first term has $L^2(\mathbb{C},\mu_{a,b})$-asymptotics at $z=0$, the second term has non-L^2-asymptotics. Thus the spectral measure ρ is supported by the set of functions of the form $\Psi_1(z)\widetilde\Psi_1(\overline z)$.

Asymptotics at $z=0$. Logarithmic case. Now let $a+b=n=2, 3,\dots$. Then the equation $\mathfrak{D}\Phi=\lambda^2\Phi$ has two holomorphic solutions,

$$\Psi_1(z)={}_2F_1[a+\lambda,a-\lambda;n,z],\qquad \Psi_2(z),$$

where $\Psi_2(z)$ is a logarithmic solution, which has form (3.39). The equation $\mathfrak{D}\Phi = \overline{\lambda}^2\Phi$ has two antiholomorphic solutions,

$$\widetilde{\Psi}_1(\overline{z}) = {}_2F_1[a+\overline{\lambda}, a-\overline{\lambda}; n, \overline{z}], \qquad \widetilde{\Psi}_2(\overline{z}).$$

A single valued solution must have a form

$$A\Psi_1(z)\,\widetilde{\Psi}_1(\overline{z}) + B\Big(\Psi_1(z)\widetilde{\Psi}_2(\overline{z}) + \Psi_2(z)\widetilde{\Psi}_1(\overline{z})\Big).$$

The asymptotics of the second summand is $(z^{-n+1} + \overline{z}^{\,-n+1}) + O(z^{-n+2})$ if $n \geqslant 3$. If $n = 2$ we have $(z^{-1} + \overline{z}^{\,-1}) + O(z^{-\varepsilon})$. We get a non-$L^2$ asymptotics.

Thus, *for $a + b > 2$ the spectral measure is supported by set of functions of the form $\Psi_1(z)\widetilde{\Psi}_1(\overline{z})$.*

Single-valuedness near $z = 1$. Non-logarithmic case. Assume that $a - b \notin \mathbb{Z}$. We apply formulas Erdélyi, et al., [7], (2.10.1), (2.10.5) and write explicit expansions of Ψ_1, $\overline{\Psi}_1$ at $z = 1$.

$$\Psi_1(z) = A_1 G_1(1-z) + A_2(1-z)^{b-a} G_2(1-z);$$
$$\widetilde{\Psi}_1(\overline{z}) = \widetilde{A}_1 \widetilde{G}_1(1-\overline{z}) + \widetilde{A}_2(1-\overline{z})^{b-a} \widetilde{G}_2(1-\overline{z}),$$

where G_1, G_2 are certain series ${}_2F_1$ and coefficients A_1, A_2 are products of gamma functions, see explicit formulas above (3.42)–(3.43). Clearly, the product $\Psi_1(z)\widetilde{\Psi}_1(\overline{z})$ can be single-valued only if $A_2 = \widetilde{A}_2 = 0$, or $A_1 = \widetilde{A}_1 = 0$. Looking to the explicit expressions for gamma-coefficients, we observe that the first case happens if both the hypergeometric series $G_1(z)$, $G_2(z)$ are terminating (i.e., $a - \lambda = 0, -1, \dots$ or $a + \lambda = 0, -1, \dots$, in particular, λ is real). The second variant holds iff both the series $G_2(1-z)$, $\widetilde{G}_2(1-\overline{z})$ are terminating (i.e., $b - \lambda = 0, -1, \dots$ or $b + \lambda = 0, -1, \dots$).

Single-valuedness near $z = 1$. Logarithmic case. Now let $b - a \in \mathbb{Z}$. The transposition $a \leftrightarrow b$ corresponds to a homographic transformation of differential operators, it preserves the condition $a+b \geqslant 2$. Therefore we can assume $m := b-a \geqslant 0$. Represent $\Psi_1(z)$, $\widetilde{\Psi}_1(\overline{z})$ as combinations of basic solutions of the hypergeometric equations at the point $z = 1$,

$$\Psi_1(z) = A\ {}_2F_1[a+\lambda, a-\lambda; b-a+1; z] + B\,\Theta(1-z);$$
$$\widetilde{\Psi}_1(\overline{z}) = \widetilde{A}\ {}_2F_1[a+\overline{\lambda}, a-\overline{\lambda}; b-a+1; \overline{z}] + \widetilde{B}\,\widetilde{\Theta}(1-\overline{z}),$$

where $\Theta(1-z)$ is a logarithmic series of the type (3.39), see Erdélyi, et al., [7], (2.10.12). A straightforward calculation shows that the product $\Psi_1(z)\,\widetilde{\Psi}_1(\overline{z})$ can be single valued near $z = 1$ only if $B = \widetilde{B} = 0$. Therefore $\Psi_1(z)$ is single valued near $z = 1$, and therefore it is a single valued solution of a hypergeometric equation on the whole plane $\dot{\mathbb{C}}$. Hence (see Erdélyi, et al., [7], Subset. 2.2.1) $\Psi_1(z)$ is a polynomial.

Behavior at infinity. Thus the spectral measure ρ is supported by generalized eigenfunctions of the following types

$$p_1(z)\,p_2(\overline{z}), \qquad (1-z)^{b-a|b-a}\,q_1(z)\,q_2(\overline{z}),$$

where p_j, q_j are polynomials. However, our density $\mu_{a,b}(z)$ has a behavior $|z|^{4a-2}$ at infinity and therefore the space L^2 can contain only finite number of orthogonal functions of such type. $\square$

5. Symmetry of differential operators

Here we show that $J^*_{a,b}$ sends $\mathcal{D}_{\mathrm{even}}(\dot\Lambda)$ to $\mathcal{R}_{a,b}$ and verify that $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ are adjoint one to another on $\mathcal{R}_{a,b}$.

In this section we denote by $D_r(u)\subset\mathbb{C}$ (resp. $\overline{D}_r(u)$) the open (resp. closed) disc in $\mathbb{C}$ of radius r with center at u. By $S_r(u)$ we denote the circle $|z-u|=r$.

5.1. The map $J^*_{a,b}$ on the space $\mathcal{D}_{\mathrm{even}}(\dot\Lambda)$. Introduce a natural topology in the space $\mathcal{R}_{a,b}(\dot{\mathbb{C}})$ defined in Subsect. 1.1. Consider the space $\mathcal{R}(0)$ of functions in $\overline{D}_{1/3}(0)$ having the form $\alpha(z)+\beta(z)|z|^{2a+2b-2}$, where $\alpha(z)$, $\beta(z)$ are smooth in $\overline{D}_{1/3}(0)$ up to the boundary. Let $C^\infty_{\mathrm{flat}}(\overline{D}_{1/3}(0))\subset C^\infty(\overline{D}_{1/3}(0))$ be the subspace consisting of all functions that are flat at 0. The space $\mathcal{R}(0)$ is a quotient space

$$\mathcal{R}(0)\simeq\Bigl[C^\infty(\overline{D}_{1/3}(0))\oplus|z|^{2a+2b-2}C^\infty(\overline{D}_{1/3}(0))\Bigr]/C^\infty_{\mathrm{flat}}(\overline{D}_{1/3}(0)).$$

We equip $\mathcal{R}(0)$ with the topology of a quotient space. In the same way we define a topology in the space $\mathcal{R}(1)$ of smooth functions in $\overline{D}_{1/3}(1)$ having the form $\gamma(z)+\delta(z)|1-z|^{2a-2b}$.

We define a topology in $\mathcal{R}_{a,b}$ as a weakest topology such that:

a) Restriction operators

$$\mathcal{R}_{a,b}\to\mathcal{R}(0),\quad \mathcal{R}_{a,b}\to\mathcal{R}(1),\quad \mathcal{R}_{a,b}\to C^\infty\Bigl(\overline{D}_2(0)\setminus(D_{1/3}(0)\cap D_{1/3}(1))\Bigr)$$

are continuous.

b) For all α, β, N the following seminorms are continuous

$$p_{\alpha,\beta,N}(f)=\sup_{\mathbb{C}\setminus D_2(0)}|z|^{2+\alpha+\beta}(\ln|z|)^N\left|\frac{\partial^{\alpha+\beta}f(z)}{\partial z^\alpha\partial\overline{z}^\beta}\right|. \tag{5.1}$$

Recall that $\dot\Lambda:=\Lambda\setminus\{(0,0)\}$.

Lemma 5.1. *For $|z|>2$, $(k,s)\in\dot\Lambda$ we have the following expansion*

$$\mathcal{K}(z;k,is)=z^{-a-\frac{k+is}{2}}\bigr|^{-a-\frac{-k+is}{2}}B(k,s;z^{-1})+ \\ +z^{-a+\frac{k+is}{2}}\bigr|^{-a+\frac{-k+is}{2}}B(-k,-s;z^{-1}), \tag{5.2}$$

where the expression $B(k,s;u)$ for fixed k is smooth s except the point $(k,s)=(0,0)$.

Proof. We refer to expansion (3.32)–(3.34). Notice that for $k=0$, $s=0$ we have a singularity in this expansion (but the kernel itself is analytic at this point). □

Proposition 5.2. a) *Let $\Phi\in\mathcal{D}_{\mathrm{even}}(\dot\Lambda)$. Then $J^*_{a,b}\Phi\in\mathcal{R}_{a,b}$.*

b) *Moreover, the operator $J^*_{a,b}$ is a continuous operator operator from $\mathcal{D}_{\mathrm{even}}(\dot\Lambda)$ to $\mathcal{R}_{a,b}$.*

Proof. Forms of asymptotics of $J^*_{a,b}\Phi$ at 0 and 1 follow from expressions (3.26), (3.29). Let us examine the asymptotics at $z\to\infty$. Without loss of generality we can assume that $|k|$ is fixed. We write

$$J^*_{a,b}\Phi(z)=z^{-a-\frac{k}{2}}|^{-a+\frac{k}{2}}\int_{\mathbb{R}}z^{-\frac{is}{2}}|^{-\frac{is}{2}}B(k,s;z^{-1})\,\Phi(k,s)\,ds+\begin{Bmatrix}\text{similar}\\ \text{term}\end{Bmatrix}.$$

Differentiating the first summand $\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \overline{z}^\beta}$ and keeping in the mind (3.32) and Lemma 3.6, we get an expression of the form

$$\begin{gathered}
z^{-a-\frac{k}{2}-\alpha|-a+\frac{k}{2}-\beta} \sum_{0\leqslant p\leqslant\alpha, 0\leqslant q\leqslant\beta} \int_{\mathbb{R}} z^{-\frac{is}{2}|-\frac{is}{2}} U^{\alpha,\beta}_{p,q}(a,b,k,s)\times \\
\times \frac{\Gamma^{\mathbb{C}}(-k-is|k+is)\,(a+\frac{k+is}{2})_p\,(a+\frac{-k+is}{2})_p\,(a+\frac{k-is}{2})_q\,(a+\frac{-k-is}{2})_q}{\Gamma^{\mathbb{C}}(b-\frac{k+is}{2}|b-\frac{-k+is}{2})\Gamma^{\mathbb{C}}(a+\frac{k+is}{2}|a+\frac{-k+is}{2})\,(a+b)_p\,(a+b)_q}\times \\
\times {}_2F_1\left[\begin{matrix}a+\frac{k+is}{2}+p, a+\frac{-k+is}{2}+p \\ a+b+p\end{matrix}; z^{-1}\right] {}_2F_1\left[\begin{matrix}a+\frac{-k-is}{2}+q, a+\frac{k-is}{2}+q \\ a+b+q\end{matrix}; \overline{z}^{-1}\right]\times \\
\times \Phi(k,s)\,ds,
\end{gathered}$$

where $U^{\alpha,\beta}_{p,q}(a,b,k,s)$ are polynomials. It is easy to verify that the integrand is a smooth compactly supported function on $\dot{\Lambda}$. Next, we write

$$|z|^{-is} = \frac{i}{\ln|z|}\frac{\partial}{\partial s}|z|^{-is},$$

integrate our expansion in parts N times and observe that $p_{\alpha,\beta,N}(J^*_{a,b}\Phi) < \infty$.

The continuity follows from the same considerations. □

As a corollary, we obtain the following lemma.

Lemma 5.3. *The operator $J^*_{a,b}$ is continuous as an operator from $\mathcal{D}_{\mathrm{even}}(\dot{\Lambda})$ to the space $L^2(\mathbb{C},\mu_{a,b})$.*

PROOF. Indeed, for $(a,b)\in\Pi$ the identical embedding $f\mapsto f$ of $\mathcal{R}_{a,b}$ to $L^2(\mathbb{C},\mu_{a,b})$ is continuous. □

Lemma 5.4. *If $f\in\mathcal{R}_{a,b}$, then $\mathfrak{D}f\in\mathcal{R}_{a,b}$.*

PROOF. Let us check a behavior of $\mathfrak{D}f$ at 0, to be definite assume that $a+b\neq 1$. Then near zero we have

$$\begin{aligned}
\mathfrak{D}f = \mathfrak{D}\big(\alpha(z)+\beta(z)|z|^{1-a-b}\big) &= \\
&= \Big\{\Big(z(1-z)\frac{d^2}{dz^2}+(a+b)\frac{d}{dz}\Big)z^{1-a-b}\Big\}\cdot\overline{z}^{1-a-b}\beta(z)+\Big\{\text{the rest}\Big\}.
\end{aligned}$$

Obviously, the rest has the form $\widetilde{\alpha}(z)+\widetilde{\beta}(z)|z|^{2-a-b}$ with smooth $\widetilde{\alpha}$, $\widetilde{\beta}$. The expression in the curly brackets[18] is $-(a+b)(a+b-1)z^{1-a-b}$. □

5.2. Symmetry of differential operators.

Proposition 5.5. *For any f, $g\in\mathcal{R}_{a,b}(\dot{\mathbb{C}})$*

$$\langle\mathfrak{D}f,g\rangle = \langle f,\overline{\mathfrak{D}}g\rangle.$$

PROOF. Let f, $g\in\mathcal{R}_{a,b}$. We wish to show that

$$\int_{\dot{\mathbb{C}}}\big(\mathfrak{D}f(z)\cdot\overline{g(z)} - f(z)\cdot\mathfrak{D}\overline{g(z)}\big)\,\mu_{a,b}\,d\overline{\overline{z}} = 0.$$

This is an improper absolutely convergent integral, we represent it as

$$\lim_{\varepsilon\to 0}\int_{D_{1/\varepsilon}(0)\setminus\big(D_\varepsilon(0)\cup D_\varepsilon(1)\big)}(\dots)\,\frac{dz\wedge d\overline{z}}{2i}.$$

[18] Cf. [9], Sect.I.2.

Next, we integrate two times by parts in z (with the Green formula) and after a simple calculation come to

$$\lim_{\varepsilon\to 0}\Bigl\{\int_{S_{1/\varepsilon}(0)} V(z)d\,\overline{z} - \int_{S_\varepsilon(0)} V(z)\,d\overline{z} - \int_{S_\varepsilon(1)} V(z)\,d\overline{z}\Bigr\}, \tag{5.3}$$

where

$$V(z) = \Bigl(\frac{\partial f}{\partial z}\cdot\overline{g(z)} - f(z)\cdot\frac{\partial\overline{g(z)}}{\partial z}\Bigr)z(1-z)\,\mu_{a,b}(z).$$

We claim that all summands in (5.3) tend to 0. For the first summand this is clear. For the second summand we represent f, $\overline{g}$ as

$$f(z) = \alpha(z) + \beta(z)\,z^{1-a-b|1-a-b}, \qquad \overline{g}(z) = \gamma(z) + \delta(z)\,z^{1-a-b|1-a-b}.$$

Then $V(z)$ transforms to an expression of the following type:

$$\Bigl(A(z) + B(z)z^{-a-b|1-a-b} + C(z)z^{2-2a-2b|2-2a-2b}\Bigr)\times \\ \times z(1-z)\cdot z^{a+b-1|a+b-1}(1-z)^{a-b|a-b},$$

where $A(z)$, $B(z)$, $C(z)$ are smooth near 0. We emphasize that the term with $z^{1-2a-2b|2-2a-2b}$ in the bracket appears with coefficient

$$(2-2a-2b)\bigl(\beta(z)\delta(z) - \beta(z)\delta(z)\bigr) = 0.$$

Thus we get summands with the following behavior at 0:

$$\sim A(0)\,z^{a+b|a+b-1}, \qquad \sim B(0)\,z^0, \qquad \sim C(0)\,z^{2-a-b|1-a-b}.$$

Since $0 < a+b < 2$ all powers > -1 and therefore[19] $\int_{|z|=\varepsilon}(\dots)\,d\overline{z}$ tends to 0. $\square$

6. The operator $J^*_{a,b}$ is an isometry

Here we prove half of Theorem 1.3.

6.1. The statement. First, denote by Λ_+ the subset of Λ consisting of $(k+is)/2$ such that $k>0$ or $k=0$ and $s>0$. We have an obvious identification $\mathcal{D}_{\mathrm{even}}(\dot\Lambda) \simeq \mathcal{D}(\Lambda_+)$.

Lemma 6.1. *Let $u(\lambda)$, $v(\lambda)$ be smooth compactly supported function on Λ_+. Then*

$$\langle J^*_{a,b}u, J^*_{a,b}v\rangle_{L^2(\dot{\mathbb{C}},\mu_{a,b})} = 2\langle u, v\rangle_{L^2(\Lambda_+,\varkappa_{a,b})}.$$

Our proof is based on heuristic arguments outlined in Berezin, Shubin [3], Section 2.6 for ordinary differential operators. However, this way is tiresome.

6.2. Preliminary remarks. Recall that

$$J^*_{a,b}u(z) = 2\int_{\Lambda_+} u(\lambda)\,\mathcal{K}(z,\lambda)\,\varkappa_{a,b}(\lambda)\widetilde{d}\lambda.$$

By Lemma 5.3, this operator is continuous as an operator $\mathcal{D}(\dot\Lambda_+) \to L^2(\dot{\mathbb{C}},\mu_{a,b})$. Therefore the sesquilinear form

$$T(u,v) := \langle J^*_{a,b}u, J^*_{a,b}v\rangle_{L^2(\dot{\mathbb{C}},\mu_{a,b})} \tag{6.1}$$

---

[19] See a discussion of a parallel situation for ordinary differential operators in [31], Section 1. However, in one-dimensional case we must impose boundary conditions in such points.

is continuous as a form $\mathcal{D}(\Lambda_+)\times\mathcal{D}(\Lambda_+)\to\mathbb{C}$. By the kernel theorem (see, e.g., [17], Sect. 5.2) it is determined by a distribution. Formally, we transform (6.1) as

$$\int_{\dot{\mathbb{C}}}\Bigl(\int_{\Lambda_+} u(\lambda)\mathcal{K}(z,\lambda)\varkappa_{a,b}(\lambda)\,\widetilde{d}\lambda\Bigr)\cdot\Bigl(\int_{\Lambda_+}\overline{v(\nu)}\mathcal{K}(z,\nu)\varkappa_{a,b}(\nu)\,\widetilde{d}\nu\Bigr)\,\mu_{a,b}(z)\,d\overline{\overline{z}}= \tag{6.2}$$

$$=\int_{\Lambda_+}\int_{\Lambda_+} u(\lambda)\overline{v(\nu)}H(\lambda,\nu)\varkappa_{a,b}(\lambda)\varkappa_{a,b}(\nu)\,\widetilde{d}\lambda\,\widetilde{d}\nu, \tag{6.3}$$

where

$$H(\lambda,\nu)=\int_{\dot{\mathbb{C}}}\mathcal{K}(z,\lambda)\,\mathcal{K}(z,\nu)\,\mu_{a,b}(z)\,d\overline{\overline{z}}. \tag{6.4}$$

Notice that all integrals in line (6.2) converge absolutely. However, the triple integral $\int_{\Lambda_+}\int_{\Lambda_+}\int_{\dot{\mathbb{C}}}$ is not absolutely convergent. The integrand in (6.4) decreases as $|z|^{-2}$ and the integral diverges.

However, we regard $H(\lambda,\nu)$ as a distribution, then Lemma 6.1 can be reformulated in the form:

Lemma 6.2. *We have the following identity of distributions on* $\mathcal{D}(\Lambda_+)\times\mathcal{D}(\Lambda_+)$*:*

$$H(\lambda,\nu)=\delta(\lambda-\nu). \tag{6.5}$$

6.3. Orthogonality of packets.

Lemma 6.3. *Let* u, $v\in\mathcal{D}(\Lambda_+)$ *and* (*closed*) *supports* $\mathrm{supp}(u)$ *and* $\mathrm{supp}(v)$ *have empty intersections. Then*

$$\langle J^*_{a,b}u, J^*_{a,b}v\rangle_{L^2(\dot{\mathbb{C}},\mu_{a,b})}=0.$$

Proof. Denote $D_+:=\frac{1}{2}(\mathfrak{D}+\overline{\mathfrak{D}})$, $D_-=\frac{1}{2i}((\mathfrak{D}-\overline{\mathfrak{D}})$. By Proposition 5.2, $J^*_{a,b}u$ is contained in the space $\mathcal{R}_{a,b}$. By Proposition 5.5, the operators D_+, D_- are formally symmetric on $\mathcal{R}_{a,b}$. Since they formally commute, for any real polynomial $p(D_+,D_-)$ we have

$$\langle p(D_+,D_-)J^*_{a,b}u, J^*_{a,b}v\rangle=\langle J^*_{a,b}u, p(D_+,D_-)J^*_{a,b}v\rangle,$$

or

$$\langle J^*_{a,b}\,p(\operatorname{Re}\lambda,\operatorname{Im}\lambda)\cdot u, J^*_{a,b}v\rangle=\langle J^*_{a,b}u, J^*_{a,b}\,p(\operatorname{Re}\lambda,\operatorname{Im}\lambda)\cdot v\rangle, \tag{6.6}$$

where $\cdot$ denotes the operator of multiplication by a function. We choose a sequence p_N of polynomials such that p_N uniformly converges to 1 on $\mathrm{supp}(u)$ with all derivatives and converges to 0 on $\mathrm{supp}(v)$. By Lemma 5.3 the map $J^*_{a,b}$ is continuous as a map $\mathcal{D}(\Lambda_+)\to L^2(\mathbb{C},\mu_{a,b})$. Replacing p by p_N in (6.6) and passing to a limit, we come to the desired statement. □

6.4. Next reduction of our statement. Let $S(u,v)$ be an Hermitian form on $\mathcal{D}(\Lambda_+)$. We say that S is C^ω*-smooth*, if it has a form

$$S(u,v)=\int_{\Lambda_+}\int_{\Lambda_+} M(\lambda,\nu)\,u(\lambda)\overline{v(\nu)}\,\widetilde{d}\lambda\,\widetilde{d}\nu,$$

where M is a real analytic function on $\Lambda_+\times\Lambda_+$.

Lemma 6.4. *We have*

$$\langle J^*_{a,b}u, J^*_{a,b}v\rangle_{L^2(\mathbb{C},\mu_{a,b})}=\langle u,v\rangle_{L^2(\Lambda,\varkappa_{a,b})}+S(u,v), \tag{6.7}$$

where $S(u,v)$ *is* C^ω*-smooth.*

This lemma together with Lemma 6.3 imply the desired statement, i.e., the identity (6.5). Indeed, for any u, v with disjoint support, we have

$$\int_{\Lambda_+}\int_{\Lambda_+} M(\lambda,\nu)\, u(\lambda)\, \overline{v}(\nu)\, \varkappa_{a,b}(\lambda)\varkappa_{a,b}(\nu)\, \widetilde{d}\lambda\, \widetilde{d}\nu = 0,$$

therefore $M(\lambda,\nu)=0$.

The rest of this section is occupied by a proof of Lemma 6.4.

6.5. The beginning of proof of Lemma 6.4. Cleaning of the problem. *Step 1.* Represent

$$u=\sum_k u_k\,\delta(\operatorname{Re}\lambda - k/2), \qquad v=\sum_l v_l\,\delta(\operatorname{Re}\lambda - l/2),$$

in fact the sums are finite and u_k, v_l depend on a real variable s. By Lemma 6.3, we have

$$\langle J^*_{a,b}u_k, J^*_{a,b}v_l\rangle = 0 \qquad \text{for } k\ne l.$$

Therefore it is sufficient to examine only inner products

$$\langle J^*_{a,b}u_k, J^*_{a,b}v_k\rangle = \int_{\dot{\mathbb{C}}} R(z)\; d\overline{\overline{z}},$$

where

$$R(z) := \int_{\Lambda_+} u_k(is)\,\mathcal{K}\bigl(z,\tfrac12(k+is)\bigr)\varkappa_{a,b}\bigl(\tfrac12(k+is)\bigr)\, ds\times \\ \times \int_{\Lambda_+} \overline{v_k(it)}\,\mathcal{K}\bigl(z,\tfrac12(k+it)\bigr)\varkappa_{a,b}\bigl(\tfrac12(k+it)\bigr)\, dt\; \mu_{a,b}(z).$$

Step 2. Represent the integral as $\int_{|z|\leqslant 2} R + \int_{|z|\geqslant 2} R$.

Let us show that the first summand is C^ω-smooth. In this case the triple integral absolutely converges and can be represented as

$$\int_{|z|\leqslant 2} R\; d\overline{\overline{z}} = \int_{\mathbb{R}}\int_{\mathbb{R}} u_k(is)\overline{v_k(it)} L(s,t)\, ds\, dt,$$

where

$$L(s,t)=\int_{|z|\leqslant 2} \mathcal{K}\bigl(z,\tfrac12(k+is)\bigr)\,\mathcal{K}\bigl(z,\tfrac12(k+it)\bigr)\,\mu_{a,b}(z)\; d\overline{\overline{z}}.$$

Integrand makes sense for complex s, t that are sufficiently close to $\mathbb{R}$ and the integral absolutely converges (singularities at $z=0$ and 1 have forms (4.4), (4.5)). Therefore $L(s,t)$ is a holomorphic function in s, t near $\mathbb{R}\times\mathbb{R}$.

Therefore our question is reduced to an examination the integral

$$\int_{|z|>2} R(z)\; d\overline{\overline{z}}$$

Step 3. *A decomposition of the kernel.* Applying Theorem 3.9.c, we represent $\mathcal{K}(z,\lambda)$ in the domain $|z|\geqslant 2$ as

$$\text{(6.8)}\quad \mathcal{K}(z,\lambda) = W_1+W_2+W_3 := \\ = A(\lambda)(-z)^{-a-\lambda|-a+\overline{\lambda}} + A(-\lambda)(-z)^{-a+\lambda|-a-\overline{\lambda}} + \Psi(z,\lambda),$$

where

$$A(\lambda)=\frac{\Gamma^{\mathbb{C}}(2\lambda|-2\overline{\lambda})}{\Gamma^{\mathbb{C}}(b-\lambda|b+\overline{\lambda})\Gamma^{\mathbb{C}}(a-\lambda|a+\overline{\lambda})}$$

and

$$\Psi(z,\lambda)=O(|z|^{-2a-1}) \qquad \text{as } z\to\infty.$$

Notice that

$$|A(\lambda)|^2=A(\lambda)\,A(-\lambda)=\varkappa_{a,b}^{-1}(\lambda). \tag{6.9}$$

Therefore the integral $\int_{|z|>2}R(z)\,d\overline{\overline{z}}$ splits into a sum of 9 summands $V_{\alpha\beta}$,

$$V_{\alpha\beta}:=\int_{|z|>2}\int_{\mathbb{R}}W_\alpha(z;k,s)u_k(is)\,\varkappa_{a,b}\big(\tfrac12(k+is)\big)\,ds\times \\ \times\int_{\mathbb{R}}W_\beta(z;k,t)v_k(it)\,\varkappa_{a,b}\big(\tfrac12(k+it)\big)\,dt\cdot\mu_{a,b}(z)\;d\overline{\overline{z}}.$$

Step 4. For five summands V_{13}, V_{23}, V_{31}, V_{32}, V_{33} we immediate get absolute convergence of triple integrals and C^ω-smoothness. For instance,

$$V_{13}=\int_{\mathbb{R}}\int_{\mathbb{R}}u_k(is)\overline{u_k(it)}\,A\big(\tfrac12(k-is)\big)^{-1}\times \\ \times\left[\int_{|z|\geqslant 2}\Big(\frac{z}{\overline{z}}\Big)^{-k}|z|^{-2a+is}\,\overline{\Psi\big(z,\tfrac12(k+is)\big)}\,\mu_{a,b}(z)\,d\overline{\overline{z}}\right]ds\,dt.$$

(we simplified the integrand with (6.9)). The expression in the square brackets is real analytic (the integrand decreases as $|z|^{-3}$).

Step 5. Non-obvious summands are V_{11}, V_{12}, V_{21}, V_{22}. We start with V_{11},

$$V_{11}:=\int_{|z|\geqslant 2}\int_{\mathbb{R}}u_k(is)A\big(\tfrac12(k+is)\big)\Big(\frac{z}{\overline{z}}\Big)^{-k/2}|z|^{-2a-is}\varkappa_{a,b}\big(\tfrac12(k+is)\big)\,ds\times \\ \times\int_{\mathbb{R}}\overline{A\big(\tfrac12(k+it)\big)\,v_k(it)}\Big(\frac{z}{\overline{z}}\Big)^{k/2}|z|^{-2a+it}\varkappa_{a,b}\big(\tfrac12(k+it)\big)\,dt\;\mu_{a,b}(z)\;d\overline{\overline{z}}.$$

For $k=0$ we must keep in mind that the integration $\int_{\mathbb{R}}$ actually is taken over a ray $[\varepsilon,\infty)$ for some $\varepsilon>0$. Applying (6.9), we come to

$$V_{11}:=\int\limits_{|z|\geqslant 2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}u_k(is)\overline{v_k(it)}A\big(\tfrac12(k-is)\big)^{-1}A\big(\tfrac12(k+it)\big)^{-1}|z|^{-4a-is+it}\,ds\,dt\times \\ \times\,\mu_{a,b}(z)\;d\overline{\overline{z}}. \tag{6.10}$$

Next, we notice that

$$\mu_{a,b}(z)=|z|^{2a+2b-2}|1-z|^{2a-2b}=|z|^{4a-2}+O(|z|^{4a-3}) \qquad \text{as } z\to\infty.$$

We write

$$\mu_{a,b}(z)=|z|^{4a-2}+(\mu_{a,b}(z)-|z|^{4a-2}), \tag{6.11}$$

substitute this to (6.10) and decompose (6.10) as a sum of two integrals. The second summand immediately gives a C^ω-smooth term. The first summand is the topic of our interest. It equals the following expression

$$I(u,v) := \int\limits_{|z|\geqslant 2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} u_k(is)\overline{v_k(it)}A\bigl(\tfrac{1}{2}(k-is)\bigr)^{-1}A\bigl(\tfrac{1}{2}(k+it)\bigr)^{-1}\times \times \boxed{|z|^{-2-is+it}}\,ds\,dt\,d\overline{\overline{z}}. \tag{6.12}$$

6.6. Application of the Sokhotski formula and disappearance of a singular term.

Step 6. Extension to the complex domain. Now consider a function $I(u,v,\varepsilon)$ obtained by replacing $s\mapsto s-i\varepsilon$ in the boxed term, $\varepsilon>0$. The new triple integral absolutely converges, we can change the order of integrations and explicitly integrate in z. We get

$$I(u,v,\varepsilon)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} u_k(is)\overline{v_k(it)}A\bigl(\tfrac{1}{2}(k-is)\bigr)^{-1}A\bigl(\tfrac{1}{2}(k+it)\bigr)^{-1}\frac{2^{-is-\varepsilon+it}}{-is-\varepsilon+it}\,ds\,dt.$$

Next, we claim that

$$I(u,v)=\lim_{\varepsilon\to+0} I(u,v,\varepsilon).$$

Indeed, we integrate $I(u,v,\varepsilon)$ two times by parts in s and come to

$$I(u,v,\varepsilon)=\int\limits_{|z|\geqslant 2}\int\limits_{\mathbb{R}}\overline{v_k(it)}A\bigl(\tfrac{1}{2}(k+it)\bigr)^{-1}\int\limits_{\mathbb{R}}\frac{\partial^2}{\partial s^2}\Bigl[u_k(is)A\bigl(\tfrac{1}{2}(k-is)\bigr)^{-1}\Bigr]\times \times\frac{|z|^{-2-\varepsilon-is+it}}{i^2\ln^2|z|}\,ds\,dt\,d\overline{\overline{z}}$$

The new triple integral absolutely converges and is continuous at $\varepsilon=+0$.

Thus we come to the so-called distribution $\frac{1}{x-i\varepsilon}$, see, e.g., [13]. Recall the Sokhotski formula

$$\lim_{\varepsilon\to+0}\int_\alpha^\beta\frac{f(y)\,dy}{x-y-i\varepsilon}=\text{p.v.}\int_\alpha^\beta\frac{f(y)\,dy}{x-y}+\pi i f(x), \tag{6.13}$$

where p.v. denotes the principal value of an integral.

Applying this formula and keeping in mind (6.9), we come to

$$I(u,v)=\text{p.v.}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} u_k(is)\overline{v_k(it)}\,A\bigl(\tfrac{1}{2}(k-is)\bigr)^{-1}A\bigl(\tfrac{1}{2}(k+it)\bigr)^{-1}\frac{2^{-is+it}}{-is+it}\,ds\,dt+ +\pi\int_{\mathbb{R}} u_k(is)\overline{v_k(is)}\,\varkappa_{a,b}\bigl(\tfrac{1}{2}(k+is)\bigr)\,ds. \tag{6.14}$$

Step 8. We elaborate V_{22} in the same way and come to

$$V_{22}=\text{p.v.}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} u_k(is)\overline{v_k(it)}A\bigl(\tfrac{1}{2}(k-it)\bigr)^{-1}A\bigl(\tfrac{1}{2}(k+is)\bigr)^{-1}\frac{2^{-it+is}}{-it+is}\,ds\,dt+ +\pi\int_{\mathbb{R}} u_k(is)\overline{v_k(is)}\varkappa_{a,b}\bigl(\tfrac{1}{2}(k+is)\bigr)\,ds+\Bigl\{\text{a } C^\omega\text{-smooth term}\Bigr\}. \tag{6.15}$$

Next, we take the sum $V_{11}+V_{22}$ modulo C^ω-smooth terms. The expression

$$\frac{A\big(\frac12(k-is)\big)^{-1}A\big(\frac12(k+it)\big)^{-1}2^{-is+it}-A\big(\frac12(k-it)\big)^{-1}A\big(\frac12(k+is)\big)^{-1}2^{-it+is}}{-i(s-t)}$$

has a form

$$\frac{L(t,s)-L(s,t)}{s-t}$$

with analytic $L(t,s)$. It has a removable singularity on the line $t=s$. Thus the first summands in (6.14) and (6.15) give us a C^ω-smooth term, the second summands give us the first term in (6.7), i.e., the desired delta-function.

6.7. End of proof of Lemma 6.4.

Step 9. Next, we examine the term V_{12}. We write the integral and apply the transformation (6.11). We get a sum of a C^ω-smooth term and the integral

$$J(u,v)=\int_{|z|\geqslant 2}\int_{\mathbb{R}}\int_{\mathbb{R}} u_k(is)\overline{v_k(it)}A\big(-\tfrac12(k+is)\big)^{-1}A\big(-\tfrac12(k+it)\big)^{-1}\times \\ \times\Big(\frac{z}{\overline{z}}\Big)^k\boxed{|z|^{-2-is-it}}\,d\overline{\overline{z}}\,dt\,ds.$$

As above, we change $s\mapsto s-i\varepsilon$ in the box and get integrals $J(u,v,\varepsilon)$ with $\varepsilon>0$. As above,

$$J(u,v;\varepsilon)=\int_{\mathbb{R}}\int_{\mathbb{R}} u_k(is)\overline{v_k(it)}A\big(-\tfrac12(k+is)\big)^{-1}A\big(-\tfrac12(k+it)\big)^{-1}\times \\ \times\Big[\int_{|z|\geqslant 2}\Big(\frac{z}{\overline{z}}\Big)^{-k}|z|^{-2-\varepsilon-is-it}\,d\overline{\overline{z}}\Big]\,dt\,ds.$$

If $k>0$, then the term in square brackets is zero (we pass to polar coordinates and get 0 after the integration with respect to the angle coordinate). If $k=0$, then we get

$$\frac{2^{-\varepsilon-is-it}}{\varepsilon+i(s+t)}.$$

However, $\mathrm{supp}(u_0)$, $\mathrm{supp}(v_0)$ are contained in domains $s>0$, $t>0$, and actually we have no singularity. Thus V_{12} is C^ω-smooth.

The same examination shows C^ω-smoothness of V_{21}. This completes a proof of Lemma 6.4. □

7. Asymptotics of the kernel in parameters

7.1. The statement. Let us modify a notation for the kernel $\mathcal{K}$. Set

$$\mathcal{K}^\circ(z;\lambda;\sigma):= \\ :=\frac{1}{\Gamma^{\mathbb{C}}(a+b|a+b)}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}a+\lambda+\frac{\sigma}{2}|a-\overline{\lambda}+\frac{\sigma}{2},\ a-\lambda-\frac{\sigma}{2}|a+\overline{\lambda}-\frac{\sigma}{2}\\ a+b|a+b\end{matrix};z\right]= \\ =\frac{1}{\Gamma^{\mathbb{C}}(a+b|a+b)}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix}a+\frac{k+\sigma+is}{2}|a+\frac{-k+\sigma+is}{2},\ a+\frac{-k-\sigma-is}{2}|a+\frac{k-\sigma-is}{2}\\ a+b|a+b\end{matrix};z\right],$$

where $\lambda\in\Lambda$, $\sigma\in\mathbb{R}$. In fact,

$$\mathcal{K}^\circ\big(z;\tfrac{k+is}{2};\sigma\big)=\mathcal{K}(z;k,\sigma+is).$$

However, in calculations of this section the variables σ and is have different meanings.

Denote

$$t_{\pm}(z) = 1 \pm \sqrt{1 - 1/z}.$$

Theorem 7.1. *Then for a fixed z we have the following asymptotic expansion*

$$(7.1)\quad \mathcal{K}^\circ(z;\lambda;\sigma) = \frac{1}{\Gamma^{\mathbb{C}}\big(a-\lambda-\frac{\sigma}{2}|a+\overline{\lambda}-\frac{\sigma}{2}\big)\,\Gamma^{\mathbb{C}}\big(b+\lambda+\frac{\sigma}{2}|b-\overline{\lambda}+\frac{\sigma}{2}\big)\cdot|\lambda|}\times$$
$$\times |1-1/z|^{-1/2}\cdot|1-z|^{b-a}\cdot|z|^{-a-b}\times$$
$$\times\Bigg[\Big(\frac{t_-(z)}{t_+(z)}\Big)^{\frac{\sigma}{2}+\lambda|\frac{\sigma}{2}-\overline{\lambda}}\sum_{k\geqslant 0,\, l\geqslant 0,\, k+l<N}\frac{\overline{\lambda}^{-k}\lambda^{-l}}{k!\,l!}A_k(\sigma,\sqrt{1-z})\,A_l(\sigma,\sqrt{1-\overline{z}})+$$
$$+\Big(\frac{t_+(z)}{t_-(z)}\Big)^{\frac{\sigma}{2}+\lambda|\frac{\sigma}{2}-\overline{\lambda}}\sum_{k\geqslant 0,\, l\geqslant 0,\, k+l<N}\frac{\overline{\lambda}^{-k}\lambda^{-l}}{k!\,l!}A_k(\sigma,-\sqrt{1-z})\,A_l(\sigma,-\sqrt{1-\overline{z}})\Bigg]+$$
$$+R_N(z,\sigma,\lambda),$$

where $A_k(\xi)$ are rational expressions in ξ (depending on parameters a, b) having poles at $\xi = 0$, ± 1 and $A_0 = 1$. The reminder $R_N(z)$ satisfies

$$R_N(z,\sigma,\lambda) = O(|\lambda|^{-N}), \qquad \text{as } \lambda\to\infty,$$

moreover $O(\cdot)$ is uniform in z and σ on compact subsets in $\dot{\mathbb{C}}\times\mathbb{R}$.

The proof occupies the rest of this section.

Remark. This formula is counterpart of Watson's [39] formulas for asymptotics of the Gauss hypergeometric functions ${}_2F_1[a-\lambda, b+\lambda; c; z]$ in parameters λ (see an exposition of Watson's results in [21], Sect. 7.2, see also a remark in [36], p.162, on typos in [39]). We do not see a way to reduce our statement to Watson's work. ☒

Remark. Lemma 2.1 gives us an asymptotics of the gamma-factor in (7.1). ☒

7.2. Stationary phase approximation. We transform $\mathcal{K}^\circ(z,\lambda,\sigma)$ as

$$(7.2)\quad \frac{1}{\Gamma^{\mathbb{C}}\big(a-\lambda-\frac{\sigma}{2}|a+\overline{\lambda}-\frac{\sigma}{2}\big)\,\Gamma^{\mathbb{C}}\big(b+\lambda+\frac{\sigma}{2}|b-\overline{\lambda}+\frac{\sigma}{2}\big)\cdot|\lambda|}\int_{\mathbb{C}} R(t,z,\sigma)\exp\Big\{Q(t,z,\lambda,\sigma)\Big\}\,d\,\overline{\overline{t}},$$

where

$$(7.3)\qquad R(t,z) := t^{a-\frac{\sigma}{2}-1|a-\frac{\sigma}{2}-1}(1-t)^{b+\frac{\sigma}{2}-1|b+\frac{\sigma}{2}-1}(1-tz)^{-a-\frac{\sigma}{2}}$$

and

$$(7.4)\quad Q(t,z,\lambda) := \lambda\ln\Big(\frac{t(1-zt)}{1-t}\Big) - \overline{\lambda}\,\overline{\ln\Big(\frac{t(1-zt)}{1-t}\Big)} =$$
$$= ik\,\mathrm{Im}\ln\Big(\frac{t(1-zt)}{1-t}\Big) + is\ln\Big(\frac{t(1-zt)}{1-t}\Big).$$

The function $\mathrm{Im}\ln(\dots)$ is ramified, however the exponent is well-defined and formulas below contain only partial derivatives of $\ln(\dots)$, which are independent of a choice of a branch.

We apply the stationary phase approximation, see, e.g., Fedoryuk [8]. Singular points are 0, 1, ∞. Stationary points are

$$t_{\pm} = 1 \pm \sqrt{1-1/z},$$

they are the same for the both summands in (7.4). This can be a fatal obstacle for an evaluation of a uniform asymptotics, however this does not happen. Also the domain of convergence of the integral (7.2) is smaller than it is necessary for our purposes.

Consider a partition of unity

$$1 = \rho_0 + \rho_1 + \rho_{z^{-1}} + \rho_\infty + \rho_{t_+} + \rho_{t_-} + \tau,$$

where ρ_α is zero outside a small neighborhood of α, and τ is zero in neighborhoods of $0,\ 1,\ z^{-1},\ \infty,\ t_\pm$. According this partition we expand (7.2) into a sum of 6 integrals,

$$I = I_0 + I_1 + I_{z^{-1}} + I_\infty + I_{t_+} + I_{t_-} + J.$$

Obviously (see [8], Lemma III.2.1), for each N we have

$$J = O(k^2+s^2)^{-N} \qquad \text{as } n+is \to \infty.$$

7.3. Preparatory statement.

Theorem 7.2. *Let Ω be a domain in $\mathbb{C}$, $f(t)$, $\varphi(t)$ be holomorphic in Ω. Let t_0 be a unique zero of $\varphi'(t)$ in Ω and $\varphi''(t_0) \neq 0$. Let $\rho(t)$ be a C^∞-smooth function compactly supported by Ω such that $\rho = 1$ in a neighborhood of t_0. Consider the integral*

$$I(\lambda) = \int_\Omega \rho(t)\, f(t)\, \overline{f(t)} \exp\Big\{ i\,\mathrm{Re}(\lambda\varphi(t) \Big\}\ d\,\overline{\overline{t}}, \tag{7.5}$$

where $\lambda \in \mathbb{C}$ is a parameter. Then

a) *For $|\lambda| > 1$ we have the following expansion*

$$I(\lambda) = \frac{1}{|f''(t_0)|\,|\lambda|} \exp\Big\{ i\,\mathrm{Re}(\lambda\varphi(t) \Big\} \times \tag{7.6}$$
$$\times \Big(\sum_{k\geqslant 0,\, j\geqslant 0,\, k+l<N} \frac{\lambda^{-k}\overline{\lambda}^{-l}}{k!\,l!} a_k(f,\varphi)\, a_l(\overline{f},\overline{\varphi}) + R_N(\lambda) \Big),$$

where a_k are rational expressions

$$a_k = a_k\big(\varphi(t_0), \varphi'(t_0), \dots; f(t_0), f'(t_0), \dots; \varphi''(t_0)^{-1}\big)$$

and $a_0 = 1$. The reminder R_N satisfies

$$R_N(\lambda) = O(|\lambda|^N) \qquad \text{as } \lambda \to \infty. \tag{7.7}$$

b) *The asymptotic expansion*

$$I(\lambda) \sim |\lambda|^{-1} \sum_{k\geqslant 0,\, l\geqslant 0} \frac{c_{kl}}{\lambda^k \overline{\lambda}^l} \qquad \text{as } \lambda \to \infty$$

can be written as

$$I(\lambda) \sim \frac{1}{|f''(t_0)|\,|\lambda|} \exp\Big\{ i\,\mathrm{Re}(\lambda\varphi(t) \Big\} \times \tag{7.8}$$
$$\times \exp\Big\{ \frac{i}{2\overline{\lambda}\,\varphi''(t_0)} \frac{\partial^2}{\partial t^2} \Big\} \Big(f(t) \exp\Big\{ \overline{\lambda}\big(\varphi(t) - \varphi'(t_0) - \tfrac12 \varphi''(t_0)(t-t_0)^2\big) \Big\} \Big) \Big|_{t=t_0} \times$$
$$\times \exp\Big\{ \frac{i}{2\lambda\,\overline{\varphi''(t_0)}} \frac{\partial^2}{\partial \overline{t}^2} \Big\} \Big(f(t) \exp\Big\{ \lambda\big(\overline{\varphi(t) - \varphi'(t_0) - \tfrac12 \varphi''(t_0)(t-t_0)^2}\big) \Big\} \Big) \Big|_{t=t_0}.$$

c) *Let $\varphi = \varphi_\alpha$, $f = f_\alpha$ smoothly depend on a parameter α, where α ranges in a compact domain $K \subset \mathbb{C}$ and the conditions of the preamble of the theorem are satisfied for all α. Then $O(\cdot)$ in (7.7) is uniform in $\alpha \in K$.*

PROOF. b) We use Fedoryuk [8], Proposition III.2.2. Let f be a smooth compactly supported function on $\mathbb{R}^n$, let S be smooth. Consider an n-dimensional integral

$$I(\sigma) := \int_{\mathbb{R}^n} f(x) \exp\{i\sigma S(x)\}\, dx, \qquad t \geqslant 1.$$

Let x_0 be a unique critical point of S on the support of f, let it be nondegenerate. Let $H(x_0)$ be the Hessian of S at x_0 (i.e., the matrix composed of second partial derivatives), let $\operatorname{sgn} H(x_0)$ denote the signature of the Hessian (the number of positive eigenvalues minus the number of negative eigenvalues). Consider the second order differential operator

$$L := \frac{i}{2} \langle H(x_0)^{-1} \nabla_x, \nabla_x \rangle,$$

where ∇_x denotes the vector column composed of $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. Denote

$$S(x, x_0) := S(x) - S(x_0) - \frac{1}{2} \langle H(x_0)(x - x_0), (x - x_0) \rangle, \tag{7.9}$$

this expression is the part of the Taylor expansion of $S(x)$ at x_0 starting cubic terms. Then the following expansion take place:

$$\begin{aligned} I(\sigma) = \Bigl(\frac{2\pi}{\sigma}\Bigr)^{n/2} |\det H(x_0)|^{-1/2} \exp\Bigl[\frac{i\pi}{4} \operatorname{sgn} H(x_0)\Bigr] \times \\ \times \Bigl(\sum_{k=0}^{N-1} \frac{\sigma^{-k}}{k!}\, L^k \bigl(f(x) \exp\{i\sigma S(x, x_0)\}\bigr)\Bigr|_{x=x_0} + \sigma^{-N+[2N/3]} V(\sigma) \Bigr), \end{aligned} \tag{7.10}$$

where $V(\sigma)$ is bounded.

Let us return to our integral (7.5). Without loss of generality, we can set $t_0 = 0$, $\varphi''(t_0) = 1$, i.e.,

$$\varphi(t) = t^2 + r(t), \qquad \text{where } r(0) = r'(0) = r''(0) = 0.$$

Set $\lambda = s e^{i\theta}$, $s > 0$. Set $z = x + iy$, then

$$\varphi(x, y) = x^2 - y^2 + 2ixy + r(x, y).$$

Thus we come to an oscillating integral in s with the parameter θ,

$$I(s, \theta) = \int \rho(x, y) f(x, y) \overline{f(x, y)} \exp\Bigl\{ is(\cos\theta \operatorname{Re} \varphi(x, y) + \sin\theta \operatorname{Im} \varphi(x, y)) \Bigr\}\, dx\, dy.$$

We wish to apply the general statement formulated above. The Hessian is given by

$$H = 2 \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \qquad H^{-1} = \frac{1}{2} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

The signature is 0. The differential operator L is

$$L = \frac{i}{4} \Bigl(\cos\theta \Bigl(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \Bigr) + 2 \sin\theta \frac{\partial^2}{\partial x \partial y} \Bigr) = \frac{i}{2} \Bigl(e^{i\theta} \frac{\partial^2}{\partial t^2} + e^{-i\theta} \frac{\partial^2}{\partial \overline{t}^2} \Bigr).$$

Next, we rewrite our phase function $S(\cdot)$ as

$$e^{-i\theta} \varphi(t) + e^{i\theta} \overline{\varphi(t)}.$$

Therefore expression (7.9) is

$$e^{-i\theta}r(t) + e^{i\theta}\overline{r(t)}.$$

Applying (7.10), we get

$$\begin{aligned}I(s,\theta) := \frac{2\pi}{s}\exp\Big\{\frac{i}{2s}\Big(e^{i\theta}\frac{\partial^2}{\partial t^2} + e^{-i\theta}\frac{\partial^2}{\partial \bar{t}^2}\Big)\Big\}\times \\ \times\Big(f(t)\overline{f(t)}\exp\big\{is\big(e^{-i\theta}r(t) + e^{i\theta}\overline{r(t)}\big)\big\}\Big)\Big|_{t=0} = \\ = \frac{2\pi}{s}\exp\Big\{\frac{i}{2se^{-i\theta}}\frac{\partial^2}{\partial t^2}\Big\}\Big(f(t)\exp\big\{ise^{-i\theta}r(t)\big\}\Big)\Big|_{t=0}\times \\ \times\exp\Big\{\frac{i}{2se^{i\theta}}\frac{\partial^2}{\partial \bar{t}^2}\Big\}\Big(\overline{f(t)}\exp\big\{ise^{i\theta}\overline{r(t)}\big\}\Big)\Big|_{t=0}.\end{aligned}$$

We obtained asymptotics in s for fixed θ. However, θ ranges in a compact set, by [8], Theorem III.2.2, we get that the term $V(\cdot)$ in (7.10) is bounded uniformly in θ.

a) follows from b).

c) We again refer to parametric version of the stationary phase approximation, see [8], Theorem III.2.2. □

7.4. Contribution of stationary points. Let us apply Theorem 7.2 to our integral (7.2). We have

$$\begin{aligned}f(t) = R(t,z) = \Big(\frac{t}{1-zt}\Big)^a (1-t)^b \Big(\frac{1-t}{t(1-zt)}\Big)^{\frac{\sigma}{2}}\big(t(t-1)\big)^{-1}; \\ \varphi(t) = 2\ln\Big(\frac{1-t}{t(1-zt)}\Big).\end{aligned}$$

Denote

$$\zeta = \sqrt{1-1/z}.$$

We substitute $t = t_+$ and transform factors of $R(t,z) = f(t)\overline{f(t)}$:

$$\Big(\frac{t}{1-zt}\Big)^{a|a}\Big|_{t=t_+} = \Big(\frac{1-\zeta^2}{-\zeta}\Big)^{a|a} = \big((z-1)z\big)^{-a/2|-a/2}; \tag{7.11}$$

$$(1-t)^{b|b}\Big|_{t=t_+} = \Big(\frac{1-z}{z}\Big)^{b/2|b/2}; \tag{7.12}$$

$$\Big(\frac{1-t}{t(1-zt)}\Big)^{\frac{\sigma}{2}}\Big|_{t=t_+} = \Big(\frac{1-\zeta}{1+\zeta}\Big)^{\frac{\sigma}{2}|\frac{\sigma}{2}} \tag{7.13}$$

$$\big(t(t-1)\big)^{-1|-1}\Big|_{t=t_+}; = \Big(\frac{-1}{\zeta(1+\zeta)}\Big)^{1|1}. \tag{7.14}$$

Next,

$$\varphi(t_+) = 2\ln\Big(\frac{1-\zeta}{1+\zeta}\Big),$$

therefore

$$\exp\Big\{i\,\mathrm{Re}\big(\varphi(t_+)\tfrac{1}{2}(k+is)\big)\Big\} = \Big(\frac{1-\zeta}{1+\zeta}\Big)^{\lambda|-\overline{\lambda}} = \Big(\frac{t_-}{t_+}\Big)^{\lambda|-\overline{\lambda}}.$$

Finally,

$$\varphi''(t) = \frac{-2}{(1-t)^2} + \frac{2}{t^2} + \frac{2z^2}{(1-tz)^2},$$

and

$$\varphi''(t_+) = \frac{-4}{\zeta(1+\zeta)^2}.$$

Uniting these data we get that the leading term at the point t_+ is

$$- |\zeta|\, |1-z|^{b-a}\, |z|^{-a-b} \Bigl(\frac{t_-}{t_+}\Bigr)^{\lambda|-\overline{\lambda}} \cdot \frac{1}{(k^2+s^2)^{1/2}}. \tag{7.15}$$

General form of the asymptotic expansion at $t = t_+$ follows from Theorem 7.2.

7.5. Contributions of singular points.

Lemma 7.3. *Contributions at singular points* 0, 1, ∞ *are* $O(|\lambda|^{-N})$ *for any* N.

Proof. To be definite examine the point 0. We have integral

$$I_0(\lambda) = \int_{\mathbb{C}} \rho_0(t)\, t^{a-1|a-1} (1-t)^{c-a-1} (1-zt)^{-a} \Bigl(\frac{t(1-zt)}{1-t}\Bigr)^{\lambda|-\overline{\lambda}}\, d\,\overline{\overline{t}},$$

defined as an analytic continuation. Keeping in mind that a support of ρ_0 can be chosen sufficiently small, we pass to a new variable in a neighborhood of 0,

$$u = \frac{t(1-zt)}{1-t}$$

and come to an integral of the form

$$I_0(\lambda) = \int_{\mathbb{C}} u^{a-\lambda-1|a+\overline{\lambda}-1} \Phi(u)\, d\,\overline{\overline{u}},$$

where Φ is a smooth compactly supported function. It remains to apply Theorem 2.2.

Argumentation for other singular points is the same. $\square$

8. Symmetry of difference operators

Here we prove Theorem 1.7, i.e., show that if $f \in \mathcal{D}(\dot{\mathbb{C}})$, then $J_{a,b} f$ is contained the space $\mathcal{W}_{a,b}$ of meromorphic functions on $\Lambda_{\mathbb{C}}$. Also we show that $\mathfrak{L}$ and $\overline{\mathfrak{L}}$ are formally adjoint one to another on $\mathcal{W}_{a,b}$, see Theorem 8.4.

8.1. Beginning of proof of Theorem 1.7. We follow to the list of properties in the definition of $\mathcal{W}_{a,b}$, see Subsect. 1.8,

a) is a corollary of the symmetry $\mathcal{K}_{a,b}(z; -k, -\sigma) = \mathcal{K}_{a,b}(z; k, \sigma)$.

b) We must examine poles of $\mathcal{K}_{a,b}(z; k, \sigma)$ as a function of variable σ for a fixed $z \in \dot{\mathbb{C}}$, $k \in \mathbb{Z}$. Let $a + b \ne 1$. We look to the expansion (3.26) of ${}_2F_1^{\mathbb{C}}[\cdot]$ at $z = 0$. The only source of poles of $\mathcal{K}$ are zeros of the denominators in (3.28), i.e., zeros of the expression

$$\begin{aligned} R(k,\sigma) := \Gamma^{\mathbb{C}}\bigl(a + \tfrac{k+\sigma}{2}\big|a + \tfrac{-k+\sigma}{2}\bigr)\, \Gamma^{\mathbb{C}}\bigl(a + \tfrac{-k-\sigma}{2}\big|a + \tfrac{-k-\sigma}{2}\bigr) \times \\ \Gamma^{\mathbb{C}}\bigl(b + \tfrac{k+\sigma}{2}\big|b + \tfrac{-k+\sigma}{2}\bigr)\, \Gamma^{\mathbb{C}}\bigl(b + \tfrac{-k-\sigma}{2}\big|b + \tfrac{-k-\sigma}{2}\bigr). \end{aligned} \tag{8.1}$$

This gives us the desired list of possible poles.

Let us examine the case $a + b = 1$. Decomposition of hypergeometric functions (3.26) at $z = 0$ produces an expression of the type

$$\mathcal{K}_{a,b}(z; k, \sigma) = \frac{u_{a,b}(z,k,\sigma) - v_{z,a,b}(z,k,\sigma)}{a+b-1} \tag{8.2}$$

with $u_{a,b}$, $v_{a,b}$ having poles at zeros of $R(k,\sigma)$. Decomposition at $z=1$ gives

$$\mathcal{K}_{a,b}(z;k,\sigma)=\frac{U_{a,b}(z,k,\sigma)-V_{a,b}(z,k,\sigma)}{a-b},$$

therefore the singularity in (8.2) at $a+b=1$ is removable.

d) Indeed, we have $\mathcal{K}_{a,b}(p,q)=\mathcal{K}_{a,b}(q,p)$, i.e.,

$$ {}_2F_1^{\mathbb{C}}\left[\begin{matrix} a+\frac{p+q}{2}\big|a+\frac{-p+q}{2},\ a+\frac{-p-q}{2}\big|a+\frac{p-q}{2} \\ a+b\big|a+b\end{matrix};z\right]= $$
$$ ={}_2F_1^{\mathbb{C}}\left[\begin{matrix} a+\frac{p+q}{2}\big|a+\frac{p-q}{2},\ a+\frac{-p-q}{2}\big|a+\frac{-p+q}{2} \\ a+b\big|a+b\end{matrix};z\right]. $$

This is a special case of the symmetry (3.36).

We also mention the similar identity for (8.1):

$$R(p,q)=R(q,p), \tag{8.3}$$

it is a special case of (3.37).

The statement c) about behavior at infinity is a corollary the expansion (7.1) and the following lemma

Lemma 8.1. *Let $t_\pm(z)$ be as in Theorem 7.1. Let $\Phi\in\mathcal{D}(\dot{\mathbb{C}})$ be a function with a simply connected support. Then for any $A>0$ for any $N>0$ in the strip $|\operatorname{Re}\sigma|<A$ we have*

$$\int_{\dot{\mathbb{C}}}\Phi(z)\Bigl(\frac{t_-(z)}{t_+(z)}\Bigr)^{(k+\sigma)/2|(-k+\sigma)/2}\,d\,\overline{\overline{z}}=O\bigl(k^2+(\operatorname{Im}\sigma)^2\bigr)^{-N}$$

as $(k^2+(\operatorname{Im}\sigma)^2)\to\infty$.

We need in a simply connected support since the integrand is ramified at points $z=0$, $z=1$. A proof of the lemma requires some preparations.

8.2. A change of variable. We define a new variable

$$p:=\frac{t_+(z)}{t_-(z)}, \tag{8.4}$$

The inverse map is done by

$$z=\zeta(p):=\frac{(p+1)^2}{4p}. \tag{8.5}$$

The map $\zeta(p)$ determines a two-sheet covering map from

$$\ddot{\mathbb{C}}:=\mathbb{C}\setminus\{0,1,-1\} \tag{8.6}$$

to $\dot{\mathbb{C}}$. Notice that

$$1-z=-\frac{(p-1)^2}{4p},\qquad \sqrt{1-1/z}=\frac{p-1}{p+1},\qquad \zeta'(p)=\frac{p^2-1}{4p^2}, \tag{8.7}$$

$$t_+=\frac{2p}{p+1},\qquad t_-=\frac{2}{p+1},\qquad \frac{t_+}{t_-}=p. \tag{8.8}$$

Also,

$$\zeta(p^{-1})=\zeta(p),\qquad \zeta'(p^{-1})\,p^{-1}=\zeta(p)\,p. \tag{8.9}$$

8.3. Proof of Theorem 1.7.c.

PROOF OF LEMMA 8.1 We substitute $z = \zeta(p)$ to the integral and get

$$\frac{1}{16}\int_{\ddot{\mathbb{C}}} p^{(k+\sigma)/2|(-k+\sigma)/2}\Big(\Phi(\zeta(p))\,|p^2-1|^2\,p^{-2}\Big)\;d\overline{\overline{p}}.$$

This is a Mellin transform of a function compactly supported by $\ddot{\mathbb{C}}$. In virtue of Theorem 2.2 the integral rapidly decreases in the union of strips $|\operatorname{Re}\sigma| < A$.

PROOF OF THE STATEMENT C) OF THEOREM 1.7. We represent $\varphi(z)$ as a sum of functions in $\mathcal{D}(\dot{\mathbb{C}})$ with simply connected supports. Next, we decompose the kernel according Theorem 7.1 and apply the lemma to each summand.

8.4. Continuity.

Corollary 8.2. *The map $J_{a,b}$ is a continuous map from $\mathcal{D}(\dot{\mathbb{C}})$ to $L^2_{\mathrm{even}}(\Lambda,\varkappa_{a,b})$.*

PROOF. Define the following seminorms on the space of smooth functions on Λ:

$$p_{\alpha,N}(F) = \sup_{\lambda\in\Lambda}\Big|\frac{\partial^N F}{\partial\sigma^N}(1+|\lambda|)^\alpha\Big|,$$

and the space $\mathcal{Y}$ defined by these seminorms. Clearly, our proof provides a continuity of $J_{a,b}$ as a map $\mathcal{D}(\dot{\mathbb{C}})$ to Y. It remains to notice that the identical embedding $f \mapsto f$ of Y to L^2 is continuous.

If $k = 0$ and $a = 1$ or $b = 1$, then elements of $\mathcal{W}_{a,b}$ have a pole of order two[20] at $k = 0$, $s = 0$. In this case we write $\lambda^2 F$ instead of F in the definition of the seminorms. $\square$

8.5. Invariance of $\mathcal{W}_{a,b}$. Consider the difference operators $\mathfrak{L}$, $\overline{\mathfrak{L}}$ defined above (1.28),

$$\mathfrak{L}F(k,\sigma) = \frac{(a+\frac{k+\sigma}{2})(b+\frac{k+\sigma}{2})}{(k+\sigma)(1+k+\sigma)}\big(F(k+1,\sigma+1)-F(k,\sigma)\big)+ \\ + \frac{(a+\frac{-k-\sigma}{2})(b+\frac{-k-\sigma}{2})}{(-k-\sigma)(1-k-\sigma)}\big(F(k-1,\sigma-1)-F(k,\sigma)\big); \tag{8.10}$$

$$\overline{\mathfrak{L}}F(k,\sigma) = \frac{(a+\frac{-k+\sigma}{2})(b+\frac{-k+\sigma}{2})}{(-k+\sigma)(1-k+\sigma)}\big(F(k-1,\sigma+1)-F(k,\sigma)\big)+ \\ + \frac{(a+\frac{k-\sigma}{2})(b+\frac{k-\sigma}{2})}{(k-\sigma)(1+k-\sigma)}\big(F(k+1,\sigma-1)-F(k,\sigma)\big). \tag{8.11}$$

Lemma 8.3. *The space $\mathcal{W}_{a,b}$ is invariant with respect to the operators $\mathfrak{L}$, $\overline{\mathfrak{L}}$.*

PROOF. Since $F(0,-1) = F(1,0) = F(-1,0) = F(0,1)$, the expressions

$$\frac{F(k+1,\sigma+1)-F(k,\sigma)}{1+k+\sigma},\qquad \frac{F(k-1,\sigma-1)-F(k,\sigma)}{1-k-\sigma}$$

have no poles at $k=-1$, $\sigma=0$ and $k=1$, $\sigma=0$ respectively.

Since a function $F(k,\sigma)$ is even, it can not have a pole of order 1 at $k=0$, $\sigma=0$.

New poles of $F(k+1,\sigma+1)$ that are not poles of $F(k,\sigma)$ are annihilated by the rational factor in (8.10).

The condition $\mathfrak{L}F(p,q) = \mathfrak{L}F(q,p)$ follows from a straightforward calculation. $\square$

8.6. Symmetry.

[20]At the same point the spectral density has a zero of order 4.

Theorem 8.4. *For $(a,b)\in\Pi$ for any F, $G\in\mathcal{W}_{a,b}$ we have*

$$\langle \mathfrak{L}F, G\rangle_{L^2(\Lambda_{\mathbb{C}},dK_{a,b})} = \langle F, \overline{\mathfrak{L}}G\rangle_{L^2(\Lambda,dK_{a,b})}. \tag{8.12}$$

Corollary 8.5. *Operators $\frac12(\mathfrak{L}+\overline{\mathfrak{L}})$, $\frac{1}{2i}(\mathfrak{L}-\overline{\mathfrak{L}})$ are symmetric on the $J_{a,b}$-image of $\mathcal{D}(\dot{\mathbb{C}})$.*

Remark. In fact, the proof uses only properties of $F\in\mathcal{W}_{a,b}$ in strips $|\operatorname{Re}\sigma|<1+\varepsilon$. So we can define operators $\mathfrak{L}$, $\overline{\mathfrak{L}}$ on meromorphic functions in the strip an obvious list of conditions. ⊠

8.7. Proof of Theorem 8.4 for the case $(a,b)\in\Pi_{\mathrm{cont}}$. First, we notice that for pure imaginary σ we have $\overline{G(k,\sigma)}=\overline{G(k,-\overline{\sigma})}$, the last function is meromorphic and also is contained in $\mathcal{W}_{a,b}$. Let $R(k,\sigma)$ be given by (8.1). Then

$$\begin{aligned} 4\pi^2 i\,\langle \mathfrak{L}F, G\rangle = \sum_k \int_{i\mathbb{R}} \Biggl\{ &\frac{(a+\frac{k+\sigma}{2})(b+\frac{k+\sigma}{2})}{(k+\sigma)(1+k+\sigma)}\bigl(F(k+1,\sigma+1)-F(k,\sigma)\bigr)+\\ &+\frac{(a+\frac{-k-\sigma}{2})(b+\frac{-k-\sigma}{2})}{(-k-\sigma)(1-k-\sigma)}\bigl(F(k-1,\sigma-1)-F(k,\sigma)\bigr)\Biggr\}\times\\ &\times\overline{G(k,-\overline{\sigma})}\,(k-\sigma)(k+\sigma)R(k,\sigma)\,d\sigma. \end{aligned}\tag{8.13}$$

It is natural to expand the expression in the curly brackets $\{\dots\}$ as a sum of 4 summands that include $F(k+1,\sigma+1)$, $F(k,\sigma)$, $F(k-1,\sigma-1)$, $F(k,\sigma)$. The whole expression $\{\dots\}$ is holomorphic near the contour of integration. The summands have simple poles on the contour, and we pass to an integration in the sense of principal values.

Let us examine the summand corresponding $F(k+1,\sigma+1)$. We get

$$\sum_k \text{v.p.}\int_{i\mathbb{R}} \frac{k-\sigma}{1+k+\sigma} F(k+1,\sigma+1)\,\overline{G(k,-\overline{\sigma})}\,\widetilde{R}(k,\sigma)\,d\sigma, \tag{8.14}$$

where

$$\begin{aligned} \widetilde{R}(k,\sigma) &:= \bigl(a+1+\tfrac{k+\sigma}{2}\bigr)\bigl(b+1+\tfrac{k+\sigma}{2}\bigr)R(k,\sigma)=\\ &=\Gamma^{\mathbb{C}}\bigl(a+1+\tfrac{k+\sigma}{2}\big|a+\tfrac{-k+\sigma}{2}\bigr)\,\Gamma^{\mathbb{C}}\bigl(a+\tfrac{-k-\sigma}{2}\big|a+\tfrac{-k-\sigma}{2}\bigr)\\ &\qquad \Gamma^{\mathbb{C}}\bigl(b+1+\tfrac{k+\sigma}{2}\big|b+\tfrac{-k+\sigma}{2}\bigr)\,\Gamma^{\mathbb{C}}\bigl(b+\tfrac{-k-\sigma}{2}\big|b+\tfrac{-k-\sigma}{2}\bigr). \end{aligned}\tag{8.15}$$

Lemma 8.6. *For $0<a<1$, $0<b<1$ the integrand in (8.14) has no poles in the strip $-1<\operatorname{Im}\sigma<0$.*

Proof. We enumerate possible (simple) poles of factors.

a) Factor $\overline{G(k,-\overline{\sigma})}$. In this case we can have poles if $k=0$. Since $a<1$, $b<1$ the poles $2-2a$, $2-2b$ are outside our strip. On other hand a pole $2a-2$ (resp. $2b-2$) is contained in the strip if $1/2<a<1$ (resp. if $1/2<b<1$).

b) Factor $F(k+1,\sigma+1)$ has a pole in our strip for $k=-1$ at $\sigma=2a-1$ (resp. $\sigma=2b-1$) if $0<a<1/2$, (resp. $0<b<1/2$).

c) Since $a>0$, $b>0$ the expression $\widetilde{R}(k,\sigma)$ has no poles at our strip.

However, the poles of $\overline{G(k,-\overline{\sigma})}$ and of $F(k+1,\sigma+1)$ are zeros of $\widetilde{R}(k,\sigma)$. Therefore the product is holomorphic. □

Lemma 8.7. *In (8.14), we can change the integration contour to $1+i\mathbb{R}$.*

Proof. We have no poles of the integrand between contours, but we have poles on contours and integrals in the sense of principal values. We have only two such poles, $\sigma=0$ on the contour $i\mathbb{R}$ for $k=-1$ and $\sigma=-1$ for $k=0$. Thus difference between two integrals is 2π by half of the sum of residues, i.e.,

$$\frac{2\pi}{2}\Biggl\{(-1-\sigma)F(0,\sigma+1)\overline{G(-1,-\overline{\sigma})}\bigl(a+1+\tfrac{-1+\sigma}{2}\bigr)\bigl(b+1+\tfrac{-1+\sigma}{2}\bigr)R(-1,\sigma)\Big|_{\sigma=0}+$$
$$+(0-\sigma)F(1,\sigma+1)\overline{G(0,-\overline{\sigma})}\bigl(a+1+\tfrac{\sigma}{2}\bigr)\bigl(b+1+\tfrac{\sigma}{2}\bigr)R(0,\sigma)\Big|_{\sigma=-1}\Biggr\}.$$

Let us show that the sum is zero. Since F, G are even and satisfy (1.25), we have

$$F(0,1)=F(1,0),\qquad \overline{G(-1,0)}=\overline{G(0,-1)}$$

By (8.3), we have

$$R(-1,0)=R(0,-1).$$

The remaining factors give

$$-(a+\tfrac12)(b+\tfrac12)\quad\text{and}\quad(a+\tfrac12)(b+\tfrac12),$$

i.e., the same expressions with different signs. $\square$

End of the proof of Theorem 8.4. Thus we can replace the integration in (8.14) by the integration over the contour $-1+i\mathbb{R}$. We change variables $l=k+1$, $t=\sigma+1$ and get

$$\sum_l \text{v.p.}\int_{i\mathbb{R}}\frac{l-t}{-1+l+t}F(l,t)\,\overline{G(l-1,-\overline{t}+1)}\,\widetilde{R}(l-1,t-1)\,dt.$$

Next,

$$\widetilde{R}(l-1,t-1)=R(l,t)\bigl(a+\tfrac{-l-t}{2}\bigr)\bigl(b+\tfrac{-l-t}{2}\bigr),$$

and we come to

$$\sum_l \text{v.p.}\int_{i\mathbb{R}}F(l,t)\Biggl[\frac{\bigl(a+\frac{-l-t}{2}\bigr)\bigl(b+\frac{-l-t}{2}\bigr)}{(-l-t)(1-l-t)}\,\overline{G(l-1,-\overline{t}+1)}\Biggr]\times$$
$$\times(l-t)(l+t)\,R(l,t)\,dt.$$

We transform the expression in the big brackets to the form $\overline{U(l,-\overline{t})}$, where

$$U(l,t)=\frac{\bigl(a+\frac{-l+t}{2}\bigr)\bigl(b+\frac{-l+t}{2}\bigr)}{(-l+t)(1-l+t)}\,G(l-1,t+1).$$

Thus we finished the transformation of the summand of the (8.13) corresponding to $F(k+1,\sigma+1)$. The transformation of the summand corresponding to $F(k-1,\sigma-1)$ is similar. The case of summands $F(k,\sigma)$ is obvious. We come to the desired expression. $\square$

8.8. End of proof of Theorem 8.4. Due to the homographic transformations, it is sufficient to examine the case $a<0$. Let Φ, $\Psi\in\mathcal{W}_{a,b}$. Denote

$$U(a,b;k,\sigma):=\Phi(k,\sigma)\,\overline{\Psi(k,-\sigma)}\,\varkappa_{a,b}(k,\sigma). \tag{8.16}$$

For $(a,b)\in\Pi_{\text{cont}}$ we have

$$4\pi^2 i\langle\Phi,\Psi\rangle_{L^2(\Lambda,\varkappa_{a,b})}=\sum_k\int_{i\mathbb{R}}U(a,b;k,\sigma)\,d\sigma. \tag{8.17}$$

We wish to write the analytic continuation of this expression to the domain $(a,b) \in \Pi$, $a \leqslant 0$.

Possible singularities of U as a function in σ in the strip $|\operatorname{Re}\sigma| < 1$ are following:

— if $b > 1/2$, then both functions Φ, Ψ have poles at $(k,\sigma) = (0, \pm(2-2b)$;

— $\varkappa_{a,b}(k,\sigma)$ has poles at $(k,\sigma) = (0, \pm 2a)$.

Due to our restrictions $2b-2 < 2a < -2a < 2-2b$.

Thus all summands of (8.17) except 0-th are holomorphic in $|a| < 1-b$.

Lemma 8.8. *Fix b. Assume that Φ, Ψ be even rapidly decreasing meromorphic functions in the strip $|\operatorname{Re}\sigma| < 1$ satisfying the condition* (1.25) *and having poles only at points* $(0, \pm(2-2b))$. *Then the following expression is holomorphic in the domain $|a| < 1-b$:*

$$\gamma^b(a) := \begin{cases} \int_{i\mathbb{R}} U(a,b;0,\sigma)\,d\sigma, & \text{if } a \geqslant 0, \\ \int_{i\mathbb{R}} U(a,b;0,\sigma)\,d\sigma + 4\pi i \operatorname{res}_{\sigma=2a} U(a,b;0,\sigma), & \text{if } a \leqslant 0. \end{cases} \tag{8.18}$$

Proof of Lemma 8.8. Denote

$$\gamma_{i\mathbb{R}}(a) = \int_{i\mathbb{R}} U(a,b;0,\sigma)\,d\sigma, \quad \Xi_\pm(a) := 2\pi i \operatorname{res}_{\sigma=\pm 2a} U(a,b;0,\sigma).$$

Since U is even in σ, we have $\Xi_-(a) = -\Xi_+(a)$. Due to the factor $(k+\sigma)(k-\sigma)$ in the Plancherel density, we have $\Xi_\pm(0) = 0$. Therefore $\Xi_\pm(a)$ are holomorphic in the disk $|a| < 1-b$.

Consider a contour L on the plane $\sigma \in \mathbb{C}$ composed of the ray $(-\infty, b-1+\varepsilon]$, the upper half of the circle $|\sigma| = 1-b-\varepsilon$ and the ray $[1-b-\varepsilon, +\infty]$. The function

$$\gamma_L(a) := \int_L U(a,b;0,\sigma)\,d\sigma$$

is holomorphic in a for $|a| < 1-b$. For $\operatorname{Re} a > 0$ we have $\gamma_L(a) = \gamma_{i\mathbb{R}}(a) - \Xi_+(a)$. For $\operatorname{Re} a < 0$ we have $\gamma_L(a) = \gamma_{i\mathbb{R}}(a) - \Xi_-(a)$. This gives us the analytic continuation. $\square$

Proof of Theorem 8.4 for $a < 0$. Thus the analytic continuation of $4\pi^2 i \langle \Phi, \Psi \rangle_{L^2(\Lambda, \varkappa_{a,b})}$ to the domain $a < 0$ is given by

$$\gamma^b(a) + \sum_{k \neq 0} \int_{i\mathbb{R}} U(a,b;k,\sigma)\,d\sigma,$$

i.e., for $a < 0$ we get $4\pi^2 i \langle \Phi, \Psi \rangle_{L^2(\Lambda_{\mathbb{C}}, dK_{a,b})}$.

Now we see that the both sides of (8.12) are real analytic in the parameter a and coincide for $a > 0$. Therefore they coincide for $a < 0$. $\square$

9. The operator $J_{a,b}$ is an isometry

Here we prove the second part of Theorem 1.3.

9.1. Statement.

Lemma 9.1. *Let f, g be smooth compactly supported functions on $\dot{\mathbb{C}}$. Then*

$$\langle J_{a,b} f, J_{a,b} g \rangle_{L^2(\Lambda, dK_{a,b})} = \langle f, g \rangle_{L^2(\dot{\mathbb{C}}, \mu_{a,b})}.$$

Here a way of proof is simpler than in Section 6. We show that $J_{a,b}$ is a perturbation of a version of the Mellin transform.

9.2. Orthogonality of packets.

Lemma 9.2. *Let f, $g \in \mathcal{D}(\dot{\mathbb{C}})$. Let* $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \varnothing$. *Then*

$$\langle J_{a,b} f, J_{a,b} g \rangle_{L^2_{\mathrm{even}}(\Lambda_{\mathbb{C}}, dK_{a,b})} = 0.$$

PROOF. By Corollary 8.2 the operator $J_{a,b}$ is continuous as an operator $\mathcal{D}(\dot{\mathbb{C}}) \to L^2(\Lambda_{\mathbb{C}}, dK_{a,b})$, by Theorem 8.4 it is symmetric on the image of $\mathcal{D}(\dot{\mathbb{C}}, \mu_{a,b})$. We consider difference operators

$$\tfrac{1}{2}(\mathfrak{L} + \overline{\mathfrak{L}}), \qquad \tfrac{1}{2i}(\mathfrak{L} - \overline{\mathfrak{L}})$$

and repeat the proof of Lemma 6.3. □

9.3. Decomposition of the kernel. Starting from this place we examine the restriction of $J_{a,b} f$ to Λ. Recall that the operator $J_{a,b}$ is defined by the formula

$$J_{a,b} f(\lambda) = \int_{\dot{\mathbb{C}}} f(z)\, \mathcal{K}(z, \lambda) \mu_{a,b}(z)\, d\,\overline{\overline{z}}. \tag{9.1}$$

Decompose the kernel $\mathcal{K}(z, \lambda)$ according (7.1) with $N = 3$. We consider $\lambda \in \Lambda$, and therefore we set $\sigma = 0$. Denote by ω the factor depending on λ in the front of the expansion. We have

$$\omega(\lambda)\, \overline{\omega(\lambda)} = \varkappa_{a,b}^{-1}(\lambda). \tag{9.2}$$

Notice also that the expression in brackets in (7.1) has a singularity at $\lambda = 0$. Denote by $\Theta(\lambda)$ a smooth function, which equals 0 for $|\lambda| \leqslant 1/3$ and 1 for $|\lambda| \geqslant 1/2$. Represent the kernel as

$$\begin{aligned}
\mathcal{K}(z, \lambda) = \omega(\lambda) |1 - z|^{b-a} |z|^{-a-b} \times \\
\times \Biggl\{ \biggl[\Bigl(\frac{t_+(z)}{t_-(z)} \Bigr)^{\lambda|-\overline{\lambda}} + \Bigl(\frac{t_-(z)}{t_+(z)} \Bigr)^{\lambda|-\overline{\lambda}} \biggr] + \\
+ \Theta(\lambda) \biggl[\Bigl(\frac{t_+(z)}{t_-(z)} \Bigr)^{\lambda|-\overline{\lambda}} \sum_{k \geqslant 0,\, l \geqslant 0,\, 1 \leqslant k+l \leqslant 2} \frac{\overline{\lambda}^{-k} \lambda^{-l}}{k!\, l!} A_k(\sqrt{1-z})\, A_l(\sqrt{1-\overline{z}}) + \\
+ \Bigl(\frac{t_-(z)}{t_+(z)} \Bigr)^{\lambda|-\overline{\lambda}} \sum_{k \geqslant 0,\, l \geqslant 0,\, 1 \leqslant k+l \leqslant 2} \frac{\overline{\lambda}^{-k} \lambda^{-l}}{k!\, l!} A_k(-\sqrt{1-z})\, A_l(-\sqrt{1-\overline{z}}) \biggr] + R_3(z, \lambda) \Biggr\},
\end{aligned}$$

where $R_3(z, \lambda)$ is a smooth function in $z \in \dot{\mathbb{C}}$ and λ,

$$R_3(z, \lambda) = O(|\lambda|^{-3}) \qquad \text{as } \lambda \to \infty.$$

uniformly on compact subsets of $\dot{\mathbb{C}}$. The summands corresponding to $k = 0$, $l = 0$ are smooth at $\lambda = 0$, so we do not multiply them by the patch function $\Theta(\lambda)$.

Next, we change the variable as in (8.4)–(8.9):

$$\zeta(p) := \frac{(p+1)^2}{4p}$$

and represent the operator $J_{a,b}$ in the form

$$J_{a,b}f(\lambda)=\omega(\lambda)\int_{\ddot{\mathbb{C}}} f\bigl(\zeta(p)\bigr)|1-\zeta(p)|^{a-b-1/2}|\zeta(p)|^{-1/2}|\xi'(p)|^2\times$$
$$\times\Biggl\{\Bigl[p^{\lambda|-\overline{\lambda}}+(p^{-1})^{\lambda|-\overline{\lambda}}\Bigr]+$$
$$+\Theta(\lambda)\Biggl[p^{\lambda|-\overline{\lambda}}\sum_{k\geqslant 0,\,l\geqslant 0,\,1\leqslant k+l\leqslant 2}\frac{\overline{\lambda}^{-k}\lambda^{-l}}{k!\,l!}A_k\Bigl(\frac{p-1}{p+1}\Bigr)A_l\Bigl(\frac{\overline{p}-1}{\overline{p}+1}\Bigr)+$$
$$+(p^{-1})^{\lambda|-\overline{\lambda}}\sum_{k\geqslant 0,\,l\geqslant 0,\,1\leqslant k+l\leqslant 2}\frac{\overline{\lambda}^{-k}\lambda^{-l}}{k!\,l!}A_k\Bigl(-\frac{p-1}{p+1}\Bigr)A_l\Bigl(-\frac{\overline{p}-1}{\overline{p}+1}\Bigr)\Biggr]+R\bigl(\zeta(p),\lambda\bigr)\Biggr\}\,d\overline{\overline{p}},$$

where $\ddot{\mathbb{C}}$ denotes $\ddot{\mathbb{C}}:=\mathbb{C}\setminus\{0,1,-1\}$ as above.

It is convenient to split the operator $J_{a,b}$ into a sum of operators,

$$J_{a,b}=[V^+_{0,0}+V^-_{0,0}]+\sum V^+_{k,l}+\sum V^-_{k,l}+V_{\mathrm{rem}}, \tag{9.3}$$

where the summands correspond to the summands of the previous formula. We also denote

$$\gamma(p):=|1-\zeta(p)|^{a-b-1/2}|\zeta(p)|^{-1/2}|\zeta'(p)|^2.$$

9.4. The main term.

Lemma 9.3. *The operator $\frac{1}{2\pi}(V^+_{0,0}+V^-_{0,0})$ is a unitary operator $L^2(\mathbb{C},\mu_{a,b})$ from $L^2_{\mathrm{even}}(\Lambda,\varkappa_{a,b})$.*

PROOF.

$$\bigl\langle (V^+_{0,0}+V^-_{0,0})f,(V^+_{0,0}+V^-_{0,0})g\bigr\rangle_{L^2(\Lambda,\varkappa)}=$$
$$=\int\limits_{\Lambda}\Bigl(\int\limits_{\ddot{\mathbb{C}}} f\bigl(\zeta(p)\bigr)\,\gamma(p)\bigl(p^{\lambda|-\overline{\lambda}}+p^{-\lambda|\overline{\lambda}}\bigr)\,d\overline{\overline{p}}\Bigr)\Bigl(\int\limits_{\ddot{\mathbb{C}}}\overline{g\bigl(\zeta(q)\bigr)}\gamma(q)\bigl(q^{-\lambda|+\overline{\lambda}}+q^{\lambda|-\overline{\lambda}}\bigr)\,d\overline{\overline{q}}\Bigr)\,\widetilde{d}\lambda$$

(we also applied (9.2). Transform this expression as

$$\int\limits_{\Lambda}\Bigl(\int\limits_{\ddot{\mathbb{C}}} f\bigl(\zeta(p)\bigr)\,\gamma(p)\,|p|^2\,\bigl(p^{\lambda-1|-\overline{\lambda}-1}+p^{-\lambda-1|\overline{\lambda}-1}\bigr)\,d\overline{\overline{p}}\Bigr)\times \tag{9.4}$$
$$\times\Bigl(\int\limits_{\ddot{\mathbb{C}}}\overline{g\bigl(\zeta(q)\bigr)}\,\gamma(q)\,|q|^2\bigl(q^{-\lambda-1|+\overline{\lambda}-1}+q^{\lambda-1|-\overline{\lambda}-1}\bigr)\,d\overline{\overline{q}}\Bigr)\,\widetilde{d}\lambda.$$

Now we apply a remark about Mellin transforms of even function from Subsect. 2.3. Keeping in mind (8.9), we observe that functions $f(\zeta(p))\gamma(p)|p|^2$ are $\times$-even. Therefore the both integrals over $\ddot{\mathbb{C}}$ in (9.4) are Mellin transforms of even functions, and we can apply the Plancherel formula for the Mellin transform. We come to

$$\int_{\ddot{\mathbb{C}}} f\bigl(\zeta(p)\bigr)\,\overline{g\bigl(\zeta(p)\bigr)}\,|\gamma(p)|^2\,|p|^4\,\frac{d\overline{\overline{p}}}{|p|^2}=$$
$$=\int_{\ddot{\mathbb{C}}} f\bigl(\zeta(p)\bigr)\,\overline{g\bigl(\zeta(p)\bigr)}\,|1-\zeta(p)|^{2a-2b}\,|\zeta(p)|^{2a+2b-2}|\zeta'(p)|^2\times$$
$$\times\Bigl(|1-\zeta(p)|^{-1}|\zeta(p)^{-1}|p|^2|\zeta'(p)|^2\Bigr)\,d\overline{\overline{p}}.$$

By (8.7)–(8.8) the expression in the big brackets is 1. Now we return to the variable $z = \zeta(p)$ and get the desired expression

$$\int_{\dot{\mathbb{C}}} f(z)\overline{g(z)}\, \mu_{a,b}(z)\; d\overline{\overline{z}}.$$

9.5. Other terms.

Lemma 9.4. *The Hermitian form*

$$\{f,g\} := \big\langle J_{a,b}f, J_{a,b}g\big\rangle_{L^2(\Lambda,\varkappa_{a,b})} - \big\langle (V^+_{0,0}+V^-_{0,0})f, (V^+_{0,0}+V^-_{0,0})g\big\rangle_{L^2(\Lambda,\varkappa_{a,b})} \tag{9.5}$$

on $\mathcal{D}(\dot{\mathbb{C}})$ *can be written as*

$$\{f,g\} = \int_{\ddot{\mathbb{C}}}\int_{\ddot{\mathbb{C}}} K(p,q)\, f(\zeta(p))\,\overline{g(\zeta(q))}\; d\overline{\overline{p}}\; d\overline{\overline{q}}, \tag{9.6}$$

where K *is a locally integrable function on* $\ddot{\mathbb{C}}\times\ddot{\mathbb{C}}$ *smooth outside the sets* $p=q$, $p=q^{-1}$.

Proof. Expanding $J_{a,b}$ according (9.3), we get many summands in (9.5). We wish to show that each summand can be written as (9.6) with its own K. Let us start discussion with the summand

(9.7)

$$\langle V^+_{0,0}f, V^-_{0,1}g\rangle_{L^2(\Lambda,\varkappa_{a,b})} = \int_\Lambda \boxed{\frac{1-\Theta(\lambda)}{\lambda}} \Big(\int_{\ddot{\mathbb{C}}} f\big(\zeta(p)\big)\,\gamma(p)\,|p|^2\, p^{\lambda-1|-\overline{\lambda}-1}\; d\overline{\overline{p}}\Big)\times$$
$$\times\Big(\int_{\ddot{\mathbb{C}}} \overline{g\big(\zeta(q)\big)}\,\gamma(q)\,|q|^2\, A_1\Big(-\frac{\overline{q}-1}{\overline{q}+1}\Big)\overline{(q^{-1})^{\lambda-1|-\overline{\lambda}-1}}\; d\overline{\overline{q}}\Big)\, \widetilde{d}\lambda.$$

The integral in the first big bracket is the Mellin transform of the function

$$F(p) := f\big(\zeta(p)\big)\,\gamma(p)\,|p|^2.$$

The integral in the second big bracket is a complex conjugate to the Mellin transform of

$$G(q) = g\big(\zeta(q^{-1})\big)\,\gamma(q^{-1})\,|q|^{-2}\,A_1\Big(-\frac{\overline{q}^{-1}-1}{\overline{q}^{-1}+1}\Big).$$

Thus we get

$$\langle V^+_{0,0}f, V^-_{0,1}g\rangle_{L^2(\Lambda,\varkappa_{a,b})} = \int_\Lambda \mathcal{M}F(\lambda)\,\overline{\mathcal{M}G(\lambda)}\;\frac{1-\Theta(\lambda)}{\lambda}\,\widetilde{d}\lambda.$$

Denote by $L(p)$ the inverse Mellin transform of $\frac{1-\Theta(\lambda)}{\lambda}$. It is easy to see that $L(p)$ is an integrable function with unique singularity of the type $1/(1-\overline{p})$ at $p=1$. We rewrite our integral as

$$\int_{\ddot{\mathbb{C}}}\int_{\ddot{\mathbb{C}}} L(pq)\,F(p)\,\overline{G(q)}\; d\overline{\overline{p}}\; d\overline{\overline{q}},$$

and it has the desired form.

For other pairs $V^\varepsilon_{k,l}$, $V^{\varepsilon'}_{k',l'}$, where $\varepsilon,\varepsilon'=\pm1$, we have similar calculations. Instead of the boxed factor in (9.7), we get

$$\frac{1-\Theta(\lambda)}{\overline{\lambda}^{k+l'}\,\lambda^{l+k'}}. \tag{9.8}$$

For $k+l+k'+l' \leqslant 2$ we repeat the same considerations, in these cases inverse Mellin transforms of functions (9.8) have (integrable) singularities[21] at $p=1$ of types

$$\frac{1}{1-p}, \quad \frac{1-p}{1-\overline{p}}, \qquad \ln|1-p|, \quad \frac{1-\overline{p}}{1-p}.$$

If $k+l+k'+l' \geqslant 3$, then this expression is integrable in λ, the triple integral is convergent, we can change order of variables and integrating in λ we immediately get expression of form (9.6) with real analytic $K(p,q)$.

For the pairs including V_{rem} we get absolutely convergent triple integrals and analytic kernels $K(p,q)$.

9.6. Proof of Lemma 9.1. Now let f, $g \in \mathcal{D}(\dot{\mathbb{C}})$ have disjoint supports. Then the both terms in (9.5) are zero (see Lemma 9.2). Therefore the kernel $K(p,q)$ satisfy the following property:

$$\{f,g\} = \int_{\ddot{\mathbb{C}}}\int_{\ddot{\mathbb{C}}} K(p,q)\,\varphi(p)\,\overline{\psi(q)}\,dp\,dq = 0$$

if φ, ψ are $\times$-even elements $\mathcal{D}(\ddot{\mathbb{C}})$ with disjoint supports.

We claim that $\{f,g\}=0$ for any $\times$-even functions f, $g \in \mathcal{D}(\ddot{\mathbb{C}})$. To observe this, we take a $\times$-even partition of unity τ_j with small supports, and decompose

$$\{f,g\} = \sum_{k,l} \{\tau_k f, \tau_l g\}.$$

Clearly, we can make this sum as close to zero as we want by refinement of a partition of unity. We omit trivial details.

10. Domains of self-adjointness

Thus $J_{a,b}$ is unitary. Clearly the multiplication operators

$$f(z) \mapsto \tfrac{1}{2}(z+\overline{z})f(z), \qquad f(z) \mapsto \tfrac{1}{2i}(z-\overline{z})f(z)$$

defined on $\mathcal{D}(\dot{\mathbb{C}})$ are essentially self-adjoint in $L^2(\mathbb{C},\mu_{a,b})$ and commute. Therefore the operators $\frac{1}{2}(\mathfrak{L}+\overline{\mathfrak{L}})$, $\frac{1}{2i}(\mathfrak{L}-\overline{\mathfrak{L}})$ are essentially self-adjoint and commute on the subspace $J_{a,b}\mathcal{D}(\dot{\mathbb{C}}) \subset L^2_{\mathrm{even}}(\Lambda_{\mathbb{C}}, dK_{a,b})$. But $\mathcal{W}_{a,b}$ contains this image. This establishes Theorem 1.2.a.

Theorem 1.8.a follows from the same argumentation.

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Vladimir Molchanov

Institute for Mathematics, Natural sciences,
and informational technologies,
Tambov State University, Tambov, Russia.

Yury Neretin

Math. Dept., University of Vienna/c.o Pauli Institute
&Institute for Theoretical and Experimental Physics (Moscow);
&MechMath Dept., Moscow State University;
&Institute for Information Transmission Problems;
URL: http://mat.univie.ac.at/∼neretin/