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# Parity flow and index: constructions and applications

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**Abstract**

This note is about the topology of the path space of linear Fredholm operators on a real Hilbert space. Fitzpatrick and Pejsachowicz introduced the parity flow (or simply parity) of such a path, based on the Leray-Schauder degree of a path of parametrics. Here an alternative and more computable analytic approach is presented which makes similarities with spectral flow apparent. Furthermore the related notion of parity index of a Fredholm pair of chiral symmetries is introduced and studied. Several non-trivial examples are provided. One of them concerns topological insulators, another an application to the bifurcation of a non-linear partial differential equation.

## 1 Introduction

Spectral flow for paths of self-adjoint Fredholm operators on a complex Hilbert space is a well-known homotopy invariant [1, 16, 10]. It plays a role in connection with index theory [1, 17, 5, 8] and bifurcation theory [10, 12]. For $\mathbb{R}$-linear operators on a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, spectral flow is still a well-defined and useful object. Moreover, for paths of arbitrary (not necessarily selfadjoint) Fredholm operators on $\mathcal{H}_{\mathbb{R}}$ a $\mathbb{Z}_2$-valued parity flow has been introduced [10] and for paths of skew-adjoint Fredholm operators another $\mathbb{Z}_2$-valued orientation flow has been studied [6]. It is the purpose of this note to present a uniform treatment for the spectral, parity and orientation flow that is close to Phillips' formulation of spectral flow on complex Hilbert spaces. As [6] already takes this perspective on the orientation flow, we here mainly focus on the parity flow and show how it can by studied by considering paths of self-adjoint operators with a supplementary so-called chiral symmetry and finite-dimensional approximations thereof. This provides a new perspective on and alternative arguments for the main properties of the parity flow as used in [12]. We also believe that the presented approach makes the parity flow more accessible in computations. To avoid misunderstandings on a notational level, we speak of "parity flow" instead of simply "parity" as was done in [10], and use "orientation flow" instead of simply "$\mathbb{Z}_2$-valued spectral flow" that was used in [6]. A new element for the parity flow is an index formula for paths between conjugate Fredholm operators, see Section 6. This corresponds to analogous results for the spectral and orientation flow [17, 6].

To further stress the similarities between spectral, orientation and parity flow, let us consider the classifying spaces for real $K$-theory as introduced by Atiyah and Singer [2]. Let $\mathcal{F}^k = \mathcal{F}^k(\mathcal{H}_\mathbb{R})$ denote the space of skew-adjoint Fredholm operators on a real separable Hilbert space $\mathcal{H}_\mathbb{R}$ which anticommute with the representations $J_1, \ldots, J_{k-1}$ of the generators of a real Clifford algebra of signature $(0, k-1)$. By reducing out these relations in a concrete representation, it is possible (but tedious) to identify each $\mathcal{F}^k$ with a set of Fredholm operators on $\mathcal{H}_\mathbb{R}$ having certain supplementary symmetry relations. Relevant for the following is that $\mathcal{F}^0$ is isomorphic to the set of all Fredholm operators on $\mathcal{H}_\mathbb{R}$, $\mathcal{F}^1$ is isomorphic to the set of skew-adjoint Fredholm operators while $\mathcal{F}^7$ is isomorphic to the self-adjoint Fredholm operators on $\mathcal{H}_\mathbb{R}$. Furthermore, $\mathcal{F}^3$ is isomorphic to the set of self-adjoint Fredholm operator which are linear over the quaternions. Atiyah and Singer [2] found that the homotopy groups of these spaces satisfy

$$\pi_j(\mathcal{F}^i) \;=\; \pi_0(\mathcal{F}^{i+j}) \;=\; \pi_{j+i}(\mathcal{F}^0) \;,$$

and are given explicitly by

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\pi_0(\mathcal{F}^i)$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $2\,\mathbb{Z}$ | 0 | 0 | 0 |
| $\pi_1(\mathcal{F}^i)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $2\,\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |

(1)

Now the spectral flow provides an explicit isomorphism from $\pi_1(\mathcal{F}^7)$ to $\mathbb{Z}$, and also from $\pi_1(\mathcal{F}^3)$ to $2\,\mathbb{Z}$. Here the factor 2 merely stresses that quaternionic operators have a Kramers' degeneracy and therefore also an even spectral flow. Furthermore the orientation flow from [6] provides the isomorphism $\pi_1(\mathcal{F}^1) \cong \mathbb{Z}_2$, and the parity flow studied here gives the isomorphism $\pi_1(\mathcal{F}^0) \cong \mathbb{Z}_2$. Hence the spectral, orientation and parity flows allow to detect the topology in the last row of (1). In view of the table (1), one does not expect there to be any other flow of interest. Let us stress though that while table (1) only concerns closed loops, the definition of spectral, orientation and parity flows apply to arbitrary (open) paths.

In Section 5 the notion of parity index of a Fredholm pair of chiral symmetries is introduced. This is the parity version of Kato's index of a Fredholm pair of projections [7] as further studied by Avron, Seiler and Simon [3]. There is closely tied to the parity flow, as explained in Section 5 and of particular interest and importance for Fredholm pairs given by unitary conjugates. This leads to an index formula proved in Section 6. Finally Sections 8 and 9 give two applications of the parity flow.

# 2 Parity flow in finite dimension

The characterizing features of the parity flow can best be understood in finite dimension. Hence let us consider a (continuous) path $t \in [0,1] \mapsto T_t$ of real $N \times N$ matrices acting on the real Hilbert space $\mathcal{H}'_\mathbb{R} = \mathbb{R}^N$. Furthermore, let the path be admissible in the sense that its endpoints $T_0$ and $T_1$ are invertible, namely are in the general linear group $\mathrm{Gl}(N, \mathbb{R})$. This group has two components, specified by either a positive or a negative determinant. The parity flow of the path $t \in [0,1] \mapsto T_t$ is simply 1 if the endpoints are in the same component and $-1$ if they are in the two different components. The following provides an analytic formula for this.

**Definition 1** *For an admissible path $t \in [0,1] \mapsto T_t$ of real $N \times N$ matrices, the parity flow is defined as*

$$\mathrm{PF}(t \in [0,1] \mapsto T_t) \;=\; \mathrm{sgn}(\det(T_1))\,\mathrm{sgn}(\det(T_0)) \;\in\; \mathbb{Z}_2\;, \tag{2}$$

*where $\mathbb{Z}_2$ is viewed as the multiplicative group $\mathbb{Z}_2 = \{-1, 1\}$. As this only depends on the endpoints, we will also simply write* $\mathrm{PF}(T_0, T_1)$.

The definition directly implies that the parity flow PF of admissible paths of real matrices is a homotopy invariant (under homotopies of the path keeping the endpoints fixed), it has a concatenation property and it is normalized in the sense that the parity of a path in the invertibles is 1. Furthermore, one has an additivity property under direct sums. Formulas describing these facts are spelled out in Theorem 2 below.

For the generalization to infinite dimension there are several possibilities. The route taken by Fitzpatrick and Pejsachowicz [12] uses the fact that $\mathrm{sgn}(\det(T))$ can, under suitable conditions, be extended to infinite dimensions as the Leray-Schauder degree, for details see Section 3 below. In this note we elaborate on another possibility which consists in first rewriting Definition 1 in terms of self-adjoint matrices on a doubled Hilbert space, just as suggested by Atiyah and Singer [2] (Atiyah and Singer chose skew-adjoints, but that is roughly equivalent). This has the advantage that tools from the spectral analysis of self-adjoint operators can be used and the similarities with the spectral flow are uncovered. Hence let us use the real Hilbert space $\mathcal{H}_\mathbb{R} = \mathcal{H}'_\mathbb{R} \oplus \mathcal{H}'_\mathbb{R}$ equipped with $J = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$. Set:

$$H_t \;=\; \begin{pmatrix} 0 & T_t \\ T_t^* & 0 \end{pmatrix}\;. \tag{3}$$

These operators have a so-called chiral symmetry:

$$J\, H_t\, J \;=\; -H_t\;. \tag{4}$$

This symmetry implies that the spectrum of such an operator satisfies $\sigma(H_t) = -\sigma(H_t) \subset \mathbb{R}$. Therefore the spectral flow of the path $t \in [0,1] \mapsto H_t$ of real chiral self-adjoints vanishes. Nevertheless, there can be non-trivial topology in the path which will be detected by the parity flow in different guise. For this purpose, let us note that the endpoints $H_0$ and $H_1$ are invertible (because the initial path was admissible) and therefore there exists an invertible $A$ such that $JAJ = A$ and $H_1 = A^* H_0 A$. One readily checks that one such $A$ is given by $A = \mathrm{diag}((T_0^*)^{-1} T_1^*, \mathbf{1})$. Thus

$$\mathrm{PF}(t \in [0,1] \mapsto H_t) \;=\; \mathrm{sgn}(\det(A)) \;\in\; \mathbb{Z}_2\;, \tag{5}$$

notably this coincides with $\mathrm{PF}(t \in [0,1] \mapsto T_t)$ as defined in Definition 1. Furthermore, one can readily check that the value on the r.h.s. of (5) does not depend on the choice of $A$. Hence this is a different formulation of the parity flow.

Let us provide some examples that illustrate the topological stability associated to the parity flow. For $N = 1$ we first consider

$$H_t \;=\; \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}\;, \qquad \widetilde{H}_t \;=\; \begin{pmatrix} 0 & |t| \\ |t| & 0 \end{pmatrix}\;, \qquad t \in [-1, 1]\;. \tag{6}$$

Clearly these two paths are isospectral $\sigma(H_t) = \sigma(\widetilde{H}_t)$. By (5) one finds $\mathrm{PF}(t \in [-1,1] \mapsto H_t) = -1$ and $\mathrm{PF}(t \in [-1,1] \mapsto \widetilde{H}_t) = 1$. This has spectral consequences. The latter can be perturbed to $\widetilde{H}_t(s)$ (within the class of real, chiral, self-adjoints) in such a way that 0 is not an eigenvalue for any $t$:

$$\widetilde{H}_t(s) \;=\; \begin{pmatrix} 0 & |t|+s \\ |t|+s & 0 \end{pmatrix} \,.$$

Indeed, the eigenvalues are then $\pm(|t|+s)$ which both never vanish for positive $s > 0$. It is not possible to construct such a perturbation for $H_t$, namely any real perturbation conserving the chiral symmetry can merely shift the eigenvalue crossing at 0.

Furthermore, let us double the non-trivial example in (6) via a direct sum to $H'_t = H_t \oplus H_t$ (with a direct sum respecting the grading of $J$). Then by the additivity of the spectral flow $\mathrm{PF}(t \in [-1,1] \mapsto H'_t) = (-1)(-1) = 1$. Again it is then possible to lift the kernel along the whole path by a real chiral self-adjoint perturbation, for example as follows:

$$H'_t(s) \;=\; \begin{pmatrix} 0 & 0 & t & -s \\ 0 & 0 & s & t \\ t & s & 0 & 0 \\ -s & t & 0 & 0 \end{pmatrix} \,.$$

Indeed, the spectrum of $H'_t(s)$ is $\{(t^2+s^2)^{\frac{1}{2}}, -(t^2+s^2)^{\frac{1}{2}}\}$ with a double degeneracy. In particular, for $s \neq 0$, $H'_t(s)$ is invertible for all $t \in [-1,1]$.

# 3 Construction of the parity flow in infinite dimension

In this section, the separable real Hilbert space $\mathcal{H}'_{\mathbb{R}}$ is now of infinite dimension and the continuous path $t \in [0,1] \mapsto T_t \in \mathcal{F}^0$ is within the Fredholm operators. For sake of simplicity, let us first suppose that it lies in the component of Fredholm operators with vanishing Noether index, the general case will then be dealt with in Section 4. In [10, 12], a parity is associated to the path using the Leray-Schauder degree which is defined as follows. One first proves that there exists a second path of real invertibles $t \in [0,1] \mapsto M_t$ such that $M_t T_t = \mathbf{1} + K_t$ with a real compact operator $K_t$. Then, if $n_t$ denotes the number of negative eigenvalues of $\mathbf{1} + K_t$, the (linear) Leray-Schauder degree is

$$\deg_2(T_t) \;=\; (-1)^{n_t} \in \mathbb{Z}_2 \,. \tag{7}$$

Let us explain how this fits together with Definition 1. If $T_t$ is a matrix, one can choose $M_t = \mathbf{1}$; the spectrum of $T_t$ is symmetric w.r.t. the reflection on the real axis; now non-real eigenvalues of $T_t$ come in complex conjugate pairs which do not contribute to $\mathrm{sgn}(\det(T_t))$; hence analyzing the real eigenvalues immediately leads to $\deg_2(T_t) = \mathrm{sgn}(\det(T_t))$. For the path, the parity flow is then as in Definition 1 given by $\mathrm{PF}(t \in [0,1] \mapsto T_t) = \deg_2(T_1)\deg_2(T_0) \in \mathbb{Z}_2$ [12]. One of the difficulties with this approach is that, in general, it is very hard to determine the path $M_t$ and therefore also parity flow by this procedure.

This work provides an alternative approach which parallels the constructions in [16, 6]. It consistently uses the passage (3) to chiral self-adjoint operators. Hence let $\mathcal{H}_{\mathbb{R}} = \mathcal{H}'_{\mathbb{R}} \oplus \mathcal{H}'_{\mathbb{R}}$ be a real Hilbert space equipped with $J = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$. Then we identify $\mathcal{F}^0 = \mathcal{F}^0(\mathcal{H}'_{\mathbb{R}})$ with

$$\mathcal{F}^0 \;\cong\; \{H \in \mathcal{B}(\mathcal{H}_{\mathbb{R}}) \,|\, H = H^* = -JHJ \;\text{ Fredholm}\} \;.$$

Now let $t \in [0,1] \mapsto H_t \in \mathcal{F}^0$ be a continuous and admissible path, namely the endpoints $H_0$ and $H_1$ are invertible. Note that this implies that the associated path $t \mapsto T_t$ lies in the component of $\mathcal{F}^0$ with vanishing Fredholm index. As already mentioned, Section 4 below addresses paths in other components. The idea in the following is to reduce the definition of the $\mathbb{Z}_2$-valued parity flow to the finite dimensional definition by projecting on the central part of the spectrum on short pieces of the path, just as in [16, 6]. For $a > 0$ set

$$P_a(t) \;=\; \chi_{(-a,a)}(H_t) \;,$$

where $\chi_I$ denotes the characteristic function on $I \subset \mathbb{R}$. Due to the symmetry of $[-a, a]$, this projection commutes with $J$:

$$P_a(t) \;=\; J P_a(t) J \;.$$

Furthermore $P_a(t)$ is of finite dimensional range for $a$ sufficiently small by the Fredholm property. Associated to these projections, one has the restrictions $P_a(t)\, H_t\, P_a(t)$ which are viewed as chiral, self-adjoint operators on $\mathcal{E}_a(t) = \mathrm{Ran}(P_a(t))$. If these operators do not have vanishing kernel, this is enforced by adding a self-adjoint, chiral perturbation $R_t$ on the kernel of $P_a(t)\, H_t\, P_a(t)$. The choice of $R_t$ is not necessarily continuous in $t$, as also the dimension of the kernel varies non-continuously with $t$. Now we introduce the following selfadjoint chiral and invertible operators on $\mathcal{E}_a(t)$:

$$H_t^{(a)} \;=\; P_a(t)\, H_t\, P_a(t) \,+\, R_t \;. \tag{8}$$

By compactness, it is possible to choose a finite partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ of $[0, 1]$ and $a_n > 0$, $n = 1, \ldots, N$, such that $t \in [t_{n-1}, t_n] \mapsto P_{a_n}(t)$ is continuous and hence with constant finite rank, and, moreover, for some $\epsilon$,

$$\|P_{a_n}(t) - P_{a_n}(t')\| \;<\; \epsilon \;, \qquad \forall\;\; t, t' \in [t_{n-1}, t_n] \;. \tag{9}$$

Let $V_n : \mathcal{E}_{a_n}(t_{n-1}) \to \mathcal{E}_{a_n}(t_n)$ be the orthogonal projection of $\mathcal{E}_{a_n}(t_{n-1})$ onto $\mathcal{E}_{a_n}(t_n)$, namely $V_n v = P_{a_n}(t_n) v$. Then $V_n$ is a bijection.

**Definition 2** *For a path $t \in [0,1] \mapsto H_t \in \mathcal{F}^0$ with endpoints having trival kernel, let $t_n$ and $a_n$ as well as $H_t^{(a)}$ and $V_n$ be as above. Then the $\mathbb{Z}_2$-valued parity flow is defined by*

$$\mathrm{PF}(t \in [0,1] \mapsto H_t) \;=\; \prod_{n=1,\ldots,N} \mathrm{PF}\big(H_{t_{n-1}}^{(a_n)}, V_n^* H_{t_n}^{(a_n)} V_n\big) \;, \tag{10}$$

*where on the r.h.s. the* PF *is the finite dimensional $\mathbb{Z}_2$-valued partiy flow on $\mathcal{E}_{a_n}(t_{n-1})$ as given in* Definition 1*, and the product is in $(\mathbb{Z}_2, \cdot)$.*

Let us stress that in infinite dimension, it is in general *not* possible to denote $\mathrm{PF}(t \in [0,1] \mapsto H_t)$ by $\mathrm{PF}(H_0, H_1)$ because the parity flow depends of the choice of the path. The basic result on the parity flow is that it is well-defined by the above procedure.

**Theorem 1** *Let $t \in [0,1] \mapsto H_t \in \mathcal{F}^0$ be an admissible path. The definition of $\mathrm{PF}(t \in [0,1] \mapsto H_t)$ is independent of the choice of the partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ of $[0,1]$ and the values $a_n > 0$ such that $t \in [t_{n-1}, t_n] \mapsto P_{a_n}(t)$ is continuous and satisfies (9), and also the choice of the $R_t$ in (8).*

**Proof.** The argument is very similar to the one leading to Theorem 4.2 in [6]. In particular, it is based on Proposition 2.7 therein which allows to identify the finite dimensional subspaces at neighboring times appering in (10). These isomorphisms also commute with $J$. Then the only new element needed to complete the proof of Theorem 1 along the lines of [6] is the following lemma which replaces Proposition 2.6 in [6]. □

**Lemma 1** *Let $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ be three $J$-invariant subspaces of $\mathcal{H}_{\mathbb{R}}$ of the same finite dimension and let $H, H', H''$ (resp.) be invertible self-adjoint chiral operators. Further let $V : \mathcal{E} \to \mathcal{E}'$, $V' : \mathcal{E}' \to \mathcal{E}''$ and $V'' : \mathcal{E}'' \to \mathcal{E}$ be three isomorphisms commuting with $J$. Then*

$$\mathrm{PF}(H, V^* H' V) \;=\; \mathrm{PF}(V H V^*, H') \;,$$

*and, provided that $\|V''((V')^*)^{-1} V - \mathbf{1}_{\mathcal{E}}\| < 1$,*

$$\mathrm{PF}(H, V'' H'' (V'')^*) \;=\; \mathrm{PF}(H, V^* H' V) \;\cdot\; \mathrm{PF}(H', (V')^* H'' V') \;.$$

**Proof.** Because the parity flow is invariant under basis change,

$$\mathrm{PF}(V H V^*, H') \;=\; \mathrm{PF}(V^* V H (V^* V)^*, V^* H' V) \;.$$

Now let us go back to the definition in (5) and compare it with that of $\mathrm{PF}(H, V^* H' V)$. The $A$ from the second one is modified to $V^* V A$ (which also commutes with $J$). As $V^* V$ is a positive operator and hence has a positive determinant, it does not modify the sign of the determinant. This implies the first claim. The second follows by the argument in the proof of Proposition 2.6 in [6]. □

The main properties of the parity flow are now collected in the following result. All of these properties are already stated in Chapter 6 of [12].

**Theorem 2** *Let $t \in [0,1] \mapsto H_t \in \mathcal{F}^0$ be an admissible path.*

(i) PF *is homotopy invariant under homotopies in the paths of self-adjoint chiral Fredholm operators keeping the endpoints fixed.*

(ii) *If $H_t$ is invertible for all $t \in [0,1]$, then* $\mathrm{PF}(t \in [0,1] \mapsto H_t) = 0$.

(iii) PF *has a concatenation property, namely if* $t \in [1,2] \mapsto H_t \in \mathcal{F}^0$ *is a second admissible path, then*

$$\mathrm{PF}(t \in [0,1] \mapsto H_t) \;\cdot\; \mathrm{PF}(t \in [1,2] \mapsto H_t) \;=\; \mathrm{PF}(t \in [0,2] \mapsto H_t)\;.$$

(iv) PF *is independent of the orientation of the path:*

$$\mathrm{PF}(t \in [0,1] \mapsto H_t) \;=\; \mathrm{PF}(t \in [0,1] \mapsto H_{1-t})\;.$$

(v) PF *is additive, namely if* $t \in [0,1] \mapsto H'_t \in \mathcal{F}^0$ *is a second admissible path,*

$$\mathrm{PF}(t \in [0,1] \mapsto H_t \oplus H'_t) \;=\; \mathrm{PF}(t \in [0,1] \mapsto H_t) \cdot \mathrm{PF}(t \in [0,1] \mapsto H'_t)\;.$$

(vi) PF *is independent under reflection of the path:*

$$\mathrm{PF}(t \in [0,1] \mapsto H_t) \;=\; \mathrm{PF}(t \in [0,1] \mapsto -H_t)\;.$$

**Proof.** For the proof of item (i), one can follow [16, 17] applying Lemma 1 instead of the corresponding property for the spectral flow. All other properties immediately transpose from the finite dimensional case. □

The following result is already stated in [12].

**Theorem 3** *The map* PF *on loops in* $\mathcal{F}^0$ *is a homotopy invariant and induces an isomorphism of* $\pi_1(\mathcal{F}^0)$ *with* $\mathbb{Z}_2$*.*

**Proof.** As $\pi_1(\mathcal{F}^0) \cong \mathbb{Z}_2$ is already known [2] and PF is homotopy invariant, one only has to check that PF takes two different values on the two different components of the based loop space in $\mathcal{F}^0$. For constant paths (and thus all contractible ones) the parity flow vanishes. An example with a parity flow equal to $-1$ is given in Section 7. □

# 4 Extensions of the notion of parity flow

Up to now, only paths $t \in [0,1] \mapsto T_t$ in the component of $\mathcal{F}^0$ with vanishing index were considered. For general paths, one has

$$\dim\big(\mathrm{Ker}(H_t)\big) \;=\; \big|\mathrm{Ind}(T_t)\big| \;+\; 2\,\mathbb{N}_0\;.$$

In particular, for non-vanishing index the dimension of the kernel of $H_t$ is positive and thus there are no admissible paths. However, there are several possibilities to reduce this case to the prior one. For that purpose, let us now call a path admissible if $\dim\big(\mathrm{Ker}(H_i)\big) = \big|\mathrm{Ind}(T_i)\big| = K$ for $i = 0, 1$. Recall that $\mathrm{Ker}(H_t)$ is $J$-invariant. Let now $t \in [0,1] \mapsto P_t$ be a continuous path of $J$-invariant orthogonal projection onto parts of the kernel of $H_t$, and being of the dimension of $\mathrm{Ker}(H_i)$ for $i = 0, 1$. Then run the constructions and arguments from Section 3 to $H_t$

restricted to the range of $\mathbf{1}-P_t$. An alternative proof uses two shifts $S_i$ of index $K$ with kernel given by $\mathrm{Ker}(H_i)$ for $i=0,1$. These shifts can be chosen such that $JS_iJ=S_i$. Finally let $t\in[0,1]\mapsto S_t$ be a continuous family of shifts with index $K$ connecting these endpoints and satisfying $JS_tJ=S_t$ (this family can be constructed explicitly starting from an isomorphism from $\mathrm{Ker}(H_0)$ onto $\mathrm{Ker}(H_1)$). Now

$$\widetilde{H}_t \;=\; S_t\,H_t\,S_t^*$$

is an admissible path. Then one can set

$$\mathrm{PF}(t\in[0,1]\mapsto H_t) \;=\; \mathrm{PF}(t\in[0,1]\mapsto \widetilde{H}_t)\;.$$

One readily checks that this definition is independent of the choice of the shifts. In both cases, all properties of Theorem 2 transpose directly, except for (ii) which now states that paths with constant nullity have a vanishing parity flow.

Another modification concerns a setting with complex Hilbert spaces and a supplementary reality condition. This is of interest in connection with applications to topological insulators, see Section 8. Suppose thus that one has a complex Hilbert space $\mathcal{H}_{\mathbb{C}}$ with a real structure given by a (anti-linear involutive) complex conjugation $\mathcal{C}:\mathcal{H}_{\mathbb{C}}\to\mathcal{H}_{\mathbb{C}}$. For any linear operator $A$ on $\mathcal{H}_{\mathbb{C}}$ let us set $\overline{A}=\mathcal{C}A\mathcal{C}$. Further suppose given a real selfadjoint involution $K$, namely $\overline{K}=K^*=K$ and $K^2=\mathbf{1}$ which, moreover, commutes with $J$. Then an operator $A$ is called $K$-real if $K^*\overline{A}K=A$. Now one considers admissible paths $t\in[0,1]\mapsto H_t$ of $K$-real self-adjoint chiral operators, namely

$$K^*\overline{H_t}K \;=\; H_t\;,\qquad H_t^* \;=\; H_t\;,\qquad J^*H_tJ \;=\; -H_t\;.$$

Also for such paths one can define the parity flow. Indeed, let $L$ be the root of $K$ which spectrum $\{1,\imath\}$. It commutes with $J$. Then set

$$\widehat{H}_t \;=\; L^*H_tL\;.$$

It can be checked that $\widehat{H}_t$ is real, self-adjoint and chiral w.r.t. $J$. Consequently, $\widehat{H}_t$ can be restricted to an $\mathbb{R}$-linear operator on $\mathcal{H}_{\mathbb{R}}=\mathrm{Ker}(\mathcal{C}-\mathbf{1})\subset\mathcal{H}_{\mathbb{C}}$. Thus it is within the class of paths considered above and the parity flow of $t\in[0,1]\mapsto H_t$ can be defined as that of $t\in[0,1]\mapsto\widehat{H}_t$.

# 5 Parity index for Fredholm pairs of chiral symmetries

The aim of this section is to construct an alternative formula for the parity flow. This will first be done for special paths between symmetries that are close in the Calkin algebra, then later on it will also be extended to general paths. Recall that a symmetry $Q$ on $\mathcal{H}_{\mathbb{R}}$ is a linear, self-adjoint and unitary operator on $\mathcal{H}_{\mathbb{R}}$. It is called chiral if, moreover, $JQJ=-Q$ for another symmetry $J$. The following definition is motivated by [4, 6], as well as Kato's pair of projections and its index [7, 3].

**Definition 3** *A pair* $(Q_0, Q_1)$ *of chiral symmetries on* $\mathcal{H}_{\mathbb{R}}$ *is called a Fredholm pair of chiral symmetries if* $\|\pi(Q_0 - Q_1)\|_{\mathcal{Q}} < 1$. *Its parity index is then defined by*

$$\mathrm{PI}(Q_0, Q_1) \;=\; \tfrac{1}{2}\,\dim_{\mathbb{R}}(\mathrm{Ker}_{\mathbb{R}}(Q_0 + Q_1)) \bmod 2 \;\in\; \mathbb{Z}_2\;. \tag{11}$$

The index is indeed well-defined because $Q_0 + Q_1 \;=\; 2\,Q_0 + (Q_1 - Q_0)$ has no essential spectrum at 0 and it will be shown in the proof of Theorem 4 that the kernel of $Q_0+Q_1$ is even dimensional. On the r.h.s. of (11) the additive version of $\mathbb{Z}_2$ was used. The following justifies this definition.

**Theorem 4** *The map* $(Q_0, Q_1) \mapsto \mathrm{PI}(Q_0, Q_1) \in \mathbb{Z}_2$ *is a homotopy invariant on the set of Fredholm pairs of chiral symmetries. Moreover, for both signs one has*

$$\mathrm{PI}(Q_0, Q_1) \;=\; \dim_{\mathbb{R}}(\mathrm{Ker}_{\mathbb{R}}(Q_0 - Q_1 \pm 2\,\mathbf{1})) \bmod 2\;. \tag{12}$$

Before going into the proof, let us elaborate on the connection to the index of a Fredholm pair of projections [3]. Here there are two projections $P_0 = \frac{1}{2}(Q_0 + \mathbf{1})$ and $P_1 = \frac{1}{2}(Q_1 + \mathbf{1})$ associated to the symmetries. The property $\|\pi(Q_0 - Q_1)\|_{\mathcal{Q}} < 1$ is equivalent to $(P_0, P_1)$ being a Fredholm pair. Furthermore, these two projections are real $P_j = \overline{P_j}$ and satisfy $J\overline{P_j}J = \mathbf{1} - P_j$. In the terminology of [13] the latter means that the $P_j$ are even Lagrangian projections. These symmetries imply that the index of the Fredholm pair $(P_0, P_1)$ vanishes, namely the two signs on the r.h.s. of (12) lead to the same dimension (compare with eq. (3.1) in [3]). Hence one sees that parity index $\mathrm{PI}(Q_0, Q_1)$ is a secondary invariant associated to the Fredholm pair $(P_0, P_1)$ which is well-defined due to Theorem 4. Finally let us note that there is another $\mathbb{Z}_2$-valued index that can be associated to pairs of symmetries (or equivalently projections) which are skewadjoint on a real Hilbert space, given by the map $j$ defined in [4, 6].

The proof of Theorem 4 will be based on the following lemma in which the chiral symmetry and reality are irrelevant. The lemma can be traced back to [4]. An equivalent algebraic fact has also been used for pairs of orthogonal projections [3, Theorem 2.1].

**Lemma 2** *Let* $Q_0$ *and* $Q_1$ *be symmetries. Set*

$$T_0 \;=\; \tfrac{1}{2}\,(Q_0 + Q_1)\;, \qquad T_1 \;=\; \tfrac{1}{2}\,(Q_0 - Q_1)\;.$$

*Then the following identities hold:*

$$T_0^2 \,+\, T_1^2 \;=\; \mathbf{1}\;, \qquad T_0 T_1 \;=\; -\,T_1 T_0\;,$$

*as well as*

$$T_0 Q_0 \;=\; Q_1 T_0\;, \qquad T_1 Q_0 \;=\; -Q_1 T_1\;, \qquad T_0^2 Q_0 \;=\; Q_0 T_0^2\;, \qquad T_0 T_1 Q_0 \;=\; -Q_0 T_0 T_1\;.$$

**Proof.** Everything is verified by straightforward computations. $\Box$

**Proof** of Theorem 4. The idea of the proof is to show that every eigenvalue of $T_0$ has even degeneracy. As $T_0$ is chiral and its spectrum satisfies $\sigma(T_0) = -\sigma(T_0)$, this implies that the nullity of $T_0$ only changes by multiples of 4 under homotopic changes of $T_0$ (induced by a homotopy of $Q_0$ and $Q_1$). The main tool is to view $JQ_0$ as a complex structure on $\mathcal{H}_{\mathbb{R}}$. Indeed, $(JQ_0)^* = Q_0 J = -JQ_0$ and $(JQ_0)^2 = -\mathbf{1}$ because $JQ_0 J = -Q_0$ and $Q_0^2 = \mathbf{1}$. Now $T_0^2 JQ_0 = JQ_0 T_0^2$ so that $T_0^2$ is a complex linear operator on $\mathcal{H}_{\mathbb{R}}$ viewed as complex Hilbert space (using the complex structure $JQ_0$). Consequently the real multiplicity of all eigenvalues of $T_0^2$ is even. In particular, $\mathrm{PI}(Q_0, Q_1)$ given by (11) indeed takes values in $\{0, 1\}$. Next let us show that for $\lambda \in (0, 1)$, the complex multiplicity of the eigenspaces of $T_0^2$ is a multiple of 2 (then the real multiplicity is a multiple of 4). Suppose that $T_0^2 v = \lambda v$ for some non-vanishing vector $v$. Then set $w = T_1 T_0 v$. First of all, its norm does not vanish:

$$\|w\|^2 \;=\; v^* T_0 T_1^2 T_0 v \;=\; v^* T_0^2 (\mathbf{1} - T_0^2) v \;=\; \lambda(1-\lambda)\|v\|^2 \,.$$

Moreover, it is complex linearly independent of $v$. In fact, suppose the contrary, namely that $w = (\mu_0 + \mu_1 JQ_0)v$ for some $\mu_0, \mu_1 \in \mathbb{R}$. Multiplying this with $T_1 T_0$ leads to

$$-\lambda(1-\lambda)v \;=\; -T_0^2 T_1^2 v \;=\; T_1 T_0 w \;=\; T_1 T_0 (\mu_0 + \mu_1 JQ_0) v \;=\; (\mu_0 - \mu_1 JQ_0) w \,,$$

where in the last equality the identity $T_1 T_0 Q_0 = -Q_0 T_1 T_0$ was used. Multiplying now by $(\mu_0 + \mu_1 JQ_0)$ shows

$$-\lambda(1-\lambda) w \;=\; (\mu_0^2 + \mu_1^2) w \,,$$

that is, a contradiction. If there are further eigenvectors of $T_0^2$ with eigenvalue $\lambda$, one can restrict to the orthogonal complement and iterate the above argument.

As to the alternative formula for $\mathrm{PI}(Q_0, Q_1)$, let us note that the kernel of $T_0^2$ coincides with the eigenspace of $T_1^2$ to the eigenvalue 1, which in turn is given by the direct sum of the eigenspaces of $T_1$ for the eigenvalues 1 and $-1$. This proves the formula. Let us comment that another proof of the homotopy invariance uses the chiral symmetry of $T_1$ and checks the double degeneracy of all eigenvalues of $T_1$ in $(0, 1)$, excluding 1. □

The following result establishes the link of $\mathrm{PI}(Q_0, Q_1)$ with the parity flow of the straight line connecting $Q_0$ and $Q_1$.

**Proposition 1** *For any Fredholm pair of chiral symmetries $(Q_0, Q_1)$ on $\mathcal{H}_{\mathbb{R}}$, one has*

$$\mathrm{PF}\big(t \in [0,1] \mapsto (1-t)\,Q_0 \,+\, t\,Q_1\big) \;=\; \mathrm{PI}(Q_0, Q_1) \,.$$

**Proof.** This is essentially identical to the argument leading to Proposition 6.2 in [6], so let us just give a sketch. The operators $T_t = (1-t)Q_0 + tQ_1$ are invertible except possibly at $t = \frac{1}{2}$. Hence in Definition 2 it is sufficient to work with three intervals $[0, \frac{1}{2} - \epsilon]$, $[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ and $[\frac{1}{2} + \epsilon, 1]$ for some $\epsilon > 0$. Only the middle interval has a possibly non-vanishing contribution coming from the parity of the nullity of $T_{\frac{1}{2}} = \frac{1}{2}(Q_0 + Q_1)$. But this is precisely the definition (11) of the parity index. □

Further following [17] or [6], one can go on and rewrite the definition of the parity flow.

**Proposition 2** *Let $t \in [0,1] \mapsto H_t \in \mathcal{F}^0$ be an admissible path. Let $Q_t$ be chiral symmetries obtained by completing the phase $H_t|H_t|^{-1}$ on the kernel. Then, for a sufficiently fine partition $0 = t_0 < t_1 \cdots < t_N = 1$ satisfying $\|\pi(Q_n - Q_{n-1})\| < 1$, one has*

$$\mathrm{PF}(t \in [0,1] \mapsto H_t) \;=\; \Big(\sum_{n=1,\ldots,N} \mathrm{PI}(Q_{t_{n-1}}, Q_{t_n})\Big) \bmod 2\;.$$

**Proof.** Let us begin by rewriting Definition 2. One can choose $R_t$ in (8) sufficiently small and the partition $t_0 = 0 < t_1 < \cdots < t_N = 1$ sufficient fine such that $a_n$ from Definition 2 is not in the spectrum of $(1-t)\,(H_{t_{n-1}} + R_{t_{n-1}}) + t\,(H_{t_n} + R_{t_n})$ for any $t \in [0,1]$. By definition

$$\mathrm{PF}\big(H^{(a_n)}_{t_{n-1}}, V_n^* H^{(a_n)}_{t_n} V_n\big) \;=\; \mathrm{PF}\big(t \in [0,1] \mapsto (1-t)\,(H_{t_{n-1}} + R_{t_{n-1}}) + t\,(H_{t_n} + R_{t_n})\big)$$

and

$$\mathrm{PF}(t \in [0,1] \mapsto H_t) \;=\; \prod_{n=1,\ldots,N} \mathrm{PF}\big(t \in [0,1] \mapsto (1-t)\,(H_{t_{n-1}} + R_{t_{n-1}}) + t\,(H_{t_n} + R_{t_n})\big)\;.$$

By identifying $H_{t_n} + R_{t_n}$ with $H_{t_n}$, we can from now on assume that $H_{t_n}$ is invertible. Next let us claim that for each $n = 1, \ldots, N$ one has

$$\mathrm{PF}\big(t \in [0,1] \mapsto (1-t)\,H_{t_{n-1}} + t\,H_{t_n}\big) \;=\; \mathrm{PF}\big(t \in [0,1] \mapsto (1-t)\,Q_{t_{n-1}} + t\,Q_{t_n}\big)\;.$$

Indeed, as $H_{t_n}$ and $H_{t_{n-1}}$ are both invertible,

$$s \in [0,1] \;\mapsto\; (1-t)\,H_{t_{n-1}}|H_{t_{n-1}}|^{-s} + t\,(H_{t_n}|H_{t_n}|^{-s}$$

deforms the initial path into the path $t \in [0,1] \mapsto (1-t)\,Q_{t_{n-1}} + t\,Q_{t_n}$. During this homotopy the endpoints remain invertible so that the parity flow remains unchanged. Proposition 1 allows to conclude. $\square$

# 6 Parity flow of paths between unitary conjugates

As above, let $\mathcal{H}_\mathbb{R} = \mathcal{H}'_\mathbb{R} \oplus \mathcal{H}'_\mathbb{R}$ be equipped with $J = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$. The orthogonal group conserving $J$ is

$$\mathcal{O}(\mathcal{H}_\mathbb{R}, J) \;=\; \{O \in \mathcal{O}(\mathcal{H}_\mathbb{R}) \mid O^* J O = J\}\;.$$

This is a subgroup of $\mathcal{O}(\mathcal{H}_\mathbb{R})$ naturally identified with $\mathcal{O}(\mathcal{H}'_\mathbb{R}) \times \mathcal{O}(\mathcal{H}'_\mathbb{R})$ because $O^* J O = J$ is equivalent to $JOJ = O$ which requires $O$ to be diagonal in the grading of $J$. For any real chiral symmetry $Q$, let us set

$$\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J) \;=\; \{O \in \mathcal{O}(\mathcal{H}_\mathbb{R}, J) \mid [O, Q] \in \mathcal{K}(\mathcal{H}_\mathbb{R})\}\;, \tag{13}$$

where $\mathcal{K}(\mathcal{H}_\mathbb{R})$ denotes the compact operators on $\mathcal{H}_\mathbb{R}$. This is a subgroup of $\mathcal{O}(\mathcal{H}_\mathbb{R}, J)$. Let us note that for $O \in \mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$ one has $\pi(O^* Q O) = \pi(Q)$ in the Calkin algebra.

**Theorem 5** *For any chiral symmetry $Q$ on $\mathcal{H}_\mathbb{R}$, the based loop space $\Omega_Q\mathcal{F}^0$ of $\mathcal{F}^0$ is homotopy equivalent to $\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$. In particular, $\pi_0(\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)) \cong \mathbb{Z}_2$.*

As a preparatory result for the proof, let us state the following.

**Proposition 3** *$\mathcal{F}^0$ is homotopy equivalent to the set*

$$\mathcal{C}(\mathcal{H}_\mathbb{R}) \;=\; \{\pi(Q) \in \mathcal{Q} \,|\, \pi(Q) \text{ chiral symmetry}\} \;.$$

**Proof.** We follow closely the proof of Theorem 7.1 of [6], which in turn is based on [2, 17]. Let $\rho : \mathcal{F}^0 \to \mathcal{F}^0$ be the (non-linear and discontinuous) map sending $H$ to the partial isometry $Q = H|H|^{-1}$ in the polar decomposition. If $\pi$ denotes the projection onto the Calkin algebra $\mathcal{Q} = \mathcal{Q}(\mathcal{H}_\mathbb{R})$ over $\mathcal{H}_\mathbb{R}$, then the map $\hat{\rho} = \pi \circ \rho$ sends $\mathcal{F}^0$ surjectively onto $\mathcal{C}(\mathcal{H}_\mathbb{R})$. Indeed, any chiral symmetry $\pi(Q) \in \mathcal{Q}$ has a chiral and self-adjoint lift $Q'$ for which $(Q')^2 - \mathbf{1}$ is compact; then the Riez projections $P'_\pm$ on the positive and negative spectral projections of $Q'$ lead to a lift $Q = P_+ - P_-$ for which $Q^2 - \mathbf{1}$ is a finite dimensional projection (on the kernel of $Q'$). The Bartle-Graves selection theorem now provides a right inverse $\theta : \mathcal{C}(\mathcal{H}_\mathbb{R}) \to \mathcal{F}^0$ to $\hat{\rho}$, namely $\hat{\rho} \circ \theta = \mathbf{1}$. Moreover, $\theta \circ \hat{\rho}$ is homotopic to the identity via $t \in [0, 1] \mapsto t\,H + (1-t)\,\theta(\hat{\rho}(H))$. As $\theta(\hat{\rho}(H)) = H|H|^{-1} + K$ for some chiral self-adjoint compact $K$, this is a homotopy in $\mathcal{F}^0$. Thus $\hat{\rho}$ is actually a homotopy equivalence so that $\mathcal{F}^0$ and $\mathcal{C}(\mathcal{H}_\mathbb{R})$ are homotopy equivalent. □

**Proof** of Theorem 5. Due to Proposition 3 it is sufficient to show the homotopy equivalence of $\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$ and $\Omega_{\hat{\rho}(Q)}\mathcal{C}(\mathcal{H}_\mathbb{R})$. Here, the chiral symmetry $Q$ on $\mathcal{H}_\mathbb{R}$ which also specifies a base point $\hat{\rho}(Q)$ in $\mathcal{C}(\mathcal{H}_\mathbb{R})$. Associated to $Q$, one can define a map $\beta_Q : \mathcal{O}(\mathcal{H}_\mathbb{R}, J) \to \mathcal{C}(\mathcal{H}_\mathbb{R})$ via $\beta_Q(O) = \hat{\rho}(OQO^*) = \pi(OQO^*)$. This map is actually a Serre fibration by the argument in Theorem 3.9 of [15]. The fiber over the base point $\hat{\rho}(Q) = \pi(Q)$ is precisely the set $\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$ from (13). Hence one can dispose of the long exact sequence of homotopy groups, which due to the triviality of the homotopy groups of $\mathcal{O}(\mathcal{H}_\mathbb{R}, J)$ implies that the set $\Omega_{\hat{\rho}(Q)}\mathcal{C}(\mathcal{H}_\mathbb{R})$ of based loops in the base space is homotopy equivalent to the fiber over the base point which here is $\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$. Because the loop functor respects homotopy, we conclude from the above that the based loop space $\Omega_Q\mathcal{F}^0$ is homotopy equivalent to $\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$. The last claim follows from $\pi_1(\mathcal{F}^0) \cong \mathbb{Z}_2$. □

It is possible to use the index map $j_Q : \mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J) \to \mathbb{Z}_2$ defined by

$$j_Q(O) \;=\; \mathrm{PI}(Q, OQO^*) \;.$$

to distinguish the two components $\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$. Furthermore, applying Theorem 4 and Proposition 1 to $Q_0 = Q$ and $Q_1 = OQO^*$ leads to the following.

**Corollary 1** *For any chiral symmetry $Q$, $j_Q$ is a homotopy invariant homomorphism labelling the two components of $\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$. One has*

$$j_Q(O) \;=\; \mathrm{PF}\big(t \in [0, 1] \mapsto (1-t)\,Q \,+\, t\,OQO^*\big) \;.$$

The (Noether) index of a Toeplitz operator associated to a given index pairing can always be expressed as a spectral flow [16, 17, 8]. The following result is the parity flow version of this result, similar as [6] contains a corresponding result for the orientation flow.

**Theorem 6** *Let $Q$ be a real chiral symmetry on $\mathcal{H}_\mathbb{R}$ and $O \in \mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$. If $P$ is the spectral projection onto the positive spectrum of $Q$, then*

$$\mathrm{PF}\big(t \in [0,1] \mapsto (1-t)Q + tOQO^*\big) \;=\; \dim_\mathbb{R}\big(\mathrm{Ker}_\mathbb{R}(POP + \mathbf{1} - P)\big) \bmod 2 \;.$$

**Proof.** First of all, the $\mathbb{Z}_2$-index on the r.h.s. is of the type $(j,d) = (1,8)$ in Theorem 1 of [13]. Indeed, $P = \overline{P}$ satisfies $J\overline{P}J = \mathbf{1} - P$ (namely, $P$ is even real and even Lagrangian in the terminology of [13]) as well as $J\overline{O}J = O$. In particular, the index pairing on the r.h.s. is a homotopy invariant under variations of $O$ and $P$ respecting all the properties mentioned above. Now given $Q$, the set $\mathcal{O}_Q(\mathcal{H}_\mathbb{R}, J)$ has two components by Theorem 5. The proof of Theorem 6 is thus remarkably simple. Both sides of the equality are homotopy invariants and lie in $\mathbb{Z}_2$. Hence it is sufficient to verify equality on both components. For $O = \mathbf{1}$, both sides vanish. For the other component, the equality is verified for a non-trivial example in the next section (which can be used after having passed to the spectral respresentation of $Q$). $\square$

# 7 A non-trivial example

Let $p$ be a one-dimensional projection on an infinite-dimensional Hilbert space $\mathcal{H}'_\mathbb{R}$. We consider

$$Q \;=\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} , \qquad O \;=\; \begin{pmatrix} \mathbf{1} - 2p & 0 \\ 0 & \mathbf{1} \end{pmatrix} .$$

Then all conditions in Theorem 6 are satisfied. One has

$$P \;=\; \frac{1}{2}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} ,$$

and

$$POP + (\mathbf{1} - P) \;=\; \mathbf{1} - \frac{1}{2}\begin{pmatrix} p & p \\ p & p \end{pmatrix} .$$

In particular, $\dim(\mathrm{Ker}(POP)) = \dim(p) = 1$. Hence the index on the r.h.s. of Theorem 6 is equal to 1. On the other hand, the straight-line path is

$$Q_t \;=\; (1-t)\,Q \,+\, t\,O^*QO \;=\; \begin{pmatrix} 0 & \mathbf{1} - 2\,t\,p \\ \mathbf{1} - 2\,t\,p & 0 \end{pmatrix} .$$

This hence contains exactly one copy of the example (6), so $\mathrm{PF}(t \in [0,1] \mapsto Q_t) = -1$ (notably, the non-trivial value).

The above path can be completed to a loop with $t \in [1,2] \mapsto Q_t = O_t^*QO_t$ where $t \in [1,2] \mapsto O_t$ is a Kuipers path connecting $O$ to $\mathbf{1}$. As this second path is in the invertible it has no parity flow. Therefore $t \in [0,2] \mapsto Q_t$ is a loop with non-trivial parity flow. This provides the example needed in the proof of Theorem 3.

Let us also calculate the parity flow of $t \in [0,1] \mapsto Q_t$ as in [12]. One needs to look at the off-diagonal entry $T_t$ as in (3), and determine an invertible operator $M_t$ such that $M_tT_t - \mathbf{1} = K_t$ is compact. Clearly, $M_t = \mathbf{1}$ will do, and then $K_0 = 0$ and $K_1 = -2p$ so that $\deg_2(T_0) = 1$ and $\deg_2(T_1) = -1$. Thus one finds again that the parity flow of the path is $-1$.

# 8 Application to a topological insulator

Let $\mathcal{H}'_{\mathbb{C}} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ and consider the following operator on $\mathcal{H}_{\mathbb{C}} = \mathcal{H}'_{\mathbb{C}} \oplus \mathcal{H}'_{\mathbb{C}}$:

$$H_t \;=\; \begin{pmatrix} 0 & (S_t)^k \otimes \mathbf{1}_N \\ (S_t^*)^k \otimes \mathbf{1}_N & 0 \end{pmatrix} \;,$$

where $k \in \mathbb{Z}$ and $S_t$ is the bilateral shift perturbed on one link from 1 to $\cos(\pi t)$, namely in Dirac notation

$$S_t \;=\; \sum_{n \neq 0} |n\rangle\langle n+1| \;+\; \cos(\pi t)\,|0\rangle\langle 1| \;.$$

The Hamiltonian has the chiral symmetry (4) and is real as well as selfadjoint for all $t$:

$$H_t \;=\; H_t^* \;=\; \overline{H_t} \;=\; -JH_tJ \;.$$

Thus $t \in [0,1] \mapsto H_t$ is a path in $\mathcal{F}^0$ and it is possible to consider the parity flow. One finds

$$\mathrm{PF}(t \in [0,1] \mapsto H_t) \;=\; (-1)^{kN} \;.$$

This property is now stable under any kind of perturbations not closing the spectral gap of $H_0$, such as a chiral disordered potential $V_\omega = -JV_\omega J$ of moderate strength. Here $\omega$ is a point in a compact W*-dynamical system $(\Omega, T, \mathbb{Z}, \mathbb{P})$ given by the shift action $T$ of $\mathbb{Z}$ and an invariant and ergodic probability measure $\mathbb{P}$. Let us comment that the non-triviality of the path $t \in [0,1] \mapsto H_t$ has nothing to do the with the strong invariants appearing in the periodic table of topological insulators. The Hamiltonian has an even time-reversal symmetry and a chiral symmetry. Hence it lies in the so-called BDI class. As such, in dimension $d = 1$ there are infinitely many distinct phases labelled by the strong invariant, which in the above example is the number $kN$ specifying the winding of the off-diagonal entry of $H_0$. For each $k$, the Hamiltonian is then in the corresponding component of Fredholm operators and stays within it along the path $t \in [0,1] \mapsto H_t$, because it resulted from a merely local perturbation of $H_0$. It is now a fact that such paths can be topologically non-trivial because the fundamental group of $\mathcal{F}^0$ is $\mathbb{Z}_2$. The parity flow detects this topology.

Let us now come to the physical implications of the non-trivial parity flow. One can directly conclude that $H_t$ has to have an eigenvalue crossing through 0 at some $t \in [0,1]$. However, more can be said, namely one such eigenvalue crossing has to take place at half flux.

**Theorem 7** *If $kN$ is odd, then $H_{\frac{1}{2}}$ has an odd number of evenly degenerate zero modes, namely the multiplicity of $0$ as eigenvalue is $2$ modulo $4$.*

**Proof.** Let us introduce the gauge transformation

$$G \;=\; \sum_{n>0} |n\rangle\langle n| \;-\; \sum_{n \leq 0} |n\rangle\langle n| \;.$$

Then $GS_tG = S_{1-t}$ so that $GH_tG = H_{1-t}$. Consequently the zero eigenvalue crossings for $t$ lead to zero eigenvalue crossings for $1-t$. As the parity flow is invariant under a change of orientation and also unitary conjugations, these eigenvalue crossing cancel and do not lead to a net parity, except at $t = \frac{1}{2}$. This implies that at $t = \frac{1}{2}$ one has to have an odd number of eigenvalue crossings. Consequently, the multiplicity of the zero eigenvalue is 2 modulo 4. $\square$

Note that at half-flux, the shift does not connect left and right half-space so that $H_{\frac{1}{2}}$ is a diret sum of a left and a right half-space Hamiltonian. Each has to have a zero mode, leading to the two-fold symmetry. Let us further add a few comments on how to interprete Theorem 7 against the background of the periodic table of topological insulators. As already stated, all the above concerns Hamiltonians are from the BDI class of chiral Hamiltonians with an even time-reversal symmetry (integer spin). Also the Hamiltonian $H_{\frac{1}{2}}$ is within this class. If one extracts only the low lying spectrum (eigenvalues in the vicinity of 0), this reduced Hamiltonian is a finite dimensional matrix and hence represents a system of dimension $d = 0$ (corresponding to the local defect induced by a half-flux). The set of 0-dimensional BDI Hamiltonians has two components which are distinguished by the parity of the zero modes (of each half-sided Hamiltonian). Theorem 7 states that $H_{\frac{1}{2}}$ is in the non-trivial component of the 0-dimensional BDI Hamiltonians always having a zero mode.

# 9 Application to bifurcation theory

The aim of this final section is to apply the parity flow in the bifurcation theory of solutions to nonlinear operator equations depending on a real parameter. It was precisely for this purpose that the parity was originally introduced and put to work [10, 11, 12]. The treatment given in this note suggests to construct examples with a selfadjoint linearization which has a chiral symmetry built in. This is essentially what is done below.

Let us begin by exposing the theoretical framework of bifurcation theory and the main result used later on. Given two real Banach spaces $X$ and $Y$ and an interval $I \subset \mathbb{R}$, one considers continuous maps $F : I \times X \to Y$ for which we assume throughout that $F(t, 0) = 0$ for all $t \in I$. In this context, one then calls the set $I \times \{0\}$ the trivial branch of solutions. A bifurcation point for the family of equations $F(t, u) = 0$, $t \in I$, is a parameter value $t^*$ where a new branch of solutions appears.

**Definition 4** *A parameter value $t^* \in I$ is a bifurcation point for the family of equations $F(t, u) = 0$ if in every neighbourhood of $(t^*, 0) \in I \times X$ there is some $(t, u)$ such that $u \neq 0$ and $F(t, u) = 0$.*

Let us now assume that the map $F$ is continuously differentiable in $u$. The implicit function theorem then implies that if the linear map $D_uF(t^*, 0) : X \to Y$ is invertible, there is a neighbourhood of $(t^*, 0)$ in $I \times X$ for which there is a unique solution of the equation $F(t, u) = 0$. As $F(t, 0) = 0$ by assumption, we see that $t^*$ cannot be a bifurcation point in this case. Consequently, $D_uF(t^*, 0)$ must be singular if $t^*$ is a bifurcation point. Let us stress, however, that not every $t^*$ for which $D_uF(t^*, 0)$ is singular, is necessarily a bifurcation point. The aim of bifurcation theory is to find sufficient conditions under which a singular point $t^*$ is a bifurcation point. While such problems have been considered for centuries, topological criteria for existence of bifurcations in an infinite dimensional set-up were only made by Krasnoselskii in the sixties [14].

One extension of his ideas is the work of Fitzpatrick and Pejsachowicz [11] which uses the parity flow and is described next. For a continuously differentiable $F$ let us the bounded linear operator $T_t = D_u F(t,0)$ and suppose that their index vanishes. As $t \in I \mapsto T_t$ is a continuous path by assumption, its parity flow is defined.

**Theorem 8 ([11])** *Suppose that $t \in I \mapsto T_t$ is an admissible path. If $\mathrm{PF}(t \in I \mapsto T_t) = -1$, then there is a bifurcation point $t^* \in I$ for the family of equations $F(t,u) = 0$.*

In the following, we provide an example of a parameter dependent system of partial differential equations for which the parity flow can be calculated explicitly. On $\Omega = (0,\pi) \times (0,\pi)$ let us consider the family of elliptic systems parametrized by $t \in \mathbb{R}$

$$\begin{cases} -\Delta u \;=\; tv + f(t,x,u,v) \;, & \text{in } \Omega \;, \\ -\Delta v \;=\; tu + g(t,x,u,v) \;, & \text{in } \Omega \;, \\ \quad u \;=\; v \;=\; 0 \;, & \text{on } \partial\Omega \;, \end{cases} \tag{14}$$

where $u, v : \Omega \to \mathbb{R}$ and $f, g : \mathbb{R} \times \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. We assume that $f(t,x,0,0) = g(t,x,0,0) = 0$ for all $(t,x) \in \mathbb{R} \times \Omega$ so that $(u,v) = (0,0)$ is a solution of (14) for all $t \in \mathbb{R}$. Moreover, all partial derivatives of $f$ and $g$ with respect to $u$ and $v$ are supposed to be bounded and satisfy

$$D_{(u,v)} f(t,x,0,0) \;=\; D_{(u,v)} g(t,x,0,0) \;=\; 0 \;, \qquad (t,x) \in \mathbb{R} \times \Omega \;. \tag{15}$$

As the Laplacian as operator on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H^1_0(\Omega)$ is invertible with compact resolvent, one can transform the first two equations of (14) to the system

$$F(t,u,v) \;=\; \begin{pmatrix} u \\ v \end{pmatrix} + t \begin{pmatrix} Kv \\ Ku \end{pmatrix} + \begin{pmatrix} Kf(t,x,u,v) \\ Kg(t,x,u,v) \end{pmatrix} \;=\; 0 \;,$$

where $K = \Delta^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact. The assumptions on $f$, $g$ and the compactness of $K$ imply that $F : \mathbb{R} \times L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega, \mathbb{R}^2)$ is differentiable. Moreover, the derivative at $(0,0)$ of the nonlinear part vanishes by (15) and therefore

$$T_t(u,v) \;=\; D_{(u,v)} F(t,0,0)(u,v) \;=\; \begin{pmatrix} u \\ v \end{pmatrix} + t \begin{pmatrix} Kv \\ Ku \end{pmatrix} \;.$$

By applying $\Delta$ to each component of the equation $T_t(u,v) = 0$, one checks that

$$(u(x), v(x)) \;=\; (\sin(x_1)\sin(x_2), \sin(x_1)\sin(x_2)) \;, \qquad x \;=\; (x_1, x_2) \;\in\; \Omega \;,$$

is in the kernel of $T_2$. To find out if $t^* = 2$ is a bifurcation point of $F(t,u,v) = 0$, let us now compute $\mathrm{PF}(t \in [2-\delta, 2+\delta] \mapsto T_t)$. First of all, one needs to consider the eigenvalue problem

$$\begin{pmatrix} 0 & T_t \\ T_t^* & 0 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \;=\; \mu \begin{pmatrix} z \\ w \end{pmatrix} \;.$$

By setting $z = (u_1, v_1)$ and $w = (u_2, v_2)$ and applying the Laplace operator in each component, this amounts to solve the system of equations

$$\begin{cases} \Delta u_2 + t v_2 \;=\; \mu \Delta u_1 \,, & \text{in } \Omega \,, \\ \Delta v_2 + t u_2 \;=\; \mu \Delta v_1 \,, & \text{in } \Omega \,, \\ \Delta u_1 + t v_1 \;=\; \mu \Delta u_2 \,, & \text{in } \Omega \,, \\ \Delta v_1 + t u_1 \;=\; \mu \Delta v_2 \,, & \text{in } \Omega \,, \\ u_1 \;=\; u_2 \;=\; v_1 \;=\; v_2 \;=\; 0 \,, & \text{on } \partial\Omega \,. \end{cases} \tag{16}$$

Setting for integer $k_j, m_j, l_j, n_j$ where $j = 1, 2$,

$$\begin{aligned} u_1(x) &= \sin(k_1 x_1)\sin(k_2 x_2), & u_2(x) &= \sin(m_1 x_1)\sin(m_2 x_2) \,, \\ v_1(x) &= \sin(l_1 x_1)\sin(l_2 x_2), & v_2(x) &= \sin(n_1 x_1)\sin(n_2 x_2) \,, \end{aligned}$$

the equations (16) are equivalent to

$$\begin{cases} -(m_1^2 + m_2^2) u_2 + t v_2 \;=\; -\mu (k_1^2 + k_2^2) u_1 \,, & \text{in } \Omega \,, \\ -(n_1^2 + n_2^2) v_2 + t u_2 \;=\; -\mu (l_1^2 + l_2^2) v_1 \,, & \text{in } \Omega \,, \\ -(k_1^2 + k_2^2) u_1 + t v_1 \;=\; -\mu (m_1^2 + m_2^2) u_2 \,, & \text{in } \Omega \,, \\ -(l_1^2 + l_2^2) v_1 + t u_1 \;=\; -\mu (n_1^2 + n_2^2) v_2 \,, & \text{in } \Omega \,. \end{cases}$$

It is readily seen that for $t$ close to 2, one can only have an eigenvalue crossing zero in the subspace of $L^2(\Omega, \mathbb{R}^2) \oplus L^2(\Omega, \mathbb{R}^2)$ spanned by $(u_1, 0, 0, 0)$, $(0, v_1, 0, 0)$, $(0, 0, u_2, 0)$ and $(0, 0, 0, v_2)$ when $k_1 = k_2 = m_1 = m_2 = l_1 = l_2 = n_1 = n_2 = 1$. The four eigenvalues in this subspace are

$$\lambda_1 \;=\; -\tfrac{1}{2}(t-2) \,, \quad \lambda_2 \;=\; \tfrac{1}{2}(t-2) \,, \quad \lambda_3 \;=\; -\tfrac{1}{2}(t+2) \,, \quad \lambda_4 \;=\; \tfrac{1}{2}(t+2) \,.$$

The eigenvalue crossing is simple and analytic, and of the type of the first example in (6). Consequently, $\mathrm{PF}(t \in [2-\delta, 2+\delta] \mapsto T_t) = -1$ for all small $\delta > 0$ and thus $t^* = 2$ is indeed a bifurcation point for (14) by Theorem 8.

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