Asymptotic behavior of the $W^{1/q,q}$ -norm of mollified BV functions and applications to singular perturbation problems

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Abstract

Motivated by results of Figalli and Jerison [8] and Hernández [7], we prove the following formula:

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|\eta_{\varepsilon} * u\|_{W^{1/q,q}(\Omega)}^q = C_0 \int_{J_u} |u^+(x) - u^-(x)|^q d\mathcal{H}^{N-1}(x),$$

where $\Omega \subset \mathbb{R}^N$ is a regular domain, $u \in BV(\Omega) \cap L^{\infty}$, q > 1 and $\eta_{\varepsilon}(z) = \varepsilon^{-N}\eta(z/\varepsilon)$ is a smooth mollifier. In addition, we apply the above formula to the study of certain singular perturbation problems.

1 Introduction

Figalli and Jerison found in [8] a relationship between the perimeter of a set and a fractional Sobolev norm of its characteristic function. More precisely, for the mollifying kernel $\eta_{\varepsilon}(z) = \varepsilon^{-N}\eta(z/\varepsilon)$, where $\eta(z)$ denotes the standard Gaussian in \mathbb{R}^N , they showed that there exist constants $C_1 > 0$ and $C_2 > 0$ such that for every set $A \subset \mathbb{R}^N$ of finite perimeter P(A) we have

$$C_1 P(A) \le \liminf_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|\eta_\varepsilon * \chi_A\|_{H^{1/2}(\mathbb{R}^N)}^2 \le \limsup_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|\eta_\varepsilon * \chi_A\|_{H^{1/2}(\mathbb{R}^N)}^2 \le C_2 P(A), \quad (1.1)$$

where χ_A is the characteristic function of A. More recently, Hernández improved this result in [7] as follows. For η_{ε} as above he showed that there exist a constant $C_0 > 0$ such that for every $u \in BV(\mathbb{R}^N) \cap L^{\infty}$ we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|\eta_{\varepsilon} * u\|_{H^{1/2}(\mathbb{R}^N)}^2 = C_0 \int_{J_u} |u^+(x) - u^-(x)|^2 d\mathcal{H}^{N-1}(x). \tag{1.2}$$

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A related result in which the same R.H.S. as in (1.2) appears, was obtained in [13]. More precisely, we showed in [13] that for every radial $\eta \in C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ there exists a constant $C = C_{\eta} > 0$ such that for every $u \in BV(\Omega, \mathbb{R}^d) \cap L^{\infty}$ we have

$$\lim_{\varepsilon \to 0^+} \varepsilon \left\| \eta_{\varepsilon} * u \right\|_{H^1(\Omega)}^2 = C_\eta \int_{J_u} \left| u^+(x) - u^-(x) \right|^2 d\mathcal{H}^{N-1}(x). \tag{1.3}$$

More recently, we showed in [14] yet another related result:

Theorem 1.1. Let $\Omega \subset \mathbb{R}^N$ be an open set with bounded Lipschitz boundary and let $u \in BV(\Omega, \mathbb{R}^d) \cap L^{\infty}(\Omega, \mathbb{R}^d)$. Then, for every q > 1 we have

$$\lim_{\varepsilon \to 0^+} \int_{\Omega} \int_{B_{\varepsilon}(x) \cap \Omega} \frac{1}{\varepsilon^N} \frac{\left| u(y) - u(x) \right|^q}{|y - x|} dy dx = C_N \int_{J_u} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x), \tag{1.4}$$

with the dimensional constant $C_N > 0$ defined by

$$C_N := \frac{1}{N} \int_{S^{N-1}} |z_1| d\mathcal{H}^{N-1}(z), \qquad (1.5)$$

where we denote $z := (z_1, \ldots, z_N) \in \mathbb{R}^N$.

In the present paper we generalize the formula (1.2) in several aspects:

- We allow a general mollifying kernel $\eta \in W^{1,1}(\mathbb{R}^N,\mathbb{R})$ (not only the Gaussian as before),
- We allow a general domain $\Omega \subset \mathbb{R}^N$, of certain regularity, while previous results required $\Omega = \mathbb{R}^N$,
- We treat the $W^{1/q,q}(\Omega)$ -norm for any q>1, while previous results were restricted to the case q=2.

Recall that the Gagliardo seminorm $||u||_{W^{1/q,q}(\Omega,\mathbb{R}^d)}$ is given by

$$||u||_{W^{1/q,q}(\Omega,\mathbb{R}^d)} := \left(\int_{\Omega} \left(\int_{\Omega} \frac{|u(x) - u(y)|^q}{|x - y|^{N+1}} dy \right) dx \right)^{\frac{1}{q}}.$$
 (1.6)

Our first main result is

Theorem 1.2. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^N, \mathbb{R}^d)$ be such that $||Du||(\partial\Omega) = 0$. For $\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})$, every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ define

$$u_{\varepsilon}(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y-x}{\varepsilon}\right) u(y) dy = (\eta_{\varepsilon} * u)(x). \tag{1.7}$$

Then, for any q > 1 we have

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{|\ln \varepsilon|} \|u_{\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} = 2 \left| \int_{\mathbb{R}^{N}} \eta(z) dz \right|^{q} \left(\int_{\mathbb{R}^{N-1}} \frac{dv}{\left(\sqrt{1+|v|^{2}}\right)^{N+1}} \right) \int_{J_{u}\cap\Omega} \left| u^{+}(x) - u^{-}(x) \right|^{q} d\mathcal{H}^{N-1}(x). \quad (1.8)$$

Theorem 1.2 enables us to prove an upper bound, in the limit $\varepsilon \to 0^+$, for the following singular perturbation functionals with differential constraints:

(i)
$$E_{\varepsilon}^{(1)}(v) := \begin{cases} \frac{1}{|\ln \varepsilon|} \|v\|_{W^{1/q,q}(\Omega,\mathbb{R}^d)}^q + \frac{1}{\varepsilon} \int_{\Omega} W(v,x) dx & \text{if } A \cdot \nabla v = 0 \\ +\infty & \text{otherwise,} \end{cases}$$
for $v : \Omega \to \mathbb{R}^d$:

(ii)
$$E_{\varepsilon}^{(2)}(v) := \begin{cases} \frac{1}{|\ln \varepsilon|} \left(\|v\|_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)}^q - \|v\|_{W^{1/q,q}(\mathbb{R}^N\setminus\overline{\Omega},\mathbb{R}^d)}^q \right) + \frac{1}{\varepsilon} \int_{\Omega} W(v,x) dx & \text{if } A \cdot \nabla v = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

$$\text{for } v : \mathbb{R}^N \to \mathbb{R}^d.$$

$$(1.10)$$

In both cases $A: \mathbb{R}^{d \times N} \to \mathbb{R}^l$ is a linear operator (possibly trivial). The most important particular cases are the following:

(a) $A \equiv 0$ (i.e., without any prescribed differential constraint),

(b)
$$d = N$$
, $l = N^2$ and $A \cdot \nabla v \equiv \operatorname{curl} v := \{\partial_k v_j - \partial_j v_k\}_{1 \le k, j \le N}$

(c)
$$l = d$$
 and $A \cdot \nabla v \equiv \operatorname{div} v$.

The Γ -limit of the functional (1.9) in the L^p -topology when $A \equiv 0$, q = 2, N = 1 and W is a double-well potential was found by Alberti, Bouchitté and Seppecher [1]. The result was generalized to any dimension $N \geq 1$, for the functional (1.10), by Savin and Valdinoci [15].

Note that the functional (1.9) resembles the energy functional in the following singular perturbation problem:

$$\hat{E}_{\varepsilon}(v) := \begin{cases} \varepsilon^{q-1} \|v\|_{W^{1,q}(\Omega,\mathbb{R}^d)}^q + \frac{1}{\varepsilon} \int_{\Omega} W(v,x) dx & \text{if } A \cdot \nabla v = 0 \\ +\infty & \text{otherwise,} \end{cases}$$
(1.11)

that attracted a lot of attention by many authors, starting from Modica and Mortola [10], Modica [9], Sternberg [16] and others, who studied the basic special case of (1.11) with $A \equiv 0$, q = 2 and W being a double-well potential. The Γ limit of (1.11) with $A \equiv 0$, q = 2 and a general $W \in C^0$ that does not depend on x, was found by Ambrosio in [2]. As an example with a nontrivial differential constraint we mention the Aviles-Giga functional, that appear in various applications. It is defined for scalar functions ψ by

$$\tilde{E}_{\varepsilon}(\psi) := \int_{\Omega} \left\{ \varepsilon |\nabla^2 \psi|^2 + \frac{1}{\varepsilon} \left(1 - |\nabla \psi|^2 \right)^2 \right\} dx \qquad (\text{see } [3, 5, 6]), \tag{1.12}$$

and the objective is to study the Γ -limit, as $\varepsilon \to 0^+$. This can be seen as a special case of (1.11) if we set $v := \nabla \psi$ and let $A \cdot \nabla v \equiv \operatorname{curl} v$, q = 2 and $W(v, x) = (1 - |v|^2)^2$.

Our second result provides an upper bound for the energies (1.9)-(1.10):

Theorem 1.3. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $W : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}$ be a Borel measurable nonnegative function, continuous and continuously differentiable w.r.t. the first argument, such that $W(0,\cdot) \in L^1(\Omega,\mathbb{R})$. Assume further that for every D > 0 there exists $C := C_D > 0$ such that

$$\left|\nabla_b W(b, x)\right| \le C_D \qquad \forall x \in \mathbb{R}^N, \ \forall \ b \in B_D(0).$$
 (1.13)

Let $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^N, \mathbb{R}^d)$ be such that W(u(x), x) = 0 a.e. in Ω , $||Du||(\partial\Omega) = 0$, and $A \cdot Du = 0$ in \mathbb{R}^N , where $A : \mathbb{R}^{d \times N} \to \mathbb{R}^l$ is a prescribed linear operator (possibly trivial). Then, for any q > 1 there exists a sequence of functions $\{\psi_{\varepsilon}\}_{\varepsilon>0} \subset C^{\infty}(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^d)$ such that $A \cdot D\psi_{\varepsilon} = 0$ in \mathbb{R}^N , $\psi_{\varepsilon}(x) \to u(x)$ strongly in $L^p(\mathbb{R}^N, \mathbb{R}^d)$ for every $p \geq 1$, and

$$\limsup_{\varepsilon \to 0^{+}} \left(\frac{1}{|\ln \varepsilon|} \left(\|\psi_{\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q} - \|\psi_{\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N}\setminus\overline{\Omega},\mathbb{R}^{d})}^{q} \right) + \frac{1}{\varepsilon} \int_{\Omega} W \left(\psi_{\varepsilon}(x), x \right) dx \right) = \\
\limsup_{\varepsilon \to 0^{+}} \left(\frac{1}{|\ln \varepsilon|} \|\psi_{\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} + \frac{1}{\varepsilon} \int_{\Omega} W \left(\psi_{\varepsilon}(x), x \right) dx \right) = \\
\left(\int_{\mathbb{R}^{N-1}} \frac{2}{\left(\sqrt{1 + |v|^{2}} \right)^{N+1}} dv \right) \int_{J_{u} \cap \Omega} |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y). \quad (1.14)$$

Moreover, in the case $A \equiv 0$ we can choose ψ_{ε} to satisfy also

$$\int_{\Omega} \psi_{\varepsilon}(x) dx = \int_{\Omega} u(x) dx \qquad \forall \varepsilon > 0.$$
 (1.15)

Unfortunately, the upper bound found in Theorem 1.3 is not sharp in the most general case with a nontrivial prescribed differential constraint. For example, in the particular case of (1.9) with N=2, $A\cdot\nabla v\equiv\operatorname{curl} v$, q>3 and $W(v,x)=(1-|v|^2)^2$, the functional on the R.H.S. of (1.14) is not lower semicontinuous, hence cannot be the Γ -limit (see [3]). However, we still hope that the result of the above theorem could provide the sharp upper bound in some cases with A=0. Indeed, the Γ -limit, computed in [1] for the special case of (1.9) with $A\equiv 0$, q=2, N=1 and W being a double well potential, coincides with the upper bound found in Theorem 1.3. Moreover, since the functional in (1.10) is superior to the functional in (1.9), the Γ -limit, found in [15] (see also [12]) for the energy (1.10) in any dimension $N\geq 1$ with $A\equiv 0$, q=2 and W being a double well potential, coincides again with our upper bound.

The paper is organized as follows. In section 2 we prove our two main results. For the convenience of the reader, in the Appendix we recall some known results on BV functions, needed for the proofs.

2 Proof of the main results

Proposition 2.1. Let q > 1, $\Omega \subset \mathbb{R}^N$ be an open set and $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^N, \mathbb{R}^d)$ be such that $||Du||(\partial\Omega) = 0$. Let $\eta \in C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ and for every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ define

$$u_{\varepsilon}(x) := \frac{1}{\varepsilon^{N}} \int_{\mathbb{R}^{N}} \eta\left(\frac{y-x}{\varepsilon}\right) u(y) dy = (\eta_{\varepsilon} * u)(x). \tag{2.1}$$

Then,

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{|\ln \varepsilon|} \|u_{\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} = 2 \left| \int_{\mathbb{R}^{N}} \eta(z) dz \right|^{q} \left(\int_{\mathbb{R}^{N-1}} \frac{1}{\left(\sqrt{1+|v|^{2}}\right)^{N+1}} dv \right) \int_{J_{u} \cap \Omega} \left| u^{+}(x) - u^{-}(x) \right|^{q} d\mathcal{H}^{N-1}(x). \quad (2.2)$$

Proof. We start with some notations. For every $\boldsymbol{\nu} \in S^{N-1}$ and $x \in \mathbb{R}^N$ set

$$H_{+}(x, \mathbf{\nu}) = \{ \xi \in \mathbb{R}^{N} : (\xi - x) \cdot \mathbf{\nu} > 0 \},$$
 (2.3)

$$H_{-}(x, \boldsymbol{\nu}) = \{ \xi \in \mathbb{R}^{N} : (\xi - x) \cdot \boldsymbol{\nu} < 0 \}$$
 (2.4)

and

$$H_0(\boldsymbol{\nu}) = \{ \xi \in \mathbb{R}^N : \xi \cdot \boldsymbol{\nu} = 0 \}.$$
(2.5)

Let R > 0 be such that supp $\eta \subset B_R(0)$. For every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ we rewrite (2.1) as:

$$u_{\varepsilon}(x) := \frac{1}{\varepsilon^{N}} \int_{\mathbb{R}^{N}} \eta\left(\frac{y-x}{\varepsilon}\right) u(y) dy = \int_{\mathbb{R}^{N}} \eta(z) u(x+\varepsilon z) dz = \int_{B_{R}(0)} \eta(z) u(x+\varepsilon z) dz. \tag{2.6}$$

By (2.6) we have

$$\frac{d}{d\varepsilon}u_{\varepsilon}(x) := -\frac{N}{\varepsilon^{N+1}} \int_{\mathbb{R}^N} \eta\left(\frac{y-x}{\varepsilon}\right) u(y) dy - \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \frac{y-x}{\varepsilon^2} \cdot \nabla \eta\left(\frac{y-x}{\varepsilon}\right) u(y) dy = \\
-\frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \operatorname{div}_y \left\{ \eta\left(\frac{y-x}{\varepsilon}\right) \frac{y-x}{\varepsilon} \right\} u(y) dy = \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y-x}{\varepsilon}\right) \frac{y-x}{\varepsilon} \cdot d\left[Du(y)\right]. \quad (2.7)$$

Moreover, by (1.6) we have

$$\|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = \|u_{\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} = \int_{\mathbb{R}^{N}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x) - u_{\varepsilon}(y) \right|^{q}}{|x - y|^{N+1}} \chi_{\Omega}(y) dy \right) \chi_{\Omega}(x) dx$$

$$= \int_{\mathbb{R}^{N}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x + z) - u_{\varepsilon}(x) \right|^{q}}{|z|^{N+1}} \chi_{\Omega}(x + z) \chi_{\Omega}(x) dz \right) dx, \quad (2.8)$$

where

$$\chi_{\Omega}(x) := \begin{cases} 1 & \forall x \in \Omega \\ 0 & \forall x \in \mathbb{R}^N \setminus \Omega \end{cases}$$
 (2.9)

Thus,

$$\frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = -\frac{1}{\ln \varepsilon} \int_{\mathbb{R}^{N}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x+z) - u_{\varepsilon}(x) \right|^{q}}{|z|^{N+1}} \chi_{\Omega}(x+z) \chi_{\Omega}(x) dz \right) dx. \tag{2.10}$$

Since $-\ln\varepsilon \to +\infty$ as $\varepsilon \to 0^+$, applying L'Hôpital's rule to the expression in (2.10) yields

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = -\lim_{\varepsilon \to 0^{+}} \int_{\mathbb{R}^{N}} \left(\int_{\mathbb{R}^{N}} \frac{\varepsilon}{|z|^{N+1}} \left(\frac{d}{d\varepsilon} \left(u_{\varepsilon}(x+z) - u_{\varepsilon}(x) \right) \right) \cdot \nabla F_{q} \left(u_{\varepsilon}(x+z) - u_{\varepsilon}(x) \right) \chi_{\Omega}(x+z) \chi_{\Omega}(x) dz \right) dx,$$
(2.11)

where $F_q \in C^1(\mathbb{R}^d, \mathbb{R})$ is defined by

$$F_q(h) := |h|^q \qquad \forall h \in \mathbb{R}^d. \tag{2.12}$$

Thus, by (2.11), (2.6) and (2.7) we get

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} =$$

$$-\lim_{\varepsilon \to 0^{+}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{\varepsilon}{|z|^{N+1}} \left\{ \frac{1}{\varepsilon^{N}} \int_{\mathbb{R}^{N}} \left(\eta \left(\frac{y - (x+z)}{\varepsilon} \right) \frac{y - (x+z)}{\varepsilon} - \eta \left(\frac{y - x}{\varepsilon} \right) \frac{y - x}{\varepsilon} \right) \cdot d[Du(y)] \right\} \times$$

$$\times \nabla F_{q} \left(\int_{\mathbb{R}^{N}} \eta(\xi) \left(u(x+z+\varepsilon\xi) - u(x+\varepsilon\xi) \right) d\xi \right) \chi_{\Omega}(x+z) \chi_{\Omega}(x) dz dx =$$

$$-\lim_{\varepsilon \to 0^{+}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{\varepsilon}{|z|^{N+1}} \frac{1}{\varepsilon^{N}} \left(\eta \left(\frac{y - (x+z)}{\varepsilon} \right) \frac{y - (x+z)}{\varepsilon} - \eta \left(\frac{y - x}{\varepsilon} \right) \frac{y - x}{\varepsilon} \right) \times$$

$$\times \nabla F_{q} \left(\int_{\mathbb{R}^{N}} \eta(\xi) \left(u(x+z+\varepsilon\xi) - u(x+\varepsilon\xi) \right) d\xi \right) \chi_{\Omega}(x+z) \chi_{\Omega}(x) dz dx \cdot d[Du(y)]. \quad (2.13)$$

Changing variable, $z/\varepsilon \to z$, in the integration on the R.H.S. of (2.13) gives

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = \\ -\lim_{\varepsilon \to 0^{+}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \frac{1}{\varepsilon^{N}} \left(\eta \left(\frac{y-x}{\varepsilon} - z \right) \left(\frac{y-x}{\varepsilon} - z \right) - \eta \left(\frac{y-x}{\varepsilon} \right) \frac{y-x}{\varepsilon} \right) \times \\ \times \nabla F_{q} \left(\int_{\mathbb{R}^{N}} \eta(\xi) \left(u(x+\varepsilon z + \varepsilon \xi) - u(x+\varepsilon \xi) \right) d\xi \right) \chi_{\Omega}(x+\varepsilon z) \chi_{\Omega}(x) dz dx \cdot d[Du(y)] = \\ -\lim_{\varepsilon \to 0^{+}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x-z)(x-z) - \eta(x)x \right) \times \\ \times \nabla F_{q} \left(\int_{\mathbb{R}^{N}} \eta(\xi) \left(u(y+\varepsilon z + \varepsilon \xi - \varepsilon x) - u(y+\varepsilon \xi - \varepsilon x) \right) d\xi \right) \chi_{\Omega}(y-\varepsilon x + \varepsilon z) \chi_{\Omega}(y-\varepsilon x) dz dx \cdot d[Du(y)].$$

$$(2.14)$$

Therefore,

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q}$$

$$= -\lim_{\varepsilon \to 0^{+}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x-z)(x-z) - \eta(x)x \right) \times$$

$$\times \nabla F_{q} \left(\int_{\mathbb{R}^{N}} \left(\eta(\xi-z) - \eta(\xi) \right) u(y + \varepsilon \xi - \varepsilon x) d\xi \right) \chi_{\Omega}(y - \varepsilon x + \varepsilon z) \chi_{\Omega}(y - \varepsilon x) dz dx \cdot d \left[Du(y) \right]$$

$$= -\lim_{\varepsilon \to 0^{+}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x-z)(x-z) - \eta(x)x \right) \times$$

$$\times \nabla F_{q} \left(\int_{\mathbb{R}^{N}} \left(\eta(\xi+x-z) - \eta(\xi+x) \right) u(y+\varepsilon \xi) d\xi \right) \chi_{\Omega}(y-\varepsilon x + \varepsilon z) \chi_{\Omega}(y-\varepsilon x) dz dx \cdot d \left[Du(y) \right].$$

$$(2.15)$$

On the other hand, by (3.1) in the Appendix, for every $x, z \in \mathbb{R}^N$ and \mathcal{H}^{N-1} -a.e. $y \in \mathbb{R}^N$ we have

$$\lim_{\varepsilon \to 0^{+}} \left\{ \int_{\mathbb{R}^{N}} \left(\eta(\xi + x - z) - \eta(\xi + x) \right) u(y + \varepsilon \xi) d\xi \right\} = u^{+}(y) \int_{H_{+}(0, \nu(y))} \left(\eta(\xi + x - z) - \eta(\xi + x) \right) d\xi + u^{-}(y) \int_{H_{-}(0, \nu(y))} \left(\eta(\xi + x - z) - \eta(\xi + x) \right) d\xi.$$
(2.16)

with $H_{\pm}(x, \nu)$ as defined in (2.3) and (2.4). Thus, since $||Du||(\partial\Omega) = 0$, by (2.16) and the Dominated Convergence Theorem we obtain:

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} =$$

$$- \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x-z)(x-z) - \eta(x)x \right) \nabla F_{q} \left(u^{+}(y) \int_{H_{+}(0,\nu(y))} \left(\eta(\xi+x-z) - \eta(\xi+x) \right) d\xi \right)$$

$$+ u^{-}(y) \int_{H_{-}(0,\nu(y))} \left(\eta(\xi+x-z) - \eta(\xi+x) \right) d\xi \right) \chi_{\Omega}^{2}(y) dz dx \cdot d \left[Du(y) \right] =$$

$$- \int_{\Omega} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x-z)(x-z) - \eta(x)x \right) \nabla F_{q} \left(u^{+}(y) \int_{H_{+}(0,\nu(y))} \left(\eta(\xi+x-z) - \eta(\xi+x) \right) d\xi \right)$$

$$+ u^{-}(y) \int_{H_{-}(0,\nu(y))} \left(\eta(\xi+x-z) - \eta(\xi+x) \right) d\xi \right) dz dx \cdot d \left[Du(y) \right].$$

$$(2.17)$$

It follows that

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = -\int_{\Omega} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x-z)(x-z) - \eta(x)x \right) \times \\
\times \nabla F_{q} \left((u^{+}(y) - u^{-}(y)) \int_{H_{+}(0,\nu(y))} \left(\eta(\xi+x-z) - \eta(\xi+x) \right) d\xi \right) \\
+ u^{-}(y) \int_{\mathbb{R}^{N}} \left(\eta(\xi+x-z) - \eta(\xi+x) \right) d\xi \right) dz dx \cdot d \left[Du(y) \right] \\
= -\int_{\Omega} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x-z)(x-z) - \eta(x)x \right) \times \\
\times \nabla F_{q} \left((u^{+}(y) - u^{-}(y)) \int_{H_{+}(0,\nu(y))} \left(\eta(\xi+x-z) - \eta(\xi+x) \right) d\xi \right) dz dx \cdot d \left[Du(y) \right], \quad (2.18)$$

where we used in the last step the fact that $\int_{\mathbb{R}^N} \eta(\xi+x-z)d\xi = \int_{\mathbb{R}^N} \eta(\xi+x)d\xi$. Next, by

(2.18) and (2.12) we infer that

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = -\int_{\Omega} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x-z)(x-z) - \eta(x)x \right) \times \\ \times \nabla F_{q} \left(\left(u^{+}(y) - u^{-}(y) \right) \left(\int_{H_{+}(x-z,\boldsymbol{\nu}(y))} \eta(\xi)d\xi - \int_{H_{+}(x,\boldsymbol{\nu}(y))} \eta(\xi)d\xi \right) \right) dz dx \cdot d \left[Du(y) \right] \\ = \int_{J_{u} \cap \Omega} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{1}{|z|^{N+1}} \left(\eta(x)x \cdot \boldsymbol{\nu}(y) - \eta(x-z)(x-z) \cdot \boldsymbol{\nu}(y) \right) \times \\ \times \frac{dG_{q}}{d\rho} \left(\int_{(x-z)\cdot\boldsymbol{\nu}(y)}^{x\cdot\boldsymbol{\nu}(y)} \int_{H_{0}(\boldsymbol{\nu}(y))} \eta(t\boldsymbol{\nu}(y) + \xi)d\mathcal{H}^{N-1}(\xi)dt \right) dx dz |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y), \quad (2.19)$$

where $G_q(\rho) \in C^1(\mathbb{R}, \mathbb{R})$ is defined by

$$G_q(\rho) := |\rho|^q \qquad \forall \rho \in \mathbb{R},$$
 (2.20)

and $H_0(\nu)$ is defined in (2.5). Therefore,

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = \int_{J_{u} \cap \Omega} \int_{\mathbb{R}} \int_{H_{0}(\boldsymbol{\nu}(y))} \frac{1}{|z|^{N+1}} \left(\eta \left(s\boldsymbol{\nu}(y) + \zeta \right) s - \eta \left(\left(s - z \cdot \boldsymbol{\nu}(y) \right) \boldsymbol{\nu}(y) + \zeta \right) \left(s - z \cdot \boldsymbol{\nu}(y) \right) \right) \times \times \frac{dG_{q}}{d\rho} \left(\int_{s-z\cdot\boldsymbol{\nu}(y)}^{s} \int_{H_{0}(\boldsymbol{\nu}(y))} \eta(t\boldsymbol{\nu}(y) + \xi) d\mathcal{H}^{N-1}(\xi) dt \right) d\mathcal{H}^{N-1}(\zeta) ds dz |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y)$$

$$= \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\left(\sqrt{\tau^{2} + |w|^{2}} \right)^{N+1}} \times \left(\int_{H_{0}(\boldsymbol{\nu}(y))} \left(\eta \left(s\boldsymbol{\nu}(y) + \zeta \right) s - \eta \left(\left(s - \tau \right) \boldsymbol{\nu}(y) + \zeta \right) \left(s - \tau \right) \right) d\mathcal{H}^{N-1}(\zeta) \right) \times \frac{dG_{q}}{d\rho} \left(\int_{s-\tau}^{s} \int_{H_{0}(\boldsymbol{\nu}(y))} \eta(t\boldsymbol{\nu}(y) + \xi) d\mathcal{H}^{N-1}(\xi) dt \right) d\tau ds dw \right) |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y) . \quad (2.21)$$

Introducing the notation

$$\Lambda(y,a,b) = \int_a^b \int_{H_0(\boldsymbol{\nu}(y))} \eta(t\boldsymbol{\nu}(y) + \xi) d\mathcal{H}^{N-1}(\xi) dt$$
 (2.22)

allows us to rewrite (2.21) as

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = \int_{J_{u} \cap \Omega} \left\{ \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\tau^{2}} \frac{1}{|\tau|^{N-1}} \frac{1}{\left(\sqrt{1+|w/|\tau||^{2}}\right)^{N+1}} \times \left(\int_{H_{0}(\boldsymbol{\nu}(y))} \left(\eta \left(s\boldsymbol{\nu}(y) + \zeta\right)s - \eta \left(\left(s - \tau\right)\boldsymbol{\nu}(y) + \zeta\right)\left(s - \tau\right) \right) d\mathcal{H}^{N-1}(\zeta) \right) \times \frac{dG_{q}}{d\rho} \left(\Lambda(y, s - \tau, s) \right) d\tau ds dw \right\} |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y). \quad (2.23)$$

The change of variables $w/|\tau| \to v$ in the R.H.S. of (2.23) gives

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} =$$

$$D_{N} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\tau^{2}} \left(\int_{H_{0}(\boldsymbol{\nu}(y))} \left(\eta(s\boldsymbol{\nu}(y) + \zeta)s - \eta((s-\tau)\boldsymbol{\nu}(y) + \zeta)(s-\tau) \right) d\mathcal{H}^{N-1}(\zeta) \right) \times \frac{dG_{q}}{d\rho} \left(\Lambda(y, s-\tau, s) \right) d\tau ds \right) |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y), \quad (2.24)$$

where D_N is the dimensional constant given by

$$D_N := \int_{\mathbb{R}^{N-1}} \frac{1}{\left(\sqrt{1+|v|^2}\right)^{N+1}} dv. \tag{2.25}$$

Then we rewrite (2.24) as

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = \lim_{M \to +\infty} \left(D_{N} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} \int_{-M}^{M} \frac{1}{\tau^{2}} \left(\int_{H_{0}(\boldsymbol{\nu}(y))} s \left(\eta \left(s \boldsymbol{\nu}(y) + \zeta \right) - \eta \left((s - \tau) \boldsymbol{\nu}(y) + \zeta \right) \right) d\mathcal{H}^{N-1}(\zeta) \right) \times \right. \\ \left. \times \frac{dG_{q}}{d\rho} \left(\Lambda(y, s - \tau, s) \right) d\tau ds \right) \left| u^{+}(y) - u^{-}(y) \right|^{q} d\mathcal{H}^{N-1}(y) \right. \\ \left. + D_{N} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} \int_{-M}^{M} \frac{1}{\tau} \left(\int_{H_{0}(\boldsymbol{\nu}(y))} \eta \left((s - \tau) \boldsymbol{\nu}(y) + \zeta \right) d\mathcal{H}^{N-1}(\zeta) \right) \times \right. \\ \left. \times \frac{dG_{q}}{d\rho} \left(\Lambda(y, s - \tau, s) \right) d\tau ds \right) \left| u^{+}(y) - u^{-}(y) \right|^{q} d\mathcal{H}^{N-1}(y) \right). \quad (2.26)$$

Integration by parts of (2.26) and using (2.20) give

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} =$$

$$-\lim_{M \to +\infty} D_{N} \int_{J_{u} \cap \Omega} |u^{+}(y) - u^{-}(y)|^{q} \left(\int_{\mathbb{R}} \int_{-M}^{M} \frac{1}{\tau^{2}} |\Lambda(y, s - \tau, s)|^{q} d\tau ds \right) d\mathcal{H}^{N-1}(y)$$

$$+\lim_{M \to +\infty} D_{N} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} \int_{-M}^{M} \frac{1}{\tau^{2}} |\Lambda(y, s - \tau, s)|^{q} d\tau ds \right) |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y)$$

$$+\lim_{M \to +\infty} \frac{D_{N}}{M} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} |\Lambda(y, s - M, s)|^{q} ds + \int_{\mathbb{R}} |\Lambda(y, s, s + M)|^{q} ds \right) |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y)$$

$$= \lim_{M \to +\infty} \frac{D_{N}}{M} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} |\Lambda(y, s - M, s)|^{q} ds + \int_{\mathbb{R}} |\Lambda(y, s, s + M)|^{q} ds \right) |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y).$$

$$(2.27)$$

Therefore, applying L'Hôpital's rule in (2.27), using (2.20), we deduce that

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = \lim_{M \to +\infty} D_{N} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} \frac{dG_{q}}{d\rho} \left(\Lambda(y, s - M, s) \right) \left(\int_{H_{0}(\boldsymbol{\nu}(y))} \eta \left((s - M) \boldsymbol{\nu}(y) + \xi \right) d\mathcal{H}^{N-1}(\xi) \right) ds + \int_{\mathbb{R}} \frac{dG_{q}}{d\rho} \left(\Lambda(y, s, s + M) \right) \left(\int_{H_{0}(\boldsymbol{\nu}(y))} \eta \left((s + M) \boldsymbol{\nu}(y) + \xi \right) d\mathcal{H}^{N-1}(\xi) \right) ds \times \left| u^{+}(y) - u^{-}(y) \right|^{q} d\mathcal{H}^{N-1}(y). \quad (2.28)$$

Changing variables of integration we rewrite (2.28) as

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} =$$

$$\lim_{M \to +\infty} D_{N} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} \frac{dG_{q}}{d\rho} \Big(\Lambda(y, s, s + M) \Big) \Big(\int_{H_{0}(\boldsymbol{\nu}(y))} \eta(s\boldsymbol{\nu}(y) + \xi) d\mathcal{H}^{N-1}(\xi) \Big) ds + \int_{\mathbb{R}} \frac{dG_{q}}{d\rho} \Big(\Lambda(y, s - M, s) \Big) \Big(\int_{H_{0}(\boldsymbol{\nu}(y))} \eta(s\boldsymbol{\nu}(y) + \xi) d\mathcal{H}^{N-1}(\xi) \Big) ds \Big)$$

$$\times |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y)$$

$$= D_{N} \int_{J_{u} \cap \Omega} \left(\int_{\mathbb{R}} \frac{dG_{q}}{d\rho} \Big(\Lambda(y, s, \infty) \Big) \Big(\int_{H_{0}(\boldsymbol{\nu}(y))} \eta(s\boldsymbol{\nu}(y) + \xi) d\mathcal{H}^{N-1}(\xi) \Big) ds + \int_{\mathbb{R}} \frac{dG_{q}}{d\rho} \Big(\Lambda(y, -\infty, s) \Big) \Big(\int_{H_{0}(\boldsymbol{\nu}(y))} \eta(s\boldsymbol{\nu}(y) + \xi) d\mathcal{H}^{N-1}(\xi) \Big) ds \Big)$$

$$\times |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y). \quad (2.29)$$

Applying Newton-Leibniz formula in (2.29) and using (2.20) we obtain that

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln \varepsilon} \|u_{\varepsilon}\|_{W^{1/q,q}}^{q} = 2D_{N} \int_{J_{u} \cap \Omega} \left| \int_{-\infty}^{\infty} \int_{H_{0}(\nu(y))} \eta(t\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) dt \right|^{q} \left| u^{+}(y) - u^{-}(y) \right|^{q} d\mathcal{H}^{N-1}(y)$$

$$= 2D_{N} \left| \int_{\mathbb{R}^{N}} \eta(z) dz \right|^{q} \int_{J_{u} \cap \Omega} \left| u^{+}(y) - u^{-}(y) \right|^{q} d\mathcal{H}^{N-1}(y), \quad (2.30)$$

and (2.2) follows.

Corollary 2.1. Let q > 1 and let $\Omega \subset \mathbb{R}^N$ be an open set. Assume $W : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function such that, $W(0,\cdot) \in L^1(\Omega,\mathbb{R})$ and for every D > 0 there exists $C := C_D > 0$ such that

$$|W(b,x) - W(a,x)| \le C_D|b-a| \qquad \forall x \in \mathbb{R}^N, \ \forall a,b \in B_D(0). \tag{2.31}$$

Let $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^N, \mathbb{R}^d)$ be such that $||Du||(\partial\Omega) = 0$ and W(u(x), x) = 0 a.e. in Ω . Let $\eta \in C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ be such that $\int_{\mathbb{R}^N} \eta(z)dz = 1$ and supp $\eta \subset B_R(0)$. For every $\rho > 0$ set

$$\eta_{\rho}(z) := \frac{1}{\rho^{N}} \eta\left(\frac{z}{\rho}\right) \quad \forall z \in \mathbb{R}^{N}.$$
(2.32)

Finally, for every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ define

$$u_{\rho,\varepsilon}(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta_{\rho} \left(\frac{y-x}{\varepsilon} \right) u(y) dy = \int_{\mathbb{R}^N} \eta(z) u(x+\varepsilon \rho z) dz = \int_{B_R(0)} \eta(z) u(x+\varepsilon \rho z) dz. \quad (2.33)$$

Then,

$$\lim_{\rho \to 0^{+}} \left\{ \limsup_{\varepsilon \to 0^{+}} \left(\frac{1}{-\ln(\varepsilon)} \left(\|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q} - \|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N}\setminus\overline{\Omega},\mathbb{R}^{d})}^{q} \right) + \frac{1}{\varepsilon} \int_{\Omega} W \left(u_{\rho,\varepsilon}(x), x \right) dx \right) \right\}$$

$$= \lim_{\rho \to 0^{+}} \left\{ \limsup_{\varepsilon \to 0^{+}} \left(\frac{1}{-\ln(\varepsilon)} \|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} + \frac{1}{\varepsilon} \int_{\Omega} W \left(u_{\rho,\varepsilon}(x), x \right) dx \right) \right\}$$

$$= \left(\int_{\mathbb{R}^{N-1}} \frac{2}{\left(\sqrt{1 + |v|^{2}} \right)^{N+1}} dv \right) \int_{J_{u} \cap \Omega} \left| u^{+}(y) - u^{-}(y) \right|^{q} d\mathcal{H}^{N-1}(y). \quad (2.34)$$

Proof. Since $\int_{\mathbb{R}^N} \eta_{\rho}(z) dz = 1$, applying Proposition 2.1, first for \mathbb{R}^N , then for $\mathbb{R}^N \setminus \overline{\Omega}$, and finally for Ω , yields, for every $\rho > 0$,

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{-\ln(\varepsilon)} \left(\|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q} - \|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N}\setminus\overline{\Omega},\mathbb{R}^{d})}^{q} \right)
= 2D_{N} \left(\int_{J_{u}} |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y) - \int_{J_{u}\cap(\mathbb{R}^{N}\setminus\overline{\Omega})} |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y) \right)
= 2D_{N} \int_{J_{u}\cap\Omega} |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y) = \lim_{\varepsilon \to 0^{+}} \left(\frac{1}{-\ln(\varepsilon)} \|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} \right), \quad (2.35)$$

where D_N is the constant defined in (2.25). On the other hand, since W(u(x), x) = 0 a.e. in Ω and $u \in L^{\infty}$, by (2.31) we get that

$$\left| \frac{1}{\varepsilon} \int_{\Omega} W \Big(u_{\rho,\varepsilon}(x), x \Big) dx \right| = \left| \frac{1}{\varepsilon} \int_{\Omega} \Big(W \Big(u_{\rho,\varepsilon}(x), x \Big) - W \Big(u(x), x \Big) \Big) dx \right| \le C \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \left| u_{\rho,\varepsilon}(x) - u(x) \right| dx
\le C \int_{B_R(0)} \left| \eta(z) \right| \left(\int_{\mathbb{R}^N} \frac{1}{\varepsilon} \left| u(x + \varepsilon \rho z) - u(x) \right| dx \right) dz
= C \rho \int_{B_R(0)} |z| |\eta(z)| \left(\int_{\mathbb{R}^N} \frac{1}{\varepsilon \rho |z|} \left| u(x + \varepsilon \rho z) - u(x) \right| dx \right) dz, \quad (2.36)$$

for some constant C > 0, independent of ε and ρ . Thus, taking into account the following well known uniform bound from the theory of BV functions,

$$\int_{\mathbb{R}^N} \frac{1}{\rho \varepsilon |z|} |u(x + \rho \varepsilon z) - u(x)| dx \le C_0 ||Du|| (\mathbb{R}^N) \quad \forall z \in \mathbb{R}^N, \, \forall \rho, \, \varepsilon > 0,$$
 (2.37)

we obtain that

$$\limsup_{\varepsilon \to 0^+} \left| \frac{1}{\varepsilon} \int_{\Omega} W\left(u_{\rho,\varepsilon}(x), x\right) dx \right| \le CC_0 \|Du\|(\mathbb{R}^N) \rho \int_{B_R(0)} |z| |\eta(z)| dz = O(\rho). \tag{2.38}$$

By (2.38) and (2.35) we finally derive (2.34).

Proof of Theorem 1.3. Let η , η_{ρ} and $u_{\rho,\varepsilon}$ be defined as in Corollary 2.1. Then $u_{\rho,\varepsilon} \in C^{\infty}(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^d)$ and by Corollary 2.1 we have

$$\lim_{\rho \to 0^{+}} \left\{ \limsup_{\varepsilon \to 0^{+}} \left(\frac{1}{-\ln(\varepsilon)} \left(\|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q} - \|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N}\setminus\overline{\Omega},\mathbb{R}^{d})}^{q} \right) + \frac{1}{\varepsilon} \int_{\Omega} W \left(u_{\rho,\varepsilon}(x), x \right) dx \right) \right\}$$

$$= \lim_{\rho \to 0^{+}} \left\{ \limsup_{\varepsilon \to 0^{+}} \left(\frac{1}{-\ln\varepsilon} \|u_{\rho,\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} + \frac{1}{\varepsilon} \int_{\Omega} W \left(u_{\rho,\varepsilon}(x), x \right) dx \right) \right\}$$

$$= \left(\int_{\mathbb{R}^{N-1}} \frac{2}{\left(\sqrt{1 + |v|^{2}} \right)^{N+1}} dv \right) \int_{J_{u} \cap \Omega} \left| u^{+}(y) - u^{-}(y) \right|^{q} d\mathcal{H}^{N-1}(y). \quad (2.39)$$

Clearly, for every $x \in \mathbb{R}^N$ we have $A \cdot \nabla u_{\rho,\varepsilon}(x) = 0$ and $u_{\rho,\varepsilon}(x) \to u(x)$ strongly in $L^p(\mathbb{R}^N, \mathbb{R}^d)$ as $\varepsilon \to 0^+$ for every fixed ρ and p. Therefore, by the above and by (2.39) we can complete the proof of the first assertion of the theorem using a standard diagonal argument.

It remains to show the second assertion of the theorem, namely, that in the case $A \equiv 0$ we can construct ψ_{ε} satisfying the additional condition (1.15). Let $\varphi \in C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ be such that $\int_{\Omega} \varphi(x) dx = 1$. Define

$$\tilde{u}_{\rho,\varepsilon}(x) := u_{\rho,\varepsilon}(x) - \varphi(x)c_{\varepsilon,\rho},$$
(2.40)

where

$$c_{\varepsilon,\rho} := \int_{\Omega} u_{\rho,\varepsilon}(y)dy - \int_{\Omega} u(y)dy. \tag{2.41}$$

In particular,

$$\int_{\Omega} \tilde{u}_{\rho,\varepsilon}(x)dx = \int_{\Omega} u(x)dx,\tag{2.42}$$

and $\lim_{\varepsilon\to 0^+} c_{\varepsilon,\rho} = 0$. On the other hand, since W(u(x),x) = 0 a.e. in Ω , W(b,x) is nonnegative and W(b,x) is differentiable with respect to the b variable, we have

$$\nabla_b W(u(x), x) = 0 \quad \text{a.e. in } \Omega. \tag{2.43}$$

Thus, since $u \in L^{\infty}$, by (2.40) we get that

$$\left| \frac{1}{\varepsilon} \int_{\Omega} \left(W \left(\tilde{u}_{\rho,\varepsilon}(x), x \right) - W \left(u_{\rho,\varepsilon}(x), x \right) \right) dx \right| = \left| \frac{c_{\varepsilon,\rho}}{\varepsilon} \cdot \int_{0}^{1} \int_{\Omega} \nabla_{b} W \left(u_{\rho,\varepsilon}(x) - s\varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds \right| \\
\leq C \left(\int_{\mathbb{R}^{N}} \frac{1}{\varepsilon} \left| u_{\rho,\varepsilon}(x) - u(x) \right| dx \right) \left| \int_{0}^{1} \int_{\Omega} \nabla_{b} W \left(u_{\rho,\varepsilon}(x) - s\varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds \right| \\
\leq C \left(\int_{B_{R}(0)} \left| \eta(z) \right| \left(\int_{\mathbb{R}^{N}} \frac{1}{\varepsilon} \left| u(x + \varepsilon \rho z) - u(x) \right| dx \right) dz \right) \times \\
\times \left| \int_{0}^{1} \int_{\Omega} \nabla_{b} W \left(u_{\rho,\varepsilon}(x) - s\varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds \right| \\
= C \rho \left(\int_{B_{R}(0)} |z| |\eta(z)| \left(\int_{\mathbb{R}^{N}} \frac{1}{\varepsilon \rho |z|} |u(x + \varepsilon \rho z) - u(x)| dx \right) dz \right) \times \\
\times \left| \int_{0}^{1} \int_{\Omega} \nabla_{b} W \left(u_{\rho,\varepsilon}(x) - s\varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds \right|. (2.44)$$

On the other hand, taking into account (2.37) and using the Dominated Convergence Theorem and (2.43), we obtain that

$$\lim_{\varepsilon \to 0+} \sup \left(\int_{B_{R}(0)} |z| |\eta(z)| \left(\int_{\mathbb{R}^{N}} \frac{1}{\varepsilon \rho |z|} |u(x + \varepsilon \rho z) - u(x)| dx \right) dz \right) \times \\
\times \left| \int_{0}^{1} \int_{\Omega} \nabla_{b} W \left(u_{\rho,\varepsilon}(x) - s\varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds \right| \leq C_{0} \left(\|Du\|(\mathbb{R}^{n}) \right) \left(\int_{B_{R}(0)} |z| |\eta(z)| dz \right) \times \\
\times \left| \int_{0}^{1} \int_{\Omega} \nabla_{b} W \left(\lim_{\varepsilon \to 0+} u_{\rho,\varepsilon}(x) - s\varphi(x) \lim_{\varepsilon \to 0+} c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds \right| \\
= C_{0} \left(\|Du\|(\mathbb{R}^{n}) \right) \left(\int_{B_{R}(0)} |z| |\eta(z)| dz \right) \left| \int_{\Omega} \nabla_{b} W \left(u(x), x \right) \varphi(x) dx \right| = 0. \quad (2.45)$$

Using (2.45) in (2.44) yields

$$\lim_{\varepsilon \to 0+} \sup \left| \frac{1}{\varepsilon} \int_{\Omega} \left(W \left(\tilde{u}_{\rho,\varepsilon}(x), x \right) - W \left(u_{\rho,\varepsilon}(x), x \right) \right) dx \right| = 0.$$
 (2.46)

Plugging (2.46) into (2.39) we get that

$$\lim_{\rho \to 0^{+}} \left\{ \limsup_{\varepsilon \to 0^{+}} \left(\frac{1}{-\ln(\varepsilon)} \left(\|\tilde{u}_{\rho,\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q} - \|\tilde{u}_{\rho,\varepsilon}\|_{W^{1/q,q}(\mathbb{R}^{N}\setminus\overline{\Omega},\mathbb{R}^{d})}^{q} \right) + \frac{1}{\varepsilon} \int_{\Omega} W \left(\tilde{u}_{\rho,\varepsilon}(x), x \right) dx \right) \right\}$$

$$= \lim_{\rho \to 0^{+}} \left\{ \limsup_{\varepsilon \to 0^{+}} \left(\frac{1}{-\ln \varepsilon} \|\tilde{u}_{\rho,\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} + \frac{1}{\varepsilon} \int_{\Omega} W \left(\tilde{u}_{\rho,\varepsilon}(x), x \right) dx \right) \right\}$$

$$= \left(\int_{\mathbb{R}^{N-1}} \frac{2}{\left(\sqrt{1 + |v|^{2}} \right)^{N+1}} dv \right) \int_{J_{u} \cap \Omega} |u^{+}(y) - u^{-}(y)|^{q} d\mathcal{H}^{N-1}(y). \quad (2.47)$$

Moreover, $\tilde{u}_{\rho,\varepsilon} \to u$ strongly in $L^p(\mathbb{R}^N, \mathbb{R}^d)$ as $\varepsilon \to 0^+$ for every fixed ρ and p. Therefore, by the above and (2.47) we complete again the proof by a standard diagonal argument. \square

The next lemma is needed for the proof of Theorem 1.2 (in the general case $\eta \in W^{1,1}$).

Lemma 2.1. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^N, \mathbb{R}^d)$. For $\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})$, every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ define

$$u_{\varepsilon}(x) := \frac{1}{\varepsilon^{N}} \int_{\mathbb{R}^{N}} \eta\left(\frac{y-x}{\varepsilon}\right) u(y) dy = \int_{\mathbb{R}^{N}} \eta(z) u(x+\varepsilon z) dz. \tag{2.48}$$

Then, for every q > 1 and for every $\varepsilon \in (0,1)$ we have

$$\frac{1}{\omega_{N-1} |\ln \varepsilon|} \int_{\Omega} \left(\int_{\Omega} \frac{\left| u_{\varepsilon}(x) - u_{\varepsilon}(y) \right|^{q}}{|x - y|^{N+1}} dy \right) dx \leq \frac{2^{q} \|u\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})}^{q-1} \|\eta\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R})}}{|\ln \varepsilon|} + \frac{\left(3\|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \right)^{q-1} \|\eta\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R})} \|Du\|(\mathbb{R}^{N})}{(q-1) |\ln \varepsilon|} + \left(3\|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \right)^{q-1} \|\eta\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R})} \|Du\|(\mathbb{R}^{N}), \quad (2.49)$$

where ω_{N-1} denotes the surface area of the unit ball in \mathbb{R}^N .

Proof. Assume first that $\eta(z) \in C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$. Then, by (2.48) we have

$$\varepsilon \nabla u_{\varepsilon}(x) = -\frac{1}{\varepsilon^{N}} \int_{\mathbb{R}^{N}} \nabla \eta \left(\frac{y-x}{\varepsilon} \right) u(y) dy = -\int_{\mathbb{R}^{N}} \nabla \eta(z) u(x+\varepsilon z) dz. \tag{2.50}$$

By (2.48) and (2.50) we get that

$$||u_{\varepsilon}||_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})} + ||\varepsilon\nabla u_{\varepsilon}||_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})} \leq ||u||_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})} ||\eta||_{W^{1,1}(\mathbb{R}^{N},\mathbb{R})} \quad \text{and}$$

$$||u_{\varepsilon}||_{L^{q}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q} \leq ||u||_{L^{1}(\mathbb{R}^{N},\mathbb{R}^{d})} ||u||_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q-1} ||\eta||_{L^{1}(\mathbb{R}^{N},\mathbb{R})}^{q} \quad \forall \varepsilon > 0, \ \forall q \in [1, +\infty). \quad (2.51)$$

Next, for every $\varepsilon \in (0,1)$ we have

$$\int_{\Omega} \left(\int_{\Omega} \frac{\left| u_{\varepsilon}(x) - u_{\varepsilon}(y) \right|^{q}}{|x - y|^{N+1}} dy \right) dx \leq \int_{\mathbb{R}^{N}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x) - u_{\varepsilon}(y) \right|^{q}}{|x - y|^{N+1}} dy \right) dx = \\
\int_{\mathbb{R}^{N}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|^{q}}{|y|^{N+1}} dy \right) dx = \int_{\mathbb{R}^{N}} \left(\int_{B_{\varepsilon}(0)} \frac{\left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|^{q}}{|y|^{N+1}} dy \right) dx \\
+ \int_{\mathbb{R}^{N}} \left(\int_{B_{1}(0) \setminus B_{\varepsilon}(0)} \frac{\left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|^{q}}{|y|^{N+1}} dy \right) dx + \int_{\mathbb{R}^{N}} \left(\int_{\mathbb{R}^{N} \setminus B_{1}(0)} \frac{\left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|^{q}}{|y|^{N+1}} dy \right) dx \\
= \int_{B_{\varepsilon}(0)} \frac{1}{|y|^{N+1-q}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|^{q}}{|y|} dx \right) dy \\
+ \int_{B_{1}(0) \setminus B_{\varepsilon}(0)} \frac{1}{|y|^{N}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|^{q}}{|y|} dx \right) dy \\
+ \int_{\mathbb{R}^{N} \setminus B_{1}(0)} \frac{1}{|y|^{N+1}} \left(\int_{\mathbb{R}^{N}} \left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|^{q} dx \right) dy. \quad (2.52)$$

On the other hand, (2.51) yields

$$\left| u_{\varepsilon}(x+y) - u_{\varepsilon}(x) \right| + \frac{\varepsilon \left| u_{\varepsilon}(x+y) - u_{\varepsilon}(x) \right|}{|x-y|} \le 3 \|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \qquad \forall \varepsilon > 0, \ \forall x, y \in \mathbb{R}^{N}.$$

$$(2.53)$$

Thus, inserting (2.53) into (2.52) we deduce that

$$\int_{\Omega} \left(\int_{\Omega} \frac{\left| u_{\varepsilon}(x) - u_{\varepsilon}(y) \right|^{q}}{|x - y|^{N+1}} dy \right) dx \leq 2^{q} \|u_{\varepsilon}\|_{L^{q}(\mathbb{R}^{N}, \mathbb{R}^{d})}^{q} \int_{\mathbb{R}^{N} \setminus B_{1}(0)} \frac{dy}{|y|^{N+1}} + \frac{\left(3\|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \right)^{q-1}}{\varepsilon^{q-1}} \int_{B_{\varepsilon}(0)} \frac{1}{|y|^{N+1-q}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|}{|y|} dx \right) dy + \left(3\|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \right)^{q-1} \int_{B_{1}(0) \setminus B_{\varepsilon}(0)} \frac{1}{|y|^{N}} \left(\int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x + y) - u_{\varepsilon}(x) \right|}{|y|} dx \right) dy. \quad (2.54)$$

Inserting (2.48) into (2.54) and using the second inequality in (2.51) we infer,

$$\int_{\Omega} \left(\int_{\Omega} \frac{\left| u_{\varepsilon}(x) - u_{\varepsilon}(y) \right|^{q}}{|x - y|^{N+1}} dy \right) dx \leq 2^{q} \|u\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})}^{q-1} \|\eta\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R})} \int_{\mathbb{R}^{N} \setminus B_{1}(0)} \frac{dy}{|y|^{N+1}} + \frac{\left(3\|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \right)^{q-1}}{\varepsilon^{q-1}} \times \\
\times \int_{B_{\varepsilon}(0)} \frac{1}{|y|^{N+1-q}} \left(\int_{\mathbb{R}^{N}} |\eta(z)| \int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x + \varepsilon z + y) - u_{\varepsilon}(x + \varepsilon z) \right|}{|y|} dx dz \right) dy + \left(3\|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \right)^{q-1} \times \\
\times \int_{B_{1}(0) \setminus B_{\varepsilon}(0)} \frac{1}{|y|^{N}} \left(\int_{\mathbb{R}^{N}} |\eta(z)| \int_{\mathbb{R}^{N}} \frac{\left| u_{\varepsilon}(x + \varepsilon z + y) - u_{\varepsilon}(x + \varepsilon z) \right|}{|y|} dx dz \right) dy. \quad (2.55)$$

Taking into account the following well known uniform bound from the theory of BV functions:

$$\int_{\mathbb{R}^N} \frac{\left| u(x+\varepsilon z+y) - u(x+\varepsilon z) \right|}{|y|} dx = \int_{\mathbb{R}^N} \frac{\left| u(x+y) - u(x) \right|}{|y|} dx \le ||Du||(\mathbb{R}^N) \quad \forall y \in \mathbb{R}^N,$$
(2.56)

we rewrite (2.55) as

$$\int_{\Omega} \left(\int_{\Omega} \frac{\left| u_{\varepsilon}(x) - u_{\varepsilon}(y) \right|^{q}}{|x - y|^{N+1}} dy \right) dx \leq 2^{q} \|u\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})}^{q-1} \|\eta\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R})} \int_{\mathbb{R}^{N} \setminus B_{1}(0)} \frac{dy}{|y|^{N+1}} + \frac{\left(3\|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \right)^{q-1}}{\varepsilon^{q-1}} \|\eta\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R})} \|Du\|(\mathbb{R}^{N}) \int_{B_{\varepsilon}(0)} \frac{dy}{|y|^{N+1-q}} + \left(3\|u\|_{L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{d})} \|\eta\|_{W^{1,1}(\mathbb{R}^{N}, \mathbb{R})} \right)^{q-1} \|\eta\|_{L^{1}(\mathbb{R}^{N}, \mathbb{R})} \|Du\|(\mathbb{R}^{N}) \int_{B_{1}(0) \setminus B_{2}(0)} \frac{dy}{|y|^{N}}. \tag{2.57}$$

Computing the integrals on the R.H.S. of (2.57) yields (2.49) in the case $\eta \in C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$.

Next consider the general case $\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})$. Thanks to the density of $C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ in $W^{1,1}(\mathbb{R}^N, \mathbb{R})$, there exists a sequence $\{\eta_n\}_{n=1}^{\infty} \subset C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ such that

$$\lim_{n \to +\infty} \| \eta_n - \eta \|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} = 0.$$
 (2.58)

Thus if we define

$$u_{n,\varepsilon}(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta_n \left(\frac{y-x}{\varepsilon} \right) u(y) dy = \int_{\mathbb{R}^N} \eta_n(z) u(x+\varepsilon z) dz, \tag{2.59}$$

then

$$\lim_{n \to +\infty} u_{n,\varepsilon}(x) = u_{\varepsilon}(x) \quad \forall x \in \mathbb{R}^N, \ \forall \varepsilon > 0.$$
 (2.60)

On the other hand, since we proved (2.49) for the case $\eta_n \in C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$, for every q > 1, for every $n = 1, 2, \ldots$ and for every $\varepsilon \in (0, 1)$ we have:

$$\frac{1}{\omega_{N-1} |\ln \varepsilon|} \int_{\Omega} \left(\int_{\Omega} \frac{\left| u_{n,\varepsilon}(x) - u_{n,\varepsilon}(y) \right|^{q}}{|x - y|^{N+1}} dy \right) dx \leq \frac{2^{q} \|u\|_{L^{1}(\mathbb{R}^{N},\mathbb{R}^{d})} \|u\|_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q-1} \|\eta_{n}\|_{L^{1}(\mathbb{R}^{N},\mathbb{R})}}{|\ln \varepsilon|} + \frac{\left(3 \|u\|_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})} \|\eta_{n}\|_{W^{1,1}(\mathbb{R}^{N},\mathbb{R})} \right)^{q-1} \|\eta_{n}\|_{L^{1}(\mathbb{R}^{N},\mathbb{R})} \|Du\|(\mathbb{R}^{N})}{(q-1) |\ln \varepsilon|} + \left(3 \|u\|_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})} \|\eta_{n}\|_{W^{1,1}(\mathbb{R}^{N},\mathbb{R})} \right)^{q-1} \|\eta_{n}\|_{L^{1}(\mathbb{R}^{N},\mathbb{R})} \|Du\|(\mathbb{R}^{N}). \tag{2.61}$$

Letting n go to infinity in (2.61), using (2.58) in the R.H.S. and (2.60) together with Fatou's Lemma in the L.H.S., we obtain (2.49) in the general case $\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})$.

Proof of Theorem 1.2. In the case $\eta \in C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ the result follows by Proposition 2.1. Next consider the general case $\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})$. As before, by the density of $C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ in $W^{1,1}(\mathbb{R}^N, \mathbb{R})$, there exists a sequence $\{\eta_n\}_{n=1}^{\infty} \subset C_c^{\infty}(\mathbb{R}^N, \mathbb{R})$ such that

$$\lim_{n \to +\infty} \|\eta_n - \eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} = 0. \tag{2.62}$$

Next, as before, define

$$u_{n,\varepsilon}(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta_n \left(\frac{y-x}{\varepsilon} \right) u(y) dy = \int_{\mathbb{R}^N} \eta_n(z) u(x+\varepsilon z) dz.$$
 (2.63)

Defining $u_{n,\varepsilon}$ as in (2.59) we get by Proposition 2.1, for all $n \ge 1$ (see (2.25)),

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{|\ln \varepsilon|} \|u_{n,\varepsilon}\|_{W^{1/q,q}(\Omega,\mathbb{R}^{d})}^{q} = 2D_{N} \left| \int_{\mathbb{R}^{N}} \eta_{n}(z) dz \right|^{q} \int_{J_{u} \cap \Omega} \left| u^{+}(x) - u^{-}(x) \right|^{q} d\mathcal{H}^{N-1}(x) := L_{n}, \tag{2.64}$$

and then

$$\lim_{n \to \infty} L_n = \bar{L} := 2D_N \left| \int_{\mathbb{R}^N} \eta(z) dz \right|^q \int_{L \cap \Omega} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x). \tag{2.65}$$

On the other hand, by Lemma 2.1, for all $n \geq 1$ and every $\varepsilon \in (0, 1/e)$ we have

$$\frac{1}{\omega_{N-1}|\ln\varepsilon|} \int_{\Omega} \left(\int_{\Omega} \frac{1}{|x-y|^{N+1}} \left| \left(u_{n,\varepsilon}(x) - u_{n,\varepsilon}(y) \right) - \left(u_{\varepsilon}(x) - u_{\varepsilon}(y) \right) \right|^{q} dy \right) dx = \\
\frac{1}{\omega_{N-1}|\ln\varepsilon|} \int_{\Omega} \left(\int_{\Omega} \frac{1}{|x-y|^{N+1}} \left| \left(u_{n,\varepsilon}(x) - u_{\varepsilon}(x) \right) - \left(u_{n,\varepsilon}(y) - u_{\varepsilon}(y) \right) \right|^{q} dy \right) dx \\
\leq 2^{q} \|u\|_{L^{1}(\mathbb{R}^{N},\mathbb{R}^{d})} \|u\|_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})}^{q-1} \|\eta_{n} - \eta\|_{L^{1}(\mathbb{R}^{N},\mathbb{R})}^{q} \\
+ \frac{\left(3\|u\|_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})} \|\eta_{n} - \eta\|_{W^{1,1}(\mathbb{R}^{N},\mathbb{R})} \right)^{q-1} \|\eta_{n} - \eta\|_{L^{1}(\mathbb{R}^{N},\mathbb{R})} \|Du\|(\mathbb{R}^{N})}{(q-1)} \\
+ \left(3\|u\|_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{d})} \|\eta_{n} - \eta\|_{W^{1,1}(\mathbb{R}^{N},\mathbb{R})} \right)^{q-1} \|\eta_{n} - \eta\|_{L^{1}(\mathbb{R}^{N},\mathbb{R})} \|Du\|(\mathbb{R}^{N}) := H_{n}. \quad (2.66)$$

Thus, by the triangle inequality we get, for every $n \ge 1$ and every $\varepsilon \in (0, 1/e)$,

$$\frac{1}{|\ln \varepsilon|^{1/q}} \Big| \|u_{n,\varepsilon}\|_{W^{1/q,q}} - \|u_{\varepsilon}\|_{W^{1/q,q}} \Big| \le \frac{\|u_{n,\varepsilon} - u_{\varepsilon}\|_{W^{1/q,q}}}{|\ln \varepsilon|^{1/q}} \le (\omega_{N-1} H_n)^{1/q}. \tag{2.67}$$

Then, by (2.67) and (2.64), for all $n \ge 1$ we obtain:

$$\limsup_{\varepsilon \to 0^{+}} \left| \frac{\|u_{\varepsilon}\|_{W^{1/q,q}}}{|\ln \varepsilon|^{1/q}} - \bar{L}^{1/q} \right| \leq \limsup_{\varepsilon \to 0^{+}} \frac{1}{|\ln \varepsilon|^{1/q}} \left| \|u_{n,\varepsilon}\|_{W^{1/q,q}} - \|u_{\varepsilon}\|_{W^{1/q,q}} \right|
+ \limsup_{\varepsilon \to 0^{+}} \left| \frac{\|u_{n,\varepsilon}\|_{W^{1/q,q}}}{|\ln \varepsilon|^{1/q}} - L_{n}^{1/q} \right| + |L_{n}^{1/q} - \bar{L}^{1/q}| \leq \left(\omega_{N-1}H_{n}\right)^{1/q} + 0 + |L_{n}^{1/q} - \bar{L}^{1/q}|. \quad (2.68)$$

Letting n go to infinity in (2.68), using (2.65), the definition of \bar{L} in (2.65) and the fact that $\lim_{n\to+\infty} H_n = 0$, we finally deduce (1.8).

3 Appendix: Some known results on BV-spaces

In what follows we present some known definitions and results on BV-spaces; some of them were used in the previous sections. We rely mainly on the book [4] by Ambrosio, Fusco and Pallara.

Definition 3.1. Let Ω be a domain in \mathbb{R}^N and let $f \in L^1(\Omega, \mathbb{R}^m)$. We say that $f \in BV(\Omega, \mathbb{R}^m)$ if the following quantity is finite:

$$\int_{\Omega} |Df| := \sup \bigg\{ \int_{\Omega} f \cdot \operatorname{div} \varphi \, dx : \, \varphi \in C_c^1(\Omega, \mathbb{R}^{m \times N}), \, |\varphi(x)| \le 1 \, \forall x \bigg\}.$$

Definition 3.2. Let Ω be a domain in \mathbb{R}^N . Consider a function $f \in L^1_{loc}(\Omega, \mathbb{R}^m)$ and a point $x \in \Omega$.

i) We say that x is an approximate continuity point of f if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \to 0^+} \frac{\int_{B_{\rho}(x)} |f(y) - z| \, dy}{\rho^N} = 0.$$

In this case we denote z by $\tilde{f}(x)$. The set of approximate continuity points of f is denoted by G_f .

ii) We say that x is an approximate jump point of f if there exist $a, b \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in S^{N-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0^{+}} \frac{\int_{B_{\rho}(x)} |f(y) - \chi(a, b, \boldsymbol{\nu})(y)| dy}{\rho^{N}} = 0, \tag{3.1}$$

where $\chi(a, b, \boldsymbol{\nu})$ is defined by

$$\chi(a, b, \boldsymbol{\nu})(y) := \begin{cases} b & \text{if } \boldsymbol{\nu} \cdot y < 0, \\ a & \text{if } \boldsymbol{\nu} \cdot y > 0. \end{cases}$$

The triple $(a, b, \boldsymbol{\nu})$, uniquely determined, up to a permutation of (a, b) and a change of sign of $\boldsymbol{\nu}$, is denoted by $(f^+(x), f^-(x), \boldsymbol{\nu}_f(x))$. We shall call $\boldsymbol{\nu}_f(x)$ the approximate jump vector and we shall sometimes write simply $\boldsymbol{\nu}(x)$ if the reference to the function f is clear. The set of approximate jump points is denoted by J_f . A choice of $\boldsymbol{\nu}(x)$ for every $x \in J_f$ determines an orientation of J_f . At an approximate continuity point x, we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.

Theorem 3.1 (Theorems 3.69 and 3.78 from [4]). Consider an open set $\Omega \subset \mathbb{R}^N$ and $f \in BV(\Omega, \mathbb{R}^m)$. Then:

- i) \mathcal{H}^{N-1} -a.e. point in $\Omega \setminus J_f$ is a point of approximate continuity of f.
- ii) The set J_f is σ - \mathcal{H}^{N-1} -rectifiable Borel set, oriented by $\boldsymbol{\nu}(x)$. I.e., the set J_f is \mathcal{H}^{N-1} σ -finite, there exist countably many C^1 hypersurfaces $\{S_k\}_{k=1}^{\infty}$ such that $\mathcal{H}^{N-1}\left(J_f\setminus\bigcup_{k=1}^{\infty}S_k\right)=0$, and for \mathcal{H}^{N-1} -a.e. $x\in J_f\cap S_k$, the approximate jump vector $\boldsymbol{\nu}(x)$ is normal to S_k at the point x. iii) $[(f^+-f^-)\otimes\boldsymbol{\nu}_f](x)\in L^1(J_f,d\mathcal{H}^{N-1})$.

Theorem 3.2 (Theorems 3.92 and 3.78 from [4]). Consider an open set $\Omega \subset \mathbb{R}^N$ and $f \in BV(\Omega, \mathbb{R}^m)$. Then, the distributional gradient Df can be decomposed as a sum of two Borel regular finite matrix-valued measures μ_f and $D^j f$ on Ω ,

$$Df = \mu_f + D^j f,$$

where

$$D^j f = (f^+ - f^-) \otimes \boldsymbol{\nu}_f \mathcal{H}^{N-1} \sqcup J_f$$

is called the jump part of Df and

$$\mu_f = (D^a f + D^c f)$$

is a sum of the absolutely continuous and the Cantor parts of Df. The two parts μ_f and D^jf are mutually singular to each other. Moreover, $\mu_f(B) = 0$ for any Borel set $B \subset \Omega$ which is \mathcal{H}^{N-1} σ -finite.

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