An analog of the Dougall formula and of the de Branges–Wilson integral

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We derive a beta-integral over $\mathbb{Z}\times\mathbb{R}$, which is a counterpart of the Dougall ${}_5H_5$-formula and of the de Branges–Wilson integral, our integral includes ${}_{10}H_{10}$-summation. For a derivation we use a two-dimensional integral transform related to representations of the Lorentz group, this transform is a counterpart of the Olevskii index transform (a synonym: Jacobi transform).

1 The statement

1.1. Gamma function of the complex field. Denote by $\Lambda_\mathbb{C}$ the set of all pairs $a|a'\in\mathbb{C}^2$ such that $a-a'\in\mathbb{Z}$. For nonzero $z\in\mathbb{C}$ we denote

$$z^{a|a'}:=z^a\,\overline{z}^{\,a'}:=|z|^a\Bigl(\frac{\overline{z}}{z}\Bigr)^{a'-a}.$$

Denote by $\Lambda\subset\Lambda_\mathbb{C}$ the set of all $a|a'\in\Lambda_\mathbb{C}$ satisfying the additional condition: $a+a'$ is pure imaginary. We have

$$\Bigl|z^{a|a'}\Bigr|=1 \qquad \text{for } a|a'\in\Lambda.$$

Elements of Λ can be represented as

$$a|a'=a|-\overline{a}=\tfrac12(k+is)\Big|\tfrac12(-k+is), \qquad \text{where } k\in\mathbb{Z},\ s\in\mathbb{R}.$$

Let z be a complex variable. Denote the Lebesgue measure by

$$d\,\overline{\overline{z}}:=d\,\mathrm{Re}\,z\,d\,\mathrm{Im}\,z.$$

Following [7], define the gamma function of the complex field by

$$\begin{aligned}\Gamma^\mathbb{C}(a|a')&:=\frac{1}{\pi}\int_\mathbb{C} z^{a-1|a'-1}e^{2i\,\mathrm{Re}\,z}\,d\,\overline{\overline{z}}=\\&=i^{a-a'}\frac{\Gamma(a)}{\Gamma(1-a')}=i^{a'-a}\frac{\Gamma(a')}{\Gamma(1-a)}=\frac{i^{a'-a}}{\pi}\Gamma(a)\Gamma(a')\sin\pi a'.\end{aligned}\tag{1.1}$$

Here $a|a'\in\Lambda_\mathbb{C}\simeq\mathbb{Z}\times\mathbb{C}$. The $\Gamma^\mathbb{C}$-function has poles at points $a|a'=-k|-l$, where $k,\,l\in\mathbb{N}$.

1.2. The statement. We derive the following beta-integral:

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Theorem 1.1 *Let a_1, a_2, a_3, a_4 satisfy the conditions $a_\alpha > 0$, $\sum a_\alpha < 1$. Then*

$$\frac{1}{4\pi^2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\Bigl|(k+is)\prod_{\alpha=1}^{4}\Gamma^{\mathbb{C}}\Bigl(a_\alpha+\tfrac{k+is}{2}\big|a_\alpha+\tfrac{-k+is}{2}\Bigr)\Bigr|^2 ds = \\ = \frac{\prod_{1\leqslant\alpha<\beta\leqslant 4}\Gamma^{\mathbb{C}}(a_\alpha+a_\beta|a_\alpha+a_\beta)}{\Gamma^{\mathbb{C}}\bigl(a_1+a_2+a_3+a_4\big|a_1+a_2+a_3+a_4\bigr)}. \quad (1.2)$$

We also can write the left hand side as

$$\frac{1}{4\pi^2}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}(k+is)(k-is)\times \\ \times\prod_{\alpha=1}^{4}\Gamma^{\mathbb{C}}\Bigl(a_\alpha+\tfrac{k+is}{2}\big|a_\alpha+\tfrac{-k+is}{2}\Bigr)\,\Gamma^{\mathbb{C}}\Bigl(a_\alpha+\tfrac{k-is}{2}\big|a_\alpha+\tfrac{-k-is}{2}\Bigr)\,ds. \quad (1.3)$$

Then the identity with the same right-hand side holds for

$$\operatorname{Re} a_\alpha > 0, \qquad \operatorname{Re}\sum a_\alpha < 1. \quad (1.4)$$

1.3. The de Branges–Wilson integral and the Dougall formula. Recall that the de Branges–Wilson integral is given by

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\left|\frac{\prod_{\alpha=1}^{4}\Gamma(a_\alpha+is)}{\Gamma(2is)}\right|^2 ds = \frac{\prod_{1\leqslant\alpha<\beta\leqslant 4}\Gamma(a_\alpha+a_\beta)}{\Gamma(a_1+a_2+a_3+a_4)}. \quad (1.5)$$

This formula was obtained by de Branges [4],[5] in 1972, a proof was not published; it was rediscovered by Wilson [18], 1980; see also [2]). The Dougall ${}_5H_5$ formula is

$$\sum_{k=-\infty}^{\infty}\frac{k+\theta}{\prod_{\alpha=1}^{4}\Gamma(b_\alpha+\theta+k)\Gamma(b_\alpha-\theta-k)} = \frac{\sin 2\pi\theta}{2\pi}\,\frac{\Gamma(b_1+b_2+b_3+b_4-3)}{\prod_{1\leqslant\alpha<\beta\leqslant 4}\Gamma(b_\alpha+b_\beta-1)} \quad (1.6)$$

Setting $b_4+\theta=1$, we get a series $\sum_{k\geqslant 0}$ of type ${}_5F_4[\dots;1]$, this result was contained in a family of identities obtained by Dougall [6], 1906. It seems that the general bilateral formula was obtained by Bailey [3], 1936; see also [2].

Let us explain a similarity of such formulas. Denote by $I(s)$ the integrand in (1.5) and extend it into a complex domain writing

$$|\Gamma(c+is)|^2 = \Gamma(c+is)\Gamma(c-is).$$

Next, consider the sum $\sum_{k=-\infty}^{\infty} I(i\theta+ik)$. We have

$$\Gamma(2is)\Gamma(-2is)\Big|_{s=i(\theta+k)} = \frac{-4\pi(\theta+k)}{\sin 2\pi\theta},$$

and we get the left-hand side of (1.6) with $b_\alpha = 1-a_\alpha$. Also, we get the same right-hand sides.

In [16] (see fromula (1.28)) there was derived a one-dimensional hybrid of (1.5) and (1.6) including both integration over the real axis and a summation over a lattice on the imaginary axis.

Our integral (1.2) can be obtained by formal replacing Γ-factors in the integrand (1.5) by similar $\Gamma^{\mathbb{C}}$-factors. The function $\Gamma_{\mathbb{C}}$ satisfies the reflection identity

$$\Gamma^{\mathbb{C}}(a|a')\,\Gamma^{\mathbb{C}}(1-a|1-a') = (-1)^{a'-a}. \qquad (1.7)$$

Therefore

$$\Gamma^{\mathbb{C}}(k+is|-k+is)\,\Gamma^{\mathbb{C}}(-k-is|k-is) = \frac{1}{(k+is)(k-is)}$$

and we come to the integrand in (1.2).

1.4. Further structure of the paper. In Section 2 we derive our integral (1.2). For a calculation we use a unitary integral transform $J_{a,b}$ defined in [13], see below Subsect. 2.3. This transform is an analog of a classical integral transform known under the names *generalized Mehler–Fock transform, Olevskii transform, Jacobi transform,* see [11], [12], [14].

We write an appropriate family of functions H_μ, and our integral (1.2) arises as the identity $\langle J_{a,b}H_\mu, J_{a,b}H_\nu\rangle = \langle H_\mu, H_\nu\rangle$.

Section 3 contains a further discussion of Theorem 1.1.

2 Calculation

2.1. Convergence of the integral.

Lemma 2.1 *Let* $a|a' \in \Lambda_{\mathbb{C}}$, $\lambda|\lambda' \in \Lambda$, *i.e.,* $\lambda' = -\overline{\lambda}$. *Then*

$$\Gamma^{\mathbb{C}}(a+\lambda|a'-\overline{\lambda}) \sim i^{a-a'+\lambda+\overline{\lambda}}\lambda^{-\frac12+a|-\frac12+a'}\cdot\frac{\lambda^{\lambda}}{\overline{\lambda}^{\overline{\lambda}}}\cdot e^{-\lambda+\overline{\lambda}}, \qquad \text{as } |\lambda|\to\infty.$$

In particular,

$$\bigl|\Gamma^{\mathbb{C}}(a+\lambda|a'-\overline{\lambda})\bigr| \sim |\lambda|^{-1+\mathrm{Re}(a+a')}, \qquad \text{as } |\lambda|\to\infty.$$

Remark. Our expression is single valued. Indeed,

$$\lambda^\lambda\overline{\lambda}^{-\overline{\lambda}} = \exp\bigl\{\lambda(\ln\lambda+2\pi iN) - \overline{\lambda}(\ln\overline{\lambda}-2\pi iN)\bigr\} = \lambda^\lambda\overline{\lambda}^{-\overline{\lambda}}\exp\bigl\{2\pi iN(\lambda+\overline{\lambda})\bigr\}.$$

But $\lambda+\overline{\lambda}\in\mathbb{Z}$, and therefore the result does not depend on a choice of a branch of $\ln\lambda$. ⊠

Proof. We use two expressions for $\Gamma^{\mathbb{C}}(a+\lambda|a'-\overline{\lambda})$, namely,

$$i^{a-a'+\lambda+\overline{\lambda}}\frac{\Gamma(a+\lambda)}{\Gamma(1-a'+\overline{\lambda})} = i^{-a+a'-\lambda-\overline{\lambda}}\frac{\Gamma(a'-\overline{\lambda})}{\Gamma(1-a-\lambda)}.$$

If $|\arg\lambda| < \pi-\varepsilon$, then we apply the Stirling formula (see e.g., [2]) to the first expression. If $|\arg(-\lambda)| < \pi-\varepsilon$, we apply it to the second expression. □

Corollary 2.2 *If the parameters a_α satisfy* (1.4)*, then the integral* (1.3) *absolutely converges.*

2.2. The Gauss hypergeometric function of the complex field. For $h|h' \in \Lambda_{\mathbb{C}}$ we denote

$$[h|h'] = \tfrac{1}{2}\,\mathrm{Re}(h+h').$$

Following [7] we define the beta function $B^{\mathbb{C}}[\cdot]$ and the Gauss hypergrometric function ${}_2F_1^{\mathbb{C}}[\cdot]$ of the complex field. Let $a|a'$, $b|b' \in \Lambda$. Then

$$B^{\mathbb{C}}(a|a', b|b') := \frac{1}{\pi}\int_{\mathbb{C}} t^{a-1|a'-1}(1-t)^{b-1|b'-1}\,d\overline{\overline{t}} = \frac{\Gamma^{\mathbb{C}}(a|a')\,\Gamma^{\mathbb{C}}(b|b')}{\Gamma^{\mathbb{C}}(a+b|a'+b')}. \qquad (2.1)$$

The integral absolutely converges iff

$$[a|a'] > 0, \quad [b|b'] > 0, \quad [a|a'] + [b|b'] < 2.$$

The right hand side gives a meromorphic continuation of $B^{\mathbb{C}}$ to the whole $\Lambda^2_{\mathbb{C}} \simeq \mathbb{Z}^2 \times \mathbb{C}^2$.

For $a|a'$, $b|b'$, $c|c' \in \Lambda$ we define the hypergeometric function

$$\begin{aligned}&{}_2F_1^{\mathbb{C}}\left[\begin{matrix}a|a', b|b'\\ c|c'\end{matrix};z\right] :=\\ &:= \frac{1}{\pi B^{\mathbb{C}}(b|b', c-b|c'-b')}\int_{\mathbb{C}} t^{b-1|b'-1}(1-t)^{c-b-1|c'-b'-1}(1-zt)^{-a|-a'}\,d\overline{\overline{t}}.\end{aligned} \qquad (2.2)$$

The integral has an open domain of convergence on any connected component of the set of parameters $\Lambda^3_{\mathbb{C}} \simeq \mathbb{Z}^3 \times \mathbb{C}^3$, it admits a meromorphic continuation to the whole set $\Lambda^3_{\mathbb{C}}$, see [13], Section 3.

The functions ${}_2F_1^{\mathbb{C}}[\cdot]$ admit explicit expressions in terms of sums of products of Gauss hypergeometric functions ${}_2F_1$. The standard properties of Gauss hypergeometric functions can be transformed to similar properties of functions ${}_2F_1^{\mathbb{C}}$, see [13], Section 3.

Below we need the following analog of the Gauss formula for ${}_2F_1[a,b;c;1]$:

$${}_2F_1^{\mathbb{C}}\left[\begin{matrix}a|a', b|b'\\ c|c'\end{matrix};z\right] = \frac{\Gamma^{\mathbb{C}}(c|c')\,\Gamma^{\mathbb{C}}(c-a-b|c'-a'-b')}{\Gamma^{\mathbb{C}}(c-a|c'-a')\,\Gamma^{\mathbb{C}}(c-b|c'-b')}, \qquad (2.3)$$

which is valid if

$$[c|c'] > [a|a'] + [b|b'], \qquad (2.4)$$

see [13], Proposition 3.2, the last condition coincides with a condition of continuity of ${}_2F_1^{\mathbb{C}}[\ldots;z]$ at $z=1$.

2.3. The index hypergeometric transform. Fix real a, b such that

$$0 \leqslant a \leqslant 1, \quad 0 \leqslant b \leqslant 1, \qquad (a,b) \neq (\pm 1, \pm 1), (\mp 1, \pm 1).$$

Consider the measure on $\mathbb{C}$ given by

$$\rho_{a,b}(z)\ d\overline{\overline{z}} = |z|^{2a+2b-2}|1-z|^{2a-2b}\ d\overline{\overline{z}}.$$

Let

$$\lambda|\lambda' = \tfrac{k+is}{2}\Big|\tfrac{-k+is}{2} \in \Lambda.$$

Consider the following function on $\Lambda \simeq \mathbb{Z}\times\mathbb{R}$:

$$\varkappa_{a,b}(\lambda|\lambda') = \varkappa_{a,b}(k,s) := \Big|\lambda\,\Gamma^{\mathbb{C}}(a-\lambda|a+\overline{\lambda}')\,\Gamma^{\mathbb{C}}(b+\lambda|b-\lambda')\Big|^2$$

and consider the space $L^2_{\mathrm{even}}(\Lambda,\varkappa_{a,b})$ of even functions on Λ with inner product

$$\langle\Phi,\Psi\rangle_{L^2_{\mathrm{even}}(\Lambda,\varkappa_{a,b})} := \sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\Phi(k,s)\,\overline{\Psi(k,s)}\varkappa_{a,b}(k,s)\,ds.$$

Next, define the kernel on $\mathbb{C}\times\Lambda$ by

$$\mathcal{K}_{a,b}(\lambda|\lambda') = \frac{1}{\Gamma^{\mathbb{C}}(a+b|a+b)}\,{}_2F_1^{\mathbb{C}}\left[\begin{matrix} a+\lambda|a-\lambda',\ a-\lambda|a+\lambda' \\ a+b|a+b\end{matrix};z\right]. \tag{2.5}$$

In [13] there was obtained the following statement:

The operator $J_{a,b}$ defined by

$$J_{a,b}f(\lambda|\lambda') := \int_{\mathbb{C}}\mathcal{K}(z;\lambda|\lambda')\,f(z)\,\rho_{a,b}(z)\,d\overline{\overline{z}}$$

is a unitary operator

$$L^2(\mathbb{C},\rho_{a,b}) \to L^2_{\mathrm{even}}(\Lambda,\varkappa_{a,b}).$$

2.4. Application of the Mellin transform. We define a Mellin transform $\mathcal{M}$ on $\mathbb{C}$ as the Fourier transform on the multiplicative group $\mathbb{C}^\times$ of $\mathbb{C}$. Since $\mathbb{C}^\times \simeq (\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R}$, the Mellin transform is reduced to the usual Fourier transform and Fourier series. We have

$$\mathcal{M}f(\xi|\xi') = \mathcal{M}f\big(\tfrac{l+\tau}{2}\big|\tfrac{-l+\tau}{2}\big) := \int_{\mathbb{C}} t^{\xi-1|\xi'-1}f(t)\ d\overline{\overline{t}}, \tag{2.6}$$

where $\xi|\xi'\in\Lambda_{\mathbb{C}}$. In the cases discussed below a function f on $\mathbb{C}\setminus 0$ is differentiable except the point $t=1$, where a singularity has a form $C_1+C_2(1-t)^{h|h'}$, $[h|h']>-1$. Also in our cases the integral (2.6) absolutely converges for σ being in a certain strip $A<[\xi|\xi']<B$, therefore the Mellin transform is holomorphic in the strip. The inversion formula is given by

$$f(t) = \frac{1}{4\pi^2 i}\sum_{l=-\infty}^{\infty}\int_{\gamma-i\infty}^{\gamma+i\infty} t^{-(l+\tau)/2|-(-l+\tau)/2}\,\mathcal{M}f\big(\tfrac{l+\tau}{2}\big|\tfrac{-l+\tau}{2}\big)\,d\tau,$$

the integration is taken over arbitrary line $\mathrm{Re}\,\sigma = \gamma$, where $A < \gamma < B$. We understand the integral (which can be non absolutely convergent) as

$$\sum_{l=-\infty}^{\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} := \lim_{N\to\infty,\, R\to\infty} \sum_{-N}^{N} \int_{\gamma-iR}^{\gamma+iR},$$

The inversion formula holds at points of differentiability of f, also it holds at points of singularities of the form $C_1 + C_2(1-t)^{h|h'}$ with $\mathrm{Re}(h+h') > -0$. In this case we can repeat the standard proof of pointwise inversion formula for the one-dimensional Fourier transform and pointwise covergence of Fourier series, see, e. g., [10], Sect. VIII.1, VIII.3; for advanced multi-dimensional versions of the Dini condition, see, e.g., [19], Sect. 9.

Convolution $f_1 * f_2$ on $\mathbb{C}^\times$ is defined by

$$f_1 * f_2(t) := \int_{\mathbb{C}} f_1(t/z) f_2(z)\, |z|^{-2}\, d\,\overline{\overline{z}}.$$

As usual, we have

$$\mathcal{M}(f_1 * f_2) = \mathcal{M}(f_1)\mathcal{M}(f_2),$$

this identity holds in intersection of strips of holomorphy $\mathcal{M}(f_1)$ and $\mathcal{M}(f_2)$. We also define a function $f^\star(t) = f(t^{-1})$. Then

$$\mathcal{M}f^\star(\xi|\xi') = \mathcal{M}f(-\xi|-\xi').$$

So we have the following corollary of the convolution formula:

$$\int_{\mathbb{C}} f_1(t)\, f_2(t)\, |t|^{-2}\, d\,\overline{\overline{t}} = \mathcal{M}(f_1^\star * f_2)(1) =$$
$$= \frac{1}{4\pi^2 i} \sum_{l=-\infty}^{\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}f_1\bigl(-\tfrac{l+\tau}{2}\bigm|-\tfrac{-l+\tau}{2}\bigr)\, \mathcal{M}f_2\bigl(\tfrac{l+\tau}{2}\bigm|\tfrac{-l+\tau}{2}\bigr)\, d\sigma, \quad (2.7)$$

where the integration contour is contained in the intersection of domains of holomorphy of $\mathcal{M}f_1^\star$ and $\mathcal{M}f_2$.

Lemma 2.3 a) *Let $[q|q'] < 0$. Then the Mellin transform sends a function*

$$t^{p|p'}(1-t)^{q|q'}$$

to

$$\Gamma^{\mathbb{C}}(q+1|q'+1) \frac{\Gamma^{\mathbb{C}}\bigl(p+\xi\bigm|p'+\xi'\bigr)}{\Gamma^{\mathbb{C}}\bigl(p+q+1+\xi\bigm|p'+q'+1+\xi'\bigr)},$$

it is holomorphic in the strip

$$-[p|p'] < [\xi|\xi'] < -[p|p'] - [q|q'].$$

b) *Assume that*

$$[c|c'] - [a|a'] - [b|b'] > -1.$$

Then the Mellin transform sends a function

$$ {}_2F_1^{\mathbb{C}}\begin{bmatrix} a|a', b|b' \\ c|c' \end{bmatrix}; t $$

to the function

$$ \frac{\Gamma^{\mathbb{C}}(a-\xi|a'-\xi')\,\Gamma^{\mathbb{C}}(b-\xi|b'-\xi')\,\Gamma^{\mathbb{C}}(\xi|\xi')\,\Gamma^{\mathbb{C}}(1-c+\xi|1-c'+\xi')}{\Gamma^{\mathbb{C}}(a|a')\,\Gamma^{\mathbb{C}}(b|b')\,\Gamma^{\mathbb{C}}(1-c|1-c')} \tag{2.8} $$

defined in the strip

$$ \max\bigl(-1, -1+[c|c']\bigr) < [\xi|\xi'] < \min\bigl([a|a'], [b|b]\bigr) \tag{2.9} $$

(*if it is non-empty*)

Proof. The statement a) follows from the definition of $B^{\mathbb{C}}$-function (2.1).

b) The function ${}_2F_1^{\mathbb{C}}[z]$ has singularities at $z = 0$, 1, ∞ with asymptotics of the form

$$ {}_2F_1^{\mathbb{C}}[z] \sim A_1 + A_2 |z|^{1-c|1-c'} \qquad \text{as } z \to 0; \tag{2.10} $$

$$ {}_2F_1^{\mathbb{C}}[z] \sim B_1 + B_2 (1-z)^{c-a-b|c'-a'-b'} \qquad \text{as } z \to 1; \tag{2.11} $$

$$ {}_2F_1^{\mathbb{C}}[z] \sim C_1 z^{-a|-a'} + C_2 z^{-b|-b'} \qquad \text{as } z \to \infty, \tag{2.12} $$

see [13], Theorem 3.9. This gives us a strip of convergence of the Mellin transform.

We must evaluate the integral

$$ \int_{\mathbb{C}} z^{\xi-1|\xi'-1} \int_{\mathbb{C}} t^{b-1|b'-1}(1-t)^{c-b-1|c'-b'-1}(1-zt)^{-a|-a'}\; d\,\overline{\overline{t}}\; d\,\overline{\overline{z}}. \tag{2.13} $$

We change an order of the integration and integrate in z:

$$ \int_{\mathbb{C}} z^{\xi-1|\xi'-1}(1-zt)^{-a|-a'}\; d\,\overline{\overline{z}} = \pi B^{\mathbb{C}}(\xi|\xi', -a+1|-a'+1)t^{-\xi|-\xi'}. $$

Integrating in t we again met a $B^{\mathbb{C}}$-function and after simple cancellations and an application of the reflection formula (1.7) we come to (2.8). The successive integration is valid under conditions (2.9).

We must justify the change of order of integrations. In fact, (2.13) is absolutely convergent as a double integral, i.e.,

$$ \int_{\mathbb{C}}\int_{\mathbb{C}} |z|^{\mathrm{Re}(\xi+\xi')-2} t^{b+b'-2} |1-t|^{\mathrm{Re}(c+c'-b-b')-2} |1-zt|^{-\mathrm{Re}(a+a')}\; d\,\overline{\overline{t}}\; d\,\overline{\overline{z}} < \infty. $$

It is a special case of integral (2.13), we integrate it successively in z and in t under the same condition as for successive integration in (2.13). $\square$

Lemma 2.4 *Let* $\lambda|\lambda' \in \Lambda$ *and*

$$a > 0, \quad b > 0, \quad \mu > 0, \quad a+\mu < 1, \quad b+\mu < 1. \tag{2.14}$$

Then

$$\int_{\mathbb{C}} z^{a+b-1|a+b-1}(1-z)^{-b-\mu|-b-\mu}\,{}_2F_1^{\mathbb{C}}\begin{bmatrix} a+\lambda|a-\lambda', a-\lambda|a+\lambda' \\ a+b|a+b \end{bmatrix}; z\Big]\, d\overline{\overline{z}} =$$

$$= \frac{\Gamma^{\mathbb{C}}(a+b|a+b)\,\Gamma^{\mathbb{C}}(\mu+\lambda|\mu+\lambda')\,\Gamma^{\mathbb{C}}(\mu-\lambda|\mu-\lambda')}{\Gamma^{\mathbb{C}}(a+\mu|a+\mu)\,\Gamma^{\mathbb{C}}(b+\mu|b+\mu)} \tag{2.15}$$

Proof. We apply formula (2.7) assuming

$$f_1 := z^{a+b-1|a+b-1}(1-z)^{-b-\mu|-b-\mu}, \qquad f_2 := {}_2F_1^{\mathbb{C}}[\dots; z].$$

We evaluate Mellin transforms of f_1, f_2 applying Lemma 2.3. In the integrand in the right hand side of (2.7) we get a product of two factors. The first factor is

$$\frac{\Gamma^{\mathbb{C}}(-b-\mu+1|-b-\mu+1)\,\Gamma^{\mathbb{C}}(a+b-\xi|a+b-\xi')}{\Gamma^{\mathbb{C}}(a-\mu+1+\xi|a-\mu+1+\xi')},$$

it is holomorphic un the strip

$$-a-b < [\xi|\xi'] < \mu - a. \tag{2.16}$$

The second factor

$$\frac{\Gamma^{\mathbb{C}}(a+b|a+b)\,\Gamma^{\mathbb{C}}(a+\lambda-\xi|a-\lambda-\xi')\Gamma^{\mathbb{C}}(a-\lambda-\xi|a+\lambda-\xi')\Gamma^{\mathbb{C}}(\xi|\xi')(-1)^{\xi-\xi'}}{\Gamma^{\mathbb{C}}(a+\lambda|a+\lambda')\Gamma^{\mathbb{C}}(a-\lambda|a-\lambda')\Gamma^{\mathbb{C}}(a+b-\xi|a+b-\xi')}$$

is holomorphic in the strip

$$a+b-1 < [\xi|\xi'] < a. \tag{2.17}$$

It a, b are sufficiently small, then strips (2.16) and (2.17) have a non-empty intersection and we can apply formula (2.7). Two factors $\Gamma^{\mathbb{C}}(a+b-\xi|a+b-\xi')$ cancel and we get a factor independent on ξ and the integral

$$\sum \int \frac{\Gamma^{\mathbb{C}}(a+\lambda-\xi|a-\lambda-\xi')\Gamma^{\mathbb{C}}(a-\lambda-\xi|a+\lambda-\xi')\Gamma^{\mathbb{C}}(\xi|\xi')(-1)^{\xi-\xi'}}{\Gamma^{\mathbb{C}}(a-\mu+1+\xi|a-\mu+1+\xi')}\, d\tau.$$

The integrand up to a constant factor is a Mellin transform of a function ${}_2F_1^{\mathbb{C}}[\dots; z]$. By the inversion formula, integral (2.15) converts to

$$\frac{\Gamma^{\mathbb{C}}(a+b|a+b)\,\Gamma^{\mathbb{C}}(-b-\mu-1|-b-\mu-1)}{\Gamma^{\mathbb{C}}(a-\mu-1|a-\mu-1)}\,{}_2F_1^{\mathbb{C}}\begin{bmatrix} a+\lambda|a-\lambda', a-\lambda|a+\lambda' \\ a-\mu+1|a-\mu+1 \end{bmatrix}; 1\Big].$$

For sufficiently small a we can apply the Gauss identity (2.3). Thus we get (2.15) for sufficiently small a, $b > 0$.

Keeping in mind (2.10)–(2.12), we can easily verify that the integral in the left-hand side of (2.15) converges for

$$\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re}(a+\mu) < 1, \quad \operatorname{Re}(b+\mu) < 1.$$

Thus, under these conditions the left hand side is holomorphic. The right hand side also is holomorphic in this domain. Therefore, they coincide. □

2.5. Proof of Theorem 1.1. Let $\in \mathbb{R}$. Consider a function $H_\mu(z)$ on $\mathbb{C}$ given by

$$H_\mu(z) := (1-z)^{-a-\mu|-a-\mu}.$$

Lemma 2.5 a) $H_\mu \in L^2(\mathbb{C}, \rho_{a,b})$ *iff*

$$0 < 2\mu < 1 - a - b.$$

b)

$$\langle H_\mu, H_\nu \rangle_{L^2(\mathbb{C},\rho_{a,b})} = \frac{\Gamma^{\mathbb{C}}(a+b|a+b)\,\Gamma^{\mathbb{C}}(\mu+\nu|\mu+\nu)}{\Gamma^{\mathbb{C}}(a+b+\mu+\nu|a+b+\mu+\nu)}.$$

The statement a) is trivial, b) is reduced to $B^{\mathbb{C}}$-function. □

The $J_{a,b}$-image of H_μ is done by Lemma 2.4. Since $J_{a,b}$ is unitary, we have

$$\langle H_\mu, H_\nu \rangle_{L^2(\mathbb{C},\rho_{a,b})} = \langle J_{a,b} H_\mu, J_{a,b} H_\nu \rangle_{L^2_{\mathrm{even}}(\Lambda, \varkappa_{a,b})}.$$

This is Theorem 1.1, where the parameters a_1, a_2, a_3, a_4 are a, b, μ, ν. Our calculation is valid for positive reals a_1, a_2, a_3, a_4 satisfying conditions $a_1 + a + 2 + 2a_3 < 1$, $a_1 + a + 2 + 2a_4 < 1$. For extending the identity to the domain (1.4) we refer to Corollary 2.2, the integral (1.3) is holomorphic in the domain (1.4), the right hand side also is holomorphic.

3 Final remarks

3.1. Barnes–Ismagilov integrals. Let $p \leqslant q$. Let $a_\alpha | a'_\alpha$, $b_\alpha | b'_\alpha \in \Lambda$. Following Ismagilov [9], we define integrals of the form

$$I_{p,q}[a,b;z] := \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} \int_{-i\infty}^{i\infty} \prod_{\alpha=1}^{p} \Gamma^{\mathbb{C}}\Bigl(a_\alpha + \tfrac{k+\sigma}{2} \Bigm| a'_\alpha + \tfrac{-k+\sigma}{2}\Bigr) \times$$
$$\times \prod_{\beta=1}^{q} \Gamma^{\mathbb{C}}\Bigl(b_\beta + \tfrac{-k-\sigma}{2} \Bigm| b'_\beta + \tfrac{k-\sigma}{2}\Bigr) z^{(k+\sigma)/2 | (-k+\sigma)/2}\, d\sigma.$$

By [9], Lemma 2, such integrals admit a representation of the form

$$\sum_{j=1}^{q} \gamma_j(\cdot)\, {}_pF_{q-1}[\cdot, z]\, {}_pF_{q-1}[\cdot, z],$$

where $\gamma_j(\cdot)$ are products of Γ-factors and parameters of hypergeometric functions ${}_pF_{q-1}[\cdot, z]$ are linear expressions in a_α, b_β and a'_α, b'_β. It is reasonable to claim that integrals $I_{p,q} =: {}_pF^{\mathbb{C}}_{q-1}$ are hypergeometric functions of the complex field.

By Lemma 2.4.b, the functions ${}_2F^{\mathbb{C}}_1$ defined by (2.2) are compatible with this definition. Ismagilov considered ${}_4F^{\mathbb{C}}_3[\dots;1]$-expressions that are counterparts of the Racah coefficicients for unitary representations of the Lorentz group $\mathrm{SL}(2,\mathbb{C})$. Our theorem is an example of a hypergeometric identity for ${}_5F^{\mathbb{C}}_4[\dots;1]$.

3.2. A difference problem. The de Branges–Wilson integral, the Dougall formula, and our integral (1.2) are representatives of beta integrals in the sense of Askey [1]. Quite often integrands $w(x)$ in beta integrals are weight functions for systems of hypergeometric orthogonal polynomials. In particular, orthogonal polynomials corresponding to the de Branges–Wilson integral are the Wilson polynomials, see [18], [2]. Recall that they are even eigenfunctions of the following difference operator

$$Lf(s) = \frac{\prod_{\alpha=1}^{4}(a_\alpha + is)}{2is(1+2is)}\bigl(f(s-i) - f(s)\bigr) + \frac{\prod_{\alpha=1}^{4}(a_\alpha - is)}{-2is(1-2is)}\bigl(f(s+i) - f(s)\bigr),$$

where $i^2 = -1$. If an integrand $w(x)$ of a beta integral decreases as a power function, then only finite number of moments $\int x^n w(x)\,dx$ converge; however in this case a beta integral can be a weight for a finite system of hypergeometric orthogonal polynomials (this phenomenon was firstly observed by Romanovski in [17]), the system of orthogonal polynomials related to the Dougall ${}_5H_5$ formula was obtained in [14]. On the other hand, such finite systems are discrete parts of spectra of explicitly solvable Sturm–Liouville problems (see, e.g., [8], [15]).

In the case of our integral (1.2) the integrand decreases as $|k+is|^{2\sum a_j - 8}$, we have no orthogonal polynomials. However a difference Sturm–Liouville problem can be formulated. We consider a space of meromorphic even functions $\Phi(\lambda|\lambda')$ on $\Lambda_{\mathbb{C}}$, a weight on $\Lambda \subset \Lambda_{\mathbb{C}}$ defined by the integrand (1.2), and the following commuting difference operators:

$$\mathfrak{L}\Phi(\lambda|\lambda') = \frac{\prod_{\alpha=1}^4 (a_\alpha + \lambda)}{2\lambda(1+2\lambda)}\bigl(\Phi(\lambda+1|\lambda') - \Phi(\lambda|\lambda')\bigr) + \\ + \frac{\prod_{\alpha=1}^4 (a_\alpha - \lambda)}{-2\lambda(1-2\lambda)}\bigl(\Phi(\lambda-1|\lambda') - \Phi(\lambda|\lambda')\bigr);$$

$$\mathfrak{L}\Phi(\lambda|\lambda') = \frac{\prod_{\alpha=1}^4 (a_\alpha + \lambda')}{2\lambda'(1+2\lambda')}\bigl(\Phi(\lambda|\lambda'-1) - \Phi(\lambda|\lambda')\bigr) + \\ + \frac{\prod_{\alpha=1}^4 (a_\alpha - \lambda')}{-2\lambda'(1-2\lambda')}\bigl(\Phi(\lambda|\lambda'+1) - \Phi(\lambda|\lambda')\bigr).$$

See a simpler pair of difference operators of this kind in [13]. On the other hand, see a one-dimensional operator with continuous spectrum similar to L in Groenevelt [8].

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