

# Causality: a decision theoretic foundation

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## Abstract

We propose a decision theoretic model akin to Savage [14] that is useful for defining causal effects. Within this framework, we define what it means for a decision maker (DM) to act as if the relation between two variables is causal. Next, we provide axioms on preferences and show that these axioms are equivalent to the existence of a (unique) Directed Acyclic Graph (DAG) that represents the DM's preference. The notion of representation has two components: the graph factorizes the conditional independence properties of the DM's subjective beliefs, and arrows point from cause to effect. Finally, we explore the connection between our representation and models used in the statistical causality literature (for example, Pearl [12]).

KEYWORDS: causality, decision theory, subjective expected utility, axioms, representation theorem, intervention preferences

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## 1 INTRODUCTION

Consider a statistician (say, Alex) that investigates the relation between intellectual ability, education level, and lifetime earnings of a particular citizen (say, Mr. Kane). As a good statistician, Alex is able to choose between the following options. A safe bet that pays \$0 for sure, or the risky bet defined below.

- If Mr. Kane has a college degree, and earns more than \$ 100K a year, Alex gets \$1
- If Mr. Kane has a college degree, and earns less than \$ 100K a year, Alex gets -\$1
- If Mr. Kane does not have a college degree, Alex gets \$0.

For concreteness, suppose Alex chooses the risky option. Her behavior reveals that, *conditional on obtaining a college degree*, Alex believes that it is more likely that Mr. Kane earns more than \$100K a year than it is that he earns less than \$100K a year. Now, assume Alex is presented with the same choice, but replacing “college degree” with “high-school degree”; moreover, assume that Alex now prefers getting \$0 for sure. Her behavior reveals that, *conditional on obtaining a high-school degree*, Alex believes that it is more likely that Mr. Kane earns less than \$100K a year than it is that he earns more than \$100K a year. Alex’s behavior reveals that she believes Mr. Kane’s education levels and lifetime earnings are qualitatively *positively correlated*: she accepts a \$1 gamble that Mr. Kane is making more than \$100K a year conditional on *observing* that Mr. Kane obtained a college degree, but not conditional on *observing* that Mr. Kane only obtained a high-school degree. Finally, if Alex is probabilistically sophisticated then we can represent her beliefs with a joint probability distribution over all relevant variables. In particular, this probability distribution is such that education and lifetime earnings are positively correlated.

Alex is now approached by a benevolent politician who wants to improve his constituents’ lifetime earnings. Since Alex believes that education and earnings are positively correlated, this politician expects that a policy which forces everyone to obtain a college degree should be useful to improve lifetime earnings. However,

Alex rejects that conclusion. While she believes Mr. Kane’s education level and lifetime earnings are positively correlated, she is of the opinion that policies which change the population’s education levels keeping all other things equal are useless for affecting lifetime earnings. Alex believes that high education levels are associated to high intellectual ability, high intellectual ability is associated to higher lifetime earnings, and that this is the only channel through which education levels and lifetime earnings are related. Thus, a policy which improves education levels but leaves intellectual ability unchanged is useless to improve lifetime earnings.

The apparent tension between Alex’s belief that education and earnings are positively correlated, while maintaining a position that policies which only affect education are useless to affect lifetime earnings, is rationalized by the adage “correlation is not causation”. In this context, *causation* has a specific meaning: a variable *subjectively causes* another variable if, holding fixed all other variables, policy interventions on the first variable affect Alex’s beliefs about the second. That Alex believes education policies are useless to affect lifetime earnings (holding fixed intellectual ability) means that she believes education levels don’t cause lifetime earnings.

The above definition of causal effect is entirely subjective. As such, this definition is not about objective truths, or uncovering laws of nature. However, this definition exactly captures how causality is understood in economics. In economics, causal relations are correlations which, *in the analysts subjective opinion*, are valid grounds for making policy recommendations. While disagreements exist with regards to *how* one arrives at the conclusion that an observed correlation is sufficient grounds for making policy recommendations, the definition of causation as the bridge between correlation and policy recommendation is undisputed. This dichotomy -when are two variables correlated versus when is it that one variable is a useful policy tool to affect the other- is the foundation of our definition of causal effect. By identifying a unique numerical representation of this definition, our paper provides a foundation for selecting models with which empirical researchers can estimate causal effects.

The purpose of axiomatic exercises like Savage’s [14] is to provide a link between

some numerical model and the way a rational Decision Maker (henceforth, DM) approaches the issue of interest (in this case, causality). The goal is to guarantee that the numerical model treats the object of study the way a rational DM would. For empirical research, the role of the DM is played by the researcher (who presumably wants to behave rationally) and the role of the DM's beliefs is played by the probability laws the researcher feeds into the numerical model. The axiomatic exercise requires that the DM/researcher can produce beliefs/probability distributions to feed into the numerical model, but how those distributions are obtained is an orthogonal question. The objective of this paper is to provide a theoretical foundation for selection of numerical models to use in empirical studies of causality in economics.

The paper is structured in three steps. First, we propose a decision problem similar to Savage's: there is a set of states, a set of acts mapping states into monetary amounts, and a DM who chooses amongst acts. The DM makes choices as if picking the best alternative according to a preference relation. This language is sufficient to talk about the subjective correlation structure in the DM's beliefs. However, to discuss causal effects, we also need a language to talk about preferences over intervention policies that affect the states. Therefore, we extend the language in the Savage model to accommodate for the possibility of choosing policies that affect the states. Section 2 describes the model and Section 3 formally defines causality. Second, we carry out a standard decision theoretic analysis: we provide axioms on preferences and show that the axioms hold if, and only if, we can numerically represent the DM's beliefs. Since we are interested in causal effects, the conditional independence properties of the DM's joint beliefs are of special interest. A useful tool to represent conditional independence properties of a probability law is the language of Bayesian Graphs (see Section 4.1 for the axioms, and Section 4.2 for the representation theorem). Thus, our representation has a graph theoretic component. In particular, Proposition 1 and Theorem 1 show that our definition of causal effect -which, as argued before, is tailored to how economists use the term causality- admits a unique representation in terms of a Bayesian Network. Finally, to make the model useful for empirical applications, we provide an additional axiom under which causal effects can be directly calculated from standard conditional

probabilities in a decision problem with no intervention policies. When this is the case we say that causal effects are *identified*. Proposition 2 and Theorem 2 in Section 5 prove the identification results.

As the reader may anticipate, there is an extensive literature in statistics, computer science, and economics that deals with causal effects. The related literature is discussed in Section 6, and we delay a discussion of it until after we present our results. This is because our results depend on a series of definitions and terms that connect to various literatures. Hence, we do not yet have the language to meaningfully discuss related work.

## 2 MODEL AND NOTATION

### 2.1 General notation

The following notation is useful, and is used through the paper. The set  $\mathcal{N} = \{1, \dots, N\}$  is a set of indexes. For each  $\mathcal{J} \subset \mathcal{N}$ , let  $\{X_j : j \in \mathcal{J}\}$  be a family of sets indexed by  $\mathcal{J}$ . We denote with  $X_{\mathcal{J}} = \prod_{j \in \mathcal{J}} X_j$  the Cartesian products of the family, and with  $x_{\mathcal{J}} = (x_j)_{j \in \mathcal{J}}$  a canonical element in  $X_{\mathcal{J}}$ . Moreover, all complements are taken with respect to  $\mathcal{N}$ : if  $\mathcal{J} \subset \mathcal{N}$ , then  $\mathcal{J}^c \equiv \mathcal{N} \setminus \mathcal{J}$ . As a final piece of notation, if  $\mathcal{J} \subset \mathcal{N}$  and  $E \subset X_{\mathcal{J}}$ , then  $\mathbb{1}_E : X_{\mathcal{J}} \rightarrow \{0, 1\}$  denotes the indicator function that event  $E$  has occurred. That is,  $\mathbb{1}_E(x_{\mathcal{J}}) = 1 \Leftrightarrow x_{\mathcal{J}} \in E$ .

The following notation refers to the graph theoretic component of the model. A directed graph is a pair  $(V, E)$  such that  $V$  is a (finite) set of nodes and  $E \subset V \times V$  is the set of edges. If two nodes,  $i$  and  $j$ , satisfy that  $(i, j) \in E$ , we simplify notation by writing  $i \rightarrow j$ . Moreover, the set of *parents* for a node  $v \in V$  is the set  $Pa(v) = \{v' \in V : (v', v) \in E\}$ . A node  $v \in V$  is a descendant of a node  $v' \in V$  whenever there is a directed path from  $v'$  to  $v$ . Formally, if there is a sequence  $(v_1, \dots, v_T) \in V^T$  such that  $v_1 = v'$ ,  $v_t$  is a parent of  $v_{t+1}$  for each  $t \in \{1, \dots, T-1\}$ , and  $v_T = v$ . Likewise,  $v'$  is an ancestor of  $v$  whenever  $v$  is a descendant of  $v'$ . A directed graph is a DAG if, and only if, for all  $v \in V$ ,  $v$  is not a descendant of  $v$ . We denote with  $D(v)$  the set of descendants of  $v$  and with  $ND(v)$  the set of non-descendants.

## 2.2 Model description

Our DM faces a variant of a standard Savage problem. The state space is  $S = \prod_{i=1}^N X_i$ , where each  $X_i$  is finite. We make this assumption for technical simplicity, because causality is orthogonal to whether state spaces are finite or infinite. We let  $\mathcal{N} = \{1, \dots, N\}$  and we call each  $i \in \mathcal{N}$  a *variable*. Set  $\mathcal{A} = \mathbb{R}^S$  is the set of Savage acts, and a decision maker has preferences  $>$  over  $\mathcal{A}$ .

However, our problem differs from Savage's since we incorporate policies that intervene the states. This added language allows us to distinguish correlations from other types of relations amongst variables. A set of *intervention policies* is a set  $\mathcal{P} = \prod_{i=1}^N (X_i \cup \{\emptyset\})$ . The interpretation is as follows. Let a policy  $p \in \mathcal{P}$  be such that  $p_i = \emptyset$  for some  $i \in \mathcal{N}$ . Then, this policy leaves variable  $i$  unaffected; that is,  $i$  will be determined as it would have been in a standard Savage world. However, if for some  $j \in \mathcal{N}$ , we have  $p_j = x_j \in X_j$ , then policy  $j$  forces variable  $j$  to take value  $x_j$ . That is, the value for variable  $j$  is not determined as it would have been in a Savage problem but has been chosen by the DM. Therefore, each policy implies a collection of interventions of the state space.

To understand the role of intervention policies in our model, consider the following example.

**Example 1.** *Let acts  $f$  and  $g$  over lifetime earnings be defined as follows. Act  $f$  pays \$1 if lifetime earnings are above \$100K per year, and  $-\$1$  otherwise. Act  $g$  is the opposite: it pays  $-\$1$  if lifetime earnings are above \$100K per year, and \$1 otherwise. Compare the following statements:*

1. *“Having observed that Mr Kane got a college degree (out of his own free will and ability), Alex prefers  $f$  to  $g$ .”*
2. *“Having forced Mr. Kane to obtain a college degree (regardless of his desire or ability to do so) Alex prefers  $f$  to  $g$ ”.*

These are clearly different statements, and don't imply one another. Therefore, we need a language to distinguish these two distinct decision problems. Intervention policies are such a language.

The discussion above implies our DM's primitive choice domain must be ex-

panded so as to include intervention policies. To do so, let  $p \in \mathcal{P}$  be any policy, and let  $\mathcal{N}(p) = \{i \in \mathcal{N} : p_i = \emptyset\}$ . That is,  $\mathcal{N}(p)$  are the variables that  $p$  leaves unaffected. Furthermore, let  $\mathcal{A}(p) \equiv \mathbb{R}^{X_{\mathcal{N}(p)}}$  be the set of acts defined over the variables that  $p$  left unaffected. Then, the primitive domain of choice for the DM is the set  $\{(p, a) : p \in \mathcal{P}, a \in \mathcal{A}(p)\}$ . That is, our DM's problem is to pick an intervention policy and a Savage act over the non-intervened variables. We endow this DM with a preference relation  $\succ$  on  $\{(p, a) : p \in \mathcal{P}, a \in \mathcal{A}(p)\}$ .

Given  $\succ$ , each  $p$  induces an *intervention preference* on  $\mathcal{A}(p)$ : for each  $p \in \mathcal{P}$  and each  $f, g \in \mathcal{A}(p)$  we say  $f \succ_p g$  if, and only if,  $(p, f) \succ (p, g)$ . Since our axioms are focused on the DM's intervention preferences, it is convenient to express intervention preferences explicitly in terms of the values at which the variables where intervened. For each policy  $p \in \mathcal{P}$ , if  $p_{\mathcal{N}(p)^c} = x_{\mathcal{N}(p)^c}$ , we use  $\succ_{x_{\mathcal{N}(p)^c}}$  to denote  $\succ_{p_{\mathcal{N}(p)^c}}$ . The special case where  $p = (\emptyset, \dots, \emptyset)$ , so that no variables are intervened, corresponds to the DM's preferences in a standard Savage world. For such a  $p$ , we use  $\succ_{(\emptyset, \dots, \emptyset)} = \succ$  for notational simplicity.

Intervention preferences look like Savage conditional preferences but have important differences. Savage conditional preferences operate as follows. First, a decision maker picks an act of the form  $\mathbb{1}_{x_{\mathcal{J}}} f$ , where  $x_{\mathcal{J}}$  is some realization of the variables in  $\mathcal{J}$  and  $f$  is some Savage act. Then, the DM goes about his life as usual as he waits for forces outside of his control (say, “Nature”) to pick a value for the state. When a state  $s \in X$  gets realized, if  $s_{\mathcal{J}} = x_{\mathcal{J}}$ , the DM is paid  $f(s)$ ; otherwise, the DM is paid 0. Thus, Savage conditional preferences refer to statements as in item [1.] of Example 1. In an intervention preference, the DM chooses a policy  $p$ , and an act  $f$  over  $\mathcal{N}(p)$ . That is, Nature is brushed aside while the DM steps in to *impose* that variables in  $\mathcal{N}^c(p)$  take the value  $p_{\mathcal{N}^c(p)}$ . After choosing  $p$  and  $f$ , the DM goes about his life as usual as he waits for Nature to pick the value of the non-intervened variables,  $\mathcal{N}(p)$ . Thus, intervention preferences refer to statements as in item [2.] of Example 1. Understanding the relation between Savage conditional preferences and intervention preferences is the core of our analysis of causal effect.

### 3 DEFINITION OF CAUSAL EFFECT

We now introduce the definition of causal effect, which formalizes the intuitive definition given in Section 1. Conceptually, we say that a variable  $j$  *causes* a variable  $i$  when policies that *intervene*  $j$  at different levels have a *ceteris paribus* effect on the DM’s beliefs about  $i$ . If the way a DM chooses acts over  $i$  is insensitive to whether or not  $j$  was intervened, we say  $j$  does not cause  $i$ . As mentioned in Section 1, this is precisely how “causation” is understood in economics: as any correlation which, subjectively for the analyst, is sufficient ground for making policy recommendations.

The definition informally described above emphasized two terms: “intervene”, and “ceteris paribus”. As discussed in Section 2, interventions are important to distinguish relations that are purely correlations (as decided by Nature) from other types of relations. The term *ceteris paribus* -meaning that variables other than  $i$  and  $j$  are intervened to a constant level- is present to guarantee that we are defining direct causal effects, rather than indirect effects.

Figure 1 illustrates the importance of using *ceteris paribus* interventions when defining causal effects. The figure captures a DM who believes *Ability* has a direct impact on *Education*, *Education* has a direct impact on *Lifetime earnings*, but *Ability* has no direct impact on *Lifetime earnings*. If  $a, a' \in A$  are two ability levels, and  $f, g \in \mathbb{R}^{L \times E}$  are to acts on lifetime earnings that are constant in  $E$ , we might have the DM behave as follows:  $f \succ_a g$ , and  $g \succ_{a'} f$ . This is because intervening  $A$  at different levels has an impact on  $E$ , and  $E$  has an impact on  $L$ . However, this captures an indirect effect of  $A$  on  $L$ , whereas we are interested in the *direct* effect of  $A$  on  $L$ . The correct way to capture this is to look at intervention preferences  $\succ_{(a,e)}$  as a function of  $a$ , for each fixed  $e \in E$ . This is the purpose of the term “*ceteris paribus*” in our informal description of causal effect. To overcome the naivety presented above, we define intervention-independence. Consider a set of variables  $\mathcal{K}$  and two variables  $i, j \notin \mathcal{K}$ . Informally,  $i$  is  $\mathcal{K}$ -independent of  $j$  if, after eliminating the possibility that  $i$  and  $j$  are mediated through variables in  $\mathcal{K}$ , the choice of acts over  $i$  is insensitive to interventions of  $j$ . Formally, we say that  $i$  is  $\mathcal{K}$ -independent of  $j$  if the following holds:  $(\forall x_{\mathcal{K}} \in X_{\mathcal{K}})$ ,  $(\forall x_j, x'_j \in X_j)$ , and



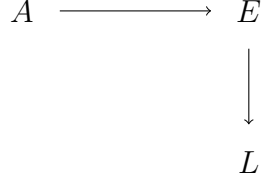


Figure 1: The naive definition of causal effect would have us wrongly assess that  $A$  causes  $E$ .

$(\forall f, g \in \mathbb{R}^{X_i}),$

$$\begin{aligned}
f >_{x_j, x_K} g &\Leftrightarrow f >_{x'_j, x_K} g, \\
f >_{x_j, x_K} g &\Leftrightarrow \mathbb{1}_{X_j} f >_{x_K} \mathbb{1}_{X_j} g.
\end{aligned}$$

The first line indicates that, having intervened  $K$ , intervening  $j$  at different values does not affect the DM's choice of act in  $\mathbb{R}^{X_i}$ . The second line indicates that, having intervened  $K$ , the ability to intervene  $j$  at all, regardless of the values at which it is intervened, does not affect the DM's choice of act in  $\mathbb{R}^{X_i}$ . Notice that the second of these conditions implies the first. Indeed, if the second condition holds then we have that, for all  $x_j, x'_j$ ,

$$f >_{x_j, x_K} g \Leftrightarrow \mathbb{1}_{X_j} f >_{x_K} \mathbb{1}_{X_j} g \Leftrightarrow f >_{x'_j, x_K} g,$$

and so the first equation holds too. For future reference, we record this in the following definition.

**Definition 1.** For all  $i, j \in \mathcal{N}$ , and  $K \subset \{i, j\}^c$  we say variable  $i$  is  $K$ -independent of variable  $j$  if for all  $f, g \in \mathbb{R}^{X_i}$ ,

$$f >_{x_j, x_K} g \Leftrightarrow \mathbb{1}_{X_j} f >_{x_K} \mathbb{1}_{X_j} g,$$

**Definition 2.** For all  $i, j \in \mathcal{N}$ , we say variable  $j$  causes variable  $i$  if  $i$  is not  $\{i, j\}^c$ -independent of  $j$ .

Let  $Ca(i) = \{j \in \mathcal{N} : j \text{ causes } i\}$  denote the causal set of  $i$ .

Finally, we say  $j$  is an indirect cause of  $i$  if there is a sequence  $j_0, \dots, j_T$  such that, for all  $t \in \{0, \dots, T-1\}$ ,  $j_t$  causes  $j_{t+1}$ ,  $j_0 = j$  and  $j_T = i$ .

We make two observations regarding Definition 2. First, it refers to direct (or unmediated) effects:  $j$  causes  $i$  if, *all other variables held constant*, interventions on  $j$  affect the DMs choice of acts in  $\mathbb{R}^{X_i}$ . This does not rule out that interventions on  $i$  *without keeping other variables constant* might also affect choices in  $\mathbb{R}^{X_i}$  (see Figure 1). However, we do not call such an effect a causal effect, since it might be an effect that  $i$  has on  $j$  *mediated* through other variables. Second, if a variable  $i$  is such that  $Ca(i) = \emptyset$ , we say  $i$  is an *exogenous primitive*; otherwise, we say it is an endogenous variable. Indeed, when a DM forms a causal model of the world, the set of primitives of such model is precisely the set of variables that are not caused by any other variable in the model. Exogenous primitives are relevant in our discussion of Axiom 2.

Finally, we define the causal graph associated to a preference,  $\succ$ . Causal graphs are an integral part of our representation, which is introduced in section 4.2. Given  $\succ$ , draw a graph by letting the set of nodes be the set of variables, and the set of arrows be defined by the causal sets. That is, by letting  $j \rightarrow i \Leftrightarrow j \in Ca(i)$ . This is well defined because  $Ca(i)$  is well defined for each  $i \in \mathcal{N}$ . We denote such graph with  $G(\succ)$ .

**Definition 3.** *Let  $\succ$  be a preference, and  $\{Ca(i) : i \in \mathcal{N}\}$  be the collection of causal sets derived from  $\succ$ . Define  $G(\succ) = (V, E)$  by setting  $V = \mathcal{N}$  and  $E = \{(j, i) : j \in Ca(i)\}$ .*

## 4 AXIOMS AND REPRESENTATION

Our main theorem is stated below. Section 4.1 states and discusses the set of axioms. Section 4.2 discusses why our notion for when a DAG represents a preference relation is reasonable.

**Theorem 1.** *Let  $\succ$  satisfy Axiom 1, and let  $\mu_p$  be the intervention beliefs elicited from  $\succ_p$ . The following are equivalent:*

- i* Axioms 2 and 3 hold,
- ii*  $(\exists G)$  such that  $G$  is a DAG, and  $G$  represents  $\succ$  in the following sense:

$(\forall x \in \Pi_{i \in \mathcal{N}} X_i)$ , and all  $\mathcal{J} \subset \mathcal{N}$

- $\mu_{x_{\mathcal{J}}}(x_{\mathcal{J}^c}) = \Pi_{i=1}^N \mu_{x_{\mathcal{J}}}(x_i | x_{ND(i) \setminus \mathcal{J}}) = \Pi_{i=1}^N \mu_{x_{\mathcal{J}}}(x_i | x_{Pa(i) \setminus \mathcal{J}})$
- $(\forall i)$   $Pa(i)$  is the smallest set such that the above property holds,
- If  $(i, j) \in E$  then  $\mu_{x_{\{i,j\}^c}}(x_j | x_i) = \mu_{x_{\{j\}^c}}(x_j)$ .

Furthermore, if  $G$  represents  $\succ$ , then  $G = G(\succ)$ .

#### 4.1 Axioms

We consider three basic axioms on  $\succ$ . Axiom 1 is standard and states that for each policy  $p$ , the DM's preferences induced by  $p$  are probabilistically sophisticated. Axioms 2 and 3 restrict how policies affect the choice of Savage acts over non-intervened variables; these restrictions convey qualitative properties of causal effects.

Axiom 1 establishes the following three properties: (i) our DM is a subjective expected utility maximizer, (ii) all relevant variables are included in the state space, and (iii) all states have positive probability. We include these properties as part of a basic axiom since they are more primitive issues than causation. Before introducing Axiom 1, we need to recall Gul's [4] axioms (Appendix D contains a formal statement of the Axioms). The reader is referred to the original paper for a normative discussions of these axioms.

**Axiom (Gul '95).** Let  $T$  be a finite set, and  $>$  a binary relation on  $\mathbb{R}^T$ , with weak part  $\succeq$  and symmetric part  $\sim$ .

G1.  $\succeq$  is complete and transitive.

G2. Independence axiom:  $(\forall f, g, h \in \mathbb{R}^T), (\forall t \in T), f > g \Leftrightarrow f' > g'$ , where  $f'$  and  $g'$  are appropriately constructed mixtures of  $f$  with  $h$  and  $g$  with  $h$  respectively. (see appendix for details on the mixing operation).

G3.  $\succeq$  is monotone increasing; if  $f$  and  $f'$  are constant acts, and  $f$  pays more than  $f'$ , then  $f > f'$ . Furthermore, the state space can be partitioned into two equally likely events; that is, there is an event  $A \subset T$  such that, for any two values  $x, y \in \mathbb{R}$ ,  $xAy \sim yAx$  (where  $xAy$  is the act that pays  $x$  in  $A$  and

$y$  elsewhere, and  $yAx$  is analogously defined)

G4.  $\succsim$  is continuous (upper and lower contour sets are closed).

**Axiom 1.** For each  $\mathcal{J} \subset \mathcal{N}$ , the following are true.

- i- For each  $p \in \mathcal{P}$ , the preferences induced by  $\succ_p$  satisfy the Gul [4] axioms. That is, for each  $\mathcal{J} \subset \mathcal{N}$ ,  $x_{\mathcal{J}} \in X_{\mathcal{J}}$ , the preference  $\succ_{x_{\mathcal{J}}}$  on  $\mathbb{R}^{X_{\mathcal{J}^c}}$ , satisfies the Gul axioms when setting  $T = X_{\mathcal{J}^c}$ .
- ii-  $(\forall i, j \in \mathcal{N})$ ,  $(\forall x_{\{i\}^c} \in X_{\{i\}^c})$ , and  $(\forall f, g \in \mathbb{R}^{X_i})$ , if  $j \in Ca(i)$  then  $f \succ_{x_{\{i\}^c}} g \Leftrightarrow \mathbb{1}_{x_j} f \succ_{x_{\{i,j\}^c}} \mathbb{1}_{x_j} g$ .
- iii- There are no null states: For all  $x \in X$ ,  $\mathbb{1}_x \succ \mathbb{1}_X 0$ .
- vi- Policies don't affect tastes:  $(\forall x, y \in \mathbb{R})$ ,  $(\forall p, p' \in \mathcal{P})$ ,  $\mathbb{1}_{X_{\mathcal{N}(p)}} x \succ_p \mathbb{1}_{X_{\mathcal{N}(p)}} y \Leftrightarrow \mathbb{1}_{X_{\mathcal{N}(p')}} x \succ_{p'} \mathbb{1}_{X_{\mathcal{N}(p')}} y$

Item [i.] in Axiom 1 is standard and implies that agents are subjective expected utility maximizers in all choice problems they face. In particular, for each possible intervention, agents are subjective expected utility maximizers in the associated choice problem. Formally, if Axiom 1 holds then for each  $\mathcal{J} \subset \mathcal{N}$ , and for each element  $x_{\mathcal{J}} \in X_{\mathcal{J}}$ , there exists an increasing and continuous function  $u_{x_{\mathcal{J}}} : \mathbb{R} \rightarrow \mathbb{R}$  and a probability distribution  $\mu_{x_{\mathcal{J}}}$  over  $X_{\mathcal{J}^c}$  such that

$$f \succ_{x_{\mathcal{J}}} g \Leftrightarrow \sum_{x_{\mathcal{J}^c}} u_{x_{\mathcal{J}}}(f(x_{\mathcal{J}^c})) \mu_{x_{\mathcal{J}}}(x_{\mathcal{J}^c}) > \sum_{x_{\mathcal{J}^c}} u_{x_{\mathcal{J}}}(g(x_{\mathcal{J}^c})) \mu_{x_{\mathcal{J}}}(x_{\mathcal{J}^c}).$$

Furthermore, each  $\mu_{x_{\mathcal{J}}}$  is unique and each  $u_{x_{\mathcal{J}}}$  is unique up to positive affine transformations. In particular, G3 implies that all Bernoulli utilities represent the same ordinal preference (namely, more money is better), but they might differ in cardinal ways, such as risk aversion coefficients. However, the central object of our study are the probability distributions, not the Bernoulli indexes. Thus, item [iv.] rules out that policies have a direct impact on the Bernoulli utility indexes.

Notice that Axiom 1 implies that all intervention preferences are probabilistically sophisticated. Henceforth, we refer to the probability distributions induced by intervention preferences as *intervention beliefs*.

Item [ii.] in Axiom 1 says that the state space is complete in the following sense:

given two variables (say,  $i$  and  $j$ ), the state space includes all variables that could mediate effects between  $i$  and  $j$ . Indeed, given  $i$  and  $j$ , two possibilities exist: either one of the variables affects the other directly (say,  $j$  is a cause of  $i$ ), or the variables are unrelated. If  $j$  causes  $i$ , since all possible confounding variables have been intervened, observing that  $x_j \in X_J$  was realized, or intervening variable  $j$  to value  $x_j$ , should lead to the same preference over  $\mathbb{R}^{X_i}$ . Violations of this axiom are only reasonable if the state space was missing some potential confounding variables. In line with Savage [14], we assume the state space is complete.

Axiom 1 states that the DM is probabilistically sophisticated, but is silent about the statistical properties of causal sets. Without further axioms to discipline how the causal sets behave we cannot guarantee that these sets will have any properties that we normatively associate with causation. Axioms 2 through 4 provide such discipline.

**Axiom 2.** *For all  $i \in \mathcal{N}$ ,  $i$  is not an indirect cause of  $i$ .*

Axiom 2 is equivalent to the following statement: for each set of variables  $\mathcal{I} \subset \mathcal{N}$ , there exists  $i \in \mathcal{I}$  such that  $Ca(i) \cap \mathcal{I} = \emptyset$ . That is, if the DM is asked to explain the relation between variables in  $\mathcal{I}$ , and only those in  $\mathcal{I}$ , the DM has an explanation that involves at least one exogenous primitive relative to  $\mathcal{I}$ . Models without primitives describe identities rather than relations amongst logically independent variables. Therefore, Axiom 2 says that the DM's state space only includes logically independent variables.

Axiom 2 could be seen as precluding DM from viewing the world as a system of recursive structural equations. As such, Axiom 2 could be seen as precluding the DM from reasoning in terms of equilibrium equations (see, for example, the critique in Heckman and Pinto [8]). This assessment stems from interpreting functional relations as causal relations. However, equations in a model (in particular, equilibrium equations) are succinct descriptions of the specific values that the variables may obtain; they say nothing of *how* those values are achieved. As such, causality and equilibrium equations are orthogonal issues.

To make the above discussion concrete, consider a general equilibrium model, with aggregate demand curve  $D$  and aggregate supply curve  $S$ . An equilibrium is defined as follows:  $(p^*, q^*)$  constitute an equilibrium if  $D(p^*) = q^*$  and  $S(p^*) = q^*$ .

Notice that this is a definition and as such, *equilibrium* price and *equilibrium* quantity are not logically independent. These equations describe the values one should expect for prices and quantities but are silent regarding the mechanism that generated them. This silence precisely motivates the equilibrium convergence literature. For example, a tâtonnement convergence process is compatible with the general equilibrium equations, without invoking feedback loops: a DM posits that prices in period  $t$  cause quantities in period  $t$  (via consumer/producer optimization), and that quantities in period  $t$  cause prices in period  $t + 1$  (through a process that increases/decreases price if there is an excess demand/supply). That the system stabilizes at a point where  $p_t = p_{t+1} = p^*$  and  $q_t = q_{t+1} = q^*$  is orthogonal to issues of causation. In short, one should not mistake functional equations -which simply describe relations between variables- with causal statements.

Another potential critique of this axiom is that certain systems are inherently cyclical. For instance, the relation between the speed of a car, the distance traveled by the car, and the time traveled by the car is inherently circular: any two determine the third. The problem with this system is that speed is not *caused* by distance and time traveled, but that speed is *defined* in terms of distance and time traveled. Therefore, the model includes variables which are not logically independent of one another. The correct model to analyze this situation is one that only variables are time and distance traveled by the car, as these are the only logically independent variables. In this world, it is sensible to assume that no causal cycles exists.

To understand Axiom 3 consider the sources of information that a DM finds useful for the purposes of choosing acts over a variable,  $i$ . Clearly, the causes of  $i$  should provide information about  $i$ ; by implication, indirect causes of  $i$  should also provide information about  $i$ . However, the information about  $i$  encoded in the indirect causes of  $i$  should already be contained in the causes of  $i$ . Therefore, conditional on the causes of  $i$ , indirect causes of  $i$  no longer provide useful information about  $i$ . Furthermore, the variables of which  $i$  is an indirect cause also provide information about  $i$ : by definition, such variables indirectly depend on  $i$ , so knowing their value is useful information about  $i$ . Axiom 3 formalizes these ideas. It states that there are only two fundamental sources of information about  $i$ : its causes and the variables that are indirectly caused by  $i$ . In particular, Axiom

3 implies that a variable is never statistically independent of its causes. Both of these properties are normative properties a set of causes should have.

Before presenting the axiom, let  $ND(i)$  denote the set of variables that are not indirectly caused by some variable  $i$ . We use this notation since this set will correspond to the set of  $i$ 's non-descendants in our representation.

$$ND(i) = \{j \in \{i\}^c : i \text{ is not an indirect cause of } j\}$$

**Axiom 3.**  $(\forall i \in \mathcal{N})$  and  $(\forall f, g \in \mathbb{R}^{X_i})$ , a set  $\mathcal{I} \subset \{i\}^c$  satisfies the condition below if, and only if,  $Ca(i) \subset \mathcal{I}$ .

$(\forall \mathcal{J} \subset ND(i) \cap \mathcal{I}^c)$ ,  $(\forall \mathcal{K} \subset \{i\}^c, \mathcal{J} \cap \mathcal{K} = \emptyset)$ , and  $(\forall x_{\mathcal{I} \cup \mathcal{J} \cup \mathcal{K}} \in X_{\mathcal{I} \cup \mathcal{J} \cup \mathcal{K}})$ ,

$$\mathbb{1}_{x_{\mathcal{J}}} \mathbb{1}_{x_{\mathcal{I} \setminus \mathcal{K}}} f \succ_{x_{\mathcal{K}}} \mathbb{1}_{x_{\mathcal{J}}} \mathbb{1}_{x_{\mathcal{I} \setminus \mathcal{K}}} g \Leftrightarrow \mathbb{1}_{x_{\mathcal{I} \setminus \mathcal{K}}} f \succ_{x_{\mathcal{K}}} \mathbb{1}_{x_{\mathcal{I} \setminus \mathcal{K}}} g$$

To understand why Axiom 3 captures the discussion presented above, fix a variable  $i$  and consider three cases separately: when  $\mathcal{K} = \emptyset$  and  $\mathcal{I} = Ca(i)$ , when  $\mathcal{K} \neq \emptyset$  and  $\mathcal{I} = Ca(i)$ , and when  $\mathcal{I} \subsetneq Ca(i)$ .

First, let  $\mathcal{K} = \emptyset$  and  $\mathcal{I} = Ca(i)$ . Axiom 3 says that the DM treats the following two problems in the same way. The first problem is to choose between two acts over  $i$  (say,  $f$  and  $g$ ), whose payments are contingent on the causes of  $i$  obtaining a certain value,  $x_{Ca(i)}$ . The second problem is to choose between the same acts  $f$  and  $g$  when payments are contingent on the causes of  $i$  obtaining the same value  $x_{Ca(i)}$ , as well as variables in  $\mathcal{J} \subset ND(i) \setminus Ca(i)$  obtaining some value,  $x_{\mathcal{J}}$ . In the example illustrated by Figure 2 below, let  $\mathcal{I} = Ca(i) = \{w, j\}$  and let  $\mathcal{J} = \{k\}$ . Thus, the first problem is to choose between  $\mathbb{1}_{x_{w,j}} f$  or  $\mathbb{1}_{x_{w,j}} g$  and the second problem is to choose between  $\mathbb{1}_{x_{w,j,k}} f$  or  $\mathbb{1}_{x_{w,j,k}} g$ . Our DM treats these problems the same way because, once the payments of  $f$  and  $g$  are contingent on the values of  $\{w, j\}$ , then the only variables that could affect his beliefs over  $X_i$  are variables that are indirectly caused by  $i$ . Since no variable in  $\mathcal{J}$  is indirectly caused by  $i$  (in Figure 2,  $\mathcal{J} = \{k\}$ ), none of these variables provide useful information about  $i$  that isn't already included in  $\{w, j\}$ . Hence, the two decision problems are equivalent. When  $\mathcal{K} \neq \emptyset$  and  $\mathcal{I} = Ca(i)$ , the logic in the above paragraph is still valid even if we intervene the variables in  $\mathcal{K}$  and work with the preference

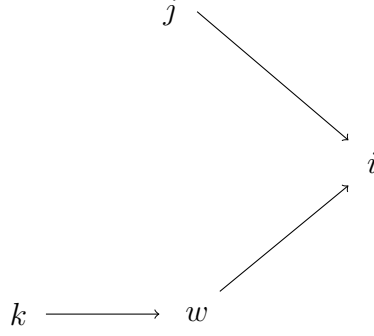


Figure 2: If  $i$  is independent of  $ND(i) \setminus \mathcal{I}$ , then  $\mathcal{I}$  must contain  $\{w, j\}$ .

induced by this intervention. Since the intervention  $\mathcal{K}$  might include variables in  $\mathcal{I}$ , we only choose contingent acts on the remaining variables,  $\mathcal{I} \setminus \mathcal{K}$ . For instance, in terms of Figure 2 let  $\mathcal{J} = \{k\}$ ,  $\mathcal{K} = \{w\}$ , and  $\mathcal{I} = Ca(i) = \{w, j\}$ . Axiom 3 states that  $\mathbb{1}_{x_j} f >_{x_w} \mathbb{1}_{x_j} g$  if, and only if,  $\mathbb{1}_{x_{j,k}} f >_{x_w} \mathbb{1}_{x_{j,k}} g$ : once  $w$  is intervened,  $k$  is uninformative about  $i$  since the only information  $k$  provides about  $i$  is that which is mediated through  $w$ .

Finally, suppose that  $\mathcal{I} \subsetneq Ca(i)$ . Since  $Ca(i) \setminus \mathcal{I} \neq \emptyset$ , and no variable in  $Ca(i) \setminus \mathcal{I}$  is indirectly caused by  $i$ , we can choose  $\mathcal{J} = Ca(i) \setminus \mathcal{I}$ . In the first decision problem, the payments of  $f$  and  $g$  are contingent on  $\mathcal{J} = Ca(i) \setminus \mathcal{I}$  and  $\mathcal{I}$ , so the payments of  $f$  and  $g$  are contingent on the full set of causes of  $i$ . In the second problem, the payments of  $f$  and  $g$  are only contingent on  $\mathcal{I}$ . Hence, they are not contingent on  $Ca(i) \setminus \mathcal{I}$ . In terms of Figure 2, let  $\mathcal{I} = \{w\} \subsetneq \{w, j\}$ ,  $\mathcal{J} = \{j\} \subset ND(i) \cap \mathcal{I}^c$  and  $\mathcal{K} = \emptyset$ . The first decision problem is to choose between  $\mathbb{1}_{x_{w,j}} f$  and  $\mathbb{1}_{x_{w,j}} g$ , whereas the second problem is to choose between  $\mathbb{1}_{x_w} f$  and  $\mathbb{1}_{x_w} g$ . Since variables in  $Ca(i) \setminus \mathcal{I}$  ( $\{j\}$  in the example) *do* provide useful information on  $i$ , then these two problems should *not* be equivalent. Thus, the equivalence established in Axiom 3 should hold when  $Ca(i) \subset \mathcal{I}$  but not when  $\mathcal{I} \subsetneq Ca(i)$ .

While Axioms 1 through 3 are our basic axioms, Axiom 4 is a supplementary axiom, which is relevant for Theorem 2. We present it here in the interest of keeping all axioms, and their corresponding discussions, contained withing a single section.

**Axiom 4.**  $(\forall i \in \mathcal{N}), (\forall \mathcal{J} \subset \{i\}^c) (\forall f, g \in \mathbb{R}^{X_i}), (\forall x_{Ca(i) \cup \mathcal{J}} \in X_{Ca(i) \cup \mathcal{J}}),$

$$\mathbb{1}_{\{x_{Ca(i)}\}} f > \mathbb{1}_{\{x_{Ca(i)}\}} g \Leftrightarrow \mathbb{1}_{x_{Ca(i) \setminus \mathcal{J}}} f >_{x_{\mathcal{J}}} \mathbb{1}_{x_{Ca(i) \setminus \mathcal{J}}} g.$$



Like Axiom 3, Axiom 4 states that two particular decision problems are equivalent. Given a variable  $i$ , and acts  $f, g \in \mathbb{R}^{X_i}$ , the first problem is to choose  $f$  or  $g$  when their payments are contingent on the causes of  $i$  obtaining a particular value,  $x_{Ca(i)}$ . In the second decision problem, the DM intervenes a subset of causes of  $i$ , (say,  $\mathcal{J} \subset Ca(i)$ ) to the value  $x_{\mathcal{J}}$ , and the payments of  $f$  and  $g$  are now contingent on the values  $x_{\mathcal{J}^c}$  being realized.

However, contrary to Axiom 3, Axiom 4 imposes restrictions *across* different intervention preferences. Indeed, for any fixed variable  $i$  and intervention  $x_{\mathcal{K}}$ , Axiom 3 provides conditions under which choice of acts over  $i$  is insensitive to observation of variables in a  $\mathcal{J}$ . Therefore, axiom 3 deals with the statistical independence properties of a *fixed* intervention belief, but is silent regarding the connection between the intervention preference  $>_{x_{\mathcal{K}}}$  and the standard Savage preference  $>$ . Axiom 4 is about the connection between any particular intervention preference,  $>_{x_{\mathcal{K}}}$ , and the conditional Savage preference,  $>_{\cdot|x_{\mathcal{K}}}$ . Thus, contrary to Axiom 3, Axiom 4 imposes restrictions *across* different intervention preferences.

To illustrate why Axiom 4 is normatively appealing, we use Figure 3 below.

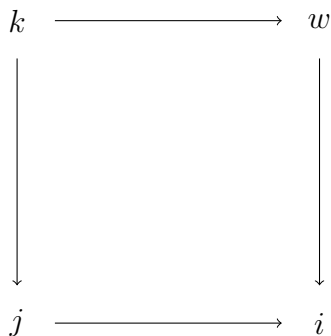


Figure 3: Observing or intervening  $j$  makes the DM update differently about  $k$ . This difference in updating may affect the DM's beliefs about  $i$ .

Suppose a DM has to choose between two acts over  $i$  (say,  $f, g \in \mathbb{R}^{X_i}$ ) whose payments are contingent on  $j$  taking value  $x_j$ . That is, the DM has to choose between  $\mathbb{1}_{x_j}f$  and  $\mathbb{1}_{x_j}g$ . Observing that  $j$  took value  $x_j$  gives the DM information about the value of  $k$ ; in turn, this information about  $k$  gives the DM information about  $w$  which, ultimately, gives the DM information about  $i$ . Thus, observing that  $j$  took value  $x_j$  is informative about  $i$  in two ways: directly, because  $j \in Ca(i)$ ,

and indirectly via  $k$  and  $w$ . If the DM intervenes  $j$  at value  $x_j$  he receives the same direct information about  $i$ , but loses the indirect information mediated via  $k$  and  $w$ . Thus, the DM could say that  $\mathbb{1}_{x_j}f > \mathbb{1}_{x_j}g$  but  $g >_{x_j} f$ . Clearly, observing  $x_j$  or intervening variable  $j$  to value  $x_j$  are different problems.

Now consider the situation above but when the payments of  $f$  and  $g$  are contingent on the values of both  $j$  and  $w$ . That is, for some  $x_j$  and  $x_w$ , the DM must choose between  $\mathbb{1}_{x_j, x_w}f$  and  $\mathbb{1}_{x_j, x_w}g$ . For concreteness, suppose that  $\mathbb{1}_{x_j, x_w}f > \mathbb{1}_{x_j, x_w}g$ . If the DM intervened  $j$  to value  $x_j$  and then had to choose between  $\mathbb{1}_{x_w}f$  and  $\mathbb{1}_{x_w}g$ , would the DM lose any information? Notice that in both problems  $w$  is observed to take value  $x_w$ . Therefore, any information  $j$  could indirectly provide about  $i$  through  $w$  is still directly captured in the observation of  $x_w$ . Thus, intervening  $j$  entails no information loss relative to simply observing that  $j$  took value  $x_j$ . Thus, the DM has the same information in both problems, so he should treat them equivalently. This is precisely what Axiom 4 requires.

To complete our discussion of Axiom 4 we must consider the case when  $\mathcal{J}$  contains non-causes of  $i$ . Axiom 4 states that, once we know the value of all the causes of  $i$ , intervening variables that are not causes of  $i$  is uninformative about  $i$ . In Figure 3, if an act's payments are contingent on  $x_w$  and  $x_j$  then intervening the value of  $k$  to some  $x_k$  is uninformative about  $i$ .

Finally, an important restriction that Axiom 4 imposes is that interventions have no structural impact on the DM's probabilistic model. That is, only the value of the causes of a variable matter, not whether those values arose by means of an observation or an intervention. In the context of Figure 3, what matters for choosing acts in  $\mathbb{R}^{X_i}$  are the numerical values obtained by  $j$  and  $w$ . Axiom 4 implies that whether the payments of an act  $f \in \mathbb{R}^{X_i}$  are contingent on the values of  $j$  and  $w$ , or if the DM intervened the values of  $j$  and  $w$ , is in itself irrelevant. Situations where interventions have an intrinsic effect are incompatible with Axiom 4.

#### 4.2 Representation

We now define our representation of  $\succsim$ . Since  $\succsim$  is associated to a collection of probability distributions, we first define what it means for a DAG to represent a family of probability distributions (see Section 2.1 for a reminder of our graph-

theoretic notation.)

Lauritzen et. al. [10], provide a definition for when a DAG represents a probability distribution. Let  $p \in \Delta(\Pi_{i \in \mathcal{N}} X_i)$  and let  $G = (\{1, \dots, N\}, E)$  be a DAG. The chain rule implies that  $(\forall x \in \Pi_{i \in \mathcal{N}} X_i)$ ,  $p(x) = \prod_{i=1}^N p(x_i | ND(i))$ . However, the only non-descendants that provide direct information about  $i$  are its parents. Thus, we have the following definition.

**Definition 4.** Let  $p \in \Delta(\Pi_{i \in \mathcal{N}} X_i)$ . A DAG  $(\{1, \dots, N\}, E)$  represents  $p$  if, and only if, the following hold:

$(\forall x \in \Pi_{i \in \mathcal{N}} X_i)$ ,

$$\begin{aligned} p(x) &= \prod_{i=1}^N p(x_i | Pa(i)) \\ (\forall (\mathcal{T}_i)_{i \in \mathcal{N}}) (\mathcal{T}_i \subset Pa(i)), \text{ if } p(x) &= \prod_{i=1}^N p(x_i | \mathcal{T}_i) \Rightarrow (\forall i \in \mathcal{N}), \mathcal{T}_i = Pa(i) \end{aligned}$$

Definition 4 says two things. First, a DAG represents a probability distribution if, and only if, the DAG summarizes the conditional independence properties of  $p$ . Namely, that conditional on its parents a variable is independent of its non-descendants. Second, it says that the set of parents is the smallest set that allows for such a decomposition. Indeed, consider a set of nodes  $V = \{1, 2, 3\}$  and a probability distribution  $p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3)$ . Since all variables are statistically independent, both DAGs in Figure 4 would represent this  $p$ . Indeed, both  $p(x_1, x_2, x_3) = p(x_1)p(x_2|x_1)p(x_3|x_1)$  and  $p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3)$  are true statements. However, the first representation includes irrelevant arrows. The minimality requirement prevents this.

$$X_1 \longrightarrow X_2 \longrightarrow X_3$$

$$X_1 \qquad X_2 \qquad X_3$$

Figure 4: Both the DAGs above represent the same probability distribution,  $p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3)$ , but the top one includes irrelevant arrows.

Notice that  $\succ$  is associated with a collection of beliefs, one for each induced  $\succ_p$ , rather than to a single belief, as in Savage's model. Thus, to define when a DAG

represents preferences  $\succ$ , we first define what it means for a DAG to represent a collection of probabilities, rather than just a single probability distribution. We do so by defining the *truncation* of a DAG. Let  $G = (V, E)$  be a DAG, and let  $W \subsetneq V$ . The  $W$  truncated DAG,  $G_W$ , is the DAG obtained by eliminating all nodes in  $W$ , together with their incoming and outgoing arrows. Formally,  $G_W = (V \setminus W, E \cap W^c \times W^c)$ . This DAG is useful to represent intervention beliefs. After variables in  $W$  are intervened, they no longer form part of the DM's statistical model; they are now deterministic objects which are uninformative about the value of their causes. Thus, we exclude them from the corresponding DAG. For example, if Alex observes Mr Kane obtained a college degree, then his education level is no longer random, but Alex can still make inference about Mr. Kane's intellectual ability. Thus, education is still a legitimate element of Alex's statistical model. However, if Mr. Kane's education is intervened to "college degree", then his education level is no longer random and, furthermore, is uninformative about his ability level. Thus, we exclude it from the DM's post-intervention model.

**Definition 5.** Let  $\succ$  satisfy Axiom 1, and let  $G = (\mathcal{N}, E)$  be a DAG. For each  $\mathcal{T} \subset \mathcal{N}$ , and each  $x_{\mathcal{T}} \in X_{\mathcal{T}}$ , let  $\mu_{x_{\mathcal{T}}}$  be the probability distribution over  $X_{\mathcal{T}^c}$  that represents the DM's beliefs after  $X_{\mathcal{T}}$  is intervened to value  $x_{\mathcal{T}}$ . We say  $G$  represents  $\succ$  if the following are true for each  $\mathcal{K} \subset \mathcal{N}$  and each  $x \in X$ :

- i  $G_{\mathcal{K}}$  represents  $\mu_{x_{\mathcal{K}}}$ ,
- ii If  $(i, j) \in E$  then  $\mu_{x_{\{i,j\}^c}}(x_j | x_i) = \mu_{x_{\{j\}^c}}(x_j)$ .

Definition 5 has two components. First, once we eliminate all intervened variables from the graphical description of the DM's probabilistic model, then the remaining graph represents (in the sense of Definition 4) the intervention beliefs over the remaining variables. However, that a graph represents a probability distribution means that the graph summarizes the conditional independence properties of that probability distribution. Since independence is a symmetric relation, this is insufficient information to identify the direction of causality.<sup>1</sup> Indeed, if  $j$  and  $i$  are two variables that are not  $\{i, j\}^c$ -independent, then either  $i$  causes  $j$  or vice versa, but our model is yet unable to identify which is the cause and which is the

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<sup>1</sup>For a discussion of how lack of uniqueness is dealt with in the basic Pearl model, see Pearl [19].

consequence. The second item in Definition 5 deals with the identification of the direction of causality. A crucial difference between cause and consequence is that *observing* the value of a consequence is informative about the value of the cause, but *intervening* the value of a consequence is uninformative about the value of the cause. Therefore, if  $i$  is the cause and  $j$  the consequence then intervening  $i$  or conditioning on  $i$  should lead to the same (ceteris paribus) beliefs on  $j$ . That is, if  $i \rightarrow j$  then  $\mu_{x_{\{i,j\}^c}}(x_j|x_i) = \mu_{x_{\{j\}^c}}(x_j)$ , which is precisely what Definition 5 requires.

Proposition 1 below is our first result.

**Proposition 1.** *Let  $\succsim$  be a DM's preferences, and let  $G(\succsim) = (\mathcal{N}, E)$  be the directed graph defined by setting  $Pa(i) = Ca(i)$  for each  $i \in \mathcal{I}$ . If  $\succsim$  satisfies Axiom 1, then the following are true:*

- i If  $G = (\mathcal{N}, F)$  is a directed graph that represents  $\succsim$ , then  $(j, i) \in F \Rightarrow j \in Ca(i)$ .*
- ii If  $G = (\mathcal{N}, F)$  is a directed graph that represents  $\succsim$ , then  $j \in Ca(i) \Rightarrow (j, i) \in F$  or  $i \in Ca(j)$ .*

We say that a graph  $G = (\mathcal{N}, E)$  has no 2-cycles if, for each pair  $i, j \in \mathcal{N}$ ,  $(i, j) \in E \Rightarrow (j, i) \notin E$ . That is, no two variables can mutually point at each other. Notice Axiom 2 implies  $G(\succsim)$  has no 2-cycles. We then have the following corollary.

**Corollary 1.** *Under the assumptions of the Proposition, if  $G(\succsim)$  has no 2-cycles, then  $G(\succsim) = G$ . In particular, if Axiom 2 holds, then  $G = G(\succsim)$ .*

Proposition 1 asserts two things. First, any representing graph includes all causal arrows, in the sense that Definition 2 assigns to the expression “causal arrow”. Second, a representing graph  $G = (\mathcal{N}, F)$  might fail to include some causal links; however, this situation only arises when two variables mutually cause each other. As in our discussion of Axiom 2, if  $i \rightarrow j$  and  $j \rightarrow i$  then  $i$  and  $j$  are not logically independent variables, so either  $i$  or  $j$  should not have been included in the state space to begin with.

**Theorem 1.** *Let  $\succsim$  satisfy Axiom 1. The following are equivalent:*

- i Axioms 2 and 3 hold,*

ii  $(\exists G)$  such that  $G$  is a DAG, and represents  $\succsim$ .

Furthermore, if  $G$  represents  $\succsim$ , then  $G = G(\succsim)$ .

Notice that for any collection of probability laws  $\{\mu_{x_{\mathcal{J}}} \Delta(X_{\mathcal{N} \setminus \mathcal{J}}) : \mathcal{J} \subset \mathcal{N}, x_{\mathcal{J}} \in X_{\mathcal{J}}\}$ , there exists a preference  $\succsim$  such that  $\mu_{x_{\mathcal{J}}}$  is the intervention belief associated to  $\succ_{x_{\mathcal{J}}}$ .<sup>2</sup> From the uniqueness claim we have the following Corollary.

**Corollary 2.** *If a DAG  $G$  represents  $\{\mu_{x_{\mathcal{J}}} \Delta(X_{\mathcal{N} \setminus \mathcal{J}}) : \mathcal{J} \subset \mathcal{N}, x_{\mathcal{J}} \in X_{\mathcal{J}}\}$  then there exists  $\succsim$  such that  $G = G(\succsim)$*

The immediate contribution of Theorem 1 is to provide a foundation for representing a DM’s causal model. In this way, Theorem 1 shows that causality can seamlessly be incorporated into the standard Savage framework, in a way which is consistent with how economists understand the term “causality”.

For practical applications, the first implication of Theorem 1 comes from Corollary 2. Consider an empirical researcher that models a causal structure by means of a set of probabilities  $\{\mu_p : p \in \mathcal{P}\}$ , and the DAG that represents said probabilities. The analyst would like to interpret each  $\mu_p$  as an intervention probability, and the arrows in a DAG as representing a reasonable definition of causality. Corollary 2 says that there must be a preference  $\succsim$  such that  $G = G(\succsim)$  and each  $\mu_p$  is the intervention belief induced by policy  $p \in \mathcal{P}$ . Thus, the researcher’s model of causal effects is capturing “causality” as defined in Definition 2 and not some other notion of “causality”. This is valuable since Definition 2 was tailored for use in economics (for a discussion of Definition 2 see Section 1 and 3). Theorem 1 states that causality models based on DAGs identify a unique and useful definition of causal effect, and this is the theorem’s first contribution.

Theorem 1 also provides a foundation for unifying and structuring our understanding of causation. The theorem states that any formal discussion of causality must begin with two items: a collection of probability laws,  $\{\mu_p \in \Delta(X_{\mathcal{N}(p)}) : p \in \mathcal{P}\}$ , and a DAG,  $G$ , that represents those laws. Models that include these ingredients can legitimately be called models of “causation”, regardless of any other details the model might include. However, models that cannot be phrased in terms of intervention beliefs and their representing DAG are not models of causality, as

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<sup>2</sup>For any  $\mathcal{J} \subset \mathcal{N}$ , any  $x_{\mathcal{J}}$ , and any  $f, g \in \mathbb{R}^{X_{\mathcal{J}^c}}$  define  $\succ_{x_{\mathcal{J}}}$  as  $f \succ_{x_{\mathcal{J}}} g$  iff  $\sum_{x_{\mathcal{J}^c}} f(x_{\mathcal{J}^c}) \mu_{x_{\mathcal{J}}}(x_{\mathcal{J}^c}) > \sum_{x_{\mathcal{J}^c}} g(x_{\mathcal{J}^c}) \mu_{x_{\mathcal{J}}}(x_{\mathcal{J}^c})$ .

understood by Definition 2. In this way, Theorem 1 provides a foundation for selecting amongst models with which to empirically study causal effects.

Finally, as discussed in Section 1, the purpose of this axiomatic exercise is to provide a link between numerical methods (in this case, DAG representations of a family of probability distributions) and the way a rational DM approaches the issue of interest (in this case, causation). The goal is to guarantee that the numerical method is treating causality the way a perfectly rational DM would. In the context of empirical research, the DM is the empirical researcher and the DMs beliefs are the probability laws the researcher feeds into the numerical model. Typically those probabilities are calculated from empirical data using some statistical method the DM/researcher deems acceptable. Outside of experimental settings where different policies  $p \in \mathcal{P}$  can be implemented, data regarding intervention probabilities is generally unavailable. Theorem 2 in Section 5 shows an extra axiom under which, for various policies  $p$ ,  $\mu_p$  can be expressed purely as a function of the non-intervention probability,  $\mu$ . In such cases we say  $\mu_p$  is *identified*. Since data to calculate non-intervention probabilities,  $\mu$ , is generally available, and identification provides a way to express intervention probabilities,  $\mu_p$ , in terms of  $\mu$ , identification provides a way to calculate  $\mu_p$  using non-intervention data.

## 5 IDENTIFICATION OF INTERVENTION BELIEFS

In this section, we ask the following question. Let  $\mu \in \Delta(X)$  be the DM's beliefs elicited from his Savage preference and  $\mu_p$  be the DM's beliefs elicited from an intervention preference  $\succ_p$ ; when can we express  $\mu_p$  as a function of  $\mu$ ? Proposition 2 and Theorem 2 in this section answer this question.

Answering the question above is useful to make the model applicable to empirical research. When  $\mu_p$  is expressed in terms of  $\mu$  (henceforth, when  $\mu_p$  is *identified*), anything that allows a DM to update his Savage beliefs,  $\mu$ , also allows the DM to update his intervention beliefs,  $\mu_p$ . In particular, if the DM is an analyst, the model allows the DM to use data from outside a controlled setting to calculate intervention beliefs, which is paramount to how empirical causal analysis is carried out.

When added to Axioms 1 through 3, Axiom 4 yields a model where different

intervention beliefs,  $\mu_p$ , can be expressed in terms of  $\mu$ . In what follows, we remind the reader of Axiom 4 and illustrate Theorem 2 through three simple examples. After doing so, we state and discuss the general form of Theorem 2.

**Axiom 4.**  $(\forall i \in \mathcal{N}), (\forall \mathcal{J} \subset \{i\}^c) (\forall f, g \in \mathbb{R}^{X_i}), (\forall x_{Ca(i) \cup \mathcal{J}} \in X_{Ca(i) \cup \mathcal{J}}),$

$$\mathbb{1}_{\{x_{Ca(i)}\}} f > \mathbb{1}_{\{x_{Ca(i)}\}} g \Leftrightarrow \mathbb{1}_{x_{Ca(i) \setminus \mathcal{J}}} f >_{x_{\mathcal{J}}} \mathbb{1}_{x_{Ca(i) \setminus \mathcal{J}}} g.$$

**Example 2.** Consider Alex from Section 1, who studies the relation between Ability, Education, and Lifetime earnings. Alex understands causal effects as defined in Definition 2 and she acknowledges that Axioms 1 through 3 are appealing normative properties of causal models. Thus, Alex models causal effects via a DAG that represents a family of intervention beliefs. In particular, Alex's causal mode is that ability causes both education and lifetime earnings, but education does not cause lifetime earnings. This is graphically depicted in Figure 5. Thus, identifying the direct causal effect of education on earnings is simple: there is none.

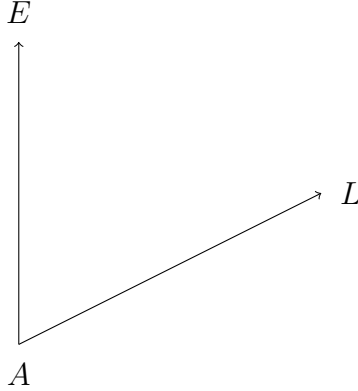


Figure 5:  $E$  has no causal effect on  $L$ .

**Example 3.** Blake is a colleague of Alex who believes that ability causes both education and lifetime earnings, but also believes education causes lifetime earnings. This is graphically depicted in Figure 6. To understand the effect of education on lifetime earnings Blake has to understand how  $\mu_{a,e}(\cdot)$  changes with  $e \in E$ , for each fixed  $a \in A$ . However, Blake cannot access a controlled environment, so Blake has no data on  $\mu_{(a,e)}$  with which to form his beliefs. Under Axiom 4 however, this data is unnecessary. By setting  $\mathcal{J} = \{A, E\}$  Axiom 4 implies  $\mu_{(a,e)}(\cdot) = \mu(\cdot|a, e)$ . Thus,



the direct causal effect of education on lifetime earnings is calculated by computing how  $\mu(\cdot|a, e)$  varies with  $e$  for each value  $a$ . Blake can therefore use data from outside a controlled environment to form his intervention beliefs.

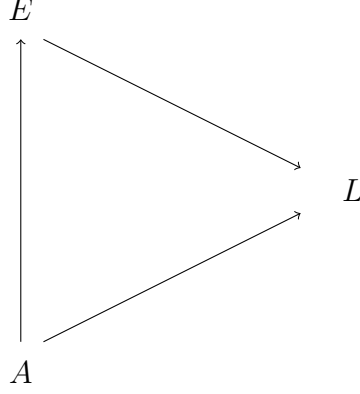


Figure 6: Causal effects are identified:  $\mu_{(a,e)}(l) = \mu(l|a, e)$ .

**Example 4.** *Charlie is another colleague of Alex. However, Charlie believes education causes ability, and that ability is the sole cause of lifetime earnings, as depicted in Figure 7. Charlie is interested in studying the indirect effect that education policies have on lifetime earnings. Applying Axiom 4 twice permits this. First, set  $\mathcal{J} = \{E\}$ ,  $i = A$  to conclude  $\mu_a(e) = \mu(e|a)$  for each  $(a, e) \in A \times E$ . Second, set  $\mathcal{J} = \{E\}$ ,  $i = L$  to conclude  $\mu_e(l|a) = \mu(l|a)$  for each  $(e, a, l) \in E \times A \times L$ . Finally, we get the following derivation.*

$$\begin{aligned}
 \mu_e(l) &= \sum_a \mu_e(l, a) \\
 &= \sum_a \mu_e(l|a) \mu_e(a) \\
 &= \sum_a \mu(l|a) \mu(a|e).
 \end{aligned}$$

Thus, calculating the indirect effect of  $E$  and  $L$  requires computing  $\mu(l|a)$  and  $\mu(a|e)$ . Even is access to a controlled environment is unavailable, identification of  $\mu_e$  implies such data is unnecessary.

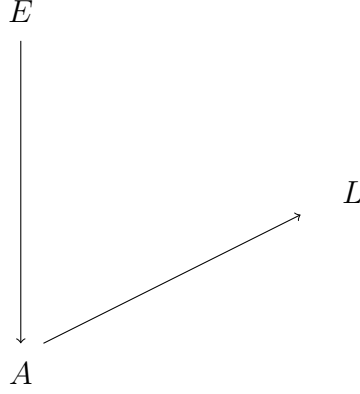


Figure 7: Indirect Causal effect of  $E$  on  $L$  is identified:  $\mu_e(l) = \sum_a \mu(l|a)\mu(a|e)$ .

The examples highlight two simple cases where intervention beliefs are identified. First, if  $j$  is a cause of  $i$ , then the direct causal effect that  $j$  has on  $i$  is identified via the formula  $\mu_{x_{\{i,j\}^c}, x_j}(x_i) = \mu(x_i|x_j, x_{Ca(i)\setminus\{j\}})$ . Thus, one can obtain the direct causal effect of  $j$  on  $i$  by conditioning on all causes of  $i$ , and analyzing how that conditional probability varies with the value  $x_j$ . Similarly, if  $j$  causes  $k$ ,  $k$  causes  $i$ , and this is the only connection between  $j$  and  $i$ , the indirect causal effect of  $j$  on  $i$  is calculated by following the chain rule:  $\mu_{x_j}(x_i) = \sum_{x_k} \mu(x_i|x_k)\mu(x_k|x_j)$ . However, other intervention beliefs may be identified as well.

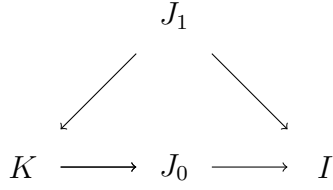
Given a family of intervention beliefs and a DAG that represents them, what is the set of all intervention beliefs that are identified, and how are they identified? Axiom 4 provides two formulas such that, for some policy intervention  $p \in \mathcal{P}$ ,  $\mu_p$  is identified if, and only if, it is identified by iterative application of those two formulas. However, to state the formulas we need two definitions: we need to define specific truncations of a DAG and we need the definition of a blocked path. We provide these definitions and then state formally the result. Appendix B discusses the intuition behind why we need these definitions.

**Definition 6.** Given  $G$ , and three disjoint sets of variables  $\mathcal{I}, \mathcal{J}, \mathcal{K} \subset \mathcal{N}$  the truncated DAGs  $G_{\mathcal{I}^{in}}$ ,  $G_{\mathcal{J}^{out}}$ , and  $G_{\mathcal{I}^{in}, \mathcal{J}(\mathcal{K})^{out}}$  are defined as follows:

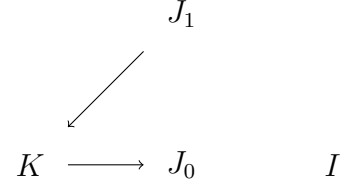
- 1  $G_{\mathcal{I}^{in}}$  is obtained from  $G$  by eliminating all arrows pointing to nodes in  $\mathcal{I}$ ,
- 2  $G_{\mathcal{I}^{in}, \mathcal{J}^{out}}$  is obtained from  $G$  by eliminating all arrows emerging from nodes in  $\mathcal{J}$  and all arrows pointing to nodes in  $\mathcal{I}$ ,

3  $G_{\mathcal{I}^{in}, \mathcal{J}(\mathcal{K})^{in}}$  is obtained by eliminating all arrows pointing to nodes in  $\mathcal{J}(\mathcal{K})$  and  $\mathcal{I}$ , where  $\mathcal{J}(\mathcal{K})$  is the set of  $\mathcal{J}$  nodes that are not ancestors of any  $\mathcal{K}$  nodes in  $G_{\mathcal{I}^{in}}$ .

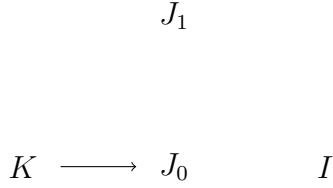
The following figures show a base DAG,  $G$ , followed by its corresponding truncations. In all cases,  $\mathcal{J} = \{J_0, J_1\}$ ,  $\mathcal{I} = \{I\}$ ,  $\mathcal{K} = \{K\}$ .



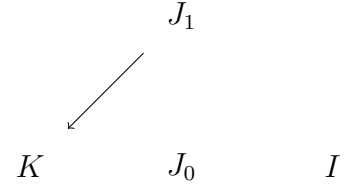
(a) A base DAG,  $G$ .



(b) DAG  $G_{\mathcal{I}^{in}}$  obtained by eliminating all arrows into  $I$ .



(c) DAG  $G_{\mathcal{I}^{in}, \mathcal{J}^{out}}$  obtained by:  
(i) eliminating arrows into  $I$ , and (ii) all arrows emerging from  $J_0$  and  $J_1$ .



(d) DAG  $G_{\mathcal{I}^{in}, \mathcal{J}(\mathcal{K})^{in}}$  obtained by:  
(i) eliminating all arrows into  $\mathcal{I}$ , and (ii) then eliminating all arrows into  $J_0$  since  $J_0$  is the only  $\mathcal{J}$  node which is not an ancestor of a  $\mathcal{K}$  node.

Figure 8: Different truncations of a DAG.

For the following definition, suppose  $Q$  is an undirected path between two nodes-*i.e.* a collection of nodes, regardless of directionality- and  $q$  is a node on  $Q$ . For example, Figure 8 shows an undirected path  $Q = (J_1, I, J_0, K)$  from  $J_1$  to  $K$ . We say that  $Q$  has *converging arrows at  $q$*  if there exist nodes  $q_0, q_1$  that are adjacent to  $q$  in  $Q$  such that  $q_0 \rightarrow q \leftarrow q_1$ . For example, path  $Q = (J_1, I, J_0, K)$  has converging arrows at  $I$ . We say that  $Q$  does not have converging arrows at  $q$  if for all nodes  $q_0, q_1$  that are adjacent to  $q$  in  $Q$ , either  $q_0 \rightarrow q \rightarrow q_1$  or  $q_0 \leftarrow q \rightarrow q_1$  holds. For example,  $Q = (J_1, I, J_0, K)$  does not have converging arrows at  $J_0$ .

**Definition 7.** Let  $\mathcal{I}, \mathcal{J}, \mathcal{K}$  be three disjoint sets of variables, and let  $Q$  be an undirected path between a node in  $\mathcal{I}$  and a node in  $\mathcal{J}$ . We say  $\mathcal{K}$  blocks  $Q$  if there is a node  $q$  on  $Q$  such that one of the following conditions holds:

- $Q$  has converging arrows at  $q$  and neither  $q$  nor any of its descendants is in  $\mathcal{K}$ , or
- $Q$  does not have converging arrows at  $q$  and  $q \in \mathcal{K}$ .

Proposition 2 below provides two rules with which to identify intervention beliefs. While the rules themselves are known in the statistical causality literature (for a discussion, see section 6), the proposition makes two contributions. First, the rules are only known to be valid in the context of a probabilistic model called the *do-probability* model, which we describe in Section 5. However, we derive those rules without reference to do-probabilities. Furthermore, it is known that the do-probability model is sufficient to guarantee that all identified intervention beliefs are identified by iterative applications of Rules 1 and 2. It is unknown whether the do-probability model is also necessary for obtaining that conclusion. The proposition states that, in the context of Axiom 1 through 3, Axiom 4 is a necessary and sufficient condition to guarantee that all identified intervention beliefs are identified by iterative applications of Rules 1 and 2.

**Proposition 2.** Let  $\succsim$  satisfy Axioms 1 through 3, let  $G$  represent  $\succsim$ , and let  $\{\mu_p : p \in \mathcal{P}\}$  be the DMs intervention beliefs. Then, the following are equivalent.

- $\succsim$  satisfies Axiom 4
- Rules 1 and 2 below hold. Furthermore, if for some  $p \in \mathcal{P}$ ,  $\mu_p$  is identified, then the identification is obtained by iterative application of these two rules.

**Rule 1.** (Exchanging intervention and observation.) Let  $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$  be disjoint sets of variables. If  $\mathcal{I}_1 \cup \mathcal{I}_3$  block all paths from  $\mathcal{I}_0$  to  $\mathcal{I}_2$  in graph  $G_{\mathcal{I}_1^{in}, \mathcal{I}_2^{out}}$  then

$$\mu_{x_{\mathcal{I}_1}, x_{\mathcal{I}_2}}(x_{\mathcal{I}_0} | x_{\mathcal{I}_3}) = \mu_{x_{\mathcal{I}_1}}(x_{\mathcal{I}_0} | x_{\mathcal{I}_2}, x_{\mathcal{I}_3}), \quad (1)$$

**Rule 2.** (Eliminating interventions.) Let  $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$  be disjoint sets of variables.

If  $\mathcal{I}_1 \cup \mathcal{I}_3$  block all paths from  $\mathcal{I}_0$  to  $\mathcal{I}_2$  in graph  $G_{\mathcal{I}_1^{\text{in}}, \mathcal{I}_2(\mathcal{I}_3)^{\text{in}}}$  then

$$\mu_{x_{\mathcal{I}_1}, x_{\mathcal{I}_2}}(x_{\mathcal{I}_0} | x_{\mathcal{I}_3}) = \mu_{x_{\mathcal{I}_1}}(x_{\mathcal{I}_0} | x_{\mathcal{I}_3}), \quad (2)$$

With Proposition 2, we can refer back to Example 4 and obtain the identification result by applying Rules 1 and 2. In Rule 2 set  $\mathcal{I}_0 = \{L\}$ ,  $\mathcal{I}_1 = \emptyset$ ,  $\mathcal{I}_2 = \{E\}$ ,  $\mathcal{I}_3 = \{A\}$ . The corresponding truncated DAG is  $G$  itself. In  $G$ ,  $A$  blocks the unique path from  $E$  to  $L$ , since there are no converging arrows at  $A$ . Thus,  $\mu_e(l|a) = \mu(l|a)$ . Likewise, in Rule 1 set  $\mathcal{I}_0 = \{A\}$ ,  $\mathcal{I}_1 = \emptyset$ ,  $\mathcal{I}_2 = \{E\}$ ,  $\mathcal{I}_3 = \emptyset$ . In the truncated graph that results,  $E$  is isolated from any other variable, so vacuously any path from  $E$  to  $A$  is blocked; thus  $\mu_e(a) = \mu(a|e)$ . These two conclusions yield identification of  $\mu_e(l) = \sum_a \mu(l|a)\mu(a|e)$ .

### 5.1 Markov representations and do-probabilities

The results in the previous section are obtained purely from adding Axiom 4 to the list of Axioms imposed on  $\succ$ . As such, they depend only on the Axioms. Moreover, Pearl [12] and Huang and Valtorta [9]) show that if a DAG represents a family of *do-probabilities* (to be formally defined shortly), then any do-probability that can be identified is identified by iterative application of Rules 1 and 2. In this section we prove that Axiom 4 is necessary and sufficient for intervention beliefs to admit a representation in terms of do-probabilities. This establishes the link between the rules of causal calculus studied in the previous section, the do-probability model, and Axiom 4. In what follows we define do-probabilities, state Theorem 2, and discuss its implications.

**Definition 8.** Let  $p \in \Delta(X)$ . For each  $i \in \mathcal{N}$  let  $p_i$  be the marginal over  $X_i$ . For each  $i \in \mathcal{N}$  let  $\varepsilon_i$  be a random variable with range  $\mathcal{E}_i$ ,  $G$  be the DAG defined by a family of sets of parents  $(Pa(i))_{i \in \mathcal{N}}$ , and  $h_i$  a function  $h_i : X_{Pa(i)} \times \mathcal{E}_i \rightarrow X_i$ . Let  $\phi$  be the joint distribution of the vector  $(\varepsilon_1, \dots, \varepsilon_N)$ . A Markov representation of  $p$  is a tuple  $(G, (h_1, \dots, h_N), (\varepsilon_1, \dots, \varepsilon_N))$  that satisfies the following:

- $(\forall i, j), \varepsilon_i$  is independent of  $\varepsilon_j$ ,

- $p$  can be recovered implicitly as a solution to the following system of equations:

$$p(x_i) = \phi(\{\varepsilon : h_i(x_{Pa(i)}, \varepsilon_i) = x_i\}), \quad (i \in \{1, \dots, N\}). \quad (3)$$

Markov representations are used in statistical causality to numerically represent causal effects (see Pearl [12]). The interpretation is as follows. Each variable  $i$  is a deterministic function of a set of variables,  $Pa(i)$ , and an idiosyncratic noise,  $\varepsilon_i$ . Each  $h_i$  is interpreted as a random production function for variable  $i$ , with  $Pa(i)$  as the set of inputs, and  $\varepsilon_i$  as the random component. The causal effect of a variable  $j$  on  $i$  is (loosely speaking) calculated by looking at how  $h_i(\cdot)$  changes as we change the value of variable  $j$ . To make this precise, we need the definition of do-probability, which we take from Pearl [12]. See examples 5 and 6 in Appendix C for a concrete illustration of how to calculate do-probabilities, and how they differ from standard conditional probabilities.

**Definition 9.** Let  $p \in \Delta(X)$  be a probability distribution, and  $((h_1, \dots, h_N), (\varepsilon_1, \dots, \varepsilon_N))$  be a Markov representation of  $p$ . Given two disjoint set of variables,  $\mathcal{I}$  and  $\mathcal{J}$ , the do-probability  $p(x_{\mathcal{I}}|do(x_{\mathcal{J}}))$  is calculated as follows:

- 1 For all  $j \in \mathcal{J}$ , eliminate from system (3) in Definition 8 all the formulas  $p(x_j) = \phi(\{\varepsilon : h_j(x_{Pa(j)}, \varepsilon_i) = x_j\})$ .
- 2 For each  $i \notin \mathcal{J}$  and for each  $j \in Pa(i) \cap \mathcal{J}$ , input value  $x_j$  into the corresponding formula in system (3) of Definition 8.
- 3 Calculate the probability of realization  $x_{\mathcal{I}}$  in the model resulting after applying steps 1 and 2 above.

While do-probabilities are commonly referred to as the causal effect of one variable on another, it is important to be cautious with the language. Do-probabilities reflect the effect that an intervention on a set of variables has on the whole system of equations. That is, do-probabilities capture both the direct and indirect effects of interventions. For example, consider the DAG in Figure 9. This DAG says that there is no direct causal effect of  $A$  on  $C$ ; however,  $Pr(x_C|do(x_A)) = Pr(x_C|x_A)$ , which is a non-trivial function of  $x_A$ . Indeed, intervening  $A$  has an effect on  $B$  which, in turn, affects  $C$ . In this example,  $Pr(x_C|do(x_A))$  captures this indirect

effect. In line with our definition of causal effect, the causal effect of  $A$  on  $C$  is given by how  $Pr(x_C|do(x_A, x_B))$  changes with  $x_A$ . In this case,  $Pr(X_c|do(x_A, x_B))$  is a constant function of  $x_A$ , which is consistent with  $A$  having no direct causal impact on  $x_C$ .



Figure 9:  $A$  has no direct causal effect on  $C$ , but  $pr(x_C|do(x_A))$  is a non-trivial function of  $x_A$ .

Having defined Markov representations and do-probabilities, we can now state Theorem 2.

**Theorem 2.** *Let  $\succ$  satisfy Axiom 1, and let  $(\mu_{x_I})_{I \subset \mathcal{N}}$  be the subjective beliefs elicited from  $\succ$ . The following are equivalent:*

- Axioms 2, 3, and 4 hold,
- $\exists$  a Markov representation of  $\mu$ ,  $(G, (h_1, \dots, h_N), (\varepsilon_1, \dots, \varepsilon_N))$ , such that
  - $(\forall \mathcal{J} \subset \mathcal{N}), (\forall x_{\mathcal{J}} \in X_{\mathcal{J}}); \mu_{x_{\mathcal{J}}} = \mu(\cdot|do(x_{\mathcal{J}})) \in \Delta(X_{\mathcal{J}^c})$ ,
  - $G$  represents  $\succ$ .

Furthermore, if  $G$  represents  $\succ$ , then  $G = G(\succ)$ .

The crucial contribution of Theorem 2 is that it clarifies the role of do-probabilities in the understanding of causal effects. Do probabilities are presented as natural representations of post-intervention beliefs. However, we show that DAGs can legitimately represent a causal model based on intervention beliefs without needing to invoke the do-probability formalism. This is analogous to the exercise Machina-Schmeidler [11] carry out: just as expected utility and probabilistic sophistication can be behaviorally separated, we show that the graph theoretic aspects of Pearl-like models can be separated from the do-probability formalism. The substantive assumptions about causality are conveyed by the DAG, while do-probabilities represent an assumption about when interventions and simple observations can be used interchangeably.

Theorem 2 further clarifies that Axiom 4 is the fundamental property that iden-

tifies do-probabilities with intervention beliefs. Indeed, as discussed after introducing Axiom 4, some types of causal effect may be incompatible with Axiom 4 and thus, incompatible with do-probabilities. As such, Axiom 4 clarifies which types of casual effects can be represented via do-probabilities.

Jointly, Proposition 2 and Theorem 2 imply that Pearl’s rules of causal calculus serve as an axiomatization of do-probability. Indeed, Huang and Valtorta [9] show that, in a do-probability model, Rules 1 and 2 summarize all obtainable identification results. To the best of our knowledge, it is unknown whether or not other probabilistic models are consistent with the aforementioned result. We show that when Rules 1 and 2 summarize all obtainable identification results, Axiom 4 must hold, so that intervention beliefs are do-probabilities. Therefore, the rules of causal calculus are a complete description of all obtainable identification results if, and only if, intervention probabilities are do-probabilities.

As a final remark on Theorem 2, notice that Definition 8 implicitly requires that the Markov representation that defines do-probabilities have a unique solution. While this has been sometimes pointed to as a limitation of the theory (see Halpern [5]), under Axiom 4, this is without loss of generality.

## 6 RELATED LITERATURE

This paper relates to two main fields: economics (both theory and applied) and work done in statistics and computer science.

In economic theory, the closest related work to ours is Spiegler ([15], [16], [17]). The main difference is the focus of the papers. Spiegler’s work does not provide any definition for the term “causal effect” except that it can be represented via a Acyclic Directed Graphs (DAG) that satisfies two properties. First, it factorizes the correlation structure in the DM’s beliefs; second, arrows in the DAG are interpreted as pointing from cause to effect. Given that assumption, he asks what kinds of mistakes a DM with a misspecified causal model might make. In our paper, we first define what a causal relation is, and then seek to understand which axioms on behavior allow us to represent causal effects in the language of DAGs. Proposition 1 provides the point of contact between both papers. If a graph  $G$  both represents a DM’s correlation structure and is interpreted causally (in the



sense that arrows point from cause to effect), then the definition of causal effect must be as in Definition 2.

In statistics and computer science there is a literature that uses graphical methods to represent the conditional independence structure of any given joint probability law (see Dawid [1], Geiger et al. [3], Lauritzen et al. [10]). In particular, Dawid [1] and Geiger et al. [3] show that, given a probability distribution over a set of variables,  $p(\cdot)$ , and given a graph  $G$  that represents  $p$ , the *D-separation* criterion for graphs (see Definitions 7 and 10) summarizes the independence structure of  $p$ . Our proofs rely on the one-to-one correspondence between variables that satisfy the D-separation criterion and variables that are conditionally independent. Lauritzen et al. [10] provide alternative graphical tests for D-separation, based on the cut-sets of  $G$ .

In causal statistics, the closest related papers are those in the Bayesian networks literature (see, Spirtes [18], Pearl [12], and follow-up work). There are two main points of contact between that literature and our paper. First, the statistical causality literature offers no formal definition for the term “causal relation” is given, and the exact meaning of this phrase is left up to the researcher’s common sense. As Pearl puts it “*The first step in this analysis is to construct a causal diagram such as the one given in Fig. [1] (sic.), which represents the investigator’s understanding of the major causal influences among measurable quantities in the domain*” and later “*The purpose of the paper is not to validate or repudiate such domain-specific assumptions but, rather, to test whether a given set of assumptions is sufficient for quantifying causal effects from non-experimental data, for example, estimating the total effect of fumigants on yields*”. Second, the numerical value for the causal effect of one variable on another (say, Education on Lifetime earnings) is given by the *do-probability* formalism. As Pearl writes in [13]: “*the definition of a “cause” is clear and crisp; variable  $X$  is a probabilistic-cause of variable  $Y$  if  $P(y|do(x)) \neq P(y)$  for some values  $x$  and  $y$ .*” In contrast, we show that, under Axioms 1 through 3 there exists a unique definition of causal effect that is both representable via a DAG, and consistent with an interventionist perspective of causality. Thus, we show that causal models based on causal diagrams implicitly impose a specific definition of what “causality” must mean. Moreover, Axioms 1

through 3 neither imply, nor are implied by, a representation of causality in terms of do-probabilities. Contrary to Pearl’s quote, do-probabilities neither define, nor are defined by, the definition of causality embodied by the causal diagram. Theorem 2 shows that under Axioms 1 through 4, causal effects are representable via a DAG that is compatible with the do-probability formulas. This makes explicit the fundamental restrictions imposed by using do-probabilities to numerically quantify causal effect.

In terms of axiomatic definitions for causal effects, Galles and Pearl [2], Halpern [5], Halpern and Pearl ([6], [7]) provide an alternative approach. In particular, Halpern [5] expands on Galles and Pearl [2] and axiomatizes a more general model. Rather than a decision theoretic approach, Halpern [5] axiomatizes causal effects through a syntactic logic approach. That is, rather than using a DM’s preferences over a suitably defined choice domain as a primitive, Halpern’s axiomatization is in terms of syntactic structures of a base language. The main results show that different axioms on the languages considered axiomatize various classes of causal models. Those papers not only axiomatize the basic Pearl [12] model -which is the model we axiomatize here- but also more general models that cannot be captured in our framework. However, the primitives in those models are not directly associated to objects that economists use to reason about causality. In particular, it is unclear from the axiomatization whether the Pearl model is a suitable model for causal analysis in economics. By providing an axiomatic foundation of the same model based on choice of Savage acts and policy interventions we show that the Pearl model is indeed a suitable choice for reasoning about causality in economics.

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## A PROOFS

**Proposition 1.** *Let  $\succ = (\succ_{\mathcal{I}})_{\mathcal{I} \subset \mathcal{N}}$  be a DM’s preferences, and let  $G(\succ) = (\mathcal{N}, E)$  be the directed graph defined by setting  $Pa(i) = Ca(i)$  for each  $i \in \mathcal{I}$ . If  $\succ$  satisfies Axiom 1, then the following are true:*

- *If  $G = (\mathcal{N}, F)$  is a directed graph that represents  $\succ$ , then  $(j, i) \in F \Rightarrow j \in Ca(i)$ .*
- *If  $G = (\mathcal{N}, F)$  is a directed graph that represents  $\succ$ , then  $j \in Ca(i) \Rightarrow (j, i) \in F$  or  $i \in Ca(j)$ .*

*Proof.* Let  $\succ$  be as in the statement of the proposition,  $G(\succ)$  be the directed graph defined by setting  $Pa(i) = Ca(i)$  for each  $i \in \mathcal{N}$ , and  $G = (\mathcal{N}, F)$  be any other directed graph that represents  $\succ$ . For each  $\mathcal{I} \subset \mathcal{N}$  and each realization  $x_{\mathcal{I}} \in X_{\mathcal{I}}$ ,

let  $\mu_{x_{\mathcal{I}}} \in \Delta(X_{\mathcal{I}}^c)$  represent beliefs obtained from  $\succ_{x_{\mathcal{I}}}$ .

We first show  $j \in Ca(i) \Rightarrow (j, i) \in F$  or  $i \in Ca(j)$ . If  $j \in Ca(i)$  then the function  $T : X_j \rightarrow \mathbb{R}$  defined as  $T(x_j) = \mu_{x_{\{i,j\}^c}, x_j}(x_i)$  is not constant in  $x_j$ . Also, by Axiom 1,  $\mu_{x_{\{i,j\}^c}}(x_i|x_j) = T(x_j)$ . Thus,  $i$  and  $j$  are not independent after intervening  $\{i, j\}^c$ . Because  $G$  represents  $\succ$  then  $G_{\{i,j\}^c}$  represents  $\succ_{\{i,j\}^c}$ . Thus, either  $(i, j) \in F$  or  $(j, i) \in F$  (if not,  $G_{\{i,j\}^c}$  would treat  $i$  and  $j$  as independent, which is a contradiction). If  $(j, i) \in F$  the proof concludes. Therefore, let  $(j, i) \notin F$  so that  $(i, j) \in F$ . Because  $G$  represents  $\succ$  this means that  $\mu_{x_{\{i,j\}^c}}(x_j|x_i) = \mu_{x_{\{j\}^c}}(x_j)$ . By definition, the above equation says  $i \in Ca(j)$ , as desired.

We now show  $(j, i) \in F \Rightarrow j \in Ca(i)$ . First, note that for all  $x \in X$ ,  $\mu_{x_{\{i,j\}^c}}(x_i, x_j) = \mu_{x_{\{i,j\}^c}}(x_j)\mu_{x_{\{i,j\}^c}}(x_i|x_j)$ . Because  $G$  represents  $\succ$ ,  $(j, i) \in F$  and the minimality condition in Definition 4, jointly imply that  $i$  and  $j$  are not independent after intervening  $\{i, j\}^c$ . That is,  $\mu_{x_{\{i,j\}^c}}(x_i|x_j)$  is not constant in  $x_j$ . Moreover, because  $G$  represents  $\succ$  and  $(j, i) \in F$ , we get that  $\mu_{x_{\{i,j\}^c}}(x_i|x_j) = \mu_{x_{\{i,j\}^c}, x_j}(x_i)$ . Therefore, there is a value of  $x_{\{j\}^c}$  for which  $T(x_j) = \mu_{x_{\{i,j\}^c}, x_j}(x_i)$  is not constant in  $x_j$ . Therefore,  $j \in Ca(i)$ .  $\square$

**Remark 1.** Without axiom 2, any representing graph must include the causal links in the sense of Definition 2 (i.e.,  $(j, i) \in F \Rightarrow j \in Ca(i)$ ) but  $F$  could omit some arrows. However, only arrows involved in 2-cycles are omitted.

Before proving Theorem 1, we need two Lemmas. Let  $i$  be a variable and  $\mathcal{I}, \mathcal{J}$  be two disjoint set of variables that do not contain  $i$ . It is known from Dawid ([1]) and Pearl ([12]) that  $i$  is independent to  $\mathcal{I}$  conditional on  $\mathcal{J}$  if, and only if,  $\mathcal{J}$  D-separates  $\{i\}$  from  $\mathcal{I}$  (see below for a definition of D-separation). The next two lemmas prove that, for each variable  $i$ ,  $Ca(i)$  D-separates  $\{i\}$  from all sets  $\mathcal{J}$  that satisfy  $\mathcal{J} \subset ND(i)$ , where  $ND(i)$  is the set of non-descendants of  $i$ . Furthermore,  $Ca(i)$  is the smallest set that has this property.

**Definition 10.** Let  $\mathcal{I}, \mathcal{J}, \mathcal{K} \subset \mathcal{N}$  be three disjoint set of variables. We say  $\mathcal{K}$  D-separates  $\mathcal{I}$  from  $\mathcal{J}$  if for each undirected path between a variable in  $\mathcal{I}$  and a variable in  $\mathcal{J}$ , one of the following properties holds:

- There is a node  $w$  along the path such that  $w$  is a collider (that is, there are nodes  $w_0, w_1$  in the path such that  $w_0 \rightarrow w \leftarrow w_1$ ), and such that  $w \notin \mathcal{K}$  and  $\mathcal{K} \subset ND(w)$ .

- There is a node  $w$  along the path such that  $w$  is not a collider, and such that  $w \in \mathcal{K}$ .

**Lemma 1.** Fix  $\mathcal{K} \subset \mathcal{N}$  and  $x_{\mathcal{K}} \in X_{\mathcal{K}}$ . Let  $G_{\mathcal{K}}$  represent  $\succ_{x_{\mathcal{K}}}$ . For each  $i \in \mathcal{N}$ ,  $Ca(i) \setminus \mathcal{K}$  D-separates  $\{i\}$  from  $ND(i) \setminus \mathcal{K} \equiv \{\hat{j} \in \mathcal{K}^c : i \text{ is not an indirect cause of } \hat{j}\}$ .

*Proof.* Pick  $j \in \{\hat{j} \in \mathcal{K}^c : i \text{ is not an indirect cause of } \hat{j}\}$ . Pick an undirected trail  $t$  from  $j$  to  $i$ . That is,  $t = (i_0, \dots, i_N)$  where  $i_0 = j$ ,  $i_N = i$ , and, for each  $n \in \{1, \dots, N\}$ , either  $(i_{n-1}, i_n) \in E$  or  $(i_n, i_{n-1}) \in E$ . First, since  $i$  is not an indirect cause of  $j$ , then  $t$  cannot be a directed path from  $i$  to  $j$ . That is,  $t$  cannot be such that  $(i_n, i_{n-1}) \in E$  for each  $n$ . Second, if  $t$  is a directed path from  $j$  to  $i$  (that is,  $(i_{n-1}, i_n) \in E$  for each  $n$ ), then  $t$  is blocked by  $i_{N-1} \in Ca(i) \setminus \mathcal{K}$ . Third, assume that  $t$  is not directed in any direction. Then,  $t$  has colliders and/or tail-to-tail nodes. Let  $i_n$  be the last node that is either a collider or a tail-to-tail node. Let  $q = (i_n, \dots, i_N)$  be the trail starting at  $i_n$ . By definition of  $i_n$ ,  $q$  must be directed. Assume that  $q$  is directed from  $i_n$  to  $i$ . Then,  $i_n$  is tail-to-tail. Then,  $t$  is blocked by  $i_{N-1}$ . Finally, assume that  $q$  is directed from  $i$  to  $i_n$ . Then,  $i_n$  is a collider. If  $i_n \in Ca(i) \setminus \mathcal{K}$  then  $(i_n, i, q)$  is a cycle. Thus,  $i_n \notin Ca(i) \setminus \mathcal{K}$ . By a similar argument, no descendants of  $i_n$  can be in  $Ca(i) \setminus \mathcal{K}$ . Therefore,  $i_n$  blocks  $t$ . Since each trail joining  $j$  to  $i$  is blocked, this concludes the proof.  $\square$

**Lemma 2.** Fix  $\mathcal{K} \subset \mathcal{N}$ ,  $x_{\mathcal{K}} \in X_{\mathcal{K}}$ , and  $i \in \mathcal{K}^c$ . Let  $G_{\mathcal{K}}$  represent  $\succ_{x_{\mathcal{K}}}$ . If  $\mathcal{T} \subset \mathcal{K}^c$  satisfies that  $\mathcal{T}$  D-separates  $\{i\}$  from  $ND(i)$ , then  $Ca(i) \setminus \mathcal{K} \subset \mathcal{T}$ .

*Proof.* Let  $\mathcal{K}$ ,  $i$ , and  $\mathcal{T}$  be as in the statement of the Lemma. Assume  $w \in Ca(i) \setminus \mathcal{K}$ . Then,  $w \in ND(i)$  because otherwise  $G_{\mathcal{K}}$  would not be acyclic. Consider the path  $w \rightarrow i$ . Then,  $\mathcal{T}$  can D-separate this path only if  $w \in \mathcal{T}$ . Thus,  $Ca(i) \setminus \mathcal{K} \subset \mathcal{T}$ .  $\square$

**Theorem 1.** Let  $\succ$  satisfy Axiom 1. The following are equivalent:

- Axioms 2 and 3 hold,
- $(\exists G)$  such that  $G$  is a DAG, and represents  $\bar{\succ}$ .

Furthermore, if  $G$  represents  $\bar{\succ}$ , then  $G = G(\bar{\succ})$ .

*Proof.* The uniqueness claim is proved in Proposition 1.

We now prove that the axioms imply the existence of a representation. Without

loss of generality, label the variables so that  $i < j$  implies  $j \in ND(i)$ . Construct  $G$  by setting  $Pa(i) = Ca(i)$ . By axiom 2,  $G$  is acyclic. Indeed, if for some length  $k \in \mathbb{N}$  there was a cycle  $e = ((i_1, i_2), (i_2, i_3), \dots, (i_k, i_1))$ , then  $i_1$  would be an indirect cause of itself. Pick any set  $\mathcal{K} \subset \mathcal{N}$  and any realization  $x_{\mathcal{K}} \in X_{\mathcal{K}}$ . Let  $K = \#\mathcal{K}$ . We need to show that  $\mu_{x_{\mathcal{K}}}(x_{\mathcal{K}^c}) = \prod_{i \in \mathcal{K}^c} \mu_{x_{\mathcal{K}}}(x_i | Ca(i) \cap \mathcal{K}^c)$ . By our enumeration,  $\{j \notin \mathcal{K} : j < i\} \subset \{j \in \mathcal{N} : i \text{ is not an indirect cause of } j\}$ . Let  $\mathcal{I} = Ca(i)$ ,  $\mathcal{J} = \{j \notin \mathcal{K} : j < i \text{ and } j \notin Ca(i)\}$ . By axiom 3 applied to these sets  $\mathcal{K}$ ,  $\mathcal{I}$ ,  $\mathcal{J}$ ,  $\mu_{x_{\mathcal{K}}}(x_i | \{j \notin \mathcal{K} : j < i\}) = \mu_{x_{\mathcal{K}}}(x_i | Ca(i) \cap \mathcal{K}^c)$ . By the chain rule, we know  $\mu_{x_{\mathcal{K}}}(x_{\mathcal{K}^c}) = \prod_{i=1, i \notin \mathcal{K}}^N \mu_{x_{\mathcal{K}}}(x_i | \{j \notin \mathcal{K} : j < i\})$ . Combining the last two claims,  $\mu_{x_{\mathcal{K}}}(x_{\mathcal{K}^c}) = \prod_{i=1, i \notin \mathcal{K}}^N \mu_{x_{\mathcal{K}}}(x_i | Ca(i) \cap \mathcal{K}^c)$ , which is what we wanted to prove. We now prove minimality of  $Ca(i)$ . Suppose  $(\mathcal{T}_i)_{i \notin \mathcal{K}}$ ,  $\mathcal{T}_i \subset \{1, \dots, i-1\} \cap \mathcal{K}^c$  satisfies

$$\mu_{x_{\mathcal{K}}}(x_{\mathcal{K}^c}) = \prod_{i \in \mathcal{K}^c} \mu_{x_{\mathcal{K}}}(x_i | \mathcal{T}_i). \quad (4)$$

We need to show that  $Ca(i) \setminus \mathcal{K} \subset \mathcal{T}_i$ . That  $\mathcal{T}_i \subset \{1, \dots, i-1\} \setminus \mathcal{K}$  satisfies 4 implies that  $\mathcal{T}_i$  D-separates  $\{i\}$  from  $ND(i)$ . By Lemma 2,  $Ca(i) \setminus \mathcal{K} \subset \mathcal{T}_i$ .

The last step is to show that if  $\mu_{x_{\{i,j\}^c}}(x_j | x_i) \neq \mu_{x_{\{j\}^c}}(x_j)$  then  $i \rightarrow j$  (or, equivalently, that  $i \notin Ca(j)$ ). This follows from the contrapositive statement of Axiom 1 item [ii].

Now, suppose  $G$  is a DAG that represents  $\succ$ . By our uniqueness claim, without loss of generality  $G$  is such that  $Pa(i) = Ca(i)$ . By contrapositive, that  $G$  is acyclic implies Axiom 2 holds. If Axiom 2 did not hold, there exists  $i$  and there exists a sequence  $(i, i_1, \dots, i_T, i)$  such that  $i \in Ca(i_1)$ , for all  $t \in \{1, \dots, T-1\}$ ,  $i_t \in Ca(i_{t+1})$ , and  $i_T \in Ca(i)$ . Thus,  $((i, i_1), \dots, (i_{t-1}, i_t), \dots, (i_T, i))$  is a cycle in  $G$ . To see that Axiom 3 holds, consider Lemmas 1 and 2. For each  $\mathcal{K}$ , each  $i \notin \mathcal{K}$ , and each  $\mathcal{J} \subset \{\hat{j} \in \mathcal{K}^c : i \text{ is not an indirect cause of } \hat{j}\}$ ,  $Ca(i) \setminus \mathcal{K}$  D-separates  $\{i\}$  from  $\mathcal{J}$ . Furthermore, it is the smallest set with this property. Therefore,  $i$  is independent of  $\mathcal{J}$  conditional on  $Ca(i) \setminus \mathcal{K}$ , and this is the smallest set with this property. Thus, Axiom 3 holds.  $\square$

**Theorem 2.** *Let  $\succ$  satisfy Axiom 1, and let  $(\mu_{x_{\mathcal{I}}})_{\mathcal{I} \subset \mathcal{N}}$  be the subjective beliefs elicited from  $\succ$ . The following are equivalent:*

- Axioms 2, 3, and 4 hold,

- $\exists$  a Markov representation of  $\mu$ ,  $(G, (h_1, \dots, h_N), (\varepsilon_1, \dots, \varepsilon_N))$ , such that
  - $(\forall \mathcal{J} \subset \mathcal{N}), (\forall x_{\mathcal{J}} \in X_{\mathcal{J}}); \mu_{x_{\mathcal{J}}} = \mu(\cdot | do(x_{\mathcal{J}})) \in \Delta(X_{\mathcal{J}^c})$ ,
  - $G$  represents  $\bar{\succ}$ .

Furthermore, if  $G$  represents  $\bar{\succ}$ , then  $G = G(\bar{\succ})$ .

*Proof.* The uniqueness claim was proven in 1.

We first show the axioms imply the representation. By Theorem 1, Axioms 2 and 3 imply there exists a DAG  $G$  such that  $G$  represents  $\bar{\succ}$ . For each  $i \in \mathcal{N}$  let  $Pa(i)$  be the set of parents of  $i$  in  $G$ . Note  $Pa(i) = Ca(i)$  by the uniqueness claim. For each  $i \in \mathcal{N}$ , let  $\varepsilon_i \sim U[0, 1]$ . For each realization  $x_i \in X_i$  and each  $x_{Pa(i)} \in X_{Pa(i)}$ , let  $I(x_i, x_{Pa(i)}) \subset [0, 1]$  be an interval of length  $\mu_{x_{Pa(i)}}(x_i)$ . Because  $\sum_{x_i \in X_i} \mu_{x_{Pa(i)}}(x_i) = 1$  for each  $x_{Pa(i)}$ , then  $I(\cdot, x_{Pa(i)})$  can be chosen to form a partition of  $[0, 1]$ . Fix any variable  $i \in \mathcal{N}$ , let  $h_i(x_{Pa(i)}, \varepsilon_i) = \sum_{x_i \in X_i} x_i \mathbb{1}_{I(x_i, x_{Pa(i)})}(\varepsilon_i)$ . By construction,  $(G, (h_1, \dots, h_N), (\varepsilon_1, \dots, \varepsilon_N))$  is a Markov representation of the beliefs elicited from  $\bar{\succ}$ . Pick any  $\mathcal{J} \subset \mathcal{N}$  and any  $i \in \mathcal{J}^c$ . By Axiom 4, for each  $x_i \in X_i$ , and each  $x_{Ca(i) \cup \mathcal{J}} \in X_{Ca(i) \cup \mathcal{J}}$ , we obtain

$$\mu_{x_{\mathcal{J}}}(x_i | x_{Ca(i) \setminus \mathcal{J}}) = \mu(x_i | x_{Ca(i)}). \quad (5)$$

Our Markov representation implies

$$\begin{aligned} \mu(x_i | x_{Ca(i)}) &= \phi(\{\varepsilon : h_i(x_{Ca(i)}, \varepsilon_i) = x_i\}) \\ &= \mu(x_i | do(x_{\mathcal{J}}), x_{Ca(i) \setminus \mathcal{J}}). \end{aligned} \quad (6)$$

By 5 and 6,  $\mu_{x_{\mathcal{J}}}(x_i | x_{Ca(i) \setminus \mathcal{J}}) = \mu(x_i | do(x_{\mathcal{J}}), x_{Ca(i) \setminus \mathcal{J}})$ . Because  $G$  represents  $\bar{\succ}$ , for each  $x \in X$ ,

$$\begin{aligned} \mu_{x_{\mathcal{J}}}(x_{\mathcal{J}^c}) &= \prod_{i=1, i \notin \mathcal{J}}^N \mu_{x_{\mathcal{J}}}(x_i | x_{Ca(i) \setminus \mathcal{J}}) \\ &= \prod_{i=1, i \notin \mathcal{J}}^N \mu(x_i | do(x_{\mathcal{J}}), x_{Ca(i) \setminus \mathcal{J}}) = \mu(x_{\mathcal{J}^c} | do(x_{\mathcal{J}})). \end{aligned}$$

Thus,  $\mu_{x_{\mathcal{J}}}(\cdot) = \mu(\cdot | do(x_{\mathcal{J}})) \in \Delta(X_{\mathcal{J}^c})$ .

We now show the representation implies the axioms. If there exists a DAG  $G$  that



represents  $\succ$  then axioms 2 and 3 hold is proven in Theorem 1. Let  $i \in \mathcal{N}$ ,  $\mathcal{J} \subset \{i\}^c$ ,  $f, g \in \mathbb{R}^{X_i}$ ,  $x_{\mathcal{J}} \in X_{\mathcal{J}}$  and  $x_{Ca(i) \setminus \mathcal{J}} \in X_{Ca(i) \setminus \mathcal{J}}$  be arbitrarily selected. We know from the Markov representation that for each  $x_i \in X_i$ ,  $\mu_{x_{\mathcal{J}}}^i(x_i | x_{Ca(i) \setminus \mathcal{J}}) = \mu^i(x_i | x_{Ca(i)})$ , where  $\mu^i$  denotes the marginal of  $\mu$  on  $X_i$ . Thus, Axiom 4 holds.  $\square$

Proposition 2 is a direct consequence of Theorem 2 and Theorem 3, which is stated and proven below.

**Theorem 3.** *Let  $\bar{\mu} = \{\mu_p : p \in \mathcal{P}\}$  be a collection of intervention beliefs, and let  $G$  be a DAG that represent  $\bar{\mu}$ . If equations 1 and 2 hold, then Axiom 4 holds.*

*Proof.* Let  $\bar{\mu}$  and  $G$  be as in the theorem. Let  $i \in \mathcal{N}$  and  $\mathcal{J} \subset \{i\}^c$ . We want to show that  $\mu(x_i | x_{Ca(i)}) = \mu_{x_{\mathcal{J}}}(x_i | x_{Ca(i) \setminus \mathcal{J}})$ . Let  $\mathcal{J}^* \equiv \mathcal{J} \cap Ca(i)$ ; that is,  $\mathcal{J}^*$  are those variables in  $\mathcal{J}$  that are direct causes of  $i$ . Thus, we need to show that  $\mu(x_i | x_{Ca(i)}) = \mu_{x_{\mathcal{J}}}(x_i | x_{Ca(i) \setminus \mathcal{J}^*})$ ; we do this in two steps.

First we show that  $\mu(x_i | x_{Ca(i)}) = \mu_{x_{\mathcal{J}^*}}(x_i | x_{Ca(i) \setminus \mathcal{J}^*})$ . To so this, notice that  $Ca(i) \setminus \mathcal{J}^*$  blocks any path from  $\{i\}$  to  $\mathcal{J}^*$  in the graph  $G_{(\mathcal{J} \setminus \mathcal{J}^*)^{in}, (\mathcal{J}^*)^{out}}$ . Indeed, let  $p$  be any path from  $i$  to some  $j \in \mathcal{J}^*$  in graph  $G_{(\mathcal{J} \setminus \mathcal{J}^*)^{in}, (\mathcal{J}^*)^{out}}$ . Write  $p = (i_0, \dots, i_T)$  where  $i_0 = i$  and  $i_T = j$ . Because  $j \in Ca(i)$ , then  $p$  cannot be a directed path from  $i$  to  $j$ , or else  $G$  would have a cycle. Likewise,  $p$  cannot be a directed path from  $j$  to  $i$  since  $G_{(\mathcal{J} \setminus \mathcal{J}^*)^{in}, (\mathcal{J}^*)^{out}}$  has no arrows emerging from  $j$ . Therefore,  $p$  has a collider or a tail-to-tail node. Let  $w$  be the first node that is either a collider or a tail-to-tail node. First, assume  $w$  is tail-to-tail. Then  $p$  is of the form  $i \leftarrow i_1(\dots) \leftarrow w \rightarrow (\dots)j$ . Then  $i_1 \in Ca(i) \setminus \mathcal{J}^*$ : indeed,  $i_1 \in Ca(i)$  and  $i_1 \notin \mathcal{J}^*$  (since there are no arrows emerging from nodes in  $\mathcal{J}^*$ ). Furthermore,  $i_1$  is not a collider. Then,  $i_1$  blocks  $p$ . Now, assume  $w$  is a collider rather than tail-to-tail. Then  $p$  is of the form  $i \rightarrow i_1(\dots) \rightarrow w \leftarrow (\dots)j$ . Then,  $w$  is a descendant of  $i$ , so neither  $w$  nor any  $w$  descendant is in  $Ca(i)$ . A fortiori, neither  $w$  nor any descendant of  $w$  is in  $Ca(i) \setminus \mathcal{J}^*$ . Thus,  $w$  blocks  $p$ . Therefore, by formula 1 we have  $\mu(x_i | x_{Ca(i)}) = \mu_{x_{\mathcal{J}^*}}(x_i | x_{Ca(i) \setminus \mathcal{J}^*})$ .

Second, we show  $\mu_{x_{\mathcal{J}^*}}(x_i | x_{Ca(i) \setminus \mathcal{J}^*}) = \mu_{x_{\mathcal{J}^* \cup (\mathcal{J} \setminus \mathcal{J}^*)}}(x_i | x_{Ca(i) \setminus \mathcal{J}^*})$ . This is because  $Ca(i)$  blocks all paths between  $i$  and  $\mathcal{J} \setminus \mathcal{J}^*$  in graph  $G_{\mathcal{J} \setminus \mathcal{J}^* (Ca(i) \setminus \mathcal{J}^*)^{in}}$ . To see this, notice that if  $\mathcal{J} \setminus \mathcal{J}^*$  contains only non-descendants of  $i$ , then the result is a direct consequence of lemma 1. Let  $p$  be a path (not necessarily directed) between

$i$  and  $j \in \mathcal{J} \setminus \mathcal{J}^*$ . By contradiction, assume that  $j \in \mathcal{J} \setminus \mathcal{J}^*$  is a descendant of  $i$ . Then,  $j \notin Ca(i)$  and  $j$  is not an ancestor of any node in  $Ca(i)$ . Therefore,  $j \in \mathcal{J} \setminus \mathcal{J}^*(Ca(i) \setminus \mathcal{J}^*)$ , so there are no arrows into  $j$ . Therefore, no path from  $i$  to  $j$  can be directed in any direction, so there is at least one collider or tail-to-tail node. Let  $w$  be the first such node, and assume  $w$  is a collider. Then,  $p$  is of the form  $i \rightarrow (\dots) \rightarrow w \leftarrow (\dots) \leftarrow j$ . Then, neither  $w$  nor any descendant of  $w$  can be in  $Ca(i)$ , so  $p$  is blocked by  $Ca(i)$ . Alternatively, say  $w$  is a tail-to-tail node. Then,  $p$  is of the form  $i \leftarrow i_1(\dots) \leftarrow w \rightarrow (\dots) \leftarrow j$  (with possibly  $w = i_1$ ). Then,  $i_1 \in Ca(i)$  and  $i_1$  is not a collider. Thus,  $Ca(i) = \mathcal{J}^* \cup (Ca(i) \setminus \mathcal{J}^*)$  blocks  $p$ . Thus, by formula 2,  $\mu_{x_{\mathcal{J}^*}}(x_i | x_{Ca(i) \setminus \mathcal{J}^*}) = \mu_{x_{\mathcal{J}^* \cup (Ca(i) \setminus \mathcal{J}^*)}}(x_i | x_{Ca(i) \setminus \mathcal{J}^*}) = \mu_{x_{\mathcal{J}}}(x_i | x_{Ca(i) \setminus \mathcal{J}})$ .

Combining this with the first step, we conclude  $\mu(x_i | x_{Ca(i)}) = \mu_{x_{\mathcal{J}}}(x_i | x_{Ca(i) \setminus \mathcal{J}})$  as desired.

□

## B THE RULES OF CAUSAL CALCULUS

In this appendix we give some intuition behind why the notion of a block is relevant for analyzing conditional independence. Furthermore, we give intuition as to why the truncations in Figure 10a are the relevant truncations for identifying intervention beliefs. We begin by reminding the reader of the definition of a block.

**Definition 11.** Let  $\mathcal{I}, \mathcal{J}, \mathcal{K}$  be three disjoint sets of variables, and let  $p$  be any path (not necessarily directed) between a node in  $\mathcal{I}$  and a node in  $\mathcal{J}$ . We say  $\mathcal{K}$  blocks  $p$  if there is a node  $K$  on  $p$  such that one of the following conditions holds:

- $K$  has converging arrows along  $p$  and neither  $K$  nor any of its descendants is in  $\mathcal{K}$ , or
- $K$  does not have converging arrows in  $p$  and  $K$  is in  $\mathcal{K}$ .

To illustrate the notion of a block, see Figure 8, replicated below for convenience. In that case, the singleton  $\{K\}$  blocks all paths from  $J_1$  to  $J_0$ . Indeed, one such path is  $J_1 \rightarrow K \rightarrow J_0$ . This path is blocked by  $\{K\}$  because (i) the path has no converging arrows at  $K$ , and (ii)  $K \in \{K\}$ . The other path from  $J_1$  to  $J_0$  is  $J_1 \rightarrow I \leftarrow J_0$ . This path is blocked by  $\{K\}$  because  $I$  is a node along the path such

that there are converging arrows at  $I$ , but neither  $I$  nor any of its descendants are in  $\{K\}$ .

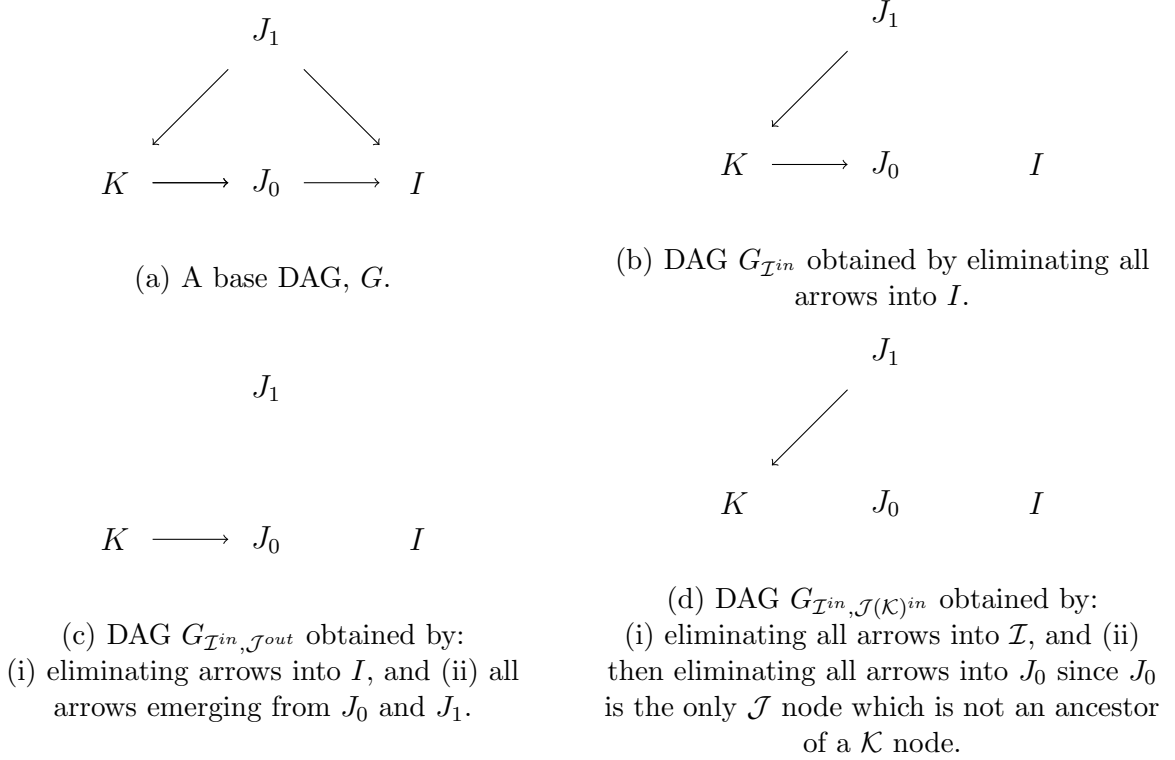


Figure 10: Different truncations of a  $DAG$ .

The notion of a block is a graphical depiction of conditional independence. Indeed, that a path exists between two sets of variables,  $\mathcal{I}$  and  $\mathcal{J}$ , implies  $\mathcal{I}$  and  $\mathcal{J}$  are (a priori) statistically dependent: any variable  $w$  present in a path from  $\mathcal{I}$  to  $\mathcal{J}$  may potentially act as a correlating device between  $\mathcal{I}$  and  $\mathcal{J}$ .

In particular, the position of a variable  $w$  in a path between  $\mathcal{I}$  and  $\mathcal{J}$  is relevant to the way in which  $w$  correlates these variables. Say that there is a path  $i \rightarrow w \leftarrow j$ , where  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ ; *i.e.* there is a path joining  $\mathcal{I}$  and  $\mathcal{J}$  that has converging arrows at  $w$ . This implies that observations of  $w$  (and its descendants) are informative about  $i$  and  $j$  simultaneously. However, interventions of  $w$  are useless for the purposes of predicting the value of either  $i$  or  $j$ , since neither  $w$  nor any of its descendants are a cause of either  $i$  nor  $j$ . By contrast, if there is

a path of the form  $i \rightarrow w \rightarrow j$  or  $i \leftarrow w \rightarrow j$  (i.e. a path with non-converging arrows) then we know that both observations *and* interventions of  $w$  are useful for predicting the values of  $i$  and  $j$ , though in different ways. In the case where  $i \leftarrow w \rightarrow j$ , observing or intervening  $w$  provides the same joint information about  $i$  and  $j$ , since  $w$  is a common direct cause of  $j$  and  $i$ . However, if  $i \rightarrow w \rightarrow j$ , intervening  $w$  provides information about  $j$  (since  $w$  is a direct cause of  $j$ ) but provides no information about  $i$  (since  $w$  is neither a direct nor an indirect cause of  $i$ ). In this case, intervening  $w$  breaks down the statistical dependence of  $i$  and  $j$  in a way that is different to simply conditioning on observations of  $w$ . This sparks a natural question: can the structure of the graph tell us something about the conditional independence properties of the underlying conditional and do-probability distributions? This is the object of study in Dawid ([1]), Geiger, Pearl, and Verma ([3]), Lauritzen et. al ([10]) and others. The rules of causal calculus are a particular way in which the structure of the graph is informative about intervention beliefs.

## C TWO EXAMPLES OF DO-PROBABILITY

**Example 5.** Consider a set  $\mathcal{N} = \{1, 2, 3\}$ , and a distribution  $p \in \Delta(X_1 \times X_2 \times X_3)$ . Suppose  $p$  has the following Markov Representation:

$$\begin{aligned} Pa(1) &= \emptyset, & h_1(\varepsilon_1) &= \varepsilon_1, \\ Pa(2) &= \{1\}, & h_2(x_1, \varepsilon_2) &= x_1 + \varepsilon_2, \\ Pa(3) &= \{1\}, & h_3(x_1, x_2, \varepsilon_3) &= x_1 - \varepsilon_3. \end{aligned}$$

Then,  $p$  can be represented as follows:

$$\begin{aligned} p(x_1) &= \phi(\{\varepsilon : \varepsilon_1 = x_1\}), \\ p(x_2) &= \phi(\{\varepsilon : x_1 + \varepsilon_2 = x_2\}) = \phi(\{\varepsilon : \varepsilon_1 + \varepsilon_2 = x_2\}), \\ p(x_3) &= \phi(\{\varepsilon : x_1 - \varepsilon_3 = x_3\}) = \phi(\{\varepsilon : \varepsilon_1 - \varepsilon_3 = x_3\}). \end{aligned}$$

Therefore, we can calculate  $p(x_3|do(x_2))$ , and  $p(x_3|x_2)$  as follows:

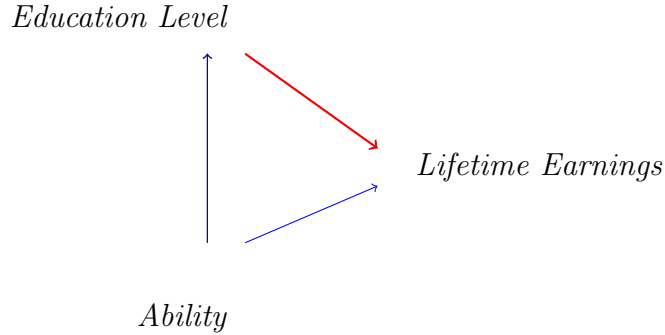
$$p(x_3|do(x_2)) = \phi(\{\varepsilon : \varepsilon_1 - \varepsilon_3 = x_3\}), \quad (7)$$

$$p(x_3|x_2) = \phi(\{\varepsilon : \varepsilon_1 - \varepsilon_3 = x_3\} | \{\varepsilon : \varepsilon_1 + \varepsilon_2 = x_2\}). \quad (8)$$

In 7, the equation determining the value  $x_2$  is eliminated from the Markov representation. This makes the value  $x_2$  uninformative about the value of  $\varepsilon_1$ . In (8), we recognize that variable 2 depends on  $\varepsilon_1$ , so the value  $x_2$  gives information about the value of  $\varepsilon_1$ . Therefore, the do-probability in (7) is independent of the value  $x_2$ , whereas the conditional probability in (8) does depend on  $x_2$ . That  $p(x_2|do(x_2))$  is a constant function (when viewed as a function of  $x_2$ ) is intended to reflect that variable 2 is not a cause of variable 3. That  $x_2$  does affect  $p(x_3|x_2)$  captures that there is a correlation between these two variables (in this example, mediated by variable 1). The difference between these two calculations highlights the difference between causation and correlation.

Example 6 below illustrates how to use do-probabilities to identify causal effects in terms of conditional probabilities only. By connecting intervention beliefs to do-probabilities, Theorem 2 effectively provides all tools for identifying causal effects from conditional probabilities. For more detail on this see section 5.

**Example 6.** Assume a DM's preferences can be represented by the DAG below.



If this DAG represents a probability distribution that admits a Markov representa-

tion then there exist functions  $h_A, h_E, h_L$  such that the following holds:

$$\begin{aligned} p(L = l|E = e) &= \Pr(\{\varepsilon : h_L(h_A(\varepsilon_A), h_E(h_A(\varepsilon_A), \varepsilon_E), \varepsilon_L) = l\} | h_E(h_A(\varepsilon_A), \varepsilon_E) = e), \\ p(L = l|do(E = e)) &= p(\{\varepsilon : h_L(h_A(\varepsilon_A), \mathbf{e}, \varepsilon_L) = l\}). \end{aligned}$$

Suppose we are interested in quantifying the direct effect that education has on earnings (graphically represented by the red arrow). However, as the graph shows,  $E$  provides information about  $L$  in two ways. The first, is the direct effect (indicated by the red path). The second is through the effect that  $A$  has on both  $E$  and  $L$ : observing the value of  $E$  provides information about  $A$ , and  $A$  provides direct information on  $L$  (as indicated by the blue path). In the first equation, which corresponds to a conditional probability, we explicitly see that  $h_L$  depends on  $A$  through  $h_E$ . In the second line, we eliminate the equation determining education, and instead directly impute a value of  $E = e$ . In this way we block the dependence of  $L$  on  $A$  via  $E$ , and only the red effect remains.

As far as quantifying this effect, algebraically manipulating the equations above yields the following:

$$\begin{aligned} p(L = l|do(E = e)) &= \sum_a p(A = a, L = l|do(E = e)), \\ &= \sum_a p(A = a|do(E = e))p(L = l|do(E = e), A = a), \\ &= \sum_a p(A = a)p(L = l|A = a, E = e), \\ &\neq p(L = l|E = e), \end{aligned} \tag{9}$$

Therefore, if we wish to elicit the direct effect that  $E$  has on  $L$ , all we need data on  $p(A = a)$ ,  $p(L = l|A = a, E = e)$ , and to apply equation (10). Notice also that the above equation array can be replicated in terms of intervention beliefs and Axiom

4:

$$\begin{aligned}
p_e(L = l) &= \sum_a p_e(A = a, L = l), \\
&= \sum_a p_e(A = a) p_e(L = l | A = a), \\
&= \sum_a p(A = a) p(L = l | A = a, E = e), \tag{10}
\end{aligned}$$

where the last line applies after applying Axiom 4 noting that (i) the marginal of  $A$  is the same regardless of whether we intervene  $E$  or not because  $Ca(A) = \emptyset$  and (ii) since  $Ca(L) = \{A, E\}$ , then conditioning on  $A$  and  $E$  or conditioning on  $A$  and intervening  $E$  yield the same marginal over  $L$ .

#### D GUL [4] AXIOMS

In this appendix we formally define Gul's ([4]) axiomatization of subjective EU for finite state spaces.

**Axiom (Gul '95).** Let  $T$  be a finite set, and  $>$  a binary relation on  $\mathbb{R}^T$ , with weak part  $\succsim$  and symmetric part  $\sim$ . For each set  $Q \subset T$ , let  $\mathbb{1}_Q$  be the indicator function of  $Q$  (that is,  $\mathbb{1}_Q(t) = 1$  if  $t \in Q$ , and  $\mathbb{1}_Q(t) = 0$  if  $t \notin Q$ ). The following are the axioms in Gul ([4])

G1.  $\succsim$  is complete and transitive.

G2.  $(\forall f, g, h \in \mathbb{R}^T), (\forall t \in T)$ , and  $\forall A \subset T$ , construct  $f'$  and  $g'$  so that

$$\begin{aligned}
\mathbb{1}_T(\cdot) f'(t) &\sim \mathbb{1}_A(\cdot) f(t) + (1 - \mathbb{1}_A(\cdot)) h(t), \\
&\text{and} \\
\mathbb{1}_T(\cdot) g'(t) &\sim \mathbb{1}_A(\cdot) g(t) + (1 - \mathbb{1}_A(\cdot)) h(t).
\end{aligned}$$

Then,  $f > g \Leftrightarrow f' > g'$ . If  $f', g'$  are impossible to construct, this item holds vacuously.

- Note: The act  $\mathbb{1}_T(\cdot) f'(t)$  is an act that pays  $f'(t)$  at each state  $t' \in T$ , and analogously for  $\mathbb{1}_T(\cdot) g'(t)$ .

- *Note: The act  $\mathbb{1}_A(\cdot)f(t) + (1 - \mathbb{1}_A(\cdot))h(t)$  is an act that pays  $f(t)$  at each state  $t' \in A$ , and  $h(t)$  at each state  $t' \notin A$  (analogously for  $\mathbb{1}_A(\cdot)g(t) + (1 - \mathbb{1}_A(\cdot))h(t)$ ).*
- G3.  $(\forall z, z' \in \mathbb{R}), z > z' \Leftrightarrow \mathbb{1}_T(\cdot)z > \mathbb{1}_T(\cdot)z'$ . Furthermore, there exists  $A \subset T$  such that  $\mathbb{1}_A(\cdot)x + (1 - \mathbb{1}_A(\cdot))y \sim \mathbb{1}_A(\cdot)y + (1 - \mathbb{1}_A(\cdot))x$  for all  $x, y \in \mathbb{R}$ .*
- G4.  $(\forall f \in \mathbb{R}^T)$ , the sets  $B(f) = \{g \in \mathbb{R}^T : g \succeq f\}$  and  $W(f) = \{g \in \mathbb{R}^T : f \succeq g\}$  are closed (please see the original paper for the topological definition of closed and open).*