Regressions with Fractional $d = 1/2$ and Weakly Nonstationary Processes

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Abstract

Despite major theoretical advances, important classes of fractional models (e.g. ARFIMA models) have not yet been fully characterised in terms of asymptotic theory. In particular, no limit theory is available for general additive functionals of fractionally integrated processes of order $d = 1/2$. Such processes cannot be handled by existing asymptotic results for either stationary or nonstationary processes, i.e. by existing LLNs or FCLTs. Their asymptotics must therefore be derived by novel arguments, such as are developed in this paper. In the course of doing so, we develop new limit theory for a broader class of linear processes lying on the boundary between stationarity and nonstationarity – what we term weakly nonstationary processes. This includes, as leading examples, both fractional processes with $d = 1/2$, and arrays of autoregressive processes with roots drifting slowly towards unity. We apply our new results to the asymptotics of both parametric and kernel regression estimators.

Keywords: Fractional process, half unit root, kernel regression, mildly integrated process, weakly nonstationary process

1 Introduction

Autoregressive Fractionally Integrated Moving Average (ARFIMA) models, and their vector generalisations, are one of the most important classes of stochastic processes in statistics. Yet the limit theory for these processes still remains to be fully characterised. In particular, no asymptotic theory is currently available for additive functionals of fractionally integrated processes of order d (hereafter '$I(d)$') when $d = 1/2$. This value is of particular significance as it demarcates the boundary between stationarity and nonstationarity for these processes.

Existing results suggest the following dichotomy in the limit theory for fractional processes. Firstly, in the stationary region ($|d| < 1/2$), laws of large numbers (LLNs) for dependent processes apply, yielding limiting results of the form

$$\frac{1}{n}\sum_{t=1}^{n} f\left(x_t\right) \xrightarrow{p} \mathbf{E} f\left(x_t\right) \tag{1}$$

for suitably integrable f (see e.g. Taqqu, 1975; Giraitis, 1985; Giraitis and Surgailis, 1985). On the other hand, when $d \in (1/2, 3/2)$, the asymptotics are driven by functional central limit theorems (FCLTs) of the form $n^{1/2-d}x_{\lfloor nr \rfloor} \xrightarrow{d} W_d(r)$, where W_d denotes a fractional Brownian motion (fBM) of order d (see e.g. Taqqu, 1975, Phillips, 1987a; Chan and Wei, 1988; Marinucci and Robinson, 2000). Together with the continuous mapping theorem, or more general results on the convergence of integral functionals, such FCLTs yield

$$\frac{1}{n}\sum_{t=1}^{n} f(n^{1/2-d}x_t) \xrightarrow{d} \int_0^1 f[W_d(r)]dr \tag{2}$$

These differences in the limit theory are not only of theoretical interest, but also practically relevant to problems of estimation and inference in regression models with fractionally integrated regressors. For example, consider the linear regression

$$y_t = \beta x_{t-1} + u_t,$$

where u_t is a martingale difference sequence (e.g. with respect to the natural filtration for (x_t, u_t)). Then for $|d| < 1/2$, a LLN together with a martingale central limit theorem (CLT) yields

$$n^{1/2}(\hat{\beta}_{LS} - \beta) \xrightarrow{d} N\left(0, \mathbf{E}\left(u_t^2\right)/\mathbf{E}\left(x_t^2\right)\right),$$

i.e. the least squares (LS) estimator $\hat{\beta}$ is $n^{-1/2}$-consistent for β and asymptotically normal. On the other hand for $d \in (1/2, 3/2)$, the weak convergence of stochastic integrals yields

$$n^d(\hat{\beta}_{LS} - \beta) \xrightarrow{d} \left[\int_0^1 W_d(r)^2 dr\right]^{-1} \int_0^1 W_d(r) dB_u(r),$$

where $B_u(r)$ is a Brownian motion (see e.g. Phillips, 1995; Robinson and Hualde, 2003). In this case, the LS estimator enjoys a faster rate of convergence, but its limiting distribution is such that conventional inferential procedures – i.e. those based on a normal approximation to the t-statistic – are no longer valid.

This paper develops new limit theory for the case where $d = 1/2$, showing that there is in fact a trichotomy in the asymptotics of fractional processes. Although $I(1/2)$ processes are nonstationary – thus ruling out the application of a LLN in the manner of (1) – FCLTs do not apply to these processes either. For example, in the case of the 'type II' fractional process: while the finite dimensional distributions of $X_n(r) := Var(x_n)^{-1/2} x_{\lfloor nr \rfloor}$ do converge, their limit is a nonseparable Gaussian 'white noise' process G, which has the property that $G(r)$ and $G(s)$ are independent for every $r \neq s$.[1] Since the sample paths of G are not a.s. bounded, this convergence cannot be strengthened to weak convergence with respect to the uniform or Skorokhod topologies; nor does $\{X_n\}$ satisfy even the much weaker regularity conditions sufficient for the convergence of integral functionals in the form of (2) (see Gikhman and Skorokhod, 1969, p. 485, Theorem 1). Despite this, we shall prove that for $I(1/2)$ type II processes

$$\frac{1}{n}\sum_{t=1}^{n} f(\beta_n^{-1} x_t) \xrightarrow{p} \int_{\mathbb{R}} f(x)\varphi_1(x)dx \tag{3}$$

where $\beta_n^2 = Var(x_n)$, and φ_{σ^2} denotes a normal density with mean zero and variance σ^2. The counterpart of (3) for type I processes is

$$\frac{1}{n}\sum_{t=1}^{n} f(\beta_n^{-1} x_t) \xrightarrow{d} \int_{\mathbb{R}} f(x + X^-)\varphi_{1/2}(x)dx \tag{4}$$

[1] For the distinction between 'type I' and 'type II' fractional processes, due to Marinucci and Robinson (1999), see Section 2.1.

where $X^- \sim N[0, 1/2]$. Results of the kind (3) and (4) for general nonlinear transformations are new to the literature – the limit theory for $I(1/2)$ processes previously developed by Shimotsu and Phillips (2005) and Hualde and Robinson (2011) being limited to specific functionals that arise in the context of memory estimation.

En route to (3) and (4), we develop a general framework that is able to handle the asymptotics of a much broader class of processes (and arrays thereof) that share two of the key characteristics of $I(1/2)$ processes: (a) they are sufficiently nonstationary to resist the application of existing LLNs; and (b) their dependence is sufficiently weak that their finite-dimensional distributions converge to those of a nonseparable Gaussian process. We term these *weakly nonstationary processes* (WNPs). We develop a set of general high-level conditions that express (a) and (b) in more precise mathematical terms, and which allow us to establish the asymptotics of

$$\frac{1}{n}\sum_{t=1}^{n} f(\beta_n^{-1} x_t) \qquad \text{and} \qquad \frac{\beta_n}{nh_n}\sum_{t=1}^{n} K\left(\frac{x_t - x}{h_n}\right) \tag{5}$$

where $\beta_n^2 = Var(x_n)$, f is locally integrable, K is integrable, and h_n denotes a bandwidth sequence. We also provide an accompanying set of lower-level sufficient conditions that may be more directly verified for linear processes.

Beyond the ARFIMA class, another important class of processes to which our low-level conditions apply are the *mildly integrated* (MI) processes considered by Giraitis and Phillips (2006) and Phillips and Magdalinos (2007, 2009). These are closely related to the Nearly Integrated (NI) processes developed by Chan and Wei (1987) and Phillips (1987), and more recently extended by Buchmann and Chan (2007). Both MI and NI processes may be defined in terms of an array as

$$x_t(n) = (1 - \kappa_n^{-1}) x_{t-1}(n) + v_t,\ x_0(n) = 0, \tag{6}$$

where v_t is a stationary process and $\kappa_n > 0$ with $\kappa_n \to \infty$, so that the autoregressive coefficient becomes increasingly proximate to unity as n grows. Both NI and MI processes thus describe wide sense autoregressive processes, which have a root in the vicinity of unity. They have accordingly been used to investigate the behaviour of various inferential procedures under local departures from unit roots (e.g. Mikusheva, 2007, and Duffy, 2017), and in the construction of robust inferential procedures (e.g. Magdalinos and Phillips, 2009; Kostakis, Magdalinos and Stamatogiannis, 2014).

The crucial difference between NI and MI processes concerns the assumed growth rate of the sequence κ_n. NI processes are defined by $\kappa_n/n \to c \neq 0$, with the consequence that $n^{-1/2}x_{\lfloor nr \rfloor}$ converges weakly to an Ornstein-Uhlenbeck process, leading to convergence results akin to (2). MI processes have $\kappa_n/n \to 0$, which tilts $x_t(n)$ closer to stationarity. As a consequence, FCLTs no longer apply, and it is shown in this paper that limits analogous to (3) hold instead. Again, these results are new to the literature. Existing results due to Giraitis and Phillips (2006) and Phillips and Magdalinos (2007) consider only quadratic transformations of MI processes driven by short memory linear processes errors. Duffy (2017) provides limit theory for bounded and Lipschitz continuous kernel functionals of MI processes driven by short memory errors. Our results extend these papers both by allowing for much more general nonlinear transformations, and for MI processes that are driven by long memory errors – an extension of MI processes analogous to that of Buchmann and Chan (2007) for NI processes.

Our general results on the asymptotics of the functionals in (5) appear in Sections 2 and 3. We then apply these to deduce the large-sample behaviour of both parametric and nonparametric estimators in regression models involving WNPs – or non-linear transformations thereof – as regressors (Section 4). In conjunction with suitable martingale central limit theorems (Hall and Heyde, 1980; and Wang, 2014), we show that both the ordinary least squares (OLS) and Nadaraya-Watson (NW) kernel regression estimators have limit distributions that are mixed normal when the regressor is $I(1/2)$ type-I, and normal when the regressor is $I(1/2)$ type-II or MI. Therefore, in regressions with WNPs conventional inferential procedures remain valid and none of the modifications familiar from inference with nonstationary processes (e.g. Phillips, 1995; Robinson and Hualde, 2003) are required. Proofs of the main results of the paper appear in Section 5; proofs of auxiliary lemmas are given in the Supplementary Material.

Notation. For two deterministic sequences $\{a_n\}$ and $\{b_n\}$, $a_n \sim b_n$ denotes $\lim_{n\to\infty} a_n/b_n = 1$. For two random variables X and Y, $X \sim Y$ and $X \overset{d}{=} Y$ denote distributional equality between X and Y. For a real number x, $\lfloor x \rfloor$ denotes its integer part. $1\{A\}$ denotes the indicator function for the set A. $\overline{\mathbb{R}}$, $\mathbb{R}_+$ and $\mathbb{R}_+^*$ are the extended, the positive, and strictly positive real numbers respectively. A $k \times k$ diagonal matrix with diagonal elements $a_1, ..., a_k$ is written $diag\,(a_1, ..., a_k)$. $:=$ denotes definitional equality. $\sigma(X)$ denotes the sigma algebra generated by random variable X. All limits are taken as $n \to \infty$ unless otherwise indicated.

2 Additive functionals of standardised WNPs

2.1 $I(1/2)$ and MI processes

We first consider the asymptotics of additive functionals of standardised WNPs: that is, quantities of the form

$$\frac{1}{n}\sum_{t=1}^{n} f(\beta_n^{-1} x_t) \tag{7}$$

where f is locally integrable, and $\beta_n^2 = Var(x_n)$.

As noted in the introduction, two classes of processes with which we are particularly concerned are the $I(1/2)$ and MI processes. To define these more formally, let $\{\xi_t\}_{t\in\mathbb{Z}}$ denote an i.i.d. sequence with mean zero and unit variance (further properties of which will be specified by **Assumption INN** in Section 2.3 below), and define the linear processes

$$x_t = \sum_{j=0}^{t-1} \phi_j v_{t-j} \qquad \text{where} \qquad v_t = \sum_{i=0}^{\infty} c_i \xi_{t-i} \tag{8}$$

for some coefficient sequence $\{c_i\}_{i\in\mathbb{Z}}$.

The definition of a 'fractionally integrated process' used this paper closely follows that of Marinucci and Robinson (1999; see also Robinson and Hualde, 2003, p. 1728-1730). These authors classify non-stationary fractional processes as type I or type II according to the 'type' of the fractional Brownian motion (fBM) to which their finite dimensional distributions converge (upon standardisation). Although these authors consider processes with a long memory component that is specified 'parametrically' – via the expansion of an autoregressive lag polynomial $(1-L)^d$ (see also **Remark 2.4(a)** below) – their classification extends straightforwardly to the case where this is instead formulated 'semi-parametrically', in terms of the decay rate of the coefficients $\{\phi_j\}$. Thus we shall say that for $d \in (1/2, 1)$, x_t is an $I(d)$ process of

- type I: if $\phi_j = 1\ \forall j$, $c_s \sim \ell(s)s^{d-2}$ and $\sum_{s=0}^{\infty} c_s = 0$; and

- type II: if $\phi_j \sim \ell(j)j^{d-1}$, $\sum_{s=0}^{\infty}|c_s| < \infty$, and $\sum_{s=0}^{\infty} c_s \neq 0$;

where ℓ is slowly varying at infinity (henceforth, 'SV') in the sense of Bingham, Goldie and Teugels (1987, p. 6). Fractional processes of this kind have been widely studied (for $d \neq 1/2$): see e.g. Jeganathan (1999, 2004, 2008) for the type I case (and also Taqqu, 1975; Astrauskas, 1983; Kasahara and Maejima, 1988); and Robinson and Hualde (2003), Marmol and Velasco (2004), Phillips and Shimotsu, (2004), Shimotsu and Phillips (2005), Hualde and Robinson (2011) for the type II case. The preceding definitions extend naturally to the case where $d = 1/2$, and give the definitions of $I(1/2)$ processes (of each type) used throughout this paper – though the standardised processes will not converge to an fBM of either type in the case. Note also that if $\phi_n \to 0$ sufficiently rapidly, then these processes may in fact be stationary (see **Remark 2.4(b)** below).

MI processes, which were defined in (6) above, can also be encompassed within the framework of (8) if we allow ϕ_j to depend on n as per

$$\phi_j = \phi_j(n) = (1 - \kappa_n^{-1})^j$$

where $\kappa_n > 0$, $\kappa_n \to \infty$ and $\kappa_n/n \to 0$. Previous work on these processes has assumed that v_t is short memory in the sense that $\sum_{s=0}^{\infty}|c_s| < \infty$ (see e.g. Giraitis and Phillips, 2006; Magdalinos and Phillips, 2007). Our results apply to this case, but we shall also allow v_t to have long memory in the sense that $c_s \sim s^{-m}$ for $m \in (1/2, 1)$, thereby extending this previous work much in the manner of Buchmann and Chan's (2007) extension of earlier work on NI processes.

Both $I(1/2)$ and MI processes may thus be regarded as instances of (arrays of) linear processes formed from the underlying i.i.d. sequence $\{\xi_t\}$, which we denote generally as

$$x_t\,(n) = \sum_{k=0}^{\infty} a_{k,t}\,(n)\,\xi_{t-k}, \tag{9}$$

where (suppressing the dependence of these quantities on n for the sake a readability, as we shall do freely below)

$$a_{k,t} = \sum_{j=0}^{(t-1)\wedge k} \phi_j c_{k-j} = \begin{cases} \sum_{j=0}^{k} \phi_j c_{k-j} =: a_k & \text{if } 0 \leq k \leq t-1, \\ \sum_{j=0}^{t-1} \phi_j c_{k-j} =: a_{k,t}^- & \text{if } k \geq t. \end{cases} \tag{10}$$

Note, in particular, that $a_{k,t}$ does not depend on t for $1 \leq k \leq t-1$, and we accordingly denote these coefficients by simply a_k. For $k \geq t$, the notation $a^-_{k,t}$ reminds us that these coefficients refer to innovations dated $t \leq 0$.

2.2 Asymptotics under high-level conditions

To allow our limit theory to cover a broad class of processes, which encompass the examples of $I(1/2)$ and MI processes as a special cases, this section provides high-level conditions (**Assumption HL**) under which the limit of (7) may be obtained. We subsequently provide sufficient conditions for these to be satisfied by general linear processes of the form (9)–(10) (**Assumption LP** in Section 2.3 below). These 'linear process conditions' are then, in turn, verified for $I(1/2)$ and MI processes under certain low-level technical conditions (**Assumption LL** in Section 2.4 below).

Assumption HL (high-level conditions)

HL0 *Let* $\{X_t(n)\}_{t=1}^n$, $n \in \mathbb{N}$ *be a random array and* $\{\mathcal{F}_t\}_{t=-\infty}^{\infty}$ *a filtration such that* $X_t(n)$ *is* $\mathcal{F}_t$*-measurable for all* t *and* n. *Let* $\{\beta_n\}$ *denote a non-negative sequence with* $\beta_n \to \infty$.

HL1 $X_t(n) = X_t(n)^+ + X_t(n)^- + R_t(n)$, *where* $X_t(n)^-$ *is* $\mathcal{F}_0$*-measurable, and* $\sup_{1\leq t\leq n} \mathbf{P}\{\beta_n^{-1}|R_t(n)| > \epsilon\} \to 0$ *for every* $\epsilon > 0$.

HL2 *There are random variables* X^+ *and* X^-, *where* X^+ *has bounded density* Φ_{X^+} *such that: for every* $\delta \in (0,1)$ and $\{t_n\}$ *with* $\lfloor n\delta \rfloor \leq t_n \leq n$

(a) $\beta_n^{-1} X_{t_n}(n)^+ \xrightarrow{d} X^+$, *conditionally on* $\mathcal{F}_0$ *in the sense that for all bounded and continuous* $h : \mathbb{R} \to \mathbb{R}$

$$\mathbf{E}\left[h\left(\beta_n^{-1} X_{t_n}(n)^+\right) \mid \mathcal{F}_0\right] \xrightarrow{p} \int_{\mathbb{R}} h(x)\, \Phi_{X^+}(x) dx; \text{ and}$$

(b) $\beta_n^{-1} X_{t_n}(n)^- \xrightarrow{d} X^-$, *and* $\beta_n^{-1}\left[X_n(n)^- - X_{t_n}(n)^-\right] \xrightarrow{p} 0$.

HL3 $\beta_t^{-1} X_t(n)$ *has Lebesgue density* $\mathcal{D}_{n,t}(x)$ *such that for some* $n_0 \geq t_0 \geq 1$,

$$\sup_{n\geq n_0, t_0\leq t\leq n} \sup_x \mathcal{D}_{n,t}(x) < \infty$$

HL4 *For every bounded and Lipschitz continuous* $g : \mathbb{R} \to \mathbb{R}$

$$\frac{1}{n}\sum_{t=1}^{n} g\left(\beta_n^{-1} X_t\left(n\right)\right) = \frac{1}{n}\sum_{t=1}^{n} \mathbf{E}\left[g\left(\beta_n^{-1} X_t\left(n\right)\right) \mid \mathcal{F}_0\right] + o_p(1).$$

HL5 *For some* $\lambda \in (0, \infty)$ *and some* $n_0 \geq 1$

(a) $\sup_{n \geq n_0, 1 \leq t \leq n} \mathbf{E}\left|\beta_n^{-1} X_t\left(n\right)\right|^{\lambda} < \infty$*; or*

(b) $\sup_{n \geq n_0, 1 \leq t \leq n} \mathbf{E} \exp\left(\lambda \left|\beta_n^{-1} X_t\left(n\right)\right|\right) < \infty$.

HL6 *For some* $n_0 \geq t_0$, $\sup_{n \geq n_0} \frac{\beta_n}{n} \sum_{t=t_0}^{n} \beta_t^{-1} < \infty$*, where* t_0 *is as in* **HL3**.

Remark 2.1. **(a)** When $X_t(n) = x_t(n)$, the linear process array in (9), condition **HL1** is trivially satisfied by splitting $x_t(n)$ into two weighted sums as per

$$x_t(n) = \sum_{k=0}^{t-1} a_k\left(n\right) \xi_{t-k} + \sum_{k=t}^{\infty} a_{k,t}^{-}\left(n\right) \xi_{t-k} =: x_t^{+}\left(n\right) + x_t^{-}\left(n\right) \qquad (11)$$

(b) **HL2** expresses one of the key properties of a WNP: that its finite dimensional distributions should converge (upon standardisation), albeit not to those of a separable process. Its requirements are best illustrated by the $I(1/2)$ type I process with $\ell(x) = 1$. In this case, the application of a CLT for weighted sums yields that for every $r, s \in (0, 1]$

$$\beta_n^{-1}(x_{\lfloor nr \rfloor}^{+}, x_{\lfloor ns \rfloor}^{+}) \xrightarrow{d} (\eta_r, \eta_s) \qquad (12)$$

where $\beta_n^2 = Var(x_n) \sim C \cdot \ln n$, and η_r and η_s are independent $N[0, 1/2]$ random variables. We thus have the marginal convergence of each coordinate of $\beta_n^{-1} x_{\lfloor nr \rfloor}^{+}$ to identical *distributional* limits, as per **HL2(a)** – and if ξ_t is i.i.d., the required conditional convergence holds trivially. In that case, (12) holds jointly with (and independently of)

$$\beta_n^{-1}(x_{\lfloor nr \rfloor}^{-}, x_{\lfloor ns \rfloor}^{-}) \xrightarrow{d} (\eta^{-}, \eta^{-}) \qquad (13)$$

where $\eta^{-} \sim N[0, 1/2]$. Note, in particular, the degeneracy in the joint distribution of the limit in (13), consistent with the second part of **HL2(b)**.

(c) **HL3** is useful for establishing L_1-approximations for functionals of WNPs: that is, they allow convergence results proved under the requirement that f in (7) is bounded and continuous to be extended to a much broader class of integrable functions. High-level conditions similar to **HL3** have been employed for similar purposes in many previous works, e.g. Akonom (1993), Jeganathan (2004, 2008), Pötscher (2004), Gao, King, Lu and Tjøstheim (2009), Wang and Phillips (2009a,b; 2012) among others.

(d) Together with **HL2**, **HL4** is the principal condition that expresses the requirement that a WNP should not be too strongly dependent. It would fail both for $I(d)$ processes with $d > 1/2$, and for NI processes – and indeed for any process for which $\beta_n^{-1} X_{\lfloor nr \rfloor}(n)$ converges weakly to a process with continuous sample paths.

Under the preceding high-level conditions, we have the following result, the proof of which appears in **Section 5**.

Theorem 2.1 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is locally Lebesgue integrable, and **HL0**–**HL4**, **HL6** hold. Further suppose f is bounded or that the following hold:*

(i) There is a $\mathcal{Y} \subseteq \mathbb{R}$ such that $\int_{\mathbb{R}} |f(x+y)| \Phi_{X^+}(x)dx < \infty$ for all $y \in \mathcal{Y}$, and $\mathbf{P}(X^- \in \mathcal{Y}) = 1$;

*(ii) $n^{-1} \sum_{t=1}^{t_0-1} f\left(\beta_n^{-1} X_t(n)\right) = o_p(1)$, for t_0 as in **HL3**;*

*(iii) For $\lambda' \in (0, \lambda)$, where λ is as in **HL5**, either:*

*(a) $|f(x)| = O(|x|^{\lambda'})$, as $|x| \to \infty$ and **HL5(a)** holds; or*

*(b) $|f(x)| = O(\exp(\lambda' |x|))$, as $|x| \to \infty$ and **HL5(b)** holds.*

Then as $n \to \infty$,

$$\frac{1}{n} \sum_{t=1}^{n} f\left(\beta_n^{-1} X_t(n)\right) \xrightarrow{d} \int_{\mathbb{R}} f\left(x + X^-\right) \Phi_{X^+}(x)dx. \tag{14}$$

Further, if $\tilde{f} : \mathbb{R} \to \overline{\mathbb{R}}$ is such that $\tilde{f}(x) = f(x)$ almost everywhere, then (14) holds with f replaced by $\tilde{f}$.

Remark 2.2. **(a)** If f is bounded, conditions (i)–(iii) hold trivially, and **HL5** is unnecessary for (14). If f is additionally Lipschitz, then **HL3** and **HL6** may also be dispensed with.

(b) Condition (ii) of **Theorem 2.1** is a technical requirement that has also been employed in other studies that develop limit theory for functionals of nonstationary processes, e.g. Jeganathan (2004) and Pötscher (2004).

2.3 Sufficient conditions for linear processes

We next provide conditions under which linear process arrays of the form (9) satisfy the preceding high-level conditions. Although we do *not* require that these processes be generated according to the model (8), we shall make one simplifying assumption consistent with that model: that the coefficients $a_{k,t}$ should not depend on t for $1 \leq k \leq t-1$. Our conditions thus envisage an array of the form

$$x_t(n) = \sum_{k=0}^{t-1} a_k(n)\,\xi_{t-k} + \sum_{k=t}^{\infty} a_{k,t}^{-}(n)\,\xi_{t-k} =: x_t^{+}(n) + x_t^{-}(n)\,. \tag{15}$$

Our first assumption concerns the innovation sequence $\{\xi_t\}$.

Assumption INN (innovations).
(i) *ξ_t is i.i.d. with $\mathbf{E}\xi_1 = 0$ and $Var(\xi_1) = \sigma_\xi^2 < \infty$.*
(ii) *ξ_1 has an absolutely continuous distribution, and a characteristic function $\psi_\xi(\lambda)$ that satisfies $\int_{\mathbb{R}} \left|\psi_\xi(\lambda)\right|^\theta d\lambda < \infty$, for some $\theta \in \mathbb{N}$.*

To state our conditions on the coefficients on the linear process in (15), we first define

$$\beta_{n,t}^2 := Var(x_t(n)) = \sigma_\xi^2 \sum_{k=0}^{t} a_k(n)^2 + \sigma_\xi^2 \sum_{k=0}^{t} a_{k,t}^{-}(n)^2 =: (\beta_{n,t}^{+})^2 + (\beta_{n,t}^{-})^2$$

and set $\beta_n := \beta_{n,n}$, for $n \in \mathbb{N}$ and $t \in \{1, \ldots, n\}$. The following conditions will always be applied in conjunction with **Assumption INN**, and are stated in terms of the θ and the i.i.d. sequence $\{\xi_t\}$ appearing in that assumption.

Assumption LP (linear process).

LP1 *$x_t(n)$, $a_k(n)$, and $a_{k,t}^{-}(n)$ are as in (15).*

LP2 (a) *$\mathbf{E}\left|\xi_1\right|^\lambda < \infty$ for some $\lambda \in [2, \infty)$ or* **(b)** *ξ_1 has a finite moment generating function (m.g.f.) in a neighbourhood of zero.*

LP3 *$\beta_{n,t} < \infty$ for all $n, t \geq 1$. Further, there are sequences $\underline{q}_t, \overline{q}_t \in \mathbb{N}$ where $\overline{q}_t - \underline{q}_t \to \infty$, $\overline{q}_t \leq t-1$ as $t \to \infty$, and a $\delta \in (0, 1]$ such that for some $n_0 \geq t_0 \in \mathbb{N}$ and all $\rho > 0$:*

(a) $\inf_{n\geq n_0, t_0\leq t\leq n} \beta_t^{-2} \sum_{k=\underline{q}_t+\theta}^{\overline{q}_t} a_k(n)^2 =: \mathcal{C}_1 > 0$;

(b) $\sup_{n\geq n_0, t_0\leq t\leq n} \beta_t^{-\delta} \sup_{\underline{q}_t\leq k\leq \overline{q}_t} |a_k(n)| \vee 1 =: \mathcal{C}_2 < \infty$;

(c) $\sup_{n\geq n_0, t_0\leq t\leq n} \exp\left\{-\rho\beta_t^{2(1-\delta)}\right\} \beta_t \sup_{\underline{q}_t\leq k\leq \underline{q}_t+\theta} |a_k(n)|^{-1} < \infty$.

LP4 *For some $q \geq \theta$, each $n \in \mathbb{N}$ and $1 \leq t \leq n$ there are $\left\{k^*_{n,t,l}\right\}_{l=1}^{q} \in \mathbb{Z}_+$ such that for all $t_0 \in \mathbb{N}$*

$$\liminf_{n\to\infty} \inf_{1\leq l\leq q, 1\leq t\leq t_0} \left|a_{k^*_{n,t,l},t}(n)\right| > 0.$$

LP5 $\lim_{n\to\infty} n^{-1/2}\beta_n^{-1} \sum_{k=0}^{n} |a_k(n)| = 0$.

LP6 *For any $\delta \in (0,1)$ and $\{t_n\}$ such that $\lfloor n\delta \rfloor \leq t_n \leq n$:*

(a) $\beta_n^{-1}\left[\beta^+_{n,t_n}, \beta^-_{n,t_n}\right] \to [\sigma_+, \sigma_-]$ *with $\sigma_+ > 0$ and $\sigma_- \geq 0$;*

(b) $\beta_n^{-1}\left(\sup_{0\leq k\leq n} |a_k(n)| + \sup_{1\leq t\leq n, k\geq t} \left|a^-_{k,t}(n)\right|\right) \to 0$;

(c) $\beta_n^{-2} \sum_{l=-\infty}^{0} \left[a^-_{t_n-l,t_n}(n) - a^-_{n-l,n}(n)\right]^2 \to 0$.

LP7 $\sup_{n\geq n_0, 1\leq t\leq n} \beta_n^{-2}\beta_{n,t}^2 < \infty$ *for some $n_0 \geq 1$.*

Remark 2.3. The principal relationships between the preceding conditions, and the high-level conditions (**Assumption HL**) as they would be applied to $X_t(n) = x_t(n)$ may be summarised as follows; these are also stated more formally as **Proposition 2.1** below.

(a) **LP2** and **LP7** imply that $\beta_n^{-1}x_t(n)$ has finite moments of a sufficient order (as per **HL5**). For $\beta_n^{-1}x_t(n)$ to have finite λ-moments, it is sufficient that ξ_t also have finite λ-moments; $\beta_n^{-1}x_t(n)$ will have finite exponential moments if ξ_t has an m.g.f. that is finite in a neighbourhood of zero.

(b) **LP3** and **LP4** ensure that $\beta_t^{-1}x_t(n)$ has a uniformly bounded density, as required by **HL3**. Under **Assumption INN**, a weighted sum involving at least θ of the innovations ξ_t will have an integrable characteristic function. $\beta_t^{-1}x_t^+(n) = \sum_{k=0}^{t-1} \beta_t^{-1}a_k(n)\xi_{t-k}$ will thus have a density bounded uniformly over n and t, provided that the L_1 norm of its characteristic function can be uniformly bounded. This in turn requires that: (i) the variance of $\beta_t^{-1}x_t^+(n)$ can be bounded away from zero; and (ii) it is never dominated by less than θ of the innovations that contribute to it. Both are ensured by **LP3**, at least for n and t sufficiently large. **LP4** entails that for *every* t, a sufficient number of coefficients $\{a_{k,t}(n)\}$ are bounded away from zero; together with **LP3** it is sufficient for **HL3** to hold with $t_0 = 1$.

(c) LP5 can be understood as a kind of weak dependence condition, which is used solely to verify **HL4**.

(d) LP6 permits a central limit theorem for weighted sums to be applied to each of $\beta_n^{-1} x_t^+(n)$ and $\beta_n^{-1} x_t^-(n)$, as required by **HL2**. **LP6(a)** determines the limiting variance of each of these two terms, while **LP6(b)** is a negligibility requirement on the linear process coefficients, akin to a Lindeberg condition. Finally, **LP6(c)** implies the second part of **HL2(b)**.

Proposition 2.1 (LP ⇒ HL) *Suppose **Assumptions LP1** and **INN** hold. Then for* $\mathcal{F}_t := \sigma(\{\xi_r\}_{r \le t})$, $X_t^+ := x_t^+(n)$, *and* $X_t^-(n) := x_t^-(n)$*:*

(i) ***LP6*** $\Rightarrow$ ***HL2*** *with* $[X^+, X^-] \sim N[\mathbf{0}, diag(\sigma_+^2, \sigma_-^2)]$.

(ii) (a) ***LP3*** $\Rightarrow$ ***HL3****;*

(b) ***LP3*** *and* ***LP4*** $\Rightarrow$ ***HL3*** *with* $t_0 = 1$.

(iii) ***LP5*** $\Rightarrow$ ***HL4***.

(iv) (a) ***LP2(a)*** *and* ***LP7*** $\Rightarrow$ ***HL5(a)*** *with* $\lambda \ge 2$*;*

(b) ***LP2(b)***, ***LP6(b)*** *and* ***LP7*** $\Rightarrow$ ***HL5(b)***.

2.4 Verification for leading examples

We turn finally to the application of our 'intermediate-level' results to $I(1/2)$ and MI processes. That is, we now specialise to the case of a linear process array of the form

$$x_t(n) = \sum_{j=0}^{t-1} \phi_j(n) v_{t-j}, \qquad \text{where} \qquad v_t = \sum_{i=0}^{\infty} c_i \xi_{t-i}, \tag{16}$$

and $\phi_j(n)$ and c_i are as specified by the 'low level conditions' given immediately below. These conditions further elaborate the definitions of $I(1/2)$ and MI processes that were introduced in Section 2.1, and impose certain technical requirements that facilitate the verification of **Assumption LP**.

Assumption LL (low-level conditions). *$x_t(n)$ is generated by (16). The function ℓ is SV such that $L(n) := \int_1^n [\ell^2(x)/x]dx < \infty$ for all $n \in \mathbb{N}$, $L(n) \to \infty$ as $n \to \infty$, and either:*

LL1 *$\phi_j \sim \ell(j)\, j^{-1/2}$, $\phi_0 \neq 0$, $\sum_{s=0}^{\infty} |c_s| < \infty$, $\sum_{s=0}^{\infty} c_s \neq 0$, and either*
(a) $\limsup_{j\to\infty} j\,|c_j| < \infty$, *or*
(b) $\lim_{n\to\infty} \ell(n)^{-1} \sum_{j=\lfloor n\delta \rfloor}^{\infty} j^{1/2}\,|c_j| = 0$ *for some $0 < \delta < 1$; or*
LL2 *$\phi_j = 1\ \forall j \geq 0$, $c_s \sim \ell(s)s^{-3/2}$, and $\sum_{s=0}^{\infty} c_s = 0$.*

Or: for $\rho_n = 1-\kappa_n^{-1}$ – where $\kappa_n > 0$, $\kappa_n = n^{\alpha_\kappa}\ell_\kappa(n)$ for ℓ_κ SV and $\alpha_\kappa \in [0,1)$, $\kappa_n \to \infty$ and $\sup_{n\geq 1, 1\leq t \leq n} \sup \kappa_t/\kappa_n < \infty$ – either:

LL3 *$\phi_j(n) = \rho_n^j$, $\sum_{s=0}^{\infty} |c_s| < \infty$, $\sum_{s=0}^{\infty} c_s \neq 0$; or*
LL4 *$\phi_j(n) = \rho_n^j$, $c_s \sim s^{-m}$, $1/2 < m < 1$.*

Remark 2.4 (a) LL1 and **LL2** respectively imply that $x_t(n)$ is an $I(1/2)$ process of types II and I, according to the definitions of these processes given in **Section 2.1** above. **LL1** also specifies some additional regularity conditions (**(a)** and **(b)**) that are used principally to ensure $x_t(n)$ satisfies condition **LP3** (implying that $\beta_t^{-1}x_t(n)$ has a uniformly bounded density). For verifying the other requirements of **Assumption LP**, these conditions can be dispensed with. (Similar conditions have appeared elsewhere in the literature, e.g. Jeganathan, 2008 and Hualde and Robinson, 2011.)

These regularity conditions are sufficiently weak to ensure that parametric ARFIMA models with $d = 1/2$ are covered by **LL1** and **LL2**, which may be demonstrated as follows.

Parametric type II: Consider the ARFIMA(1/2) type II model

$$(1-L)^{1/2}x_t = v_t 1\{t > 1\}, \qquad \text{where} \qquad A(L)v_t = B(L)\xi_t, \tag{17}$$

where A and B denote finite-order polynomials in the lag operator L. In this case, it is possible to write $x_t = \sum_{j=0}^{t-1} \phi_j v_{t-j}$, where $\{\phi_j\}_{j\geq 0}$ are the coefficients in the power series expansion of $(1-L)^{-1/2}$; and so $\phi_0 = 1$ and $\phi_j \sim C \cdot j^{-1/2}$ for some $C \in (0,\infty)$ (see e.g. p. 673 in Johansen and Nielsen, 2012). Further, if all the roots of A lie outside the unit circle, $v_t = A(L)^{-1}B(L)\xi_t$ is a linear process with geometrically decaying coefficients (e.g. Theorem 3.1.1 in Brockwell and Davis, 1991). Thus **LL1(a)** and **LL1(b)** are easily satisfied. Since $\ell(x) = 1$, we have trivially that $L(n) \to \infty$.

Parametric type I: Under **LL2**,

$$x_t = x_{t-1} + v_t, \qquad x_0 = 0 \tag{18}$$

i.e. a random walk driven by a fractional error process v_t, integrated of order $\delta = -1/2$. Under **LL2**, we may specify that $v_t = \sum_{j=0}^{\infty} \pi_j \varepsilon_{t-j}$, where $\varepsilon_t = \sum_{j=0}^{\infty} \theta_j \xi_{t-j}$, $|\theta_j| = O\left(j^{-q}\right)$ for $q \geq 3/2$, and $\sum_{j=0}^{\infty} \theta_j \neq 0$. This encompasses the case where ε_t is a stationary ARMA process. Further, suppose that π_j are the coefficients in the series expansion of $(1-x)^{1/2}$ (i.e. $(I-L)^{-1/2} v_t = \varepsilon_t$). Then $\pi_j \sim C \cdot j^{-3/2}$ and $\sum_{j=0}^{\infty} \pi_j = 0$. In this case we can write $v_t = \sum_{i=0}^{\infty} c_i \xi_{t-i}$ with $c_i = \sum_{j=0}^{i} \theta_j \pi_{i-j}$. To see that $\{c_i\}$ satisfies **LL2**, note that: (a) $c_i \sim \pi_i \sum_{j=0}^{\infty} \theta_j$ (see **Lemma 7.1(ii)** in **Appendix B**); and (b) $\sum_{i=0}^{\infty} c_i = 0$ by Mertens' Theorem for Cauchy products (see Thm 8.46 in Apostol, 1981).

(b) Since only the values taken by $\ell(x)$ for sufficiently large $x \in \mathbb{N}$ are relevant to **LL1** and **LL2**, we may require $L(n) := \int_1^n [\ell^2(x)/x] dx < \infty$ for all $n \in \mathbb{N}$ without loss of generality.[2]

It may be shown that $Var(x_n) \sim C \cdot L(n)$ under **LL1** and **LL2**, where $L(n)$ is itself an SV function. The divergence or convergence of $L(n)$ effectively demarcates the boundary between the WNPs considered in this paper, and stationary linear processes. Either is possible: e.g. $\ell(n) = 1$ gives $L(n) \sim \ln n$ and $\ell(n) = (\ln n)^{-1/2}$ gives $L(n) \sim \ln \ln n$, whereas $\ell(n) = (\ln n)^{-1}$ gives a bounded $L(n)$. When $L(n)$ diverges, $L(n)^{-1/2} x_n$ will obey a CLT (under **Assumption INN**) – and x_n will thus satisfy **Assumption HL2** above. But when $L(n)$ is bounded, no CLT applies: and indeed in this case we have under **LL1** that $\sum_{j=0}^{\infty} \phi_j^2 < \infty$, so that x_t is stationary.

(c) The requirements (in **LL3** and **LL4**) that κ_n be regularly varying and $\sup_{n \geq 1, 1 \leq t \leq n} \kappa_t / \kappa_n < \infty$ are helpful in unifying notation and simplifying some derivations – most particularly in the context of verifying **Assumption HL3**. They can be dispensed with at the expense of a considerably more involved exposition.

Proposition 2.2 next shows that under **Assumption LL** and **Assumption INN**, the WNPs specified by (16) satisfy **Assumption LP**.

[2]This follows the fact that a SV function is locally bounded on $[x_0, \infty)$ for any x_0 sufficiently large: see Lemma 1.3.2 in Bingham et al. (1987).

Proposition 2.2 (LL $\Rightarrow$ LP) *Suppose that **Assumption INN** and **Assumption LL** hold. Then **LP3**, **LP5**, **LP6** and **LP7** hold. If $\theta = 1$ then **LP4** also holds.*

Our main low-level result is essentially a direct consequence of **Theorem 2.1** and **Propositions 2.1-2.2**. Recall that $\varphi_{\sigma^2}(x)$ denotes the $N[0, \sigma^2]$ density, and $\beta_n^2 = Var(x_n(n))$.

Theorem 2.2 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is locally Lebesgue integrable, and **Assumptions INN** and **LL** hold. Further suppose:*

*(i) Either: (a) for each $t' \in \mathbb{N}$ fixed, $n^{-1} \sum_{t=1}^{t'} f\left(\beta_n^{-1} x_t(n)\right) = o_p(1)$; or (b) **Assumption INN** holds with $\theta = 1$.*

(ii) For some $\lambda' \in (0, \infty)$, as $|x| \to \infty$ either

(a) $|f(x)| = O(|x|^{\lambda'})$, and $\mathbf{E}\left|\xi_1\right|^{2\vee\lambda} < \infty$ for some $\lambda > \lambda'$; or

(b) $|f(x)| = O\left(\exp\left(\lambda' |x|\right)\right)$, and ξ_1 has a finite m.g.f. in a neighborhood of zero.

Then

$$\frac{1}{n} \sum_{t=1}^{n} f(\beta_n^{-1} x_t(n)) \xrightarrow{d} \begin{cases} \int_{\mathbb{R}} f(x)\varphi_1(x)dx & \text{under } \textbf{\textit{LL(1,3,4)}} \\ \int_{\mathbb{R}} f(x + X^-)\varphi_{1/2}(x)dx & \text{under } \textbf{\textit{LL2}} \end{cases} \tag{19}$$

where $X^- \sim N[0, 1/2]$. Further, suppose that $\tilde{f} : \mathbb{R} \to \overline{\mathbb{R}}$ is such that $\tilde{f}(x) = f(x)$ almost everywhere. Then (19) holds with f replaced by $\tilde{f}$.

Remark 2.5 (a) Theorem 2.2 bridges existing asymptotic results for $I(d)$ processes of order $|d| < 1/2$ and $d \in (1/2, 3/2)$. There are similarities and differences between (19) and the limit theory that applies in these two cases. Firstly, there is some analogy with the LLN results that hold when $|d| < 1/2$. As in that case, the limit term in (19) is determined by an expectation, but with the difference that the expectation in (19) is with respect to a limiting distribution, rather than the invariant distribution of a strictly stationary process. Secondly, the limiting density (φ_1 or $\varphi_{1/2}$) is obtained via application of a CLT, and in this respect the limit theory is analogous to that for $d \in (1/2, 3/2)$, which involves the application of FCLTs. In that case, the weak limits of additive functionals are stochastic, being a functional of an fBM as in (2) above. This nondegeneracy of the limit carries over to the $I(1/2)$ type I process (**LL2**), as evinced by the presence of $X^{-} \sim N[0, 1/2]$ on the r.h.s. of (19). (The type II process can be regarded as a truncated type I process, and the additional variability of the latter appears to make its behaviour closer to that of an $I(d)$ processes with $d > 1/2$).

(b) Theorem 2.2 generalises the limit theory of Giraitis and Phillips (2006) and Phillips and Magdalinos (2007) for MI processes to general nonlinear functionals. Those two papers consider quadratic functions (i.e. $f(x) = x^2$) of MI processes driven by short memory linear processes errors. Using a direct approach they show, in particular, that $(n\beta_n^2)^{-1} \sum_{t=1}^{n} x_t\,(n)^2 \xrightarrow{p} 1$. **Theorem 2.2** provides a more general characterisation of the aforementioned limit term; indeed the result in (19) specialises to

$$\int_{\mathbb{R}} f(x)\varphi_1 dx = \int_{\mathbb{R}} x^2 \varphi_1 dx = 1$$

in this case.

(c) (19) is formulated in terms of the standardised process $\beta_n^{-1} x_t\,(n)$. The growth rate of β_n is of interest insofar as it affects the convergence rate of parametric and nonparametric regression estimators, as will be discussed in the next section. It may be shown (see **Lemma LVAR** in **Appendix A1**) that $\beta_n \sim \gamma_n^2 \mathcal{V}_\infty^2$, where

$$[\gamma_n^2, \mathcal{V}_\infty^2] := \begin{cases} [L(n), & \sigma_\xi^2 \left(\sum_{s=0}^{\infty} c_s\right)^2] & \text{under } \mathbf{LL1} \\ [L(n), & 8\sigma_\xi^2] & \text{under } \mathbf{LL2} \\ [\kappa_n, & \sigma_\xi^2 \left(\sum_{s=0}^{\infty} c_s\right)^2 /2] & \text{under } \mathbf{LL3} \\ [\kappa_n^{3-2m}, & \Gamma\left(2-2m\right) B\left(1-m, 2m-1\right)] & \text{under } \mathbf{LL4} \end{cases} \tag{20}$$

(Γ, B are Gamma and Beta functions respectively). (19) may accordingly be restated as

$$\frac{1}{n}\sum_{t=1}^{n} f(\gamma_n^{-1} x_t(n)) \xrightarrow{d} \begin{cases} \int_{\mathbb{R}} f(x)\varphi_{\mathcal{V}_\infty^2}(x)dx & \text{under } \mathbf{LL(1,3,4)} \\ \int_{\mathbb{R}} f(x+\mathcal{V}_\infty X^{-})\varphi_{\mathcal{V}_\infty^2/2}(x)dx & \text{under } \mathbf{LL2} \end{cases}$$

(see also **Lemma CLT** in **Appendix A1**).

(d) The existence of moments of $\{\xi_j\}$ under **Assumption INN**, ensures that with appropriate scaling $x_{\lfloor nr \rfloor}(n)$ satisfies a CLT. As a result the limiting distributions that appear in (19) are normal. Some preliminary work of the authors' shows that **Theorem 2.2** can be extended to the case where $\left\{\xi_j\right\}$ is in the domain of attraction of an α-stable law with parameter $0 < \alpha \leq 2$ (e.g. Astrauskas, 1983; Kasahara and Maejima, 1988; Jeganathan, 2004), in which case other stable distributions (rather than the normal distribution) will appear in the limit. We leave extensions of this kind for future work.

(e) Under the requirement that $\lim_{n\to\infty} n^{-1} f\left(\beta_n^{-1} z_n\right) = 0$ for any finite convergent sequence $\{z_n\}$, condition (i.a) of **Theorem 2.2** is satisfied. For example, suppose **Assumption INN** and **LL3** or **LL4** hold. Then

$$x_t(n) = \sum_{j=0}^{t-1} \rho_n^j(n)\, v_{t-j} \overset{a.s.}{\to} \sum_{j=0}^{t-1} v_{t-j}$$

where the r.h.s. is a.s. finite. Hence condition (i.a) holds under the stated requirement on f.

3 Kernel functionals of WNPs

We now turn to the limiting behaviour of kernel functionals of the form

$$\frac{\beta_n}{h_n n} \sum_{t=1}^{n} K\left(\frac{x_t(n) - x}{h_n}\right), \tag{21}$$

where $x \in \mathbb{R}$, h_n is a bandwidth sequence, and K is an integrable kernel function satisfying

Assumption K (kernel). $K : \mathbb{R} \to \overline{\mathbb{R}}$ is such that K and K^2 are Lebesgue integrable.

Whereas in (7) the nonlinear transformation f is applied to the standardised process $\beta_n^{-1} x_t(n)$, in (21) K is applied to the unstandardised process $x_t(n)$. This leads to a different limit theory, which is partly reflected in the different normalisations of the sums in (7) and (21). Nonetheless, the results of the preceding section turn out to be highly relevant for the asymptotics of (21). To explain why this is the case, we return to the setting of **Assumption HL** above, which we recall refers to a general triangular array $\{X_t(n)\}$ that is not required to conform to a specific time series model. We shall augment this assumption by the following additional smoothness conditions on the density of the increments of $X_t(n)$. To state these, let

$$\Omega_n(\eta) := \left\{\{s,t\} \in \mathbb{N}^2 : \lfloor \eta n \rfloor \leq s \leq \lfloor 1-\eta \rfloor n,\ \lfloor \eta n \rfloor + s \leq t \leq n\right\},$$

for $\eta \in (0,1)$.

HL7 *Let $X_0(n) = 0$ and $t > s \geq 0$. Conditionally on $\mathcal{F}_s$, $\beta_{t-s}^{-1}(X_t(n) - X_s(n))$ has density $\mathcal{D}_{t,s,n}(x)$ such that for some $n_0, t_0 \geq 1$*

$$\sup_{n \geq n_0, 0 \leq s < t \leq n, t-s \geq t_0} \sup_x \mathcal{D}_{t,s,n}(x) < \infty.$$

HL8 *For all $q_0, q_1 > 0$*

$$\lim_{\eta \downarrow 0} \lim \sup_{n \to \infty} \sup_{(s,t) \in \Omega_n(\eta)} \sup_{|x| \leq q_0 \eta^{q_1}} |\mathcal{D}_{t,s,n}(x) - \mathcal{D}_{t,s,n}(0)| = 0$$

HL9 *For t_0 as in **HL7**:*

(a) $\lim_{\eta \to 0} \lim_{n \to \infty} \frac{\beta_n}{n} \sum_{t=t_0}^{\lfloor \eta n \rfloor} \beta_t^{-1} = 0$;

(b) $\lim_{\eta \to 0} \lim_{n \to \infty} \frac{\beta_n}{n} \sum_{t=\lfloor (1-\eta) n \rfloor}^{n} \beta_t^{-1} = 0$;

(c) $\lim_{\eta \to 0} \lim_{n \to \infty} \frac{\beta_n}{n} \sup_{0 \leq s \leq (1-\eta)n} \sum_{t=s+t_0}^{s+\lfloor \eta n \rfloor} \beta_{t-s}^{-1} = 0$;

(d) $\limsup_{n \to \infty} \frac{\beta_n}{n} \sup_{0 \leq s \leq n-1} \sum_{t=s+t_0}^{n} \beta_{t-s}^{-1} < \infty$;

(e) *for each $\eta \in (0,1)$, there exist $l_0 > 0$ and $l_1 \in (0,1)$ such that* $\liminf_{n \to \infty} \beta_n^{-1} \inf_{(s,t) \in \Omega_n(\eta)} \beta_{t-s}^{-1} \geq \eta^{l_1}/l_0$.

Conditions **HL7** and **HL9** may be regarded as strengthened versions of **HL3** and **HL6**, and are closely related to Assumptions 2.3 in Wang and Phillips (2009a). Indeed, under these conditions, an L_1-approximation argument developed by those authors and Jeganathan (2004) yields that, for t_0 as in **HL7**,

$$\frac{\beta_n}{h_n n} \sum_{t=t_0}^{n} K\left(\frac{X_t(n) - x}{h_n}\right) = \frac{1}{n} \sum_{t=t_0}^{n} \varphi_{\varepsilon^2}(\beta_n^{-1} X_t(n)) \int_{\mathbb{R}} K(u) du + o_p(1)$$

as $n \to \infty$ and then $\varepsilon \to 0$ (see **Lemma 5.3** below); recall that φ_{σ^2} denotes the density of a normal random variable with mean zero and variance σ^2. The leading order term on the r.h.s. clearly has the same form as the l.h.s. of (14) and is thus is amenable to a direct application of **Theorem 2.1** – which under **HL0–HL4** and **HL6** entails

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^{n} \varphi_{\varepsilon^2}(\beta_n^{-1} X_t(n)) &\xrightarrow{d} \int \varphi_{\varepsilon^2}(x + X^-) \Phi_{X^+}(x) dx, \quad \text{as } n \to \infty \\ &\xrightarrow{p} \Phi_{X^+}(-X^-), \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{22}$$

by dominated convergence. We thus have the following high-level result, which is the counterpart of **Theorem 2.1** for kernel functionals.

Theorem 3.1 *Suppose that, in addition to **Assumptions K**, **HL0–HL2**, **HL4** and **HL6–HL9**, the following hold:*

(i) $\{h_n\}$ *is a positive sequence with* $\beta_n^{-1}h_n + \beta_n(nh_n)^{-1} \to 0$*; and*

(ii) for each $x \in \mathbb{R}$ *and* t_0 *as in **HL7**,* $\frac{\beta_n}{nh_n}\sum_{t=1}^{t_0-1} K\left(\frac{X_t(n)-x}{h_n}\right) = o_p(1)$.

Then

$$\frac{\beta_n}{h_n n}\sum_{t=1}^{n} K\left(\frac{X_t(n)-x}{h_n}\right) \xrightarrow{d} \Phi_{X^+}(-X^-)\int_{\mathbb{R}} K(u)du.$$

To apply the preceding to the processes covered by **Assumption LL** above, it is simply a matter of verifying that these also satisfy **HL7–HL9** – the other requirements of **Assumption HL** having already been verified by **Propositions 2.1** and **2.2**. This leads to the following counterpart of **Theorem 2.2**.

Theorem 3.2 *Suppose that, in addition to **Assumption K**:*

(i) $x_t(n)$ *is given by (16), and satisfies **Assumptions INN and LL**;*

(ii) $\{h_n\}$ *satisfies condition (i) of **Theorem 3.1**; and*

(iii) for each $x \in \mathbb{R}$ *and* $t' \in \mathbb{N}$, $\frac{\beta_n}{nh_n}\sum_{t=1}^{t'} K\left(\frac{x_t(n)-x}{h_n}\right) = o_p(1)$.

Then

$$\frac{\beta_n}{h_n n}\sum_{t=1}^{n} K\left(\frac{x_t(n)-x}{h_n}\right) \xrightarrow{d} \int_{\mathbb{R}} K(u)du \cdot \begin{cases} \varphi_1(0) & \text{under } \textbf{\textit{LL(1,3,4)}} \\ \varphi_{1/2}(X^-) & \text{under } \textbf{\textit{LL2}} \end{cases}$$

where $X^- \sim N[0, 1/2]$.

Remark 3.1 (a) Theorem 3.2 fills the gap in existing asymptotic theory for kernel functionals of linear processes. A general theory for stationary linear processes, including $I(d)$ processes with $0 \leq d < 1/2$, is given in Wu and Mielniczuk (2002). Supposing that $\int_{\mathbb{R}} K = 1$ for simplicity, under their conditions kernel functionals converge to the invariant density of the stationary linear process. Jeganathan (2004, 2008) provides limit theorems for kernel functionals of $I(d)$ processes with $1/2 < d < 3/2$. In that case, kernel functionals converge to the local time of a fractional Brownian motion – so their limit is an occupation density rather than the invariant density of some strictly stationary process. The limiting behaviour of kernel functionals of $I(1/2)$ processes is intermediate between these two cases. These converge to the density of a random variable, rather than to an occupation density, but the density corresponds to a limiting random variate, rather than the invariant density of a stationary process.

(b) Under **Assumption INN**, **Assumption LL** and some additional requirements on $x_t(n)$, similar to **LP4** (see also Wang and Phillips 2009a), it is possible to show that **HL7** and **HL9** hold with $t_0 = 1$. In this case, condition (iii) of **Theorem 3.2** is redundant. We do not pursue extensions of this kind here, in order to keep the paper to a manageable length.

(c) **Theorem 3.2** nests a similar result provided by Duffy (2017) for bounded kernel functionals of MI processes driven by short memory errors (**Assumption LL3**), which unlike **Assumption K** requires K to be bounded and Lipschitz continuous.

4 Application to regression estimators

The limit theory developed in the preceding sections is fundamental to the asymptotics of both parametric and nonparametric least squares estimators, in models involving WNPs as regressors. To illustrate some of the potential applications of our results, in this section we consider: (a) parametric estimation of $\beta \in \mathbb{R}$ in the model

$$y_t = \beta g(x_{t-1}(n)) + u_t \tag{23}$$

where g is a known nonlinear transformation, using the least squares (LS) estimator

$$\hat{\beta} = \frac{\sum_{t=2}^{n} g(x_{t-1}(n)) y_t}{\sum_{t=2}^{n} g^2(x_{t-1}(n))} \tag{24}$$

(**Section 4.1**); and (b) nonparametric estimation of the unknown function g in the model

$$y_t = g(x_{t-1}(n)) + u_t \tag{25}$$

by the kernel regression (Nadaraya–Watson; NW) estimator

$$\hat{g}(x) = \frac{\sum_{t=2}^{n} K[(x_{t-1}(n) - x)/h_n] y_t}{\sum_{t=2}^{n} K[(x_{t-1}(n) - x)/h_n]}$$

(**Section 4.2**). In both (23) and (25) u_t is assumed to be m.d.s. with respect to the $\{\mathcal{F}_t\}$, the natural filtration for $\{(u_t, \xi_t)\}$, and $x_t(n)$ an $I(1/2)$ or MI process (as per **Assumption LL**). [3]

Since the regressor in (23) and (25) is predetermined (i.e. $\mathcal{F}_{t-1}$-measurable) relative to the error u_t, both are instances of so-called 'predictive' or 'reduced form' regression models. Correlation between $x_t(n)$ and u_t is not precluded here: such correlation does not affect the consistency of either the LS or NW estimators – and if $x_t(n)$ is stationary, both estimators will be asymptotically normal. If $x_t(n)$ is nonstationary – being e.g. $I(d)$ with $1/2 < d < 3/2$ or near integrated – then the NW estimator is mixed normal, and standard methods of inference remain valid (see Wang and Phillips, 2009a, 2012). However, the results of Park and Phillips (1999, 2001) show that the *parametric* LS estimator of β has a non-standard limiting distribution, unless either g is itself integrable, or $x_t(n)$ and u_t satisfy a very restrictive 'long-run orthogonality' condition. This necessitates either non-standard inferential procedures, or the use of suitably modified estimators that enjoy mixed normal limiting distributions, such as are developed by Phillips (1995) and Robinson and Hualde (2003).

When $x_t(n)$ is a WNP, **Theorems 4.1** and **4.2** below respectively establish that the large-sample behaviour of both the LS and NW estimators is similar to the case where $x_t(n)$ is stationary. Both estimators are either asymptotically normal or mixed normal, ensuring the validity of conventional methods of inference (e.g. t-statistics will be asymptotically standard normal). These results in turn follow from the application of the results of the preceding sections and suitable martingale central limit theorems to quantities of the form

$$\left(\sum_{t=2}^{n} G_{n,t-1}^{j}\right)^{-1} \sum_{t=2}^{n} G_{n,t-1} u_t \tag{26}$$

upon a suitable rescaling, where: (i) for the LS estimator, $j = 2$, $G_{n,t} = H_g[\beta_n^{-1} x_t(n)]$ for H_g as in **Definition AHF** below and $\beta_n = Var(x_n(n))$; (ii) for the NW estimator, $j = 1$ and $G_{n,t} = K[(x_t(n) - x)/h_n]$. Under the assumption that $\mathbf{E}[u_t^2 \mid \mathcal{F}_{t-1}] = \sigma_u^2$ a.s., the limiting behaviour of both the

[3] The model in (23) is linear in the parameters. It would also be possible to extend our results to cover the estimation of models in which g is non-linearly parametrised, along the lines of Park and Phillips (2001) and Chan and Wang (2015).

denominator in (26) and the conditional variance of the martingale in the numerator,

$$\sum_{t=2}^{n} \mathbf{E}\{[G_{n,t-1}u_t] \mid \mathcal{F}_{t-1}\} = \sigma_u^2 \sum_{t=2}^{n} G_{n,t-1}^2, \tag{27}$$

can be deduced directly from **Theorems 2.2** and **3.2**. In cases where, upon rescaling, (27) converges to a deterministic limit (as under **LL1**, **LL3** and **LL4**), Theorem 3.2 in Hall and Heyde (1980) establishes the asymptotic normality of (26); whereas when (27) converges in distribution to a random limit (as under **LL2**), it is necessary to invoke an extension of that result due to Wang (2014).

4.1 Parametric regression

The connection between the LS estimator in (24), which involves nonlinear (and generally non-integrable) transformations of the unstandardised regressor $x_t(n)$, and the limit theory developed in **Section 4.1** above, where these regressors appear standardised by $\beta_n = Var(x_n(n))$, may be established via an argument due to Park and Phillips (1999, 2001). For that purpose, we need the following

Definition AHF (asymptotically homogeneous function). Let $\{x_t(n)\}$ denote a random array and $\beta_n^2 = Var(x_n(n))$. A function $g : \mathbb{R} \to \overline{\mathbb{R}}$ is *asymptotically homogeneous* for $x_t(n)$, if for each $\lambda > 0$, g admits the decomposition

$$g(x) = \kappa_g(\lambda) H_g(x/\lambda) + R_g(x, \lambda),$$

where $\kappa_g : \mathbb{R}_+^* \to \mathbb{R}_+^*$, $H_g : \mathbb{R} \to \overline{\mathbb{R}}$, $R_g : \mathbb{R} \times \mathbb{R}_+^* \to \overline{\mathbb{R}}$ and

$$R_{g,n}^2 := \frac{1}{\kappa_g(\beta_n) n} \sum_{t=1}^{n} \mathbf{E}\left|R_g(x_t(n), \beta_n)\right|^2 = o(1). \tag{28}$$

AHFs encompass a wide range of commonly used regression functions, such as polynomial functions, cumulative distribution functions (with $H_g(u) = 1\{u > 0\}$), and logarithmic functions (with $H_g(u) = 1$); see Park and Phillips (1999, 2001) for some further examples. Such a condition as

$$\lim_{\lambda \to \infty} \kappa_g(\lambda)^{-1} \sup_x |R_g(x, \lambda)| = 0$$

is sufficient, but not necessary, for (28) to hold.

The relevance of AHFs for LS estimators can be seen from the fact that if g is an AHF with limit homogeneous component H_g, such that H_g^2 and $\{x_t(n)\}$ satisfies satisfy requirements of **Theorem 2.2**, then

$$\frac{1}{\kappa_g^2(\beta_n)n}\sum_{t=1}^{n} g^2(x_t(n)) = \frac{1}{n}\sum_{t=1}^{n} H_g^2(\beta_n^{-1}x_t(n)) + o_p(1)$$
$$\xrightarrow{d} \begin{cases} \int_{\mathbb{R}} H_g^2(x)\varphi_1(x)dx & \text{under } \mathbf{LL(1,3,4)} \\ \int_{\mathbb{R}} H_g^2(x+X^-)\varphi_{1/2}(x)dx & \text{under } \mathbf{LL2} \end{cases} \tag{29}$$

where $X^- \sim N[0,1/2]$, by (28) and the Cauchy-Schwarz inequality, and then **Theorem 2.2**. Reasoning along these lines thus yields our main result on parametric LS estimators. To state it, let $MN[0,\varsigma^2]$ denote a mixed normal distribution with mixing variate ς^2 (i.e. which has characteristic function $\mu \mapsto \mathbf{E}e^{-\varsigma^2\mu^2/2}$).

Theorem 4.1 *Let $\{y_t\}_{t=1}^n$ be generated by (23), $\mathcal{F}_t := \sigma(\{\xi_s, u_s\}_{s\leq t})$, and suppose that:*

*(i) $x_t(n)$ satisfies (16) and **Assumptions INN** and **LL**;*

(ii) $\{u_t,\mathcal{F}_t\}_{t\geq 1}$ is a martingale difference sequence such that $\mathbf{E}[u_t^2 \mid \mathcal{F}_{t-1}] = \sigma_u^2$ a.s. for some constant $\sigma_u^2 \in \mathbb{R}_+^*$;

(iii) $\sup_{1\leq t\leq n}\mathbf{E}[u_t^2 1\{|u_t| \geq A_n\} \mid \mathcal{F}_{t-1}] = o_p(1)$, for all deterministic $0 < A_n \to \infty$; and

*(iv) $g(x)$ is AHF for the array $\{x_t(n)\}$, with limit homogeneous component H_g such that H_g^2 satisfies the requirements of **Theorem 2.2**.*

Then $\kappa_g(\beta_n)\sqrt{n}(\hat{\beta}_{LS} - \beta) \xrightarrow{d} MN[0,\varsigma^2]$, where

$$\varsigma^2 = \sigma_u^2 \cdot \begin{cases} \left[\int_{\mathbb{R}} H_g^2(x)\varphi_1(x)dx\right]^{-1} & \textit{under } \textbf{\textit{LL(1,3,4)}}, \\ \left[\int_{\mathbb{R}} H_g^2(x+X^-)\varphi_{1/2}(x)dx\right]^{-1} & \textit{under } \textbf{\textit{LL2}}; \end{cases} \tag{30}$$

where $X^- \sim N[0,1/2]$.

Remark 4.1 (a) For $I(1/2)$ type II and MI processes $\hat{\beta}_n$ is asymptotically normal; for $I(1/2)$ type I processes it is merely mixed normal, because of the limiting variate X^-. In either case, the associated t statistic will be asymptotically standard normal.

(b) In a linear regression model, i.e. $g(x) = x$, we may take $H_g(u) = u$ and $\kappa_g(\lambda) = \lambda$, and it follows that the LS estimator has convergence rate $\beta_n\sqrt{n}$, which is faster than the $\sqrt{n}$-convergence rate that obtains when the regressor is stationary. For $I(1/2)$ processes the gain in convergence rate is given by the slowly varying factor $L(n)$ (see **Remark 2.4(b)**).

(c) If $g(x) = F(x)$ for some cumulative distribution function F, then $H_g = 1\{x > 0\}$ as noted, and so

$$\int_{\mathbb{R}} H_g^2(x)\varphi_1(x)dx = \int_{\mathbb{R}} 1\{x > 0\}\varphi_1(x)dx = 1/2.$$

If $g(x) = \ln|x|$, then $H_g = 1$ and so in all cases the r.h.s. of (30) is $N[0, \sigma_u^2]$.

4.2 Kernel regression

We conclude with the asymptotics of the NW estimator. As noted above, fundamental limit theorems for kernel functionals of nonstationary fractional processes with $d > 1/2$ can be found in Jeganathan (2004, 2008; see also Borodin and Ibragimov, 1995 and the references therein for some earlier results). The following theorem is a direct consequence of **Theorem 3.2** and certain martingale central limit theorems, and is complementary to the recent work of Wang and Phillips (2009a,b; 2012) who develop estimation and testing procedures in the context of nonparametric regression with fractional $d > 1/2$ processes.

Theorem 4.2 *Let $\{y_t\}_{t=1}^n$ be generated by (25) and suppose that:*

*(i) conditions (i)–(iii) of **Theorem 4.1** hold;*

*(ii) K and K^2 satisfy **Assumption K**, with $\int_{\mathbb{R}} K(u)du = 1$, and $h_n + \beta_n/nh_n \to 0$;*

(iii) for each $t' \in \mathbb{N}$ and $x \in \mathbb{R}$, $\frac{\beta_n}{nh_n}\sum_{t=1}^{t'} K[(x_t(n) - x)/h_n] = o_p(1)$; and

(iv) there is a real function $\overline{g}(x, z)$ such that $|g(x + hz) - g(x)| \leq h^\mu \overline{g}(x, z)$ for some $\mu \in (0, 1]$, for all h sufficiently small, and $\int_{\mathbb{R}} \overline{g}(x, z)|K(z)|dz < \infty$ for every $x \in \mathbb{R}$.

Then $\hat{g}(x) \xrightarrow{p} g(x)$. *If in addition* $nh_n^{1+2\mu}/\beta_n \to 0$, *then*

$$\sqrt{\frac{nh_n}{\beta_n}}\left(\hat{g}(x)-g(x)\right) \xrightarrow{d} MN\left[0, \frac{\sigma_u^2}{\varsigma^2 \int_{\mathbb{R}} K^2(u)du}\right],$$

where $\varsigma^2 = \varphi_1(0)$ *under* ***LL1***, ***LL3*** *and* ***LL4***, *and* $\varsigma^2 = \varphi_{1/2}(X^-)$ *for* $X^- \sim N[0,1/2]$ *under* ***LL2***.

Remark 4.2 (a) Condition (iv) of **Theorem 4.2** is a smoothness requirement on the regression function g that is also utilised by Wang and Phillips (2009a,b).

(b) Notice that because $\beta_n \to \infty$, the convergence rate is slower than that for stationary processes. In particular, for $I(1/2)$ processes the reduction in the convergence speed is given by a slowly varying factor.

5 Proofs of main results

This Section provides proofs of our main theorems under high level conditions (i.e. **Theorems 2.1** and **3.1**). (The proofs of the remaining results are given in **Appendices A**, **B** and **C**.) Throughout $C \in (0,\infty)$ denotes a generic constant which may take different values at each appearance. We first state three technical lemmas.

Lemma 5.1 *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is locally integrable, and let* $\varepsilon, \eta > 0$. *Then there is a Lipschitz continuous* $f_{\varepsilon,\eta}$ *such that* $\int_{|x|\le\eta} |f(x) - f_{\varepsilon,\eta}(x)|\, dx < \varepsilon$ *and* $f_{\varepsilon,\eta}(x) = 0$ *for* $|x| > \eta$.

Lemma 5.2 *Let* X_n *and* Y_n *be real valued random sequences on some probability space* $(\Omega, \mathcal{A}, \mathbf{P})$ *and* $\mathcal{F} \subset \mathcal{A}$ *a* σ*-field, for which*

(i) $X_n \xrightarrow{d} X \sim F_X$, *conditionally on* $\mathcal{F}$, *in the sense that* $\mathbf{E}\left(h(X_n) \mid \mathcal{F}\right) \xrightarrow{p} \int_{\mathbb{R}} h(x)dF_X(x)$ *for all* $h : \mathbb{R} \to \mathbb{R}$ *bounded and continuous; and*

(ii) $Y_n \xrightarrow{d} Y$, *where* Y_n *is* $\mathcal{F}$*-measurable for each* n.

Then for all $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *bounded and Lipschitz continuous,*

$$\mathbf{E}\left(g(X_n, Y_n) \mid \mathcal{F}\right) \xrightarrow{d} \int_{\mathbb{R}} g(x, Y)dF_X(x).$$

Lemma 5.3 *Suppose that:*

(i) ***HL0****, and* ***HL7-HL9*** *hold;*

(ii) K *satisfies* ***Assumption K****; and*

(iii) $\{h_n\}$ *is such that* $\beta_n^{-1}h_n + \beta_n(nh_n)^{-1} \to 0$.

Then for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \mathbf{E}\left|\frac{\beta_n}{h_n n}\sum_{t=t_0}^{n} K\left(\frac{X_t(n)-x}{h_n}\right) - \frac{1}{n}\sum_{t=t_0}^{n}\varphi_{\varepsilon^2}\left(\beta_n^{-1}X_t(n)\right)\right| = 0. \quad (31)$$

The proof of **Lemma 5.1** follows from Theorem 2.26 in Folland (1999), while the proof **Lemma 5.2** is given in **Appendix A3**. Finally, **Lemma 5.3** follows from Lemma 7 in Jeganathan (2004) and arguments similar to those used in Wang and Phillips (2009a, pp. 725-728).

Proof of Theorem 2.1. Let $\varepsilon > 0$. By **Lemma 5.1** for each $\varepsilon > 0$ there is a Lipschitz continuous $f_\varepsilon(x)$ such that $\int_{|x|\leq\varepsilon^{-1}} |f(x) - f_\varepsilon(x)|\, dx < \varepsilon$ and $f_\varepsilon(x) = 0$ for $|x| > \varepsilon^{-1}$. We shall prove that:

$$\frac{1}{n}\sum_{t=1}^{n} f\left(\beta_n^{-1}X_t(n)\right) = \frac{1}{n}\sum_{t=1}^{n} f_\varepsilon\left(\beta_n^{-1}X_t(n)\right) + o_p(1), \quad (32)$$

as $n \to \infty$ then $\varepsilon \to 0$; that for each $\varepsilon > 0$

$$\frac{1}{n}\sum_{t=1}^{n} f_\varepsilon\left(\beta_n^{-1}X_t(n)\right) \xrightarrow{d} \int_{\mathbb{R}} f_\varepsilon\left(x + X^-\right)\Phi_{X^+}(x)dx, \quad (33)$$

as $n \to \infty$; and that

$$\int_{\mathbb{R}} f_\varepsilon\left(x + X^-\right)\Phi_{X^+}(x)dx \xrightarrow{a.s.} \int_{\mathbb{R}} f\left(x + X^-\right)\Phi_{X^+}(x)dx. \quad (34)$$

as $\varepsilon \to 0$, where the integral on the r.h.s. exists a.s. due to condition (i) of **Theorem 2.1**. In view of (32)-(34), (14) then follows from Theorem 4.2 in Billingsley (1968). The final part of the theorem (for $\tilde{f} : \mathbb{R} \to \overline{\mathbb{R}}$ with $\tilde{f} = f$ a.e.) in turn follows straightforwardly from each $X_t(n)$ having a density (as per **HL3**).

Proof of (32). We prove (32) under condition (iii.a) of **Theorem 2.1**. The proof under (iii.b) is identical and therefore omitted. By condition (iii.a) we may choose $\varepsilon > 0$ sufficiently small that $|f(x)| \leq C|x|^{\lambda'}$ for all $|x| \geq \varepsilon^{-1}$. Decompose

$$f(x) = f(x) 1\left\{|x| \leq \varepsilon^{-1}\right\} + f(x) 1\left\{|x| > \varepsilon^{-1}\right\} =: f_{1,\varepsilon}(x) + f_{2,\varepsilon}(x),$$

where $\int_{|x|\leq\varepsilon^{-1}} |f_{1,\varepsilon}(x) - f_\varepsilon(x)| \, dx < \varepsilon$, and $|f_{2,\varepsilon}(x)| \leq C|x|^{\lambda'} 1\{|x| > \varepsilon^{-1}\}$. Letting $\tilde{X}_t(n) := \beta_n^{-1} X_t(n)$, in view of condition (ii) of **Theorem 2.1**, (32) will follow once we have shown that

$$\frac{1}{n}\sum_{t=t_0}^{n} \mathbf{E}|f_{1,\varepsilon}(\tilde{X}_t(n)) - f_\varepsilon(\tilde{X}_t(n))| + \frac{1}{n}\sum_{t=1}^{n} \mathbf{E}|f_{2,\varepsilon}(\tilde{X}_t(n))| \to 0, \tag{35}$$

as $n \to \infty$ and then $\varepsilon \to 0$, for t_0 as in condition (ii).

To that end, note that by **HL5(a)** and condition (iii.a) of **Theorem 2.1**, there is an $n_0 \geq 1$ and a $\lambda > \lambda'$ such that $\sup_{n\geq n_0, 1\leq t\leq n} \mathbf{E}|\tilde{X}_t(n)|^\lambda < \infty$. Hence $|\tilde{X}_t(n)|^{\lambda'}$ is uniformly integrable, and so for $n \geq n_0$

$$\frac{1}{n}\sum_{t=1}^{n} \mathbf{E}|f_{2,\varepsilon}(\tilde{X}_t(n))| \leq \sup_{n\geq n_0, 1\leq t\leq n} \mathbf{E}|\tilde{X}_t(n)|^{\lambda'} 1\{|\tilde{X}_t(n)| > \varepsilon^{-1}\} \to 0,$$

as $\varepsilon \to 0$. This gives the required negligibility of the second l.h.s. term in (35). For the first l.h.s. term, we note that by **HL3** that

$$\begin{aligned}
&\frac{1}{n}\sum_{t=t_0}^{n} \mathbf{E}|f_{1,\varepsilon}(\tilde{X}_t(n)) - f_\varepsilon(\tilde{X}_t(n))| \\
&\qquad = \frac{1}{n}\sum_{t=t_0}^{n} \int_{\mathbb{R}} \left| f_{1,\varepsilon}\left(\frac{\beta_t}{\beta_n}x\right) - f_\varepsilon\left(\frac{\beta_t}{\beta_n}x\right) \right| \mathcal{D}_{n,t}(x) \, dx \\
&\qquad \leq \sup_{n\geq n_0, t_0\leq t\leq n} \sup_x \mathcal{D}_{n,t}(x) \int_{\mathbb{R}} |f_{1,\varepsilon}(x) - f_\varepsilon(x)| dx \cdot \frac{\beta_n}{n}\sum_{t=t_0}^{n} \beta_t^{-1} \\
&\qquad \leq C \int_{|x|\leq\varepsilon^{-1}} |f_{1,\varepsilon}(x) - f_\varepsilon(x)| \, dx \\
&\qquad \leq C\varepsilon \to 0
\end{aligned}$$

as $\varepsilon \to 0$, where the second inequality holds by **HL6**, for n sufficiently large.

Proof of (33). Let $\mathbf{E}_0(.) := \mathbf{E}\,(.\mid \mathcal{F}_0)$, and $\delta \in (0,1)$. By **HL4** and the boundedness of f_ε, we have

$$\frac{1}{n}\sum_{t=1}^{n} f_\varepsilon(\tilde{X}_t(n)) = \frac{1}{n}\sum_{t=\lfloor n\delta\rfloor+1}^{n} \mathbf{E}_0[f_\varepsilon(\tilde{X}_t(n))] + o_p(1) \tag{36}$$

as $n \to \infty$ and then $\delta \to 0$. Now let $\tilde{X}_t^+(n) := \beta_n^{-1}X_t^+(n)$ and $\tilde{X}_t^-(n) := \beta_n^{-1}X_t^-(n)$. Since f_ε is bounded and Lipschitz, it follows from **HL1** that

$$\begin{aligned}\frac{1}{n}\sum_{t=\lfloor n\delta\rfloor+1}^{n} \mathbf{E}\left|\mathbf{E}_0[f_\varepsilon(\tilde{X}_t(n))] - \mathbf{E}_0[f_\varepsilon(\tilde{X}_t^+(n)+\tilde{X}_t^-(n))]\right| \\ \leq C\sup_{1\leq t\leq n}\mathbf{E}(\beta_n^{-1}|R_t(n)|\wedge 1) \to 0 \end{aligned}\tag{37}$$

as $n \to \infty$ for each $\delta \in (0,1)$; and from **HL2(b)** that

$$\begin{aligned}\frac{1}{n}\sum_{t=\lfloor n\delta\rfloor+1}^{n} \mathbf{E}\left|\mathbf{E}_0[f_\varepsilon(\tilde{X}_t^+(n)+\tilde{X}_t^-(n))] - \mathbf{E}_0[f_\varepsilon(\tilde{X}_t^+(n)+\tilde{X}_n^-(n))]\right| \\ \leq C\sup_{\lfloor n\delta\rfloor+1\leq t\leq n}\mathbf{E}(|\tilde{X}_t^-(n)-\tilde{X}_n^-(n)|\wedge 1) \\ = C\mathbf{E}(|\tilde{X}_{l_n}^-(n)-\tilde{X}_n^-(n)|\wedge 1) \to 0 \end{aligned}\tag{38}$$

as $n \to \infty$ for each $\delta \in (0,1)$, where $l_n \in \{\lfloor n\delta\rfloor+1,\ldots,n\}$ may always be chosen such that the final equality holds. Finally, by **HL2(a)**, Theorem 2.1(ii) in Billingsley (1968) and **Lemma 5.2** we have as

$$\begin{aligned}\frac{1}{n}\sum_{t=\lfloor n\delta\rfloor+1}^{n} \mathbf{E}_0\left[f_\varepsilon\left(\beta_n^{-1}\left[X_t^+(n)+X_n^-(n)\right]\right)\right] \\ \xrightarrow{d} (1-\delta)\int_{\mathbb{R}} f_\varepsilon\left(x+X^-\right)\Phi_{X^+}(x)dx \end{aligned}\tag{39}$$

as $n \to \infty$, for each $\delta \in (0,1)$. Hence (33) follows from (36)-(39) and Theorem 4.2 in Billingsley (1968).

Proof of (34). Let $y \in \mathcal{Y}$ for $\mathcal{Y}$ as in condition (i) of **Theorem 2.1**.

Noting $f_\varepsilon(x) = 0$ for $|x| > \varepsilon$, and Φ_{X^+} is bounded under **HL2,** we have

$$
\begin{aligned}
&\int |f_\varepsilon(x+y) - f(x+y)|\Phi_{X^+}(x)dx \\
&\qquad \leq \sup_u \Phi_{X^+}(u) \int_{|x|\leq\varepsilon^{-1}} |f_\varepsilon(x) - f(x)|dx + \int_{|x|>\varepsilon^{-1}} f(x)\Phi_{X^+}(x-y)dx \\
&\qquad = C\varepsilon + o(1)
\end{aligned}
$$

as $\varepsilon \to 0$, where the negligibility of the second r.h.s. term follows by condition (i) of **Theorem 2.1**. Noting that $\mathbf{P}\{X^- \in \mathcal{Y}\} = 1$ completes the proof. ■

Proof of Theorem 3.1 By **Lemma 5.3** and condition (ii) of the theorem,

$$
\frac{\beta_n}{h_n n} \sum_{t=1}^{n} K\left(\frac{x_t(n) - x}{h_n}\right) = \frac{1}{n} \sum_{t=1}^{n} \varphi_{\varepsilon^2}(\beta_n^{-1} X_t(n)) + o_p(1)
$$

as $n \to \infty$ and then $\varepsilon \to 0$. Noting that **HL7** implies **HL3**, we have by **Theorem 2.1** that

$$
\frac{1}{n} \sum_{t=1}^{n} \varphi_{\varepsilon^2}(\beta_n^{-1} X_t(n)) \xrightarrow{d} \int_{\mathbb{R}} \varphi_{\varepsilon^2}(x + X^-)\Phi_{X^+}(x)dx,
$$

as $n \to \infty$. Finally, as argued in (22) above,

$$
\int_{\mathbb{R}} \varphi_{\varepsilon^2}(x + X^-)\Phi_{X^+}(x)dx \to \Phi_{X^+}(-X^-)
$$

as $\varepsilon \to 0$. The result then follows by Theorem 4.2 in Billingsley (1968). ■

6 Appendix A:

Appendix A provides proofs for the rest of results of Section 2 (see **Appendix A2**), and a proof for **Lemma 5.2** (see **Appendix A3**). **Appendix A1** states some useful lemmas required for the aforementioned proofs. The proofs of these lemmas are given in **Appendix B**. Recall that throughout $C \in (0, \infty)$ denotes a generic constant which may take different values at each appearance.

Appendix A1a (technical lemmas)

Lemma 6.1 *Let $\xi \sim (0, \sigma^2)$ and $i = \sqrt{-1}$. There is some $0 < \rho_* < \infty$ such that*

$$\left|\mathbf{E}\left(e^{i\xi\lambda}\right)\right| \leq e^{-\rho_*\left(\lambda^2 \wedge 1\right)}.$$

We introduce some notation that is utilised in the subsequent lemma. Set

$$\begin{aligned} \left\{\overline{q}_t, \underline{q}_t, \eta_t\right\} &:= \begin{cases} \left\{t, \lfloor L(t) \rfloor, \sqrt{t}/\ell(t)\right\}, \text{ under } \mathbf{LL1\text{-}LL2} \\ \left\{\lfloor \kappa_t \rfloor, \ \lfloor \kappa_t/2 \rfloor, 1/\sqrt{2}\right\}, \text{ under } \mathbf{LL3\text{-}LL4} \end{cases} \\ \delta_t &:= \begin{cases} 1, \text{ under } \mathbf{LL1\text{-}LL3} \\ \kappa_t^{1-m}, \text{ under } \mathbf{LL4} \end{cases} \end{aligned} \tag{40}$$

Lemma 6.2 *Suppose that **Assumption LL** holds. There are $n_0, n_1 > 0$ and constants $0 < D_1 \leq D_2 < \infty$ such that for all $n \geq n_0$ and $n_1 \leq t \leq n$ the following hold*

(i) $D_1 \leq \delta_t^{-1} \inf_{\underline{q}_t + \theta \leq k \leq \overline{q}_t} \eta_k \left|a_k(n)\right| \leq \delta_t^{-1} \sup_{0 \leq k \leq \overline{q}_t} \eta_k \left|a_k(n)\right| \leq D_2$, *for all $\theta \in \mathbb{N}$.*

(ii) $\delta_n^{-1} \sup_{0 \leq k \leq n} \eta_k \left|a_k(n)\right| \leq D_2$.

(iii) $\lim_{n\to\infty} \gamma_n^{-1} \sup_{1 \leq t \leq n, k \geq t} \left|a_{k,t}^-(n)\right| = 0$.

Remark A1. Note that **Lemma 6.2(ii,iii)** implies $\lim_{n\to\infty} \gamma_n^{-1} \sup_{1 \leq t \leq n, k \geq 0} \left|a_{k,t}(n)\right| = 0$.

Lemma 6.3 *Suppose that $\varphi_j \in \mathbb{R}$ for all $j \in \mathbb{N}$, ς is SV (in the sense of Bingham et al., 1987; p. 6) and $x_0 \in \mathbb{N}$ is such that $\varsigma(x)$ is locally bounded for all $x \geq x_0$. The following hold:*

(i) Suppose that $\varphi_j \sim j^l \varsigma(j)$, where $l > -1$. Then for all $0 \leq s < r < \infty$ as $n \to \infty$

$$\frac{1}{n^{1+l}\varsigma(n)} \sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nr \rfloor} \varphi_j \to \int_s^r x^l dx.$$

(ii) Suppose that $\varphi_j \sim j^l \varsigma(j)$, where $l < -1$. Then as $n \to \infty$

$$\frac{1}{n^{1+l}\varsigma(n)} \sum_{j=n}^{\infty} \varphi_j \to \int_1^{\infty} x^l dx.$$

(iii) *Suppose that* $S(x) := \int_{x_0}^{x} \varsigma(u)/udu$ *is such that* $S(x) \to \infty$ *as* $x \to \infty$. *Then as* $n \to \infty$

$$\sum_{j=x_0}^{n} \frac{\varsigma(j)}{j} \sim S(n).$$

Lemma 6.4 *Suppose that* $\rho_n = 1 + \frac{c}{\kappa_n}$ *with* $c < 0$ *and* $q \in \mathbb{N}$.

(i) *For* n *large*

$$(a) \quad \left| \rho_n^{qj} - \exp\left\{ qj \frac{c}{\kappa_n} \right\} \right| \leq 2 \exp\left\{ \frac{c}{2\kappa_n} qj \right\} qj \left(\frac{c}{\kappa_n} \right)^2.$$

$$(b) \quad \sup_{1 \leq j \leq \tau_n} \left| \rho_n^{qj} - \exp\left\{ qj \frac{c}{\kappa_n} \right\} \right| = O\left(\frac{\tau_n}{\kappa_n^2} \right)$$

(ii) *For* $\kappa_n/\lambda_n \to 0$ *and all* $0 \leq \delta < |c|$ *we have*

$$\kappa_n e^{q\delta \frac{\lambda_n}{\kappa_n}} \left| \rho_n^{q\lambda_n} - \exp\left\{ c \frac{q\lambda_n}{\kappa_n} \right\} \right| = o(1).$$

(iii) *For* $k_n > \tau_n \geq 0$, *we have*

$$\sum_{t=1}^{k_n} \rho_n^{qt} - \sum_{t=1}^{\tau_n} \rho_n^{qt} = O\left(\kappa_n \rho_n^{q\tau_n} \right).$$

If in addition $\tau_n^{-1} \kappa_n \ln(\kappa_n) \to 0$, *then* $\kappa_n \rho_n^{q\tau_n} = o(1)$.

Appendix A1b (intermediate lemmas)

Lemma UBD (uniformly bounded density) *Suppose that **Assumption INN**, **LP1**, and **LP3** with* $\beta_n \to \infty$ *as* $n \to \infty$.
(i) *The following hold:*
(a) $\beta_t^{-1} x_t(n)$ *possesses density* $\mathcal{D}_{t,n}(x)$ *such that for some* $n_0, t_0 \in \mathbb{N}$,

$$\sup_{n \geq n_0, t_0 \leq t \leq n} \sup_{x} \mathcal{D}_{t,n}(x) < \infty;$$

(b) *If in addition **LP4** holds, then the above holds with* $t_0 = 1$.
(ii) *Let* $i = \sqrt{-1}$ *and* $\psi_{t,n}(\lambda) := \mathbf{E} \exp(i\lambda x_t(n)/\beta_t)$. *The following hold:*

(a) *For some $n_0, t_0 \in \mathbb{N}$,*

$$\lim_{A\to\infty} \sup_{n\geq n_0, t_0\leq t\leq n} \int_{|\lambda|\geq A} \left|\psi_{t,n}(\lambda)\right| d\lambda = 0;$$

(b) *If in addition **LP4** holds, then the above holds with $t_0 = 1$.*

Remark A.2 (a) Note that the results of parts (ii.a) and (ii.b) imply those of (i.a) and (i.b) respectively. Recall that uniform integrability (of the characteristic function of $x_t(n)/\beta_t$) implies integrability, which in turn implies uniform bounded density e.g. Feller (1971, p. 516). **Lemma UBD(i)** is utilised for the proof of **Theorem 2.1**, whilst its stronger version (i.e. **Lemma UBD(ii)**) is employed for the proof of **Theorem 3.2**.

(b) It follows easily from the arguments used for the proof of **Lemma UBD** that parts (i.a) and (i.b) also apply to $\beta_t^{-1}x_t^+(n)$.

Lemma LVAR (limit variance) *Suppose that $x_t(n)$ is given by (16), and **Assumption INN(i)** and **Assumption LL** hold. Let $t_n \in [\lfloor rn\rfloor, n]$, where $0 < r \leq 1$ and γ_n, $\mathcal{V}_\infty$ as in (20). The following hold:*

*Under **LL1** or **LL3** or **LL4** as $n\to\infty$*

$$Var\left(\gamma_n^{-1}x_{t_n}(n)\right) = Var\left(\gamma_n^{-1}x_{t_n}^+(n)\right) + o(1) \to \mathcal{V}_\infty;$$

*Under **LL2** as $n\to\infty$*

$$Var\left(\gamma_n^{-1}x_{t_n}^+(n)\right), Var\left(\gamma_n^{-1}x_{t_n}^-(n)\right) \to \mathcal{V}_\infty/2.$$

Further, let $0 < s \leq r \leq 1$. Then as $n\to\infty$

$$Var\left(\gamma_n^{-1}\left(x_{\lfloor rn\rfloor}^-(n) - x_{\lfloor sn\rfloor}^-(n)\right)\right) \to 0.$$

Lemma CLT (central limit theorem). *Suppose that $x_t(n)$ is given by (16), and **Assumption INN(i)** and **Assumption LL** hold. Let $t_n \in [\lfloor rn\rfloor, n]$, where $0 < r \leq 1$ and γ_n, $\mathcal{V}_\infty$ as in (20). Then as $n\to\infty$*

$$\gamma_n^{-1}\left[x_{t_n}^+(n), x_{t_n}^-(n)\right] \xrightarrow{d} N(\mathbf{0}, \Sigma),$$

*where $\Sigma = diag(\mathcal{V}_\infty/2, \mathcal{V}_\infty/2)$ under **LL2**, and $\Sigma = diag(\mathcal{V}_\infty, 0)$ otherwise.*

Remark A.3 The proofs of **Lemma LVAR** and **Lemma CLT** do not utilise the restrictions on coefficients $\{c_j\}$ given by conditions **LL1(a,b)**.

Appendix A2

Proof of Proposition 2.1 (i) For $\{\xi_k^*\} \overset{d}{=} \{\xi_k\}$ and $\{\xi_k^*\} \perp \{\xi_k\}$ using (15) write

$$\beta_n^{-1}\left[x_{t_n}^+(n), x_{t_n}^-(n)\right] \overset{d}{=} \beta_n^{-1}\left[\sum_{k=0}^{t_n-1} a_k(n)\xi_k, \sum_{k=t_n}^{\infty} a_{k,t_n}^-(n)\xi_k^*\right].$$

Next, note that due to **LP6(b)** as $n \to \infty$

$$\beta_n^{-1}\left[\sup_{0\le k\le t_n-1}|a_k(n)| + \sup_{k\ge t_n}\left|a_{k,t_n}^-(n)\right|\right] = o(1).$$

In view of the above and **LP6(a)**, the weak limit result follows immediately from Lemma 2.1 of Abadir, Distaso, Giraitis and Koul (2014). Further, for the second part of **HL2(b)** note that due to condition **LP6(c)**

$$\begin{aligned}\beta_n^{-2}\mathbf{E}\left[\sum_{k=t_n}^{\infty} a_{k,t_n}^-(n)\xi_{t_n-k} - \sum_{k=n}^{\infty} a_{k,n}^-(n)\xi_{n-k}\right]^2 &= \beta_n^{-2}\mathbf{E}\left[\sum_{l=-\infty}^{0} a_{t_n-l,t_n}^-(n)\xi_l - \sum_{l=-\infty}^{0} a_{n-l,n}^-(n)\xi_l\right]^2 \\ l &= t_n - k \to k = t_n - l\end{aligned}$$

$$= \beta_n^{-2}\sigma_\xi^2\sum_{l=-\infty}^{0}\left[a_{t_n-l,t_n}^-(n) - a_{n-l,n}^-(n)\right]^2 = o(1).$$

(ii) This part follows from **Lemma UBD** (**Appendix A1b**, above).

(iii) The proof uses arguments similar to those used in Wu and Mielniczuk (2002, p. 1443) and Duffy (2017, Lemma C.3)). Set $\mathbf{E}_t(.) = \mathbf{E}(. \mid \mathcal{F}_t)$, $t \in \mathbb{Z}$. Recall that

$$x_t(n) = \sum_{k=0}^{t-1} a_k(n)\xi_{t-k} + x_t^-(n).$$

Write

$$g(x_t(n)) - \mathbf{E}_0 g(x_t(n)) = \sum_{s=0}^{t-1}\left\{\mathbf{E}_{t-s}g(x_t(n)) - \mathbf{E}_{t-s-1}g(x_t(n))\right\}.$$

Let $g_{t,n} := g\left(x_t(n)/\beta_n\right)$. In view of the above and the fact that $\left\{\mathbf{E}_r g_{t,n}, \mathcal{F}_r\right\}_{r\in\mathbb{Z}}$ is a martingale we get

$$I_n := \frac{1}{n}\mathbf{E}\left|\sum_{t=1}^{n} g_{t,n} - \sum_{t=1}^{n}\mathbf{E}_0 g_{t,n}\right|$$

$$= \frac{1}{n}\mathbf{E}\left|\sum_{t=1}^{n}\sum_{s=0}^{t-1}\left\{\mathbf{E}_{t-s}g_{t,n} - \mathbf{E}_{t-s-1}g_{t,n}\right\}\right| = \frac{1}{n}\mathbf{E}\left|\sum_{s=0}^{n}\sum_{t=s+1}^{n}\left\{\mathbf{E}_{t-s}g_{t,n} - \mathbf{E}_{t-s-1}g_{t,n}\right\}\right|$$

$$\left[\begin{array}{ccc} 1\le t\le n & \rightarrow & 0\le s\le n-1 \\ 0\le s\le t-1 & & s+1\le t\le n \end{array}\right]$$

$$\le \frac{1}{n}\sum_{s=0}^{n}\mathbf{E}\left|\sum_{t=s+1}^{n}\left\{\mathbf{E}_{t-s}g_{t,n} - \mathbf{E}_{t-s-1}g_{t,n}\right\}\right| \le \frac{1}{n}\sum_{s=0}^{n}\sqrt{\mathbf{E}\left|\sum_{t=s+1}^{n}\left\{\mathbf{E}_{t-s}g_{t,n} - \mathbf{E}_{t-s-1}g_{t,n}\right\}\right|^2}$$

$$= \frac{1}{n}\sum_{s=0}^{n}\sqrt{\sum_{t=s+1}^{n}\mathbf{E}\left\{\mathbf{E}_{t-s}g_{t,n} - \mathbf{E}_{t-s-1}g_{t,n}\right\}^2}.$$

Next, write

$$x_t\left(n\right) = \sum_{k=0}^{s-1} a_k\left(n\right)\xi_{t-k} + a_s\left(n\right)\xi_{t-s} + \sum_{k=s+1}^{t-1} a_k\left(n\right)\xi_{t-k} + x_t^-\left(n\right)$$

$$\stackrel{d}{=} \sum_{k=0}^{s-1} a_k\left(n\right)\xi_{t-k} + a_s\left(n\right)\xi^* + \sum_{k=s+1}^{t-1} a_k\left(n\right)\xi_{t-k} + x_t^-\left(n\right) =: x_{t,s}^*,$$

where $\xi^* \stackrel{d}{=} \xi_1$ and $\xi^* \perp \left\{\xi_j\right\}$. Therefore, $\left\{\mathbf{E}_{t-s}g\left(x_t(n)/\beta_n\right), \mathbf{E}_{t-s-1}g\left(x_t(n)/\beta_n\right)\right\}$ $\stackrel{d}{=} \left\{\mathbf{E}_{t-s}g\left(x_t(n)/\beta_n\right), \mathbf{E}_{t-s}g\left(x_{t,s}^*(n)/\beta_n\right)\right\}$ which implies that

$$I_n \le \frac{1}{n}\sum_{s=0}^{n}\left\{\sum_{t=s+1}^{n}\mathbf{E}\left|\mathbf{E}_{t-s}g_{t,n} - \mathbf{E}_{t-s-1}g_{t,n}\right|^2\right\}^{1/2}$$

$$= \frac{1}{n}\sum_{s=0}^{n}\left\{\sum_{t=s+1}^{n}\mathbf{E}\left|\mathbf{E}_{t-s}g_{t,n} - \mathbf{E}_{t-s}g\left(\tilde{x}_{t,s}(n)/\beta_n\right)\right|^2\right\}^{1/2}$$

$$\leq \frac{1}{n}\sum_{s=0}^{n}\left\{\sum_{t=s+1}^{n}\mathbf{E}\mathbf{E}_{t-s}\left|g_{t,n}-g\left(x_{t,s}^{*}(n)/\beta_n\right)\right|^2\right\}^{1/2}$$

$$=\frac{1}{n}\sum_{s=0}^{n}\left\{\sum_{t=s+1}^{n}\mathbf{E}\left|g\left(x_t\left(n\right)/\beta_n\right)-g\left(x_{t,s}^{*}/\beta_n\right)\right|^2\right\}^{1/2}$$

$$\leq C\sqrt{\mathbf{E}\left|\xi_1-\xi^*\right|^2}\frac{1}{n}\sum_{s=0}^{n}\left\{\beta_n^{-2}\sum_{t=1}^{n}\left|a_s\left(n\right)\right|^2\right\}^{1/2}=\frac{C}{\beta_n\sqrt{n}}\sum_{s=0}^{n}\left|a_s\left(n\right)\right|=o(1),$$

where we have used the Lipschitz continuity of $g(.)$ and **LP5**.

(iv) Recall $\beta_{n,t}^2 = Var\left(x_t\left(n\right)\right)$. For part (a) without loss of generality suppose that $\lambda \geq 2$. By Theorem 2 of Whittle (1960) and **LP1** we have for $n \geq n_0$

$$\sup_{1\leq t\leq n}\mathbf{E}\left|\frac{x_t\left(n\right)}{\beta_n}\right|^{\lambda}\leq C_\lambda \mathbf{E}\left(\left|\xi_1\right|^{\lambda}\right)\sup_{1\leq t\leq n}\left(\beta_n^{-2}\sum_{k=0}^{\infty}a_{k,t}^2\left(n\right)\right)^{\lambda/2}$$

$$=C_\lambda \mathbf{E}\left|\xi_1\right|^{\lambda}\sup_{1\leq t\leq n}\left(\frac{\beta_n^{-2}}{\mathbf{E}\left(\xi_1^2\right)}Var\left(x_t\left(n\right)\right)\right)^{\lambda/2}\leq\frac{C_\lambda \mathbf{E}\left|\xi_1\right|^{\lambda}}{\left[\mathbf{E}\left(\xi_1^2\right)\right]^{1/\lambda}}\left(\sup_{n\geq n_0,1\leq t\leq n}\beta_n^{-2}\beta_{n,t}^2\right)^{\lambda/2}<\infty,$$

where $0 < C_\lambda < \infty$, and the last inequality follows from **LP2(a)** and **LP7**.

Next, we show part (b). For $\lambda \in (0,\infty)$, consider

$$\mathbf{E}e^{\beta_n^{-1}\lambda\left|x_t(n)\right|}=\mathbf{E}e^{\beta_n^{-1}\lambda x_t(n)}1\left\{x_t\left(n\right)\geq 0\right\}+\mathbf{E}e^{-\beta_n^{-1}\lambda x_t(n)}1\left\{x_t\left(n\right)<0\right\}$$

$$\leq \mathbf{E}e^{\beta_n^{-1}\lambda x_t(n)}+\mathbf{E}e^{-\beta_n^{-1}\lambda x_t(n)}.$$

We shall prove the result for the first term on the r.h.s. above. The proof for the second term is identical. Note that **LP2(b)** implies that ξ_t is sub-exponential i.e. there are $\varsigma_\xi^2, b_\xi > 0$ such that $\mathbf{E}\exp\left(\mu\xi_t\right) \leq \exp\left(\mu^2\varsigma_\xi^2/2\right)$ for all $|\mu| < b_\xi^{-1}$. Next, for each $\lambda \in (0,\infty)$ there is some $n_0 \geq 1$ such that $\sup_{n\geq n_0,1\leq t\leq n}\sup_{k\geq 0}\lambda\beta_n^{-1}\left|a_{k,t}\left(n\right)\right| < b_\xi^{-1}$ due to **LP6(b)**. To see this note that from (15) and **LP6(b)** we get as $n\to\infty$

$$\beta_n^{-1}\sup_{1\leq t\leq n}\sup_{k\geq 0}\left|a_{k,t}\left(n\right)\right|\leq\beta_n^{-1}\left[\sup_{0\leq k\leq n}\left|a_k\left(n\right)\right|+\sup_{1\leq t\leq n,k\geq t}\left|a_{k,t}^{-}\left(n\right)\right|\right]=o(1).$$

In view of the above for n large and Fatou's lemma we have

$$\mathbf{E}e^{\beta_n^{-1}\lambda x_t(n)} = \mathbf{E}\exp\left(\beta_n^{-1}\lambda\sum_{k=0}^{\infty}a_{k,t}(n)\,\xi_{t-k}\right) \le \lim\inf_{M\to\infty}\prod_{k=0}^{M}\mathbf{E}\exp\left(\beta_n^{-1}\lambda a_{k,t}(n)\,\xi_{t-k}\right)$$

$$\le \lim\inf_{M\to\infty}\prod_{k=0}^{M}\exp\left(\beta_n^{-2}\lambda^2 a_{k,t}(n)^2\varsigma_\xi^2/2\right) = \exp\left(\beta_n^{-2}\lambda^2\sum_{k=0}^{\infty}a_{k,t}(n)^2\varsigma_\xi^2/2\right)$$

$$= \exp\left(\beta_n^{-2}\lambda^2 Var\left(x_t(n)\right)\varsigma_\xi^2/2\sigma_\xi^2\right) \le \exp\left(\frac{\varsigma_\xi^2\lambda^2}{2\sigma_\xi^2}\sup_{n\ge n_0,1\le t\le n}\beta_n^{-2}\beta_{n,t}^2\right) < \infty,$$

where the last inequality follows from **LP7**. ■

Proof of Proposition 2.2. Without loss of generality set $\mathbf{E}\xi_1^2 = 1$.

Proof of **LP3**. Part (a): Let γ_n and $\mathcal{V}_\infty$ as in (20). By **Lemma LVAR** as $t\to\infty$

$$\gamma_t^{-2}\beta_t^2 \to \mathcal{V}_\infty. \tag{41}$$

In view of this for $n\ge t$ and as $t\to\infty$ we get

$$\beta_t^{-2}\sum_{k=\underline{q}_t+\theta}^{\overline{q}_t}a_k(n)^2 = \mathcal{V}_\infty^{-1}\gamma_t^{-2}\sum_{k=\underline{q}_t+\theta}^{\overline{q}_t}a_k(n)^2 + o(1).$$

Suppose that (40) holds. By **Lemma 6.2(i)** as $n,t\to\infty$ we have

$$\gamma_t^{-2}\sum_{k=\underline{q}_t+\theta}^{\overline{q}_t}a_k(n)^2 = \gamma_t^{-2}\delta_t^2\sum_{k=\underline{q}_t+\theta}^{\overline{q}_t}\eta_k^2\delta_t^{-2}a_k^2(n)/\eta_k^2$$

$$\ge D_1^2\gamma_t^{-2}\delta_t^2\sum_{k=\underline{q}_t+\theta}^{\overline{q}_t}\eta_k^{-2} =_{(1)} D_1^2\left(1+o(1)\right),$$

and the result follows from the above if we show $=_{(1)}$ which what we set out to do next. First, note that under **Assumption LL1-LL2** by **Lemma 6.3(iii)** as $t\to\infty$

$$\frac{\delta_t^2}{\gamma_t^2}\sum_{k=\underline{q}_t+\theta}^{\overline{q}_t}\eta_k^{-2} = L(t)^{-1}\sum_{k=\lfloor L(t)\rfloor+\theta}^{t}\frac{\ell(k)^2}{k} = 1+o(1).$$

Next, under **Assumption LL3-LL4** as $t \to \infty$

$$\frac{\delta_t^2}{\gamma_t^2} \sum_{k=\underline{q}_t+\theta}^{\overline{q}_t} \eta_k^{-2} = \frac{2\delta_t^2 \left(\overline{q}_t - \underline{q}_t - \theta + 1\right)}{\gamma_t^2} = \frac{2\delta_t^2 \left(\lfloor \kappa_t \rfloor - \lfloor \kappa_t/2 \rfloor - \theta + 1\right)}{\gamma_t^2} = 1+o(1),$$

where we have used the fact that $\kappa_t \delta_t^2 = \gamma_t^2$.

Part (b): Set $\delta = (1-m)/(3/2-m)$ under **Assumption LL4**, and $\delta = 0$ otherwise. Note that in this case δ_t introduced in (40) becomes $\delta_t = \gamma_t^\delta$ (recall that $\gamma_n = \kappa_n^{3/2-m}$ under **Assumption LL4**). In view of this, by (41) and **Lemma 6.2(i)** there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $n_0 \leq t \leq n$ we have

$$\beta_t^{-\delta} \sup_{0 \leq k \leq \overline{q}_t} |a_k(n)| \leq C\gamma_t^{-\delta} \sup_{0 \leq k \leq \overline{q}_t} |a_k(n)| \leq C \cdot D_2.$$

Part (c): Let $\rho > 0$. By (41) we have $C/2 \leq \beta_t/\gamma_t \leq C$ for some $0 < C < \infty$ and t large. In view of this and **Lemma 6.2(i)**, as $t \to \infty$

$$\exp\left\{-\rho\beta_t^{2(1-\delta)}\right\} \beta_t \sup_{\underline{q}_t \leq k \leq \underline{q}_t+\theta} |a_k(n)|^{-1} \leq C \exp\left\{-\rho (C\gamma_t/2)^{2(1-\delta)}\right\} \gamma_t \sup_{\underline{q}_t \leq k \leq \underline{q}_t+\theta} |a_k(n)|^{-1}$$

$$= C \exp\left\{-\rho (C\gamma_t/2)^{2(1-\delta)}\right\} \gamma_t^{1-\delta} \sup_{\underline{q}_t \leq k \leq \underline{q}_t+\theta} \left|\gamma_t^{-\delta} a_k(n)\right|^{-1}$$

$$= C \exp\left\{-\rho (C\gamma_t/2)^{2(1-\delta)}\right\} \gamma_t^{1-\delta} \sup_{\underline{q}_t \leq k \leq \underline{q}_t+\theta} \left|\delta_t^{-1} \frac{\eta_k}{\eta_k} a_k(n)\right|^{-1}$$

$$\leq D_1^{-1} C \exp\left\{-\rho (C\gamma_t/2)^{2(1-\delta)}\right\} \gamma_t^{1-\delta} \sup_{\underline{q}_t \leq k \leq \underline{q}_t+\theta} \eta_k := T_t.$$

Now recall that by (20) and (40), under **Assumption LL1-LL2**, $\gamma_t = \sqrt{L(t)}$, $\underline{q}_t = \lfloor L(t) \rfloor$, $\delta = 0$ and $\eta_t = \sqrt{t}/\ell(t)$. Therefore, it can be easily seen that as $t \to \infty$

$$T_t \leq D_1^{-1} C \exp\left\{-\rho C^2 L(t)/4\right\} \sqrt{L(t)} \left(L(t) + \theta\right) = o(1).$$

Further, note that under **Assumption LL3-LL4** $\eta_t = \sqrt{1/2}$. In view of this and the fact that $\gamma_t^{1-\delta} \to \infty$, we get as $t \to \infty$,

$$T_t \leq \sqrt{2} D_1^{-1} C \exp\left\{-\rho (C\gamma_t/2)^{2(1-\delta)}\right\} \gamma_t^{1-\delta} = o(1).$$

Proof of **LP4** ($\theta = 1$) For a sequence $\{\pi_j\}_{j=0}^{\infty}$ define $b_t(\{\pi_j\}, k) := \sum_{j=0}^{t\wedge k} \pi_j c_{k-j}$. Hence, under **Assumption LL** we can also write $x_t(n)$ as

$$x_t(n) = \sum_{k=0}^{\infty} b_t\left(\left\{\phi_j(n)\right\}, k\right) \xi_{t-k},\ \phi_j(n) = \begin{cases} \phi_j, \text{ under } \textbf{LL1-LL2} \\ \rho_n^j, \text{ under } \textbf{LL3-LL4} \end{cases}$$

First, suppose that **LL1-LL2** hold. Then for each $t \geq 1$ there is some $k_t^* \geq 0$ such that $b_t\left(\left\{\phi_j\right\}, k_t^*\right) \neq 0$. Note that under our assumptions $\phi_0 \neq 0$. To see that the aforementioned statement is true, suppose that there some $t \geq 1$ such that $b_t\left(\left\{\phi_j\right\}, k\right) = 0$ for all k i.e.

$$\left.\begin{array}{l} b_t\left(\left\{\phi_j\right\}, 0\right) = \phi_0 c_0 \\ b_t\left(\left\{\phi_j\right\}, 1\right) = \phi_0 c_1 + \phi_1 c_0 \\ b_t\left(\left\{\phi_j\right\}, 2\right) = \phi_0 c_2 + \phi_1 c_1 + \phi_2 c_0 \\ . \\ . \\ b_t\left(\left\{\phi_j\right\}, t-1\right) = \phi_0 c_{t-1} + ... + \phi_{t-2} c_1 + \phi_{t-1} c_0 \\ b_t\left(\left\{\phi_j\right\}, t\right) = \phi_0 c_t + ... + \phi_{t-1} c_1 + \phi_t c_0 \\ b_t\left(\left\{\phi_j\right\}, t+1\right) = \phi_0 c_{t+1} + \phi_1 c_t + ... + \phi_{t-1} c_2 + \phi_t c_1 \\ b_t\left(\left\{\phi_j\right\}, t+2\right) = \phi_0 c_{t+2} + \phi_1 c_{t+1} + ... + \phi_{t-1} c_3 + \phi_t c_2 \\ . \\ . \end{array}\right\} = 0.$$

This implies that $c_j = 0$ for all $j \geq 0$, which contradicts **Assumptions LL1-LL2** i.e. the facts that either $\sum_{j=0}^{\infty} c_j \neq 0$ or $c_j \sim \ell(j) j^{-3/2}$. Hence, indeed for each $t \geq 1$ there is some $k_t^* \geq 0$ such that $b_t\left(\left\{\phi_j\right\}, k_t^*\right) \neq 0$. Therefore, for any $t_0 \in \mathbb{N}$.

$$\inf_{1 \leq t \leq t_0} \left| b_t\left(\left\{\phi_j\right\}, k_t^*\right)\right| > 0.$$

Next, suppose that **LL3-LL4** hold. Using the same argument given above we can show that for each $t \geq 1$ there is some $k_t^* \geq 0$ such that $b_t(\{1\}, k_t^*) \neq 0$. Then

$$\liminf_{n\to\infty} \inf_{1 \leq t \leq t_0} \left| b_t\left(\left\{\rho_n^j\right\}, k_t^*\right)\right| = \inf_{1 \leq t \leq t_0} \left| b_t(\{1\}, k_t^*)\right| > 0,$$

as required.

Proof of **LP5** Set $R_n := \frac{1}{\beta_n \sqrt{n}} \sum_{k=0}^{n} |a_k(n)|$. Under **Assumption LL1-LL2** (recall in this case $\eta_k = \sqrt{k}/\ell(k)$, $\delta_n = 1$) by **Lemma 6.2(ii)**, **Lemma**

LVAR and **Lemma 6.3(i)** we get as $n \to \infty$

$$R_n \le \frac{\delta_n D_2}{\beta_n \sqrt{n}} \sum_{k=0}^{n} \eta_k^{-1} = \frac{D_2}{\beta_n \sqrt{n}} \sum_{k=1}^{n} \ell(k) k^{-1/2} + o(1)$$

$$\le \frac{C}{\sqrt{L(n)}\sqrt{n}} \sum_{k=1}^{n} \ell(k) k^{-1/2} = \frac{C\ell(n)}{\sqrt{L(n)}} \left(\int_0^1 x^{-1/2} dx + o(1) \right) = o(1),$$

where the last approximation is due to Proposition 1.5.9a in Bingham et al. (1987). Next,

$$R_n \le \frac{1}{\beta_n \sqrt{n}} \sum_{k=0}^{n} \sum_{j=0}^{k} \left| \rho_n^j c_{k-j} \right|$$

$$= \frac{1}{\beta_n \sqrt{n}} \sum_{j=0}^{n} \left| \rho_n^j \right| \sum_{k=j}^{n} \left| c_{k-j} \right| = \frac{1}{\beta_n \sqrt{n}} \sum_{j=0}^{n} \left| \rho_n^j \right| \sum_{l=0}^{n-j} \left| c_l \right|$$

$$[l = k - j]$$

$$\le_{(1)} \frac{C\kappa_n}{\beta_n \sqrt{n}} \sum_{l=0}^{\infty} |c_l| =_{(2)} \frac{C\kappa_n}{\gamma_n \sqrt{n}} =_{(3)} \frac{C\sqrt{\kappa_n}}{\sqrt{n}} = o(1),$$

where $\le_{(1)}$ is due to **Lemma 6.4(iii)**, $=_{(2)}$ due to **Lemma 6.4(iii),** and $=_{(3)}$ due to **LL3**. Similarly, under **Assumption LL4** using **Lemma LVAR**, **Lemma 6.3(i)** and **Lemma 6.4(iii)** we get

$$R_n \le \frac{C}{\beta_n \sqrt{n}} \sum_{j=0}^{n} \left| \rho_n^j \right| \sum_{k=0}^{\infty} |c_k| \le \frac{C\kappa_n}{\kappa_n^{3/2-m} \sqrt{n}} \sum_{q=0}^{n} |c_q| = \frac{C\kappa_n^{m-1/2}}{\sqrt{n}} \left(\sum_{q=1}^{n} q^{-m} + o(1) \right)$$

$$= \frac{C\kappa_n^{m-1/2} n^{1-m}}{\sqrt{n}} \left(\int_0^1 x^{-m} dx + o(1) \right) = O\left((\kappa_n / n)^{m-1/2} \right) = o(1).$$

Proof of **LP6** Part (a) It is a direct consequence of **Lemma LVAR**. Part (b) is a direct consequence of **Lemma 6.2(ii,iii)** and **Lemma LVAR**. For part (c) note that by **Lemma LVAR** under **LL2**

$$\beta_n^{-2} Var\left[x_{t_n}^-(n) - x_n^-(n) \right] \to 0.$$

Further, under **LL1**, **LL3** and **LL4** again by **Lemma LVAR** we have

$$\beta_n^{-2} \left[Var\left(x_{t_n}^-(n) \right) + Var\left(x_n^-(n) \right) \right] \to 0.$$

In view of the above the result follows from the fact that

$$\beta_n^{-2}\sum_{l=-\infty}^{0}\left[a_{t_n-l,t_n}^{-}(n)-a_{n-l,n}^{-}(n)\right]^2=\beta_n^{-2}Var\left[x_{t_n}^{-}(n)-x_n^{-}(n)\right]$$

$$\leq\beta_n^{-2}\left\{Var\left(x_{t_n}^{-}(n)\right)+Var\left(x_n^{-}(n)\right)+2\sqrt{Var\left(x_{t_n}^{-}(n)\right)Var\left(x_n^{-}(n)\right)}\right\}.$$

Proof of **LP7** First note that under **Assumption LL** and **INN**, it can be easily checked that $Var\left(x_t\left(n\right)\right)$ is a finite sequence i.e. $Var\left(x_t\left(n\right)\right)\in\mathbb{R}_+$ for all $1\leq t\leq n$, $n\in\mathbb{N}$.

Suppose that **LL1** or **LL2** holds. Without loss of generality we assume that $\gamma_n^2>0$. Otherwise we can set $\gamma_n^2=1\vee L(n)$ which is strictly positive and (eventually) increasing. By **Lemma LVAR,** we have as $n\to\infty$

$$\beta_n^{-2}\sup_{1\leq t\leq n}Var\left(x_t\right)\leq\beta_n^{-2}\sup_{1\leq t\leq n}\gamma_t^2\sup_{1\leq t\leq n}\gamma_t^{-2}Var\left(x_t\right)$$

$$=\beta_n^{-2}\gamma_n^2\sup_{1\leq t\leq n}\gamma_t^{-2}Var\left(x_t\right)\leq C\beta_n^{-2}\gamma_n^2=C\mathcal{V}_\infty^{-1}+o(1),$$

where $\mathcal{V}_\infty>0$, and the last inequality above follows from the fact that $\gamma_t^{-2}Var\left(x_t\right)$ is a finite convergent sequence (**Lemma LVAR**).

Next, suppose that **LL3** or **LL4** holds. Note that in this case $x_t\left(n\right)$ is an array. We have $Var\left(x_t\left(n\right)\right)=Var\left(x_t^{+}\left(n\right)\right)+Var\left(x_t^{-}\left(n\right)\right)$. For $1\leq t\leq n$, the first term

$$Var\left(x_t^{+}\left(n\right)\right)=\sum_{k=0}^{t-1}\left(\sum_{j=0}^{k}\rho_n^jc_{k-j}\right)^2\leq\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\rho_n^jc_{k-j}\right)^2=Var\left(x_n^{+}\left(n\right)\right).$$

In view of the above and **Lemma LVAR** we have as $n\to\infty$

$$\beta_n^{-2}\sup_{1\leq t\leq n}Var\left(x_t^{+}\left(n\right)\right)\leq\beta_n^{-2}Var\left(x_n^{+}\left(n\right)\right)=1+o(1).$$

Hence, $\beta_n^{-2}\sup_{1\leq t\leq n}Var\left(x_t^{+}\left(n\right)\right)$ is bounded for n large. Next, we show that $\beta_n^{-2}\sup_{1\leq t\leq n}Var\left(x_t^{-}\left(n\right)\right)$ is bounded for n large enough. Consider

$$\sup_{1\leq t\leq n}Var\left(x_t^{-}\left(n\right)\right)=\sup_{1\leq t\leq n}\sum_{k=t}^{\infty}\left(\sum_{j=0}^{t-1}\rho_n^jc_{k-j}\right)^2\leq\sup_{1\leq t\leq n}\sum_{k=t}^{\infty}\left(\sum_{j=0}^{k\wedge n}\left|\rho_n^jc_{k-j}\right|\right)^2$$

$$\leq \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k\wedge n}\left|\rho_n^j c_{k-j}\right|\right)^2 = \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\left|\rho_n^j c_{k-j}\right|\right)^2 + \sum_{k=n}^{\infty}\left(\sum_{j=0}^{n}\left|\rho_n^j c_{k-j}\right|\right)^2$$
$$=: C_n + D_n.$$

Notice that C_n and D_n resemble $Var\left(x_n^+(n)\right)$ and $Var\left(x_n^-(n)\right)$ respectively. Therefore, using the same arguments as those used for the proof of **Lemma LVAR**, we can show that $\gamma_n^{-2}D_n = o(1)$. Further,

$$\gamma_n^{-2}C_n \to \begin{cases} \left(\sum_{s=0}^{\infty}|c_s|\right)^2/2, & \text{under } \mathbf{LL3} \\ \mathcal{V}_\infty, & \text{under } \mathbf{LL4} \end{cases}$$

where $\mathcal{V}_\infty$ is as in (20). The requisite result follows from the above and the fact that $\beta_n^{-2}\gamma_n^2 \to \mathcal{V}_\infty^{-1}$ where $\mathcal{V}_\infty > 0$. ■

Proof of Theorem 2.2 We show that the assumptions of **Theorem 2.1** hold true, under the assumptions of **Theorem 2.2** with $X_t(n) = x_t(n)$, $t \geq 1$. We start with the result for f.

(i) **HL0** & **HL1**: Let $\mathcal{F}_t = \sigma\left(\{\xi_r\}_{r\leq t}\right)$, $t \in \mathbb{Z}$, and for $t \geq 1$, $\left[X_t^+(n), X_t^-(n)\right] .= \left[x_t^+(n), x_t^-(n)\right]$ where $x_t^+(n)$, $x_t^-(n)$ are given in (15). It can be easily seen that in this case **HL0** and **HL1** hold due **Assumption INN** and **Assumption LL**.

(ii) **HL2**: By **Lemma LVAR**, **LP6(a)** holds with $\left[\sigma_+^2, \sigma_-^2\right] = [1, 0]$ under **LL(1,3,4)** and $\left[\sigma_+^2, \sigma_-^2\right] = [1/2, 1/2]$ otherwise. In view of this, **HL2** by **Propositions 2.1-2.2**.

(iii) **HL3-HL4**: These conditions are direct consequences of **Propositions 2.1-2.2**.

(iv) **HL6**: In view of **Assumption LL**, **Lemma LVAR** (see also eq. (20)) implies that $\beta_n \sim n^\alpha \varsigma(n)$ where $0 \leq a < 1$ and $\varsigma(x)$ SV in some neighborhood of infinity. Therefore, for some $t_0 \geq 1$ and as $n \to \infty$ **Lemma 6.3(i)** yields

$$\frac{\beta_n}{n}\sum_{t=t_0}^{n}\beta_t^{-1} = \int_0^1 x^{-\alpha}dx + o(1),$$

and therefore **HL6** holds.

We have established **HL0-HL4** and **HL6** under the assumptions of the current Theorem. Hence, by **Theorem 2.1**, (14) holds for f bounded. For f unbounded we also need validating conditions (i)-(iii) of **Theorem 2.1**, which is what we set out to do next. First, notice that **HL2** holds with

$[X^+, X^-] \sim N\left[0, diag(\sigma_+^2, \sigma_-^2)\right]$, where σ_+^2, σ_-^2 are as in part (ii) above. Hence, $\mathbf{E}\int_{\mathbb{R}} |f(x + X^-)|\, \Phi_{X^+}(x)dx = \int_{\mathbb{R}} |f(x)|\, \varphi_1(x)dx$ and the latter is finite due to condition (ii) of **Theorem 2.2**. This in turn implies that condition (i) of **Theorem 2.1** holds. Next, condition (ii) of **Theorem 2.1** holds due to condition (i) of the current Theorem. Finally, condition (iii) of **Theorem 2.1** holds by condition (ii) of the current Theorem, and **Propositions 2.1-2.2**.

Next, we show the result for $\tilde{f}$. Note that under our assumptions, for all n large enough and each $1 \le t \le n$, there is at least one coefficient $a_{k,t}(n)$ in $x_t(n) = \sum_{k=0}^{\infty} a_{k,t}(n)\, \xi_{t-k}$ that is bounded away from zero, as shown in the proof **Proposition 2.2**. In view of this and the fact that each $\{\xi_t\}$ has a density (**Assumption INN**), it follows that $\{x_t(n)\}_{t=1}^{n}$ also has a density for n large (e.g. Lukacs, 1970; Theorem 3.3.2). Hence, for n large $\frac{1}{n}\sum_{t=1}^{n}\left\{f\left(\beta_n^{-1}x_t(n)\right) - \tilde{f}\left(\beta_n^{-1}x_t(n)\right)\right\} = 0$ $a.s.$, and the result follows. ■

Proof of Theorem 3.2 We check that the assumptions of **Theorem 3.1** hold true, under the conditions of **Theorem 3.2** with $X_t(n) = x_t(n)$, $t \ge 1$.

(i) **HL0-HL2**, **HL4** and **HL6** can be established using identical arguments as those used in the proof of **Theorem 2.2**.

(ii) Next, we show **HL7**. Note that from (15) we can write

$$\begin{aligned} x_t^+(n) &= \sum_{k=0}^{t-1} a_k(n)\, \xi_{t-k} = \sum_{l=1}^{t} a_{t-l}(n)\, \xi_l. \\ & [l = t-k \to k = t-l] \end{aligned}$$

Hence, for $t > s$

$$x_t(n) - x_s(n) = \sum_{l=1}^{t} a_{t-l}(n)\, \xi_l - \sum_{l=1}^{s} a_{s-l}(n)\, \xi_l + x_t^-(n) - x_s^-(n)$$

$$= \sum_{l=s+1}^{t} a_{t-l}(n)\, \xi_l + \sum_{l=1}^{s}\left[a_{t-l}(n) - a_{s-l}(n)\right]\xi_l + x_t^-(n) - x_s^-(n) =: \sum_{l=s+1}^{t} a_{t-l}(n)\, \xi_l + \mathcal{X}_{t,s}(n)$$

The first term on the r.h.s. in the last line above is

$$\begin{aligned} \sum_{l=s+1}^{t} a_{t-l}(n)\, \xi_l &= \sum_{k=0}^{t-s-1} a_k(n)\, \xi_{t-k} \overset{d}{=} x_{t-s}^+(n). \\ & [l = t-k \to k = t-l] \end{aligned}$$

It follows from **Lemma UBD(ia)** (see also **Remark A.2**) that $\beta_t^{-1} x_t^+ (n)$ possesses a density, $\mathcal{D}_{t,n}^+(x)$ say, such that $\sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \mathcal{D}_{t,n}^+(x) < \infty$, for some $n_0 \geq t_0 \geq 1$. In view of the above, conditionally on $\mathcal{F}_s$, $\beta_{t-s}^{-1} [x_t(n) - x_s(n)]$ has a density $\mathcal{D}_{t,s,n}(x)$ such that $\mathcal{D}_{t,s,n}(x) = \mathcal{D}_{t-s,n}^+(x - \mathcal{X}_{t,s}(n))$ and

$$\sup_{n \geq n_0, 0 \leq s < t \leq n, t-s \geq t_0} \sup_x \mathcal{D}_{t-s,n}^+(x - \mathcal{X}_{t,s}(n)) \leq \sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \mathcal{D}_{t,n}^+(x) < \infty.$$

(iii) **HL8** follows from **Lemma CLT**, **Lemma UBD(ii)** and arguments similar to those used in Wang and Phillips (2009a), p. 729-731.

(vi) Next, we show **HL9**. **HL9(a,b)** follows from similar arguments as those used for establishing **HL6** in the proof of **Theorem 2.2**. For **HL9(c)** recall that $\beta_n \sim n^\alpha \varsigma(n)$ where $0 \leq \alpha < 1$ and $\varsigma(x)$ SV. Again using similar arguments as those used in the proof of **Theorem 2.2** we get as $n \to \infty$ first and then as $\eta \downarrow 0$

$$\frac{\beta_n}{n} \sup_{0 \leq s \leq (1-\eta)n} \sum_{t=s+t_0}^{s+\lfloor \eta n \rfloor} \beta_{t-s}^{-1} = \underset{[l = t-s]}{\frac{\beta_n}{n} \sup_{0 \leq s \leq (1-\eta)n} \sum_{l=t_0}^{\lfloor \eta n \rfloor} \beta_l^{-1}} \overset{n \to \infty}{=} \int_0^\eta x^{-\alpha} dx + o(1) \overset{\eta \downarrow 0}{=} o(1).$$

HL9(d), follows using similar arguments as those used for **HL9(c)**. We finally show **HL9(e)**. Let $0 < \eta < 1$. As $n \to \infty$

$$\begin{aligned}
\sup_{(t,s) \in \Omega_n(\eta)} \left| \frac{\beta_{t-s}}{\beta_n} - \frac{(t-s)^\alpha}{n^\alpha} \right| &=_{(1)} \sup_{(t,s) \in \Omega_n(\eta)} \left| \frac{(t-s)^\alpha \varsigma(t-s)}{n^\alpha \varsigma(n)} - \frac{(t-s)^\alpha}{n^\alpha} \right| + o(1) \\
&\leq \sup_{\frac{\lfloor \eta n \rfloor}{n} \leq x \leq 1} \left| \frac{(nx)^\alpha \varsigma(nx)}{n^\alpha \varsigma(n)} - x^\alpha \right| \leq \sup_{\eta/2 \leq x \leq 1} \left| \frac{(nx)^\alpha \varsigma(nx)}{n^\alpha \varsigma(n)} - x^\alpha \right| \\
&= \sup_{\eta/2 \leq x \leq 1} x^\alpha \left| \frac{\varsigma(nx)}{\varsigma(n)} - 1 \right| \leq \sup_{\eta/2 \leq x \leq 1} \left| \frac{\varsigma(nx)}{\varsigma(n)} - 1 \right| =_{(2)} o(1),
\end{aligned}$$

where $=_{(1)}$ follows from arguments similar to those used for the proof of eq. (43) and $=_{(2)}$ is due to Theorem 1.2.1 in Bingham et al. (1987). Hence, as $n \to \infty$

$$\beta_n^{-1} \inf_{(t,s) \in \Omega_n(\eta)} \left| \beta_{t-s} \right| = \inf_{(t,s) \in \Omega_n(\eta)} (t-s)^\alpha / n^\alpha + o(1),$$

where $\inf_{(t,s)\in\Omega_n(\eta)} (t-s)^{\alpha}/n^{\alpha} \geq \lfloor n\eta \rfloor^{\alpha}/n^{\alpha} \geq \eta^{\alpha}/2$, and the result follows. ■

Appendix A3

Proof of Lemma 5.2. By Theorems 6.3 and 6.4 in Kallenberg (2001), there is a probability kernel ν_n from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mathbf{E}\left(g(X_n, Y_n) \mid \mathcal{F}\right) \overset{a.s.}{=} \int_{\mathbb{R}} g(x, Y_n) d\nu_n(x) =: h_n\left(Y_n\right).$$

Now using the above and condition (i), for each $y \in \mathbb{R}$ fixed we have

$$h_n\left(y\right) \overset{a.s.}{=} \mathbf{E}\left(g(X_n, y) \mid \mathcal{F}\right) \overset{p}{\to} \int_{\mathbb{R}} g(x, y) dF_X(x) =: h\left(y\right).$$

Moreover, by the Lipschitz continuity of g

$$\left|h_n\left(y\right) - h_n\left(y'\right)\right| \leq \int_{\mathbb{R}} \left|g(x, y) - g(x, y')\right| d\nu_n(x) \leq C\left|y - y'\right|.$$

Hence, $\{h_n\left(y\right)\}$ is stochastically equicontinuous on $\mathbb{R}$, whence if follows that $h_n\left(y\right) \overset{p}{\to} h\left(y\right)$ uniformly on every compact subset of $\mathbb{R}$ (see for example Theorem 1 and Lemma 1 in Andrews, 1992).

Finally, fix $\varepsilon > 0$ and choose M_{ε} such that $\limsup_{n\to\infty} \mathbf{P}\left(|Y_n| > M_{\varepsilon}\right) < \varepsilon$, which is possible since $Y_n \overset{d}{\to} Y$. Then as $n \to \infty$

$$\mathbf{P}\left(\left|h_n\left(Y_n\right) - h\left(Y_n\right)\right| > \varepsilon\right) \leq \mathbf{P}\left(\left\{\left|h_n\left(Y_n\right) - h\left(Y_n\right)\right| > \varepsilon\right\} \cap \left\{|Y_n| \leq M_{\varepsilon}\right\}\right)$$

$$+\mathbf{P}\left(|Y_n| > M_{\varepsilon}\right) \leq \mathbf{P}\left(\sup_{|y|\leq M_{\varepsilon}} \left|h_n\left(y\right) - h\left(y\right)\right| > \varepsilon\right) + \varepsilon \to \varepsilon,$$

by the uniform convergence in probability of h_n on compacta. In view of the above,

$$\mathbf{E}\left(g(X_n, Y_n) \mid \mathcal{F}\right) \overset{a.s.}{=} h_n\left(Y_n\right) = h\left(Y_n\right) + o_p(1) \overset{d}{\to} \int_{\mathbb{R}} g(x, Y) dF_X(x),$$

where the last approximation is due to condition (ii) and the fact that $h\left(y\right)$ is continuous in y. ■

7 Appendix B

Appendix B1 states some technical lemmas that are subsequently utilised for the proofs of the preliminary lemmas of **Appendix A1**. The proofs of the preliminary lemmas of **Appendix A**, are given in **Appendix B2a** whilst **Appendix B2b** contains proofs for the techincal lemmas of **Appendix A1.** The proofs of the of the technical lemmas of **Appendix B1** are given in **Appendix B3**.

Appendix B1 (technical lemmas)

Lemma 7.1 *The following hold.*

(i) Suppose that $\phi_j \sim \ell(j) j^{-1/2}$*, where* $\ell(x)$ *SV,* $\sum_{j=0}^{\infty} |c_j| < \infty$*, and one of the following holds:*

(a) $\limsup_{j\to\infty} j |c_j| < \infty$*; or*

(b) $\lim_{s\to\infty} \ell^{-1}(s) \sum_{j=\lfloor s\delta \rfloor}^{\infty} j^{1/2} |c_j| = 0$ *for some* $0 < \delta < 1$*.*

Then as $s \to \infty$

$$\sum_{j=0}^{s} \phi_j c_{s-j} = \phi_s \sum_{j=0}^{\infty} c_j + o\left(\ell(s)/s^{1/2}\right).$$

(ii) Suppose that $\pi_j \sim \ell_\pi(j) j^{-q_1}$*,* $\theta_j \sim \ell_\theta(j) j^{-q_2}$ *for some* $q_1, q_2 \geq 1$*,* $\sum_{j=0}^{\infty} (|\pi_j| + |\theta_j|) < \infty$*, and* $\ell_\pi(x)$*,* $\ell_\theta(x)$ *are SV. Then as* $s \to \infty$

$$\sum_{j=0}^{s} \pi_j \theta_{s-j} = \pi_s \sum_{j=0}^{\infty} \theta_j + o(\pi_s) + \theta_s \sum_{j=0}^{\infty} \pi_j + o(\theta_s)$$

If in addition $\sum_{j=0}^{\infty} \pi_j = 0$ *and* $\theta_j = O(\pi_j)$*, then as* $s \to \infty$

$$\sum_{j=0}^{s} \pi_j \theta_{s-j} = \pi_s \sum_{j=0}^{\infty} \theta_j + o(\pi_s).$$

Remark. **Lemma 7.1(i)** implies that as $s \to \infty$

$$\frac{s^{1/2}}{\ell(s)} \sum_{j=0}^{s} \phi_j c_{s-j} = \sum_{j=0}^{\infty} c_j + o(1).$$

Therefore, if $\sum_{j=0}^{\infty} c_j \neq 0$, $(\sqrt{s}/\ell(s)) \sum_{j=0}^{s} \phi_j c_{s-j}$, is bounded away from zero for s large.

Lemma 7.2 *The following hold.*

(i) Under ***Assumption LL4***

(a) $\sup_{n\geq t,0\leq k\leq \kappa_t}\left|\frac{1}{\kappa_t^{1-m}}a_k\left(n\right)-\int_0^{k/\kappa_t}e^{-\left(\frac{k}{\kappa_n}-\frac{\kappa_t}{\kappa_n}x\right)}x^{-m}dx\right|=o(1)$*, as* $t\to\infty$*;*

(b) $\sup_{0\leq k\leq n}\left|\frac{1}{\kappa_n^{1-m}}a_k\left(n\right)-\int_0^{k/\kappa_n}e^{-\left(\frac{k}{\kappa_n}-x\right)}x^{-m}dx\right|=o(1)$*, as* $n\to\infty$*;*

(c) $\sup_{1\leq t\leq n,k\geq t}\left|\frac{1}{\kappa_n^{1-m}}a_{k,t}^{-}\left(n\right)-\int_{(k-t)/\kappa_n}^{k/\kappa_n}e^{-\left(\frac{k}{\kappa_n}-x\right)}x^{-m}dx\right|=o(1)$*, as* $n\to\infty$*.*

(ii) Under ***Assumption LL4,*** *there are* $n_0,n_1\in\mathbb{N}$ *such that*

$$\inf_{n\geq n_0,n_1\leq t\leq n,\lfloor\kappa_t/2\rfloor\leq k\leq\kappa_t}\int_0^{k/\kappa_t}e^{-\left(\frac{k}{\kappa_n}-\frac{\kappa_t}{\kappa_n}x\right)}x^{-m}dx>0.$$

Lemma 7.3 *For* $1/2<m<1$ *we have the representation*

$$\int_0^\infty e^{2y}\left(\int_0^y e^{-x}x^{-m}dx\right)^2dy=\Gamma\left(2-2m\right)B\left(1-m,2m-1\right),$$

where $\Gamma\left(.\right)$ *and* $B\left(.,.\right)$ *are Gamma and Beta functions respectively.*

Appendix B2a (proofs of preliminary lemmas of Appendix A)

Proof of Lemma UBD. We only prove part (i). Part (ii) follows easily from minor modifications of the arguments used below.

(i.a) We shall prove that the characteristic function of $x_t\left(n\right)/\beta_t$ has a uniformly bounded L_1-norm i.e. $\sup_{n\geq n_0,t_0\leq t\leq n}\int_\lambda\left|\psi_{t,n}(\lambda)\right|d\lambda<\infty$ for some $n_0\geq t_0\geq 1$. This implies the conclusion of part (i.a). Consider

$$\int_{\mathbb{R}}\left|\mathbf{E}\left(e^{i\lambda x_t(n)/\beta_t}\right)\right|d\lambda\leq\int_{\mathbb{R}}\left|\mathbf{E}\left(e^{i\lambda x_t^+(n)/\beta_t}\right)\right|d\lambda=\int_{\mathbb{R}}\left|\mathbf{E}\left(e^{i\lambda\left(\sum_{k=0}^{t-1}a_k(n)\xi_{t-k}\right)/\beta_t}\right)\right|d\lambda$$

$$=\left(\int_{|\lambda|\leq\beta_t^{1-\delta}\mathcal{C}_2^{-1}}+\int_{|\lambda|\geq\beta_t^{1-\delta}\mathcal{C}_2^{-1}}\right)\prod_{k=0}^{t-1}\left|\psi_\xi\left(\frac{\lambda}{\beta_t}a_k\left(n\right)\right)\right|d\lambda=:I_{1,n}\left(t\right)+I_{2,n}\left(t\right),$$

where δ and $\mathcal{C}_2$ are given in **Assumption LP3(b)**. By **Lemma 6.1** and in view of **LP3(b)** for n and t large enough the first term is

$$I_{1,n}(t) \leq \int_{|\lambda|\leq\beta_t^{1-\delta}\mathcal{C}_2^{-1}} \prod_{k=\underline{q}_t}^{\overline{q}_t} \left|\psi_\xi\left(\frac{\lambda}{\beta_t}a_k(n)\right)\right| d\lambda$$

$$\leq \int_{|\lambda|\leq\beta_t^{1-\delta}\mathcal{C}_2^{-1}} \prod_{k=\underline{q}_t}^{\overline{q}_t} \exp\left(-\rho_*\frac{\lambda^2}{\beta_t^2}a_k^2(n)\right) d\lambda = \int_{|\lambda|\leq\beta_t^{1-\delta}\mathcal{C}_2^{-1}} \exp\left(-\rho_*\frac{\lambda^2}{\beta_t^2}\sum_{k=\underline{q}_t}^{\overline{q}_t} a_k^2(n)\right) d\lambda$$

$$\left[\mu = \left(\beta_t^{-2}\sum_{k=\underline{q}_t}^{\overline{q}_t} a_k^2(n)\right)^{1/2}\lambda\right]$$

$$\leq \left(\inf_{n\geq n_0, t_0\leq t\leq n}\beta_t^{-2}\sum_{k=\underline{q}_t}^{\overline{q}_t} a_k^2(n)\right)^{-1/2} \int_{\mathbb{R}} \exp\left(-\rho_*\mu^2\right) d\mu < \infty,$$

where the last inequality follows from **LP3(a)**. Hence, $\sup_{n\geq n_0, t_0\leq t\leq n} I_{1,n}(t) < \infty$ for some $n_0 \geq t_0 \in \mathbb{N}$.

Next, we show that $I_{2,n}(t)$ is bounded. For $n \geq t$ and t large enough we get

$$I_{2,n}(t) \leq \int_{|\lambda|\geq\beta_t^{1-\delta}\mathcal{C}_2^{-1}} \prod_{k=\underline{q}_t}^{\overline{q}_t} \left|\psi_\xi\left(\frac{\lambda}{\beta_t}a_k(n)\right)\right| d\lambda$$

$$= \int_{|\lambda|\geq\beta_t^{1-\delta}\mathcal{C}_2^{-1}} \prod_{k=\underline{q}_t}^{\underline{q}_t+\theta-1} \left|\psi_\xi\left(\frac{\lambda}{\beta_t}a_k(n)\right)\right| \prod_{k=\underline{q}_t+\theta}^{\overline{q}_t} \left|\psi_\xi\left(\frac{\lambda}{\beta_t}a_k(n)\right)\right| d\lambda,$$

with θ is as in **Assumption INN**. For $|\lambda| \geq \beta_t^{1-\delta}\mathcal{C}_2^{-1}$ and $\underline{q}_t + \theta \leq k \leq \overline{q}_t$ we get

$$\left|\psi_\xi\left(\frac{\lambda}{\beta_t}a_k(n)\right)\right| \leq_{(1)} \exp\left\{-\rho_*\left[\left(\frac{\lambda}{\beta_t}a_k(n)\right)^2 \wedge 1\right]\right\}$$

$$\leq \exp\left\{-\rho_*\left[\left(\frac{\beta_t^{1-\delta}\mathcal{C}_2^{-1}}{\beta_t}a_k(n)\right)^2 \wedge 1\right]\right\} \leq \exp\left\{-\rho_*\left(\frac{\mathcal{C}_2^{-1}}{\beta_t^{\delta}}a_k(n)\right)^2\right\},$$

where $0 < \rho_* < \infty$, and $\leq_{(1)}$ is due to **Lemma 6.1**. It follows from the above and **LP3(a)** that

$$\sup_{|\lambda| \geq \beta_t^{1-\delta} \mathcal{C}_2^{-1}} \prod_{k=\underline{q}_t+\theta}^{\overline{q}_t} \left| \psi_\xi \left(\frac{\lambda}{\beta_t} a_k(n) \right) \right| \leq \prod_{k=\underline{q}_t+\theta}^{\overline{q}_t} \exp \left\{ -\rho_* \left(\frac{\mathcal{C}_2^{-1}}{\beta_t^\delta} a_k(n) \right)^2 \right\}$$

$$= \exp \left\{ -\frac{\rho_* \sum_{k=\underline{q}_t+\theta}^{\overline{q}_t} a_k^2(n)}{\beta_t^{2\delta} \mathcal{C}_2^2} \right\} = \exp \left\{ -\frac{\rho_* \beta_t^{-2} \sum_{k=\underline{q}_t+\theta}^{\overline{q}_t} a_k^2(n)}{\beta_t^{2(\delta-1)} \mathcal{C}_2^2} \right\}$$

$$\leq \exp \left\{ -\rho_* \mathcal{C}_1^2 \beta_t^{2(1-\delta)} / \mathcal{C}_2^2 \right\} =: Q_t,$$

where the last inequality follows from **LP3(a)**. Note that $Q_t \to 0$ as $t \to \infty$, because $\beta_t \to \infty$. Hence, for $n \geq t$ and t large we get

$$I_{2,n}(t) \leq Q_t \int_{|\lambda| \geq \beta_t^{1-\delta} \mathcal{C}_2^{-1}} \prod_{k=\underline{q}_t}^{\underline{q}_t+\theta-1} \left| \psi_\xi \left(\frac{\lambda}{\beta_t} a_k(n) \right) \right| d\lambda$$

$$\leq_{(2)} Q_t \prod_{k=\underline{q}_t}^{\underline{q}_t+\theta-1} \left(\int_{|\lambda| \geq \beta_t^{1-\delta} \mathcal{C}_2^{-1}} \left| \psi_\xi \left(\frac{\lambda}{\beta_t} a_k(n) \right) \right|^\theta d\lambda \right)^{1/\theta}$$

$$\left[\mu = \frac{\lambda}{\beta_t} a_k(n) \right]$$

$$\leq Q_t \beta_t \prod_{k=\underline{q}_t}^{\underline{q}_t+\theta-1} \left(|a_k(n)|^{-1} \int_{\mathbb{R}} \left| \psi_\xi(\lambda) \right|^\theta d\lambda \right)^{1/\theta}$$

$$\leq_{(3)} \sup_{n \geq n_0, t_0 \leq t \leq n} Q_t \beta_t \sup_{\underline{q}_t \leq k \leq \underline{q}_t+\theta-1} |a_k(n)|^{-1} \int_{\mathbb{R}} \left| \psi_\xi(\lambda) \right|^\theta d\lambda < \infty,$$

where $\leq_{(2)}$ follows from Lemma 7 in Jeganathan (2008), $\leq_{(3)}$ is due to **Assumption INN** and **LP3(c)**, and the result follows.

(i.b) For the second part let $1 \leq t \leq t_0 < \infty$. We have for n large

$$\int_{\mathbb{R}} \left| \mathbf{E} \left(e^{i\lambda x_t(n)/\beta_t} \right) \right| d\lambda \leq \int_{\mathbb{R}} \prod_{l=1}^{q} \left| \psi \left(\frac{\lambda}{\beta_t} a_{k^*_{n,t,l},t}(n) \right) \right| d\lambda$$

$$\leq \prod_{l=1}^{q}\left\{\int_{\mathbb{R}}\left|\psi_{\xi}\left(\frac{\lambda}{\beta_t}a_{k^*_{n,t,l},t}(n)\right)\right|^{q}d\lambda\right\}^{1/q}=\prod_{l=1}^{q}\left\{\int_{\mathbb{R}}\frac{\beta_t}{\left|a_{k^*_{n,t,l},t}(n)\right|}\left|\psi_{\xi}\left(\lambda\right)\right|^{q}d\lambda\right\}^{1/q}$$

$$\leq \frac{\sup_{1\leq t\leq t_0}\beta_t}{\inf_{1\leq l\leq q,1\leq t\leq t_0}\left|a_{k^*_{n,t,l},t}(n)\right|}\int_{\mathbb{R}}\left|\psi_{\xi}\left(\lambda\right)\right|^{q}d\lambda<\infty,$$

where the last inequality is due to **Assumption INN**, **LP4** and the fact that β_t is a finite sequence (**LP3**). ■

Proof of Lemma LVAR. For convenience set $\sigma_{\xi}^2=1$. Throughout we also use the convention $\sum_{j=b}^{a}u_j=0$ for $a<b$. Further, under **LL1-LL2** without loss of generality set $t_n=n$. The result for the general case follows immediately from the arguments used below and the fact that $L(t_n)/L(n)\to 1$ (see **Lemma 6.3(iii)**).

(i) LL1: We prove the result under **LL1**, without utilising conditions (a) or (b) i.e. the only restrictions imposed on $\{c_j\}$ in this proof are $\sum_{j=0}^{\infty}|c_j|<\infty$ and $\sum_{j=0}^{\infty}c_j\neq 0$. Recall that $\ell(x)$ is a SV on $[\overline{\ell},\infty)$, $\overline{\ell}>0$. For convenience and without loss of generality extend the domain of definition by setting $\ell(x)=1$ for all integer $x\leq\overline{\ell}$. Further, given that ϕ_j can be approximated by $\ell(j)j^{-1/2}$ for j large, there is no loss of generality if we assume that $0<\ell(j)<\infty$ for all j. Note that all SV functions are eventually non zero and finite (e.g. Bingham et al. 1987, Lemma 1.3.2). These conventions will simplify the derivations below. Write $x_t=x_t(n)$, and consider

$$\gamma_n^{-2}Var(x_n^{+})=\gamma_n^{-2}\sum_{s=0}^{n-1}\left(\sum_{l=0}^{s}\phi_l c_{s-l}\right)^2=\gamma_n^{-2}\sum_{s=0}^{n-1}\left(\sum_{l=0}^{s}\phi_{s-l}c_l\right)^2$$

$$=\gamma_n^{-2}\sum_{s=0}^{n-1}\sum_{l=0}^{s}\phi_j^2c_{s-l}^2+2\gamma_n^{-2}\sum_{s=0}^{n-1}\sum_{l=1}^{s}\sum_{k=0}^{l-1}\phi_{s-l}\phi_{s-k}c_lc_k=:A_n+2B_n.$$

We shall show that as $n\to\infty$

$$B_n=\gamma_n^{-2}\sum_{s=1}^{n-1}\ell\left(s\right)^2s^{-1}\sum_{k=0}^{n-1}\sum_{l=k+1}^{n}c_kc_l+o(1). \tag{42}$$

Similarly, it can be shown that

$$A_n=\gamma_n^{-2}\sum_{s=1}^{n-1}\ell\left(s\right)^2s^{-1}\sum_{k=0}^{n}c_k^2+o(1).$$

Further, noting that $\gamma_n^2 = L(n)$, **Lemma 6.3(iii)** gives $\gamma_n^{-2} \sum_{s=1}^{n-1} \ell(s)^2 s^{-1} = 1 + o(1)$. In view of the above, we get as $n \to \infty$

$$\gamma_n^{-2} Var(x_n^+) = \sum_{k=0}^{\infty} c_k^2 + 2 \sum_{k=0}^{\infty} \sum_{l=k+1}^{\infty} c_k c_l + o(1) = \left(\sum_{k=0}^{\infty} c_k \right)^2 .$$

Subsequently we shall show that

$$\gamma_n^{-2} Var(x_n^-) = o(1).$$

First, we set out to show (42). Write

$$B_n = \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=l}^{n-1} \phi_{s-l} \phi_{s-k} c_l c_k = \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \left(\sum_{s=l+\sqrt{L(n)}+1}^{n-1} + \sum_{s=l}^{l+\sqrt{L(n)}} \right) \phi_{s-l} \phi_{s-k} c_l c_k.$$

Note that as $n \to \infty$

$$\gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=l}^{l+\sqrt{L(n)}} \left| \phi_{s-l} \phi_{s-k} c_l c_k \right| \leq \left(\sup_{s \geq 0} |\phi_s| \right)^2 \sqrt{L(n)} \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k|$$

$$\leq C \left(\sum_{l=1}^{\infty} |c_l| \right)^2 \sqrt{L(n)} \gamma_n^{-2} = O\left(L(n)^{-1/2} \right) = o(1).$$

Set $\lambda(k) := \ell(k) k^{-1/2}$. Hence, as $n \to \infty$

$$B_n = \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=l+\sqrt{L(n)}+1}^{n} \phi_{s-l} \phi_{s-k} c_l c_k + o(1)$$

$$= \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=l+\sqrt{L(n)}+1}^{n} c_l c_k \lambda(s-l) \lambda(s-k) + o(1) =: B_n' + o(1), \quad (43)$$

where the approximation in the last line above will be demonstrated in detail later. Set

$$G_n := \sup_{\sqrt{L(n)} \leq x \leq n} \sup_{1 \leq y < \infty} \sum_{j=1}^{2} \left| \frac{\lambda(xy)^j}{\lambda(x)^j} - y^{-j/2} \right|, \quad (44)$$

$$G_{s,k,l} := \left| \left(\frac{s-k}{s-l} \right)^{-1/2} - 1 \right| + \left| \left(\frac{s}{s-l} \right)^{-1} - 1 \right|.$$

Note that for $j=\{1,2\}$, $\lambda(x)^j$ is regularly varying with index $-j/2$, and $L(n)\to\infty$ as $n\to\infty$. Therefore, by Theorem 1.5.2. in Bingham et al. (1987) we have $\lim_{n\to\infty} G_n = 0$. In view of this, as $n\to\infty$

$$|B'_n - B''_n| := \left|\gamma_n^{-2}\sum_{l=1}^{n-1}\sum_{k=0}^{l-1}\sum_{s=l+\sqrt{L(n)}+1}^{n}\left\{c_l c_k\left[\lambda(s-l)\lambda(s-k)-\lambda(s)^2\right]\right\}\right|$$

$$\leq \gamma_n^{-2}\sum_{l=1}^{n-1}\sum_{k=0}^{l-1}\sum_{s=l+\sqrt{L(n)}}^{n}\left|c_l c_k\left[\lambda(s-l)\lambda(s-k)-\lambda(s)^2\right]\right|$$

$$\leq \gamma_n^{-2}\sum_{l=1}^{n-1}\sum_{k=0}^{l-1}\sum_{s=l+\sqrt{L(n)}}^{n}|c_l c_k|\left\{\left|\lambda(s-l)\lambda(s-k)-\lambda(s-l)^2\right|+\left|\lambda(s-l)^2-\lambda(s)^2\right|\right\}$$

$$= \gamma_n^{-2}\sum_{l=1}^{n-1}\sum_{k=0}^{l-1}|c_l c_k|\sum_{s=l+\sqrt{L(n)}}^{n}\lambda(s-l)^2\left\{\left|\frac{\lambda(s-k)}{\lambda(s-l)}-1\right|+\left|\frac{\lambda(s)^2}{\lambda(s-l)^2}-1\right|\right\}$$

$$\leq \gamma_n^{-2}\sum_{l=1}^{n-1}\sum_{k=0}^{l-1}|c_l c_k|\sum_{s=l+\sqrt{L(n)}}^{n}\lambda(s-l)^2\left\{\left|\frac{\lambda(s-k)}{\lambda(s-l)}-\left(\frac{s-k}{s-l}\right)^{-1/2}\right|+\left|\frac{\lambda(s)^2}{\lambda(s-l)^2}-\left(\frac{s}{s-l}\right)^{-1}\right|\right.$$

$$\left.+\left|\left(\frac{s-k}{s-l}\right)^{-1/2}-1\right|+\left|\left(\frac{s}{s-l}\right)^{-1}-1\right|\right\}$$

$$\leq \gamma_n^{-2}\sum_{l=1}^{n-1}\sum_{k=0}^{l-1}|c_l c_k|\sum_{s=l+\sqrt{L(n)}}^{n}\lambda(s-l)^2\left[G_n+G_{s,k,l}\right] =: R_{1,n}+R_{2,n},$$

where we have used the fact that for $j=\{1,2\}$

$$\sup_{1\leq l\leq n}\sup_{0\leq k\leq l-1}\sup_{l+\sqrt{L(n)}\leq s\leq n}\left|\frac{\lambda(s-k)^j}{\lambda(s-l)^j}-\left(\frac{s-k}{s-l}\right)^{-j/2}\right| \leq \sup_{1\leq l\leq n}\sup_{0\leq k\leq l-1}\sup_{\sqrt{L(n)}\leq s-l\leq n}\left|\frac{\lambda\left((s-l)\,\frac{s-k}{s-l}\right)^j}{\lambda(s-l)^j}\right.$$

$$\left.-\left(\frac{s-k}{s-l}\right)^{-j/2}\right| \leq \sup_{\sqrt{L(n)}\leq x\leq n}\sup_{1\leq y<\infty}\left|\frac{\lambda(xy)^j}{\lambda(x)^j}-y^{-j/2}\right|.$$

The first term

$$R_{1,n} \leq G_n \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \sum_{q=1+\sqrt{L(n)}}^{n-l} \lambda(q)^2$$

$$\leq G_n \gamma_n^{-2} \left(\sum_{l=1}^{\infty} |c_l| \right)^2 \sum_{q=1}^{n} \lambda(q)^2 = G_n \left(\sum_{l=1}^{\infty} |c_l| \right)^2 (1 + o(1)) = o(1),$$

where the first approximation shown above is due to **Lemma 6.3(iii)**. The second term

$$R_{2,n} = \gamma_n^{-2} \left(\sum_{l=1}^{\sqrt{L(n)}} + \sum_{l=1+\sqrt{L(n)}}^{n-1} \right) \sum_{k=0}^{l-1} |c_l c_k| \sum_{q=1+\sqrt{L(n)}}^{n-l} \lambda(q)^2 \left[\left| \left(\frac{q}{q+(l-k)} \right)^{1/2} - 1 \right| \right.$$

$$\left. + \left| \frac{q}{q+l} - 1 \right| \right] := R'_{2,n} + R''_{2,n}.$$

Note that

$$\left| \left(\frac{q}{q-(l-k)} \right)^{1/2} - 1 \right| + \left| \frac{q}{q+l} - 1 \right| \leq \left(\frac{l-k}{q+(l-k)} \right)^{1/2} + \frac{l}{q+l}.$$

Hence,

$$R'_{2,n} \leq \gamma_n^{-2} \sum_{l=1}^{\sqrt{L(n)}} \sum_{k=0}^{l-1} |c_l c_k| \sum_{q=1+\sqrt{L(n)}}^{n} \lambda(q)^2 \left[\frac{\sqrt[4]{L(n)}}{q^{1/2}} + \frac{\sqrt{L(n)}}{q} \right]$$

$$\leq C \gamma_n^{-2} \sqrt{L(n)} \left(\sum_{l=1}^{\infty} |c_l| \right)^2 \sum_{q=1}^{\infty} \lambda(q)^2 q^{-1/2} = O\left(L(n)^{-1/2} \right) = o(1),$$

where we have used the fact that $\gamma_n^2 = L(n)$. Further,

$$R''_{2,n} \leq 2\gamma_n^{-2} \sum_{l=1+\sqrt{L(n)}}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \sum_{q=1+\sqrt{L(n)}}^{n} \lambda(q)^2 \leq 2 \sum_{l=\sqrt{L(n)}}^{\infty} |c_l| \sum_{k=0}^{\infty} |c_k| (1 + o(1)) = o(1).$$

Therefore, as $n \to \infty$ we get $|B'_n - B''_n| = o(1)$.

Next, we get as $n \to \infty$

$$|B'''_n - B''_n| := \gamma_n^{-2} \left| \sum_{l=1}^{n-1} c_l \sum_{k=0}^{l-1} c_k \left(\sum_{s=1}^{n} \lambda(s)^2 - \sum_{s=l+1+\sqrt{L(n)}}^{n} \lambda(s)^2 \right) \right|$$

$$\leq \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \left(\sum_{s=1}^{l} \lambda(s)^2 + \sum_{s=l+1}^{l+\sqrt{L(n)}} \lambda(s)^2 \right)$$

$$\leq \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \sum_{s=1}^{l} \lambda(s)^2 + o(1) + \sqrt{L(n)} \gamma_n^{-2} \left(\sum_{l=0}^{\infty} |c_l| \right)^2 \sup_{s \geq 1} \lambda(s)^2$$

$$\leq \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \left[\left| \sum_{s=1}^{l} \lambda(s)^2 - L(l) \right| + L(l) \right] + O(L(n)^{-1/2})$$

$$=_{(1)} \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \, L(l) + o(1) \leq C \gamma_n^{-2} \sum_{l=1}^{n} |c_l| \, L(l) =_{(2)} o(1),$$

where $=_{(1)}$ follows from **Lemma 6.3(iii)**, and $=_{(2)}$ from Kronecker's lemma e.g. Hall and Heyde (1980), p. 31 (recall that $L(n) = \gamma_n^2$). In view of the above, and using again **Lemma 6.3(iii),** we have as $n \to \infty$

$$B_n = B'''_n + o(1) = \sum_{l=1}^{n} \sum_{k=0}^{l-1} c_l c_k + o(1) = \sum_{k=0}^{n-1} \sum_{l=k+1}^{n} c_k c_l,$$

which yields (42).

Next, we show (43). Note that given $\varepsilon > 0$ and some integer N_ε we have $\left| \frac{\phi_{s-l}\phi_{s-k}}{\lambda(s-l)\lambda(s-k)} - 1 \right| < \varepsilon$ for $s-l$, $s-k \geq N_\varepsilon$. Hence, as $n \to \infty$ first and then as $\varepsilon \downarrow 0$

$$|B_n - B'_n| \leq \gamma_n^{-2} \sum_{l=1}^{n} \sum_{k=0}^{l-1} \sum_{s=l+1+N_\varepsilon}^{n} \left| \frac{\phi_{s-l}\phi_{s-k}}{\lambda(s-l)\,\lambda(s-k)} - 1 \right| \lambda(s-l)\,\lambda(s-k)\,|c_l c_k|$$

$$\leq \varepsilon \gamma_n^{-2} \sum_{l=1}^{n} \sum_{k=0}^{l-1} \sum_{s=l+1}^{n} \lambda(s-l)\,\lambda(s-k)\,|c_l c_k| \leq \varepsilon C \overset{\varepsilon\downarrow 0}{\to} 0$$

where the last inequality follows from the fact that

$$\gamma_n^{-2}\sum_{l=1}^{n}\sum_{k=0}^{l-1}\sum_{s=l+1}^{n}\lambda\left(s-l\right)\lambda\left(s-k\right)\left|c_l c_k\right|$$

resembles B_n' which is convergent.

Finally, consider

$$\gamma_n^{-2}Var\left(x_n^{-}\right)=\sum_{s=n}^{\infty}\left(\sum_{j=0}^{n-1}\phi_j c_{s-j}\right)^2\leq\sum_{s=n}^{\infty}\left(\sum_{j=0}^{n}\left|\phi_j c_{s-j}\right|\right)^2$$

$$=\gamma_n^{-2}\left(\sum_{s=n}^{\infty}\sum_{j=0}^{n}\phi_j^2 c_{s-j}^2+2\sum_{s=n}^{\infty}\sum_{j=0}^{n-1}\sum_{j'=j+1}^{n}\left|\phi_j c_{s-j}\phi_{j'}c_{s-j'}\right|\right)=:E_n+2F_n.$$

Recall that $\lambda(x)=\ell(x)x^{-1/2}$. The first term

$$E_n=\gamma_n^{-2}\sum_{s=n}^{\infty}\left(\sum_{j=0}^{\lfloor n/2\rfloor}+\sum_{j=\lfloor n/2\rfloor+1}^{n}\right)\phi_j^2 c_{s-j}^2=:E_n'+E_n''.$$

Next, as $n\to\infty$

$$\begin{aligned}E_n' &= \gamma_n^{-2}\sum_{j=0}^{\lfloor n/2\rfloor}\phi_j^2\sum_{s=n}^{\infty}c_{s-j}^2=\gamma_n^{-2}\sum_{j=0}^{\lfloor n/2\rfloor}\phi_j^2\sum_{l=n-j}^{\infty}c_l^2\leq\gamma_n^{-2}\sum_{j=0}^{\lfloor n/2\rfloor}\phi_j^2\sum_{l=\lfloor n/2\rfloor}^{\infty}c_l^2=o(1).\\ & [l=s-j]\end{aligned}$$

The second term as $n\to\infty$

$$E_n''=\gamma_n^{-2}\sum_{j=\lfloor n/2\rfloor+1}^{n}\phi_j^2\sum_{s=n}^{\infty}c_{s-j}^2\leq\gamma_n^{-2}\sum_{j=\lfloor n/2\rfloor+1}^{n}\phi_j^2\sum_{s=0}^{\infty}c_s^2$$

$$=C\gamma_n^{-2}\sum_{j=\lfloor n/2\rfloor+1}^{n}\phi_j^2=C\gamma_n^{-2}\sum_{j=\lfloor n/2\rfloor+1}^{n}\frac{\ell\left(j\right)^2}{j}+o(1)=C\gamma_n^{-2}\left[\sum_{j=1}^{n}\frac{\ell\left(j\right)^2}{j}-\sum_{j=1}^{\lfloor n/2\rfloor}\frac{\ell\left(j\right)^2}{j}\right]$$

$$=C\left[1+o(1)-\frac{\gamma_{\lfloor n/2\rfloor}^2}{\gamma_n^2}\left(1+o(1)\right)\right]=o(1),$$

where the first approximation shown above follows from similar arguments as those used for the proof of (43), whilst the second approximation is due **Lemma 6.3(iii)** and the fact that $\gamma_n^2 = L(n)$. The final term is

$$F_n = \gamma_n^{-2}\sum_{s=n}^{\infty}\sum_{j=0}^{n-1}\sum_{j'=j+1}^{n}\left|\phi_j\phi_{j'}c_{s-j}c_{s-j'}\right| \leq \gamma_n^{-2}\sum_{j=\sqrt{L(n)}+1}^{n-1}\sum_{j'=j+1}^{n}\sum_{s=n}^{\infty}\left|\phi_j\phi_{j'}c_{s-j}c_{s-j'}\right|$$

$$+\gamma_n^{-2}\sqrt{L(n)}\left(\sup_{j\in\mathbb{N}}\left|\phi_j\right|\sum_{s=0}^{\infty}\left|c_s\right|\right)^2$$

$$= \gamma_n^{-2}\sum_{j=\sqrt{L(n)}+1}^{n-1}\sum_{j'=j+1}^{n}\lambda\left(j\right)\lambda\left(j'\right)\sum_{s=n}^{\infty}\left|c_{s-j}c_{s-j'}\right| + o(1) + O\left(L(n)^{-1/2}\right), \quad (45)$$

where the last approximation will be demonstrated in detail later. Set G_n as in (44) and note that

$$\sup_{\sqrt{L(n)}\leq j\leq n}\sup_{j\leq j'\leq n}\left|\frac{\lambda\left(j\frac{j'}{j}\right)}{\lambda\left(j\right)} - \left(\frac{j'}{j}\right)^{-1/2}\right| = \sup_{\sqrt{L(n)}\leq x\leq n}\sup_{1\leq y<\infty}\left|\frac{\lambda\left(xy\right)}{\lambda\left(x\right)} - y^{-1/2}\right| \leq G_n.$$

In view of this consider

$$\left|\gamma_n^{-2}\sum_{j=\sqrt{L(n)}+1}^{n-1}\sum_{j'=j+1}^{n}\lambda\left(j\right)\lambda\left(j'\right)\sum_{s=n}^{\infty}\left|c_{s-j}c_{s-j'}\right| - F_n'\right| :=$$

$$\gamma_n^{-2}\left|\sum_{j=\sqrt{L(n)}+1}^{n-1}\sum_{j'=j+1}^{n}\left[\lambda\left(j\right)\lambda\left(j'\right) - \lambda\left(j\right)^2\right]\sum_{s=n}^{\infty}\left|c_{s-j}c_{s-j'}\right|\right|$$

$$\leq \gamma_n^{-2}\sum_{j=\sqrt{L(n)}+1}^{n-1}\sum_{j'=j+1}^{n}\lambda\left(j\right)^2\left|\frac{\lambda\left(j'\right)}{\lambda\left(j\right)} - 1\right|\sum_{s=n}^{\infty}\left|c_{s-j}c_{s-j'}\right|$$

$$\leq \gamma_n^{-2}\sum_{j=\sqrt{L(n)}+1}^{n-1}\lambda\left(j\right)^2\sum_{s=n}^{\infty}\sum_{j'=j+1}^{n}\left|c_{s-j}c_{s-j'}\right|\left[G_n + \left|\left(\frac{j'}{j}\right)^{-1/2} - 1\right|\right]$$

$$\leq [G_n + 2]\, \gamma_n^{-2} \sum_{j=\sqrt{L(n)}+1}^{n-1} \lambda(j)^2 \sum_{s=n}^{\infty} \sum_{j'=j+1}^{n} |c_{s-j} c_{s-j'}| = [G_n + 2]\, F_n'.$$

Note that $G_n = o(1)$ from before. Further, it is shown below that $F_n' = o(1)$. In view of the above this in turn implies $F_n = F_n' + o(1) = o(1)$, as required. Next, we show that $F_n' = o(1)$ in detail. Note that

$$\begin{aligned} F_n' &\leq \gamma_n^{-2} \sum_{j=1}^{n} \lambda(j)^2 \sum_{s=n}^{\infty} \sum_{j'=j+1}^{n} |c_{s-j} c_{s-j'}| \\ &\quad [q = s - j'] \end{aligned}$$

$$\begin{aligned} &= \gamma_n^{-2} \sum_{j=1}^{n} \lambda(j)^2 \sum_{s=n}^{\infty} |c_{s-j}| \sum_{q=s-n}^{s-j-1} |c_q| \leq \sum_{q=0}^{\infty} |c_q|\, \gamma_n^{-2} \sum_{j=1}^{n} \lambda(j)^2 \sum_{l=n-j}^{\infty} |c_l| \\ &\quad [l = s - j] \end{aligned}$$

$$= C\gamma_n^{-2} \left(\sum_{j=1}^{\lfloor n/2 \rfloor} + \sum_{j=\lfloor n/2 \rfloor+1}^{n} \right) \lambda(j)^2 \sum_{l=n-j}^{\infty} |c_l| \leq C\gamma_n^{-2} \left[\sum_{j=1}^{\lfloor n/2 \rfloor} \lambda(j)^2 \sum_{l=\lfloor n/2 \rfloor}^{\infty} |c_l| + \sum_{j=\lfloor n/2 \rfloor}^{n} \lambda(j)^2 \right]$$

$$= C \left[(1 + o(1)) \sum_{l=\lfloor n/2 \rfloor}^{\infty} |c_l| + 1 + o(1) - \frac{\gamma_{\lfloor n/2 \rfloor}^2}{\gamma_n^2} (1 + o(1)) \right] = o(1),$$

where the last line above is due to **Lemma 6.3(iii)**. Finally, we show that the approximation in (45) holds. As before, note that $\left|\phi_j \phi_{j'} / \lambda(j)\, \lambda(j') - 1\right| < \varepsilon$ for $j, j' \geq N_\varepsilon$. Hence, for n large enough such that $\sqrt{L(n)} > N_\varepsilon$ we have

$$\begin{aligned} F_n'' &:= \left| \gamma_n^{-2} \sum_{j=1+\sqrt{L(n)}}^{n-1} \sum_{j'=j+1}^{n} \left[\phi_j \phi_{j'} - \lambda(j)\, \lambda(j') \right] \sum_{s=n}^{\infty} |c_{s-j} c_{s-j'}| \right| \\ &\leq \varepsilon \gamma_n^{-2} \sum_{j=1+\sqrt{L(n)}}^{n-1} \sum_{j'=j+1}^{n} \lambda(j)\, \lambda(j') \sum_{s=n}^{\infty} |c_{s-j} c_{s-j'}|. \end{aligned}$$

Noting that the term above resembles that in (45), which in convergent, we get as $n \to \infty$ first and then as $\varepsilon \downarrow 0$, $F_n'' = o(1)$.

(ii) LL2: Without loss of generality set $\ell(j) = 1$ (i.e. $c_j \sim j^{-3/2}$). The result for the general case follows from **Lemma 6.3(ii)** and arguments similar to those used in the previous part. In view of the fact that $\phi_j = 1$, write

$$x_t = \sum_{s=0}^{\infty} \sum_{j=0}^{(t-1)\wedge s} \phi_j c_{s-j} \xi_{t-s} = \sum_{s=0}^{t-1} \sum_{j=0}^{s} c_{s-j} \xi_{t-s} + \sum_{s=t}^{\infty} \sum_{j=0}^{t-1} c_{s-j} \xi_{t-s} =: x_t^+ + x_t^-.$$

Using the fact that $\sum_{j=0}^{\infty} c_j = 0$, the first term

$$\gamma_n^{-2} Var(x_n^+) = \gamma_n^{-2} \sum_{s=0}^{n-1} \left(\sum_{j=0}^{s} c_{s-j} \right)^2 = \gamma_n^{-2} \sum_{s=0}^{n-1} \left(\sum_{r=0}^{s} c_r \right)^2$$
$$[r = s - j]$$

$$= \gamma_n^{-2} \sum_{s=0}^{n-1} \left(- \sum_{r=s+1}^{\infty} c_r \right)^2 = \frac{\gamma_n^{-2}}{n} \sum_{s=0}^{n-1} \left(\frac{1}{n} \sum_{r=s+1}^{\infty} \left(\frac{r}{n} \right)^{-3/2} \right)^2 + o(1),$$

where the approximation above can be established using arguments similar to those used in part (i), eq. (43). Hence, in view of the above Euler summation (**Lemma 6.3(ii)-(iii)**) gives

$$\gamma_n^{-2} Var(x_n^+) = \frac{\gamma_n^{-2}}{n} \sum_{s=0}^{n-1} \left(\int_{\frac{s+1}{n}}^{\infty} x^{-3/2} dx \right)^2 + o(1)$$

$$= \frac{\gamma_n^{-2}}{n} \sum_{s=0}^{n-1} \left(2 \left[x^{-1/2} \right]_{\frac{s+1}{n}}^{\infty} \right)^2 = 4 \frac{\gamma_n^{-2}}{n} \sum_{s=0}^{n-1} \left(\frac{s+1}{n} \right)^{-1} = 4 + o(1).$$

Similarly we get as $n \to \infty$,

$$\gamma_n^{-2} Var(x_n^-) = \gamma_n^{-2} \sum_{s=n}^{\infty} \left(\sum_{j=0}^{n-1} c_{s-j} \right)^2 = \gamma_n^{-2} \sum_{s=n}^{\infty} \left(\sum_{j=s+1-n}^{s} c_l \right)^2 = \gamma_n^{-2} \sum_{s=n+1}^{\infty} \left(\sum_{j=s-n}^{s} l^{-3/2} \right)^2 + o(1)$$
$$[l = s - j]$$

$$= \frac{\gamma_n^{-2}}{n} \sum_{s=n+1}^{\infty} \left(\frac{1}{n} \sum_{j=s-n}^{s} \left(\frac{l}{n} \right)^{-3/2} \right)^2 = \frac{\gamma_n^{-2}}{n} \sum_{s=n+1}^{\infty} \left(\int_{\frac{s}{n}-1}^{\frac{s}{n}} x^{-3/2} dx \right)^2 + o(1)$$

$$= \frac{4\gamma_n^{-2}}{n} \sum_{s=n+1}^{\infty} \left(\left[x^{-1/2} \right]_{\frac{s}{n}-1}^{\frac{s}{n}} \right)^2 = \frac{4\gamma_n^{-2}}{n} \sum_{s=n+1}^{\infty} \left(\left(\frac{s}{n} \right)^{-1/2} - \left(\frac{s}{n} - 1 \right)^{-1/2} \right)^2$$

$$\begin{aligned} = \quad & 4\gamma_n^{-2} \int_{1+\frac{1}{n}}^{\infty} \left(x^{-1/2} - (x-1)^{-1/2} \right)^2 dx + o(1) = 4\gamma_n^{-2} \int_{\frac{1}{n}}^{\infty} \left((u+1)^{-1/2} - u^{-1/2} \right)^2 du \\ & [u = x - 1] \end{aligned}$$

$$= 4\gamma_n^{-2} \left(\int_{\frac{1}{n}}^{1} + \int_{1}^{\infty} \right) \left((u+1)^{-1/2} - u^{-1/2} \right)^2 du =: 4\left(A_n + B_n \right).$$

Using the mean value theorem the second term

$$\gamma_n^{-2} B_n = \gamma_n^{-2} \int_1^{\infty} \left((u+1)^{-1/2} - u^{-1/2} \right)^2 du \leq \gamma_n^{-2} \int_1^{\infty} u^{-3} du = o(1).$$

The first term

$$A_n = \gamma_n^{-2} \int_{\frac{1}{n}}^{1} \left((u+1)^{-1} + u^{-1} - 2\,(u+1)^{-1/2}\, u^{-1/2} \right) du$$

$$= 1 + o(1) - 2\gamma_n^{-2} \int_{\frac{1}{n}}^{1} (u+1)^{-1/2}\, u^{-1/2} du = 1 + o(1),$$

where we have used the fact that

$$\gamma_n^{-2} \int_{\frac{1}{n}}^{1} (u+1)^{-1/2}\, u^{-1/2} du \leq \gamma_n^{-2} \int_0^1 u^{-1/2} du = o(1).$$

Proof of the increments result: Write

$$\begin{aligned} x_t^- \quad = \quad & \sum_{k=t}^{\infty} \sum_{j=0}^{t-1} c_{k-j} \xi_{t-k} = \sum_{i=-\infty}^{0} \sum_{j=0}^{t-1} c_{t-i-j} \xi_i = \sum_{i=-\infty}^{0} \sum_{l=1}^{t} c_{l-i} \xi_i \\ & [i = t - k \to k = t - i, l = t - j] \end{aligned}$$

Let $0 < s \leq r \leq 1$. Consider

$$\gamma_n^{-2} \mathbf{E} \left(x_{\lfloor nr \rfloor}^- - x_{\lfloor ns \rfloor}^- \right)^2 = \gamma_n^{-2} \mathbf{E} \left(\sum_{i=-\infty}^{0} \sum_{l=1}^{\lfloor nr \rfloor} c_{l-i} \xi_i - \sum_{i=-\infty}^{0} \sum_{l=1}^{\lfloor ns \rfloor} c_{l-i} \xi_i \right)^2$$

$$= \gamma_n^{-2} \sum_{i=-\infty}^{0} \left(\sum_{l=\lfloor ns \rfloor+1}^{\lfloor nr \rfloor} c_{l-i} \right)^2 = \gamma_n^{-2} \sum_{i=-\infty}^{0} \left(\sum_{l=\lfloor ns \rfloor+1}^{\lfloor nr \rfloor} (l-i)^{-3/2} \right)^2 + o(1).$$

where the approximation shown above follows from similar arguments as those used in the proof of (43). Further, using arguments similar to those used in part (ii) above we also get $\int_s^\infty \left[(r-s+u)^{-1/2} - u^{-1/2} \right]^2 du < \infty$. In view of this, Euler summation yields

$$\gamma_n^{-2} \mathbf{E} \left(x^-_{\lfloor nr \rfloor} - x^-_{\lfloor ns \rfloor} \right)^2 = \frac{\gamma_n^{-2}}{n} \sum_{i=-\infty}^{0} \left(\frac{1}{n} \sum_{l=\lfloor ns \rfloor+1}^{\lfloor nr \rfloor} \left(\frac{l-i}{n} \right)^{-3/2} \right)^2 + o(1)$$

$$= \frac{\gamma_n^{-2}}{n} \sum_{i=-\infty}^{0} \left(\int_{\frac{\lfloor ns \rfloor+1}{n}}^{\frac{\lfloor nr \rfloor}{n}} \left(x - \frac{i}{n} \right)^{-3/2} dx \right)^2 + o(1) = 4 \frac{\gamma_n^{-2}}{n} \sum_{i=-\infty}^{0} \left(\left[\left(x - \frac{i}{n} \right)^{-1/2} \right]_{\frac{\lfloor ns \rfloor+1}{n}}^{\frac{\lfloor nr \rfloor}{n}} \right)^2$$

$$= 4 \frac{\gamma_n^{-2}}{n} \sum_{i=0}^{\infty} \left(\left[\left(\frac{\lfloor nr \rfloor}{n} + \frac{i}{n} \right)^{-1/2} - \left(\frac{\lfloor ns \rfloor + 1}{n} + \frac{i}{n} \right)^{-1/2} \right] \right)^2$$

$$= \quad 4\gamma_n^{-2} \int_0^\infty \left[\left(\frac{\lfloor nr \rfloor}{n} + x \right)^{-1/2} - \left(\frac{\lfloor ns \rfloor + 1}{n} + x \right)^{-1/2} \right]^2 dx + o(1)$$

$$\left[u = \frac{\lfloor ns \rfloor + 1}{n} + x \to x = u - \frac{\lfloor ns \rfloor + 1}{n} \right]$$

$$= 4\gamma_n^{-2} \int_{\frac{\lfloor ns \rfloor+1}{n}}^{\infty} \left[\left(\frac{\lfloor nr \rfloor - \lfloor ns \rfloor - 1}{n} + u \right)^{-1/2} - u^{-1/2} \right]^2 du$$

$$\leq 4\gamma_n^{-2} \int_s^\infty \left[(r-s+u)^{-1/2} - u^{-1/2} \right]^2 du + o(1) = O\left(\gamma_n^{-2}\right).$$

(iii) Consider

$$\gamma_n^{-2} Var(x^+_{t_n}(n)) = \gamma_n^{-2} \sum_{s=0}^{t_n-1} \left(\sum_{j=0}^{s} \rho_n^j c_{s-j} \right)^2 = \gamma_n^{-2} \sum_{s=0}^{t_n} \left(\sum_{l=0}^{s} \rho_n^{s-l} c_l \right)^2 + o(1)$$

$$= \gamma_n^{-2} \sum_{s=0}^{t_n-1} \sum_{l=0}^{s} \rho_n^{2(s-l)} c_l^2 + 2\gamma_n^{-2} \sum_{s=0}^{t_n-1} \sum_{l=1}^{s} \sum_{k=0}^{l-1} \rho_n^{s-l} c_l \rho_n^{s-k} c_k =: A_n + 2B_n.$$

We shall show that as $n \to \infty$

$$B_n = \frac{1}{2} \sum_{l=1}^{\infty} c_l \sum_{k=0}^{l-1} c_k + o(1) \tag{46}$$

Further, it can be shown that

$$A_n = \frac{1}{2} \sum_{j=0}^{\infty} c_j^2 + o(1). \tag{47}$$

The result follows immediately from (46) and (47). We only consider in detail B_n which is more complicated than A_n. Recall that $\gamma_n^2 = \kappa_n$ and write

$$\begin{aligned} B_n &= \gamma_n^{-2} \sum_{l=1}^{t_n-1} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} \sum_{s=l}^{t_n-1} \rho_n^{2(s-l)} = \gamma_n^{-2} \sum_{l=1}^{t_n-1} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} \sum_{q=0}^{t_n-1-l} \rho_n^{2q} \\ & [q = s - l] \\ &= \frac{\gamma_n^{-2}}{1-\rho_n^2} \sum_{l=1}^{t_n} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} \left(1 - \rho_n^{2(t_n-l)}\right) + o(1) \\ &= \frac{1}{2} \left(\sum_{l=1}^{t_n} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} - \sum_{l=1}^{t_n} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} \rho_n^{2(t_n-l)} \right) =: \frac{1}{2} \left(B_n' - B_n''\right) \end{aligned}$$

Let $m_n > 0$ integer valued sequence such that $\frac{1}{m_n} + \frac{m_n}{\kappa_n} \to 0$. Consider first

$$B_n' = \sum_{l=1}^{t_n} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} = \left(\sum_{l=1}^{m_n} + \sum_{l=m_n+1}^{t_n} \right) c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} =: C_n' + C_n''.$$

In view of **Lemma 6.4(i)**

$$C_n' = \sum_{l=1}^{m_n} c_l \sum_{k=0}^{l-1} c_k e^{-\frac{l-k}{\kappa_n}} + O\left(m_n/\kappa_n^2\right) = \sum_{l=1}^{\infty} c_l \sum_{k=0}^{l-1} c_k + o(1),$$

where the last approximation follows from the fact that $\sup_{1 \le l \le m_n, 0 \le k \le l-1} \left| e^{-\frac{l-k}{\kappa_n}} - 1 \right| \le \frac{m_n}{\kappa_n}$. Next, using the fact that $|\rho_n| \le 1$ for n large we get

$$|C_n''| \le \sum_{l=m_n+1}^{\infty} |c_l| \sum_{k=0}^{\infty} |c_k| = o(1).$$

Further, for n large enough

$$|B_n''| \le \left(\sum_{l=1}^{m_n} + \sum_{l=m_n+1}^{t_n}\right) \sum_{k=0}^{l-1} \left|c_l c_k \rho_n^{l-k} \rho_n^{2(t_n-l)}\right| \le \sum_{l=1}^{m_n} |c_l| \sum_{k=0}^{l-1} |c_k| \left|\rho_n\right|^{2(t_n-l)}$$

$$+ \sum_{l=m_n+1}^{\infty} |c_l| \sum_{k=0}^{\infty} |c_k| \le \left(\sum_{k=0}^{\infty} |c_k|\right)^2 \left|\rho_n\right|^{2(t_n-m_n)} + o(1) = o(1),$$

where the last approximation is due to **Lemma 6.4(ii)**, and hence (46) holds.

Finally, consider

$$\gamma_n^{-2} Var\left(x_{t_n}^-(n)\right) \le \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left(\sum_{j=0}^{t_n} \left|\rho_n^j c_{s-j}\right|\right)^2$$

$$= \gamma_n^{-2} \left(\sum_{j=0}^{t_n} \sum_{s=t_n}^{\infty} \rho_n^{2j} c_{s-j}^2 + 2 \sum_{s=t_n}^{\infty} \sum_{j=0}^{t_n-1} \sum_{j'=j+1}^{t_n} \left|\rho_n^j c_{s-j} \rho_n^{j'} c_{s-j'}\right|\right) =: D_n + 2E_n$$

Consider

$$D_n \le \gamma_n^{-2} \sum_{j=0}^{t_n} \rho_n^{2j} \sum_{l=t_n-j}^{\infty} c_l^2 = \gamma_n^{-2} \left(\sum_{j=0}^{\lfloor t_n/2 \rfloor} + \sum_{j=\lfloor t_n/2 \rfloor+1}^{t_n}\right) \rho_n^{2j} \sum_{l=t_n+1-j}^{\infty} c_l^2 =: D_n' + D_n''.$$

Lemma 6.4(iii) is utilised in the subsequent approximations. First in view of the fact that $\gamma_n^2 = \kappa_n$, the first term

$$D_n' \le \gamma_n^{-2} \sum_{j=0}^{\lfloor t_n/2 \rfloor} \rho_n^{2j} \sum_{l=t_n+1-\lfloor t_n/2 \rfloor}^{\infty} c_l^2 \le C \sum_{l=\lfloor t_n/2 \rfloor}^{\infty} c_l^2 = o(1).$$

Next, for some $0 < \delta < 1$ **Lemma 6.4(ii)-(iii)** yields

$$D_n'' \le \gamma_n^{-2} \sum_{j=\lfloor t_n/2 \rfloor+1}^{t_n} \rho_n^{2j} \sum_{l=0}^{\infty} c_l^2 = O\left(e^{-2\frac{\lfloor t_n/2 \rfloor}{\kappa_n}}\right) + o\left(\kappa_n^{-1} e^{-2\delta\frac{\lfloor t_n/2 \rfloor}{\kappa_n}}\right) = o(1).$$

Finally, for n large we have

$$\begin{aligned} |E_n| &= \gamma_n^{-2} \sum_{j=0}^{t_n-1} \rho_n^{2j} \sum_{j'=j+1}^{t_n} \sum_{s=t_n}^{\infty} |c_{s-j} c_{s-j'}| = \gamma_n^{-2} \sum_{j=0}^{t_n-1} \rho_n^{2j} \sum_{s=t_n}^{\infty} |c_{s-j}| \sum_{l=s-t_n}^{s-j-1} |c_l| \\ [l &= s-j'] \end{aligned}$$

$$\leq \gamma_n^{-2}\sum_{j=0}^{t_n}\rho_n^{2j}\sum_{s=t_n}^{\infty}|c_{s-j}|\sum_{l=0}^{\infty}|c_l| \leq \gamma_n^{-2}\sum_{j=0}^{t_n}\rho_n^{2j}\sum_{q=t_n-j}^{\infty}|c_q|\sum_{l=0}^{\infty}|c_l|$$

$$[q = s-j]$$

$$= C\gamma_n^{-2}\left(\sum_{j=0}^{\lfloor t_n/2\rfloor} + \sum_{j=1+\lfloor t_n/2\rfloor}^{t_n}\right)\rho_n^{2j}\sum_{q=t_n-j}^{\infty}|c_q| \leq C\left(\sum_{q=\lfloor t_n/2\rfloor}^{\infty}|c_q| + \gamma_n^{-2}\sum_{j=1+\lfloor t_n/2\rfloor}\rho_n^{2j}\sum_{q=0}^{\infty}|c_q|\right)$$

$$= o(1) + C\gamma_n^{-2}\sum_{j=1+\lfloor t_n/2\rfloor}^{t_n}\rho_n^{2j} = O\left(e^{-2\frac{\lfloor t_n/2\rfloor}{\kappa_n}}\right) + o\left(\kappa_n^{-1}e^{-2\delta\frac{\lfloor t_n/2\rfloor}{\kappa_n}}\right) = o(1).$$

(iv) LL4: Recall that in this case $\gamma_n = \kappa_n^{3/2-m}$, $1/2 < m < 1$. Set $\varepsilon_n := t_n \wedge \delta_n$ with $\delta_n := \lfloor\kappa_n^{\beta}\rfloor$, $\beta \in \left(1, \frac{7-2m}{5-2m}\right)$. Notice that $1 < \frac{7-2m}{5-2m} < 2$ and therefore $\kappa_n \ln(\kappa_n)/\delta_n + \delta_n/\kappa_n^2 \to 0$. Further, for any sequence η_n set $Q_{\eta_n}(t_n) := \sum_{s=t_n}^{\infty}\left(\sum_{j=0}^{\eta_n-1}\rho_n^j c_{s-j}\right)^2$. We shall show first that

$$\gamma_n^{-2}Var\left(x_{t_n}^{+}(n)\right) = \gamma_n^{-2}Var\left(x_{\varepsilon_n}^{+}\right) + o(1), \tag{48}$$

and

$$\gamma_n^{-2}Var\left(x_{t_n}^{-}(n)\right) = \gamma_n^{-2}Q_{\varepsilon_n}(t_n) + o(1). \tag{49}$$

Subsequently, we show that

$$\gamma_n^{-2}Var\left(x_{\varepsilon_n}^{+}(n)\right) = \int_{y=0}^{\infty} e^{-2y}\left(\int_{x=0}^{y} e^x x^{-m}dx\right)^2 dy + o(1) \tag{50}$$

and

$$\gamma_n^{-2}Q_{\varepsilon_n}(t_n) = o(1). \tag{51}$$

In view of (48)-(51), the result follows from **Lemma 7.3** that shows that

$$\int_{y=0}^{\infty} e^{-2y}\left(\int_{x=0}^{y} e^x x^{-m}dx\right)^2 dy = \Gamma(2-2m)\,B(1-m, 2m-1) < \infty,$$

where $B(x,y)$ and $\Gamma(x)$ are Beta and Gamma function respectively.

We start with the proof of (48). Notice that

$$Var\left(x_{t_n}^{+}(n)\right) - Var\left(x_{\varepsilon_n}^{+}(n)\right) = \left[Var\left(x_{t_n}^{+}(n)\right) - Var\left(x_{\delta_n}^{+}(n)\right)\right] I\{\delta_n < t_n\}.$$

Now for $\delta_n < t_n$,

$$\gamma_n^{-2}\left[Var\left(x_{t_n}^{+}(n)\right) - Var\left(x_{\varepsilon_n}^{+}(n)\right)\right] = \gamma_n^{-2}\sum_{s=\delta_n}^{t_n-1}\left(\sum_{j=0}^{s}\rho_n^j c_{s-j}\right)^2$$

$$\le \gamma_n^{-2}\sum_{s=\delta_n}^{t_n}\sum_{j=0}^{s}\rho_n^{2j}c_{s-j}^2 + 2\gamma_n^{-2}\sum_{s=\delta_n}^{t_n}\sum_{j=0}^{s-1}\sum_{k=j+1}^{s}\left|\rho_n^j c_{s-j}\rho_n^k c_{s-k}\right| =: A_n + 2B_n.$$

Let $\bar{\delta}_n := \lfloor \delta_n/2 \rfloor$. As $n\to\infty$, the second term

$$\gamma_n^2 B_n = \sum_{j=0}^{t_n}\sum_{s=\delta_n\vee(j+1)}^{t_n}\sum_{k=j+1}^{s}\left|\rho_n^j\rho_n^k c_{s-j}c_{s-k}\right| = \sum_{j=0}^{t_n}\sum_{k=j+1}^{t_n}\sum_{s=\delta_n\vee k}^{t_n}\left|\rho_n^j\rho_n^k c_{s-j}c_{s-k}\right|$$

$$\left[\begin{array}{c} \delta_n \le s \le t_n \\ 0 \le j \le s-1 \\ j+1 \le k \le s \end{array} \rightarrow \begin{array}{c} 0 \le j \le t_n - 1 \\ \delta_n \vee (j+1) \le s \le t_n \\ j+1 \le k \le s \end{array}\right.$$

$$\left.\rightarrow \begin{array}{c} 0 \le j \le t_n - 1 \\ j+1 \le k \le t_n \\ \delta_n \vee (j+1) \vee k \le s \le t_n \end{array} \rightarrow \begin{array}{c} 0 \le j \le t_n - 1 \\ j+1 \le k \le t_n \\ \delta_n \vee k \le s \le t_n \end{array}\right]$$

$$\le \sum_{j=0}^{t_n}\sum_{k=j+1}^{t_n}\sum_{s=\delta_n\vee k}^{t_n}\left|\rho_n^j\rho_n^k c_{s-j}c_{s-k}\right| \le \left(\sum_{j=0}^{\bar{\delta}_n}\sum_{k=j+1}^{\bar{\delta}_n} + \sum_{j=0}^{\bar{\delta}_n}\sum_{k=\bar{\delta}_n+1}^{t_n}\right.$$

$$\left. + \sum_{j=\bar{\delta}_n+1}^{t_n}\sum_{k=j+1}^{t_n}\right)\sum_{s=\delta_n\vee k}^{t_n}\left|\rho_n^j\rho_n^k c_{s-j}c_{s-k}\right| =: \gamma_n^2\left(B_n' + B_n'' + B_n'''\right).$$

The first term

$$\gamma_n^2 B_n' = \sum_{j=0}^{\bar{\delta}_n}\sum_{k=j+1}^{\bar{\delta}_n}\sum_{s=\delta_n\vee k}^{t_n}\left|\rho_n^j\rho_n^k c_{s-j}c_{s-k}\right| = \sum_{j=0}^{\bar{\delta}_n}\sum_{k=j+1}^{\bar{\delta}_n}\sum_{l=[\delta_n\vee k]-k}^{t_n-k}\left|\rho_n^j\rho_n^k c_{l+k-j}c_l\right|$$

$$[l = s-k \rightarrow s = l+k]$$

$$\le \sum_{j=0}^{\bar{\delta}_n}\sum_{k=j+1}^{\bar{\delta}_n}\sum_{l=[\delta_n\vee k]-k}^{\infty}\left|\rho_n^j\rho_n^k c_l^2\right| \le \sum_{j=0}^{\bar{\delta}_n}\sum_{k=j+1}^{\bar{\delta}_n}\sum_{l=\delta_n-\bar{\delta}_n}^{\infty}\left|\rho_n^j\rho_n^k c_l^2\right| \le \sum_{j=0}^{\bar{\delta}_n}\rho_n^j\sum_{k=j}^{\bar{\delta}_n}\rho_n^k\sum_{l=\bar{\delta}_n}^{\infty}c_l^2.$$

Hence, using **Lemmas 6.3(iii)** and **6.4(iii)** we get

$$B_n' \le C\frac{\kappa_n}{\gamma_n^2}\sum_{j=0}^{\bar{\delta}_n}\rho_n^{2j}\sum_{l=\bar{\delta}_n}^{\infty}c_l^2 \le C\frac{\kappa_n^2}{\gamma_n^2}\sum_{l=\bar{\delta}_n}^{\infty}c_l^2$$

$$= C\frac{\kappa_n^2}{\gamma_n^2}\sum_{l=\bar{\delta}_n}^{\infty}l^{-2m} + o(1) = C\frac{\kappa_n^2\bar{\delta}_n^{1-2m}}{\gamma_n^2}\frac{1}{\bar{\delta}_n}\sum_{l=\bar{\delta}_n}^{\infty}\left(\frac{l}{\bar{\delta}_n}\right)^{-2m}$$

$$= C\frac{\kappa_n^2\bar{\delta}_n^{1-2m}}{\gamma_n^2}\left[\int_1^{\infty}x^{-2m}dx + o(1)\right] = o(1),$$

where the last approximation follows from the fact that

$$\frac{2^{1-2m}\kappa_n^2\bar{\delta}_n^{1-2m}}{\gamma_n^2} \sim \frac{\kappa_n^2\delta_n^{1-2m}}{\kappa_n^{3-2m}} = \frac{\delta_n^{1-2m}}{\kappa_n^{1-2m}} = o(1).$$

Similarly, as $n \to \infty$

$$\begin{aligned}
B_n'' &= \gamma_n^{-2}\sum_{j=0}^{\bar{\delta}_n}\sum_{k=\bar{\delta}_n+1}^{t_n}\sum_{s=\delta_n\vee k}^{t_n}\left|\rho_n^j\rho_n^k c_{s-j}c_{s-k}\right| = \gamma_n^{-2}\sum_{j=0}^{\bar{\delta}_n}\sum_{k=\bar{\delta}_n+1}^{t_n}\sum_{l=(\delta_n\vee k)-k}^{t_n-k}\left|\rho_n^j\rho_n^k c_{l+k-j}c_l\right| \\
l &= s-k \to s = l+k
\end{aligned}$$

$$\le \sum_{l=0}^{\infty}\left|c_l^2\right|\gamma_n^{-2}\sum_{j=0}^{\bar{\delta}_n}\sum_{k=\bar{\delta}_n+1}^{t_n}\rho_n^j\rho_n^k \le C\frac{\kappa_n\rho_n^{\bar{\delta}_n}}{\gamma_n^2}\sum_{j=0}^{\bar{\delta}_n}\rho_n^j \le C\frac{\kappa_n^2\rho_n^{\bar{\delta}_n}}{\gamma_n^2} = C\frac{\kappa_n\rho_n^{\bar{\delta}_n}}{\kappa_n^{2(1-m)}} = o(1),$$

where the last approximation above follows from **Lemma 6.3(iii)** and the fact that $\bar{\delta}_n^{-1}\kappa_n\ln(\kappa_n) \to 0$ (which imply $\kappa_n\rho_n^{\bar{\delta}_n} = o(1)$). Similarly,

$$\left|B_n'''\right| = \gamma_n^{-2}\sum_{j=\bar{\delta}_n+1}^{t_n}\sum_{k=j+1}^{t_n}\sum_{s=\delta_n\vee k}^{t_n}\left|\rho_n^j\rho_n^k c_{s-j}c_{s-k}\right| \le \gamma_n^{-2}\sum_{j=\bar{\delta}_n+1}^{t_n}\rho_n^j\sum_{k=j+1}^{t_n}\rho_n^k\sum_{s=0}^{\infty}c_s^2$$

$$\le C\frac{\kappa_n}{\gamma_n^2}\sum_{j=\bar{\delta}_n}^{t_n}\rho_n^{2j} = C\frac{\kappa_n^2}{\gamma_n^2}\rho_n^{2\bar{\delta}_n} = C\frac{\left(\kappa_n\rho_n^{\bar{\delta}_n}\right)^2}{\gamma_n^2} = o(\gamma_n^{-2}).$$

Finally, using similar arguments we can also show that $\gamma_n^{-2}A_n = o(1)$. Therefore, (48) holds. Next, we show (49). Consider

$$Var\left(x_{t_n}^-(n)\right) - Q_{\varepsilon_n}(t_n) = \left[Var\left(x_{t_n}^-(n)\right) - Q_{\delta_n}(t_n)\right] I\left\{\delta_n < t_n\right\}.$$

Recall that $Q_{\delta_n}(t_n) := \sum_{s=t_n}^{\infty}\left(\sum_{j=0}^{\delta_n-1}\rho_n^j c_{s-j}\right)^2$ and therefore $Var\left(x_{t_n}^-(n)\right) = Q_{t_n}(t_n)$. For $\delta_n < t_n$, write

$$\gamma_n^{-2}Var\left(x_{t_n}^-(n)\right) = \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left(\sum_{j=0}^{\delta_n-1}+\sum_{j=\delta_n}^{t_n-1}\right)\rho_n^{2j}c_{s-j}^2$$

$$+\gamma_n^{-2}2\left(\sum_{j=0}^{\delta_n-1}\sum_{k=j+1}^{\delta_n-1}+\sum_{j=0}^{\delta_n-1}\sum_{k=\delta_n}^{t_n-1}+\sum_{j=\delta_n}^{t_n-1}\sum_{k=j+1}^{t_n-1}\right)\sum_{s=t_n}^{\infty}\rho_n^j c_{s-j}\rho_n^k c_{s-k}$$

Hence,

$$\gamma_n^{-2}Var\left(x_{t_n}^-(n)\right) - \gamma_n^{-2}Q_{\delta_n}(t_n) = \gamma_n^{-2}\sum_{s=t_n}^{\infty}\sum_{j=\delta_n}^{t_n-1}\rho_n^{2j}c_{s-j}^2$$

$$+2\gamma_n^{-2}\left(\sum_{j=0}^{\delta_n-1}\sum_{k=\delta_n}^{t_n-1}+\sum_{j=\delta_n}^{t_n-1}\sum_{k=j+1}^{t_n-1}\right)\sum_{s=t_n}^{\infty}\rho_n^j c_{s-j}\rho_n^k c_{s-k} := C_n + 2D_n.$$

As $n\to\infty$, the second term above

$$|D_n| \le \gamma_n^{-2}\left(\sum_{j=0}^{\delta_n}\sum_{k=\delta_n}^{t_n}+\sum_{j=\delta_n}^{t_n}\sum_{k=j+1}^{t_n}\right)\sum_{s=t_n}^{\infty}\left|\rho_n^j c_{s-j}\rho_n^k c_{s-k}\right|$$

$$\le \sum_{s=0}^{\infty}c_s^2\gamma_n^{-2}\left[\sum_{j=0}^{\delta_n}\rho_n^j\sum_{k=\delta_n}^{t_n}\rho_n^k+\sum_{j=\delta_n}^{t_n}\rho_n^j\sum_{k=j+1}^{t_n}\rho_n^k\right]$$

$$\le C\frac{\kappa_n}{\gamma_n^2}\left[\rho_n^{\delta_n}\sum_{j=0}^{\delta_n}\rho_n^j+\sum_{j=\delta_n}^{t_n}\rho_n^{2j}\right] \le C\frac{\kappa_n}{\gamma_n^2}\left[\kappa_n\rho_n^{\delta_n}+\kappa_n\rho_n^{2\delta_n}\right] = o(1),$$

where we have used **Lemma 6.3(iii)** and the fact that for n large, $\kappa_n\rho_n^{2\delta_n} \le \kappa_n\rho_n^{\delta_n} = o(1)$. Similarly, we can show that $C_n = o(1)$. Therefore, (49) holds.

Next, we show (50). Write

$$\gamma_n^{-2}\left|Var\left(x_{\varepsilon_n}^+(n)\right) - \sum_{s=0}^{\varepsilon_n-1}\left(\sum_{j=0}^{s}e^{-\frac{j}{\kappa_n}}c_{s-j}\right)^2\right| = \gamma_n^{-2}\left|\sum_{s=0}^{\varepsilon_n-1}\left\{\left(\sum_{j=0}^{s}\rho_n^j c_{s-j}\right)^2-\left(\sum_{j=0}^{s}e^{-\frac{j}{\kappa_n}}c_{s-j}\right)^2\right\}\right|$$

$$= \gamma_n^{-2}\left|\sum_{s=0}^{\varepsilon_n-1}\left\{\sum_{j=0}^{s}\rho_n^j c_{s-j}-\sum_{j=0}^{s}e^{c\frac{j}{\kappa_n}}c_{s-j}\right\}\left\{\sum_{j=0}^{s}\rho_n^j c_{s-j}+\sum_{j=0}^{s}e^{c\frac{j}{\kappa_n}}c_{s-j}\right\}\right|$$

$$= \gamma_n^{-2} \left| \sum_{s=0}^{\varepsilon_n - 1} \left\{ \sum_{j=0}^{s} \left(\rho_n^j - e^{-\frac{j}{\kappa_n}} \right) c_{s-j} \right\} \left\{ \sum_{j=0}^{s} \rho_n^j c_{s-j} + \sum_{j=0}^{s} e^{-\frac{j}{\kappa_n}} c_{s-j} \right\} \right|$$

$$\leq \gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} \left\{ \sum_{j=0}^{s} \left| \left(\rho_n^j - e^{-\frac{j}{\kappa_n}} \right) c_{s-j} \right| \right\} \left\{ \sum_{j=0}^{s} \left| \left(\rho_n^j - e^{-\frac{j}{\kappa_n}} \right) c_{s-j} \right| + 2 \sum_{j=0}^{s} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right| \right\}$$

$$= \gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} \left(\sum_{j=0}^{s} \left| \left(\rho_n^j - e^{-\frac{j}{\kappa_n}} \right) c_{s-j} \right| \right)^2 + 2\gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} \sum_{j=0}^{s} \left| \left(\rho_n^j - e^{-\frac{j}{\kappa_n}} \right) c_{s-j} \right| \sum_{j=0}^{s} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right|$$

$$=: E_n + 2F_n.$$

By **Lemma 6.4(i)**, Euler summation and using similar arguments to those used for the proof of (43) in part (i), the first term is

$$E_n \leq \left(\frac{\varepsilon_n}{\kappa_n^2} \right)^2 \gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} \left(\sum_{l=0}^{s} |c_l| \right)^2 = \left(\frac{\varepsilon_n}{\kappa_n^2} \right)^2 \left(\frac{\varepsilon_n}{\kappa_n} \right)^{3-2m} \frac{1}{\varepsilon_n} \sum_{s=0}^{\varepsilon_n} \left(\frac{1}{\varepsilon_n} \sum_{l=0}^{s} \left(\frac{l}{\varepsilon_n} \right)^{-m} \right)^2 + o(1)$$

$$= \frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}} \left[\int_{y=0}^{1} \left(\int_0^y x^{-m} dx \right)^2 dy + o(1) \right] = O\left(\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}} \right) =_{(1)} o(1), \quad (52)$$

where $=_{(1)}$ follows from the definition of ε_n. Similarly,

$$F_n \leq \frac{\varepsilon_n}{\kappa_n^2} \gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} \sum_{j=0}^{s} |c_{s-j}| \sum_{j=0}^{s} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right|$$

$$\leq \sqrt{ \left(\frac{\varepsilon_n}{\kappa_n^2} \right)^2 \gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} \left(\sum_{l=0}^{s} |c_l| \right)^2 \gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} \left(\sum_{j=0}^{s} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right| \right)^2 } = \sqrt{ O\left(\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}} \right) } = o(1),$$

where we have used (52) the fact that as $n \to \infty$

$$\gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} \left(\sum_{j=0}^{s} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right| \right)^2 \to \int_{y=0}^{\infty} e^{-2y} \left(\int_{x=0}^{y} e^x x^{-m} dx \right)^2 dy,$$

which is justified by the arguments that lead to (53) below. In view of the above Euler summation gives as $n \to \infty$

$$\gamma_n^{-2} Var\left(x_{\varepsilon_n}^{+}(n)\right) = \gamma_n^{-2}\sum_{s=0}^{\varepsilon_n-1}\left(\sum_{j=0}^{s} e^{-\frac{j}{\kappa_n}} c_{s-j}\right)^2 + o(1) = \gamma_n^{-2}\sum_{s=0}^{\varepsilon_n} e^{-\frac{2s}{\kappa_n}}\left(\sum_{k=0}^{s} e^{\frac{k}{\kappa_n}} c_k\right)^2 + o(1)$$

$$[k = s-j \to j = s-k]$$

$$= \kappa_n^{-1}\sum_{s=0}^{\varepsilon_n} e^{-2\frac{s}{\kappa_n}}\left(\frac{1}{\kappa_n}\sum_{k=1}^{s} e^{\frac{k}{\kappa_n}}\left(\frac{k}{\kappa_n}\right)^{-m}\right)^2 + o(1) = \int_0^{\varepsilon_n/\kappa_n} e^{-2y}\left(\int_{1/\kappa_n}^{y} e^x x^{-m} dx\right)^2 dy + o(1)$$

$$= \int_0^{\infty} e^{-2y}\left(\int_0^{y} e^x x^{-m} dx\right)^2 dy + o(1), \tag{53}$$

and this shows (50). Next, we show (51). Using (49) write

$$\gamma_n^{-2} Var\left(x_{t_n}^{-}(n)\right) - \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left(\sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}\right)^2 = \gamma_n^{-2} Q_{\varepsilon_n}(t_n) - \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left(\sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}\right)^2 + o(1)$$

$$= \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left[\sum_{j=0}^{\varepsilon_n-1}\rho_n^j c_{s-j} - \sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}\right]\left[\sum_{j=0}^{\varepsilon_n-1}\rho_n^j c_{s-j} + \sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}\right]$$

$$= \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left[\sum_{j=0}^{\varepsilon_n-1}\left(\rho_n^j c_{s-j} - e^{-\frac{j}{\kappa_n}} c_{s-j}\right)\right]\left[\sum_{j=0}^{\varepsilon_n-1}\left(\rho_n^j c_{s-j} - e^{-\frac{j}{\kappa_n}} c_{s-j}\right) + 2\sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}\right]$$

$$= \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left[\sum_{j=0}^{\varepsilon_n-1}\left(\rho_n^j c_{s-j} - e^{-\frac{j}{\kappa_n}} c_{s-j}\right)\right]^2 + 2\gamma_n^{-2}\sum_{s=t_n}^{\infty}\left[\sum_{j=0}^{\varepsilon_n-1}\left(\rho_n^j c_{s-j} - e^{-\frac{j}{\kappa_n}} c_{s-j}\right)\right]\sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}$$

$$:= G_n + 2H_n.$$

Using again **Lemma 6.4(i)** and Euler summation we have

$$G_n \le \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left[\sum_{j=0}^{\varepsilon_n-1}\left(\rho_n^j c_{s-j} - e^{-\frac{j}{\kappa_n}} c_{s-j}\right)\right]^2 \le C\left(\frac{\varepsilon_n}{\kappa_n^2}\right)^2 \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left(\sum_{j=0}^{\varepsilon_n} |c_{s-j}|\right)^2$$

$$\le C\left(\frac{\varepsilon_n}{\kappa_n^2}\right)^2 \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left(\sum_{l=s-\varepsilon_n}^{s} |c_l|\right)^2 = C\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}}\frac{1}{\varepsilon_n}\sum_{s=t_n}^{\infty}\left(\frac{1}{\varepsilon_n}\sum_{l=s-\varepsilon_n}^{s}\left(\frac{l}{\varepsilon_n}\right)^{-m}\right)^2 + o(1)$$

$$[l = s-j]$$

$$= C\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}}\left[\int_{y=\frac{t_n}{\varepsilon_n}}^{\infty}\left(\int_{x=y-1}^{y} x^{-m}dx\right)^2 dy + o(1)\right] \le C\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}}\int_{y=1}^{\infty}\left(\int_{x=y-1}^{y} x^{-m}dx\right)^2 dy + o(1)$$

$$= C\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}}\int_{u=0}^{\infty}\left[(1+u)^{1-m} - u^{1-m}\right]^2 du = O\left(\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}}\right) = o(1).$$

Similarly,

$$|H_n| \le \sqrt{G_n\gamma_n^{-2}\sum_{s=t_n}^{\infty}\left[\sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}\right]^2} \le \sqrt{O\left(\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}}\right)\gamma_n^{-2}\sum_{s=t_n}^{\infty}\left[\sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}\right]^2}$$

$$= \sqrt{o\left(\frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}}\right)} = o(1),$$

where we have used the facts that $\varepsilon_n^{5-2m}\left(\kappa_n^{7-2m}\right)^{-1} = o(1)$ and

$$\gamma_n^{-2}\sum_{s=t_n}^{\infty}\left[\sum_{j=0}^{\varepsilon_n-1} e^{-\frac{j}{\kappa_n}} c_{s-j}\right]^2 = \int_{y=t_n/\kappa_n}^{\infty} e^{-2y}\left(\int_{x=0}^{y} e^x x^{-m}dx\right)^2 dy + o(1) = o(1),$$

which is justified by the arguments that lead to (54) below. In view of the above, Euler summation gives as $n \to \infty$

$$\begin{aligned}
\gamma_n^{-2}Var\left(x_{t_n}^{-}(n)\right) &= \gamma_n^{-2}Q_{\varepsilon_n}(t_n) + o(1) = \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left(\sum_{j=0}^{\varepsilon_n} e^{-\frac{j}{\kappa_n}} c_{s-j}\right)^2 + o(1) \\
&\quad [k = s-j \to j = s-k]
\end{aligned}$$

$$= \gamma_n^{-2}\sum_{s=t_n}^{\infty}\left(\sum_{k=s-\varepsilon_n}^{s} e^{-\frac{s-k}{\kappa_n}} c_k\right)^2 = \kappa_n^{-(3-2m)}\sum_{s=t_n}^{\infty} e^{-2\frac{s}{\kappa_n}}\left(\sum_{k=s-\varepsilon_n}^{s} e^{\frac{k}{\kappa_n}} c_k\right)^2 + o(1)$$

$$= \frac{1}{\kappa_n}\sum_{s=t_n}^{\infty} e^{-2\frac{s}{\kappa_n}}\left(\frac{1}{\kappa_n}\sum_{k=s-\varepsilon_n}^{s} e^{\frac{k}{\kappa_n}}\left(\frac{k}{\kappa_n}\right)^{-m}\right)^2$$

$$= \int_{y=t_n/\kappa_n}^{\infty} e^{-2y}\left(\int_{x=y-\varepsilon_n/\kappa_n}^{y} e^x x^{-m}dx\right)^2 dy + o(1)$$

$$\leq \int_{y=t_n/\kappa_n}^{\infty} e^{-2y} \left(\int_{x=0}^{y} e^x x^{-m} dx \right)^2 dy = o(1), \tag{54}$$

where the last approximation shown above is due to dominated convergence and the fact that $t_n/\kappa_n \to \infty$ (recall that $\int_{y=0}^{\infty} e^{-2y} \left(\int_{x=0}^{y} e^x x^{-m} dx \right)^2 dy < \infty$, by **Lemma 7.3**). ■

Proof of Lemma CLT. The result follows from **Lemma LVAR** and arguments similar to those used in the proof of **Proposition 2.1(i)**. ■

Appendix B2b (proofs of technical lemmas of Appendix A)

Proof of Lemma 6.1. Fix, $\rho \in (0, 1/2)$. Then there is some $0 < \delta_\rho < 1$ and some $\vartheta_\rho > 0$ (e.g. Feller, 1971, p. 516) such that

$$\left|\psi_\xi (\lambda)\right| \leq \begin{cases} e^{-\rho\lambda^2}, & |\lambda| \leq \delta_\rho \\ e^{-\vartheta_\rho}, & |\lambda| > \delta_\rho \end{cases} . \tag{55}$$

Set $\rho_* := \rho \wedge \vartheta_\rho$. It follows from (55) that

$$\begin{aligned} \left|\psi_\xi (\lambda)\right| &\leq e^{-\rho\lambda^2} 1 \{|\lambda| \leq \delta_\rho\} + e^{-\vartheta_\rho} 1 \{|\lambda| > \delta_\rho\} \\ &\leq e^{-\rho_*\lambda^2} 1 \{|\lambda| \leq \delta_\rho\} + e^{-\rho_*} 1 \{|\lambda| > \delta_\rho\} \\ &= e^{-\rho_*\lambda^2} 1 \{|\lambda| \leq \delta_\rho\} + e^{-\rho_*} 1 \{1 \geq |\lambda| > \delta_\rho\} + e^{-\rho_*} 1 \{|\lambda| > 1\} \\ &\leq e^{-\rho_*\lambda^2} 1 \{|\lambda| \leq 1\} + e^{-\rho_*} 1 \{|\lambda| > 1\} = e^{-\rho_*\left(\lambda^2 \wedge 1\right)}, \end{aligned}$$

as required. ■

Proof of Lemma 6.2. **Part (i) & (ii):** Without loss of generality set $\theta = 0$. It can be easily seen that the arguments below hold true for any $\theta \geq 0$. First, suppose that **Assumption LL1** holds. By **Lemma 7.1** as $k \to \infty$

$$\delta_t^{-1} \eta_k \left| a_k (n) \right| = \frac{\sqrt{k}}{\ell(k)} \left| \sum_{j=0}^{k} \phi_j c_{k-j} \right| = \left| \sum_{j=0}^{\infty} c_j \right| + o(1).$$

In view of this and the fact that $\sum_{j=0}^{\infty} |c_j| < \infty$ and $\left|\sum_{j=0}^{\infty} c_j\right| > 0$, the result for part (i) and (ii) follows immediately.

Under **Assumption LL2** using **Lemma 6.3(ii)**, we get as $k \to \infty$

$$\delta_t^{-1}\eta_k a_k(n) = \frac{k^{1/2}}{\ell(k)}\sum_{j=0}^{k} c_j = -\frac{k^{1/2}}{\ell(k)}\sum_{j=k+1}^{\infty} c_j = -\frac{k^{1/2}}{\ell(k)}\sum_{j=k+1}^{\infty}\ell(j)j^{-3/2} + o(1)$$

$$= -\int_1^{\infty} x^{-3/2}dx + o(1) = -\frac{1}{1-3/2}\left[x^{-1/2}\right]_1^{\infty} + o(1) = -2,$$

and the result follows. Notice that the approximation in the first line above is justified as follows. Fix $\varepsilon > 0$. Then for k large enough $\left|\frac{c_j}{\ell(j)j^{-3/2}} - 1\right| < \varepsilon$. Hence, as $k \to \infty$ first and then as $\varepsilon \to 0$

$$\frac{k^{1/2}}{\ell(k)}\sum_{j=k+1}^{\infty}\left|c_j - \ell(j)j^{-3/2}\right| = \frac{k^{1/2}}{\ell(k)}\sum_{j=k+1}^{\infty}\left|\frac{c_j}{\ell(j)j^{-3/2}} - 1\right|\ell(j)j^{-3/2}$$

$$= \varepsilon\frac{k^{1/2}}{\ell(k)}\sum_{j=k+1}^{\infty}\ell(j)j^{-3/2} \overset{k\to\infty}{=} \varepsilon\int_1^{\infty} x^{-3/2}dx + o(1) \overset{\varepsilon\to 0}{=} o(1),$$

as required.

Suppose that **Assumption LL3** holds and without loss of generality set $\eta_k = 1$. Choose n_0 large enough such that $0 < \rho_n < 1$ for $n \geq n_0$. Hence, $0 < \rho_n^{k-j} \leq 1$ for all $0 \leq k - j$ and $n \geq n_0$. We get

$$\delta_t^{-1}\sup_{0\leq k}\eta_k\left|a_k(n)\right| = \sup_{0\leq k}\left|\sum_{j=0}^{k}\rho_n^{k-j}c_j\right| \leq \sup_{0\leq k}\sum_{j=0}^{k}\left|\rho_n^{k-j}\right|\left|c_j\right| \leq \sum_{j=0}^{\infty}\left|c_j\right| < \infty,$$

and this proves the upper bound in both part (i) and (ii).

For the lower bound note again that $0 < \rho_n < 1$ for n large enough. Therefore as $n \to \infty$

$$\Delta_n := \sup_{n\geq t\geq 1, 0\leq k\leq \overline{q}_t}\delta_t^{-1}\eta_k\left|a_k(n) - \rho_n^k\sum_{j=0}^{k}c_j\right| = \sup_{t\geq 1, 0\leq k\leq \overline{q}_t}\left|\sum_{j=0}^{k}\rho_n^{k-j}c_j - \rho_n^k\sum_{j=0}^{k}c_j\right|$$

$$\leq \sup_{n\geq t\geq 1, 0\leq k\leq \overline{q}_t}\left|\rho_n^k\right|\left|\sum_{j=0}^{k}\rho_n^{-j}c_j - \sum_{j=0}^{k}c_j\right|$$

$$\leq \sup_{n\geq t\geq 1}\sum_{j=0}^{\kappa_t}\left|\rho_n^{-j} - 1\right|\left|c_j\right| \leq \sum_{j=0}^{\infty}\left|\rho_n^{-j} - 1\right| 1\left\{j \leq \kappa_*\kappa_n\right\}\left|c_j\right| = o(1), \quad (56)$$

where $\infty > \kappa_* := \sup_{n \geq 1, 1 \leq t \leq n} \kappa_t/\kappa_n$ and the last approximation is due to dominated convergence. To see this note that for n large enough, $|\rho_n^{-j} - 1| \, 1\{j \leq \kappa_n\}$ is a finite. Further, it is bounded because as $n \to \infty$

$$\left|\rho_n^{-j} - 1\right| 1\{j \leq \kappa_n\} \leq \left|\rho_n^{-\kappa_n} + 1\right| = |e + 1| + o(1).$$

Therefore, noting that $\lim_{n\to\infty} \rho_n^{-j} = 1$ for all j fixed, dominated convergence yields $\Delta_n \to 0$. Next, recall that $\sum_{j=0}^{\infty} c_j \neq 0$. Hence, there is $k_1 > 0$ such for all $k \geq k_1$, $\left|\sum_{j=0}^{k} c_j\right| \geq \left|\sum_{j=0}^{\infty} c_j\right| /2 > 0$. Choose n_1 large enough such that $\lfloor \kappa_t/2 \rfloor \geq k_1$ for $t \geq n_1$. Hence, in view of the uniform convergence result of (56) as $n \to \infty$ we have

$$\inf_{n_1 \leq t \leq n, \lfloor \kappa_t/2 \rfloor \leq k \leq \lfloor \kappa_t \rfloor} \delta_t^{-1} \eta_k \left|a_k(n)\right| = \inf_{n_1 \leq t \leq n, \lfloor \kappa_t/2 \rfloor \leq k \leq \lfloor \kappa_t \rfloor} \left|\rho_n^k\right| \left|\sum_{j=0}^{k} c_j\right| + o(1)$$

$$\geq |\rho_n^{\kappa_* \kappa_n}| \inf_{k_1 \leq k} \left|\sum_{j=0}^{k} c_j\right| \geq |\rho_n^{\kappa_* \kappa_n}| \left|\sum_{j=0}^{\infty} c_j\right| /2 = e^{-\kappa_*} \left|\sum_{j=0}^{\infty} c_j\right| /2 + o(1),$$

and the result follows from the fact that the limit term above is strictly positive.

Finally, suppose that **Assumption LL4** holds. As before without loss of generality, set $\eta_k = 1$. We start with the proof for the upper bound of part (i). Again note that $0 < \rho_n < 1$ for n large. Hence, for $n \geq t$ as we get as $t \to \infty$

$$\delta_t^{-1} \sup_{0 \leq k \leq \overline{q}_t} \eta_k \left|a_k(n)\right| = \kappa_t^{-(1-m)} \sup_{0 \leq k \leq \lfloor \kappa_t \rfloor} \left|\sum_{j=0}^{k} \rho_n^{k-j} c_j\right| \leq \kappa_t^{-(1-m)} \sup_{0 \leq k \leq \lfloor \kappa_t/2 \rfloor} \sum_{j=0}^{k} \left|\rho_n^{k-j}\right| |c_j|$$

$$\leq \kappa_t^{-(1-m)} \sum_{j=0}^{\lfloor \kappa_t \rfloor} |c_j| = \kappa_t^{-1} \sum_{j=1}^{\lfloor \kappa_t \rfloor} \left(\frac{j}{\kappa_t}\right)^{-m} + o(1) = \int_0^1 x^{-m} dx + o(1),$$

where the last approximation above follows from **Lemma 6.3(i)**. For the upper bound of part (ii) we have from **Lemma 7.2**

$$\delta_n^{-1} \sup_{0 \leq k \leq n} \eta_k \left|a_k(n)\right| = \kappa_n^{-(1-m)} \sup_{0 \leq k \leq n} \left|\sum_{j=0}^{k} \rho_n^{k-j} c_j\right| = \kappa_n^{-(1-m)} \sup_{0 \leq k \leq n} \int_0^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n} - x\right)} x^{-m} dx + o(1)$$

$$\leq \kappa_n^{-(1-m)} \left(\sup_{0\leq k\leq \kappa_n} + \sup_{\kappa_n \leq k\leq n} \right) \int_0^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n}-x\right)} dx$$

$$\leq \kappa_n^{-(1-m)} \left[\int_0^1 e x^{-m} dx + \kappa_n^{-(1-m)} \sup_{k\geq \kappa_n} \int_1^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n}-x\right)} x^{-m} dx \right]$$

$$= \kappa_n^{-(1-m)} \left[C + \sup_{k\geq\kappa_n} \left[e^{-\left(\frac{k}{\kappa_n}-x\right)} \right]_1^{k/\kappa_n} \right] = O\left(\kappa_n^{-(1-m)}\right) = o(1). \tag{57}$$

For the lower bound note that by **Lemma 7.2** for $n \geq t$ and as $t \to \infty$ we have

$$\inf_{\underline{q}_t \leq k\leq \overline{q}_t} \delta_t^{-1} \eta_k \left| a_k(n) \right| = \inf_{\lfloor \kappa_t/2 \rfloor \leq k \leq \lfloor \kappa_t \rfloor} \left| \frac{1}{\kappa_t^{1-m}} \sum_{j=0}^{k} \rho_n^{k-j} c_j \right|$$

$$= \inf_{\lfloor \kappa_t/2 \rfloor \leq k \leq \lfloor \kappa_t \rfloor} \int_0^{k/\kappa_t} e^{-\left(\frac{k}{\kappa_n} - \frac{\kappa_t}{\kappa_n} x\right)} x^{-m} dx + o(1) \geq C > 0.$$

Part (iii): Under **Assumption LL1-LL3** we have as $n \to \infty$

$$\gamma_n^{-1} \sup_{1\leq t\leq n, k\geq t} \left| a_{k,t}^{-}(n) \right| \leq \gamma_n^{-1} \sup_{1\leq t\leq n, k\geq t} \sum_{j=0}^{t} \left| \phi_j(n) c_{k-j} \right| \leq \gamma_n^{-1} C \sum_{j=0}^{\infty} |c_j| = o(1).$$

Under **Assumption LL4** using **Lemma 7.2** we get

$$\gamma_n^{-1} \sup_{1\leq t\leq n, k\geq t} \left| a_{k,t}^{-}(n) \right| = \gamma_n^{-1} \sup_{1\leq t\leq n, k> t} \left| \sum_{j=0}^{t} \phi_j(n) c_{k-j} \right|$$

$$= \kappa_n^{-1/2} \left(\sup_{1\leq t\leq n, k\geq t} \int_{(k-t)/\kappa_n}^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n}-x\right)} x^{-m} dx + o(1) \right)$$

$$\leq \kappa_n^{-1/2} \left(\sup_{1\leq k\leq \kappa_n} + \sup_{k\geq \kappa_n} \right) \int_0^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n}-x\right)} x^{-m} dx = O(\kappa_n^{-1/2}) = o(1),$$

where the last equality follows from the same argument that leads to (57) above. ■

Proof of Lemma 6.3. Note that by Lemma 1.3.2. In Bingham et a. (1987) there is some $x_0 > 0$ such that $\varsigma(x)$ is locally bounded for all $x \geq x_0$. We start with the proof of part **(i)**.

(i) Without loss of generality set $s = 0$. The result for $s > 0$ follows immediately from the arguments shown below. Note that $\left|\varphi_j / j^l \varsigma(j) - 1\right| < \varepsilon$ for all $j \geq N_\varepsilon$ and some $N_\varepsilon \geq x_0$. Further, for all j large enough ($j \geq N_\varepsilon$ say) $\varsigma(j)$ is strictly positive and finite. Hence, as $n \to \infty$ first, and then as $\varepsilon \to 0$

$$\left| \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=1}^{\lfloor nr \rfloor} \varphi_j - \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=1+N_\varepsilon}^{\lfloor nr \rfloor} j^l \varsigma(j) \right|$$

$$\leq \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=1+N_\varepsilon}^{\lfloor nr \rfloor} \left|\varphi_j - j^l \varsigma(j)\right| + \frac{\max_{1 \leq j \leq N_\varepsilon} \left|\varphi_j\right|}{n^{1+l}\varsigma(n)} \overset{n\to\infty}{=} \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=1+N_\varepsilon}^{\lfloor nr \rfloor} \left|\varphi_j - j^l \varsigma(j)\right| + o(1)$$

$$\leq \frac{\varepsilon}{n^{1+l}\varsigma(n)} \sum_{j=1+N_\varepsilon}^{\lfloor nr \rfloor} j^l \varsigma(j) \overset{n\to\infty}{=_{(1)}} \varepsilon \left(\int_0^r x^l dx + o(1) \right) \overset{\varepsilon\to 0}{=} o(1),$$

where $=_{(1)}$ is demonstrated by (58) below. Fix δ such that $r > \delta > 0$ and consider

$$\frac{1}{n^{1+l}\varsigma(n)} \sum_{j=1+N_\varepsilon}^{\lfloor nr \rfloor} j^l \varsigma(j) = \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=\lfloor n\delta \rfloor+1}^{\lfloor nr \rfloor} j^l \varsigma(j)$$

$$+ \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=1+N_\varepsilon}^{\lfloor n\delta \rfloor} j^l \varsigma(j) =: S_{n,\delta} + T_{n,\delta}.$$

First, consider

$$\left| S_{n,\delta} - \frac{1}{n^{1+l}} \sum_{j=\lfloor n\delta \rfloor+1}^{\lfloor nr \rfloor} j^l \right| \leq \frac{1}{n} \sum_{j=\lfloor n\delta \rfloor+1}^{\lfloor nr \rfloor} \left(\frac{j}{n}\right)^l \left| \frac{\varsigma(j)}{\varsigma(n)} - 1 \right|.$$

Now notice that by the uniform convergence Theorem for regularly varying functions (e.g. Theorem 1.5.2. in Bingham et al., 1987) as $n \to \infty$

$$\sup_{\lfloor n\delta \rfloor+1 \leq j \leq n} \left| \frac{\varsigma(j)}{\varsigma(n)} - 1 \right| = \sup_{\frac{\lfloor n\delta \rfloor+1}{n} \leq x \leq 1} \left| \frac{\varsigma(xn)}{\varsigma(n)} - 1 \right| \leq \sup_{\delta \leq x \leq 1} \left| \frac{\varsigma(xn)}{\varsigma(n)} - 1 \right| = o(1).$$

In view of this Euler summation gives as $n \to \infty$ first, and then as $\delta \to 0$

$$S_{n,\delta} = \int_\delta^r x^l dx + o(1) \overset{\delta\to 0}{=} \int_0^r x^l dx + o(1). \tag{58}$$

Hence, it suffices showing that $T_{n,\delta}$ is negligible. Set $0 < \eta < 1 + l$. By Potter's Theorem (e.g. Theorem 1.5.6 in Bingham et al., 1987) there is a positive integer X_η such that for all $n \geq j \geq X_\eta$

$$\varsigma(j)/\varsigma(n) \leq 2(n/j)^\eta.$$

Therefore, as $n \to \infty$ first, and then as $\delta \to 0$

$$|T_{n,\delta}| \leq \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=1+N_\varepsilon}^{\lfloor n\delta \rfloor} j^l |\varsigma(j)| = \frac{1}{n^{1+l}\varsigma(n)} \left(\sum_{j=1+N_\varepsilon}^{X_\eta} + \sum_{j=X_\eta+1}^{\lfloor n\delta \rfloor} \right) j^l |\varsigma(j)|$$

$$= \frac{\max_{\{j\in\mathbb{N}:1+N_\varepsilon \leq j \leq X_\eta\}} |\varsigma(j)|}{n^{1+l}\varsigma(n)} + \frac{2}{n} \sum_{j=X_\eta+1}^{\lfloor n\delta \rfloor} \left(\frac{j}{n}\right)^{l-\eta} \overset{n\to\infty}{=} o(1) + 2\int_0^\delta x^{l-\eta} dx \overset{\delta\to 0}{=} o(1).$$

(ii) Using arguments similar those used above we have

$$\frac{1}{n^{1+l}\varsigma(n)} \sum_{j=n}^{\infty} \varphi_j = \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=n}^{\infty} j^l \varsigma(j) + o(1).$$

Next, let $R_p(x) := x^p \varsigma(x)$ (i.e. regularly varying with index p). For $l < -1$, set $l_1 = l - l_2$ with $l_2 = (1+l)/2$. Notice that $l_2 < 0$ and $l_1 < -1$. Therefore, as $n \to \infty$

$$\left| \frac{1}{n^{1+l}\varsigma(n)} \sum_{j=n}^{\infty} j^l \varsigma(j) - \frac{1}{n} \sum_{j=n}^{\infty} \left(\frac{j}{n}\right)^l \right| \leq \frac{1}{n} \sum_{j=n}^{\infty} \left| \frac{j^{l_2}\varsigma(j)}{n^{l_2}\varsigma(n)} - \left(\frac{j}{n}\right)^{l_2} \right| \left(\frac{j}{n}\right)^{l_1}$$

$$= \lim_{M\to\infty} \frac{1}{n} \sum_{j=n}^{M} \left| \frac{R_{l_2}(j)}{R_{l_2}(n)} - \left(\frac{j}{n}\right)^{l_2} \right| \left(\frac{j}{n}\right)^{l_1} \leq \sup_{1 \leq y < \infty} \left| \frac{R_{l_2}(yn)}{R_{l_2}(n)} - y^{l_2} \right| \lim_{M\to\infty} \frac{1}{n} \sum_{j=n}^{M} \left(\frac{j}{n}\right)^{l_1}$$

$$= \sup_{1 \leq y < \infty} \left| \frac{R_{l_2}(yn)}{R_{l_2}(n)} - y^{l_2} \right| \frac{1}{n} \sum_{j=n}^{\infty} \left(\frac{j}{n}\right)^{l_1} =_{(1)} \sup_{1 \leq y < \infty} \left| \frac{R_{l_2}(yn)}{R_{l_2}(n)} - y^{l_2} \right| \left(\int_1^\infty x^{l_1} dx + o(1) \right) =_{(2)} o(1),$$

where $=_{(1)}$ is due to Euler summation and $=_{(2)}$ due to Theorem 1.5.2. in Bingham et al. (1987).

(iii) Set $s(x) := \varsigma(x)/x$. Recall that for all $x \geq x_0$, $s(x)$ is locally bounded. Due to the measurability and positiveness of $s(x)$ (c.f. p. 6 in Bingham et al., 1987), the integral

$$S(x) = \int_{x_0}^{x} \frac{\varsigma(u)}{u} du,$$

is defined in the Lebesgue sense and is finite due to the local boundedness of $\varsigma(x)/x$ on finite intervals far enough to the right. Further, $S(x)$ is SV due to Theorem 1.5.9a in Bingham et al. (1987).

Define $\underline{s}(x) := \inf_{x_0 \leq u \leq x} s(u)$. Clearly, $\underline{s}(x)$ is decreasing. Further, by Bingham et al. (1987), Theorem 1.5.3.

$$\underline{s}(x) \sim s(x)$$

and $\underline{s}(x)$ is regularly varying of index -1. The latter implies that $\underline{s}(x)$ is of the form $\underline{\varsigma}(x)/x$, for some SV function $\underline{\varsigma}(x)$ (e.g. Theorem 1.4.1. in Bingham et al., 1987). We first show that

$$\underline{S}(n) := \int_{x_0}^{n} \underline{s}(x)dx \sim S(n) \tag{59}$$

and then that

$$\sum_{j=x_0}^{n} \frac{\varsigma(j)}{j} \sim \underline{S}(n), \tag{60}$$

which are sufficient for the requisite result. We start with the proof of (59). First note that

$$S(n)^{-1}\left|\int_{x_0}^{n} \underline{s}(u)du - \int_{x_0}^{n} s(u)du\right| =_{(1)} S(n)^{-1}\left|\int_{S(n)}^{n} \underline{s}(u)du - \int_{S(n)}^{n} s(u)du\right| + o(1)$$

$$\leq S(n)^{-1}\int_{S(n)}^{n}\left|\frac{\underline{s}(u)}{s(u)} - 1\right| s(u)du \leq \sup_{S(n)\leq u\leq n}\left|\frac{\underline{s}(u)}{s(u)} - 1\right| S(n)^{-1}\int_{S(n)}^{n} s(u)du$$

$$=_{(2)} o(1)\left[1 + o(1)\right],$$

where $=_{(1)}$ follows from the fact that $S(n)^{-1}\left[S(S(n)) + \underline{S}(S(n))\right] = o(1)$ (recall that S and $\underline{S}$ are SV) and $=_{(2)}$ is due to $\underline{s}(x) \sim s(x)$.

For the proof of (60) note that

$$\left|\underline{S}(n)^{-1}\sum_{j=x_0}^{n}\frac{\varsigma(j)}{j} - 1\right| \leq \underline{S}(n)^{-1}\sum_{j=x_0}^{n}\left|\frac{\varsigma(j) - \underline{\varsigma}(j)}{j}\right| + \left|\underline{S}(n)^{-1}\sum_{j=x_0}^{n}\frac{\underline{\varsigma}(j)}{j} - 1\right|$$

$$= \underline{S}(n)^{-1}\sum_{j=x_0}^{n}\left|\frac{\varsigma(j)}{\underline{\varsigma}(j)} - 1\right|\frac{\underline{\varsigma}(j)}{j} + \left|\underline{S}(n)^{-1}\sum_{j=x_0}^{n}\frac{\underline{\varsigma}(j)}{j} - 1\right| =: T_{1,n} + T_{2,n}$$

First, we show that the second term $T_{2,n} = o(1)$. For a function f denote its total variation on the interval $[a, b]$ as $\mathbf{V}(f)_a^b$. Using integration by parts the second term is

$$T_{2,n} = \left| \underline{S}(n)^{-1} \sum_{j=x_0}^{n} \frac{\underline{\varsigma}(j)}{j} - 1 \right| = \underline{S}(n)^{-1} \left| \sum_{j=x_0}^{n} \frac{\underline{\varsigma}(j)}{j} - \int_{x_0}^{n} \frac{\underline{\varsigma}(x)}{x} dx \right| + o(1)$$

$$= \underline{S}(n)^{-1} \left| \int_{x_0}^{n} \frac{\underline{\varsigma}(x)}{x} d(\lfloor x \rfloor - x) \right| + o(1) = \left| \underline{S}(n)^{-1} \int_{x_0}^{n} (x - \lfloor x \rfloor)\, d\left(\frac{\underline{\varsigma}(x)}{x} \right) \right|$$

$$\leq \underline{S}(n)^{-1} \mathbf{V}\left(\frac{\underline{\varsigma}(x)}{x} \right)_{x_0}^{n}$$

$$=_{(3)} \underline{S}(n)^{-1} \left(\frac{\underline{\varsigma}(x_0)}{x_0} - \frac{\underline{\varsigma}(n)}{n} \right) = o(1)\,(1 + o(1)) = o(1),$$

where $=_{(3)}$ follows from the fact that $\underline{\varsigma}(x)/x$ is decreasing. Next, fix $\varepsilon > 0$ and choose an integer $N_\varepsilon \geq x_0$ such that $\left| \frac{\varsigma(j)}{\underline{\varsigma}(j)} - 1 \right| < \varepsilon$ for $j > N_\varepsilon$. Then as $n \to \infty$ first and then as $\varepsilon \to 0$

$$T_{1,n} \leq o(1) + \varepsilon \underline{S}(n)^{-1} \sum_{j=N_\varepsilon}^{n} \frac{\underline{\varsigma}(j)}{j} =_{(4)} o(1) + \varepsilon\,(1 + o(1)) \overset{\varepsilon \to 0}{\to} 0,$$

where $=_{(4)}$ is due to the fact that $T_{2,n} = o(1)$. ■

Proof of Lemma 6.4: **(i)** Choose n large enough such that $|c|/\kappa_n < 1/2$. Further, let $\bar{x}$ be a mean value such that $0 \geq \bar{x} \geq c/\kappa_n > -1/2$, for n large. Hence, by the mean value for n large enough we get

$$\rho_n^{qj} = \exp\{qj \ln \rho_n\} = \exp\left\{ qj \ln\left(1 + \frac{c}{\kappa_n}\right) \right\} = \exp\left\{ qj \left[\frac{c}{\kappa_n} - \frac{1}{2!\,(1 + \bar{x})^2} \left(\frac{c}{\kappa_n} \right)^2 \right] \right\}.$$

In view of the above and the fact that $|e^x - 1| \leq |x|$ for all $x \leq 0$ we get for n large enough

$$\left| \rho_n^{qj} - \exp\left\{ qj \frac{c}{\kappa_n} \right\} \right| = \left| \exp\left\{ qj \frac{c}{\kappa_n} - qj \frac{1}{2!\,(1 + \bar{x})^2} \left(\frac{c}{\kappa_n} \right)^2 \right\} - \exp\left\{ qj \frac{c}{\kappa_n} \right\} \right|$$

$$= \exp\left\{qj\frac{c}{\kappa_n}\right\}\left|\exp\left\{\frac{-jq}{2!\,(1+\bar{x})^2}\left(\frac{c}{\kappa_n}\right)^2\right\}-1\right|$$

$$\leq \exp\left\{qj\frac{c}{\kappa_n}\right\}\frac{jq}{2!\,(1+\bar{x})^2}\left(\frac{c}{\kappa_n}\right)^2 \leq \exp\left\{qj\frac{c}{\kappa_n}\right\}2qj\left(\frac{c}{\kappa_n}\right)^2.$$

This shows part (a). Part (b) is an immediate consequence of the above.

(ii) It follows from part (i) that for $j=\lambda_n$ and n large

$$\kappa_n e^{q\delta\frac{\lambda_n}{\kappa_n}}\left|\rho_n^{q\lambda_n}-\exp\left\{c\frac{q\lambda_n}{\kappa_n}\right\}\right| \leq \kappa_n e^{q\delta\frac{\lambda_n}{\kappa_n}}2\exp\left\{\frac{q\lambda_n c}{\kappa_n}\right\}q\lambda_n\left(\frac{c}{\kappa_n}\right)^2$$

$$= 2qc^2\exp\left\{q\,(c+\delta)\,\frac{\lambda_n}{\kappa_n}+\ln\left(\frac{\lambda_n}{\kappa_n}\right)\right\} = 2qc^2\exp\left\{\frac{\lambda_n}{\kappa_n}\left[q\,(c+\delta)+\frac{\ln\left(\lambda_n/\kappa_n\right)}{\lambda_n/\kappa_n}\right]\right\}$$

$$= 2qc^2\exp\left\{\frac{\lambda_n}{\kappa_n}\left[q\,(c+\delta)+o(1)\right]\right\} = o(1).$$

(iii) Let $\tau_n \leq k_n$. As $n\to\infty$

$$\sum_{t=1}^{k_n}\rho_n^{qt}-\sum_{t=1}^{\tau_n}\rho_n^{qt} = \rho_n^q\frac{1-\rho_n^{qk_n}}{1-\rho_n^q}-\rho_n^q\frac{1-\rho_n^{q\tau_n}}{1-\rho_n^q}$$

$$= \frac{\rho_n^q}{1-\rho_n^q}\left(\rho_n^{q\tau_n}-\rho_n^{qk_n}\right) = O\left(\kappa_n\rho_n^{q\tau_n}\right),$$

where the last approximation is due to the binomial theorem i.e. $\frac{\rho_n^q}{1-\rho_n^q} = \rho_n^q\left(-\begin{pmatrix} q \\ 1 \end{pmatrix}\frac{c}{\kappa_n}+O(\kappa_n^{-2})\right)^{-1} = O\left(\kappa_n\right)$, and the fact that $\rho_n^{q\tau_n}\geq\rho_n^{qk_n}$ for n large (recall that $\tau_n\leq k_n$).

Now suppose that $\tau_n^{-1}\kappa_n\ln\left(\kappa_n\right)\to 0$. By part (ii) for all $|c|>\delta>0$

$$\rho_n^{q\tau_n} = e^{cq\frac{\tau_n}{\kappa_n}}+o\left(\kappa_n^{-1}e^{-\delta q\frac{\tau_n}{\kappa_n}}\right),$$

Hence, as $n\to\infty$

$$\kappa_n\rho_n^{q\tau_n} = \kappa_n\left[e^{cq\frac{\tau_n}{\kappa_n}}+o\left(\kappa_n^{-1}e^{-\delta q\frac{\tau_n}{\kappa_n}}\right)\right] = \kappa_n e^{cq\frac{\tau_n}{\kappa_n}}+o\left(e^{-\delta q\frac{\tau_n}{\kappa_n}}\right)$$

$$= e^{\frac{\tau_n}{\kappa_n}\left(cq+\frac{\ln(\kappa_n)\kappa_n}{\tau_n}\right)} + o(1) = e^{\frac{\tau_n}{\kappa_n}(cq+o(1))} = o(1),$$

as required. ■

Appendix B3 (proofs of technical lemmas of Appendix B)

Proof of Lemma 7.1. **(i)** Set $\lambda(s) := \ell(s)s^{-1/2}$. Note that for all $\varepsilon > 0$, there some N_ε such that $\left|\phi_{s-l}/\lambda(s-l) - 1\right| < \varepsilon$ when $s - l > N_\varepsilon$ and $\sup_{l>N_\varepsilon} \lambda(l) < \varepsilon$. We have

$$\sum_{j=0}^{s} \phi_j c_{s-j} =_{(1)} \sum_{l=0}^{s-1} \phi_{s-l} c_l + o(\lambda(s)) =_{(2)} \sum_{l=0}^{s-1-N_\varepsilon} \lambda(s-l) c_l + o(\lambda(s)), \quad (61)$$

where $=_{(1)}$ follows from condition (a) or (b), whilst $=_{(2)}$ is demonstrated in detail later. Note that under condition (a), $=_{(1)}$ follows immediately. Under (b) as $s \to \infty$ large

$$|c_s| \le \sum_{j=s}^{\infty} |c_j| = s^{-1/2} \sum_{j=s}^{\infty} j^{1/2} |c_j| = \lambda(s)\ell(s)^{-1} \sum_{j=s}^{\infty} j^{1/2} |c_j| = o(\lambda(s)). \quad (62)$$

Next, set $m_s := \lfloor s\delta \rfloor$ where $0 < \delta < 1$. Consider

$$\sum_{l=0}^{s-1-N_\varepsilon} \lambda(s-l) c_l = \left(\sum_{l=0}^{m_s} + \sum_{l=m_s+1}^{s-1-N_\varepsilon}\right) \lambda(s-l) c_l := T_1(s) + T_2(s).$$

The first term as $s \to \infty$

$$\left|T_1(s) - \lambda(s)\sum_{l=0}^{m_s} c_l\right| \le \sum_{l=0}^{m_s} |\lambda(s) - \lambda(s-l)| |c_l|$$

$$\le \sum_{l=0}^{m_s} \left|\frac{\lambda(s-l)}{\lambda(s)} - 1\right| \lambda(s) |c_l| \le \sum_{l=0}^{m_s} \left[\left|\frac{\lambda\left(s\frac{s-l}{s}\right)}{\lambda(s)} - \left(\frac{s-l}{s}\right)^{-1/2}\right| + \left|\left(\frac{s-l}{s}\right)^{-1/2} - 1\right| \lambda(s) |c_l|\right]$$

$$\leq_{(1)} \sup_{1-\delta\leq x\leq 1}\left|\frac{\lambda(sx)}{\lambda(s)}-x^{-1/2}\right|\lambda(s)\sum_{l=0}^{\infty}|c_l|+\lambda(s)\sum_{l=0}^{m_s}\frac{l^{1/2}}{(s-l)^{1/2}}|c_l|$$

$$=_{(2)} o(\lambda(s))+\lambda(s)\frac{1}{(s-m_s)^{1/2}}\sum_{l=0}^{m_s}l^{1/2}|c_l| =_{(3)} o(\lambda(s)),$$

where $\leq_{(1)}$ follows from the fact that

$$\left(\frac{s}{s-l}\right)^{1/2}-1=\frac{s^{1/2}-(s-l)^{1/2}}{(s-l)^{1/2}}\leq\frac{l^{1/2}}{(s-l)^{1/2}},$$

$=_{(2)}$ from the uniform convergence theorem for regularly varying functions (e.g. Bingham et al. 1987, Theorem 1.5.2.), and $=_{(3)}$ from Kronecker's lemma (e.g. p. 31 in Hall and Heyde, 1980).

Next, using condition (i.a) and **Lemma 6.3(i)** as $s\to\infty$ and then as $\delta\to 1$ we have

$$|T_2(s)|\leq\sum_{l=m_s}^{s-1}\lambda(s-l)\,l^{-1}l\,|c_l|$$

$$\leq Cm_s^{-1}\sum_{l=m_s}^{s-1}\lambda(s-l)=Cm_s^{-1}\sum_{l=1}^{s-m_s+1}\lambda(l)$$

$$=C\frac{\ell(s)\sqrt{s}}{\lfloor\delta s\rfloor}\left(\int_0^{1-\delta}x^{-1/2}dx+o(1)\right)=\frac{C\lambda(s)}{\delta}\int_0^{1-\delta}x^{-1/2}dx\,(1+o(1))=o(\lambda(s)).$$

Next, we consider $T_2(s)$ under condition (ib). Using the same arguments that lead to (62) we get as $s\to\infty$

$$|T_2(s)|\leq\sup_{l>N_\varepsilon}\lambda(l)\sum_{l=m_s}^{s-1-N_\varepsilon}|c_l|\leq\frac{C}{\delta^{1/2}}\lambda(s)\ell(s)^{-1}\sum_{l=m_s}^{\infty}l^{1/2}|c_l|=o(\lambda(s)).$$

Finally, we show that the second approximation in (61) holds. Consider

$$R_s:=\sum_{l=0}^{s-1}\phi_{s-l}c_l-\sum_{l=0}^{s-1-N_\varepsilon}\lambda(s-l)\,c_l$$

$$=\sum_{l=0}^{s-1-N_\varepsilon}\left[\phi_{s-l}-\lambda(s-l)\right]c_l+\sum_{l=s-N_\varepsilon}^{s-1}\phi_{s-l}c_l=:R_{1,s}+R_{2,s}.$$

The first term

$$R_{1,s} \leq \sum_{l=0}^{s-1-N_\varepsilon} \left| \left[\frac{\phi_{s-l}}{\lambda(s-l)} - 1 \right] \lambda(s-l) c_l \right| \leq \varepsilon \sum_{l=0}^{s-1-N_\varepsilon} \lambda(s-l) |c_l| .$$

It follows from similar arguments to those used above that $\sum_{l=0}^{s-1-N_\varepsilon} \lambda(s-l) |c_l|$ $= O(\lambda(s))$. Hence, as $s \to \infty$ first, and then as $\varepsilon \downarrow 0$ we have $\lambda(s)^{-1} R_{1,s} = o(1)$. Further, it can be easily seen (e.g. from (62)) that

$$R_{2,s} \leq \sum_{l=s-N_\varepsilon}^{s-1} \left|\phi_{s-l} c_l\right| \leq \sup_{j\in\mathbb{N}} \left|\phi_j\right| \sum_{l=s-N_\varepsilon}^{s-1} |c_l| = o(\lambda(s)).$$

(ii) Without loss of generality assume that $\pi(j) = \ell_\pi(j) j^{-q_1}$ and $\theta(j) = \ell_\theta(j) j^{-q_2}$. Using similar arguments as in those in used in part (i) we get

$$\sum_{l=0}^{s} \pi_{s-l}\theta_l = \sum_{l=0}^{s-1} \pi_{s-l}\theta_l + o(1),$$

As before, set $m_s := \lfloor s\delta \rfloor$ where $0 < \delta < 1$. Then

$$\sum_{l=0}^{s-1} \pi_{s-l}\theta_l = \left(\sum_{l=0}^{m_s} + \sum_{l=m_s+1}^{s-1} \right) \pi_{s-l}\theta_l := T_1(s) + T_2(s).$$

Again, using similar arguments as those used in the part (i) as $s \to \infty$ the first term

$$\left| T_1(s) - \sum_{l=0}^{m_s} \pi_s \theta_l \right| \leq \sum_{l=0}^{m_s} |\pi_{s-l} - \pi_s| |\theta_l|$$

$$\leq \sum_{l=0}^{m_s} \left| \frac{\pi(s-l)}{\pi(s)} - 1 \right| |\pi(s)\theta_l| \leq \sum_{l=0}^{m_s} \left[\left| \frac{\pi\left(s\frac{s-l}{s}\right)}{\pi(s)} - \left(\frac{s-l}{s}\right)^{-q_1} \right| + \left| \left(\frac{s-l}{s}\right)^{-q_1} - 1 \right| \right] |\pi(s)\theta_l|$$

$$\leq \sup_{1-\delta\leq x\leq 1} \left| \frac{\pi(sx)}{\pi(s)} - x^{-q_1} \right| |\pi(s)| \sum_{l=0}^{\infty} |\theta_l| + |\pi(s)| \sum_{l=0}^{m_s} \frac{s^{q_1} - (s-l)^{q_1}}{(s-l)^{q_1}} |\theta_l|$$

$$\leq_{(3)} o(\pi(s)) + q_1 \frac{\pi(s)\, s^{q_1-1}}{(s-m_s)^{q_1}} \sum_{l=0}^{m_s} l\,|\theta_l| =_{(4)} o(\pi(s)),$$

where $\leq_{(3)}$ follows from the mean value theorem for

$$\left|(s-l)^{q_1} - s^{q_1}\right| = \left|q_1 (s-\bar{l})^{q_1-1} l\right| \leq q_1 s^{q_1-1} l$$

with $0 \leq \bar{l} \leq m_s$ being a mean value, and $=_{(4)}$ from Kronecker's lemma. Next, noting that $T_2(s) = \sum_{l=1}^{s-m_s-1} \pi(l)\,\theta(s-l)$, and using the same arguments as those used above we get as $s \to \infty$

$$\left|T_2(s) - \theta(s) \sum_{l=1}^{s-m_s} \pi(l)\right| \leq |\theta(s)| \sum_{l=1}^{s-m_s} \left|\frac{\theta(s-l)}{\theta(s)} - 1\right| |\pi(l)|$$

$$\leq |\theta(s)| \sum_{l=1}^{s-m_s} \left|\frac{\theta\left(s\frac{s-l}{s}\right)}{\theta(s)} - \left(\frac{s-l}{s}\right)^{-q_2}\right| + \left|\left(\frac{s-l}{s}\right)^{-q_2} - 1\right| |\pi(l)|$$

$$\leq |\theta(s)| \sup_{\delta \leq x \leq 1} \left|\frac{\theta(sx)}{\theta(s)} - x^{-q_2}\right| \sum_{l=1}^{s-m_s} |\pi(l)| + |\theta(s)| \sum_{l=1}^{s-m_s} \frac{s^{q_2} - (s-l)^{q_2}}{(s-l)^{q_2}} |\pi(l)|$$

$$\leq o(\theta(s)) + |\theta(s)| \frac{q_2 s^{q_2-1}}{m_s^{q_2}} \sum_{l=1}^{s-m_s} l\,|\pi(l)|$$

$$= o(\theta(s)) + |\theta(s)| \left(\delta^{-q_2} q_2 + o(1)\right) s^{-1} \sum_{l=1}^{s-m_s} l\,|\pi(l)| = o(\theta(s)),$$

as required for the first part of (ii). The second part of (ii) follows easily from the above. ■

Proof of Lemma 7.2. **(i)** We start with part (a). Recall that under **LL4** $a_k(n) = \sum_{j=0}^{k} \rho_n^{k-j} c_j$, for $0 \leq k \leq t-1$. Next, for $t \leq n$, as $t \to \infty$ **Lemma 6.4(ia)** and Euler summation yield

$$\kappa_t^{-(1-m)} \sup_{0 \leq k \leq \kappa_t} \left|\sum_{j=0}^{k} \rho_n^{k-j} c_j - \sum_{j=0}^{k} \exp\left\{-\frac{k-j}{\kappa_n}\right\} c_j\right| \leq \kappa_t^{-(1-m)} \sup_{0 \leq k \leq n} \sum_{l=0}^{k} \left|\rho_n^l - \exp\left\{\frac{-l}{\kappa_n}\right\}\right| |c_{k-l}|$$

$$[l = k-j \leftrightarrow j = k-l]$$

$$\leq C \kappa_t^{-(1-m)} \sup_{0 \leq k \leq n} \sum_{l=0}^{k} \exp\left\{\frac{-l}{2\kappa_n}\right\} \frac{l}{\kappa_n^2} |c_{k-l}| \leq C \sup_{j \geq 0} |c_j|\, \kappa_t^{-(1-m)} \sum_{l=0}^{n} \exp\left\{\frac{-l}{2\kappa_n}\right\} \frac{l}{\kappa_n^2}$$

$$\overset{n\to\infty}{=} C\kappa_t^{-(1-m)}\left[\int_0^{n/\kappa_n} e^{-x/2}xdx + o(1)\right] \le C\kappa_t^{-(1-m)}\int_0^{\infty} e^{-x/2}xdx \overset{t\to\infty}{=} o(1)$$

Next, as $t\to\infty$ we get

$$\sup_{1\le k\le\kappa_t}\left|\kappa_t^{-(1-m)}\sum_{j=0}^{k} e^{-\frac{k-j}{\kappa_n}}c_j - \int_0^{k/\kappa_t} e^{\frac{-(k-\kappa_t x)}{\kappa_n}}x^{-m}dx\right|$$

$$\le \sup_{1\le k\le n}\left|\frac{1}{\kappa_t}\sum_{j=1}^{k} e^{-\frac{k-j}{\kappa_n}}\left(\frac{j}{\kappa_t}\right)^{-m} - \int_0^{k/\kappa_t} e^{\frac{-(k-\kappa_t x)}{\kappa_n}}x^{-m}dx\right|$$

$$+\sup_{1\le k\le\kappa_t}\left|\frac{1}{\kappa_t^{1-m}}\sum_{j=1}^{k} e^{-\frac{k-j}{\kappa_n}}c_j - \frac{1}{\kappa_t}\sum_{j=1}^{k} e^{-\frac{k-j}{\kappa_n}}\left(\frac{j}{\kappa_t}\right)^{-m}\right| + o(1) = A_{n,t}+B_{n,t}$$

Recall that for a function $g(x)$, $\mathbf{V}\left(g(x)\right)_a^b$ is its total variation on the interval $[a,b]$. Set $\tilde{c} := -\kappa_t/\kappa_n$, $g(x) := e^{-\tilde{c}x}x^{-m}$ and $g(x)' := \partial g(x)/\partial x$. Using standard arguments, the first term is

$$A_{n,t} \le \kappa_t^{-1}\sup_{1\le k\le n}\mathbf{V}\left(e^{\frac{-k}{\kappa_n}}g(x)\right)_{1/\kappa_t}^{k/\kappa_t} = \kappa_t^{-1}\sup_{1\le k\le n} e^{\frac{-k}{\kappa_n}}\int_{1/\kappa_t}^{k/\kappa_t}\left|g(x)'\right|dx,$$

where $g(x)'$ is the derivative of g. Note that

$$g(x)' := -\tilde{c}e^{-\tilde{c}x}x^{-m} - me^{-\tilde{c}x}x^{-m-1} = e^{-\tilde{c}x}x^{-m-1}\left[-\tilde{c}x - m\right]$$

Hence,

$$\left|g(x)'\right| = \begin{cases} g(x)', & x\ge -m/\tilde{c} \\ -g(x)', & 0<x<-m/\tilde{c}\end{cases}.$$

Consider the term

$$\int_{1/\kappa_t}^{k/\kappa_t}\left|g(x)'\right|dx = -\int_{1/\kappa_t}^{-m/\tilde{c}} g(x)'dx + \int_{-m/\tilde{c}}^{k/\kappa_t} g(x)'dx = -\left[g(x)\right]_{1/\kappa_t}^{-m/\tilde{c}} + \left[g(x)\right]_{-m/\tilde{c}}^{k/\kappa_t}.$$

For $1\le k\le n$ and using the fact that $\sup_{n\ge 1, 1\le t\le n}\kappa_t/\kappa_n<\infty$ we have t,n large

$$\mathbf{V}\left(e^{\frac{-k}{\kappa_n}}g(x)\right)_{1/\kappa_t}^{k/\kappa_t} \le e^{-\frac{k}{\kappa_n}}\left[2g(-m/\tilde{c}) + g(1/\kappa_t) + g\left(k/\kappa_t\right)\right]$$

$$= e^{\frac{-k}{\kappa_n}}\left[2e^{-\tilde{c}\frac{-m}{\tilde{c}}}\left(\frac{\tilde{c}}{-m}\right)^m + e^{\frac{-\tilde{c}}{\kappa_t}}\kappa_t^m + e^{\frac{-\tilde{c}k}{\kappa_t}}\left(\kappa_t/k\right)^m\right]$$

$$
\begin{aligned}
&= e^{\frac{-k}{\kappa_n}} \left[2e^m \left(\frac{\kappa_t}{m\kappa_n} \right)^m + e^{\frac{\kappa_t}{\kappa_n}\frac{1}{\kappa_t}} \kappa_t^m + e^{\frac{\kappa_t}{\kappa_n}\frac{k}{\kappa_t}} (\kappa_t/k)^m \right] \\
&= e^{\frac{-k}{\kappa_n}} \left[2e^m \left(\frac{\kappa_t}{m\kappa_n} \right)^m + e^{\frac{1}{\kappa_n}} \kappa_t^m + e^{\frac{k}{\kappa_n}} (\kappa_t/k)^m \right] \\
&\leq C (\kappa_t/\kappa_n)^m + C\kappa_t^m + (\kappa_t/k)^m \leq C\kappa_t^m.
\end{aligned}
$$

Hence, for $n \geq t$ as $t \to \infty$

$$A_{n,t} \leq C\kappa_t^{m-1} = o(1).$$

For the second term note that for each $\varepsilon > 0$ there $N_\varepsilon > 0$ such that $|c_j/j^{-m} - 1| < \varepsilon$ for $j > N_\varepsilon$. Hence, for $n \geq t$ as $t \to \infty$ first and then as $\varepsilon \to 0$

$$B_{n,t} \leq \kappa_t^{-(1-m)} \sup_{1\leq k\leq \kappa_t} \sum_{j=1}^{k} e^{-\frac{k-j}{\kappa_n}} \left|c_j - j^{-m}\right| \leq \kappa_t^{-(1-m)} \left(\sup_{N_\varepsilon < k \leq \kappa_t} \sum_{j=N_\varepsilon+1}^{k} e^{-\frac{k-j}{\kappa_n}} \left|c_j - j^{-m}\right| + CN_\varepsilon \right)$$

$$
\begin{aligned}
&= \kappa_t^{-(1-m)} \sup_{N_\varepsilon < k \leq \kappa_t} \sum_{j=1+N_\varepsilon}^{k} e^{-\frac{k-j}{\kappa_n}} \left| \frac{c_j}{j^{-m}} - 1 \right| j^{-m} + o(1) \leq \varepsilon \kappa_t^{-(1-m)} \sup_{1\leq k\leq n} \sum_{j=1}^{k} e^{-\frac{k-j}{\kappa_n}} j^{-m} \\
&\leq \varepsilon \left[A_{n,t} + \sup_{1\leq k\leq \kappa_t} \int_0^{k/\kappa_t} e^{-\frac{k-\kappa_t x}{\kappa_n}} x^{-m} dx \right] \leq \varepsilon C \overset{\varepsilon\to 0}{\to} 0,
\end{aligned}
$$

where the last inequality is justified as follows. First, note that from before $A_{n,t} = o(1)$ for $n \geq t \to \infty$. Further, using again the fact that $\sup_{n\geq 1, 1\leq t\leq n} \kappa_t/\kappa_n < \infty$, we get

$$\sup_{1\leq k\leq \kappa_t} \int_0^{k/\kappa_t} e^{-\frac{(k-\kappa_t x)}{\kappa_n}} x^{-m} dx \leq e^{\frac{\kappa_t}{\kappa_n}} \int_0^1 x^{-m} dx \leq C \int_0^1 x^{-m} dx < \infty,$$

and the result follows.

For part (b) first, note that by **Lemma 6.4(ia)** as $n \to \infty$

$$\kappa_n^{-(1-m)} \sup_{0\leq k\leq n} \left| \sum_{j=0}^{k} \rho_n^{k-j} c_j - \sum_{j=0}^{k} \exp\left\{ -\frac{k-j}{\kappa_n} \right\} c_j \right| \leq C\kappa_n^{-(1-m)} \left(\int_0^\infty e^{-x/2} x dx + o(1) \right) = o(1).$$

In view of this the result follows from similar arguments as those used for the proof of part (a).

We next show part (c). Recall that (e.g. eq. (15)) $a^{-}_{k,t}(n) = \sum_{j=0}^{t-1} \rho_n^j c_{k-j} 1\{t \le k\}$ $= \sum_{l=k-t+1}^{k} \rho_n^{k-l} c_l 1\{t \le k\}$. Using arguments similar to those used above we have as $n \to \infty$

$$\sup_{1 \le t \le n, k \ge t} \left| \frac{1}{\kappa_n^{1-m}} a^{-}_{k,t}(n) - \int_{(k-t)/\kappa_n}^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n} - x\right)} x^{-m} dx \right| \le \kappa_n^{-1} \sup_{1 \le t \le n, k \ge t} \mathbf{V}\left(e^{\frac{-k}{\kappa_n}} q(x)\right)_{(k-t+1)/\kappa_n}^{k/\kappa_n},$$

with $q(x) := e^x x^{-m}$. As before, for $1 \le t \le n$ and $k > t$ we have as $n \to \infty$

$$\kappa_n^{-1} \mathbf{V}\left(e^{\frac{-k}{\kappa_n}} q(x)\right)_{(k-t)/\kappa_n}^{k/\kappa_n} = \kappa_n^{-1} e^{\frac{-k}{\kappa_n}} \int_{(k-t)/\kappa_n}^{k/\kappa_n} |q(x)'|\, dx = \kappa_n^{-1} e^{\frac{-k}{\kappa_n}} \left(\int_m^{k/\kappa_n} - \int_{(k-t)/\kappa_n}^{m} \right) q(x)' dx$$

$$= \kappa_n^{-1} e^{\frac{-k}{\kappa_n}} \left([q(x)]_m^{k/\kappa_n} - [q(x)]_{(k-t)/\kappa_n}^{m} \right)$$

$$= \kappa_n^{-1} e^{\frac{-k}{\kappa_n}} \left[e^{\frac{k}{\kappa_n}} \left(\frac{\kappa_n}{k}\right)^m - 2e^m \left(\frac{1}{m}\right)^m + e^{\frac{k-t}{\kappa_n}} \left(\frac{\kappa_n}{k-t}\right)^m \right]$$

$$\le \kappa_n^{-1} \left[\left(\frac{\kappa_n}{k}\right)^m + e^{\frac{-t}{\kappa_n}} \left(\frac{\kappa_n}{k-t}\right)^m + 2e^{\frac{-k}{\kappa_n}} e^m \left(\frac{1}{m}\right)^m \right]$$

$$\le 2\kappa_n^{m-1} + 2e^m \left(\frac{1}{m}\right)^m \kappa_n^{-1} = o(1).$$

(ii) First, note that $\lfloor \kappa_t/2 \rfloor \le k \le \kappa_t$. Then for $n \ge t$ we get as $t \to \infty$

$$\int_0^{k/\kappa_t} e^{-\left(\frac{k}{\kappa_n} - \frac{\kappa_t}{\kappa_n} x\right)} x^{-m} dx \ge \int_0^{\lfloor \kappa_t/2 \rfloor/\kappa_t} e^{-\left(\frac{k}{\kappa_n} - \frac{\kappa_t}{\kappa_n} x\right)} dx \ge \int_0^{\lfloor \kappa_t/2 \rfloor/\kappa_t} e^{-\frac{k}{\kappa_n}} dx$$

$$= e^{-\frac{k}{\kappa_n}} \int_0^{1/2} dx + o(1) \ge \frac{1}{2} e^{-\frac{\kappa_t}{\kappa_n}} \ge \frac{1}{2} \exp\left(- \sup_{n \ge 1, 1 \le t \le n} \frac{\kappa_t}{\kappa_n}\right) > 0,$$

as required. ■

Proof of Lemma 7.3. The second equality above follows directly from the definition of Gamma and Beta function (note that $B(x,y) = \int_{l=0}^{\infty} l^{x-1} (1+l)^{-x-y} dl$). We shall prove the first equality. Consider

$$= \int_{y=0}^{\infty} \exp\{2cy\} \left(\int_{x=0}^{y} \exp\{-cx\} x^{-m} dx \right)^2 dy$$

$$= \int_{y=0}^{\infty}\int_{x=0}^{y}\int_{x_*=0}^{y} \exp\left\{c\left(2y-x-x_*\right)\right\} x^{-m}x_*^{-m}dxdx_*dy$$

$$= \int_{x=0}^{\infty}\int_{x_*=0}^{\infty}\int_{y=x\vee x_*}^{\infty} \exp\left\{c\left(2y-x-x_*\right)\right\} x^{-m}x_*^{-m}dxdx_*dy$$

$$= \int_{x=0}^{\infty}\int_{x_*=0}^{\infty}\int_{y=x\vee x_*}^{\infty} e^{2cy}e^{-c(x+x_*)}x^{-m}x_*^{-m}dydxdx_*$$

$$= \frac{1}{2c}\int_{x=0}^{\infty}\int_{x_*=0}^{\infty}\left[e^{2cy}\right]_{y=x\vee x_*}^{\infty} e^{-c(x+x_*)}x^{-m}x_*^{-m}dxdx_*$$

$$= \frac{-1}{2c}\int_{x=0}^{\infty}\int_{x_*=0}^{\infty} e^{2c(x\vee x_*)}e^{-c(x+x_*)}x^{-m}x_*^{-m}dx_*dx$$

$$= \frac{-1}{2c}\int_{x=0}^{\infty}\left[\int_{x_*=0}^{x} e^{2c(x\vee x_*)}e^{-c(x+x_*)}x^{-m}x_*^{-m}dx_* + \int_{x_*=x}^{\infty} e^{2c(x\vee x_*)}e^{-c(x+x_*)}x^{-m}x_*^{-m}dx_*\right]dx$$

$$= \frac{-1}{2c}\int_{x=0}^{\infty}\left[\left(\int_{x_*=0}^{x} + \int_{x_*=x}^{\infty}\right) e^{2cx_*}e^{-c(x+x_*)}x^{-m}x_*^{-m}dx_*\right]dx$$

$$= \frac{-1}{2c}\int_{x=0}^{\infty}\left[\int_{x_*=0}^{x} e^{cx}e^{-cx_*}x^{-m}x_*^{-m}dx_* + \int_{x_*=x}^{\infty} e^{cx_*}e^{-cx}x^{-m}x_*^{-m}dx_*\right]dx$$

$$= \frac{-1}{2c}\left[\int_{x=0}^{\infty} e^{cx}x^{-m}\int_{x_*=0}^{x} e^{-cx_*}x_*^{-m}dx_*dx + \int_{x=0}^{\infty} e^{-cx}x^{-m}\int_{x_*=x}^{\infty} e^{cx_*}x_*^{-m}dx_*dx\right] =: T_1+T_2$$

The first term

$$T_1 = \frac{-1}{2c}\int_{y=0}^{\infty} e^{cy}y^{-m}\int_{x=0}^{y} e^{-cx}x^{-m}dxdy$$

$$[x = y-z \to z = y-x]$$

$$= -\frac{-1}{2c}\int_{y=0}^{\infty} e^{cy}y^{-m}\int_{z=y}^{0} e^{-c(y-z)}\left(y-z\right)^{-m}dzdy = \frac{-1}{2c}\int_{y=0}^{\infty} y^{-m}\int_{z=0}^{y} e^{cz}\left(y-z\right)^{-m}dzdy$$

$$= \frac{-1}{2c}\int_{z=0}^{\infty} e^{cz}\int_{y=z}^{\infty} y^{-m}\left(y-z\right)^{-m}dzdy$$

$$[q = y-z \to y = q+z]$$

$$= \frac{-1}{2c}\int_{z=0}^{\infty} e^{cz}\int_{q=0}^{\infty}\left(q+z\right)^{-m}q^{-m}dqdz = \frac{-1}{2c}\int_{z=0}^{\infty} e^{cz}\int_{q=0}^{\infty} z^{-2m}\left(\frac{q}{z}+1\right)^{-m}\left(\frac{q}{z}\right)^{-m}dqdz$$

$$\left[u = \frac{q}{z} \rightarrow zdu = dq\right]$$

$$= \frac{-1}{2c} \int_{z=0}^{\infty} e^{cz} z^{1-2m} \int_{u=0}^{\infty} (u+1)^{-m} u^{-m} du dz$$

$$= \frac{-1}{2c} (-c)^{2m-1} \int_{z=0}^{\infty} e^{cz} (-cz)^{1-2m} dz \int_{u=0}^{\infty} (u+1)^{-m} u^{-m} du$$

$$\left[v = -cz \rightarrow dz = \frac{1}{-c} dv\right]$$

$$= \frac{-1}{2c} (-c)^{2m-1} \frac{1}{-c} \int_{v=0}^{\infty} e^{-v} v^{1-2m} dv \int_{u=0}^{\infty} (u+1)^{-m} u^{-m} du$$

$$= \frac{1}{2} (-c)^{2m-3} \int_{v=0}^{\infty} e^{-v} v^{1-2m} dv \int_{u=0}^{\infty} (u+1)^{-m} u^{-m} du$$

The second term

$$T_2 = \frac{-1}{2c} \int_{y=0}^{\infty} e^{-cy} y^{-m} \int_{x=y}^{\infty} e^{cx} x^{-m} dx dy$$

$$[z = x - y \rightarrow x = z + y]$$

$$= \frac{-1}{2c} \int_{y=0}^{\infty} e^{-cy} y^{-m} \int_{z=0}^{\infty} e^{c(z+y)} (z+y)^{-m} dz dy = \frac{-1}{2c} \int_{y=0}^{\infty} \int_{z=0}^{\infty} e^{cz} y^{-m} (z+y)^{-m} dz dy$$

$$= \frac{-1}{2c} \int_{y=0}^{\infty} \int_{z=0}^{\infty} e^{cz} y^{-m} (z+y)^{-m} dz dy = \frac{-1}{2c} \int_{z=0}^{\infty} e^{cz} \int_{y=0}^{\infty} y^{-m} (z+y)^{-m} dy dz$$

$$= \frac{-1}{2c} \int_{z=0}^{\infty} e^{cz} z^{-2m} \int_{y=0}^{\infty} \left(z^{-1} y\right)^{-m} \left(1 + z^{-1} y\right)^{-m} dy dz$$

$$\left[l = z^{-1} y \rightarrow dy = z dl\right]$$

$$= \frac{-1}{2c} \int_{z=0}^{\infty} e^{cz} z^{1-2m} \int_{l=0}^{\infty} l^{-m} (1+l)^{-m} dl dz = \frac{-1}{2c} \int_{z=0}^{\infty} e^{cz} z^{1-2m} dz \left(\int_{l=0}^{\infty} l^{-m} (1+l)^{-m} dl\right)$$

$$= \frac{-1}{2c} (-c)^{2m-1} \int_{z=0}^{\infty} e^{cz} (-cz)^{1-2m} dz \left(\int_{l=0}^{\infty} l^{-m} (1+l)^{-m} dl\right)$$

$$s = -cz \rightarrow dz = -\frac{1}{c} ds$$

$$= \frac{-1}{2c} (-c)^{2m-1} \left(-\frac{1}{c}\right) \int_{s=0}^{\infty} e^{-s} s^{1-2m} ds \left(\int_{l=0}^{\infty} l^{-m} (1+l)^{-m} dl\right)$$

$$= \frac{1}{2\left(-c\right)^{3-2m}} \Gamma\left(2-2m\right) \left(\int_{l=0}^{\infty} l^{-m} \left(1+l\right)^{-m} dl \right)$$

Therefore,

$$T_1 + T_2 = \frac{1}{\left(-c\right)^{3-2m}} \Gamma\left(2-2m\right) \left(\int_{l=0}^{\infty} l^{-m} \left(1+l\right)^{-m} dl \right).$$

$$\frac{1}{\left(-c\right)^{3-2m}} \Gamma\left(2-2m\right) B\left(1-m, 2m-1\right)$$

Note that

$$B\left(x,y\right) = \int_{l=0}^{\infty} l^{x-1} \left(1+l\right)^{-x-y} dl$$

Hence,

$$\int_{l=0}^{\infty} l^{-m} \left(1+l\right)^{-m} dl = \int_{l=0}^{\infty} l^{(1-m)-1} \left(1+l\right)^{-(1-m)+(1-m)-m} dl$$

$$= \int_{l=0}^{\infty} l^{(1-m)-1} \left(1+l\right)^{-(1-m)-(2m-1)} dl = B\left(1-m, 2m-1\right).$$

■

8 Appendix C

Appendix C provides proofs for the Theorems of Section 3.

Proof of Theorem 4.1. The LS is

$$\kappa_g(\beta_n)\sqrt{n}\left(\hat{\beta}-\beta\right) = \frac{\frac{1}{\sqrt{n}}\sum_{t=2}^{n} H_g\left(\beta_n^{-1}x_{t-1}(n)\right)u_t + \frac{1}{\kappa_g(\beta_n)\sqrt{n}}\sum_{t=2}^{n} R_g\left(x_{t-1}(n),\beta_n\right)u_t}{\frac{1}{n}\sum_{t=2}^{n} H_g\left(\beta_n^{-1}x_{t-1}(n)\right)^2 + o_p(1)}$$

$$= \frac{\frac{1}{\sqrt{n}}\sum_{t=2}^{n} H_g\left(\beta_n^{-1}x_{t-1}(n)\right)u_t}{\frac{1}{n}\sum_{t=2}^{n} H_g\left(\beta_n^{-1}x_{t-1}(n)\right)^2} + o_p(1), \tag{63}$$

where we have used the fact that (see (28))

$$\mathbf{E}\left(\frac{1}{\kappa_g(\beta_n)\sqrt{n}}\sum_{t=2}^{n} R_g\left(x_{t-1}(n),\beta_n\right)u_t\right)^2 = \frac{\sigma_u^2}{\kappa_g(\beta_n)^2 n}\sum_{t=2}^{n} \mathbf{E}R_g\left(x_{t-1}(n),\beta_n\right)^2 = o(1).$$

Next, using **Theorem 2.2** we get as $n \to \infty$

$$\frac{1}{n}\sum_{t=2}^{n} H_g\left(\beta_n^{-1}x_{t-1}(n)\right)^2 \xrightarrow{d} \int_{\mathbb{R}} H_g\left(x+X^-\right)^2 \varphi_{\sigma_+^2}(x)dx =: V_{H_g^2}.$$

Therefore, by Wang (2014, Theorem 3.2 and Remark 5) it suffices showing that

$$\sup_{1\le t\le n}\left|H_g\left(\beta_n^{-1}x_t(n)\right)\right|/\sqrt{n} = o_p(1), \tag{64}$$

for obtaining the joint limit

$$\left[\frac{1}{n}\sum_{t=2}^{n} H_g\left(\beta_n^{-1}x_{t-1}(n)\right)^2, \frac{1}{\sqrt{n}}\sum_{t=2}^{n} H_g\left(\beta_n^{-1}x_{t-1}(n)\right)u_t\right] \xrightarrow{d} \left[V_{H_g^2}, MN\left(0, \sigma_u^2 V_{H_g^2}\right)\right],$$

which in view of (63) is sufficient for the requisite result. Note that the mixing variate X^- is non degenerate only under **LL2** (recall that under **LL1**, **LL3** and **LL4** $X^- = 0$), in which case X^- is the distribution limit of some functional of $\left\{\xi_0, \xi_{-1}, ...\right\}$ that is $\mathcal{F}_0$-measurable (see **Lemma CLT**). Therefore, it can be easily seen that eq. (2.3) in Wang (2014) is unnecessary (see p. 523-525 in Wang, 2014). Finally, note that the following Lindeberg condition is sufficient for (64) (e.g. e.q. (3.5) in Hall and Heyde, 1980): For all $\eta > 0$, as $n \to \infty$

$$\frac{1}{n}\sum_{t=1}^{n} H_g\left(\beta_n^{-1}x_t(n)\right)^2 1\left\{\left|H_g\left(\beta_n^{-1}x_t(n)\right)\right| > \eta\sqrt{n}\right\} = o_p(1). \tag{65}$$

We set out to prove (65) next. By condition (iia) of **Theorem 2.2** for large x, $H_g(x)^2 \le C|x|^{\lambda'}$ for some $\lambda' > 0$. Further, under the conditions of **Theorem 4.1**, **Assumptions HL0-HL5(a)** and **HL6** hold with $X_t(n)$ $= x_t(n)$ (as demonstrated by **Propositions 2.1** and **2.2**). In particular, $\sup_{n\ge n_0, 1\le t\le n}\mathbf{E}\left|\beta_n^{-1}x_t(n)\right|^{\lambda} < \infty$, for some $\lambda > \lambda'$ (**HL5(a)**). Hence, $\left|\beta_n^{-1}x_t(n)\right|^{\lambda'}$ is uniformly integrable. Further, $\beta_t^{-1}x_t(n)$ possesses density $\mathcal{D}_{t,n}(x)$ such that $\sup_{n\ge n_0, t_0\le t\le n}\mathcal{D}_{t,n}(x) < \infty$, for some $n_0 \ge t_0 \ge 1$ (**HL3**). Without loss of generality suppose that $t_0 = 1$. Next, note that due to condition (vi), $H_g(x)^2$ is locally integrable. In view of the above, for all $\eta > 0$ and for $\varepsilon > 0$ large enough we have as $n \to \infty$

$$\frac{1}{n}\sum_{t=1}^{n}\mathbf{E}\left[H_g\left(\beta_n^{-1}x_t(n)\right)^2 1\left\{\left|H_g\left(\beta_n^{-1}x_t(n)\right)\right| > \eta\sqrt{n}\right\}\right]$$

$$\leq \frac{1}{n}\sum_{t=1}^{n}\mathbf{E}\left[H_g\left(\beta_n^{-1}x_t(n)\right)^2 1\left\{\left|\beta_n^{-1}x_t(n)\right|<\varepsilon\right\}1\left\{\left|H_g\left(\beta_n^{-1}x_t(n)\right)\right|>\eta\sqrt{n}\right\}\right]$$

$$+C\frac{1}{n}\sum_{t=1}^{n}\mathbf{E}\left[\left|\beta_n^{-1}x_t(n)\right|^{\lambda'}1\left\{\left|\beta_n^{-1}x_t(n)\right|^{\lambda'/2}>\eta\sqrt{n}\right\}1\left\{\left|\beta_n^{-1}x_t(n)\right|>\varepsilon\right\}\right]$$

$$\leq \frac{1}{n}\sum_{t=1}^{n}\int_x H_g\left(\frac{\beta_t}{\beta_n}x\right)^2 1\left\{\left|\frac{\beta_t}{\beta_n}x\right|<\varepsilon\right\}1\left\{\left|H_g\left(\frac{\beta_t}{\beta_n}x\right)\right|>\eta\sqrt{n}\right\}\mathcal{D}_{t,n}(x)dx$$

$$+C\sup_{1\leq t\leq n}\mathbf{E}\left[\left|\beta_n^{-1}x_t(n)\right|^{\lambda'}1\left\{\left|\beta_n^{-1}x_t(n)\right|^{\lambda'}>\eta^2 n\right\}\right]$$

$$\leq_{(1)} C\frac{\beta_n}{n}\sum_{t=1}^{n}\beta_t^{-1}\int_{|x|<\varepsilon}H_g\left(x\right)^2 1\left\{\left|H_g\left(x\right)\right|>\eta\sqrt{n}\right\}dx+o(1)$$

$$\leq_{(2)} C\int_{|x|<\varepsilon}H_g\left(x\right)^2 1\left\{\left|H_g\left(x\right)\right|>\eta\sqrt{n}\right\}dx+o(1)=o(1),$$

where the approximation in $\leq_{(1)}$ is due to the fact $\left|\beta_n^{-1}x_t(n)\right|^{\lambda'}$ is uniformly integrable and the approximation $\leq_{(2)}$ is due to **HL6** and dominated convergence (recall that by condition (vi), $f(x)=H_g\left(x\right)^2$ *a.e.* where $f(x)\in\mathbb{R}$ and locally integrable). ■

Proof of Theorem 4.2. Part (a) follows from **Theorem 3.2** and arguments similar to those used in Wang and Phillips (2009a,b). For part (b) note that by **Theorem 4.2** we have as $n\to\infty$

$$\frac{\beta_n}{h_n n}\sum_{t=1}^{n}K\left(\frac{x_t(n)-x}{h_n}\right)^2\xrightarrow{d}\varphi_{\sigma_+^2}\left(X^-\right)\int_{\mathbb{R}}K(x)^2dx=:V_{K^2},\tag{66}$$

$$\frac{\beta_n}{h_n n}\sum_{t=1}^{n}K\left(\frac{x_t(n)-x}{h_n}\right)\xrightarrow{d}\varphi_{\sigma_+^2}\left(X^-\right)\int_{\mathbb{R}}K(x)dx=:V_K.\tag{67}$$

In fact it can be seen from the arguments used in the proof of **Theorem 2.1**, that (66)-(67) hold jointly. Next, the NW estimator is

$$\sqrt{\frac{h_n n}{\beta_n}}\left(\widehat{g}\left(x\right)-g(x)\right)=\frac{\sqrt{\frac{\beta_n}{h_n n}}\sum_{t=1}^{n}K\left(\frac{x_{t-1}(n)-x}{h_n}\right)u_t}{\frac{\beta_n}{h_n n}\sum_{t=1}^{n}K\left(\frac{x_{t-1}(n)-x}{h_n}\right)}+O_p\left(\frac{nh_n^{1+2\mu}}{\beta_n}\right),\tag{68}$$

where the approximation above follows from condition (iv) and arguments similar as those used by Wang and Phillips (2009a,b). By (66) and Wang (2014, Theorem 3.2 and Remark 5) showing that

$$\sqrt{\frac{\beta_n}{h_n n}} \sup_{1\le t\le n} \left| K\left(\frac{x_{t-1}(n)-x}{h_n}\right)\right| = o_p(1), \tag{69}$$

is sufficient for the joint limit

$$\left[\frac{\beta_n}{h_n n}\sum_{t=1}^{n} K\left(\frac{x_t(n)-x}{h_n}\right)^2,\right.$$

$$\left.\sqrt{\frac{\beta_n}{h_n n}}\sum_{t=1}^{n} K\left(\frac{x_{t-1}(n)-x}{h_n}\right)u_t\right] \xrightarrow{d} \left[V_{K^2}, MN\left(0,\sigma_u^2 V_{K^2}\right)\right]. \tag{70}$$

Next, we show (69) by establishing a Lindeberg condition as in the previous proof. Note that under our assumptions, **HL3** and **HL6** hold with $X_t(n) = x_t(n)$. Without loss of generality suppose that **HL3** holds with $t_0 = 1$. Then for all $\eta > 0$, as $n \to \infty$ we have

$$\frac{\beta_n}{h_n n}\sum_{t=1}^{n} \mathbf{E}\left[K\left(\frac{x_t(n)-x}{h_n}\right)^2 1\left\{\left|K\left(\frac{x_t(n)-x}{h_n}\right)\right| > \eta\sqrt{\frac{h_n n}{\beta_n}}\right\}\right]$$

$$\le \left(\eta\sqrt{\frac{h_n n}{\beta_n}}\right)^{-2} \frac{\beta_n}{h_n n}\sum_{t=1}^{n} \mathbf{E}\left[K\left(\frac{x_t(n)-x}{h_n}\right)^4\right]$$

$$= \left(\eta\sqrt{\frac{h_n n}{\beta_n}}\right)^{-4} \frac{\beta_n}{h_n n}\sum_{t=1}^{n} \int_{\mathbb{R}}\left[K\left(\frac{\beta_t y}{h_n}-\frac{x}{h_n}\right)\right]^4 \mathcal{D}_{t,n}(y)dy$$

$$\le_{(1)} C\left(\eta\sqrt{\frac{h_n n}{\beta_n}}\right)^{-4} \frac{\beta_n}{n}\sum_{t=1}^{n} \beta_t^{-1}\int_{\mathbb{R}} K(y)^4\, dy \le_{(2)} C\left(\eta\sqrt{\frac{h_n n}{\beta_n}}\right)^{-4} \to 0,$$

where $\le_{(1)}$ follows from **HL3** and $\le_{(2)}$ from **HL6** and the fact that $\int_{\mathbb{R}} K(y)^4\, dy < \infty$ (see condition (ii)). Now using (66)-(68) and (70) we get

$$\sqrt{\frac{h_n n}{\beta_n}}\left(\widehat{g}(x) - g(x)\right) = \frac{\frac{\beta_n}{h_n n}\sum_{t=1}^{n} K\left(\frac{x_{t-1}(n)-x}{h_n}\right)^2}{\frac{\beta_n}{h_n n}\sum_{t=1}^{n} K\left(\frac{x_{t-1}(n)-x}{h_n}\right)} \frac{\sqrt{\frac{\beta_n}{h_n n}}\sum_{t=1}^{n} K\left(\frac{x_{t-1}(n)-x}{h_n}\right)u_t}{\frac{\beta_n}{h_n n}\sum_{t=1}^{n} K\left(\frac{x_{t-1}(n)-x}{h_n}\right)^2} + o_p(1)$$

$$= \frac{\int_{\mathbb{R}} K(x)^2 dx}{\int_{\mathbb{R}} K(x) dx} \frac{\sqrt{\frac{\beta_n}{h_n n}} \sum_{t=1}^n K\left(\frac{x_{t-1}(n)-x}{h_n}\right) u_t}{\frac{\beta_n}{h_n n} \sum_{t=1}^n K\left(\frac{x_{t-1}(n)-x}{h_n}\right)^2} + o_p(1) \xrightarrow{d} MN\left(0, \frac{\sigma_u^2 \int_{\mathbb{R}} K(x)^2 dx}{\varphi_{\sigma_+^2}(X^-) \left[\int_{\mathbb{R}} K(x) dx\right]^2}\right),$$

as required. ■

9 References

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