Energy-constrained diamond norms and quantum dynamical semigroups

M.E. Shirokov, A.S. Holevo

Steklov Mathematical Institute, Moscow, Russia

Abstract

In the developing theory of infinite-dimensional quantum channels the relevance of the energy-constrained diamond norms was recently corroborated both from physical and information-theoretic points of view. In this paper we study necessary and sufficient conditions for differentiability with respect to these norms of the strongly continuous semigroups of quantum channels (quantum dynamical semigroups). We show that these conditions can be expressed in terms of the generator of the semigroup. We also analyze conditions for representation of a strongly continuous semigroup of quantum channels as an exponential series converging w.r.t. the energy-constrained diamond norm. Examples of semigroups having such a representation are presented.

1 Introduction

In the developing theory of infinite-dimensional quantum channels the relevance of the energy-constrained diamond norms was recently corroborated both from physical and information-theoretic points of view [3, 21, 24, 34]. The choice of an appropriate metric on the set of quantum channels is important for analysis of characteristics of these channels and for the study of quantum dynamical semigroups. When dealing with finite-dimensional quantum channels it is natural to use the metric induced by the norm of complete boundedness (usually called "the diamond norm" [1, 31, 32]). This metric can be considered as a measure of distinguishability of quantum channels via quantum measurements [32, Ch.9]. However the topology generated by the diamond norm is generally too strong for description of physically interesting perturbations of infinite-dimensional quantum channels [34]. Mathematically, the inadequacy of the diamond norm metric in infinite dimensions can be seen from the Kretschmann-Schlingemann-Werner theorem which says that closeness of channels in this metric means closeness of the Stinespring isometries of these channels in the operator norm [17]. So, the diamond norm metric can not properly reflect "deformations" of the Stinespring isometry of a

channel in the weaker topologies on the set of isometric operators, in particular, in the strong operator topology.

More adequate is the topology of the strong (pointwise) convergence generated by the family of seminorms $\Phi \mapsto \|\Phi(\rho)\|_1$, where ρ runs over all input states of the channel. The strong convergence of a sequence $\{\Phi_n\}$ of quantum channels to a channel Φ_0 means that

$$\lim_{n \to \infty} \Phi_n(\rho) = \Phi_0(\rho) \quad \text{for any state} \quad \rho.$$

The strong convergence topology naturally appears in the study of strongly continuous quantum dynamical semigroups (QDS) – the semigroups $\{\Phi_t\}_{t \geq 0}$ of quantum channels satisfying

$$\lim_{t \to 0^+} \Phi_t(\rho) = \rho \quad \text{for any state} \quad \rho.$$

These semigroups play a central role in the theory of open quantum systems, where they are used as a basic dynamical model for irreversible evolution [8, 9, 14, 18].

It was observed recently that the strong convergence topology on the set of infinite-dimensional quantum channels is generated by the energy-constrained diamond norm (ECD norm) introduced independently in [24, 34] (a slightly different version of ECD norm was used in [21]). These norms appeared to be a useful tool for quantitative continuity analysis of the basic capacities of energy-constrained infinite-dimensional channels [24, 34]. In the recent work [3] it is shown that the ECD norms can be used effectively in the study of strongly continuous QDS. In particular, the ECD norms allow to obtain sharp estimates for the rate of convergence for strongly continuous QDS, which give the new lower bounds on the minimal time needed for a quantum system to evolve from one quantum state to another (quantum speed limits [6, 19]).

In fact, any strongly continuous QDS is continuous w.r.t. the ECD norm induced by a positive operator with discrete spectrum of finite multiplicity [24, Proposition 3]. However, since the ECD norm is not an operator norm (in the sense of the Banach space theory), one can not apply directly the Banach space theory of norm continuous semigroups to study the strongly continuous QDS, although one can expect that certain facts from this advanced theory can be used in our more general situation.

One of the differences of the ECD norm continuous semigroups as compared to the operator norm continuous semigroups is that the ECD norm continuity *does not imply* the differentiability w.r.t. to this norm, i.e. the existence of the ECD norm bounded generator which provides, in particular, the exponential representation of the semigroup. The aim of this paper is to obtain necessary and sufficient conditions for differentiability of the strongly continuous QDS with respect to the ECD norm and to analyze the conditions for a decomposition into an exponential series converging w.r.t. the ECD norm.

2 Preliminaries

2.1 Basic notations

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, $\mathfrak{B}(\mathcal{H})$ – the algebra of all bounded operators on $\mathcal{H}$ with the operator norm $\|\cdot\|$ and $\mathfrak{T}(\mathcal{H})$ – the Banach space of all trace-class operators on $\mathcal{H}$ with the trace norm $\|\cdot\|_1$ (the Schatten class of order 1) [5, 22]. Let $\mathfrak{T}_+(\mathcal{H})$ be the cone of positive operators in $\mathfrak{T}(\mathcal{H})$. Trace-class operators will be usually denoted by the Greek letters ρ, σ, ω, ... The closed convex subsets

$$\mathfrak{T}_{+,1}(\mathcal{H}) = \{\rho \in \mathfrak{T}_+(\mathcal{H}) \,|\, \mathrm{Tr}\rho \le 1\} \quad \text{and} \quad \mathfrak{S}(\mathcal{H}) = \{\rho \in \mathfrak{T}_+(\mathcal{H}) \,|\, \mathrm{Tr}\rho = 1\}$$

of the cone $\mathfrak{T}_+(\mathcal{H})$ are complete separable metric spaces with the metric defined by the trace norm. Operators in $\mathfrak{S}(\mathcal{H})$ are called density operators or *states* [5, 13]. Extreme points of $\mathfrak{S}(\mathcal{H})$ are 1-rank projectors called *pure states*.

Denote by $I_{\mathcal{H}}$ the unit operator on a Hilbert space $\mathcal{H}$ and by $\mathrm{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

If quantum systems A and B are described by the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ then the composite system AB is described by the tensor product of these spaces $\mathcal{H}_{AB} \doteq \mathcal{H}_A \otimes \mathcal{H}_B$. A state in $\mathfrak{S}(\mathcal{H}_{AB})$ is denoted by ρ_{AB}, its partial states $\rho_A = \mathrm{Tr}_B \rho_{AB}$ and $\rho_B = \mathrm{Tr}_A \rho_{AB}$ (here and in what follows Tr_A denotes $\mathrm{Tr}_{\mathcal{H}_A}$, etc.)

We will consider unbounded densely defined positive operators on $\mathcal{H}$ having discrete spectrum of finite multiplicity. Following [34] we will call such operators *discrete*. In Dirac's notations any discrete operator G can be represented as

$$G = \sum_{k=0}^{+\infty} E_k |\tau_k\rangle\langle\tau_k| \tag{1}$$

with the domain $\mathcal{D}(G) = \{\varphi \in \mathcal{H} \,|\, \sum_{k=0}^{+\infty} E_k^2 |\langle\tau_k|\varphi\rangle|^2 < +\infty\}$, where $\{\tau_k\}_{k=0}^{+\infty}$ is the orthonormal basis of eigenvectors of G corresponding to the nondecreasing sequence $\{E_k\}_{k=0}^{+\infty}$ of eigenvalues tending to $+\infty$.

A linear transformation Φ of the Banach space $\mathfrak{T}(\mathcal{H})$ will be called *superoperator*. A superoperator Φ is called *Hermitian preserving* if $\Phi(\rho^*) = [\Phi(\rho)]^*$ for any ρ in $\mathfrak{T}(\mathcal{H})$ [31]. A superoperator Φ is called *quantum channel* if it is trace preserving and completely positive [13, 31, 32]. A one parameter semigroup Φ_t of quantum channels such that

$$\lim_{t\to0^+} \Phi_t(\rho) = \rho, \quad \forall \rho \in \mathfrak{T}(\mathcal{H}),$$

in the trace norm is called *(strongly continuous) quantum dynamical semigroup*.

2.2 Energy-constrained diamond norms

The norm of complete boundedness of a linear transformation of the algebra $\mathfrak{B}(\mathcal{H}_A)$ (cf. [20]) induces (by duality) the "diamond norm"

$$\|\Phi\|_\diamond \doteq \sup_{\rho \in \mathfrak{T}(\mathcal{H}_{AR}), \|\rho\|_1 \le 1} \|\Phi \otimes \mathrm{Id}_R(\rho)\|_1 \tag{2}$$

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on the set of all superoperators on $\mathfrak{T}(\mathcal{H}_A)$, where $\mathcal{H}_R$ is a separable Hilbert space and $\mathcal{H}_{AR} = \mathcal{H}_A \otimes \mathcal{H}_R$ [1]. If Φ is a Hermitian preserving superoperator then the supremum in (2) can be taken over the set $\mathfrak{S}(\mathcal{H}_{AR})$ [31, Ch.3].

The diamond norm is widely used in the quantum information theory, but generally the convergence induced by this norm is too strong for description of physical perturbations of infinite-dimensional quantum channels: there exist quantum channels with close physical parameters such that the diamond norm distance between them is equal to the maximal value 2 [34]. The reason of this inconsistency is pointed out briefly in the Introduction. By taking it into account the energy-constrained diamond norms were introduced independently in [24, 34].[1]

Let G be a positive operator on $\mathcal{H}_A$ with a dense domain $\mathcal{D}(G)$ such that

$$\inf \left\{ \langle \varphi | G | \varphi \rangle \, | \, \varphi \in \mathcal{D}(G), \|\varphi\| = 1 \right\} = 0. \tag{3}$$

For any positive trace class operator ρ the value of $\mathrm{Tr}\rho G$ (finite or infinite) is defined as $\sup_n \mathrm{Tr}\rho P_n G$, where P_n is the spectral projector of G corresponding to the interval $[0, n]$. If $\rho = \sum_k |\varphi_k\rangle\langle\varphi_k|$ then $\mathrm{Tr}\rho G = \sum_k \|\sqrt{G}\varphi_k\|^2$, where we assume that $\|\sqrt{G}\varphi_k\| = +\infty$ if φ_k does not lie in $\mathcal{D}(\sqrt{G})$.

In applications G is usually the Hamiltonian (energy observable) of the quantum system A. So, we will call the quantity $\mathrm{Tr}\rho G$ the energy of a state ρ.[2]

Let $E > 0$. The energy-constrained diamond norm (ECD norm in what follows) of a Hermitian-preserving superoperator Φ on $\mathfrak{T}(\mathcal{H}_A)$ is defined as

$$\|\Phi\|_{\diamond,E}^G \doteq \sup_{\rho\in\mathfrak{S}(\mathcal{H}_{AR}),\mathrm{Tr}\rho_A G\leq E} \|\Phi \otimes \mathrm{Id}_R(\rho)\|_1, \tag{4}$$

where R is an infinite-dimensional quantum system (since all separable Hilbert spaces are isomorphic, this definition does not depend on R). It is shown in [34] that the nonnegative non-decreasing function $E \mapsto \|\Phi\|_{\diamond,E}^G$ is concave on $[0, +\infty)$ for any given Φ and hence

$$\|\Phi\|_{\diamond,E_1}^G \leq \|\Phi\|_{\diamond,E_2}^G \leq (E_2/E_1)\|\Phi\|_{\diamond,E_1}^G \quad \text{for any } E_2 > E_1 > 0.$$

Thus, for given operator G all the norms (4) are equivalent on the set of all Hermitian preserving superoperators on $\mathfrak{T}(\mathcal{H}_A)$.

The quantity $\|\Phi\|_{\diamond,E}^G$ can be defined by formula (4) (as a positive number or $+\infty$) for any unbounded superoperator Φ on $\mathfrak{T}(\mathcal{H}_A)$ provided that the superoperator $\Phi\otimes\mathrm{Id}_R$ is well defined on the set of states ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\mathrm{Tr}\rho_A G < +\infty$ (here R is an infinite-dimensional quantum system). We will denote the set of all such superoperators by $\mathfrak{F}_G(\mathcal{H}_A)$. The arguments in [34] showing concavity of the function $E \mapsto \|\Phi\|_{\diamond,E}^G$ remain valid for any Φ in $\mathfrak{F}_G(\mathcal{H}_A)$.

We will use the following simple observations.

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[1]Slightly different energy-constrained diamond norm is used in [21].
[2]Sometimes, it is reasonable to assume that G is some power of the Hamiltonian [3].

Lemma 1. *Let Φ be a superoperator in $\mathfrak{F}_G(\mathcal{H}_A)$ and $E > 0$.*

A) *The supremum in definition (4) can be taken over all operators in $\mathfrak{T}_{+,1}(\mathcal{H}_{AR})$ satisfying the condition $\mathrm{Tr}\rho_A G \leq E$, i.e.*

$$\|\Phi\|_{\diamond,E}^G = \sup_{\rho \in \mathfrak{T}_{+,1}(\mathcal{H}_{AR}), \mathrm{Tr}\rho_A G \leq E} \|\Phi \otimes \mathrm{Id}_R(\rho)\|_1. \tag{5}$$

B) *Let G_R be a positive operator on $\mathcal{H}_R$ unitarily equivalent to the operator G. Then*

$$\|\Phi\|_{\diamond,E}^G = \sup_{\rho \in \widehat{\mathfrak{S}}_{G,G_R,E}} \|\Phi \otimes \mathrm{Id}_R(\rho)\|_1, \tag{6}$$

where $\widehat{\mathfrak{S}}_{G,G_R,E} \doteq \{\rho \in \mathfrak{S}(\mathcal{H}_{AR}) \,|\, \mathrm{Tr}\rho_A G \leq E, \, \mathrm{Tr}\rho_R G_R \leq E, \, \mathrm{rank}\,\rho = 1\}$.

Proof. A) Since $\mathfrak{S}(\mathcal{H}_{AR}) \subset \mathfrak{T}_{+,1}(\mathcal{H}_{AR})$, it suffices to show that " $\geq$ " holds in (5). Let ρ be an operator in $\mathfrak{T}_{+,1}(\mathcal{H}_{AR})$ such that $\mathrm{Tr}\rho_A G \leq E$ and $r = \mathrm{Tr}\rho$. Then $\hat{\rho} \doteq r^{-1}\rho$ is a state such that $\mathrm{Tr}\hat{\rho}_A G \leq E/r$. So, by using concavity of the function $E \mapsto \|\Phi\|_{\diamond,E}^G$ on $\mathbb{R}_+$ and Lemma 2 below we obtain

$$\|\Phi \otimes \mathrm{Id}_R(\rho)\|_1 = r\|\Phi \otimes \mathrm{Id}_R(\hat{\rho})\|_1 \leq r\|\Phi\|_{\diamond,E/r}^G \leq \|\Phi\|_{\diamond,E}^G.$$

B) It suffices to show that " $\leq$ " holds in (6). Since the system R in definition (4) is assumed arbitrary and any mixed state in $\mathfrak{S}(\mathcal{H}_{AR})$ can be considered as a partial state of some pure state in $\mathfrak{S}(\mathcal{H}_{AR'})$, where R' is an extension of R [13], the supremum in (4) can be taken over all pure states ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ satisfying the condition $\mathrm{Tr}\rho_A G \leq E$. Since for any such pure state ρ the partial states ρ_A and ρ_B have the same nonzero spectrum, by applying local partial isometry transformation of the system R this state can be transformed into a state ρ' belonging to the set $\widehat{\mathfrak{S}}_{G,G_R,E}$. It suffices to note that $\|\Phi \otimes \mathrm{Id}_R(\rho)\|_1 = \|\Phi \otimes \mathrm{Id}_R(\rho')\|_1$. $\square$

Lemma 2. [33] *If f is a concave nonnegative function on $[0,+\infty)$ then for any positive $x < y$ and any $z \geq 0$ the inequality $xf(z/x) \leq yf(z/y)$ holds.*

The convergence on the set of quantum channels generated by any of the ECD norms implies the strong convergence:

$$\lim_{n\to\infty} \|\Phi_n - \Phi_0\|_{\diamond,E}^G = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \Phi_n(\rho) = \Phi_0(\rho) \text{ for all } \rho \in \mathfrak{T}(\mathcal{H}_A). \tag{7}$$

If G is a discrete operator (1) then " $\Leftrightarrow$ " holds in (7) [24, Proposition 3].

We will use the following

Lemma 3. *If Φ is a superoperator in $\mathfrak{F}_G(\mathcal{H}_A)$ such that $\|\Phi\|_{\diamond,E}^G = o(\sqrt{E})$ as $E \to +\infty$ then for any separable Hilbert space $\mathcal{H}_R$ and any $E > 0$ the superoperator $\Phi \otimes \mathrm{Id}_R$ is uniformly continuous on the set of pure states ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\mathrm{Tr}\rho_A G \leq E$. Quantitatively,*

$$\|\Phi \otimes \mathrm{Id}_R(\rho - \sigma)\|_1 \leq 2\varepsilon\|\Phi\|_{\diamond,2E/\varepsilon^2}^G \tag{8}$$

for any pure states ρ and σ in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\frac{1}{2}\|\rho-\sigma\|_1 \leq \varepsilon$ and $\mathrm{Tr}\rho_A G, \mathrm{Tr}\sigma_A G \leq E$.

Proof. The assumption of the lemma implies that the r.h.s. of the inequality (8) tends to zero as $\varepsilon \to 0$. So, it suffices to prove this inequality.

Let ρ and σ be pure states in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\mathrm{Tr}\rho_A G, \mathrm{Tr}\sigma_A G \leq E$ and $\frac{1}{2}\|\rho-\sigma\|_1 \leq \varepsilon$. One can find pure states ϱ and ς in some system R' such that $\varepsilon = \frac{1}{2}\|\hat\rho - \hat\sigma\|_1$, where $\hat\rho = \rho \otimes \varrho$ and $\hat\sigma = \sigma \otimes \varsigma$. Then $\alpha_- = \varepsilon^{-1}[\hat\rho - \hat\sigma]_-$ and $\alpha_+ = \varepsilon^{-1}[\hat\rho - \hat\sigma]_+$ are pure states such that $\mathrm{Tr}[\alpha_\pm]_A G \leq 2E/\varepsilon^2$ [28]. Since $\hat\rho - \hat\sigma = \varepsilon(\alpha_+ - \alpha_-)$, we have

$$\|\Phi \otimes \mathrm{Id}_R(\rho - \sigma)\|_1 \leq \|\Phi \otimes \mathrm{Id}_{RR'}(\hat\rho - \hat\sigma)\|_1 = \varepsilon\|\Phi \otimes \mathrm{Id}_{RR'}(\alpha_+ - \alpha_-)\|_1$$

$$\leq \varepsilon\|\Phi \otimes \mathrm{Id}_{RR'}(\alpha_+)\|_1 + \varepsilon\|\Phi \otimes \mathrm{Id}_{RR'}(\alpha_-)\|_1 \leq 2\varepsilon\|\Phi\|^G_{\diamond,2E/\varepsilon^2}.$$

The first and second inequalities follow from the properties of the trace norm (non-increasing under partial trace and the triangle inequality), the third one – from the definition of the ECD norm. $\square$

2.3 Operator E-norms

Let G be a positive operator on $\mathcal{H}$ with a dense domain $\mathcal{D}(G)$ satisfying the condition (3) and $E > 0$. The corresponding operator E-norm on $\mathfrak{B}(\mathcal{H})$ is defined as

$$\|A\|^G_E \doteq \sup_{\rho \in \mathfrak{S}(\mathcal{H}):\mathrm{Tr}\rho G \leq E} \sqrt{\mathrm{Tr}A\rho A^*}, \quad A \in \mathfrak{B}(\mathcal{H}), \tag{9}$$

where the supremum is over all states ρ in $\mathfrak{S}(\mathcal{H})$ such that $\mathrm{Tr}\rho G \leq E$. These norms are studied in detail in [26], where different applications of these norms are described.

One of the basic properties of the family of E-norms is the concavity of the nondecreasing function $E \mapsto \left[\|A\|^G_E\right]^2$ on $\mathbb{R}_+$ implying that

$$\|A\|^G_{E_1} \leq \|A\|^G_{E_2} \leq \sqrt{E_2/E_1}\|A\|^G_{E_1} \quad \text{for any } E_2 > E_1 > 0. \tag{10}$$

Hence for given operator G all the norms $\|\cdot\|^G_E$, $E > 0$, are equivalent on $\mathfrak{B}(\mathcal{H})$.

Different operators G induce the E-norms generating different topologies on $\mathfrak{B}(\mathcal{H})$. If G is a discrete unbounded operator (1) then for any $E > 0$ the norm $\|\cdot\|^G_E$ generates the strong operator topology on bounded subsets of $\mathfrak{B}(\mathcal{H})$ [26, Proposition 2].

If A is an unbounded linear operator with domain containing the set $\mathcal{D}(\sqrt{G})$ then its E-norm $\|A\|^G_E$ can be defined (as a nonnegative number or $+\infty$) by the same formula (9), where the supremum is taken over all *finite rank* states ρ satisfying the inequality $\mathrm{Tr}\rho G \leq E$ and $A\rho A^*$ is replaced by

$$\sum_i |\alpha_i\rangle\langle\alpha_i|, \qquad |\alpha_i\rangle = A|\varphi_i\rangle, \tag{11}$$

where $\rho = \sum_i |\varphi_i\rangle\langle\varphi_i|$ is a finite decomposition of ρ.[3]

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[3]By using Schrodinger's mixture theorem (see [4, Ch.8]) it is easy to show that the operator in (11) does not depend on this decomposition of ρ.

This extension of the E-norm to unbounded operators is closely related to the notion of $\sqrt{G}$-relatively bounded operators [16, 29]. An operator A is called relatively bounded w.r.t. the operator $\sqrt{G}$ (briefly, $\sqrt{G}$-bounded) if $\mathcal{D}(\sqrt{G}) \subseteq \mathcal{D}(A)$ and

$$\|A\varphi\|^2 \le a^2\|\varphi\|^2 + b^2\|\sqrt{G}\varphi\|^2, \quad \forall\varphi \in \mathcal{D}(\sqrt{G}), \tag{12}$$

for some nonnegative numbers a and b. The $\sqrt{G}$-bound of A (denoted by $b_{\sqrt{G}}(A)$ in what follows) is defined as the infimum of the values b for which (12) holds with some a. If the $\sqrt{G}$-bound is equal to zero then A is called $\sqrt{G}$-infinitesimal operator (infinitesimally bounded w.r.t. $\sqrt{G}$).

It is easy to show that $E \mapsto \|A\|_E^G$ is a finite function on $\mathbb{R}_+$ if and only if A is a $\sqrt{G}$-bounded operator. Moreover, it is proved in [27] that this function coincides with the greatest lower bound of the functions $E \mapsto \sqrt{a^2 + b^2E}$ over all pairs (a, b) for which (12) holds and that the $\sqrt{G}$-bound can be expressed as follows

$$b_{\sqrt{G}}(A) = \inf_{E>0} \|A\|_E^G/\sqrt{E} = \lim_{E\to+\infty} \|A\|_E^G/\sqrt{E}. \tag{13}$$

The set of all $\sqrt{G}$-bounded operators equipped with the norm $\|.\|_E^G$ is a Banach space denoted by $\mathfrak{B}_G(\mathcal{H})$ in [26] (we identify operators coinciding on the set $\mathcal{D}(\sqrt{G})$). The set $\mathfrak{B}(\mathcal{H})$ is naturally embedded into $\mathfrak{B}_G(\mathcal{H})$ as a linear subspace, its closure $\mathfrak{B}_G^0(\mathcal{H})$ (i.e. the completion of $\mathfrak{B}(\mathcal{H})$ w.r.t. the norm $\|.\|_E^G$) coincides with the set of all $\sqrt{G}$-infinitesimal operators [26, 27].

We will use the following lemma proved in [25].

Lemma 4. *Let A and B be any $\sqrt{G}$-infinitesimal operators and*

$$\mathfrak{C}_{G,E} \doteq \{\rho \in \mathfrak{T}_+(\mathcal{H}) \,|\, \mathrm{Tr}\rho \le 1, \mathrm{Tr}\rho G \le E\}, \quad E > 0. \tag{14}$$

A) For any ρ in $\mathfrak{C}_{G,E}$ the operator $A\rho B^* \in \mathfrak{T}(\mathcal{H})$ is correctly defined by the formula

$$A\rho B^* \doteq \sum_i |A\varphi_i\rangle\langle B\varphi_i|, \tag{15}$$

where $\rho = \sum_i |\varphi_i\rangle\langle\varphi_i|$ is any decomposition of ρ into 1-rank operators, and

$$|\mathrm{Tr}A\rho B^*| \le \|A\|_E^G\|B\|_E^G. \tag{16}$$

B) The function $\rho \mapsto A\rho B^*$ is affine and uniformly continuous on the set $\mathfrak{C}_{G,E}$ for any $E > 0$. Quantitatively,

$$\|A\rho B^* - A\sigma B^*\|_1 \le \|A\|_E^G f_B(E, \varepsilon) + \|B\|_E^G f_A(E, \varepsilon) \tag{17}$$

for any ρ and σ in $\mathfrak{C}_{G,E}$ such that $\|\rho - \sigma\|_1 \le \varepsilon$, where $f_X(E, \varepsilon) = \sqrt{\varepsilon}\|X\|_{4E/\varepsilon}^G$ is a function vanishing as $\varepsilon \to 0^+$ for any $\sqrt{G}$-infinitesimal operator X.

The function $f_X(E, \varepsilon)$ tends to zero as $\varepsilon \to 0^+$, since formula (13) implies that $\|X\|_E^G = o(\sqrt{E})$ as $E \to +\infty$ for any $\sqrt{G}$-infinitesimal operator X.

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Remark 1. If A and B are $\sqrt{G}$-bounded operators then inequality (16) holds for any finite rank operator ρ in $\mathfrak{C}_{G,E}$ (provided that $A\rho B^*$ is defined by formula (15)). If, in addition, A and B are closable operators then inequality (16) holds for any ρ in $\mathfrak{C}_{G,E}$ [26].

We will also use the following lemmas.

Lemma 5. [26] *If $\mathcal{K}$ is any separable Hilbert space then $\|A \otimes I_{\mathcal{K}}\|_E^{G \otimes I_{\mathcal{K}}} = \|A\|_E^G$.*

Lemma 6. *Let A be a self-adjoint operator on $\mathcal{H}$. If the operator A is $\sqrt{G}$-bounded then the operator A^p is $\sqrt{G}$-infinitesimal for any $p \in [0,1)$ and $\|A^p\|_E^G \leq \left[\|A\|_E^G\right]^p$.*

Proof. By using the spectral representation of A^2, concavity of the function x^p and the Jensen's inequality we obtain $\mathrm{Tr}\rho A^{2p} \leq [\mathrm{Tr}\rho A^2]^p$ for any finite rank state ρ with finite energy $\mathrm{Tr}\rho G$. So, $\|A^p\|_E^G \leq \left[\|A\|_E^G\right]^p$. By using (13) and concavity of the function $E \mapsto \left[\|A\|_E^G\right]^2$ it is easy to show that A^p is a $\sqrt{G}$-infinitesimal operator. $\square$

Lemma 7. *Let A be a self-adjoint operator with domain containing the set $D(\sqrt{G})$. If $\mathrm{Tr}\rho A^2$ is finite for any state ρ with finite energy $\mathrm{Tr}\rho G$ then the operator A is $\sqrt{G}$-bounded.*

Proof. Suppose that the operator A is not $\sqrt{G}$-bounded. Then $\|A\|_E^G = +\infty$ for any given $E > 0$. Hence, there is a sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$ such that $\mathrm{Tr}\rho_n G \leq E$ and $\mathrm{Tr}\rho_n A^2 \geq 2^n$. Consider the state $\rho_* = \sum_{n=1}^{+\infty} 2^{-n}\rho_n$. By lower semicontinuity and convexity of the function $\rho \mapsto \mathrm{Tr}\rho G$ we have $\mathrm{Tr}\rho_* G \leq E$, while by using concavity and nonnegativity of the function $\rho \mapsto \mathrm{Tr}\rho A^2$ it is easy to show that $\mathrm{Tr}\rho_* A^2 \geq \sum_{n=1}^{+\infty} 2^{-n}\mathrm{Tr}\rho_n A^2 = +\infty$. $\square$

3 Continuity and differentiability of quantum dynamical semigroups w.r.t. the energy-constrained diamond norm

Let Φ_t be a (strongly continuous) quantum dynamical semigroup on $\mathfrak{T}(\mathcal{H}_A)$. We will explore analytical properties of this semigroup w.r.t. the metric induced by the ECD norm (4) assuming that G is a discrete operator (1) with $E_0 = 0$.

The generator S of a semigroup Φ_t is the superoperator

$$S : \rho \mapsto \lim_{t \to 0^+} t^{-1}(\Phi_t(\rho) - \rho) \tag{18}$$

defined on the set $\mathcal{D}(S)$ of all operators $\rho \in \mathfrak{T}(\mathcal{H}_A)$ such that the limit in (18) exists w.r.t. to the trace norm. The set $\mathcal{D}(S)$ is the domain of the generator S.

Continuity of a quantum dynamical semigroup Φ_t w.r.t. the ECD norms means that[4]

$$\lim_{t \to 0^+} \|\Phi_t - \mathrm{Id}_A\|_{\diamond,E}^G = 0. \tag{19}$$

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[4]The definition of the ECD norm implies that $\|\Phi_{s+t} - \Phi_s\|_{\diamond,E}^G \leq \|\Phi_t - \mathrm{Id}_A\|_{\diamond,E}^G$ for any $t, s > 0$.

Proposition 1. *Any quantum dynamical semigroup $\{\Phi_t\}$ on $\mathfrak{T}(\mathcal{H}_A)$ is continuous w.r.t the ECD norm induced by a discrete positive operator G. If $\|S\|_{\diamond,E}^G < +\infty$ then[5]*

$$\|\Phi_t - \mathrm{Id}_A\|_{\diamond,E}^G \leq t\|S\|_{\diamond,E}^G \quad \forall t > 0. \tag{20}$$

Remark 2. In contrast to the standard semigroup theory the condition $\|S\|_{\diamond,E}^G < +\infty$ is not necessary for continuity of $\{\Phi_t\}$ w.r.t. the ECD norm (see examples in Section 3). Estimates for $\|\Phi_t - \mathrm{Id}_A\|_{\diamond,E}^G$ in the case $\|S\|_{\diamond,E}^G = +\infty$ can be obtained by the method used by Winter in [34]. More detailed investigation of this problem had been made recently by Becker and Datta in [3].

Proof. Since the convergence generated by any of the ECD norms coincides with the strong convergence on subsets of Hermitian preserving superoperators bounded w.r.t. the diamond norm [24, Proposition 3], any strongly continuous semigroup $\{\Phi_t\}$ of quantum channels is continuous w.r.t. the ECD norm.

If $\|S\|_{\diamond,E}^G < +\infty$ then Lemma 8 below implies that

$$\sup_{\rho \in \widehat{\mathfrak{S}}_{G\otimes I_R,E}^0} \|\Phi_t \otimes \mathrm{Id}_R(\rho) - \rho\|_1 \leq t\|S\|_{\diamond,E}^G,$$

where $\widehat{\mathfrak{S}}_{G\otimes I_R,E}^0$ is the set of all pure states in $\mathfrak{S}(\mathcal{H}_{AR})$ having finite Schmidt rank such that $\mathrm{Tr}\rho_A G \leq E$. Since the family $\{\Phi_t\}$ is bounded w.r.t. the diamond norm and the set $\mathfrak{S}_{G\otimes I_R,E}^0$ is dense in the set all pure states in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\mathrm{Tr}\rho_A G \leq E$, the above relation and Lemma 1B imply (20). $\square$

Lemma 8. *Let Φ_t be a quantum dynamical semigroup on $\mathfrak{T}(\mathcal{H}_A)$ such that the domain of its generator S contains the set $\mathfrak{S}_G$ of states ρ with finite energy $\mathrm{Tr}\rho G$. Let $\mathcal{H}_R$ be a separable Hibert space. Then*

$$\Phi_t \otimes \mathrm{Id}_R(\rho) - \rho = \int_0^t \Phi_s \otimes \mathrm{Id}_R(S \otimes \mathrm{Id}_R(\rho))ds \tag{21}$$

for any pure state ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ having finite Schmidt rank such that $\mathrm{Tr}\rho_A G < +\infty$,[6] where

$$S \otimes \mathrm{Id}_R(\rho) \doteq \sum_{i,j} S(|\varphi_i\rangle\langle\varphi_j|) \otimes |\psi_i\rangle\langle\psi_j| \tag{22}$$

provided that $\rho = \sum_{i,j} |\varphi_i\rangle\langle\varphi_j| \otimes |\psi_i\rangle\langle\psi_j|$ is the Schmidt representation of ρ.

Proof. It suffices to note that the assumption $\mathfrak{S}_G \subseteq \mathcal{D}(S)$ implies that $S \otimes \mathrm{Id}_R(\rho)$ is well defined by formula (22) and that

$$\lim_{t\to 0^+} \frac{\Phi_t \otimes \mathrm{Id}_R(\rho) - \rho}{t} = S \otimes \mathrm{Id}_R(\rho) \tag{23}$$

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[5]This implies that for any separable Hilbert space $\mathcal{H}_R$ the superoperator $S \otimes \mathrm{Id}_R$ is well defined on the set of states ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\mathrm{Tr}\rho_A G < +\infty$.

[6]The integral in (21) and in all the formulae below is in the Bochner sense.

for any pure state ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ having finite Schmidt rank such that $\mathrm{Tr}\rho_A G < +\infty$.
$\square$

Remark 3. Relation (21) holds for any $\rho \in \mathfrak{T}(\mathcal{H}_{AR})$ for which (23) is valid.

Consider the question of differentiability of strongly continuous semigroups of quantum channels w.r.t. the ECD norm.

Let Φ_t be a quantum dynamical semigroup on $\mathfrak{T}(\mathcal{H}_A)$ such that the domain of its generator S contains the set $\mathfrak{S}_G$ of states ρ in $\mathfrak{S}(\mathcal{H}_A)$ with finite energy $\mathrm{Tr}\rho G$. It means that

$$\lim_{t\to 0^+} \|(\Phi_t(\rho) - \rho)/t - S(\rho)\|_1 = 0 \quad \text{for all } \rho \text{ such that } \mathrm{Tr}\rho G < +\infty. \quad (24)$$

We will say that the semigroup Φ_t is *differentiable w.r.t. the ECD norm* if for a separable Hilbert space $\mathcal{H}_R$ the superoperator $S \otimes \mathrm{Id}_R$ is well defined on the set of states ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\mathrm{Tr}\rho_A G < +\infty$ and

$$\lim_{t\to 0^+} \|(\Phi_t - \mathrm{Id}_A)/t - S\|^G_{\diamond,E} = 0 \quad \text{for some} \quad E > 0. \quad (25)$$

This implies, in particular, that

$$\sup_{\rho\in\mathfrak{S}(\mathcal{H}_A):\mathrm{Tr}\rho G\leq E} \|\Phi_t(\rho) - [\rho + tS(\rho)]\|_1 = o(t) \quad \text{as } t \to 0^+.$$

It is clear that (25) is substantially stronger than (24). On the other hand, the property (25) is weaker than the differentiability of the semigroup Φ_t w.r.t. the diamond norm (which is equivalent to the boundedness of the generator S and the representation $\Phi_t = e^{tS}$). So, the semigroups having property (25) form a proper subclass of the class of all strongly continuous semigroups which is *substantially larger* than the subclass of uniformly continuous semigroups (see Corollary 1 and Section 3).

Remark 4. Proposition 3 in [24] does not imply that (25) follows from (24), since the family of superoperators $(\Phi_t - \mathrm{Id}_A)/t$ may not be bounded. This conclusion is confirmed by the examples considered in Section 3.

Remark 5. In contrast to the well known results concerning uniformly continuous semigroups, the condition $\|S\|^G_{\diamond,E} < +\infty$ is necessary but not sufficient for differentiability of the semigroup Φ_t (see Section 3).

It turns out that necessary and sufficient conditions for the differentiability of a semigroup Φ_t w.r.t. the ECD norm can be expressed in terms of the generator S of this semigroup. For given separable infinite-dimensional Hibert space $\mathcal{H}_R$ and $E > 0$ introduce the sets

$$\widehat{\mathfrak{S}}_{G\otimes I_R,E} \doteq \{\rho \in \mathfrak{S}(\mathcal{H}_{AR}) \,|\, \mathrm{Tr}\rho_A G \leq E, \mathrm{rank}\rho = 1\} \quad (26)$$

and

$$\mathfrak{T}_{G\otimes I_R,E} \doteq \{\rho \in \mathfrak{T}_+(\mathcal{H}_{AR}) \,|\, \mathrm{Tr}\rho_A G \leq E, \mathrm{Tr}\rho \leq 1\}. \quad (27)$$

Theorem 1. *Let Φ_t be a quantum dynamical semigroup on $\mathfrak{T}(\mathcal{H}_A)$ such that the domain of its generator S contains all the states ρ with finite energy $\operatorname{Tr}\rho G$. The following properties are equivalent:*

(i) *the semigroup Φ_t is differentiable w.r.t. the ECD norm, i.e. relation (25) holds;*

(ii) *the superoperator $S \otimes \operatorname{Id}_R$ is continuous on the set $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$ defined in (26) for some $E > 0$;*

(iii) *the superoperator $S \otimes \operatorname{Id}_R$ is uniformly continuous on the set $\mathfrak{T}_{G \otimes I_R, E}$ defined in (27) for any $E > 0$ and $\|S\|_{\diamond, E}^G = o(E)$ as $E \to +\infty$.*

Properties (i)-(iii) hold provided that $\|S\|_{\diamond, E}^G = o(\sqrt{E})$ as $E \to +\infty$.

Proof. It is clear that (25) implies that the superoperator S belongs to the completion of the set of all Hermitian preserving completely bounded superoperators w.r.t. the ECD norm. So, the implication (i) $\Rightarrow$ (iii) directly follows from the results of Section 5 in [25]. The implication (iii) $\Rightarrow$ (ii) is trivial.

To prove the implication (ii) $\Rightarrow$ (i) consider the superoperator

$$\Upsilon_t(\rho) = (1/t) \int_0^t (\Phi_s - \operatorname{Id}_A) \otimes \operatorname{Id}_R(S \otimes \operatorname{Id}_R(\rho)) ds$$

well defined on the set $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$. Since

$$\|\Upsilon_t(\rho) - \Upsilon_t(\sigma)\|_1 \leq 2\|S \otimes \operatorname{Id}_R(\rho) - S \otimes \operatorname{Id}_R(\sigma)\|_1, \tag{28}$$

the superoperator Υ_t is continuous on the set $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$.

By Lemma 8 we have

$$\Upsilon_t(\rho) = t^{-1}(\Phi_t \otimes \operatorname{Id}_R(\rho) - \rho) - S \otimes \operatorname{Id}_R(\rho) \tag{29}$$

for any state ρ in $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$ with finite Schmidt rank. Since the set of all such states is dense in $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$, the continuity of Υ_t implies that (29) holds for all states in $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$.

Thus, by Lemma 1B we have

$$\left\| t^{-1}(\Phi_t - \operatorname{Id}_A) - S \right\|_{\diamond, E}^G = \sup_{\rho \in \widehat{\mathfrak{S}}_{G, G_R, E}} \Upsilon_t(\rho), \tag{30}$$

where $\widehat{\mathfrak{S}}_{G, G_R, E} = \{\rho \in \mathfrak{S}(\mathcal{H}_{AR}) \,|\, \operatorname{Tr}\rho_A G \leq E, \operatorname{Tr}\rho_R G_R \leq E, \operatorname{rank}\rho = 1\}$ and G_R is an operator on $\mathcal{H}_R$ unitarily equivalent to G.

By the Lemma in [11] and Corollary 6 in [15] the set $\widehat{\mathfrak{S}}_{G, G_R, E}$ is compact. Hence the superoperator $S \otimes \operatorname{Id}_R$ is uniformly continuous on $\widehat{\mathfrak{S}}_{G, G_R, E}$ and the continuity bound (28) implies that

$$\|\Upsilon_t(\rho) - \Upsilon_t(\sigma)\|_1 \leq f(\delta) \tag{31}$$

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for any ρ and σ in $\widehat{\mathfrak{S}}_{G,G_R,E}$ such that $\|\rho - \sigma\|_1 \leq \delta$ and all $t > 0$, where $f(\delta)$ is a function vanishing as $\delta \to 0^+$.

By using the compactness of $\widehat{\mathfrak{S}}_{G,G_R,E}$ and the continuity bound (31) it is easy to show that the r.h.s. of (30) tends to zero provided that $\Upsilon_t(\rho)$ tends to zero for any ρ in $\widehat{\mathfrak{S}}_{G,G_R,E}$. The last property follows from the inequality

$$\|\Upsilon_t(\rho)\|_1 \leq (1/t) \int_0^t \|(\Phi_s - \mathrm{Id}_A) \otimes \mathrm{Id}_R(S \otimes \mathrm{Id}_R(\rho))\|_1 ds, \quad \rho \in \widehat{\mathfrak{S}}_{G \otimes I_R, E},$$

since $\|(\Phi_s - \mathrm{Id}_A) \otimes \mathrm{Id}_R(\sigma)\|_1$ tends to zero as $t \to 0^+$ for any $\sigma \in \mathfrak{T}(\mathcal{H}_{AR})$.

If $\|S\|_{\diamond,E}^G = o(\sqrt{E})$ as $E \to +\infty$ then the superoperator $S \otimes \mathrm{Id}_R$ is continuous on the set $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$ by Lemma 3 in Section 2.2. $\square$

It is well known that the generator of any uniformly continuous quantum dynamical semigroup has the standard (GKLS-) form

$$S(\rho) = \sum_k V_k \rho V_k^* + K\rho + \rho K^*, \tag{32}$$

where V_k and K are operators in $\mathfrak{B}(\mathcal{H}_A)$ such that

$$\sum_k \|V_k \varphi\|^2 = -2\Re\langle\varphi|K\varphi\rangle \tag{33}$$

for any $\varphi \in \mathcal{H}_A$ [9, 18]. Different generalizations of this representation to unbounded generators of quantum dynamical semigroups are considered in [7, 10, 2, 23].

The r.h.s. of (32) is well defined on the set $\mathfrak{S}_G^0$ of finite rank states ρ with finite energy $\mathrm{Tr}\rho G$ provided that all the operators V_k and K are defined on the domain of the operator $\sqrt{G}$ (it suffices to define the term $V_k \rho V_k^*$ by the formula similar to (15) and to rewrite the term ρK^* as $[K\rho^*]^*$, cf.[2, 23]). So, in this case we may expect that the r.h.s. of (32) can be extended to a superoperator defined on the set $\mathfrak{S}_G$ of all states with finite energy (which may be a generator of a strongly continuous semigroup differentiable w.r.t. the ECD norm).

Theorem 1 implies the following

Corollary 1. *Let Φ_t be a quantum dynamical semigroup on $\mathfrak{T}(\mathcal{H}_A)$ with the generator S defined by formula (32) on the set $\mathfrak{S}_G^0$, where V_k and K are operators defined on $\mathcal{D}(\sqrt{G})$ and satisfying condition (33) for any $\varphi \in \mathcal{D}(\sqrt{G})$. Assume that the domain of the generator S contains[7] the set $\mathfrak{S}_G$ of all states with finite energy.*

A) If the operator K is $\sqrt{G}$-infinitesimal then the semigroup Φ_t is differentiable w.r.t. the ECD norm. In this case $\|S\|_{\diamond,E}^G \leq \|\{V_k\}\|_E^G + 2\|K\|_E^G \leq 4\|K\|_E^G$, where

$$\|\{V_k\}\|_E^G \doteq \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A): \mathrm{Tr}\rho G \leq E} \sum_k \mathrm{Tr} V_k \rho V_k^*. \tag{34}$$

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[7]It means that the condition (24) holds.

B) *If the operator K is $\sqrt{G}$-bounded and the semigroup Φ_t is differentiable w.r.t. the ECD norm then the operator K and all the operators V_k are $\sqrt{G}$-infinitesimal.*

Proof. B) Let $S_1(\rho) = \sum_k V_k \rho V_k^*$ and $S_2(\rho) = K\rho + [K\rho^*]^*$ be the positive and no-event parts of the superoperator S [23]. If the operator K is $\sqrt{G}$-bounded then condition (33) shows that

$$\|\{V_k\}\|_E^G \leq 2\|K\|_E^G = o(E) \quad \text{as} \quad E \to +\infty, \tag{35}$$

where the equality follows from the concavity of the function $E \mapsto [\|K\|_E^G]^2$. This implies, by formula (13), that all the operators V_k are $\sqrt{G}$-infinitesimal. By Proposition 1 in [25] relation (35) shows that the superoperator S_1 belongs to the completion of the cone of completely positive superoperators w.r.t. the ECD norm. It follows that S_1 has property (iii) in Theorem 1. The differentiability of the semigroup Φ_t w.r.t. the ECD norm implies, by Theorem 1, that the same continuity property holds for the superoperator $S_2 = S - S_1$. Thus, Lemma 9 below shows that the operator K is $\sqrt{G}$-infinitesimal.

A) If the operator K is $\sqrt{G}$-infinitesimal then, by the above observation, all the operators V_k are $\sqrt{G}$-infinitesimal as well. So, inequality (16) with Lemma 5 and the inequality in (35) imply that

$$\|S\|_{\diamond,E}^G \leq \|\{V_k\}\|_E^G + 2\|K\|_E^G \leq 4\|K\|_E^G = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty,$$

where the equality follows from (13). By the last assertion of Theorem 1 the semigroup Φ_t is differentiable w.r.t. the ECD norm. $\square$

Remark 6. If K is not $\sqrt{G}$-bounded then it is easy to show that $\|S_2\|_{\diamond,E}^G$ can not be finite. So, if we assume differentiability of the semigroup Φ_t in Corollary 1 w.r.t. the ECD norm in this case, then $\|S_1\|_{\diamond,E}^G$ can not be finite as well (since $\|S\|_{\diamond,E}^G$ must be finite). Formally, we can not exclude this possibility, but we have not managed to construct the corresponding examples.

Lemma 9. *Let A be an operator defined on the set $\mathcal{D}(\sqrt{G})$ such that for any separable Hilbert space $\mathcal{H}_R$ the operator $A \otimes I_R$ is well defined on the set $\mathcal{D}(\sqrt{G} \otimes I_R)$.*

If one of the superoperators $\Phi_\pm(\rho) = (A \otimes I_R)\rho \pm [(A \otimes I_R)\rho^*]^*$ is continuous on the set $\mathfrak{T}_{G \otimes I_R, E}$ defined in (27) for some $E > 0$ then the operator A is $\sqrt{G}$-infinitesimal.

Proof. Assume that the operator G has the form (1) and $P_n = \sum_{k=0}^{n-1} |\tau_k\rangle\langle\tau_k|$ is the projector on the subspace spanned by the vectors $\tau_0, ..., \tau_{n-1}$.

If the operator A does not lie in the space $\mathfrak{B}_G^0(\mathcal{H}_A)$ of $\sqrt{G}$-infinitesimal operators then the sequence

$$X_n = \sup\left\{\|(A\bar{P}_n) \otimes I_R \varphi\| \,\Big|\, \varphi \in \mathcal{H}_{AR}, \|\sqrt{G} \otimes I_R \varphi\|^2 \leq E, \|\varphi\| = 1\right\},$$

where $\bar{P}_n = I_A - P_n$ does not tend to zero. Indeed, otherwise the sequence $\{AP_n\}$ of bounded operators tends to the operator A w.r.t. the norm $\|\cdot\|_E^G$ and hence A belongs to the space $\mathfrak{B}_G^0(\mathcal{H}_A)$ [26, Remark 5]. So, there is a sequence $\{\varphi_n\}$ of vectors in the unit

ball of $\mathcal{H}_{AR}$ such that $\|\sqrt{G} \otimes I_R \varphi_n\|^2 \leq E$ for all n and the sequence $\|(A\bar{P}_n) \otimes I_R \varphi_n\|$ does not tend to zero as $n \to \infty$. We may assume that the last sequence is bounded.

Let $|\psi_n\rangle = \bar{P}_n \otimes I_R |\varphi_n\rangle$. Since $\|\sqrt{G} \otimes I_R \psi_n\|^2 \leq E$ and $(\sqrt{G}\bar{P}_n) \otimes I_R \geq \sqrt{E_n}\bar{P}_n \otimes I_R$ for all n, we have $\|\psi_n\| \leq \sqrt{E/E_n}$. Hence, $\{\psi_n\}$ is a sequence tending to zero as $n \to \infty$. Let $\eta = \tau_0 \otimes \upsilon$, where υ is a unit vector in $\mathcal{H}_R$. We will assume that the vectors $|\alpha_n\rangle = A \otimes I_R |\psi_n\rangle$ do not converge to the vector $i|\eta\rangle$ (otherwise we can replace υ). Consider the sequence of operators $\rho_n = \frac{1}{2}|\psi_n + \eta\rangle\langle\psi_n + \eta|$ in $\mathfrak{T}_{G\otimes I_R,E}$ converging to the operator $\rho_0 = \frac{1}{2}|\eta\rangle\langle\eta|$. We have

$$2A \otimes I_R \rho_n + 2[A \otimes I_R \rho_n]^* = |\alpha_n + \beta\rangle\langle\psi_n + \eta| + |\psi_n + \eta\rangle\langle\alpha_n + \beta| =$$

$$|\alpha_n\rangle\langle\psi_n| + |\psi_n\rangle\langle\alpha_n| + |\beta\rangle\langle\psi_n| + |\psi_n\rangle\langle\beta| + [|\alpha_n\rangle\langle\eta| + |\eta\rangle\langle\alpha_n|] + |\beta\rangle\langle\eta| + |\eta\rangle\langle\beta|,$$

where $|\beta\rangle = A \otimes I_R |\eta\rangle$. The first four terms here tend to zero as $n \to \infty$, since $\{\psi_n\}$ tends to zero and the sequence $\{\alpha_n\}$ is bounded. But the term in the square bracket does not tend to zero. So, $\Phi_+(\rho_n)$ does not tend to

$$\Phi_+(\rho_0) = \tfrac{1}{2}[|\beta\rangle\langle\eta| + |\eta\rangle\langle\beta|].$$

Similarly, one can prove the discontinuity of the superoperator Φ_- on the set $\mathfrak{T}_{G\otimes I_R,E}$ if the operator A is not $\sqrt{G}$-infinitesimal. $\square$

4 Higher order differentiability w.r.t. the ECD norm and the exponential representation

In this section we obtain necessary and sufficient conditions for n order differentiability ($n \geq 2$) of a (strongly continuous) quantum dynamical semigroups w.r.t. the metric induced by the ECD norm (4) assuming that G is a discrete operator (1) with $E_0 = 0$.

Theorem 2. *Let Φ_t be a quantum dynamical semigroup on $\mathfrak{T}(\mathcal{H}_A)$ and S the generator of this semigroup. If $S^{k-1}(\rho) \in \mathcal{D}(S)$, $k = \overline{2,n}$, for any pure state ρ with finite energy $\mathrm{Tr}\rho G$ and the superoperators $S^k \otimes \mathrm{Id}_R$, $k = \overline{1,n}$, are continuous on the set $\widehat{\mathfrak{S}}_{G\otimes I_R,E}$ defined in (26) for some $E > 0$ then*

$$\left\|\Phi_t - \left[\mathrm{Id}_A + tS + \frac{t^2}{2}S^2 + ... + \frac{t^k}{k!}S^k\right]\right\|_{\diamond,E}^G = o(t^k) \ as \ t \to 0^+, \ k = \overline{1,n}, \qquad (36)$$

for any $E > 0$ and the l.h.s. of (36) is bounded above by $2t^k\|S^k\|_{\diamond,E}^G/k!$.

If (36) holds then the superoperators $S^k \otimes \mathrm{Id}_R$, $k = 1, 2, ..., n$, are uniformly continuous on the set $\mathfrak{T}_{G\otimes I_R,E}$ defined in (27) and $\|S^k\|_{\diamond,E}^G = o(E)$, $k = \overline{1,n}$, as $E \to +\infty$.

Proof. By sequentially applying Lemma 10 below one can obtain

$$\widehat{\Phi}_t(\rho) = \rho + t\widehat{S}(\rho) + \frac{t^2}{2}\widehat{S}^2(\rho) + .. + \frac{t^{k-1}}{(k-1)!}\widehat{S}^{k-1}(\rho) + \int_0^t \widehat{\Phi}_s(\widehat{S}^k(\rho))\frac{(t-s)^{k-1}}{(k-1)!} \, ds, \ k = \overline{1,n},$$

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for any state ρ in $\widehat{\mathfrak{S}}_{G\otimes I_R,E}$, where $\widehat{\Phi}_t$ and $\widehat{S}^k$ denote, respectively, the superoperators $\Phi_t \otimes \mathrm{Id}_R$ and $S^k \otimes \mathrm{Id}_R$. By using the superoperator

$$\Gamma_{t,k}(\rho) = \widehat{\Phi}_t(\rho) - \rho - t\widehat{S}(\rho) - \frac{t^2}{2}\widehat{S}^2(\rho) - .. - \frac{t^k}{k!}\widehat{S}^k(\rho)$$

well defined on the set $\widehat{\mathfrak{S}}_{G\otimes I_R,E}$ the above relation can be written as

$$\Gamma_{t,k}(\rho) = \int_0^t (\Phi_s - \mathrm{Id}_A) \otimes \mathrm{Id}_R(S^k \otimes \mathrm{Id}_R(\rho)) \frac{(t-s)^{k-1}}{(k-1)!}\, ds, \quad k = \overline{1,n},$$

It follows that for any states ρ and σ in $\widehat{\mathfrak{S}}_{G\otimes I_R,E}$ we have

$$t^{-k}\|\Gamma_{t,k}(\rho) - \Gamma_{t,k}(\sigma)\|_1 \leq (2/k!)\|S^k \otimes \mathrm{Id}_R(\rho) - S^k \otimes \mathrm{Id}_R(\sigma)\|_1. \qquad (37)$$

Lemma 1B implies that

$$\left\|\Phi_t - \left[\mathrm{Id}_A + tS + \frac{t^2}{2}S^2 + ... + \frac{t^k}{k!}S^k\right]\right\|_{\diamond,E}^G = \sup_{\rho\in\widehat{\mathfrak{S}}_{G,G_R,E}} \Gamma_{t,k}(\rho), \qquad (38)$$

where $\widehat{\mathfrak{S}}_{G,G_R,E} = \{\rho \in \mathfrak{S}(\mathcal{H}_{AR}) \,|\, \mathrm{Tr}\rho_A G \leq E, \mathrm{Tr}\rho_R G_R \leq E, \mathrm{rank}\rho = 1\}$ and G_R is an operator on $\mathcal{H}_R$ unitarily equivalent to G.

By the Lemma in [11] and Corollary 6 in [15] the set $\widehat{\mathfrak{S}}_{G,G_R,E}$ is compact. Hence the superoperator $S^k \otimes \mathrm{Id}_R$ is uniformly continuous on this set and continuity bound (37) implies that

$$t^{-k}\|\Gamma_{t,k}(\rho) - \Gamma_{t,k}(\sigma)\|_1 \leq g_k(\delta) \qquad (39)$$

for any ρ and σ in $\widehat{\mathfrak{S}}_{G,G_R,E}$ such that $\|\rho - \sigma\|_1 \leq \delta$ and all $t > 0$, where $g_k(\delta)$ is a function vanishing as $\delta \to 0^+$.

The compactness of $\widehat{\mathfrak{S}}_{G,G_R,E}$ and continuity bound (39) imply that the r.h.s. of (38) is $o(t^k)$ as $t \to 0^+$ if and only if $t^{-k}\|\Gamma_{t,k}(\rho)\|_1$ tends to zero as $t \to 0^+$ for any ρ in $\widehat{\mathfrak{S}}_{G,G_R,E}$. The last property can be easily proved by using the inequality

$$\|\Gamma_{t,k}(\rho)\|_1 \leq \int_0^t \|(\Phi_t - \mathrm{Id}_A) \otimes \mathrm{Id}_R(S^k \otimes \mathrm{Id}_R(\rho))\|_1 \frac{(t-s)^{k-1}}{(k-1)!}\, ds,$$

since $\|(\Phi_s - \mathrm{Id}_A) \otimes \mathrm{Id}_R(\sigma)\|_1$ tends to zero as $t \to 0^+$ for any $\sigma \in \mathfrak{T}(\mathcal{H}_{AR})$.

By this inequality $\|\Gamma_{t,k}(\rho)\|_1 \leq (2t^k/k!)\|S^k \otimes \mathrm{Id}_R(\rho)\|_1$ for any state ρ in $\widehat{\mathfrak{S}}_{G\otimes I_R,E}$. So, it follows from (38) that the l.h.s. of (36) is bounded above by $2t^k\|S^k\|_{\diamond,E}^G/k!$.

If (36) holds then it is easy to show iteratively that the superoperators S^k, $k = \overline{1,n}$, belong to the completion of the set of all Hermitian preserving completely bounded superoperators w.r.t. the ECD norm. So, the last assertion of the theorem directly follows from the results of Section 5 in [25]. $\square$

Lemma 10. *If the assumptions of Theorem 2 hold then*

$$\Phi_t \otimes \mathrm{Id}_R(S^{k-1} \otimes \mathrm{Id}_R(\rho)) - S^{k-1} \otimes \mathrm{Id}_R(\rho) = \int_0^t \Phi_s \otimes \mathrm{Id}_R(S^k \otimes \mathrm{Id}_R(\rho))ds \qquad (40)$$

for $k = \overline{1,n}$ and any state ρ in $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$.

Proof. Let ρ be a state in $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$ with finite Schmidt rank n. It can be represented as

$$\rho = \sum_{i,j=1}^n |\varphi_i\rangle\langle\varphi_j| \otimes |\psi_i\rangle\langle\psi_j|,$$

where $\{\varphi_i\}$ and $\{\psi_i\}$ are orthogonal sets of vectors in $\mathcal{D}(\sqrt{G})$ and $\mathcal{H}_R$ correspondingly. Then

$$S^{k-1} \otimes \mathrm{Id}_R(\rho) = \sum_{i,j=1}^n S^{k-1}(|\varphi_i\rangle\langle\varphi_j|) \otimes |\psi_i\rangle\langle\psi_j|.$$

Since $S^{k-1}(|\varphi_i\rangle\langle\varphi_j|) \in \mathcal{D}(S)$ for all i and j by the assumption, it follows that

$$\lim_{t \to 0^+} (1/t)(\Phi_t \otimes \mathrm{Id}_R(S^{k-1} \otimes \mathrm{Id}_R(\rho)) - S^{k-1} \otimes \mathrm{Id}_R(\rho)) = S^k \otimes \mathrm{Id}_R(\rho).$$

So, by Remark 3, Lemma 8 implies that (40) holds for the state ρ. Since the set of states in $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$ with finite Schmidt rank is dense in $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$, the validity of equality (40) for any state in $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$ follows from the continuity of both sides of this equality on $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$. $\square$

Corollary 2. *Let Φ_t be a quantum dynamical semigroup on $\mathfrak{T}(\mathcal{H}_A)$ with the generator S having the properties:*

- *$S^{n-1}(\rho) \in \mathcal{D}(S)$, $n \in \mathbb{N}$, for any pure state ρ with finite energy $\mathrm{Tr}\rho G$;*

- *the superoperators $S^n \otimes \mathrm{Id}_R$, $n \in \mathbb{N}$, are continuous on the set $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$ defined in (26) for some $E > 0$;*

- *$\|S^n\|_{\diamond,E}^G / n!$ tends to zero as $n \to +\infty$ for some $E > 0$.*

Then

$$\Phi_t|_{\mathfrak{S}_G} = e^{tS}|_{\mathfrak{S}_G}, \quad t > 0, \qquad (41)$$

where $\Lambda|_{\mathfrak{S}_G}$ is the restriction of Λ to the set $\mathfrak{S}_G \doteq \{\rho \in \mathfrak{S}(\mathcal{H}_A) \,|\, \mathrm{Tr}\rho G < +\infty\}$ and e^{tS} denotes the series

$$\mathrm{Id}_A + tS + \frac{t^2}{2}S^2 + ... + \frac{t^k}{k!}S^k + ...$$

converging w.r.t. the norm $\|\cdot\|_{\diamond,E}^G$.

5 Examples

5.1 The unitary group e^{-iAt}

Consider the group of unitary channels $\Lambda_t^A(\rho) = e^{-iAt}\rho\, e^{iAt}$, where A is a self-adjoint operator on $\mathcal{H}$. To specify the generator of this semigroup and its domain we will need the following

Definition 1. [12, II.8] A densely defined operator A is called *square-summable* w.r.t. a state ρ if the operator $\sqrt{\rho}A$ is (extended to) a Hilbert-Schmidt operator.

It is easy to see that an operator A is *square-summable* w.r.t. a state ρ if and only if the operator $A\rho A$ is (extended to) a trace-class operator. The last property is equivalent to the finiteness of $\mathrm{Tr}\rho A^2 \doteq \sup_n \mathrm{Tr}\rho P_n A^2$, where P_n is the spectral projector of A corresponding to the interval $[0,n]$.

Lemma 11. [12, Proposition VI.3.1] *The domain of the generator S_A of the semigroup Λ_t^A contains all the states ρ such that the operator A is square-summable w.r.t. the state ρ. The action of S_A on such a state ρ is given by*

$$S_A(\rho) = i\left((\sqrt{\rho}A)^*\sqrt{\rho} - \sqrt{\rho}(\sqrt{\rho}A)\right). \tag{42}$$

The operator on the right-hand side of (42) is apparently trace-class, giving a well-defined version for the expression

$$i(A\rho - \rho A) = i[A,\rho],$$

which holds literally in the case of bounded operator A.

Lemma 11 and Lemma 7 in Section 2.3 show that $\mathcal{D}(S_A)$ contains all states ρ with finite energy $\mathrm{Tr}\rho G$ if and only if the operator A is $\sqrt{G}$-bounded. By using the inequality (16) with Lemma 5 and Remark 1 it is easy to show that in this case $\|S_A\|_{\diamond,E}^G \le 2\|A\|_E^G$. So, the results of the previous sections imply the following

Proposition 2. *Let G be a positive discrete unbounded operator (1) with $E_0 = 0$.*

A) *The group $\{\Lambda_t^A\}$ is continuous w.r.t. the ECD norm induced by G. If the operator A is $\sqrt{G}$-bounded then[8]*

$$\|\Lambda_t^A - \mathrm{Id}\|_{\diamond,E}^G \le 2t\|A\|_E^G \quad \forall t > 0. \tag{43}$$

B) *The group $\{\Lambda_t^A\}$ is differentiable w.r.t. the ECD norm induced by G if and only if A is a $\sqrt{G}$-infinitesimal operator.*

C) *If A^n is a $\sqrt{G}$-infinitesimal operator for some $n \in \mathbb{N}$ then*

$$\left\|\Lambda_t^A - \left[\mathrm{Id} + tS_A + \frac{t^2}{2}S_A^2 + ... + \frac{t^k}{k!}S_A^k\right]\right\|_{\diamond,E}^G = o(t^k) \ \ as \ t \to 0^+, \ \ k = \overline{1,n}; \tag{44}$$

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[8]If $A = G$ then the r.h.s. of (43) is equal to $+\infty$. The estimate for $\|\Lambda_t^G - \mathrm{Id}\|_{\diamond,E}^G$ is obtained in [34]. It is refined in [3], where the estimates for $\|\Lambda_t^G - \mathrm{Id}\|_{\diamond,E^{2\alpha}}^{G^{2\alpha}}$, $\alpha \in (0,1]$, are also obtained. If $A = \sqrt{G}$ then (43) coincides with the estimate in Proposition 3.2 in [3] for $\alpha = 1$ (since $\|\sqrt{G}\|_E^G = \sqrt{E}$).

for any $E > 0$ and the l.h.s. of (44) is bounded above by $2(2t)^k \|A^k\|_E^G / k!$.

D) *If A^n is a $\sqrt{G}$-bounded operator for any $n \in \mathbb{N}$ and $\|A^n\|_E^G \le C[n!]^p$ for some $E, C > 0$ and $p < 1$ then*

$$\Lambda_t^A|_{\mathfrak{S}_G} = e^{tS_A}|_{\mathfrak{S}_G} \quad \text{for any} \quad t > 0, \tag{45}$$

i.e. the restriction of the group Λ_t^A to the set $\mathfrak{S}_G \doteq \{\rho \in \mathfrak{S}(\mathcal{H}) \,|\, \mathrm{Tr}\rho G < +\infty\}$ is represented by the series

$$\mathrm{Id} + tS_A + \frac{t^2}{2}S_A^2 + ... + \frac{t^k}{k!}S_A^k + ...$$

converging w.r.t. the norm $\| \cdot \|_{\diamond,E}^G$.

Proposition 2 shows that:

- the group $\Lambda_t^{G^\alpha}(\rho) = e^{-iG^\alpha t}\rho\, e^{iG^\alpha t}$ is differentiable w.r.t the ECD norm induced by a positive operator G **if and only if** $\alpha < 1/2$;

- if $\alpha < 1/2^n$ then relations (44) hold for the group $\Lambda_t^{G^\alpha}$.

To construct a group Λ_t^A having the exponential representation (45) take $A = \sqrt{\ln G}$. Since $\ln^n G \le n!G$, we have

$$\|[\sqrt{\ln G}]^n\|_E^G \le \sqrt{n!}\|\sqrt{G}\|_E^G = \sqrt{n!E} < +\infty \quad \forall n.$$

Hence in this case formula (13) and Lemma 6 imply that A^n is a $\sqrt{G}$-infinitesimal operator for any $n \in \mathbb{N}$. Thus, all the conditions of Proposition 2D hold.

Proof of Proposition 2. A) This assertion directly follows from Proposition 1 and the above upper bound on $\|S_A\|_{\diamond,E}^G$.

B) This assertion follows from Corollary 1 and the remark before the proposition.

C) If A^n is a $\sqrt{G}$-infinitesimal operator then $\varphi \in \mathcal{D}(A^n)$ for any pure state $\rho = |\varphi\rangle\langle\varphi|$ with finite energy $\mathrm{Tr}\rho G = \|\sqrt{G}\varphi\|^2$. Since

$$S_A(|\psi\rangle\langle\psi|) = \frac{1}{2}(B|\psi\rangle\langle\psi|B^* - B^*|\psi\rangle\langle\psi|B), \qquad B = A - iI,$$

for any vector $\psi \in \mathcal{D}(A)$, it is easy to show that for any pure state ρ with finite energy we have $S_A^{k-1}(\rho) = X_{k,+}^\rho - X_{k,-}^\rho$, where $X_{k,+}^\rho$ and $X_{k,-}^\rho$ are operators in $\mathfrak{T}_+(\mathcal{H})$ such that $\mathrm{Tr}X_{k,\pm}^\rho A^2 < +\infty$, $k = \overline{2,n}$. So, Lemma 11 implies that $S_A^{k-1}(\rho) \in \mathcal{D}(S_A)$, $k = \overline{2,n}$ for any such state ρ. By using inequality (16) with Lemma 5 and Lemma 6 we obtain

$$\|S_A^k\|_{\diamond,E}^G \le \sum_{i=0}^k \binom{i}{k}\|A^i\|_E^G\|A^{k-i}\|_E^G \le 2^k\|A^k\|_E^G = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty, \tag{46}$$

where the last equality follows from (13), since A^k is a $\sqrt{G}$-infinitesimal operator for each $k = \overline{1,n}$ by Lemma 6. By Lemma 3 the superoperators $S_A^k \otimes \mathrm{Id}_R$, $k = \overline{1,n}$,

are continuous on the set $\widehat{\mathfrak{S}}_{G\otimes I_R,E}$ defined in (26). Thus, this assertion follows from Theorem 2.

D) This assertion follows from Corollary 2 and Lemma 6, since the condition $\|A^n\|_E^G \leq C[n!]^p$ and estimate (46) imply that $\|S_A^n\|_{\diamond,E}^G/n!$ tends to zero as $n \to +\infty$ for any $E > 0$. $\square$

5.2 The Gaussian convolutional semigroup

Let A be a self-adjoint operator on $\mathcal{H}$. Consider the semigroup of quantum channels

$$\Xi_t^A(\rho) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-iAx} \rho\, e^{iAx} e^{-\frac{x^2}{2t}}\, dx. \tag{47}$$

We will use the following lemma proved in the Appendix.

Lemma 12. *The domain of the generator Z_A of the semigroup Ξ_t^A contains all the states ρ such that the operator A^2 is square-summable w.r.t. the state ρ.[9] The action of Z_A on such a state ρ is given by*

$$Z_A(\rho) = (\sqrt{\rho}A)^*\sqrt{\rho}A - \frac{1}{2}\left[(\sqrt{\rho}A^2)^*\sqrt{\rho} + \sqrt{\rho}(\sqrt{\rho}A^2)\right]. \tag{48}$$

The operator on the right-hand side of (48) is apparently trace-class, giving a well-defined version for the expression

$$A\rho A - \frac{1}{2}\left(A^2\rho + \rho A^2\right) = \frac{1}{2}[A,[\rho,A]],$$

which holds literally in the case of bounded operator A.

Lemma 12 and Lemma 7 in Section 2.3 show that $\mathcal{D}(Z_A)$ contains all states ρ with finite energy $\mathrm{Tr}\rho G$ if and only if the operator A^2 is $\sqrt{G}$-bounded. By using inequality (16) with Lemma 5 and Remark 1 it is easy to show that in this case

$$\|Z_A\|_{\diamond,E}^G \leq [\|A\|_E^G]^2 + \|A^2\|_E^G \leq 2\|A^2\|_E^G.$$

So, the results of the previous sections imply the following

Proposition 3. *Let G be a positive discrete unbounded operator (1) with $E_0 = 0$.*

A) *The semigroup $\{\Xi_t^A\}$ is continuous w.r.t. the ECD norm induced by G. If the operator A^2 is $\sqrt{G}$-bounded then[10]*

$$\|\Xi_t^A - \mathrm{Id}\|_{\diamond,E}^G \leq t\left[[\|A\|_E^G]^2 + \|A^2\|_E^G\right] \quad \forall t > 0. \tag{49}$$

B) *The semigroup $\{\Xi_t^A\}$ is differentiable w.r.t. the ECD norm induced by G if and only if A^2 is a $\sqrt{G}$-infinitesimal operator.*

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[9]See Definition 1 in the previous subsection.

[10]Estimates for $\|\Xi_t^A - \mathrm{Id}\|_{\diamond,E}^G$ in the case $\|A^2\|_E^G \leq +\infty$ can be obtained by using [3, Theorem 1].

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C) *If A^{2n} is $\sqrt{G}$-infinitesimal operator for some $n \in \mathbb{N}$ then*

$$\left\| \Xi_t^A - \left[\mathrm{Id} + tZ_A + \frac{t^2}{2}Z_A^2 + ... + \frac{t^k}{k!}Z_A^k \right] \right\|_{\diamond,E}^G = o(t^k) \ \ as \ t \to 0^+, \ \ k = \overline{1, n}; \quad (50)$$

for any $E > 0$ and the l.h.s. of (50) is bounded above by $2(2t)^k \|A^{2k}\|_E^G/k!$.

D) *If A^n is $\sqrt{G}$-bounded operator for any $n \in \mathbb{N}$ and $\|A^{2n}\|_E^G \leq C[n!]^p$ for some $E, C > 0$ and $p < 1$ then*

$$\Xi_t^A|_{\mathfrak{S}_G} = e^{tZ_A}|_{\mathfrak{S}_G} \quad for \ any \ \ t > 0, \quad (51)$$

i.e. the restriction of the semigroup Ξ_t^A to the set $\mathfrak{S}_G \doteq \{ \rho \in \mathfrak{S}(\mathcal{H}) \mid \mathrm{Tr}\rho G < +\infty \}$ is represented by the series

$$\mathrm{Id} + tZ_A + \frac{t^2}{2}Z_A^2 + ... + \frac{t^k}{k!}Z_A^k + ...$$

converging w.r.t. the norm $\| \cdot \|_{\diamond,E}^G$.

Proposition 3 shows that:

- the semigroup $\Xi_t^{G^\alpha}$ is differentiable w.r.t. the ECD norm indiced by a positive operator G **if and only if** $\alpha < 1/4$;

- if $\alpha < 1/4^n$ then relations (50) hold for the semigroup $\Xi_t^{G^\alpha}$.

By repeating the arguments after Proposition 2 one can show that the semigroup Ξ_t^A satisfies the condition of Proposition 3D if $A = \sqrt[4]{\ln G}$. So, in this case the exponential representation (51) is valid.

Proof of Proposition 3. A) This assertion directly follows from Proposition 1 and the above upper bound on $\|Z_A\|_{\diamond,E}^G$.

B) This assertion follows from Corollary 1 and the remark before the proposition.

C) Note first that $Z_A(\rho) = S_A^2(\rho)/2$ and hence $Z_A^n(\rho) = S_A^{2n}(\rho)/2^n$, where S_A is the superoperator defined in (42).

If A^{2n} is a $\sqrt{G}$-infinitesimal operator then, by using the arguments from the proof of Proposition 2C, it is easy to show that for any pure state ρ with finite energy $\mathrm{Tr}\rho G$ we have $Z_A^{k-1}(\rho) = 2^{1-k}S_A^{2(k-1)}(\rho) = Y_{k,+}^\rho - Y_{k,-}^\rho$, where $Y_{k,+}^\rho$ and $Y_{k,-}^\rho$ are operators in $\mathfrak{T}_+(\mathcal{H})$ such that $\mathrm{Tr}Y_{k,\pm}^\rho A^4 < +\infty$, $k = \overline{2, n}$. So, Lemma 12 implies that $Z_A^{k-1}(\rho) \in \mathcal{D}(Z_A)$, $k = \overline{2, n}$, for any such state ρ. By using inequality (16) with Lemma 5 and Lemma 6 we obtain

$$\|Z_A^k\|_{\diamond,E}^G = 2^{-k}\|S_A^{2k}\|_{\diamond,E}^G \leq 2^{-k} \sum_{i=0}^{2k} \binom{i}{2k} \|A^i\|_E^G \|A^{2k-i}\|_E^G \quad (52)$$

$$\leq 2^k \|A^{2k}\|_E^G = o(\sqrt{E}) \quad as \quad E \to +\infty,$$

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where the last equality follows from (13), since A^{2k} is a $\sqrt{G}$-infinitesimal operator for each $k = \overline{1,n}$ by Lemma 6. By Lemma 3 the superoperators $Z_A^k \otimes \mathrm{Id}_R$, $k = \overline{1,n}$, are continuous on the set $\widehat{\mathfrak{S}}_{G \otimes I_R, E}$ defined in (26). Thus, this assertion follows from Theorem 2.

D) This assertion follows from Corollary 2 and Lemma 6, since the condition $\|A^{2n}\|_E^G \leq C[n!]^p$ and the estimate (52) imply that $\|Z_A^n\|_{\diamond, E}^G / n!$ tends to zero as $n \to +\infty$ for any $E > 0$. $\square$

Appendix: Proof of Lemma 12

By making change of variable $u = x/\sqrt{t}$ in the integral (47), we obtain

$$t^{-1}[\Xi_t^A(\rho) - \rho] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^{-1} \left[e^{-i\sqrt{t}uA} \rho\, e^{i\sqrt{t}uA} - \rho \right] e^{-\frac{u^2}{2}} du. \tag{53}$$

Expanding the exponent, we have

$$e^{i\sqrt{t}uA} = I + i\sqrt{t}uA - \frac{1}{2}tu^2 A^2 + tu^2 A^2 F(\sqrt{t}uA), \tag{54}$$

where the function

$$F(u) = u^{-2}\left(e^{iu} - 1 - iu + \frac{1}{2}u^2 \right)$$

is uniformly bounded and $F(u) \to 0$ as $u \to 0$. It follows that operator-valued function $F(\sqrt{t}uA)$ is uniformly bounded in the operator norm and $F(\sqrt{t}uA) \to 0$ strongly as $t \to 0$ for any fixed u (cf. the proof of Proposition VI.3.1 in [12]) .

By inserting the expansion (54) into (53) we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^{-1} \left[(h.c.) \left(\sqrt{\rho} + i\sqrt{t}u\sqrt{\rho}A - \frac{1}{2}tu^2\sqrt{\rho}A^2 + tu^2\sqrt{\rho}A^2 F(\sqrt{t}uA) \right) - \rho \right] e^{-\frac{u^2}{2}} du,$$

where $(h.c.)$ is the Hermitian conjugate to the subsequent expression in the round bracket. Let $\sigma_1 = \sqrt{\rho}A$ and $\sigma_2 = \sqrt{\rho}A^2$ be Hilbert-Schmidt operators. Taking into account that the odd moments of the standard normal distribution vanish, second moment is 1 and fourth moment is 3, we obtain

$$Z_A(\rho) + \frac{3t}{4}\sigma_2^* \sigma_2 + \frac{t}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\overline{F}(\sqrt{t}uA)\sigma_2^* \sigma_2 F(\sqrt{t}uA) \right) u^4 e^{-\frac{u^2}{2}} du$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\overline{F}(\sqrt{t}uA)\sigma_2^* \right) \left(\sqrt{\rho} + i\sqrt{t}u\sigma_1 - \frac{1}{2}tu^2\sigma_2 \right) u^2 e^{-\frac{u^2}{2}} du + h.c.,$$

where now $h.c.$ denotes the Hermitian conjugate of the second integral. Taking into account the uniform boundedness of $\left\| F(\sqrt{t}uA) \right\|$, we conclude that the limit of this whole expression in the trace norm as $t \to 0$ is

$$Z_A(\rho) + \lim_{t \to 0} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \overline{F}(\sqrt{t}uA)\sigma_2^* \sqrt{\rho}\, u^2 e^{-\frac{u^2}{2}} du + h.c. \right].$$

But $\left\|\overline{F}(\sqrt{t}uA)\sigma_2^*\sqrt{\rho}\right\|_1 \leq \left\|\overline{F}(\sqrt{t}uA)\right\| \left\|\sigma_2^*\sqrt{\rho}\right\|_1$ is uniformly bounded and

$$\lim_{t\to 0}\left\|\overline{F}(\sqrt{t}uA)\sigma_2^*\sqrt{\rho}\right\|_1 = 0$$

for each u (this follows from the strong convergence $F(\sqrt{t}uA) \to 0$). So, by the dominated convergence theorem the limit of the integral in the trace norm is equal to zero. Thus

$$\lim_{t\to 0} t^{-1}[\Xi_t^A(\rho) - \rho] = Z_A(\rho)$$

in the trace norm for any state ρ satisfying the condition of Lemma 12.

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