SO(4)-SYMMETRY OF MECHANICAL SYSTEMS WITH 3 DEGREES OF FREEDOM

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Abstract. We answered the old question: does there exist a mechanical system with 3 degrees of freedom, except for the Coulomb system, which has 6 first integrals generating the Lie group $SO(4)$ of canonical transformations in some domain of phase space? We presented a system which is not centrally symmetric, but has $SO(4)$ symmetry of canonical transformations. We showed also that not every mechanical system with 3 degrees of freedom possesses such symmetry.

1. Introduction

It is well-known (see, e.g., [LLm]) that in the Coulomb field, i.e., in the $3d$ mechanical system with the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \text{ where } \mathbf{p}^2 := \sum_{i=1,2,3} p_i^2, \ r := \sqrt{\sum_{i=1,2,3} q_i^2}, \tag{1}$$

the symmetry group of canonical transformations has a subgroup isomorphic to SO(4) acting in the domain $H < 0$. This fact, found by V. Fock [Fock], helps to explain the structure of the spectrum of the hydrogen atom.

An important property of this $SO(4)$ is that the Casimirs of its Lie algebra $\mathfrak{o}(4)$ restore the Hamiltonian.

The Hamiltonian in Eq. (1) describes, for example, the motion of two particles interacting via gravity, and the motion of light charged particle in the field of a heavier nucleus. So, the number of works investigating this Hamiltonian is not much fewer than infinity. It is therefore astonishing that we could not find in the literature the definite answer to a natural question: "does there exist a mechanical system with 3 degrees of freedom, except for the Coulomb system, which has 6 first integrals generating the Lie algebra $\mathfrak{o}(4)$?" posed, e.g., in [Mu], [SzW].

Two different answers to the question were given fifty years ago:

1) Mukunda [Mu] claimed that every mechanical system with n degrees of freedom has a subgroup of canonical transformations locally isomorphic to $O(n+1)$.

2) Szymacha and Werle [SzW] claimed that there are no other mechanical system with the same property, except for the oscillator, if $\mathfrak{o}(4)$ contains the Lie algebra of spatial rotations of $\mathbb{R}^3$.

Here we answer a more precise question: "does there exist a mechanical system with 3 degrees of freedom, except for the Kepler system, each of these systems having 6 first integrals F_α generating the Lie algebra $\mathfrak{o}(4)$ in some domain of the phase space invariant with respect to both the Hamilton equations of motion of the system and all Hamiltonian flows generated by these first integrals F_α?". For details, see Section 2.

To prove that not any system with 3 degrees of freedom has $SO(4)$ symmetry, in Section 4 we offer a simple necessary condition for existence of $\mathfrak{o}(4)$ symmetry, and in Section 5 we give an example for which this condition is violated.

In Section 6 we consider the Hamiltonian of a charged particle in homogeneous electric field. For this Hamiltonian, there exists a family of sextuples of first integrals such that by means of the Poisson bracket every sextuple generates the Lie algebra $\mathfrak{o}(4)$. We proved that the corresponding Lie group $SO(4)$ of canonical transformations is well-defined and preserves the domain of definition of these first integrals.

The Casimirs of certain of thus constructed Lie algebras $\mathfrak{o}(4)$ do not allow, however, to recover the Hamiltonian.

2. Generalities

Recall the definition of the symmetry group of canonical transformations and its Lie algebra.

Let $H(q_i,\, p_i)$, where $i = 1, 2, 3$, be a Hamiltonian of some mechanical system[1].

Let the first integral F of this system be a real function on some domain $U_F \subset \mathbb{R}^6$. Let $(q, p) \in U_F$; the case $F = H$ is not excluded. Then F generates a 1-dimensional Lie group $\mathcal{L}_F$ of canonical transformations $(q, p) \mapsto (q^F(\tau|q, p), p^F(\tau|q, p))$ leaving the Hamiltonian H and the domain U_F invariant if

$$(q^F(\tau|q, p), p^F(\tau|q, p)) \in U_F \text{ for any } \tau \in \mathbb{R}.$$

The transformations are defined by the relations

$$\frac{dq_i^F}{d\tau} = \{q_i^F,\, F\} = \frac{\partial F(q^F,\, p^F)}{\partial p_i^F}, \tag{2}$$

$$\frac{dp_i^F}{d\tau} = \{p_i^F,\, F\} = -\frac{\partial F(q^F,\, p^F)}{\partial q_i^F}, \tag{3}$$

$$q_i^F(0|q, p) = q_i, \quad p_i^F(0|q, p) = p_i, \tag{4}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket[2] in $\mathbb{R}^6$:

$$\{F, G\} := \sum_{i=1,2,3} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right) = \sum_{\alpha,\beta=1}^{6} \frac{\partial F}{\partial z_\alpha}\omega_{\alpha\beta}\frac{\partial G}{\partial z_\beta}. \tag{5}$$

Here the symplectic form ω is of the shape $\omega = \begin{pmatrix} 0_3 & 1_3 \\ -1_3 & 0_3 \end{pmatrix}$, where 1_3 and 0_3 are 3×3 matrices.

We call the transformations Eq. (2) – (4) the *Hamiltonian flow*, generated by the Hamiltonian F.

If a certain set of first integrals $\mathcal{F} = \{F_\alpha\}$ has the same domain U invariant under the action of all Hamiltonian flows $\mathcal{L}_{F_\alpha}$, then these flows generate the Lie group whose Lie algebra is the space of first integrals generated by the set $\mathcal{F}$ by means of (5).

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[1]Sometimes we will denote the whole set of the q_i and p_i for $i = 1, 2, 3$ by z_α, where α =1, …, 6.

[2]The definition (5) has the opposite sign as compared with the one given in [LLm], but coincides with the definition of the Poisson bracket given in [App], [T-H], [G], [Arn].

3. The case of the Coulomb field

Here we briefly consider the mechanical system with the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \text{ where } \mathbf{p}^2 := \sum_{i=1,2,3} p_i^2,\ r := \sqrt{\sum_{i=1,2,3} q_i^2}\,. \tag{6}$$

This Hamiltonian has two well-known triples of first integrals: one consists of the coordinates L_i of the angular momentum vector, the other one consists of the coordinates of the Runge-Lenz vector R_i, where $i = 1,2,3$, defined in the domain

$$U = \{z \in \mathbb{R}^6 \mid H(z) < 0\} \text{ (or in any of the domains } E_{min} < H < E_{max} < 0))$$

as

$$L_i = \sum_{j,k=1,2,3} \varepsilon_{ijk} q_j p_k, \tag{7}$$

$$R_i = (-2H)^{-1/2}\left(\sum_{j,k=1,2,3} \varepsilon_{ijk} L_j p_k + \frac{q_i}{r}\right), \tag{8}$$

where ε_{ijk} is an anti-symmetric tensor such that $\varepsilon_{123} = 1$.

These first integrals satisfy the following commutation relations:

$$\begin{aligned} \{H, L_i\} = 0,\\ \{H, R_i\} = 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \{L_i, L_j\} &= \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} L_k,\\ \{R_i, R_j\} &= \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} L_k,\\ \{L_i, R_j\} &= \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} R_k. \end{aligned} \tag{10}$$

Due to relations (9) and by definition of the domain U, the later is invariant under the action of Hamiltonian flows generated by the first integrals L_i and R_i.

The relations (10) shows that these first integrals generate the Lie algebra $\mathfrak{o}(4)$.

Since $\mathfrak{o}(4) \simeq \mathfrak{o}(3) \oplus \mathfrak{o}(3)$, we can introduce two commuting triples of first integrals

$$\begin{aligned} G_i &:= \tfrac{1}{2}(L_i + R_i), \quad \text{where } i = 1,2,3,\\ G_{3+i} &:= \tfrac{1}{2}(L_i - R_i), \quad \text{where } i = 1,2,3, \end{aligned} \tag{11}$$

satisfying the commutation relations

$$\begin{aligned} &\{G_i,\, G_j\} = \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} G_k, \quad \text{where } i,j = 1,2,3,\\ &\{G_{3+i},\, G_{3+j}\} = \textstyle\sum_{k=1,2,3} \varepsilon_{ijk} G_{3+k}, \quad \text{where } i,j = 1,2,3,\\ &\{G_i,\, G_{3+j}\} = 0, \quad \text{where } i,j = 1,2,3. \end{aligned} \tag{12}$$

4. Restrictions on the rank

Let some $3d$ mechanical system have the Hamiltonian H and 6 first integrals G_α satisfying the commutation relations Eq. (12).

Consider two 6×6 matrices: the Jacoby matrix J with elements

$$J_\alpha^\beta := \frac{\partial G_\alpha}{\partial z_\beta} \tag{13}$$

and the matrix P with elements

$$P_{\alpha\beta} := \{G_\alpha,\, G_\beta\}. \tag{14}$$

Then definitions (13) and (5) imply that

$$P_{\alpha\beta} = \sum_{\gamma,\delta=1}^{6} J_\alpha^\gamma \omega_{\gamma\delta} J_\beta^\delta\,. \tag{15}$$

Suppose that $G_1^2 + G_2^2 + G_3^2 \neq 0$ and $G_4^2 + G_5^2 + G_6^2 \neq 0$. Then the matrix P has two independent null-vectors

$$(G_1,\, G_2,\, G_3,\, 0,\, 0,\, 0) \quad\text{and}\quad (0,\, 0,\, 0,\, G_4,\, G_5,\, G_6) \tag{16}$$

due to relations (12) and so $\operatorname{rank}(P) = 4$.

Since symplectic form ω is nondegenerate, the relation Eq. (15) and degeneracy of the matrix P imply that

$$\operatorname{rank}(P) \leq \operatorname{rank}(J) < 6. \tag{17}$$

So $\operatorname{rank}(J)$ is equal to either 4, or 5. Both these cases can be realized: $\operatorname{rank}(J) = 5$ for the Coulomb system and $\operatorname{rank}(J) = 4$ for the system described in Section 6.

5. Not all $3d$ systems have $\mathfrak{o}(4)$ symmetry

To give an example of a $3d$ mechanical system without $\mathfrak{o}(4)$ symmetry, consider the Hamiltonian

$$H = H_1 + H_2 + H_3, \quad\text{where}\quad H_i = \frac{1}{2}p_i^2 + \frac{\omega_i^2}{2}q_i^2 \tag{18}$$

and where the ω_i for $i = 1, 2, 3$ are incommensurable.

Evidently, each of the functions H_i is a first integral.

Let us show that each first integral of this system is a function of the H_i, where $i = 1, 2, 3$. Indeed, let F be a first integral. So, F is constant on every trajectory defined for the system under consideration by relations

$$q_i = r_i \sin(\omega_i t + \varphi_i), \qquad p_i = r_i \omega_i \cos(\omega_i t + \varphi_i) \quad\text{for } i = 1, 2, 3, \tag{19}$$

where the r_i and φ_i are constants specifying the trajectory. Since every trajectory given by Eq. (19) is everywhere dense on the torus

$$T(r_1, r_2, r_3) := \left\{ z \in \mathbb{R}^6 \mid \omega_i^{-2} p_i^2 + q_i^2 = r_i^2 \quad\text{for } i = 1, 2, 3 \right\}, \tag{20}$$

it follows that F is constant on every torus $T(r_1, r_2, r_3)$, and hence $F = F(H_1, H_2, H_3)$.

Note, that every invariant domain U_H can be represented in the form

$$U_H = \cup_{(r_1,r_2,r_3)\in\mathcal{M}} T(r_1, r_2, r_3), \tag{21}$$

where $\mathcal{M}$ is some set in $\mathbb{R}^3$.

Now suppose that the system has 6 first integrals G_α satisfying commutation relations (12) of the Lie algebra $\mathfrak{o}(4)$. Then, since $G_\alpha = G_\alpha(H_1, H_2, H_3)$, it follows that the Jacoby matrix J (see Eq. (13)) is of rank ≤ 3, and so due to Eq. (15) the matrix P (see Eq. (14)) is of rank ≤ 3. But this fact contradicts the easy to verify fact that if $G_1^2 + G_2^2 + G_3^2 \neq 0$ and $G_4^2 + G_5^2 + G_6^2 \neq 0$, then $\operatorname{rank}(P) = 4$.

So, the system under consideration has no $\mathfrak{o}(4)$ symmetry in any domain invariant with respect to Hamiltonian flow generated by H.

6. An example of non-Coulomb $3d$ mechanical system with $\mathfrak{o}(4)$ symmetry

We consider a system with 3 degrees of freedom with potential $-q_3$ and show that in the domain

$$U := \{z \in \mathbb{R}^6 \mid p_1^2 < a_1^2,\ p_2^2 < a_2^2\} \tag{22}$$

where a_s are any smooth function of Hamiltonian H, see Eq. (23), there exists a system of 6 first integrals generating the Lie algebra $\mathfrak{o}(4)$ and this U is invariant under the action of all six Hamiltonian flows. Namely, consider a particle in an homogeneous field with Hamiltonian

$$H = \frac{\mathbf{p}^2}{2} - q_3. \tag{23}$$

Then the real functions

$$\begin{aligned} G_1 &= p_1, \\ G_2 &= \sqrt{a_1^2 - p_1^2}\, \cos(q_1 - p_1 p_3), \\ G_3 &= \sqrt{a_1^2 - p_1^2}\, \sin(q_1 - p_1 p_3), \\ G_4 &= p_2, \\ G_5 &= \sqrt{a_2^2 - p_2^2}\, \cos(q_2 - p_2 p_3), \\ G_6 &= \sqrt{a_2^2 - p_2^2}\, \sin(q_2 - p_2 p_3), \end{aligned} \tag{24}$$

are the first integrals and form a basis of the Lie algebra of the group of canonical transformations of the phase space acting in the domain U.

It is subject to a direct verification that the integrals (24) satisfy the relations (12).

The Casimirs, defined as

$$K_1 = \sum_{i=1,2,3} G_i^2, \qquad K_2 = \sum_{i=1,2,3} G_{3+i}^2$$

are equal to

$$K_1 = a_1^2 \qquad K_2 = a_2^2$$

and don't define the Hamiltonian only in the case where a_s are constant. In this case of constant a_s, the Jacoby matrix (13) for the functions (24) has rank 4 at the generic point. Otherwise, $\operatorname{rank}(J) = 5$ at the generic point.

The next thing we have to prove is the invariance of the domain (22) under the flows generated by the first integrals (24).

6.1. Invariance of the domain U under the flow $\mathcal{L}_{G_1}$.

The Hamiltonian flow generated by the first integral G_1 is defined by the equations

$$\begin{aligned} &\tfrac{d}{d\tau} z_\alpha = \{z_\alpha, p_1\}, \quad \textit{i.e.}, \\ &\tfrac{d}{d\tau} p_i = 0, \\ &\tfrac{d}{d\tau} q_1 = 1, \quad \tfrac{d}{d\tau} q_i = 0 \ \text{ for } \ i = 2, 3, \end{aligned} \tag{25}$$

which implies

$$p_i(\tau) = p_i(0), \tag{26}$$

so, since H is constant on each trajectory, it follows that $(p_1(\tau))^2 < a_1^2$ if $|p_1(0)| < a_1$, i.e., the domain U is invariant under the flow $\mathcal{L}_{G_1}$.

6.2. **Invariance of the domain U under the flow $\mathcal{L}_G$, where $G := \lambda_2 G_2 + \lambda_3 G_3$.** For λ_2 and λ_3 real, $G := \lambda_2 G_2 + \lambda_3 G_3$ is of the shape

$$G = \lambda Q \cos(q_1 - p_1 p_3 + \varphi), \tag{27}$$

where $Q := \sqrt{a_1^2 - p_1^2}$, and real parameters λ and φ depend on λ_2 and λ_3.

Consider the quantity

$$Q_H := \frac{dQ}{dH} = \frac{a_1}{Q}\frac{da_1}{dH}$$

such that

$$\{z_\alpha,\, Q\} = Q_H\{z_\alpha,\, H\} - \frac{p_1}{Q}\{z_\alpha,\, p_1\},$$

$$\{z_\alpha,\, H\} = \sum_i \{z_\alpha,\, p_i\}p_i - \{z_\alpha,\, q_3\}.$$

The equations of the Hamiltonian flow $\mathcal{L}_G$ are of the form

$$\begin{array}{rcl} \frac{d}{d\tau} z_\alpha &=& \{z_\alpha, G\}, \quad i.e., \\ \frac{d}{d\tau} p_3 &=& \lambda Q_H \cos u \\ \frac{d}{d\tau} q_3 &=& \lambda Q p_1 \sin(u) + \lambda Q_H p_3 \cos u \\ \frac{d}{d\tau} p_2 &=& 0, \qquad \frac{d}{d\tau} q_2 = \lambda Q_H p_2 \cos u \\ \frac{d}{d\tau} p_1 &=& \lambda Q \sin(u), \\ \frac{d}{d\tau} q_1 &=& \lambda Q p_3 \sin(u) - \frac{\lambda p_1}{Q}\cos(u) + \lambda Q_H p_1 \cos u. \end{array} \tag{28}$$

where we introduce a new variable u instead of q_1:

$$u := q_1 - p_1 p_3 + \varphi. \tag{29}$$

Let us prove that if $p_1^2(0) < a_1^2$, then $p_1^2(\tau) < a_1^2$ for any $\tau \in \mathbb{R}$.

We have

$$\frac{d}{d\lambda\tau} u = -\frac{p_1}{Q}\cos(u), \tag{30}$$

$$\frac{d}{d\lambda\tau} p_1 = Q \sin(u), \tag{31}$$

$$\frac{d}{d\lambda\tau} Q = -p_1 \sin(u), \tag{32}$$

$$\frac{d}{d\lambda\tau} H = 0, \tag{33}$$

and hence

$$\frac{d^2}{d(\lambda\tau)^2} p_1 = -p_1. \tag{34}$$

So

$$p_1 = p_1^{max} \sin(\lambda\tau + \psi), \tag{35}$$

where $p_1^{max} \geq 0$ and ψ are some constants.

Further, we have

$$\begin{aligned} (p_1^{max})^2 &= p_1^2 + \left(\frac{d}{d\lambda\tau} p_1\right)^2 = p_1^2 + (a_1^2 - p_1^2)\sin^2(u) \\ &= a_1^2 \sin^2(u) + p_1^2 \cos^2(u) = a_1^2 - (a_1^2 - p_1^2)\cos^2(u) \end{aligned}$$

and

$$(p_1^{max})^2 = a_1^2 - (a_1^2 - p_1^2(\tau))\cos^2(u(\tau)) \text{ for an arbitrary } \tau. \tag{36}$$

Since $p_1^2(0) < a_1^2$, there exists $\varepsilon > 0$ such that $p_1^2(\tau) < a_1^2$ for any τ in the interval $(-\varepsilon, \varepsilon)$. From equations Eq. (28) it is easy to see that $u \neq const$ in the interval $(-\varepsilon, \varepsilon)$. So, there exists a $\tau_0 \in (-\varepsilon, \varepsilon)$ such that $p_1^2(\tau_0) < a_1^2$ and $\cos(u(\tau_0)) \neq 0$. Hence,

$$(p_1^{max})^2 = a_1^2 - (a_1^2 - p_1^2(\tau_0))\cos^2(u(\tau_0)) < a_1^2 \tag{37}$$

which implies $p_1^2(\tau) < a_1^2$ for any τ.

The invariance of the domain U under the Hamiltonian flows generated by G_α, where $\alpha = 4, 5, 6$ is proved analogously.

Acknowledgments

S.B. is grateful for support to the grant NYUAD-065, S.K. is grateful to Russian Fund for Basic Research (grant No. 17-02-00317) for partial support of this work. S.K. is grateful also to I.V.Tyutin and A.E.Shabad for useful discussions.

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