A robust hierarchical nominal classification method based on similarity and dissimilarity

Ana Sara Costa[a,b,*], Salvatore Corrente[c], Salvatore Greco[c,d], José Rui Figueira[a], José Borbinha[b]

[a] *CEG-IST, Instituto Superior Técnico, Universidade de Lisboa, Portugal*
[b] *INESC-ID, Instituto Superior Técnico, Universidade de Lisboa, Portugal*
[c] *Department of Economics and Business, University of Catania, Italy*
[d] *CORL, Portsmouth Business School, University of Portsmouth, United Kingdom*

Abstract

Cat-SD (CATegorization by Similarity-Dissimilarity) is a multiple criteria decision aiding method for dealing with nominal classification problems (predefined and non-ordered categories). Actions are assessed according to multiple criteria and assigned to one or more categories. A set of reference actions is used for defining each category. The assignment of an action to a given category depends on the comparison of the action to the reference set according to a likeness threshold. Distinct sets of criteria weights, interaction coefficients, and likeness thresholds can be defined per category. When applying Cat-SD to complex decision problems, considering a hierarchy of criteria may help to decompose them. We propose to apply Multiple Criteria Hierarchy Process (MCHP) to Cat-SD. An adapted MCHP is proposed to take into account possible interaction effects between criteria structured in a hierarchical way. We also consider an imprecise elicitation of parameters. With the purpose of exploring the assignments obtained by Cat-SD considering possible sets of parameters, we propose to apply the Stochastic Multicriteria Acceptability Analysis (SMAA). The SMAA methodology allows to draw statistical conclusions on the classification of the actions. The proposed method, SMAA-hCat-SD, helps the decision maker to check the effects of the variation of parameters on the classification at different levels of the hierarchy. We propose also a procedure to obtain a final classification fulfilling some requirements by taking into account the hierarchy of criteria and the probabilistic assignments obtained applying SMAA. The application of the proposed method is showed through an example.

Keywords: Multiple criteria decision aiding, Hierarchy of criteria, Interaction effects, Robust optimization, Stochastic models.

1. Introduction

In several decision situations, we face a classification problem involving the assessment of a set of actions (or alternatives), according to multiple criteria (usually conflicting), and their assign-

*Corresponding author. CEG-IST, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001, Lisboa, Portugal. Tel.: +351 218 417 729. Date: November 30, 2018.

Email addresses: anasaracosta@tecnico.ulisboa.pt (Ana Sara Costa), salvatore.corrente@unict.it (Salvatore Corrente), salgreco@unict.it (Salvatore Greco), figueira@tecnico.ulisboa.pt (José Rui Figueira), jlb@tecnico.ulisboa.pt (José Borbinha)

ment to categories defined in a nominal way. In fact, the wide range of potential real-world applications in various areas (e.g., human resources management, finance, medicine, etc.) has motivated researchers to develop MCDA methods for dealing with multiple criteria nominal classification problems. In this kind of classification problems, categories are predefined and no order exists among them (nominal categories). In opposition, in sorting problems (or ordinal classification problems), there is a preference order among the categories. Recent proposals for handling classification problems are mainly based on operations research and artificial intelligence techniques (Doumpos and Zopounidis, 2002; Zopounidis and Doumpos, 2002). In this paper, we focus on the multiple criteria decision aiding (MCDA) approach. In the literature, we can find proposals for nominal classification mainly using outranking-based procedures (Belacel, 2000; Henriet, 2000; Léger and Martel, 2002; Perny, 1998; Rigopoulos et al., 2010), rough set theory (Słowiński and Vanderpooten, 2000), and verbal decision analysis (Furems, 2013).

The majority of existing MCDA nominal classification methods are based on outranking relations (see, for example, Belacel 2000; Perny 1998). While for choice, ranking and sorting problems outranking binary relations are acceptable, for nominal classification problems, they may be questionable. One may argue that in nominal classification the aim of the pairwise comparison should be to know if the two actions are similar and not if one action is preferred to the other one. None of the current methods proposed a way to model preference information related to similarity concepts when comparing actions, neither to deal with criteria hierarchy and interactions between criteria. In addition, robustness concerns have not been considered, and it has been pointed out as an important issue in nominal classification (Zopounidis and Doumpos, 2002). The Cat-SD (CATegorization by Similarity-Dissimilarity) method has been recently proposed as a new MCDA method, covering some of these issues (Costa et al., 2018). This method allows to assign actions to nominal categories, based on similarity and dissimilarity between actions, using reference actions to define the categories. Multiple criteria and possible interactions in some pairs of criteria are considered. In Cat-SD, for each category, a particular set of preference parameters can be chosen (e.g., criteria weights and interaction coefficients), which means that distinct parameter sets can be defined among categories. Thus, Cat-SD has been designed to model subjective judgments of the Decision Maker (DM) in pairwise comparison of actions in terms of similarity and dissimilarity between the two actions. Then, likeness binary relations are constructed taking into account the preferences of the DM. Moreover, to the best of our knowledge, Cat-SD is the first MCDA nominal classification method that permits to model interactions between criteria. As stated in Costa et al. (2018), there are still aspects that need further research related to Cat-SD, namely considering a hierarchy structure of criteria and robustness analysis, while different vectors of parameter sets are taken into consideration.

In several decision aiding scenarios, complex multiple criteria decision problems arise involving a great number of criteria for assessing actions (Belton and Stewart, 2002; Greco et al., 2016; Ishizaka and Nemery, 2013). The heterogeneity and the high number of criteria are the main reasons for the complexity of the decision problems. Structuring the criteria in a hierarchical way can be a useful approach for dealing with such decision problems. Multiple Criteria Hierarchy Process (MCHP) has been proposed to handle the decision problems in which the considered criteria are hierarchically structured (Corrente et al., 2012, 2013, 2016, 2017a). MCHP imposes a hierarchical structure of criteria, which means that all criteria are not considered at the same level, and criteria are grouped into subsets according to distinct points of view. In this way, the elicitation of preferences of the DM related to the criteria can be easier than considering a great number of heterogeneous criteria at the same level. Indeed, MCHP has been applied, for example, to a ranking method, Electre III (Corrente et al., 2017b), and to sorting methods, such as Electre Tri-B, Tri-C and Tri-nC (Corrente et al., 2016), and Promethee methods (Corrente et al., 2013). MCHP has been applied also to the aggregation of interacting criteria by means of the Choquet integral in Angilella et al. (2016). To the best of our knowledge, there is no research work adopting such an approach to multiple criteria nominal classification methods. In this paper, we propose to apply MCHP to the Cat-SD method.

We introduce an adapted MCHP to handle the three types of interaction between pairs of criteria considered in Cat-SD: mutual-strengthening effect, mutual-weakening effect, and antagonistic effect. In particular, we say that criteria two criteria present a mutual-strengthening effect, if the importance assigned to these two criteria together is greater than the sum of the importance assigned to them singularly. Two criteria present a mutual-weakening effect, if the importance assigned to them together is lower than the sum of their importance when considered alone. If a given criterion is in favor of the similarity of two actions, and another one is in favor of the dissimilarity of these actions, we say that the latter criterion exercises an antagonistic effect over the first, if the importance of the first criterion is reduced by the presence of the other (for more details on the meaning of these effects in case of outranking relations, see Figueira et al., 2009). Moreover, an imprecise elicitation of criteria weights is considered. For that, we adopt an extension of the Simos-Roy-Figueira (SRF) (Figueira and Roy, 2002) by considering imprecise preference information provided by the DM to assign values to the criteria weights (Corrente et al., 2017b).

The assignment results provided by the Cat-SD method can include multiple assignments of an action, i.e., a given action can be assigned to several categories. It is interesting to know the accuracy of the assignment of each action, considering then the robustness of the recommendations with respect to the assignment results. In this sense, to take into account all sets of weights and interaction coefficients compatible with the information provided by the DM, we propose to apply the Stochastic Multicriteria Acceptability Analysis (SMAA) (Lahdelma et al., 1998; Lahdelma and Salminen, 2010) to draw conclusions with respect to the assignments of each action (Corrente et al., 2017b). Indeed, SMAA has been applied to several multiple criteria methods (see Tervonen and Figueira, 2008, for a survey on the use of the SMAA methodology). Application of SMAA allows to obtain, for each action, the probability of its assignment to each category (or a set of categories), not only when considering the whole set of criteria, but also when considering a particular node in the hierarchy structure. Of course, this can be a relevant information for the DM. Our aim is to present a comprehensive method for dealing with these interrelated issues by adopting an integrated approach. Thus, we can take advantage of the main characteristics of the methods that we propose to integrate:

- Hierarchy of criteria: the use of the MCHP is beneficial for the user from two perspectives. On one hand, the DM can provide information not only at comprehensive level but considering a particular aspect of the problem. Indeed, it can be a bit upset in comparing two actions considering all criteria simultaneously but the DM can feel more confident in expressing the preferences taking into account one or some aspects he knows more. On the other hand, the DM can get information not only at global level but at partial one, and this is an added value since the DM can discover the weak and strong points of the actions at hand;

- The imprecise SRF method: asking to the DM to provide exact values for all the parameters involved in the model is meaningless even for one expert in MCDA. In general, the DM is more confident in exercising the preferences than in justifying them. For this reason, we use the imprecise SRF method to obtain the criteria weights by asking to the DM to provide preference information in an imprecise way;

- SMAA: the motivations on the basis of the use of SMAA are strictly connected to the previous point. Indeed, in general, more than one set of values of parameters can be compatible with the information provided by the DM and choosing only one or some of them to get the final recommendations on the problem at hand can be considered arbitrary to some extent. A recommendation built taking into account the plurality of preference parameters compatible with the information provided by the DM is more robust and, consequently, more trustworthy;

- Robustness concerns: The DM is interested in a final recommendation that takes into account the robustness concerns represented by the results of SMAA. Therefore, we propose a procedure that, starting from the probability to be classified to different categories supplied by SMAA, provides a comprehensive classification fulfilling some possible requirements given by the DM.

The main objectives of this paper are:

1. To apply MCHP to the nominal classification method CAT-SD;
2. To use the imprecise SRF method for each category taking into account the hierarchy of criteria and the possible interactions between criteria;
3. To apply SMAA to the hierarchical CAT-SD method by sampling several sets of parameters compatible with some preferences provided by the DM;
4. To propose a procedure that starting from the probabilistic assignments obtained by SMAA provides a final classification that fulfills some requirements given by the DM.

This paper is organized as follows. Section 2 introduces the CAT-SD method. Section 3 is related to our proposal of applying MCHP to the CAT-SD method, in order to construct the hierarchical CAT-SD method, hCAT-SD. Section 4 presents a way for dealing with imprecise information to determine the criteria weights when considering the hCAT-SD method. Section 5 is devoted to the application of SMAA to the hCAT-SD method, building the comprehensive method SMAA-hCAT-SD. Section 6 proposes a procedure to obtain the final nominal classification results according to some requirements indicated by the DM. Section 7 provides a numerical example of application of the SMAA-hCAT-SD method. Section 8 presents some concluding remarks and future lines of research.

2. The CAT-SD method

In this section, we briefly introduce the CAT-SD method (for more details, see Costa et al. 2018). This method deals with decision problems where categories are defined in a nominal way (they are not ordered). Each category is defined *a priori* and characterized by a set of reference actions. Each action is assessed on several criteria, and assigned to a category or a set of categories. The assignment of actions is based on the concepts of similarity and dissimilarity between two actions. The main notation, concepts and definitions are presented.

2.1. Main notation

In the Cat-SD method, the following notation is used:

- $\{a, ..., a_i...\}$ is the set of actions (or alternatives) not necessarily known *a priori;*
- $\{g_1, ..., g_j, ..., g_n\}$ is the set of all criteria;
- $\{C_1, ..., C_h, ..., C_q, C_{q+1}\}$ is the set of nominal categories, where C_{q+1} is a dummy one considered to receive actions not assigned to the other categories;
- $\{B_1, ..., B_h, ..., B_{q+1}\}$ is the set of all reference actions, where $B_{q+1} = \emptyset$;
- $\{b_{h1}, \ldots, b_{h\ell}, ..., b_{h|B_h|}\}$ is the set of (representative) reference actions chosen to define category C_h, for $h = 1, ..., q$;
- k_j^h is the weight of criterion g_j for category C_h, for $j = 1, ..., n$ and $h = 1, ..., q$;
- $k_{j\ell}^h$ is a mutual-strengthening (or mutual-weakening) coefficient of the pair criteria $\{g_j, g_\ell\}$, with $k_{j\ell}^h > 0$ (or $k_{j\ell}^h < 0$), for $h = 1, ..., q$;
- $k_{j|p}^h$ is an antagonistic coefficient for the ordered pair of criteria (g_j, g_p), with $k_{j|p}^h < 0$, for $h = 1, ..., q$;
- $k(C_h)$ is the set of all criteria weights and interaction coefficients of category C_h, for $h = 1, ..., q$;
- λ^h is a likeness threshold of category C_h, for $h = 1, ..., q$.

2.2. Modeling similarity-dissimilarity

The Cat-SD method was designed to help the DM to assign actions to a single nominal category or to a set of categories. The method is based on the concepts of similarity and dissimilarity. Cat-SD is more focused on similarity between actions than on their dissimilarity, since likeness between actions is usually what counts most when categorizing actions. According to a given criterion, when an action a (the subject) is compared to an action b (the referent or the reference action), similarity-dissimilarity between them can be assessed. Indeed, the preferences of the DM with respect to the similarity-dissimilarity between the two actions on a criterion can be modeled through a function.

In what follows, let E_j denote the scale of criterion g_j, $j = 1, ..., n$ (generally bounded from below by $g_j^{\min}$ and from above by $g_j^{\max}$). Consider the difference of performances of actions a and b, $\Delta_j(a, b) = diff\,\{g_j(a), g_j(b)\}$. Let E_{Δ_j} denote the scale of such a difference. For ratio and interval scales, $diff\,\{g_j(a), g_j(b)\} = g_j(a) - g_j(b)$, and for ordinal scales, $diff\,\{g_j(a), g_j(b)\}$ corresponds to the number of performance levels between $g_j(a)$ and $g_j(b)$. Without loss of generality, we assume that criteria are to be maximized. A *per-criterion similarity-dissimilarity function* is a real-valued function $f_j\ :\ E_{\Delta_j} \to [-1, 1]$ such that:

1. f_j is a non-decreasing function of $\Delta_j(a, b)$, if $\Delta_j(a, b) \in [-diff\{g_j^{\max}, g_j^{\min}\}, 0]$;
2. f_j is a non-increasing function of $\Delta_j(a, b)$, if $\Delta_j(a, b) \in [0, diff\{g_j^{\max}, g_j^{\min}\}]$;
3. $f_j > 0$ iff criterion g_j contributes to similarity;
4. $f_j < 0$ iff criterion g_j contributes to dissimilarity.

This function defines:

- A *per-criterion similarity function* $s_j(a,b) = f_j\big(\Delta_j(a,b)\big)$, if $f_j\big(\Delta_j(a,b)\big) > 0$, and $s_j(a,b) = 0$, otherwise;

- A *per-criterion dissimilarity function* $d_j(a,b) = f_j\big(\Delta_j(a,b)\big)$, if $f_j\big(\Delta_j(a,b)\big) < 0$, and $d_j(a,b) = 0$, otherwise.

The CAT-SD method was designed to take into account interaction effects between pairs of criteria when computing likeness between two actions. In general, in real-world problems, we can find the following three types of interactions between criteria can be considered (Figueira et al., 2009):

1. *Mutual-strengthening effect* between the pair of criteria $\{g_j, g_\ell\}$. This synergy effect between the two criteria, when both criteria are in favor of similarity between actions a and b, can be modeled through a positive coefficient $k^h_{j\ell}$ ($k^h_{j\ell} = k^h_{\ell j}$), which is added to the sum of the weights $k^h_j + k^h_\ell$;
2. *Mutual-weakening effect* between the pair of criteria $\{g_j, g_\ell\}$. This redundancy effect between the two criteria, when both criteria are in favor of similarity between actions a and b, can be modeled through a negative coefficient $k^h_{j\ell}$ ($k^h_{j\ell} = k^h_{\ell j}$), which is added to the sum of the weights $k^h_j + k^h_\ell$;
3. *Antagonistic effect* between the ordered pair of criteria (g_j, g_p). This antagonistic effect between the two criteria, when criterion g_j is in favor of similarity and criterion g_p is in favor of dissimilarity between actions a and b, can be modeled through a negative coefficient $k^h_{j|p}$, which is added to the weight k^h_j (in general, $k^h_{j|p}$ is not equal to $k^h_{p|j}$ or, even more, one of the two antagonistic effects there could not be).

It should be remarked that distinct sets of weights and interaction coefficients, $k(C_h)$, can be defined among categories, $h = 1, ..., q$. For example, let us consider a problem in which some cars have to be assigned to categories "family car" and "sport car", and that criteria cost, safety, maximum speed and acceleration have to be taken into account. One can imagine that, on one hand, cost and safety are the most important criteria when assigning a car to the "family car" category, while, on the other hand, maximum speed and acceleration become the most important criteria in assigning a car to the "sport car" category.

To guarantee that the contribution of each criterion to the comprehensive similarity is not negative when considering the interaction effects, the following condition has to be fulfilled (Figueira et al., 2009):

$$k^h_j \quad - \sum_{\big\{\{j,\ell\}\in M^h \,:\, k^h_{j\ell}<0\big\}} |k^h_{j\ell}| \;-\; \sum_{(j,p)\in O^h} |k^h_{j|p}| \;\geqslant\; 0, \text{ for all } j \in G; h = 1, ..., q,$$

where

- M^h is the set of all pairs of criteria $\{j, \ell\}$ such that $f_j\big(\Delta_j(a,b)\big) > 0$, $f_\ell\big(\Delta_\ell(a,b)\big) > 0$, and there is mutual-interaction effect between them, for category C_h, $h = 1, ..., q$;

- O^h is the set of all ordered pairs of criteria (j, p) such that $f_j\big(\Delta_j(a,b)\big) > 0$, $f_p\big(\Delta_p(a,b)\big) < 0$, and g_j exercises an antagonistic effect on g_p, for category C_h, $h = 1, ..., q$.

Considering a similarity-dissimilarity function for each criterion, the set of criteria weights, and the interaction coefficients defined for each category C_h, $h = 1, ..., q$, a comprehensive similarity aggregation function can be defined. Such a function measures the strength of the arguments in favor of likeness of action a with respect to action b. A *comprehensive similarity function* is a real-valued function $f^s : [0,1]^n \times [-1,0]^n \to [0,1]$ defined as follows:

$$s^h(a,b) = f^s\big(s_1(a,b), \ldots, s_j(a,b), \ldots, s_n(a,b), d_1(a,b), \ldots, d_j(a,b), \ldots, d_n(a,b), k(C_h)\big) =$$

$$= \frac{1}{K^h(a,b)} \left(\sum_{j \in G} k_j^h s_j(a,b) + \sum_{\{j,\ell\} \in M^h} z\big(s_j(a,b), s_\ell(a,b)\big) k_{j\ell}^h + \sum_{(j,p) \in O^h} z\big(s_j(a,b), |d_p(a,b)|\big) k_{j|p}^h \right), \tag{1}$$

and

$$K^h(a,b) = \sum_{j \in G} k_j^h + \sum_{\{j,\ell\} \in M^h} z\big(s_j(a,b), s_\ell(a,b)\big) k_{j\ell}^h + \sum_{(j,p) \in O^h} z\big(s_j(a,b), |d_p(a,b)|\big) k_{j|p}^h,$$

for $h = 1, ..., q$, and $z : [0,1]^2 \to [0,1]$ is a real-valued function that takes the form $z(x,y) = xy$.

A comprehensive dissimilarity function, $d(a,b)$, can also be defined to measure the strength of the arguments in favor of dissimilarity between actions a and b, i.e., in opposition to likeness. The function considers only the dissimilarity values obtained from all per-criterion dissimilarity functions. A *comprehensive dissimilarity function is a* real-valued function $f^d : [-1,0]^n \to [-1,0]$ as follows:

$$d(a,b) = f^d\big(d_1(a,b), \ldots, d_j(a,b), \ldots, d_n(a,b)\big) = \prod_{j=1}^{n} \big(1 + d_j(a,b)\big) - 1. \tag{2}$$

In order to calculate a likeness degree that aggregates similarity and dissimilarity, for each pair of actions (a,b) (a represents a given action and b a reference action), it is necessary to use an aggregation function. A *comprehensive likeness* function is a real-valued function $f : [0,1] \times [-1,0] \to [0,1]$ as follows:

$$\delta(a,b) = f\big(s^h(a,b), d(a,b)\big) = s^h(a,b)\big(1 + d(a,b)\big). \tag{3}$$

Thus, it is possible to assess the degree of likeness of action a with respect to action b. Then, $\delta(a,b)$ is called *likeness degree* between a and b.

2.3. Relation between actions and reference actions

In order to assign actions to category C_h, $h = 1, \ldots, q$, each action has to be compared to each reference action, $b_{h\ell}$, $\ell = 1, ..., |B_h|$, computing the likeness degree, i.e., $\delta(a, b_{h\ell})$, between a and $b_{h\ell}$. A *likeness degree between an action a and a reference set B_h* can be defined as follows:

$$\delta(a, B_h) = \max_{\ell = 1, \ldots, |B_h|} \{\delta(a, b_{h\ell})\}. \tag{4}$$

A *likeness threshold*, λ^h, can be chosen by the DM for each category C_h, $h = 1, ..., q$. This preference parameter is the minimum likeness degree considered necessary to say that an action a is similar to the set B_h, $h = 1, ..., q$, taking all criteria into account. It can be interpreted as a

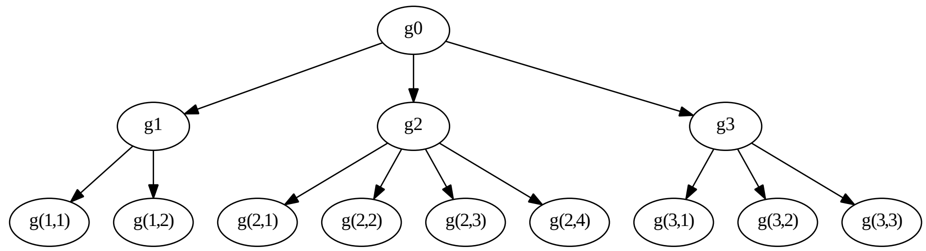

Figure 1: An example of criteria structured in a hierarchical way

majority measure of likeness allowing an action to be assigned to the most adequate categories, if any. Then, λ^h takes a value within the range $[0.5, 1]$. A *likeness binary relation*, $S(\lambda^h)$, is defined as follows:

$$aS(\lambda^h)B_h \Leftrightarrow \delta(a, B_h) \geqslant \lambda^h. \tag{5}$$

2.4. Assignment procedure

The CAT-SD assignment procedure provides at least one category to which an action a can be assigned. Each category C_h, $h = 1, ..., q$, is defined to receive actions to be processed in an identical way, at least in a first step. Given $\lambda^h \in [0.5, 1]$, $h = 1, \ldots, q$, the *likeness assignment procedure* was designed for CAT-SD as follows:

i) Compare action a with set B_h, $h = 1, \ldots, q$;

ii) Identify $U = \{u \ : \ aS(\lambda^u)B_u\}$;

iii) Assign action a to category C_u, for all $u \in U$;

iv) If $U = \emptyset$, assign action a to category C_{q+1}.

The assignment of an action to a given category is independent from the assignment to another category. Accordingly, a given action a can be assigned to:

– a single category (including C_{q+1}), in the case of a being only suitable to one category C_h, $h = 1, ..., q$ (or any).

– a set of categories (excluding C_{q+1}), in the case of a being suitable for more than one category.

3. MCHP and the hierarchical CAT-SD method

In some real-world problems, criteria are not all at the same level but they can be structured in a hierarchical way as shown, for example, in Fig. 1. It is therefore possible to consider a root criterion g_0, some macro-criteria descending from the root criterion and so on until the last level where the elementary criteria are placed.

The MCHP has been recently introduced in literature to deal with problems in which actions are evaluated on criteria structured in a hierarchical way (Corrente et al., 2012, 2013, 2016). The application of the MCHP permits to decompose the problem in small sub-problems giving to the DM the possibility to focus on a particular aspect of the problem at hand. In this way, the DM can provide information at partial level, that is considering a single criterion in the hierarchy and, at the same time, the DM can get information on the comparisons between alternatives taking into account the node on which he is more interested. In this section, we shall detail the extension of the CAT-SD method to the hierarchical case. Therefore, the MCHP and the CAT-SD method will be put together within a unified framework giving arise to the hCAT-SD method. To this aim, regarding the MCHP, we shall use the following notation:

- $\mathcal{G}$ is the set composed of all criteria in the hierarchy;
- $\mathcal{I}_{\mathcal{G}}$ is the set of the indices of criteria in $\mathcal{G}$;
- $EL \subseteq \mathcal{I}_{\mathcal{G}}$ is the set of all elementary criteria;
- $g_{\mathbf{r}}$, with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \backslash EL$, is a generic criterion in the hierarchy and it will be called *non-elementary criterion*;
- Given a non-elementary criterion $g_{\mathbf{r}}$, $E(g_r) \subseteq EL$ is the set of the indices of the elementary criteria descending from $g_{\mathbf{r}}$.

Given a non-elementary criterion $g_{\mathbf{r}}$, to perform the classification of the actions on $g_{\mathbf{r}}$, a *partial similarity function* $f^s_{\mathbf{r}} : [0,1]^{|E(g_{\mathbf{r}})|} \times [-1,0]^{|E(g_{\mathbf{r}})|} \to [0,1]$, with $E(g_{\mathbf{r}}) = \{\mathbf{t}_1, \ldots, \mathbf{t}_r\}$, needs to be defined:

$$s^h_{\mathbf{r}}(a,b) = f^s_{\mathbf{r}}\left(s_{\mathbf{t}_1}(a,b), \ldots, s_{\mathbf{t}_r}(a,b), d_{\mathbf{t}_1}(a,b), \ldots, d_{\mathbf{t}_r}(a,b), k(C_h)\right) =$$

$$= \frac{1}{K^h_{\mathbf{r}}(a,b)}\left(\sum_{\mathbf{t}\in E(g_{\mathbf{r}})} k^h_{\mathbf{t}} s_{\mathbf{t}}(a,b) + \sum_{\substack{\{\mathbf{t}_1,\mathbf{t}_2\}\in M^h:\\ \mathbf{t}_1,\mathbf{t}_2\in E(g_{\mathbf{r}})}} z\left(s_{\mathbf{t}_1}(a,b), s_{\mathbf{t}_2}(a,b)\right) k^h_{\mathbf{t}_1\mathbf{t}_2} + \sum_{\substack{(\mathbf{t}_1,\mathbf{t}_2)\in O^h:\\ \mathbf{t}_1,\mathbf{t}_2\in E(g_{\mathbf{r}})}} z\left(s_{\mathbf{t}_1}(a,b), |d_{\mathbf{t}_2}(a,b)|\right) k^h_{\mathbf{t}_1|\mathbf{t}_2}\right) \tag{6}$$

and

$$K^h_{\mathbf{r}}(a,b) = \sum_{\mathbf{t}\in E(g_{\mathbf{r}})} k^h_{\mathbf{t}} + \sum_{\substack{\{\mathbf{t}_1,\mathbf{t}_2\}\in M^h:\\ \mathbf{t}_1,\mathbf{t}_2\in E(g_{\mathbf{r}})}} z\left(s_{\mathbf{t}_1}(a,b), s_{\mathbf{t}_2}(a,b)\right) k^h_{\mathbf{t}_1\mathbf{t}_2} + \sum_{\substack{(\mathbf{t}_1,\mathbf{t}_2)\in O^h:\\ \mathbf{t}_1,\mathbf{t}_2\in E(g_{\mathbf{r}})}} z\left(s_{\mathbf{t}_1}(a,b), |d_{\mathbf{t}_2}(a,b)|\right) k^h_{\mathbf{t}_1|\mathbf{t}_2}.$$

In this way, the partial similarity function $s^h_{\mathbf{r}}(a,b)$ computes the similarity between the actions a and b taking into account the elementary criteria descending from $g_{\mathbf{r}}$ only.

As already done for the partial similarity function, the *partial dissimilarity function* $f^d_{\mathbf{r}} : [-1,0]^{|E(g_{\mathbf{r}})|} \to [-1,0]$ has to be defined for each non-elementary criterion $g_{\mathbf{r}}$ as follows:

$$d_{\mathbf{r}}(a,b) = f^d_{\mathbf{r}}\left(d_{\mathbf{t}_1}(a,b), \ldots, d_{\mathbf{t}_r}(a,b)\right) = \prod_{\mathbf{t}\in E(g_{\mathbf{r}})} \left(1 + d_{\mathbf{t}}(a,b)\right) - 1. \tag{7}$$

On the basis of the partial similarity and dissimilarity functions defined in eqs. (6) and (7), respectively, for each non-elementary criterion $g_{\mathbf{r}}$ we can define a *partial likeness* function $f_{\mathbf{r}} : [0,1] \times [-1,0] \to [0,1]$ as follows (also called likeness degree):

$$\delta_{\mathbf{r}}(a,b) = f_{\mathbf{r}}\left(s^h_{\mathbf{r}}(a,b), d_{\mathbf{r}}(a,b)\right) = s^h_{\mathbf{r}}(a,b)(1+d_{\mathbf{r}}(a,b)). \tag{8}$$

In order to assign the actions to the different categories on the non-elementary criterion $g_{\mathbf{r}}$, these have to be compared with the reference actions belonging to the reference set of the considered categories. Therefore, on the basis of eq. (8), the *partial likeness degree between action a and the reference set B_h* on $g_{\mathbf{r}}$ can be defined:

$$\delta_{\mathbf{r}}(a,B_h) = \max_{l=1,\ldots,|B_h|}\{\delta_{\mathbf{r}}(a,b_{hl})\}. \tag{9}$$

As a consequence, we say that a is alike to B_h on $g_{\mathbf{r}}$, and we write $aS_{\mathbf{r}}(\lambda^h_{\mathbf{r}})B_h$, iff $\delta_{\mathbf{r}}(a,B_h) \geqslant \lambda^h_{\mathbf{r}}$, where $\lambda^h_{\mathbf{r}} \in [0.5,1]$ is the likeness threshold. Pay attention to the fact that $\lambda^h_{\mathbf{r}}$ can be dependent on criterion $g_{\mathbf{r}}$ we are considering.

The *partial classification* of $a \in A$ on $g_{\mathbf{r}}$ is therefore performed following these steps:

i) Compare a with the set B_h on criterion $g_{\mathbf{r}}$, $h=1,\ldots,q$,

ii) Identify $U_{\mathbf{r}} = \{u : aS_{\mathbf{r}}(\lambda^u_{\mathbf{r}})B_u\}$,

iii) Assign a to the category C_u for all $u \in U_{\mathbf{r}}$,

iv) If $U_{\mathbf{r}} = \emptyset$, assign a to C_{q+1}, being a fictitious category collecting all non-assigned actions.

The added value of the application of MCHP to the Cat-SD method is that one can get the classifications of the actions not only at comprehensive level, therefore considering simultaneously all criteria, but also at a partial level by considering a particular aspect of the problem at hand only. In this way, the DM can have a deeper knowledge of the decision making problem he is dealing with.

4. The hierarchical and imprecise SRF method

As described in the previous section, the classification procedure used in the hCat-SD method is based on the knowledge of the weights of elementary criteria $g_{\mathbf{t}}$ $(k_{\mathbf{t}})$, the knowledge of the values representing the mutual strengthening and weakening effects between elementary criteria $g_{\mathbf{t}_1}, g_{\mathbf{t}_2}$, $(k_{\mathbf{t}_1\mathbf{t}_2})$, and the knowledge of the values representing the antagonistic effect exercised from elementary criterion $g_{\mathbf{t}_2}$ over elementary criterion $g_{\mathbf{t}_1}$ $\left(k_{\mathbf{t}_1|\mathbf{t}_2}\right)$. Anyway, asking the DM to provide all these parameters is unreasonable for their huge number as well as for the cognitive burden related to the complexity of their meaning. Therefore, the application of an indirect technique is preferable in this case.

To get the weights of criteria involved in the decision problem at hand, in Figueira and Roy (2002) the SRF method was proposed. The procedure, known as cards method, extended the proposal of Simos (Simos, 1990b,a) by permitting the DM to introduce the value z representing the ratio between the weight of the most important and the weight of the least important criteria. A further extension of the SRF method was recently introduced in Corrente et al. (2017b), permitting the DM to provide imprecise information regarding both the number of cards that should be introduced between two successive subsets of criteria and the z-value introduced in the SRF method. The method was also applied to hierarchical structures of criteria. In the following, we shall briefly recall the main steps involved in the application of the SRF method to the set $\{g_{(\mathbf{r},1)},\ldots,g_{\mathbf{r},n(\mathbf{r})}\}$ composed of the immediate sub-criteria of the non-elementary criterion $g_{\mathbf{r}}$:

1. Rank the criteria from the least important $L_1^{\mathbf{r}}$, to the most important $L_v^{\mathbf{r}}$, where $v \leq n(\mathbf{r})$, with the possibility of some ex-aequo;
2. Define an interval $[low_s^{\mathbf{r}}, upp_s^{\mathbf{r}}]$ in which $e_s^{\mathbf{r}}$ can vary, where $e_s^{\mathbf{r}}$ is the number of blank cards that have to be introduced between $L_s^{\mathbf{r}}$ and $L_{s+1}^{\mathbf{r}}$, with $s = 1, \ldots, v-1$. The greater the number of blank cards between $L_s^{\mathbf{r}}$ and $L_{s+1}^{\mathbf{r}}$, the more important are criteria in $L_{s+1}^{\mathbf{r}}$ with respect to criteria in $L_s^{\mathbf{r}}$;
3. Define an interval $\left[z_{low}^{\mathbf{r}}, z_{upp}^{\mathbf{r}}\right]$ in which $z^{\mathbf{r}}$ can vary, where $z^{\mathbf{r}}$ is the ratio between weights of criteria in $L_v^{\mathbf{r}}$ and criteria in $L_1^{\mathbf{r}}$.

Denoting by $K_{L_w^{\mathbf{r}}}$ the weight of a criterion in $L_w^{\mathbf{r}}$, with $w = 1, \ldots, v$, and by $C_{\mathbf{r}}$ the importance of a blank card introduced between two successive subsets of criteria, the previous preference information is translated into the following set of linear constraints (see Corrente et al., 2017b, for more details):

$$E_{\mathbf{r}} \left\{ \begin{array}{l} \left.\begin{array}{l} K_{L_{s+1}^{\mathbf{r}}} \geqslant K_{L_s^{\mathbf{r}}} + (low_s^{\mathbf{r}} + 1)C_{\mathbf{r}}, \\ K_{L_{s+1}^{\mathbf{r}}} \leqslant K_{L_s^{\mathbf{r}}} + (upp_s^{\mathbf{r}} + 1)C_{\mathbf{r}}, \\ C_{\mathbf{r}} > 0, \end{array}\right\} \text{for all } s = 1, \ldots, v-1, \\ z_{low}^{\mathbf{r}} K_{L_v^{\mathbf{r}}} - K_{L_1^{\mathbf{r}}} \leqslant 0, \\ K_{L_1^{\mathbf{r}}} - z_{upp}^{\mathbf{r}} K_{L_v^{\mathbf{r}}} \leqslant 0, \\ K_{L_1^{\mathbf{r}}} > 0. \end{array} \right.$$

Let us observe that constraints in $E_{\mathbf{r}}$ can be expressed in terms of the weights of elementary criteria assuming that, for each non-elementary criterion $g_{\mathbf{r}}$, $K_{\mathbf{r}} = \sum_{\mathbf{t} \in E(g_{\mathbf{r}})} k_{\mathbf{t}}$. Moreover, for each $s = 1, \ldots, v$ and for each $g_{(\mathbf{r},j)} \in L_s^{\mathbf{r}}$, $K_{(\mathbf{r},j)} = K_{L_s^{\mathbf{r}}}$.

Concerning the parameters $k_{\mathbf{t}_1\mathbf{t}_2}$ and $k_{\mathbf{t}_1|\mathbf{t}_2}$, with $\mathbf{t}_1, \mathbf{t}_2 \in EL$, the following constraints translate the preferences of the DM:

$$E_{int} \left\{ \begin{array}{l} k_{\mathbf{t}_1\mathbf{t}_2} > 0 \text{ if } g_{\mathbf{t}_1} \text{ and } g_{\mathbf{t}_2} \text{ present a mutual-strengthening effect,} \\ k_{\mathbf{t}_1\mathbf{t}_2} < 0 \text{ if } g_{\mathbf{t}_1} \text{ and } g_{\mathbf{t}_2} \text{ present a mutual-weakening effect,} \\ k_{\mathbf{t}_1|\mathbf{t}_2} < 0 \text{ if } g_{\mathbf{t}_2} \text{ presents an antagonistic effect over } g_{\mathbf{t}_1}. \end{array} \right.$$

The following technical constraints have also to be satisfied:

$$(E_{Norm}) \; \sum_{\mathbf{t} \in EL} k_{\mathbf{t}} + \sum_{\{\mathbf{t}_1,\mathbf{t}_2\} \subseteq EL} k_{\mathbf{t}_1\mathbf{t}_2} = 100,$$

$$(E_{Net}) \; k_{\mathbf{t}_1} + \sum_{\substack{\{\mathbf{t}_1,\mathbf{t}_2\} \subseteq EL: \\ k_{\mathbf{t}_1\mathbf{t}_2} < 0}} k_{\mathbf{t}_1\mathbf{t}_2} + \sum_{\mathbf{t}_3 \in EL} k_{\mathbf{t}_1|\mathbf{t}_3} \geqslant 0 \text{ for all } g_{\mathbf{t}_1} \text{ such that } \mathbf{t}_1 \in EL.$$

Let us observe that E_{Norm} is a technical constraint used only to put an upper bound on the coefficients. This will be useful in the sampling procedure that we will describe in the following section. Anyway, if one uses the direct technique, that is the DM is asked to provide directly the values of the coefficients involved in the computations, then this constraint can be neglected. The space of the parameters involved in the hierarchical and imprecise SRF method is therefore defined by the constraints in the set:

$$E = \cup_{\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL} E_{\mathbf{r}} \cup E_{int} \cup E_{Norm} \cup E_{Net}.$$

To check if there exists at least one set of parameters compatible with the preferences provided by the DM, one has to solve the following LP problem:

$$\varepsilon^{*} = \max \varepsilon, \text{ subject to } E^{'} \tag{10}$$

where $E^{'}$ is obtained by E converting the strict inequality constraints in weak ones by using an auxiliary variable ε. For example, constraint $C_{\mathbf{r}} > 0$ is converted into $C_{\mathbf{r}} \geqslant \varepsilon$, while $k_{\mathbf{t}_1\mathbf{t}_2} < 0$ is converted into $k_{\mathbf{t}_1\mathbf{t}_2} \leqslant -\varepsilon$. If $E^{'}$ is feasible and $\varepsilon^{*} > 0$, then the space of parameters is not empty while, in the opposite case, the set of constraints $E^{'}$ is infeasible and the cause of the infeasibility can be checked by using one of the methods proposed in Mousseau et al. (2003).

Let us observe that the hierarchical and imprecise SRF method involves the application of the imprecise SRF method to each node of the hierarchy. For example, if one deals with a hierarchical structure of criteria such that one shown in Fig. 1, the imprecise SRF method has to be applied at first on the set of criteria $\{g_1, g_2, g_3\}$, and then to the three sets of elementary criteria $\{g_{(1,1)}, g_{(1,2)}\}, \{g_{(2,1)}, g_{(2,2)}, g_{(2,3)}, g_{(2,4)}\}$ and $\{g_{(3,1)}, g_{(3,2)}, g_{(3,3)}\}$. The application of the hierarchical and imprecise SRF method will be carefully described and illustrated in Section 7.

5. SMAA and the SMAA-hCAT-SD method

As already stated in the previous section, the set of constraints E defines the space of vectors of parameters compatible with the preferences provided by the DM. Anyway, in general, if there exists at least one vector of parameters compatible with the preferences of the DM, then there exists more than one of these vectors. Therefore, using only one of them could be considered arbitrary or meaningless, so that it seems reasonable to take into consideration the whole set of set of compatible vectors of preference parameters. To avoid this choice, in this paper we shall apply the Stochastic Multicriteria Acceptability Analysis (SMAA) (see Lahdelma et al., 1998, for the paper introducing SMAA, and Tervonen and Figueira, 2008, for a survey on SMAA methods). In this section we shall describe the application of SMAA to the hCat-SD method building, therefore, the SMAA-hCat-SD method. It starts from the sampling of several sets of compatible parameters. Since the constraints in E define a convex space of parameters, one can use the Hit-And-Run (HAR) method to sample them (Smith, 1984; Tervonen et al., 2012; Van Valkenhoef et al., 2014). Of course, for each sampled set of parameters, a classification of the actions at hand on the considered macro-criteria can be performed. Denoting by $\mathcal{K}$ the space of the parameters compatible with the preferences provided by the DM, for each $k \in \mathcal{K}$, $a \in A$, $g_{\mathbf{r}}$ and C_h, writing $a \underset{k,\mathbf{r}}{\longrightarrow} C_h$ we mean that, considering the parameters in k, alternative a is assigned to class C_h on criterion $g_{\mathbf{r}}$. One can therefore define the set $\mathcal{K}^{h}_{\mathbf{r}}(a) \subseteq \mathcal{K}$ composed of the sets of compatible parameters for which a is classified on C_h with respect to $g_{\mathbf{r}}$:

$$\mathcal{K}^{h}_{\mathbf{r}}(a) = \left\{ k \in \mathcal{K} : a \underset{k,\mathbf{r}}{\longrightarrow} C_h \right\}. \tag{11}$$

As observed in Section 3, each action could be classified in more than one category. Consequently, for each $\mathcal{C} \subseteq \{C_1, \ldots, C_q\}$, we can define also the set

$$\mathcal{K}^{\mathcal{C}}_{\mathbf{r}}(a) = \left\{ k \in \mathcal{K} : \ \forall C_h \in \mathcal{C}, \ a \underset{k,\mathbf{r}}{\longrightarrow} C_h \right\}. \tag{12}$$

SMAA applied to the hCat-SD method permits therefore to calculate the give an approximate estimation of the probability with which an action is classified in a single category (or a set of categories) on criterion $g_{\mathbf{r}}$. Formally,

$$b_{\mathbf{r}}^{h}(a) = \frac{|\mathcal{K}_{\mathbf{r}}^{h}(a)|}{|\mathcal{K}|} \qquad \text{and} \qquad b_{\mathbf{r}}^{\mathcal{C}}(a) = \frac{|\mathcal{K}_{\mathbf{r}}^{\mathcal{C}}(a)|}{|\mathcal{K}|}.$$

In this way, it is possible to analyze not only the probability of the assignments when all elementary criteria are taken into account, but also when a particular macro-criterion is considered.

6. Hierarchy and robustness

The two new aspects of the approach we are proposing with respect to the basic model presented in Costa et al. (2018) are the probabilistic nature of the classification and the hierarchy of criteria. Let us discuss their implications and their advantages. The idea of probabilistic classification has gained a great success in the domain of data mining and machine learning (see, for example, Taskar et al. 2001; Williams and Barber 1998). The probabilistic aspect of the classification we are considering regards the imprecision related to the weights representing the importance of criteria, but, of course, other types of imprecision could be considered, such as values of other parameters of the model, as the likeness thresholds or the shape of the per-criterion similarity $s_j(a,b)$ through the function $f_j(\Delta_j(a,b))$.

These probabilistic results are very natural and they allow to take into account some gradualness in handling the uncertainty of the nominal classification. Also, the hierarchy of criteria is a natural approach, especially when many criteria have to be considered. For example, hierarchy of criteria has gained a great consideration in the domain of cognitive diagnostic assessment (see, for example, Leighton et al. 2004). Observe that, beyond being closer to human natural reasoning, consideration of a probabilistic classification as well as hierarchy in the criteria can result very useful and effective in real-world applications. Indeed,

(i) We can assign actions to categories taking into account different thresholds of similarity;

(ii) We can use similarity in the hierarchy of criteria to refine the decision with respect to different macro-criteria, for example, assigning to a category actions that could not be assigned taking into account the overall similarity, but that could be assigned taking into account a macro-criterion that is considered particularly important.

The above points (i) and (ii) can be used to define a progressive nominal classification procedure articulated as described in the following. Suppose that DM wants to obtain a classification that satisfies some requirements, such as:

Requirement (1) There should be a percentage not greater than $nc\%$ of non-classified actions;

Requirement (2) At least $m_h^{\geqslant}$ actions should be assigned to category C_h, $h = 1, \ldots, q$;

Requirement (3) No more than $m_h^{\leqslant}$ actions should be assigned to category C_h , $h = 1, \ldots, q$.

The basic element of the nominal classification procedure is the probability of assignment to nominal category C_h with respect to criterion $\mathbf{r}$ in the hierarchy, that is $b_{\mathbf{r}}^{h}(a)$. Each step of the procedure is associated with a positive decision rule, ρ^{+}, or a negative decision rule, ρ^{-}, with the following syntax:

(ρ^+) If action a has not already been assigned to category C_h and $b^h_{r_1}(a) \geqslant t^\rho_{h,r_1}$ and ... $b^h_{r_p}(a) \geqslant t^\rho_{h,r_p}$, then a is assigned to category C_h, $h = 1, ..., q, r_{h1}, ..., r_{hp} \in \mathcal{G} - El$, that is $r_{h1}, ..., r_{hp}$ have to be non-elementary criteria in the hierarchy;

(ρ^-) If action a has already been assigned to category C_h and $b^h_{r_1}(a) \leqslant t^\rho_{h,r_1}$ and ... $b^h_{r_p}(a) \leqslant t^\rho_{h,r_p}$, then a is unassigned from category C_h, , $h = 1, ..., q, r_{h1}, ..., r_{hp} \in \mathcal{G} - El$, that is $r_{h1}, ..., r_{hp}$ have to be non-elementary criteria in the hierarchy.

A set of positive and negative rules, ρ^+ and ρ^-, are defined and ordered in terms of their priority in a sequence Ξ. The algorithm starts by assigning to the respective category C_h the actions for which $b^h_{\mathbf{r}}(a) \geqslant t_h$, where t_h is a predefined threshold. Then, we check if all the requirements of the nominal classification are satisfied. If not, the first rule $\hat{\rho}$ in the sequence Ξ that meets one of the unsatisfied requirements is applied, so that:

- The actions satisfying the conditions of the rule are assigned to the considered category if $\hat{\rho}$ is a positive rule;
- The actions satisfying the conditions of the rule are unassigned from the considered category if $\hat{\rho}$ is a negative rule.

The algorithm stops if all the requirements are satisfied. If not all the requirements are satisfied and there is no other rule in the sequence Ξ, then some *ad-hoc* algorithm can be applied to fulfill all the remaining requirements. For example, for a given category C_h not yet containing the fixed number of actions, the criteria in the hierarchy are taken sequentially into consideration in order of their average importance with respect to the category. Denoting by $g_{\mathbf{r}}$ the most important criterion for the assignment to category C_h, the actions having the greatest probability $b^h_{\mathbf{r}}(a)$ are selected provided that $b^h_{\mathbf{r}}(a)$ is not lower than a meaningful threshold. If the requirement is not yet satisfied, we pass to the second most important criterion, and so on.

Let us remark that this procedure can be applied to other classification methods, also ordinal, such as Electre Tri (Yu, 1992) and its variants (Almeida Dias et al., 2010, 2012).

7. Illustrative example

In this section, we shall apply the SMAA-hCat-SD method presented in the previous sections extending the numerical example presented in Costa et al. (2018). In particular, the section is split in three parts. In the first part, we shall describe, in detail, how to perform the assignments at comprehensive level as well as on each macro-criterion, highlighting the importance of taking into account the MCHP. In the second part, we shall apply the SMAA method to the hierarchical hCat-SD method commenting the obtained results. In the third part, we apply the classification procedure described in Section 6 to the numerical example.

7.1. Introduction of the case study and description of the computations

Seven soldiers ($a_1, \ldots, a_7$) have to be assigned to five categories ($C_1, \ldots, C_5$): snipers (C_1), breachers (C_2), communications operators (C_3), heavy weapons operators (C_4), and non-assigned candidates (C_5). Their evaluation is performed considering several criteria structured in a hierarchical way as shown in Fig. 2.

The hierarchy of criteria is composed of three macro-criteria that are Mental Sharpness (MS), Mental Resilience (MR) and Physical and other Features (PoF). Each of these macro-criteria has

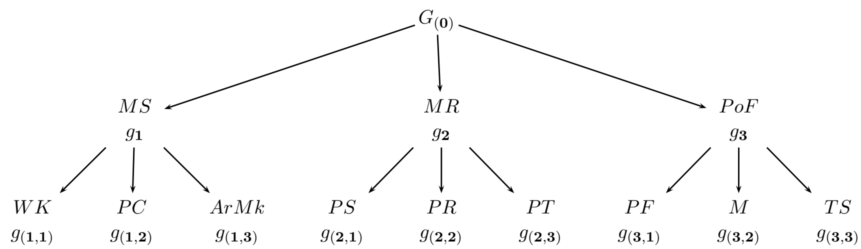

Figure 2: Hierarchical structure of criteria considered in the case study

three elementary criteria descending from them. In particular, World Knowledge (WK), Paragraph Comprehension (PC) and Arithmetic reasoning and Mathematics knowledge ($ArMk$) descend from MS; Performance Strategies (PS), Psychological Resilience (PR) and Personality Traits (PT) descend from MR; finally, Physical Fitness (PF), Motivation (M) and Teamwork Skills (TS) are sub-criteria of PoF. The description of the nine considered elementary criteria is given in Table 1.

Table 1: Description of the elementary criteria

Macro-criterion	**Elementary criterion**	**Elementary criterion description**
MS	WK	Identification of word synonyms and right definition of words in a given context
	PC	Identification of the meaning of texts
	$ArMk$	Solving arithmetic problems and knowledge of mathematics principles (algebra and geometry)
MR	PS	Goal setting, self-talk, and emotional control
	PR	Acceptance of life situations, and ability for dealing with cognitive challenges and threats
	PT	Character traits such as adaptability, dutifulness, social orientation, self-reliance, stress tolerance, vigilance, and impulsivity
PoF	PF	Physical ability with respect to aerobic fitness and strength
	M	Self motivation, persistence and dedication
	TS	Communication skills and camaraderie

The performance of the seven soldiers on the nine elementary criteria is given in Table 2.

Table 2: Performance of the considered soldiers on the elementary criteria at hand

Soldier	$g_{(1,1)}$	$g_{(1,2)}$	$g_{(1,3)}$	$g_{(2,1)}$	$g_{(2,2)}$	$g_{(2,3)}$	$g_{(3,1)}$	$g_{(3,2)}$	$g_{(3,3)}$
a_1	74	77	88	4	4	3	740	4	6
a_2	82	80	96	2	3	3	950	4	4
a_3	58	66	69	3	3	4	720	5	5
a_4	78	88	75	2	3	2	920	5	5
a_5	66	59	70	3	2	3	850	6	5
a_6	70	77	82	4	3	4	1100	5	6
a_7	73	79	78	3	4	4	710	5	6
Function	f_2	f_2	f_2	f_3	f_3	f_3	f_1	f_3	f_3

Each reference set B_h is composed of one reference action only. Their evaluations are provided in Table 3.

The three per-criterion similarity-dissimilarity functions used in the illustrative example are the following:

Table 3: Performance of the reference soldiers on the elementary criteria at hand

Reference set	**Reference action**	$g_{(1,1)}$	$g_{(1,2)}$	$g_{(1,3)}$	$g_{(2,1)}$	$g_{(2,2)}$	$g_{(2,3)}$	$g_{(3,1)}$	$g_{(3,2)}$	$g_{(3,3)}$
B_1	b_{11}	80	75	85	4	4	4	700	6	4
B_2	b_{21}	70	70	75	3	3	3	800	6	6
B_3	b_{31}	80	90	85	2	2	3	950	4	4
B_4	b_{41}	60	65	65	3	3	3	700	5	6

$$f_1\big(\Delta_1(a,b)\big) = \begin{cases} 1, & \text{if } |\Delta_1(a,b))| \leqslant 50; \\ \frac{100-|\Delta_1(a,b)|}{50}, & \text{if } 50 < |\Delta_1(a,b)| \leqslant 100; \\ 0, & \text{if } 100 < |\Delta_1(a,b)| \leqslant 150; \\ \frac{150-|\Delta_1(a,b)|}{50}, & \text{if } 150 < |\Delta_1(a,b)| \leqslant 200; \\ -1, & \text{if } |\Delta_1(a,b)| > 200. \end{cases}$$

$$f_2\big(\Delta_2(a,b)\big) = \begin{cases} 1, & \text{if } |\Delta_2(a,b)| \leqslant 5; \\ \frac{10-|\Delta_2(a,b)|}{5}, & \text{if } 5 < |\Delta_2(a,b)| \leqslant 10; \\ 0, & \text{if } -20 < \Delta_2(a,b)) < -10 \text{ or } 10 < \Delta_2(a,b) \leqslant 15; \\ \frac{20+\Delta_2(a,b)}{5}, & \text{if } -25 < \Delta_2(a,b) \leqslant -20; \\ \frac{15-\Delta_2(a,b)}{5}, & \text{if } 15 < \Delta_2(a,b) \leqslant 20; \\ -1, & \text{if } \Delta_2(a,b) \leqslant -25 \text{ or } \Delta_2(a,b) > 20. \end{cases}$$

$$f_3\big(\Delta_3(a,b)\big) = \begin{cases} 1, & \text{if } |\Delta_3(a,b)| = 0; \\ 0, & \text{if } |\Delta_3(a,b)| = 1; \\ -1, & \text{if } |\Delta_3(a,b)| \geqslant 2. \end{cases}$$

To get the weights of the elementary criteria, the hierarchical and imprecise SRF method has been applied for each category. For this reason the imprecise SRF method is applied to a set composed by the three macro-criteria as well as to three subsets of the elementary criteria descending from each macro-criterion. In the following we shall provide the details on the application of the hierarchical and imprecise SRF method with respect to each one of the four categories:

1. Application of the imprecise SRF method to category C_1:
    - PoF is less important than MR that, in turn, is less important than MS. The number of blank cards to be inserted between MR and MS belongs to the interval $[1, 2]$, while there is one blank card between PoF and MR. Moreover, the ratio between the weight of MS and the weight of PoF should belong to the interval $[4, 6]$;
    - With respect to macro-criterion MS, WK is less important than PC that is less important than $ArMk$. The number of blank cards inserted between $ArMk$ and PC belongs to the interval $[0, 2]$, while the number of blank cards inserted between PC and WK has to belong to the interval $[0, 1]$. Moreover the ratio between the weight of $ArMk$ and the weight of WK is in the interval $[3, 4]$;
    - Considering macro-criterion MR, PS is less important than PR that, in turn, is less important than PT. There is one blank card between PT and PR, while the number of blank cards to be inserted between PR and PS has to belong to the interval $[1, 2]$. Finally, the ratio between the weight of PT and that one of PS has to belong to the interval $[2, 3]$;
    - On macro-criterion PoF, PF is less important than M being less important than TS. The number of blank cards between TS and PF as well as between PF and M should be in the interval $[1, 2]$. The ratio between the weight of the most important criterion (TS) and the least important one (M) should be in the interval $[2, 4]$.
2. Application of the imprecise SRF method to category C_2:
    - PoF is less important than MS that, in turn, is less important than MR. The number of blank cards to be inserted between PoF and MS belongs to the interval $[3, 5]$, while there is one blank card between MR and MS. Moreover, the ratio between the weight of MR and the weight of PoF should belong to the interval $[3, 5]$;
    - With respect to macro-criterion MS, WK is less important than PC that is less important than $ArMk$. The number of blank cards inserted between WR and PC belong to the interval $[1, 2]$, while there is not any blank card between PC and $ArMK$. Moreover the ratio between the weight of $ArMk$ and the weight of WK is in the interval $[3, 5]$;
    - Considering macro-criterion MR, PS is less important than PR that, in turn, is less important than PT. There is not any blank card between PS and PR, while the number of blank cards to be inserted between PR and PT has to belong to the interval $[0, 1]$. Finally, the ratio between the weight of PT and that one of PS has to belong to the interval $[3, 4]$;
    - On macro-criterion PoF, PF is less important than M being less important than TS. The number of blank cards between PF and M should belong to the interval $[2, 3]$, while there is one blank card only between M and TS. The ratio between the weight of the most important criterion (TS) and the least important one (PF) should be in the interval $[3, 4]$.
3. Application of the imprecise SRF method to category C_3:

- MS is less important than MR that, in turn, is less important than PoF. The number of blank cards to be inserted between MS and MR belongs to the interval $[2,3]$, while there is one blank card between MR and PoF. Moreover, the ratio between the weight of PoF and the weight of MS should belong to the interval $[4,6]$;
- With respect to macro-criterion MS, PC is less important than WK that is as important as $ArMk$. There is only one blank card between PC and the set $\{WK, ArMk\}$. Moreover the ratio between the weight of $ArMk$ and the weight of PC is in the interval $[2,4]$;
- Considering macro-criterion MR, PR is less important than PS that has the same importance of PT. There is one blank card between PR and the set composed of PS and PT. Finally, the ratio between the weight of PT and that one of PR has to belong to the interval $[2,4]$;
- On macro-criterion PoF, TS is less important than PF having the same importance of M. The number of blank cards between TS and the weights PF and M should belong to the interval $[0,1]$. The ratio between the weight of the most important criterion (M) and the least important one (TS) should be in the interval $[3,5]$.

4. Application of the imprecise SRF method to category C_4:
   - PoF is less important than MS that, in turn, is less important than MR. The number of blank cards to be inserted between PoF and MS belongs to the interval $[2,3]$, while there are two blank cards between MS and MR. Moreover, the ratio between the weight of MR and the weight of PoF should belong to the interval $[2,5]$;
   - With respect to macro-criterion MS, PC and WK are equally important but they are less important than $ArMk$. The number of blank cards that should be inserted between $ArMk$ and the set of criteria $\{PC, WK\}$ belongs to the interval $[1,2]$. The ratio between the weight of $ArMk$ and the weight of PC is in the interval $[3,5]$;
   - Considering macro-criterion MR, PS is less important than PR being less important than PT. There is not any blank card between PS and PR while the number of blank cards between PR and PT should belong to the interval $[0,1]$. Finally, the ratio between the weight of PT and that one of PS has to belong to the interval $[2,4]$;
   - On macro-criterion PoF, PF is less important than M having the same importance of TS. The number of blank cards between the set $\{M, TS\}$ and PF belong to the interval $[1,3]$ while the ratio between the weight of the most important criterion (TS) and the least important one (PF) should be in the interval $[3,4]$.

Regarding the mutual-strengthening and mutual-weakening effects as well as the antagonistic effects, we suppose that their values are defined in a precise way and they hold for all four categories:

- There is a mutual-strengthening effect between $ArMk$ and PR quantified in the value 10;
- There is a mutual-weakening effect between PF and TS quantified in the value -4;
- There is an antagonistic effect exercised by PF over PS quantified in the value -3.

All other parameters representing mutual-strengthening, mutual-weakening effects, and antagonistic effects, but that are not specified by the DM, are supposed to be null.

Introducing all the constraints translating the preferences provided by the DM, we solved the LP problem (10) obtaining $\varepsilon^* > 0$. Therefore, there exists at least one set of parameters compatible with the preferences provided by the DM and, consequently, we applied the HAR method to sample 100,000 sets of compatible parameters.

Now, we shall present in detail all the steps necessary to perform the considered assignments, highlighting the meaning of using the MCHP. For this reason, we consider the soldier a_3 and the set of sampled weights in Table 4

Table 4: Weights considered in the first part of the example

	$g_{(1,1)}$	$g_{(1,2)}$	$g_{(1,3)}$	$g_{(2,1)}$	$g_{(2,2)}$	$g_{(2,3)}$	$g_{(3,1)}$	$g_{(3,2)}$	$g_{(3,3)}$
$k^1_{\mathbf{t}}$	0.465	35.175	21.595	3.075	1.487	17.847	6.914	0.122	7.320
$k^2_{\mathbf{t}}$	6.222	11.655	12.925	11.005	11.082	18.625	4.384	5.385	12.717
$k^3_{\mathbf{t}}$	8.229	2.910	0.514	14.196	5.848	14.208	21.392	21.618	5.982
$k^4_{\mathbf{t}}$	7.746	7.746	15.627	8.410	12.260	16.119	4.961	10.566	10.566

The steps that have to be performed to in the assignment procedure are the following:

1. *Compute the similarity-dissimilarity:* For each elementary criterion and using the three per-criterion similarity-dissimilarity functions introduced above, we compute the similarity-dissimilarity between a_3 and the four reference actions. The values are shown in Table 5.

Table 5: Similarity-dissimilarity values for each elementary criterion

	$g_{(1,1)}$	$g_{(1,2)}$	$g_{(1,3)}$	$g_{(2,1)}$	$g_{(2,2)}$	$g_{(2,3)}$	$g_{(3,1)}$	$g_{(3,2)}$	$g_{(3,3)}$
$f_{\mathbf{t}}(a_3, b_{11})$	-0.4	0.2	0	0	0	1	1	0	0
$f_{\mathbf{t}}(a_3, b_{21})$	0	1	0.8	1	1	0	0.4	0	0
$f_{\mathbf{t}}(a_3, b_{31})$	-0.4	-0.8	0	0	0	0	-1	0	0
$f_{\mathbf{t}}(a_3, b_{41})$	1	1	1	1	1	0	1	1	0

2. *Compute the comprehensive likeness:* Following eqs. (6)-(8), for each non-elementary criterion $g_{\mathbf{r}}$ in the hierarchy, we compute the partial similarity and dissimilarity functions as well as the partial likeness degree between a_3 and the considered reference actions. The values are shown in Table 6.

Table 6: Partial similarity, dissimilarity and likeness degree

	$s^h_{\mathbf{1}}(a_3,\cdot)$	$d_{\mathbf{1}}(a_3,\cdot)$	$\delta_{\mathbf{1}}(a_3,\cdot)$	$s^h_{\mathbf{2}}(a_3,\cdot)$	$d_{\mathbf{2}}(a_3,\cdot)$	$\delta_{\mathbf{2}}(a_3,\cdot)$	$s^h_{\mathbf{3}}(a_3,\cdot)$	$d_{\mathbf{3}}(a_3,\cdot)$	$\delta_{\mathbf{3}}(a_3,\cdot)$	$s^h_{\mathbf{0}}(a_3,\cdot)$	$d_{\mathbf{0}}(a_3,\cdot)$	$\delta_{\mathbf{0}}(a_3,\cdot)$
b_{11}	0.123	-0.400	0.074	0.796	0	0.796	0.482	0	0.482	0.338	-0.4	0.203
b_{21}	0.714	0	0.714	0.543	0	0.543	0.078	0	0.078	0.528	0	0.528
b_{31}	0	-0.880	0	0	0	0	0	-1	0	0	-1	0
b_{41}	1	0	1	0.562	0	0.562	0.595	0	0.595	0.743	0	0.743

For example, to compute $s^h_{\mathbf{1}}(a_3, b_{11})$, that is the similarity between a_3 and b_{11} on MS $(g_{\mathbf{1}})$, we have to take into account only the first three elementary criteria. In particular, observing that $d_{\mathbf{t}}(a_3, b_{11}) = f_{\mathbf{t}}(a_3, b_{11})$ if $f_{\mathbf{t}}(a_3, b_{11}) < 0$ and 0 otherwise, and that $s_{\mathbf{t}}(a_3, b_{11}) = f_{\mathbf{t}}(a_3, b_{11})$ if $f_{\mathbf{t}}(a_3, b_{11}) > 0$ and 0 otherwise, we have that $d_{(1,1)}(a_3, b_{11}) = f_{(1,1)}(a_3, b_{11})$ and $s_{(1,2)}(a_3, b_{11}) = f_{(1,2)}(a_3, b_{11})$[1]. Moreover, since there is not any mutual or antagonistic effect between the

[1] All other values are equal to zero

elementary criteria descending from $g_{\mathbf{1}}$, we obtain that:

- $K_1^{\mathbf{1}}(a,b_{11}) = k^1_{(1,1)} + k^1_{(1,2)} + k^1_{(1,3)} = 0.465 + 35.175 + 21.595 = 57.235$;
- $s^h_{\mathbf{1}}(a_3,b_{11}) = \frac{k^1_{(1,2)}\cdot f_{(1,2)}(a_3,b_{11})}{K_1^1(a,b_{11})} = \frac{35.175\cdot 0.2}{57.235} = 0.123$;
- $d_{\mathbf{1}}(a_3,b_{11}) = \big(1 + d_{(1,1)}(a_3,b_{11})\big) - 1 = d_{(1,1)}(a_3,b_{11}) = -0.4$;
- $\delta_{\mathbf{1}}(a_3,b_{11}) = s^h_{\mathbf{1}}(a_3,b_{11})\,(1 + d_{\mathbf{1}}(a_3,b_{11})) = 0.123 \cdot (1-0.4) = 0.074$.

The other values in Table 6 are computed analogously.

3. *Assignment procedure:* For each non-elementary criterion $g_{\mathbf{r}}$, and for each category C_h, a likeness threshold $\lambda^h_{\mathbf{r}}$ has to be defined. In this case, we are assuming that the likeness thresholds are the same for each $g_{\mathbf{r}}$ and these values are shown in Table 7.

Table 7: Likeness threshold for the four categories

	$h=1$	$h=2$	$h=3$	$h=4$
$\lambda^h_{\mathbf{r}}$	0.75	0.6	0.65	0.6

Comparing the partial likeness degree $\delta_{\mathbf{r}}(a_3,\cdot)$ with the corresponding likeness threshold $\lambda^h_{\mathbf{r}}$ for each non-elementary criterion $g_{\mathbf{r}}$, soldier a_3 can be assigned to the categories shown in Table 8.

Table 8: Assignments of a_3 on each non-elementary criterion

	$\delta_{\mathbf{1}}(a_3,\cdot)$	$\lambda^h_{\mathbf{1}}$		$\delta_{\mathbf{2}}(a_3,\cdot)$	$\lambda^h_{\mathbf{2}}$		$\delta_{\mathbf{3}}(a_3,\cdot)$	$\lambda^h_{\mathbf{3}}$		$\delta_{\mathbf{0}}(a_3,\cdot)$	$\lambda^h_{\mathbf{0}}$	
b_{11}	0.074	0.75		0.796	0.75	✓	0.482	0.75		0.203	0.75	
b_{21}	0.714	0.6	✓	0.543	0.6		0.078	0.6		0.528	0.6	
b_{31}	0	0.65		0	0.65		0	0.65		0	0.65	
b_{41}	1	0.6	✓	0.562	0.6		0.595	0.6		0.743	0.6	✓

To conclude this part, we shall briefly comment the assignments in Table 8, underlying the importance of taking into account the MCHP. Applying the traditional Cat-SD method to the considered case study, we had obtained the assignments shown in the last column of Table 8 and, therefore, that a_3 can be assigned only to the heavy weapon operator category (C_4). One could expect that all considered elementary criteria and, consequently, all the three macro-criteria could provide the same recommendation but it was not the case. Indeed, a_3 can be assigned to the same category only on criterion $g_{\mathbf{1}}$, that is MS. At the same time, on the same macro-criterion he could be assigned also to C_2 since its likeness degree is greater than the corresponding likeness threshold. Completely different recommendations can be gathered on $g_{\mathbf{2}}$ (MR) and $g_{\mathbf{3}}$ (PoF). In fact, on MR, a_3 can be assigned to C_1, while on PoF the competencies are not enough for being assigned to any of the considered categories.

*7.2. Application of the SMAA to the h*Cat-SD *method*

Considering the likeness thresholds for each category shown in Table 7[2], we applied the hCat-SD method for each sampled set of compatible parameters. Therefore, we were able to compute the probability of classification of each soldier to the considered categories reported in Table 9.

[2]Let us suppose that they are not dependent on the considered criterion

Table 9: Probability of assignments expressed in percentage

(a) Comprehensive level

Soldier	C_1	C_2	C_3	C_4	$\{C_2, C_4\}$	C_5
a_1	0	0	0	0	0	100
a_2	0	0	100	0	0	0
a_3	0	0.959	0	100	0.959	0
a_4	0	0	0	0	0	100
a_5	0	84.627	0	0.227	0.205	15.351
a_6	0	0	0	0	0	100
a_7	0	0	0	0.283	0	99.717

(b) Mental Sharpness (MS)

Soldier	C_1	C_2	C_3	C_4	$\{C_1, C_2\}$	$\{C_1, C_3\}$	$\{C_2, C_4\}$	$\{C_1, C_2, C_3\}$	C_5
a_1	100	0	71.452	0	0	71.452	0	0	0
a_2	29.846	0	23.877	0	0	8.608	0	0	54.885
a_3	0	100	0	100	0	0	100	0	0
a_4	0	0	58.452	0	0	0	0	0	41.548
a_5	0	79.443	0	100	0	0	79.443	0	0
a_6	100	100	10.219	0	89.781	0	0	10.219	0
a_7	100	100	0	0	100	0	0	0	0

(c) Mental Resilience (MR)

Soldier	C_1	C_2	C_3	C_4	$\{C_1, C_2\}$	$\{C_1, C_4\}$	$\{C_2, C_4\}$	$\{C_1, C_2, C_4\}$	$\{C_2, C_3, C_4\}$	C_5
a_1	0	0	0	0	0	0	0	0	0	100
a_2	0	100	100	100	0	0	0	0	100	0
a_3	24.574	11.405	0	37.397	1.258	7.595	4.508	0.926	0	41.837
a_4	0	0	0	0	0	0	0	0	0	100
a_5	0	100	0	100	0	0	100	0	0	0
a_6	78.623	0	0	0	0	0	0	0	0	21.377
a_7	100	0	0	0	0	0	0	0	0	0

(d) Physical and other Features (PoF)

Soldier	C_1	C_2	C_3	C_4	$\{C_2, C_4\}$	C_5
a_1	0	0	0	0	0	100
a_2	0	0	100	0	0	0
a_3	0	0	0	23.101	0	76.899
a_4	0	0	0	0	0	100
a_5	0	6.442	0	0	0	93.558
a_6	0	0	0	0	0	100
a_7	0	37.565	0	100	37.565	0

Looking at Tables 9(a)-9(d), one can observe the following:

- At comprehensive level (Table 9(a)), four out of the seven candidates, that are a_1, a_4, a_6 and a_7, are almost always assigned to C_5 representing a dummy category containing all actions that are not assigned to any other category. Regarding the remaining three candidates, a_2 is assigned with certainty to category C_3; a_3 is always assigned to category C_4 and with a probability very close to 1% (0.959%) to category C_2; a_5 is assigned to C_2 with a great probability (84.627%), while with a really marginal probability she/he can be assigned to C_4 (0.227%) or to categories C_2 and C_4 simultaneously (0.205%);
- With respect to MS, a great uncertainty can be observed. a_1 is always assigned to category

C_1 but very often (71.452%), he is assigned to C_3 too; a_2 is assigned to C_1 and C_3 with similar probability (29.846% and 23.877%, respectively) but in most of the cases it is not assigned to any category (54.885%); a_3 is assigned with certainty to both categories C_2 and C_4; a_3 is assigned in a bit more than half of the cases (58.452%) to category C_3 and in the remaining cases it is not assigned to any category; a_5 is always assigned to C_4 and very often to C_2 (79.443%); a_6 is assigned to categories $\{C_1, C_2, C_3\}$ in 10.219% of the cases and with certainty to categories C_1 and C_2; a_7 is always assigned to categories C_1 and C_2;

- On MR, a_1 and a_4 are never assigned to any category; a_2 is always assigned to categories C_2, C_3 and C_4 simultaneously; a_3 is assigned to categories C_4, C_1 and C_2 with probability of 37.397%, 24.574% and 11.405% respectively; a_5 is always assigned to categories C_2 and C_4 simultaneously; a_6 is assigned C_1 with probability of 78.623%, a_7 is always assigned to category C_1;

- Considering PoF, a_1, a_4, a_5 and a_6 are almost always assigned to any category; a_2 is assigned to C_3 with certainty; a_3 is assigned to C_4 in few cases (23.101%), while in most of the cases it not assigned to any category; a_7 is always assigned to C_4 and sometimes assigned to C_2 and C_4 simultaneously.

Let us observe that looking at Tables 9(a)-9(d), one can appreciate the usefulness of the hCat-SD method. Indeed, considering all criteria simultaneously, one can only conclude that, in all cases, a_6 is never assigned to any of the considered categories. Anyway, this statement remains unchanged in considering PoF, but it is completely different analyzing the assignments on the other macro-criteria. Indeed, as previously observed, on one hand, considering MS, a_6 is always assigned to categories C_1 and C_2, and, in some cases, to category C_3 too. On the other hand, considering criterion MR, a_6 is assigned to C_1 with a probability of 78.623%, while only in 21.377% of the cases to category C_5 (non-assigned candidates). This gives the possibility to the DM to analyze more in detail the decision problem at hand.

7.3. Robust assignments

To conclude this section, we shall show how the classification procedure described in Section 6 can be applied to this problem. As already explained, some classification requirements have to be taken into account. In this case, the requirements used to perform the final classification are the following:

Requirement (1) At least one soldier should be assigned to each category;

Requirement (2) At most two soldiers should be assigned to each category;

Requirement (3) At most 2 soldiers should not be assigned.

As second step, positive ρ^+ and negative ρ^- rules are defined and listed in order of their priority. In this case, we defined some rules taking into account the importance of criteria in assigning the soldiers to the different categories. To this aim, we computed the barycenter of the vectors of weights sampled to apply the SMAA methodology and we ordered the criteria from the most to the least important one sequentially as shown in Table 10.

From Table 10 we can conclude that, on average, $\mathbf{g_1}$ (MS) is the most important criterion in assigning soldiers to category C_1; $\mathbf{g_2}$ (MR) is the most important criterion in assigning soldiers

Table 10: Barycenter of the vector of weights sampled and used in applying the SMAA methodology. Numbers labeled by $*$ are used to highlight the most important criterion

	g_1	g_2	g_3
$k_{\mathbf{r}}^1$	53.78*	24.92	15.29
$k_{\mathbf{r}}^2$	23.79	46.60*	23.59
$k_{\mathbf{r}}^3$	10.33	34.82	49.06*
$k_{\mathbf{r}}^4$	21.95	39.57*	32.46

to categories C_2 and C_4; finally, $g_{\mathbf{3}}$ (PoF) is the most important criterion in assigning soldiers to category C_4.

The considered rules are the following:

(ρ_1^+) If $b_{\mathbf{0}}^h(a) \geqslant 90\%$, then a is assigned to category C_h;

(ρ_2^+) If $b_{\mathbf{r}}^h(a) \geqslant 90\%$, where $g_{\mathbf{r}}$ is the most important non-elementary criterion in assigning soldiers to C_h, then a is assigned to category C_h;

(ρ_1^-) If $b_{\mathbf{0}}^h(a) < 70\%$, then a is removed from category C_h, for each criterion $g_{\mathbf{r}}$;

(ρ_2^-) If $b_{\mathbf{r}}^h(a) < 70\%$, where $g_{\mathbf{r}}$ is the most important criterion in assigning soldiers to C_h, then a is removed from category C_h, for each criterion $g_{\mathbf{r}}$.

The assignment procedure is therefore composed of the following steps (at the end of each step we shall include among parenthesis the new obtained classifications; writing $a \to C$, we mean that a is assigned to C):

Step 1: Considering rule ρ_1^+ and looking at Table 9(a), soldiers a_2 and a_3 are assigned to categories C_2 and C_4, respectively, since $b_{\mathbf{0}}^3(a_2) = b_{\mathbf{0}}^4(a_2) = 100\% \geqslant 90\%$ ($a_2 \to C_3$; $a_3 \to C_4$); requirements 1 and 3 are not fulfilled;

Step 2 To assign a soldier to C_1, the most important criterion is $g_{\mathbf{1}}$. Then, looking at Table 9(b), a_1, a_6 and a_7 should be assigned to C_1 since $b_{\mathbf{1}}^1(a_1) = b_{\mathbf{1}}^1(a_6) = b_{\mathbf{1}}^1(a_7) = 100\% \geqslant 90\%$ ($a_1, a_6, a_7 \to C_1$); no requirement is fulfilled;

Step 3 To assign a soldier to C_2, the most important criterion is $g_{\mathbf{2}}$. Then, looking at Table 9(c), a_2 and a_5 should be classified to C_2 since $b_{\mathbf{2}}^2(a_2) = b_{\mathbf{2}}^2(a_5) = 100\% \geqslant 90\%$. Anyway, a_2 was already assigned to category C_3 in the first step, while a_5 can now be assigned to C_2 ($a_5 \to C_2$); the second requirement is not fulfilled;

Step 4 To assign a soldier to C_3, the most important criterion is $g_{\mathbf{3}}$. Then, looking at Table 9(d), we get, again, that a_2 should be assigned to C_3 ($b_{\mathbf{3}}^3(a_2) = 100\% \geqslant 90\%$) as already done in Step 1; the only fulfilled requirement, again, is the third one;

Step 5 To assign a soldier to C_4, the most important criterion is $g_{\mathbf{2}}$. Then, looking at Table 9(c), we can observe that a_2 and a_5 should be assigned to C_4 since $b_{\mathbf{2}}^4(a_2) = b_{\mathbf{2}}^4(a_5) = 100\% \geqslant 90\%$. Anyway, both of them have been already assigned before.

Let us observe that after the application of all positive rules, requirement 2 is not yet fulfilled since 3 actions have been assigned to category C_1. Therefore, at least one of them, should be removed. For this reason, we apply negative rules.

Step 6 Following rule ρ_1^-, the soldiers a_1, a_6 and a_7 are removed from category C_1. At this point, the only satisfied requirement is the second, since category C_1 is now empty and 4 soldiers (a_1, a_4, a_6, a_7) are not assigned. Rule ρ_2^- is not applied, since the requirements are related to the adding of soldiers to class C_1.

Step 7 Since all the rules have been applied, we have to consider some *ad-hoc* rule:

- We add the soldiers presenting the highest probability to be assigned to C_1 with respect to the most important criterion for that category, that is MS. Looking at Table 9(b), soldiers a_1, a_6 and a_7 are again included in C_1 $(a_1, a_6, a_7 \to C_1)$; the second requirement is not yet fulfilled;
- Among a_1, a_6 and a_7, we remove the soldiers presenting the lowest probability to be assigned to C_1 according to the second most important criterion, that is MR. In this way, a_1 is removed. Therefore, all the requirements are now fulfilled.

At the end, the final classifications are the following: $a_1 \to C_5$; $a_2 \to C_3$; $a_3 \to C_4$; $a_4 \to C_5$; $a_5 \to C_2$; $a_6 \to C_1$; $a_7 \to C_1$.

8. Conclusions

In this paper, we proposed a comprehensive method extending a recently proposed nominal classification, the Cat-SD method. Firstly, we applied MCHP to the Cat-SD method. Thus, we have introduced the hierarchical Cat-SD, hCat-SD. The hierarchical decomposition of a complex multiple criteria nominal classification problem is then possible when applying Cat-SD. Secondly, interactions and antagonistic effects between criteria structured in a hierarchical way were handled in our method. Then, to get the values of the criteria weights as well as the interactions and antagonistic effects used in the hCat-SD method, we adopted the hierarchical and imprecise SRF method. We applied SMAA to the hCat-SD method with the aim of obtaining the probablity with which an action is assigned to a category (or categories) at a comprehensive level and at a macro-criterion level. Finally, we proposed a procedure that starting from the probabilistic assignments obtained by SMAA provides a final classification that fulfills some requirements given by the DM. Putting together all these aspects, we therefore built the SMAA-hCat-SD method. We presented a numerical example to illustrate the application of SMAA-hCat-SD.

The proposed method gives to the DM the possibility:

– To structure the set of criteria in a hierarchical way (logical subsets of criteria can be created in the hierarchy);

– To provide imprecise information for obtaining the criteria weights by using the imprecise SRF method;

– To analyze, for several sets of compatible parameters, the probability of the assignment results provided by the Cat-SD, considering all criteria or one macro-criterion only;

– To obtain a final assignment that takes into account robustness concerns as represented by the probabilistic classification provided by SMAA.

Several advantages can be underlined with respect to the application of the proposed method. The main features can be stated as follows:

1. In situations in which the DM has to handle a great number of criteria to assess actions, adopting hCat-SD is a more adequate approach than applying Cat-SD considering all criteria at the same level;
2. For the elicitation of the criteria weights, and interaction and antagonistic coefficients, it is easier for the DM thinking about a small number of related criteria than a large number;
3. Besides the possibility of eliciting criteria weights for subsets of criteria, our method gives to the DM the possibility to provide imprecise information during the process of determining them;
4. Applying SMAA to the hCat-SD, the DM can better understand the decision problem at hand exploring the problem more in deep.

To sum up, in this work, we have considered robustness concerns by taking into account the set of all weights, and interaction and antagonistic coefficients compatible with preference information provided by the DM, while taking advantage of the hierarchical structure of criteria. Let us remark that the procedure permitting to pass from the probabilistic classification provided by SMAA to the final assignment can be applied to other probabilistic versions of classification methods, also ordinal, such as Electre Tri and its variants.

Future research can rely on applying the SMAA-hCat-SD method to real-world nominal classification problems. Extending the method to group decision making is also an interesting direction of research. It could also be interesting to study procedures for aiding the elicitation of preference information given by the DM to reduce the cognitive effort required during the decision aiding process.

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