ON C*-COMPLETIONS OF DISCRETE QUANTUM GROUP RINGS

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Abstract. We discuss just infiniteness of C*-algebras associated to discrete quantum groups and relate it to the C*-uniqueness of the quantum groups in question, i.e. to the uniqueness of a C*-completion of the underlying Hopf *-algebra. It is shown that duals of q-deformations of simply connected semisimple compact Lie groups are never C*-unique. On the other hand we present an example of a discrete quantum group which is not locally finite and yet is C*-unique.

1. Introduction

The following definition, modelled on the notion of just infiniteness for groups, was introduced by R. Grigorchuk, M. Musat and M. Rørdam in the recent paper [GMR]: a C*-algebra A is called *just infinite* if all its proper quotients are finite-dimensional. Just infinite C*-algebra were classified in [GMR] in terms of their ideal spaces. For group C*-algebras it was shown that this notion is closely related to the existence of 'nontrivial' C*-completions of group rings (see Corollary 2.8 below). In particular it was observed in [GMR] that the only obvious obstruction for the existence of different C*-completions of a group ring is local finiteness of the underlying group. This naturally raises a question whether for any discrete group Γ which is not locally finite one can construct different C*-completions of $\mathbb{C}[\Gamma]$; if the latter holds we say for short that Γ is not C*-unique. The preprint [AK] takes some steps towards verifying this conjecture, producing different completions of group rings for various classes of discrete groups, such as infinite groups of polynomial growth or groups with a central element of infinite order.

Our note considers analogous questions in the setup of discrete *quantum* groups. If $\mathbb{\Gamma}$ is a discrete quantum group, one can still consider the associated quantum group ring $\mathbb{C}[\mathbb{\Gamma}]$ and its reduced and universal C*-completions $C^*_r(\mathbb{\Gamma})$ and $C^*(\mathbb{\Gamma})$. Again one can ask about just infiniteness of these C*-algebras; as in the classical case it is not difficult to see that these are connected first to amenability of $\mathbb{\Gamma}$, and then to the uniqueness of C*-completions of $\mathbb{C}[\mathbb{\Gamma}]$. This naturally raises the question which discrete quantum groups are C*-unique and whether the equivalence of C*-uniqueness and local finiteness has a chance to be true in the quantum realm (the backward implication obviously holds). We show by an explicit construction that the duals of so-called q-deformations of classical compact Lie groups, such as the Woronowicz's $SU_q(2)$, are never C*-unique. On the other hand we answer the above question in the negative, by showing that a certain crossed product construction, combining $SU_q(2)$ and the noncommutative torus, yields a discrete quantum group which is not locally finite and yet is C*-unique.

The plan of the paper is as follows: in Section 2 we recall some quantum group terminology, establish equivalence of amenability of a discrete quantum group $\mathbb{\Gamma}$ with $C^*_r(\mathbb{\Gamma})$ admitting a finite dimensional representation, discuss some basic facts related to just infiniteness of quantum group operator algebras and note that locally finite discrete quantum groups are C*-unique. In Section

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3 we show that the duals of q-deformations are never C*-unique. Finally in Section 4 we produce an example of a C*-unique discrete quantum group which is not locally finite.

All *-algebras we study will be unital, and by a representation of a *-algebra A we will understand a unital *-homomorphism $\pi : A \to B(\mathsf{H})$, where H is a Hilbert space. We write $\mathbb{N}_0$ for $\mathbb{N}\cup\{0\}$. Scalar products will be linear on the right. When we talk about a C*-norm on a *-algebra we are strictly speaking of a pre-C*-norm.

2. Discrete quantum groups and just infiniteness of their group C*-algebras

Throughout the paper $\mathbb{G}$ will denote a compact quantum group in the sense of Woronowicz, and $\mathbb{\Gamma} := \widehat{\mathbb{G}}$ will be the discrete quantum group dual to $\mathbb{G}$. For precise definitions and all the related terminology we refer (for example) to Section 2 of [DKSS]; we will follow the conventions of that paper. We will be mainly interested in the quantum group ring $\mathbb{C}[\mathbb{\Gamma}]$ (in other words the Hopf *-algebra $\mathrm{Pol}(\mathbb{G})$), and its reduced and universal completions $\mathrm{C}^*_r(\mathbb{\Gamma})$ and $\mathrm{C}^*(\mathbb{\Gamma})$ (in other words $\mathrm{C}(\mathbb{G})$ and $\mathrm{C}^u(\mathbb{G})$), with the latter being the universal enveloping C*-algebra of $\mathbb{C}[\mathbb{\Gamma}]$. The reduced algebra $\mathrm{C}^*_r(\mathbb{\Gamma})$ acts on the Hilbert space $\ell^2(\mathbb{\Gamma})$, as does the 'algebra of bounded functions on $\mathbb{\Gamma}$', the von Neumann algebra $\ell^\infty(\mathbb{\Gamma})$. The predual of the latter is denoted $\ell^1(\mathbb{\Gamma})$.

Recall that

$$\ell^\infty(\mathbb{\Gamma}) = \prod_{\alpha\in\mathrm{Irr}_{\mathbb{G}}} M_{n_\alpha},$$

where $\mathrm{Irr}_{\mathbb{G}}$ denotes the set of equivalence classes of irreducible unitary representations of $\mathbb{G}$. For each $\alpha \in \mathrm{Irr}_{\mathbb{G}}$ we choose a representative, i.e. a unitary matrix $U^\alpha = (u^\alpha_{i,j})_{i,j=1}^{n_\alpha} \in M_{n_\alpha}(\mathbb{C}[\mathbb{\Gamma}])$. The matrix units in $M_{n_\alpha} \subset \ell^\infty(\mathbb{\Gamma})$ will be denoted by $e^\alpha_{i,j}$.

The multiplicative unitary of $\mathbb{\Gamma}$ is the unitary $W \in B(\ell^2(\mathbb{\Gamma}) \otimes \ell^2(\mathbb{\Gamma}))$ given by the formula:

$$W = \sum_{\alpha\in\mathrm{Irr}_{\mathbb{G}}} \sum_{i,j=1}^{n_\alpha} e^\alpha_{j,i} \otimes (u^\alpha_{i,j})^*$$

The coproduct of $\mathbb{\Gamma}$, a coassociative normal unital *-homomorphism $\Delta : \ell^\infty(\mathbb{\Gamma}) \to \ell^\infty(\mathbb{\Gamma})\overline{\otimes}\ell^\infty(\mathbb{\Gamma})$, is implemented by W via the following formula:

$$\Delta(x) = W^*(1 \otimes x)W, \quad x \in \ell^\infty(\mathbb{\Gamma}). \tag{2.1}$$

Given a functional $\phi \in \ell^1(\mathbb{\Gamma})$ we define the (normal, bounded) maps $L_\phi : \ell^\infty(\mathbb{\Gamma}) \to \ell^\infty(\mathbb{\Gamma})$ and $R_\phi : \ell^\infty(\mathbb{\Gamma}) \to \ell^\infty(\mathbb{\Gamma})$ via the formulas

$$L_\phi = (\phi \otimes \mathrm{id}) \circ \Delta, \quad R_\phi = (\mathrm{id} \otimes \phi) \circ \Delta.$$

A discrete quantum group $\mathbb{\Gamma}$ is called *amenable* if it admits a bi-invariant mean, i.e. a state $m \in \ell^\infty(\mathbb{\Gamma})^*$, such that for all $\phi \in \ell^1(\mathbb{\Gamma})$ there is

$$m \circ L_\phi = m \circ R_\phi = \phi(1)m.$$

By [DQV] a discrete quantum group $\mathbb{\Gamma}$ is amenable if it admits a left invariant mean $m \in \ell^\infty(\mathbb{\Gamma})^*$: a state such that for each $\phi \in \ell^1(\mathbb{\Gamma})$ there is $m \circ L_\phi = \phi(1)m$. In fact it suffices to check the last formula for the functionals of the form $\widehat{e^\alpha_{i,j}}$, $\alpha \in \mathrm{Irr}_{\mathbb{G}}$, $i, j = 1, \ldots, n_\alpha$, as the latter are linearly dense in $\ell^1(\mathbb{\Gamma})$, and the map $\phi \mapsto L_\phi$ is a (complete) isometry. Thus we will need the following explicit form of the map L_ϕ for $\phi = \widehat{e^\alpha_{i,j}}$:

$$L_\phi(x) = \sum_{p=1}^{n_\alpha} u^\alpha_{i,p} x (u^\alpha_{j,p})^*, \qquad x \in l^\infty(\mathbb{\Gamma}). \tag{2.2}$$

A pre-C*-norm on $\mathbb{C}[\mathbb{\Gamma}]$ is a norm such that the closure of $\mathbb{C}[\mathbb{\Gamma}]$ is a C*-algebra.

Definition 2.1. We call a discrete quantum group $\mathbb{\Gamma}$ C*-unique if $\mathbb{C}[\mathbb{\Gamma}]$ (i.e. $\mathrm{Pol}(\widehat{\mathbb{\Gamma}})$) has a unique pre-C*-norm. We call $\mathbb{\Gamma}$ C^*_r-unique if there is no pre-C*-norm on $\mathbb{C}[\mathbb{\Gamma}]$ that is properly majorized by the norm of $\mathrm{C}^*_r(\mathbb{\Gamma})$.

We have the canonical quotient map $\Lambda : \mathrm{C}^*(\mathbb{\Gamma}) \to \mathrm{C}^*_r(\mathbb{\Gamma})$. The following is a combination of results in [BMT] and [Tom].

Theorem 2.2. *Let $\mathbb{\Gamma}$ be a discrete quantum group. The following conditions are equivalent:*

(i) *$\mathbb{\Gamma}$ is amenable;*
(ii) *the quotient map $\Lambda : \mathrm{C}^*(\mathbb{\Gamma}) \to \mathrm{C}^*_r(\mathbb{\Gamma})$ is an isomorphism;*
(iii) *the algebra $\mathrm{C}^*_r(\mathbb{\Gamma})$ admits a character.*

We will need the following lemma, due to Biswarup Das, Matt Daws and Pekka Salmi; the proof is based on the idea of the quantum Bohr compactification from [Sol].

Lemma 2.3 (Das, Daws and Salmi)**.** *Suppose that $\mathbb{\Gamma}$ is a discrete quantum group, $n \in \mathbb{N}$ and $\pi : \mathrm{C}[\mathbb{\Gamma}] \to M_n$ is a representation. Then π is invariant under the scaling group: $\pi = \pi \circ \tau_t$, $t \in \mathbb{R}$.*

The above lemma has a simple corollary, using the properties of the unitary and the usual antipode of $\mathbb{C}[\Gamma]$.

Corollary 2.4. *Suppose that $\mathbb{\Gamma}$ is a discrete quantum group, $n \in \mathbb{N}$, $\pi : \mathrm{C}[\mathbb{\Gamma}] \to M_n$ is a representation, and $U = (u_{ij})_{i,j=1}^k \in M_k(\mathbb{C}[\Gamma])$ is a finite-dimensional unitary representation of $\widehat{\mathbb{\Gamma}}$. Then we have the following equality ($i, j = 1, \ldots, k$):*

$$\pi\left(\sum_{p=1}^k u_{j,p}^* u_{i,p}\right) = \delta_{ij} 1_{M_n}.$$

Proof. By Lemma 2.3 the representation π is invariant under the scaling group action. This implies that if we restrict π to $\mathbb{C}[\mathbb{\Gamma}]$ then we have $\pi \circ R = \pi \circ S$, where R and S denote respectively the unitary and the usual antipode of $\mathbb{C}[\mathbb{\Gamma}]$. We then use the fact that $R^2 = \mathrm{id}$ and both antipodes commute to calculate (say for $i, j, p = 1, \ldots, k$)

$$\pi(u_{i,p}) = \pi \circ R^2(u_{i,p}) = \pi \circ S \circ R(u_{i,p}) = \pi \circ R \circ S(u_{i,p}) = \pi \circ R(u_{p,i}^*) = \pi \circ S(u_{p,i}^*),$$

so that

$$\pi(u_{j,p}^* u_{i,p}) = \pi(u_{j,p}^*)\pi(u_{i,p}) = \pi \circ S(u_{p,j})\pi \circ S(u_{p,i}^*) = \pi(S(u_{p,j})S(u_{p,i}^*)) = \pi(S(u_{p,i}^* u_{p,j}))$$

and finally

$$\pi(\sum_{p=1}^k u_{j,p}^* u_{i,p}) = \pi \circ S(\sum_{p=1}^k u_{p,i}^* u_{p,j}) = \pi \circ S(\delta_{ij} 1_{\mathbb{C}[\mathbb{\Gamma}]}) = \delta_{ij} 1_{M_n},$$

where we used the unitarity of U and the fact that the antipode is unital. □

The following result is related to $\mathrm{C}^*_r(\mathbb{\Gamma})$ being just infinite; it extends Theorem 2.8 of [BMT], i.e. the implication (iii)$\Longrightarrow$(i) of Theorem 2.2. Note that the proof of that theorem in [BMT] does not extend to the matrix case considered below; we use rather the idea of an Arveson extension, as in Section 2.6 of [BO].

Proposition 2.5. *A discrete quantum group $\mathbb{\Gamma}$ is amenable if and only if $\mathrm{C}^*_r(\mathbb{\Gamma})$ admits a finite-dimensional representation.*

Proof. The forward implication is well-known (and follows for example from Theorem 2.8 of [BMT] mentioned above).

Suppose then that $\pi : \mathrm{C}^*_r(\mathbb{\Gamma}) \to M_n$ is a representation. By Lemma 2.3 we know that π is invariant under the scaling group. We view $\mathrm{C}^*_r(\mathbb{\Gamma})$ as a subalgebra of $B(\ell^2(\mathbb{\Gamma}))$ and consider a unital completely positive extension of π to the latter algebra, denoted by Φ, so that we have $\Phi : B(\ell^2(\mathbb{\Gamma})) \to M_n$; note that by a standard multiplicative domain argument we know that Φ is a $\mathrm{C}^*_r(\mathbb{\Gamma})$-bimodule map in the obvious sense. We claim that the state $m : \ell^\infty(\mathbb{\Gamma}) \to \mathbb{C}$ defined by the formula $m = \mathrm{tr} \circ \Phi|_{\ell^\infty(\mathbb{\Gamma})}$ is the desired left invariant mean.

Fix then a functional $\phi \in \ell^1(\mathbb{\Gamma})$ of the form $\widehat{e^{\alpha}_{i,j}}$, $\alpha \in \mathrm{Irr}_{\mathbb{G}}$, $i, j = 1, \ldots, n_\alpha$ and $x \in \ell^\infty(\mathbb{\Gamma})$ and compute, using the formula (2.2):

$$
\begin{aligned}
m(L_\phi(x)) &= m\left(\sum_{p=1}^{n_\alpha} u^\alpha_{i,p} x (u^\alpha_{j,p})^*\right) = \sum_{p=1}^{n_\alpha} (\mathrm{tr} \circ \Phi)\left(u^\alpha_{i,p} x (u^\alpha_{j,p})^*\right) = \sum_{p=1}^{n_\alpha} \mathrm{tr}\left(\Phi(u^\alpha_{i,p})\Phi(x)\Phi((u^\alpha_{j,p})^*)\right) \\
&= \sum_{p=1}^{n_\alpha} \mathrm{tr}\left(\Phi(x)\Phi((u^\alpha_{j,p})^*)\Phi(u^\alpha_{i,p})\right) = \sum_{p=1}^{n_\alpha} \mathrm{tr}\left(\Phi(x)\pi((u^\alpha_{j,p})^*)\pi(u^\alpha_{i,p})\right) \\
&= \mathrm{tr}\left(\Phi(x) \sum_{p=1}^{n_\alpha} \pi((u^\alpha_{j,p})^* u^\alpha_{i,p})\right).
\end{aligned}
$$

Using now Corollary 2.4 we get immediately

$$m(L_\phi(x)) = \mathrm{tr}\left(\Phi(x)\delta_{ij} 1_{M_n}\right) = m(x)\delta_{ij} = m(x)\phi(1_{\ell^\infty(\mathbb{\Gamma})}).$$

□

We are ready to present some basic facts concerning the just infiniteness of group C*-algebras of discrete quantum groups, essentially following Section 6 of [GMR] and Section 2 of [AK].

Proposition 2.6. *If $\mathrm{C}^*_r(\mathbb{\Gamma})$ is just infinite then either $\mathbb{\Gamma}$ is C*-simple (i.e. $\mathrm{C}^*_r(\mathbb{\Gamma})$ is simple), or $\mathbb{\Gamma}$ is amenable.*

Proof. Immediate consequence of Proposition 2.5. □

We call a *-algebra $\mathcal{A}$ *-just infinite if any representation of $\mathcal{A}$ on a Hilbert space is either injective or has a finite-dimensional image.

Proposition 2.7. *Let $\mathcal{A}$ be an infinite-dimensional unital *-algebra admitting a maximal C*-norm; denote by A_u the corresponding universal C*-completion. Then A_u is just infinite if and only if $\mathcal{A}$ is *-just infinite and admits a unique C*-completion. Furthermore $\mathcal{A}$ admits a unique C*-completion if and only if every non-trivial (closed, two-sided) ideal of A_u has a non-trivial intersection with $\mathcal{A}$.*

Proof. The first part follows as in Proposition 6.3 of [GMR]; we present the proof below for completeness.

Assume first that A_u is just infinite. Consider a representation $\pi : \mathcal{A} \to B(H)$ and its extension to A_u. The latter is either injective or has finite-dimensional image. But this means that the original π was either injective or had a finite-dimensional image. Hence $\mathcal{A}$ is *-just infinite. Then note that for any pre-C*-norm on $\mathcal{A}$ we have a representation $\pi : A_u \to B(H)$, injective on $\mathcal{A}$, such that the norm in question equals $\|\cdot\|_\pi$. As the image of π is infinite-dimensional, π must be injective, so that $\|\cdot\|_\pi = \|\cdot\|_{A_u}$.

Suppose then that $\mathcal{A}$ is *-just infinite and admits a unique C*-completion. Consider a non-injective representation $\pi : A_u \to B(H)$. Then it is either injective on $\mathcal{A}$, in which case it would give a different C*-completion of A_u, or it is non-injective on $\mathcal{A}$, in which case $\pi(\mathcal{A})$ is finite-dimensional. But then $\pi(A_u)$ is finite-dimensional and we showed that A_u is just infinite.

The last statement follows very easily (see Lemma 2.2 in [AK]). □

The first part of the last result implies immediately the following corollary.

Corollary 2.8. *Let $\mathbb{\Gamma}$ be infinite. Then $\mathrm{C}^*(\mathbb{\Gamma})$ is just infinite if and only if $\mathbb{C}[\mathbb{\Gamma}]$ is *-just infinite and C*-unique. Thus if $\mathrm{C}^*(\mathbb{\Gamma})$ is just infinite then $\mathbb{\Gamma}$ is amenable.*

The group $\mathbb{Z}$ is not C*-unique; in fact, as noted in [GMR] $\mathbb{C}[\mathbb{Z}]$ does not admit a minimal pre-C*-norm. This fact (or rather reasons behind it) can be used to prove the following statement (for classical groups shown in Proposition 2.4 of [AK]).

Proposition 2.9. *Suppose that $\mathbb{\Gamma}$ is a discrete quantum group, containing $\mathbb{Z}$ in its centre (in other words, $\mathbb{C}[\mathbb{Z}]$ is a Hopf *-subalgebra of the centre of $\mathbb{C}[\mathbb{\Gamma}]$). Then $\mathbb{\Gamma}$ is not C^*_r-unique.*

Proof. Consider the Haar state h on $\mathbb{C}[\mathbb{\Gamma}]$. By the uniqueness its restriction to $\mathbb{C}[\mathbb{Z}] = \mathrm{Pol}(\mathbb{T})$ is given by the integration with respect to the Lebesgue measure on the circle, and we have the inclusions $\mathrm{C}^*_r(\mathbb{Z}) \subset Z(\mathrm{C}^*_r(\mathbb{\Gamma}))$ and $\mathrm{VN}(\mathbb{Z}) \subset Z(\mathrm{VN}(\mathbb{\Gamma}))$. Arguing as in Proposition 2.4 of [AK] (see also the following section) we obtain a projection $p \in \mathrm{VN}(\mathbb{Z})$ such that the restricted representation $\pi : \mathrm{C}^*_r(\mathbb{\Gamma}) \to B(p\ell^2(\mathbb{\Gamma}))$ is faithful on $\mathrm{Pol}(\mathbb{T})$ and is not faithful on $\mathrm{C}^*_r(\mathbb{Z})$ (so also not on $\mathrm{C}^*_r(\mathbb{\Gamma})$). It remains to note that it is faithful on $\mathbb{C}[\mathbb{\Gamma}]$. This is however easy: consider the h-preserving conditional expectation $\mathbb{E} : \mathrm{VN}(\mathbb{\Gamma}) \to \mathrm{VN}(\mathbb{Z})$, whose existence follows from the Takesaki's theorem [Tak] and the fact that the latter algebra is obviously contained in the centralizer of h. As $\mathbb{C}[\mathbb{Z}] \subset \mathbb{C}[\mathbb{\Gamma}]$ can be viewed as spanned by certain (one-dimensional) irreducible representations of $\widehat{\mathbb{\Gamma}}$, Woronowicz-Peter-Weyl formulae show that $\mathbb{E}(\mathbb{C}[\mathbb{\Gamma}]) = \mathbb{C}[\mathbb{Z}]$. Then one can conclude the proof of faithfulness of π on $\mathbb{C}[\mathbb{\Gamma}]$ as in Proposition 2.4 of [AK].

The result now follows from the second statement in Proposition 2.7, if we consider the kernel of the representation π.

□

In fact the proof above shows that $\mathbb{\Gamma}$ is even not C^*_r-unique. Theorem 2.2 shows that for amenable discrete quantum groups C^*_r-uniqueness is the same as C*-uniqueness, so we will focus on the latter concept.

Definition 2.10. We call a discrete quantum group $\mathbb{\Gamma}$ locally finite if each finite subset $I \subset\subset \mathrm{Irr}(\widehat{\mathbb{\Gamma}})$ generates a finite fusion ring inside $\mathrm{Irr}(\widehat{\mathbb{\Gamma}})$.

Lemma 2.11. *Consider a discrete quantum group $\mathbb{\Gamma}$. The following are equivalent:*

- *(i) $\mathbb{\Gamma}$ is locally finite;*
- *(ii) every finite subset of $\mathbb{C}[\mathbb{\Gamma}]$ is contained in a finite-dimensional sub Hopf*-algebra of $\mathbb{C}[\mathbb{\Gamma}]$;*
- *(iii) every finite subset of $\mathbb{C}[\mathbb{\Gamma}]$ is contained in a finite-dimensional unital *-subalgebra of $\mathbb{C}[\mathbb{\Gamma}]$.*

Proof. The equivalence of (i) and (ii) is a straightforward consequence of the Woronowicz-Peter-Weyl theory. The implication (ii)$\Longrightarrow$(iii) is trivial. Assume then that (iii) holds. Consider a finite set $F \subset \mathbb{C}[\mathbb{\Gamma}]$. By the fundamental theorem on coalgebras there is a finite-dimensional subalgebra (which we may assume to be unital and self-adjoint) $C \subset \mathbb{C}[\mathbb{\Gamma}]$ containing F. Consider then the algebra generated by C. By (iii) it is finite-dimensional (we can choose a finite linear basis in C); it is easy to check that it is in fact a *-Hopf subalgebra of $\mathbb{C}[\mathbb{\Gamma}]$. □

Locally finite discrete quantum groups are C*-unique for essentially trivial reasons.

Proposition 2.12. *If a discrete quantum group $\mathbb{\Gamma}$ is locally finite, then $\mathbb{C}[\mathbb{\Gamma}]$ is* C**-unique.*

Proof. The easy proof follows exactly as in Proposition 6.7 of [GMR], using part (iii) of the equivalence in the lemma above. □

In Section 4 we will exhibit an example showing that the converse implication does not hold.

3. Non-unique completions for q-deformations

In this section we prove that the quantum groups that arise as q-deformations of simply connected semisimple compact Lie groups never have a unique C*-closure of their polynomial algebra. We prove this explicitly for $SU_q(2)$ and then treat the general case. Recall that duals of all such q-deformations are amenable, as was first shown in [Ban] and then reproved via methods related to Theorem 2.2 in the Appendix of [FST]. In particular they are C*-unique if and only if they are C$^*_r$-unique.

Let $\mathbb{G}_q = SU_q(2), q \in (-1,1)\backslash\{0\}$. Algebraically $\mathrm{Pol}(\mathbb{G}_q)$ is defined as the $*$-algebra generated by operators α and γ satisfying the relations,

$$\gamma^*\gamma = \gamma\gamma^*, \qquad q\gamma^*\alpha = \alpha\gamma^*, \qquad q\gamma\alpha = \alpha\gamma, \qquad \alpha^*\alpha + \gamma^*\gamma = 1, \qquad \alpha\alpha^* + q^2\gamma\gamma^* = 1, \tag{3.1}$$

with comultiplication extending

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma \qquad \text{and} \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma. \tag{3.2}$$

As before, we write $\mathrm{C}(\mathbb{G}_q)$ for the C*-closure under the GNS-representation of the Haar state (equivalently, the universal closure, as $\widehat{\mathbb{G}_q}$ is amenable). It is isomorphic to the C*-algebra generated by the concrete operators acting on $L_2(\{0\} \cup \bigcup_{k=0}^\infty q^k\mathbb{T})$:

$$(\widetilde{\alpha}f)(z) = \sqrt{1-|z|^2}f(q^{-1}z), \qquad (\widetilde{\gamma}f)(z) = zf(z), \qquad z \in \mathbb{T}.$$

Here we take the convention that $f(q^{-1}z) = 0, z \in \mathbb{T}$. So $\widetilde{\alpha}$ and $\widetilde{\gamma}$ are the GNS-images of α and γ and the $*$-algebra generated by $\widetilde{\alpha}$ and $\widetilde{\gamma}$ is isomorphic to $\mathrm{Pol}(\mathbb{G}_q)$. Write $\mathbb{T}_\uparrow = \{z \in \mathbb{T} \mid \Im(z) \geq 0\}$ and $\mathbb{T}_\downarrow = \mathbb{T}\backslash\mathbb{T}_\downarrow$ for the upper half circle and (open) lower half circle respectively.

Theorem 3.1. *The discrete quantum group $\widehat{\mathbb{G}_q}$ is not* C**-unique.*

Proof. Consider $A = \mathrm{C}^*\langle\widetilde{\gamma}\rangle$ be the (commutative) unital C*-subalgebra of $\mathrm{C}(\mathbb{G}_q)$ generated by the normal element $\widetilde{\gamma}$. The spectrum of $\widetilde{\gamma}$ is $\{0\} \cup \bigcup_{k=0}^\infty q^k\mathbb{T}$, so that we can identify A with $C(\{0\} \cup \bigcup_{k=0}^\infty q^k\mathbb{T})$. Let then $f \in C(\{0\} \cup \bigcup_{k=0}^\infty q^k\mathbb{T})$ be a non-zero function supported on $\bigcup_{k=0}^\infty q^k\mathbb{T}_\uparrow$, and denote by x the corresponding element of A. Let $\pi : A \to B(L_2(\bigcup_{k=0}^\infty q^k\mathbb{T}_\downarrow))$ be the representation of A that is given simply by pointwise multiplication.

We define a representation ρ of $\mathrm{Pol}(\mathbb{G}_q)$ on $L_2(\bigcup_{k=0}^\infty q^k\mathbb{T}_\downarrow)$ as follows. Set

$$\rho(\gamma)(f)(z) = zf(z), \quad \rho(\alpha)(f)(z) = \sqrt{1-|z|^2}f(q^{-1}z), \qquad z \in \bigcup_{k=0}^\infty q^k\mathbb{T}_\downarrow, \quad f \in L_2(\bigcup_{k=0}^\infty q^k\mathbb{T}_\downarrow).$$

So $\rho(\gamma)$ and $\rho(\alpha)$ are the restrictions of $\widetilde{\gamma}$ and $\widetilde{\alpha}$ to $L_2(\bigcup_{k=0}^\infty q^k\mathbb{T}_\downarrow)$. Therefore these prescriptions for ρ define a representation of $\mathrm{Pol}(\mathbb{G}_q)$ on $L_2(\bigcup_{k=0}^\infty q^k\mathbb{T}_\downarrow)$. The representation ρ extends to a representation of $\mathrm{C}(\mathbb{G}_q)$ and we have $\rho(x) = 0$.

It remains to show that ρ is injective on $\mathrm{Pol}(\mathbb{G}_q)$. In order to do this, note that by the defining relations of $\mathbb{G}_q$ a linear basis of $\mathrm{Pol}(\mathbb{G}_q)$ is given by

$$\alpha^k\gamma^m(\gamma^*)^n, \qquad k,m,n \in \mathbb{N}_0.$$

Take a finite linear combination $y = \sum_{k,m,n} c_{k,m,n}\alpha^k\gamma^m(\gamma^*)^n$. Suppose that $\rho(y) = 0$. Then the explicit description of ρ shows that for all k we must have $\rho(\sum_{m,n} c_{k,m,n}\alpha^k\gamma^m(\gamma^*)^n) = 0$ which is equivalent to the property that for all k we have that $\rho(\sum_{m,n} c_{k,m,n}\gamma^m(\gamma^*)^n) = 0$. As the functions $z \mapsto z^m\overline{z}^n$ are linearly independent in $C(\bigcup_{k=0}^{\infty} q^k\mathbb{T}_\downarrow)$ we see that this can only happen only if for all k, m and n we have $c_{k,m,n} = 0$.

So ρ is an injective representation of $\mathrm{Pol}(\mathbb{G}_q)$ and the associated C*-norm is properly majorized by the C*-norm of $\mathrm{C}(\mathbb{G}_q)$ as $\rho(x) = 0$ but $x \neq 0$. □

Next we generalize this result. Let now $q \in (0, 1)$, let K be a simply connected semisimple compact Lie group and let $\mathbb{K}_q$ denote the q-deformation of K in the sense of [KS], with $\mathrm{Pol}(\mathbb{K}_q)$ the corresponding polynomial algebra. In [KS, Theorem 6.2.7] the irreducible representations of $\mathrm{Pol}(\mathbb{K}_q)$ are classified.

Theorem 3.2 (Theorem 6.2.7 of [KS])**.** *We have the following:*

1. *There is group W generated by a set S, called the Weyl group, and a maximal torus $\mathbb{T}^d$ in $\mathbb{K}_q$ such that for every $w \in W$ with reduced expression $w = s_1 \cdots s_n$, $s_i \in S$, and $\tau \in \mathbb{T}^d$ there is a representation*
$$\pi_{w,\tau} := (\pi_{s_1} \otimes \cdots \otimes \pi_{s_n} \otimes \pi_\tau) \circ \Delta_{(n+1)},$$
of $\mathrm{Pol}(\mathbb{K}_q)$ on the Hilbert space $H_w := \ell^2(\mathbb{N}_0)^{\otimes n} \otimes \mathbb{C}$. If w has another reduced expression, then the corresponding representation is unitarily equivalent and we set $\pi_w = (\pi_{s_1} \otimes \ldots \otimes \pi_{s_n}) \circ \Delta_{(n)}$. Further all the representations indexed by W and $\mathbb{T}^d$ form a complete set of mutually inequivalent irreducible representations of $\mathrm{Pol}(\mathbb{K}_q)$; and $\mathrm{C}(\mathbb{K}_q)$ is a type I C-algebra.*
2. *The representations $\pi_s, s \in S$ factor through the (s-dependent) projections $\mathrm{Pol}(\mathbb{K}_q) \twoheadrightarrow \mathrm{Pol}(\mathbb{G}_q)$ and representations $\pi_\tau, \tau \in \mathbb{T}^d$ factor through the canonical projection $\mathrm{Pol}(\mathbb{K}_q) \twoheadrightarrow \mathrm{Pol}(\mathbb{T}^d)$.*

We now need to complement the above theorem further, using also the topological characterisation of the spectrum of $\mathrm{C}(\mathbb{K}_q)$, due to Neshveyev and Tuset ([NT]).

Proposition 3.3. *Let $\mathbb{K}_q$ be as above. Then the following hold.*

1. *Let μ be the normalized Lebesgue measure on $\mathbb{T}^d$. Consider the representation*
$$\pi = \bigoplus_{w \in W} \int_{\mathbb{T}^d}^{\oplus} \pi_w \otimes \pi_\tau \, d\mu(\tau). \tag{3.3}$$
Then π extends from $\mathrm{Pol}(\mathbb{K}_q)$ to a representation of $\mathrm{C}(\mathbb{K}_q)$ and this representation is moreover faithful. The image $\pi(\mathrm{C}(\mathbb{K}_q))$ is contained in $L_\infty(\mathbb{T}^d, \oplus_w B(H_w)) \simeq \oplus_w B(H_w) \otimes L_\infty(\mathbb{T}^d)$.
2. *We have $\pi(\mathrm{Pol}(\mathbb{K}_q)) \subseteq \oplus_w B(H_w) \odot \mathrm{Pol}(\mathbb{T}^d)$. Furthermore there exists a Borel set $X \subseteq \mathbb{T}^d$ with non-empty interior and a non-zero element $x \in \overline{\pi(\mathrm{Pol}(\mathbb{K}_q))}$ such that for every $w \in W$ the space $H_w \otimes L_2(X)$ is in the kernel of x.*

Proof. By amenability of $\widehat{\mathbb{K}_q}$ every representation of $\mathrm{Pol}(\mathbb{K}_q)$ extends to $\mathrm{C}(\mathbb{K}_q)$. Further, as (1) in Theorem 3.2 characterizes all irreducible representations of $\mathrm{Pol}(\mathbb{K}_q)$ we see that π must be a faithful representation of $\mathrm{C}(\mathbb{K}_q)$. It follows directly from the integral decomposition (3.3) that the image of $\mathrm{Pol}(\mathbb{K}_q)$ is contained in $L_\infty(\mathbb{T}^d; \oplus_w B(H_w)) \simeq \oplus_w B(H_w) \otimes L_\infty(\mathbb{T}^d)$. The first statement of (2) follows from (2) of Theorem 3.2. For the second statement put $X := \mathbb{T}^d_\uparrow = \mathbb{T}_\uparrow \times \ldots \times \mathbb{T}_\uparrow$, the

d-fold Cartesian product of $\mathbb{T}_\uparrow$. Suppose that the second statement does not hold for this set. This would mean that the $\bigcap_{w\in W, t\in\mathbb{T}_\uparrow^d} \mathrm{Ker}\, \pi_{w,t} = \{0\}$, or in other words, the set $\{\pi_{w,t} : w \in W, t \in \mathbb{T}_\uparrow^d\}$ would be dense in the spectrum of $\mathrm{C}(\mathbb{K}_q)$, equipped with Jacobson topology. This however is false, as follows from the special case of results of [NT]. Indeed, in the notation of that paper, considering the situation where $S = \prod$ and L is trivial we are reduced to the study of $\mathrm{Pol}(\mathbb{K}_q)$. Thus Theorem 4.1 (ii) of [NT] shows in particular that (again, using the notation of that paper)

$$\mathrm{cl}\{\pi_{w,t} : w \in W, t \in \mathbb{T}_\uparrow^d\} \subset \bigcup_{w\in W} \mathrm{cl}\{\pi_{w,t} : t \in \mathbb{T}_\uparrow^d\} = \bigcup_{w\in W} \{\pi_{\sigma,t} : \sigma \in W, \sigma \leq w, t \in T_{\sigma,w}\mathbb{T}_\uparrow^d\}.$$

Then if say $w_{max} \in W$ is the longest element of the Weyl group we see that $\pi_{w_{max},t} \in \mathrm{cl}\{\pi_{w,t} : w \in W, t \in \mathbb{T}_\uparrow^d\}$ if and only if $t \in \mathbb{T}_\uparrow^d$ and the density statement fails. □

We are ready to formulate the main result of this section.

Theorem 3.4. *Let $q \in (0,1)$, let K be a simply connected semisimple compact Lie group and let $\mathbb{K}_q$ denote the q-deformation of K in the sense of* [KS]. *Then $\widehat{\mathbb{K}_q}$ is not* C^**-unique.*

Proof. Let $H = \oplus_w H_w$. Let $\mathbb{T}_\uparrow^d = \mathbb{T}_\uparrow \times \ldots \times \mathbb{T}_\uparrow$ be the d-fold Cartesian product, as in the proof of the last proposition. Suppose that f and g are in $B(H) \odot \mathrm{Pol}(\mathbb{T}^d)$ and act on $H \otimes L_2(\mathbb{T}^d)$. Let $f_\uparrow$ and $g_\uparrow$ be their restrictions to the subspace $H \otimes L_2(\mathbb{T}_\uparrow^d)$. Then $f = g$ if and only if $f_\uparrow = g_\uparrow$.

Now consider the representation,

$$\pi_\uparrow = \bigoplus_{w\in W} \int_{\mathbb{T}_\uparrow}^{\oplus} \pi_w \otimes \pi_\tau \, d\mu(\tau).$$

Note that $\pi_\uparrow(x) = \pi(x)|_{H\otimes L_2(\mathbb{T}_\uparrow)}$. We have from Proposition 3.3 that π maps $\mathrm{Pol}(\mathbb{K}_q)$ injectively to $B(H) \odot \mathrm{Pol}(\mathbb{T}^d)$ and therefore $\pi_\uparrow$ is injective by the first paragraph. Hence $\|x\|_\uparrow := \|\pi_\uparrow(x)\|$ defines a C^*-norm on $\mathrm{Pol}(\mathbb{K}_q)$ which is majorized by the reduced/universal C^*-norm. The majorization is proper; indeed, by Theorem 3.3 (2) take non-zero $x \in \overline{\pi(\mathrm{Pol}(\mathbb{K}_q))} \subseteq B(H) \otimes L_\infty(\mathbb{T})$ such that the space $H \otimes L_2(\mathbb{T}_\uparrow)$ is in the kernel of x. Consider a sequence x_n in $\mathrm{Pol}(\mathbb{K}_q)$ such that $\pi(x_n)$ converges to x in the norm of $\mathrm{C}(\mathbb{K}_q)$. Then $\pi_\uparrow(x_n)$ converges to 0 in norm. This concludes that the reduced norm properly majorizes the $\| \, \|_\uparrow$-norm. □

Remark 3.5. The last theorem holds also for $q = 1$. More precisely, if G is an infinite compact linear group, then its dual is not C^*-unique. In other words, the unital *-algebra $\mathrm{Pol}(G)$ of coefficients of finite-dimensional unitary representations of G admits non-unique C^*-completions. To see it, it suffices to embed concretely $G \subset U(n)$ by choosing a fundamental representation and choose a non-empty open subset of G, say X, whose complement has non-empty interior. Then we can repeat the trick as before, simply using as the relevant Hilbert space $L^2(X)$ (with the Haar measure of G), on which $\mathrm{Pol}(G)$ acts by multiplication. We leave the details to the reader.

Finally we discuss the corresponding problems for algebras of functions on quantum homogeneous spaces. Consider a compact quantum group $\mathbb{G}$ and a compact quantum subgroup $\mathbb{K}$, so that we have a surjective Hopf *-map $q : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{K})$. We can define then the unital *-algebra $\mathrm{Pol}(\mathbb{G}/\mathbb{K}) := \{a \in \mathrm{Pol}(\mathbb{G}) : (\mathrm{id} \otimes q)(\Delta(a)) = a \otimes 1\}$. Note that $\mathrm{Pol}(\mathbb{G}/\mathbb{K})$ admits a maximal C^*-norm. This in fact extends to any 'algebraic core' of an action of a compact quantum group, as is shown for example in Proposition 4.1 and Theorem 4.2 of [DC]. This means that Proposition 2.7 applies to algebras of such type and naturally we can consider the uniqueness of their C^*-completions. One could then ask whether the theorem on non-C^*-uniqueness of duals of

q-deformations extends to the appropriate function algebras of quantum homogeneous spaces, as studied for example in [NT]. The proposition below answers it in the negative, already for the example of the Podleś sphere.

Proposition 3.6. *The algebra of (polynomial) functions on the Podleś sphere, i.e. the unital* **-algebra arising as* $\mathrm{Pol}(SU_q(2)/\mathbb{T})$*, is* C**-unique.*

Proof. The result can be proved directly – using the presentation of $\mathrm{Pol}(SU_q(2)/\mathbb{T})$ via generators and relations (see [Pod] or [Dab]). Alternatively, we can use the fact that the universal completion of $\mathrm{Pol}(SU_q(2)/\mathbb{T})$ is the unitisation of the algebra of compact operators (hence a just infinite C*-algebra, as an extension of a simple C*-algebra by a finite one, see [GMR]) and appeal to Proposition 2.7. □

4. An example of a C*-unique discrete quantum group which is not locally finite

In this section we give an example of a compact quantum group $\mathbb{G}_q^\theta$ for which $\mathrm{Pol}(\mathbb{G}_q^\theta)$ has a unique C*-completion. The corresponding discrete quantum group $\widehat{\mathbb{G}_q^\theta}$ is not locally finite (see Definition 2.10). In Question 6.8 of [GMR] the authors ask whether C*-uniqueness of a discrete group implies that the group is locally finite. Our example shows that in the theory of discrete quantum groups the answer to this question is negative.

We construct the example as follows. Throughout this section we fix $q \in (-1,1)\setminus\{0\}$ and an irrational number $\theta \in (0,1)$. Let $\mathbb{G}_q = SU_q(2)$, with generators α and γ in $\mathrm{Pol}(\mathbb{G}_q)$ satisfying the relations (3.1). Consider the automorphism ρ_θ of $\mathrm{Pol}(\mathbb{G}_q)$ given by:

$$\alpha \mapsto \alpha, \qquad \gamma \mapsto e^{2\pi i\theta}\gamma.$$

Analysing the defining relations of $\mathrm{Pol}(\mathbb{G}_q)$ we find that ρ_θ indeed extends to a $*$-automorphism $\rho_\theta : \mathrm{Pol}(\mathbb{G}_q) \to \mathrm{Pol}(\mathbb{G}_q)$. Further, the equalities (3.2) imply that $\Delta \circ \rho_\theta = (\rho_\theta \otimes \rho_\theta) \circ \Delta$ so that ρ_θ is a Hopf-$*$-automorphism of $\mathrm{Pol}(\mathbb{G}_q)$. Let $\mathbb{G}_q^\theta = \mathbb{G}_q \rtimes_{\rho_\theta} \mathbb{Z}$ be the resulting crossed product quantum group (see Section 6 of [FMP] for the details of the construction). More precisely, let $\mathrm{Pol}(\mathbb{G}_q^\theta)$ be the algebraic crossed product $\mathrm{Pol}(\mathbb{G}_q) \rtimes_{\rho_\theta} \mathbb{Z}$ which is the $*$-algebra generated by $\mathrm{Pol}(\mathbb{G}_q)$ and a unitary u_θ subject to the relations

$$u_\theta^* \gamma u_\theta = e^{2\pi i\theta}\gamma, \qquad u_\theta^* \alpha u_\theta = \alpha.$$

The coproduct on $\mathrm{Pol}(\mathbb{G}_q^\theta)$ is the extension of the coproduct from $\mathrm{Pol}(\mathbb{G}_q)$ obtained by setting

$$\Delta(u_\theta) = u_\theta \otimes u_\theta.$$

It follows from the defining relations that indeed Δ extends to a $*$-homomorphism $\Delta : \mathrm{Pol}(\mathbb{G}_q^\theta) \to \mathrm{Pol}(\mathbb{G}_q^\theta) \odot \mathrm{Pol}(\mathbb{G}_q^\theta)$. In fact $\mathrm{Pol}(\mathbb{G}_q^\theta)$ is a Hopf-$*$-algebra, and the elements $\alpha^k \gamma^m (\gamma^*)^n u_\theta^l$ with $k, m, n \in \mathbb{N}_0, l \in \mathbb{Z}$ form a linear basis of $\mathrm{Pol}(\mathbb{G}_q^\theta)$. As explained for example in [FMP] the universal completion of $\mathrm{Pol}(\mathbb{G}_q^\theta)$ yields the C*-algebra $\mathrm{C}(\mathbb{G}_q^\theta)$ fitting into the Woronowicz quantum group theory (and arising simply as the C*-algebraic crossed product $\mathrm{C}(\mathbb{G}_q) \rtimes_{\rho_\theta} \mathbb{Z}$). Basic properties of crossed products by $\mathbb{Z}$ and Theorem 2.2 show that $\widehat{\mathbb{G}_q^\theta}$ is amenable (this will also follow from the results below).

At this point we also recall the definition of the non-commutative torus $\mathbb{T}_\theta$. We define $\mathrm{Pol}(\mathbb{T}_\theta)$ as the $*$-algebra generated by two unitaries v_θ and w_θ such that $w_\theta v_\theta = e^{2\pi i\theta} v_\theta w_\theta$. Its universal completion is a C*-algebra denoted by $\mathrm{C}(\mathbb{T}_\theta)$. Recall that $\theta \in (0,1)$ is irrational.

Lemma 4.1. *The $*$-algebra generated by u_θ and γ is isomorphic to* $\mathrm{Pol}(\mathbb{T}_\theta)$ *and is* C**-unique.*

Proof. This is a consequence of the simplicity of the universal C*-algebra $\mathrm{C}(\mathbb{T}_\theta)$. Indeed, $\mathrm{C}(\mathbb{T}_\theta)$ admits a crossed product decomposition $\mathrm{C}(\mathbb{T}) \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts by means of a rotation by θ. Since this action is topologically transitive as θ is irrational, we find that $\mathrm{C}(\mathbb{T}_\theta)$ is simple, c.f. [Rie] for an overview of relevant results. Now suppose that $\pi : \mathrm{Pol}(\mathbb{T}_\theta) \to B(H)$ is an injective $*$-homomorphism. It extends to a $*$-homomorphism $\mathrm{C}(\mathbb{T}_\theta) \to B(H)$ which by simplicity of $\mathrm{C}(\mathbb{T}_\theta)$ is isometric. Therefore, the norm on $\pi(\mathrm{Pol}(\mathbb{T}_\theta))$ equals the norm of $\mathrm{C}(\mathbb{T}_\theta)$. □

For a self-adjoint operator x and a set $A \subseteq \mathbb{R}$, we let $p_A(x)$ be its spectral projection. We let $B_\delta(\lambda)$ be the open ball centered at $\lambda \in \mathbb{R}$ with radius $\delta > 0$.

Lemma 4.2. *Suppose that $x, y \in B(H)$, x is normal and $xy = qyx$ for some $q \in \mathbb{R}\backslash\{0\}$. If $\lambda \in \sigma(x)$ then $q\lambda \in \sigma(x)$ or $yp_{B_\delta(\lambda)}(x) \to 0$ as $\delta \to 0$.*

Proof. Suppose that we do not have that $yp_{B_\delta(\lambda)}(x) \to 0$ as $\delta \to 0$. Then since $\lambda \in \sigma(x)$ there exists a sequence of unit vectors $\xi_n \in H$ such that $(x - \lambda)\xi_n \to 0$ and such that $\|y\xi_n\| > \delta$ for some $\delta > 0$. Then $(x - q\lambda)y\xi_n = yq(x - \lambda)\xi_n \to 0$. But this can only happen if $x - q\lambda$ is not invertible. So $q\lambda \in \sigma(x)$. □

Lemma 4.3. *Suppose that $\mathrm{Pol}(\mathbb{G}_q^\theta)$ is represented on a Hilbert space H. We have $\sigma(\gamma^*\gamma) = \{0\} \cup \bigcup_{n\in\mathbb{N}_0} q^{2n}$ and $\sigma(\gamma) = \{0\} \cup \bigcup_{n\in\mathbb{N}_0} q^n\mathbb{T}$.*

Proof. Take $\lambda \in \sigma(\gamma^*\gamma)$. As $\alpha\gamma^*\gamma = q^2\gamma^*\gamma\alpha$ we find by Lemma 4.2 that $q^{-2}\lambda \in \sigma(\gamma^*\gamma)$ or $\alpha p_{B_\delta(\lambda)}(\gamma^*\gamma) \to 0$ as $\delta \searrow 0$. Suppose that $n \in \mathbb{N}_0$ is maximal such that $\lambda q^{-2n} \in \sigma(\gamma)$. We find that $\alpha p_{B_\delta(\lambda q^{-2n})}(\gamma^*\gamma) \to 0$ in norm as $\delta \searrow 0$. Therefore, for $\varepsilon > 0$ we may pick $\delta > 0$ small enough so that

$$\begin{aligned} |1 - \lambda q^{-2n}| =& \|(1 - \lambda q^{-2n}) p_{B_\delta(\lambda q^{-2n})}(\gamma^*\gamma)\| \\ \leq& \|(-\alpha^*\alpha + 1 - \lambda q^{-2n}) p_{B_\delta(\lambda q^{-2n})}(\gamma^*\gamma)\| + \varepsilon \\ =& \|(\gamma^*\gamma - \lambda q^{-2n}) p_{B_\delta(\lambda q^{-2n})}(\gamma^*\gamma)\| + \varepsilon \leq 2\varepsilon. \end{aligned}$$

Therefore, $\lambda q^{-2n} = 1$. This shows that $\sigma(\gamma^*\gamma) \subseteq \cup_{0\leq n\leq N} q^{2n}$ for some $N \in \mathbb{N}_0 \cup \{\infty\}$. Applying the same argument to the commutation relation $q^2\alpha^*\gamma^*\gamma = \gamma^*\gamma\alpha^*$ shows that in fact $N = \infty$. Indeed, suppose that N is finite. It follows that if $\lambda \in \sigma(\gamma^*\gamma)$ then $q^2\lambda \in \sigma(\gamma^*\gamma)$ or $\alpha^* p_{B_\delta(\lambda)}(\gamma^*\gamma) \to 0$ as $\delta \searrow 0$. So we must have $\alpha^* p_{B_\delta(q^{2N})}(\gamma^*\gamma) \to 0$ as $\delta \searrow 0$. But this entails that for $\varepsilon > 0$ we may pick $\delta > 0$ small, so that

$$1 = \|(\alpha\alpha^* + q^2\gamma^*\gamma) p_{B_\delta(q^{2N})}(\gamma^*\gamma)\| \leq q^{2+2N} + \varepsilon.$$

This is a contradiction, so $N = \infty$.

Secondly, since γ is normal it generates a commutative C*-algebra with spectrum $\sigma(\gamma)$. By the first part and the spectral mapping theorem it follows that $\sigma(\gamma) \subseteq \{0\} \cup \bigcup_{n\in\mathbb{N}_0} q^n\mathbb{T}$ and that for every $n \in \mathbb{N}_0$ we have that $\sigma(\gamma) \cap q^n\mathbb{T}$ is non-empty. Since $\gamma u_\theta = e^{2\pi i\theta} u_\theta \gamma$ and u_θ is unitary it follows by Lemma 4.2 that if $\lambda \in \sigma(\gamma)$ then $e^{2\pi i\theta}\lambda \in \sigma(\gamma)$. Since θ is irrational and the spectrum is closed this implies that $\mathbb{T}\lambda \in \sigma(\gamma)$. Hence $\sigma(\gamma) = \{0\} \cup \bigcup_{n\in\mathbb{N}_0} q^n\mathbb{T}$. □

Lemma 4.4. *Suppose that $\mathrm{Pol}(\mathbb{G}_q^\theta)$ is represented on a Hilbert space H. For $n \in \mathbb{N}_0$, let p_n be the spectral projection of γ corresponding to the circle $q^n\mathbb{T}$ and let $K_n = p_nH$. For every $n \in \mathbb{N}_0$ there exists a unitary $v_n : K_0 \to K_n$ such that $v_n^* u_\theta v_n = u_\theta p_0$, $v_n^*\gamma v_n = q^n\gamma p_0$. Further, for each $m, n, k \in \mathbb{N}_0$ we have $p_m(\alpha^*)^k p_n = \delta_{n+k,m} c_q(m, n) v_{m,n}$, where $v_{m,n} = v_m v_n^*$ and for $m > n$,*

$$c_q(m, n) = \sqrt{(1 - q^{2m})(1 - q^{2m-2}) \cdots (1 - q^{2n+2})}. \tag{4.1}$$

Proof. Let $\alpha^* = u|\alpha^*|$ be the polar decomposition of α^* and set $u_n = up_n$. The operators $\alpha\alpha^*$ and $\gamma\gamma^*$ commute and hence generate a commutative C*-algebra. γ is contractive, c.f. Lemma 4.3, so that by the relation $\alpha\alpha^* + q^2\gamma^*\gamma = 1$ we see that the support projection $u^*u = \operatorname{supp}(\alpha\alpha^*) = 1$. By the relation $\alpha^*\alpha + \gamma^*\gamma = 1$ we see that $uu^* = \operatorname{supp}(\alpha^*\alpha) = 1 - p_0$.

Fix now $n \in \mathbb{N}$. We get $u_n^* u_n = p_n u^* u p_n = p_n$. So that u_n is a partial isometry with range projection p_n. Further,

(4.2)
$$\gamma^*\gamma u p_n = \gamma^*\gamma\alpha^*|\alpha^*|^{-1}p_n = q^2\alpha^*\gamma^*\gamma|\alpha^*|^{-1}p_n = q^2\alpha^*\gamma^*\gamma|\alpha^*|^{-1}p_n = q^2\alpha^*|\alpha^*|^{-1}\gamma^*\gamma p_n = q^{2+2n}up_n,$$

so that u_n maps p_nH into $p_{n+1}H$. Similarly, u^* maps $p_{n+1}H$ into p_nH and further $uu^*p_{n+1} = (1-p_0)p_{n+1} = p_{n+1}$. This shows that $u_n : p_nH \to p_{n+1}H$ is unitary. The operators α and $\gamma^*\gamma$ commute with u_θ so that also u, p_n and u_n commute with u_θ. So $u_n^* u_\theta u_n = u_\theta p_n$. Further $u_n^*\gamma u_n = q\gamma p_n$ by essentially the same computation as (4.2). Setting $v_n = u_{n-1}\cdots u_0$ then completes the proof of the first statement.

Now, to show $p_m(\alpha^*)^k p_n = \delta_{n+k,m}c_q(m,n)v_{m,n}$ it suffices by induction to show this for $k=1$. We already concluded that $|\alpha^*|$ commutes with p_n and $p_m\alpha^*p_n = p_m u|\alpha^*|p_n = p_m u p_n|\alpha^*| = \delta_{n+1,m}|\alpha^*|p_n$. Further $|\alpha^*|^2 + q^2\gamma\gamma^* = 1$, so that $|\alpha^*|^2p_n + q^{2+2n}p_n = p_n$ and we conclude that $|\alpha^*|p_n = \sqrt{1-q^{2+2n}}\,p_n$. □

Since $\gamma^*\gamma$ and u_θ commute we see that p_n and u_θ commute and that the spaces K_n in Lemma 4.4 are invariant subspaces for u_θ. Further $q^{-n}\gamma$ restricted to K_n is unitary as its spectrum equals $\mathbb{T}$. This shows that the restrictions of u_θ and $q^{-n}\gamma$ to K_n satisfy the relations of the non-commutative torus $\mathbb{T}_\theta$ and that the prescription $v_\theta \mapsto u_\theta p_n$ and $w_\theta \mapsto q^{-n}\gamma p_n$ gives a non-trivial representation π_n of $\operatorname{Pol}(\mathbb{T}_\theta)$. As $\mathrm{C}(\mathbb{T}_\theta)$ is simple, each π_n is faithful.

Corollary 4.5. *All the representations in the family $\{\pi_n : \operatorname{Pol}(\mathbb{T}_\theta) \to B(K_n), n \in \mathbb{N}\}$ described above are unitarily conjugate.*

Proof. This is a consequence of Lemma 4.4. □

We are ready to state and prove the main result of this section.

Theorem 4.6. *The algebra $\operatorname{Pol}(\mathbb{G}_q^\theta)$ is* C*-*unique.*

Proof. Let π be a representation of $\operatorname{Pol}(\mathbb{G}_q^\theta)$ on a Hilbert space H. As in Lemma 4.4 we decompose $H = \oplus_{n=0}^\infty K_n$. We may moreover assume that all Hilbert spaces K_n are isomorphic, and that respective unitaries conjugate the actions of $\operatorname{Pol}(\mathbb{T}_\theta) \simeq \operatorname{Pol}\langle q^{-n}\gamma, u_\theta\rangle p_n$ on $K_n, n \in \mathbb{N}_0$. So we assume that $K = K_n$ and hence $H = \oplus_{n=0}^\infty K$. Moreover, there is a single representation $\pi_{\mathbb{T}_\theta} : \operatorname{Pol}(\mathbb{T}_\theta) \to B(K)$ such that for $x \in \operatorname{Pol}(\mathbb{T}_\theta)$ under the identification $H = \oplus_{n=0}^\infty K$ we have $\pi(x) = \oplus_{n=0}^\infty \pi_{\mathbb{T}_\theta}(x)$. For each $n \in \mathbb{N}_0$ let, as in Lemma 4.4, p_n be the spectral projection of $\pi(\gamma^*\gamma)$ corresponding to the spectral set $q^n\mathbb{T}$. Then p_n is by construction the projection onto the n-th summand in $H = \oplus_{n=0}^\infty K$ and we set $p_{-1} = 0$. We also find from Lemma 4.4 that $\pi(\alpha)p_n = p_{n-1}\pi(\alpha) = \sqrt{1-q^{2n}}\,vp_n$ where $v : \oplus_{n=0}^\infty K \to \oplus_{n=0}^\infty K$ is the backwards shift $(\xi_n)_{n\in\mathbb{N}_0} \mapsto (\xi_{n+1})_{n\in\mathbb{N}_0}$. In particular, it follows that

$$\begin{aligned} p_k\pi(\alpha^m)p_n =& \delta_{n-m,k}c_q(n,k), && m \geq 0,\\ p_k\pi((\alpha^*)^m)p_n =& \delta_{n+m,k}c_q(k,n), && m > 0. \end{aligned}$$

with $c_q(k,n)$ as described in (4.1) and $c_q(n,n) = 1$. For simplicity write α^m for $(\alpha^*)^{-m}$ in case $m < 0$. Also set $c_q(m,n) = c_q(n,m)$ in case $m < n$. Let $Q = \sum_{m=-\infty}^\infty P_m(u_\theta, \gamma, \gamma^*)\alpha^m$ with

$P_m(u_\theta,\gamma,\gamma^*)$ a linear combination of basis vectors $u_\theta^k(\gamma^*)^i\gamma^j$ where $k\in\mathbb{Z}, i,j\in\mathbb{N}_0$. We may identify $B(H)$ with $B(K)\otimes B(\ell_2)$ and then we see that

$$\pi(Q)=\sum_{m,n=0}^{\infty}\pi_{\mathbb{T}_\theta}(P_m(u_\theta,\gamma,\gamma^*))\otimes c_q(m,n)e_{m,n}\in B(K)\otimes B(\ell_2).$$

Suppose now that we have two representations π_1 and π_2 of $\mathrm{Pol}(\mathbb{G}_q^\theta)$, on H^1 and H^2 respectively. Then we get decompositions $H^1=\oplus_{n=1}^\infty K^1$ and $H^2=\oplus_{n=1}^\infty K^2$ with representations $\pi^1_{\mathbb{T}_\theta}$ and $\pi^2_{\mathbb{T}_\theta}$ of $\mathrm{Pol}(\mathbb{T}_\theta)$ on K^1 and K^2 respectively, such that,

$$\pi^i(Q)=\sum_{m,n=0}^{\infty}\pi^i_{\mathbb{T}_\theta}(P_m(u_\theta,\gamma,\gamma^*))\otimes c_q(m,n)e_{m,n}\in B(K^i)\otimes B(\ell_2). \tag{4.3}$$

Since $\mathrm{C}(\mathbb{T}_\theta)$ is simple we find that

$$\pi^1_{\mathbb{T}_\theta}(\mathrm{C}(\mathbb{T}_\theta))\simeq_{\pi^1_{\mathbb{T}_\theta}}\mathrm{C}(\mathbb{T}_\theta)\simeq_{\pi^2_{\mathbb{T}_\theta}}\pi^2_{\mathbb{T}_\theta}(\mathrm{C}(\mathbb{T}_\theta))$$

are isomorphic. From (4.3) we see that the complete isometry $(\pi^1_{\mathbb{T}_\theta}\circ(\pi^2_{\mathbb{T}_\theta})^{-1}\otimes\mathrm{id}_{\ell_2})$ maps $\pi^1(\mathrm{Pol}(\mathbb{G}_q^\theta))$ bijectively to $\pi^2(\mathrm{Pol}(\mathbb{G}_q^\theta))$. Therefore the norms on $\pi^i(\mathrm{Pol}(\mathbb{G}_q^\theta))$ with $i=1,2$ are equal and we conclude that $\mathrm{Pol}(\mathbb{G}_q^\theta)$ is C*-unique. □

Theorem 4.7. *The discrete quantum group $\widehat{\mathbb{G}_q^\theta}$ is* C**-unique and not locally finite.*

Proof. C*-uniqueness is proved in Theorem 4.6. This quantum group is not locally finite as the quantum subgroup generated by u_θ is not finite. In fact one may check that any non-empty choice of finitely many generators of the fusion ring of $\widehat{\mathbb{G}_q^\theta}$ generates an infinite quantum group. □

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