Testing multivariate uniformity based on random geometric graphs

B. Ebner, F. Nestmann and M. Schulte

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Abstract

We present new families of goodness-of-fit tests of uniformity on a full-dimensional set $W \subset \mathbb{R}^d$ based on statistics related to edge lengths of random geometric graphs. Asymptotic normality of these statistics is proven under the null hypothesis as well as under fixed alternatives. The derived tests are consistent and their behaviour for some contiguous alternatives can be controlled. A simulation study suggests that the procedures can compete with or are better than established goodness-of-fit tests.

1 Introduction

The analysis of point patterns in a given study area is of particular interest in a wide variety of fields, such as astronomy (e.g. occurrence of high energetic events in a sky map), biology (e.g. locations of sightings of threatened species) or geology (e.g. locations of raw materials). The concept of uniformity of the observations stands for the absence of structure in the data. Thus, testing uniformity of random vectors is a natural starting point for serious statistical inference involving any cluster analysis or multimodality assumption. To be specific, let $n \in \mathbb{N}$ and

$$\mathscr{X}_n := \{X_1, \dots, X_n\}$$

be the data set, where $X_1, \dots, X_n$ are independent identically distributed (i.i.d.) random vectors taking values in a given measurable set $W \subset \mathbb{R}^d$, $d \geq 1$, of positive finite volume, called

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the observation window. We want to test the *null hypothesis*

$$H_0 : X \sim \mathcal{U}(W) \tag{1}$$

with X being an independent copy of X_1 and $\mathcal{U}(W)$ denoting the uniform distribution on W against general alternatives. Further applications are testing pseudo random number generators, see e.g. [19, Section 3.3], or testing if i.i.d. random vectors in $\mathbb{R}^d$ follow a given absolutely continuous distribution, which is, by the Rosenblatt transformation, see [28], theoretically equivalent to testing uniformity on the d-dimensional unit cube $[0,1]^d$, although this transformation is hard to compute in many cases. The problem of testing uniformity has been investigated in classical papers in the univariate case, see [24] for a survey and [3] for a recent article, and, hitherto far less studied, in the multivariate setting, see [4, 5, 7, 12, 18, 22, 30, 31, 33], for which an empirical study was conducted in [26]. The cited methods include classical goodness-of-fit testing approaches as the Kolmogorov-Smirnov test, see [18], nearest neighbour concepts, see [12] and the references therein, the distances of the data points to the boundary of the observation window, see [7], or the volume of the largest ball that can be placed in the observation window and does not cover any data point, see [5]. The related problem of testing for complete spatial randomness of a point pattern (i.e., the points are a realisation of a homogeneous Poisson point process) is also of ongoing interest, see e.g. monographs like [2, 10] or the recent publications [11, 15].

We approach the testing problem (1) by examining the local properties of the data by means of random graphs. Using random graphs for testing uniformity is a known but not widely used concept, see [14, 20, 26]. Our new approach is to consider statistics of the random geometric graph $RGG(\mathscr{X}_n, r_n)$, $r_n > 0$: It has the realisations of the random vectors in $\mathscr{X}_n$ as vertices, and any two distinct vertices $x, y \in \mathscr{X}_n$ are connected by an edge if $\|x-y\| \le r_n$, where $\|\cdot\|$ stands for the Euclidean norm. This random graph model was introduced by Gilbert for an underlying Poisson point process in [13] and is thus also called Gilbert graph. For further details see [25] and the references cited therein. Our test statistics are related to the edge lengths of $RGG(\mathscr{X}_n, r_n)$ and are defined by

$$L_n(\beta) := \frac{1}{2} \sum_{(x,y)\in\mathscr{X}_{n,\neq}^2} \mathbf{1}\{\|x-y\| \le r_n\} \|x-y\|^\beta, \quad \beta \in \mathbb{R}.$$

Here $\sum_{(x,y)\in\mathscr{X}_{n,\neq}^2}$ stands for the sum over all pairs of distinct points of $\mathscr{X}_n$ (such sums are called U-statistics), and $\mathbf{1}\{\cdot\}$ is the indicator function. Notice that $L_n(0)$ counts the number of edges and $L_n(1)$ is the total edge length of $RGG(\mathscr{X}_n, r_n)$. These statistics differ from nearest neighbour methods, see e.g. [8, 12] and the references therein, as such that they rely on all interpoint distances not exceeding r_n, whereas nearest neighbour methods take only distances between points and their k-nearest neighbours into account. An extensive theory of properties and the asymptotic behaviour of $L_n(\beta)$ in the complete spatial randomness setting can be found in [27]. Figure 1 provides a visualisation of different point models and selected random geometric graphs. For definitions of the CLU and CON alternatives we refer to Section 5.

Based on the asymptotically standardised statistics $L_n(\beta)$, we propose the test statistics

$$T_{e,n}(\beta) := \left(\frac{L_n(\beta) - \frac{1}{2}n(n-1)\int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\} \|x-y\|^\beta \,\mathrm{d}(x,y)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}} n r_n^{\beta+d/2}} \right)^2$$

and

$$T_{a,n}(\beta) := \left(\frac{L_n(\beta) - \frac{d\kappa_d}{2(\beta+d)}\, n(n-1) r_n^{\beta+d}}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}} n r_n^{\beta+d/2}} \right)^2,$$

where $\beta > -d/2$ and rejection of H_0 will be for large values of $T_{j,n}(\beta)$, $j \in \{e, a\}$.

In order to derive distributional limit theorems for $L_n(\beta)$, $T_{e,n}(\beta)$ and $T_{a,n}(\beta)$, we apply a central limit theorem from [17] for triangular schemes of U-statistics. For $\beta = 0$ the statistic $L_n(\beta)$ was considered as application in [17]. Here, we generalise these findings to $\beta \in (-d/2, \infty)$, which is technical for $\beta \in (-d/2, 0)$, and present them in more detail. Moreover, the focus of the present paper is on statistical tests based on $L_n(\beta)$ and their properties, which even for $\beta = 0$ goes clearly beyond what was studied in [17].

In [33] some U-statistics based on interpoint distances are proposed as test statistics for uniformity on the unit cube (beside two other statistics based on data depth and normal quantiles). In contrast to $L_n(\beta)$, these U-statistics take all interpoint distances into account and not only the small ones, whence their kernels do not depend on n (i.e., the summand associated with two given points from the sample is the same for all $n \in \mathbb{N}$). The tests for multivariate uniformity studied in [4, 30] are also based on U-statistics with fixed kernels, which are more involved to compute than the distances between the sample points. For U-statistics with fixed kernels as considered in [4, 30, 33], the asymptotic behaviour is much easier to analyse than for $L_n(\beta)$, where the kernels depend on the parameters n and r_n and their interplay.

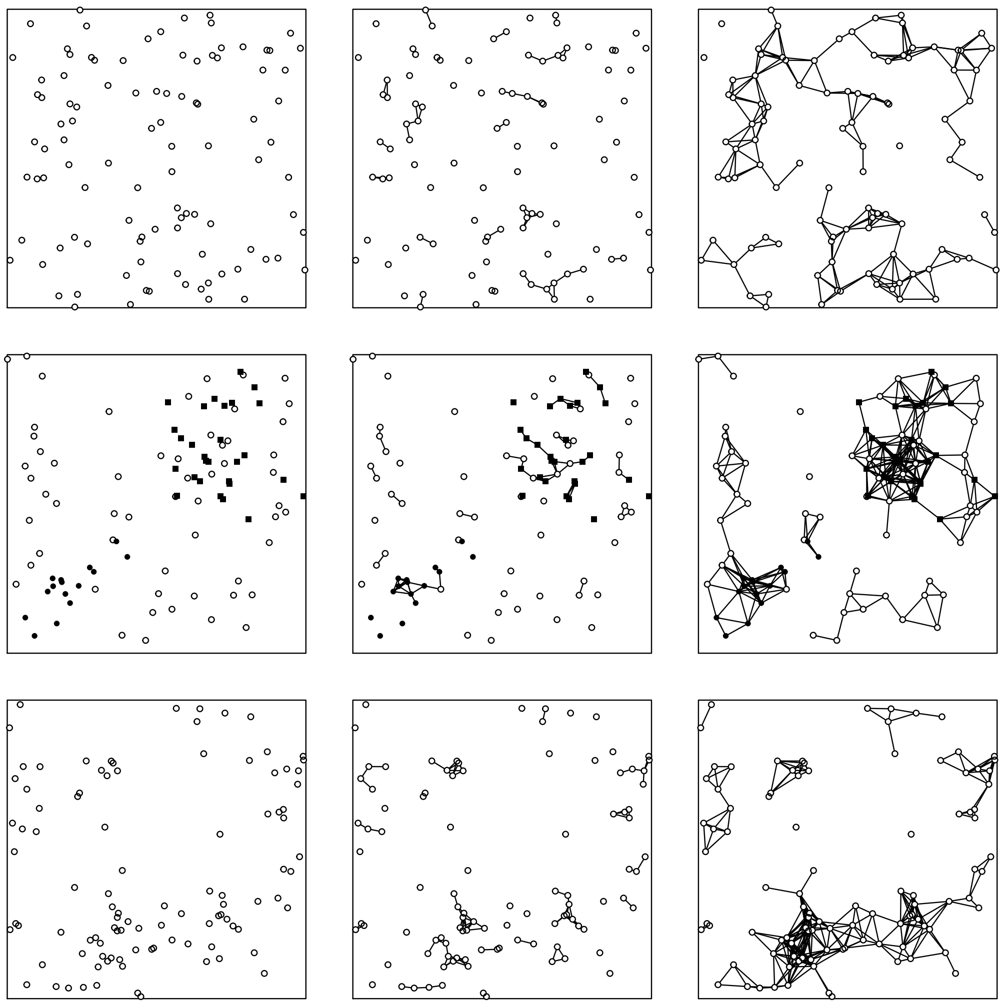

Figure 1: Realisations of uniform data in $W = [0,1]^2$ (first row), the CON alternative (second row) and the CLU alternative (third row). Point data (first column), $RGG(\mathscr{X}_n, 0.03)$ (second column) and $RGG(\mathscr{X}_n, 0.06)$ (third column), $n = 100$.

This paper is organised as follows. In Section 2 we derive the theory for $L_n(\beta)$, including formulae for the mean and the variance as well as central limit theorems, in a general setting. The two families of test statistics $T_{j,n}(\beta)$, $j \in \{e,a\}$, are discussed in Section 3, and their limiting behaviour is given under H_0 and under fixed alternatives. The behaviour for some contiguous alternatives is studied in Section 4. Section 5 provides a simulation study and a comparison to existing methods. We finish the paper with comments on open problems and research perspectives in Section 6.

2 Properties of $L_n(\beta)$

Let $\mathscr{X}_n = \{X_1, \dots, X_n\}$, where $n \geq 2$ and $X_1, \dots, X_n$ are i.i.d. random vectors distributed according to a density f, whose support is contained in a measurable set $W \subset \mathbb{R}^d$ of positive finite volume. In the following, we assume without loss of generality that $\mathrm{Vol}(W) = 1$, i.e., W has volume one. For some of our results we need the additional assumption that

$$\limsup_{r \to 0} \frac{\mathrm{Vol}(\{x \in W : d(x, \partial W) \leq r\})}{r} < \infty. \tag{2}$$

Here, we use the notation $d(x, A) := \inf_{y \in A} \|x - y\|$ for $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$. The assumption (2) requires that the volume of the set of points in W that are in the r-neighbourhood of the boundary of W is at most of order r and seems to be no significant restriction. For many sets W, for example all compact and convex W, the limit superior in (2) equals the surface area of W. The expression in (2) is related to the so-called (outer) Minkowski content. For a definition as well as some results on its finiteness we refer to [1].

Let (r_n) be a sequence of positive real numbers such that $r_n \to 0$ as $n \to \infty$. In the following $B^d(x, r)$ stands for the d-dimensional closed ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$, and $\kappa_d := \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of the d-dimensional unit ball $B^d(0, 1)$. We denote by $L^2(W)$ the space of all square-integrable functions on W. For the special case $\beta = 0$ the formulae of the following theorem can be also found in [17, Equations (4.2) and (4.3)].

Theorem 2.1 *For $\beta > -d$ and $f \in L^2(W)$,*

$$\mathbb{E} L_n(\beta) = \frac{n(n-1)}{2} \int_{W^2} \mathbf{1}\{\|x - y\| \leq r_n\} \, \|x - y\|^\beta \, f(x) \, f(y) \, \mathrm{d}(x, y) \tag{3}$$

and

$$\lim_{n \to \infty} \frac{\mathbb{E} L_n(\beta)}{n^2 r_n^{\beta + d}} = \frac{d \kappa_d}{2(\beta + d)} \int_W f(x)^2 \, \mathrm{d}x. \tag{4}$$

Proof: Equation (3) follows from

$$\mathbb{E} L_n(\beta) = \frac{n(n-1)}{2} \mathbb{E} \mathbf{1}\{\|X - Y\| \leq r_n\} \|X - Y\|^\beta,$$

where X and Y are independent random vectors distributed according to the density f. Notice that

$$\begin{aligned}\mathbb{E}\mathbf{1}\{\|X-Y\|\le r_n\}\|X-Y\|^\beta &= \int_{W^2}\mathbf{1}\{\|x-y\|\le r_n\}\,\|x-y\|^\beta\, f(x)f(y)\,\mathrm{d}(x,y)\\ &\le \int_{W^2}\mathbf{1}\{\|x-y\|\le r_n\}\,\|x-y\|^\beta\, f(x)^2\,\mathrm{d}(x,y)\\ &\le \frac{d\kappa_d}{\beta+d} r_n^{\beta+d}\int_W f(x)^2\,\mathrm{d}x,\end{aligned}$$

where we used the inequality of arithmetic and geometric means and spherical coordinates. This yields

$$\limsup_{n\to\infty}\frac{\mathbb{E}L_n(\beta)}{n^2 r_n^{\beta+d}} \le \frac{d\kappa_d}{2(\beta+d)}\int_W f(x)^2\,\mathrm{d}x. \tag{5}$$

In the following we use the shorthand notation $f_C(x) := \min\{f(x), C\}$ for $C > 0$ and $x \in W$. It follows from Lemma A.1 that, for any $C > 0$,

$$\lim_{n\to\infty}\frac{1}{r_n^{\beta+d}}\int_{B^d(x,r_n)}\|x-y\|^\beta f_C(y)\,\mathrm{d}y = \frac{d\kappa_d}{\beta+d} f_C(x)$$

for almost all $x \in W$. Now the dominated convergence theorem yields

$$\lim_{n\to\infty}\frac{1}{r_n^{\beta+d}}\int_{W^2}\mathbf{1}\{\|x-y\|\le r_n\}\,\|x-y\|^\beta\, f_C(x)f_C(y)\,\mathrm{d}(x,y) = \frac{d\kappa_d}{\beta+d}\int_W f_C(x)^2\,\mathrm{d}x.$$

Together with

$$\begin{aligned}&\int_{W^2}\mathbf{1}\{\|x-y\|\le r_n\}\,\|x-y\|^\beta\, f(x)f(y)\,\mathrm{d}(x,y)\\ &\ge \int_{W^2}\mathbf{1}\{\|x-y\|\le r_n\}\,\|x-y\|^\beta\, f_C(x)f_C(y)\,\mathrm{d}(x,y)\end{aligned}$$

we obtain

$$\liminf_{n\to\infty}\frac{\mathbb{E}L_n(\beta)}{n^2 r_n^{\beta+d}} \ge \frac{d\kappa_d}{2(\beta+d)}\int_W f_C(x)^2\,\mathrm{d}x.$$

Now letting $C\to\infty$ and the monotone convergence theorem yield

$$\liminf_{n\to\infty}\frac{\mathbb{E}L_n(\beta)}{n^2 r_n^{\beta+d}} \ge \frac{d\kappa_d}{2(\beta+d)}\int_W f(x)^2\,\mathrm{d}x.$$

Combining this with (5) proves (4). □

Theorem 2.1 states exact formulae for the mean and easy to compute asymptotic approximations under fairly general assumptions including the behaviour of $\mathbb{E}L_n(\beta)$ under H_0, which is a direct consequence. We write $g \equiv h$ to indicate that two functions $g, h : W \to \mathbb{R}$ are identical almost everywhere.

Corollary 2.2 *If $\beta > -d$ and $f \equiv \mathbf{1}_W$, then*

$$\mathbb{E}L_n(\beta) = \frac{n(n-1)}{2} \int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\} \, \|x-y\|^\beta \, \mathrm{d}(x,y)$$

and

$$\lim_{n\to\infty} \frac{\mathbb{E}L_n(\beta)}{n^2 r_n^{\beta+d}} = \frac{d\kappa_d}{2(\beta+d)}.$$

Recall that the degree of a vertex in a graph is the number of edges emanating from it. The average degree $\bar{D}_n$ of the vertices in $\mathrm{RGG}(\mathscr{X}_n, r_n)$ is given by $\bar{D}_n = 2L_n(0)/n$. Thus, it follows from Theorem 2.1 that $\mathbb{E}\bar{D}_n$ is of the same order as nr_n^d as $n \to \infty$. For the special choice $f \equiv \mathbf{1}_W$ Corollary 2.2 implies

$$\lim_{n\to\infty} \frac{\mathbb{E}\bar{D}_n}{\kappa_d n r_n^d} = 1. \tag{6}$$

In the next theorem we present exact and asymptotic formulae for the variance of $L_n(\beta)$, which generalise the findings from [17, Section 4] for $\beta = 0$.

Theorem 2.3 *Let $\beta > -d/2$ and $f \in L^3(W)$.*

(a) Then,

$$\begin{aligned} \operatorname{Var} L_n(\beta) = {} & \frac{n(n-1)}{2} \int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\} \, \|x-y\|^{2\beta} \, f(x) \, f(y) \, \mathrm{d}(x,y) \\ & + n(n-1)(n-2) \int_W \left(\int_W \mathbf{1}\{\|x-y\| \le r_n\} \, \|x-y\|^\beta \, f(y) \, \mathrm{d}y \right)^2 f(x) \, \mathrm{d}x \\ & - n(n-1)(n-3/2) \left(\int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\} \, \|x-y\|^\beta \, f(x) \, f(y) \, \mathrm{d}(x,y) \right)^2. \end{aligned} \tag{7}$$

(b) For $f \not\equiv \mathbf{1}_W$,

$$\lim_{n\to\infty} \frac{\operatorname{Var} L_n(\beta)}{\sigma^{(1)}_{\beta,f} n^2 r_n^{2\beta+d} + \sigma^{(2)}_{\beta,f} n^3 r_n^{2\beta+2d}} = 1, \tag{8}$$

where

$$\sigma^{(1)}_{\beta,f} := \frac{d\kappa_d}{2(2\beta+d)} \int_W f(x)^2 \, \mathrm{d}x$$

and

$$\sigma^{(2)}_{\beta,f} := \frac{d^2\kappa_d^2}{(\beta+d)^2} \left(\int_W f(x)^3 \, \mathrm{d}x - \left(\int_W f(x)^2 \, \mathrm{d}x \right)^2 \right).$$

(c) If $f \equiv \mathbf{1}_W$, W satisfies (2) and $nr_n^{d+1} \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} \frac{\operatorname{Var} L_n(\beta)}{n^2 r_n^{2\beta+d}} = \frac{d\kappa_d}{2(2\beta+d)}. \tag{9}$$

Notice that the orders of the two terms in the denominator in (8) differ by nr_n^d, which is the order of the expected average degree. For $\sigma_{\beta,f}^{(1)}, \sigma_{\beta,f}^{(2)} > 0$ this means that the first (second) term dominates if $\mathbb{E}\bar{D}_n \to 0$ ($\mathbb{E}\bar{D}_n \to \infty$) as $n \to \infty$, while both terms contribute to the limit if $\mathbb{E}\bar{D}_n \to c \in (0,\infty)$ as $n \to \infty$. For all choices of $f \in L^3(W)$ we have $\sigma_{\beta,f}^{(1)} > 0$. The Cauchy-Schwarz inequality implies

$$\left(\int_W f(x)^2 \,\mathrm{d}x \right)^2 \leq \int_W f(x)^3 \,\mathrm{d}x \int_W f(x) \,\mathrm{d}x = \int_W f(x)^3 \,\mathrm{d}x$$

with equality if and only if $f \equiv \mathbf{1}_W$. So $\sigma_{\beta,f}^{(2)} \geq 0$ with equality if and only if $f \equiv \mathbf{1}_W$.

The formula (9) coincides with (8) for $f \equiv \mathbf{1}_W$. Nevertheless we have to impose for (9) additional conditions on the boundary of W and on the sequence (r_n). They ensure that the sum of the second and the third term in (7) does not have an asymptotic order that is less than $n^3 r_n^{2\beta+2d}$ but still larger than $n^2 r_n^{2\beta+d}$. The following example shows that this can happen due to boundary effects (see also [17, Section 4]). For $W = [0,1]$, $f \equiv \mathbf{1}_W$, $\beta = 0$ and $r_n < 1/2$, we have

$$\begin{aligned}\int_0^1 \left(\int_0^1 \mathbf{1}\{|x-y| \leq r_n\} \,\mathrm{d}y \right)^2 \mathrm{d}x &= 2\int_0^{r_n} (r_n + x)^2 \,\mathrm{d}x + (1-2r_n)4r_n^2 \\ &= \frac{2}{3}(8r_n^3 - r_n^3) + (1-2r_n)4r_n^2 = 4r_n^2 - \frac{10}{3} r_n^3\end{aligned}$$

and

$$\begin{aligned}\int_{[0,1]^2} \mathbf{1}\{|x-y| \leq r_n\} \,\mathrm{d}(x,y) &= 2\int_0^{r_n} r_n + x \,\mathrm{d}x + (1-2r_n)2r_n \\ &= 4r_n^2 - r_n^2 + (1-2r_n)2r_n = 2r_n - r_n^2.\end{aligned}$$

Thus, the sum of the second and the third term in (7) equals

$$n(n-1)(n-2)(4r_n^2 - \frac{10}{3} r_n^3 - 4r_n^2 + 4r_n^3 - r_n^4) - \frac{1}{2} n(n-1)(2r_n - r_n^2)^2.$$

If $nr_n^2 \to \infty$ as $n \to \infty$, this is of a higher order than the first term in (7).

Theorem 3.3 in [27] states asymptotic variances for the same statistics $L_n(\beta)$ with an underlying homogeneous Poisson point process of intensity n (i.e., $f \equiv \mathbf{1}_W$ and the number of points is Poisson-distributed with mean n). In contrast to (9), these formulae show the same phase transition depending on the behaviour of nr_n^d as we have in (8) for $f \not\equiv \mathbf{1}_W$.

Proof of Theorem 2.3: A straightforward computation shows that

$$\begin{aligned}\mathbb{E}L_n(\beta)^2 = {} & \frac{n(n-1)}{2}\mathbb{E}\mathbf{1}\{\|X_1-X_2\| \le r_n\}\|X_1-X_2\|^{2\beta} \\ & + n(n-1)(n-2)\mathbb{E}\mathbf{1}\{\|X_1-X_2\|, \|X_1-X_3\| \le r_n\}\|X_1-X_2\|^{\beta}\|X_1-X_3\|^{\beta} \\ & + \frac{(n)_4}{4}\mathbb{E}\mathbf{1}\{\|X_1-X_2\|, \|X_3-X_4\| \le r_n\}\|X_1-X_2\|^{\beta}\|X_3-X_4\|^{\beta}.\end{aligned}$$

Here, $X_1, \dots, X_4$ are independent random vectors with density f and $(\cdot)_k$ denotes the kth descending factorial. Combining this with (3) yields (7).

Observe that the asymptotic behaviour of the first and the third term in (7) follows immediately from Theorem 2.1. By the inequality of arithmetic and geometric means and spherical coordinates, we obtain

$$\begin{aligned}& \frac{1}{r_n^{2\beta+2d}} \int_W \left(\int_W \mathbf{1}\{\|x-y\| \le r_n\}\, \|x-y\|^{\beta}\, f(y)\, \mathrm{d}y \right)^2 f(x)\, \mathrm{d}x \\ & \le \frac{1}{3r_n^{2\beta+2d}} \int_{W^3} \mathbf{1}\{\|x_1-x_2\|, \|x_1-x_3\| \le r_n\}\|x_1-x_2\|^{\beta}\|x_1-x_3\|^{\beta} \\ & \qquad\qquad (f(x_1)^3 + f(x_2)^3 + f(x_3)^3)\, \mathrm{d}(x_1,x_2,x_3) \\ & \le \frac{d^2\kappa_d^2}{(\beta+d)^2} \int_W f(x)^3\, \mathrm{d}x.\end{aligned} \tag{10}$$

On the other hand, Lemma A.1 and the dominated convergence theorem imply

$$\lim_{n\to\infty} \frac{1}{r_n^{2\beta+2d}} \int_W \left(\int_W \mathbf{1}\{\|x-y\| \le r_n\}\, \|x-y\|^{\beta}\, f_C(y)\, \mathrm{d}y \right)^2 f_C(x)\, \mathrm{d}x = \frac{d^2\kappa_d^2}{(\beta+d)^2} \int_W f_C(x)^3\, \mathrm{d}x$$

for each $C > 0$. Recall that $f_C(x) = \min\{f(x), C\}$ for $x \in W$. Now letting $C \to \infty$ and the monotone convergence theorem yield

$$\liminf_{n\to\infty} \frac{1}{r_n^{2\beta+2d}} \int_W \left(\int_W \mathbf{1}\{\|x-y\| \le r_n\}\, \|x-y\|^{\beta}\, f(y)\, \mathrm{d}y \right)^2 f(x)\, \mathrm{d}x \ge \frac{d^2\kappa_d^2}{(\beta+d)^2} \int_W f(x)^3\, \mathrm{d}x.$$

This, together with (10) and the observation that $\sigma^{(1)}_{\beta,f}, \sigma^{(2)}_{\beta,f} > 0$, completes the proof of (8).

For the proof of (9) we define $W_{-r_n} := \{x \in W : d(x, \partial W) \ge r_n\}$. Now straightforward computations yield

$$\frac{d^2\kappa_d^2}{(\beta+d)^2} r_n^{2\beta+2d}\, \mathrm{Vol}(W_{-r_n}) \le \int_W \left(\int_W \mathbf{1}\{\|x-y\| \le r_n\}\|x-y\|^{\beta}\, \mathrm{d}y \right)^2 \mathrm{d}x \le \frac{d^2\kappa_d^2}{(\beta+d)^2} r_n^{2\beta+2d}\, \mathrm{Vol}(W)$$

and

$$\frac{d^2\kappa_d^2}{(\beta+d)^2} r_n^{2\beta+2d}\, \mathrm{Vol}(W_{-r_n})^2 \le \left(\int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\}\|x-y\|^{\beta}\, \mathrm{d}(x,y) \right)^2 \le \frac{d^2\kappa_d^2}{(\beta+d)^2} r_n^{2\beta+2d}\, \mathrm{Vol}(W)^2.$$

It follows from (2) that there exists a constant $C_W \in (0,\infty)$ such that

$$0 \le \mathrm{Vol}(W) - \mathrm{Vol}(W_{-r_n}) \le \mathrm{Vol}(\{x \in W : d(x,\partial W) \le r_n\}) \le C_W r_n. \tag{11}$$

Together with $\mathrm{Vol}(W) = 1$ this means that the absolute value of the sum of the second and the third term in (7) can be bounded by

$$\frac{3d^2\kappa_d^2}{(\beta+d)^2} C_W n^3 r_n^{2\beta+2d+1} + \frac{d^2\kappa_d^2}{2(\beta+d)^2} n^2 r_n^{2\beta+2d}.$$

Together with the asymptotic order of the first term in (7), which is as in the proof of (8), this proves (9). □

In the following we use the abbreviation

$$\sigma_{\beta,f,n} := \sqrt{\sigma_{\beta,f}^{(1)} n^2 r_n^{2\beta+d} + \sigma_{\beta,f}^{(2)} n^3 r_n^{2\beta+2d}}$$

with $\sigma_{\beta,f}^{(1)}, \sigma_{\beta,f}^{(2)}$ as in Theorem 2.3 for $\beta > -d/2$ and $n \in \mathbb{N}$. Moreover, we write $\xrightarrow{\mathcal{D}}$ for convergence in distribution and $N_m(\mu,\Sigma)$ for an m-dimensional Gaussian random vector with mean vector $\mu \in \mathbb{R}^m$ and positive semidefinite covariance matrix $\Sigma \in \mathbb{R}^{m\times m}$. In the univariate case the index m is omitted.

Theorem 2.4 *Let $\beta > -d/2$, $f \in L^3(W)$ and assume that $n^2 r_n^d \to \infty$ as $n \to \infty$. If $f \not\equiv \mathbf{1}_W$ or if $f \equiv \mathbf{1}_W$, W satisfies (2) and $n r_n^{d+1} \to 0$ as $n \to \infty$, then*

$$\frac{L_n(\beta) - \mathbb{E}L_n(\beta)}{\sigma_{\beta,f,n}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad n \to \infty.$$

For $\beta = 0$ a central limit theorem as Theorem 2.4 is established in [17, Section 4]; see also [32] and the references therein. In [25, Section 3.5] central limit theorems for subgraph counts of random geometric graphs are derived, which include the number of edges $L_n(0)$ as special case. Notice that $n^2 r_n^d \to \infty$ as $n \to \infty$ means that the expected number of edges goes to infinity as $n \to \infty$ (see Theorem 2.1), which is a reasonable assumption for a central limit theorem involving edge lengths. The additional assumptions for $f \equiv \mathbf{1}_W$ are the same as in Theorem 2.3(c) and are used to ensure that the rescaled variances converge to one. Theorem 2.4 is proved next to the following corollary concerning the behaviour under H_0.

Corollary 2.5 *Let $\beta > -d/2$, $f \equiv \mathbf{1}_W$ and assume that W satisfies (2).*

(a) If $n^2 r_n^d \to \infty$ and $n r_n^{d+1} \to 0$ as $n \to \infty$, then

$$\frac{L_n(\beta) - \frac{n(n-1)}{2}\int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\}\,\|x-y\|^\beta \,\mathrm{d}(x,y)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}\, n r_n^{\beta+d/2}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad n \to \infty.$$

(b) If $n^2 r_n^d \to \infty$ and $n^2 r_n^{d+2} \to 0$ as $n \to \infty$, then

$$\frac{L_n(\beta) - \frac{d\kappa_d}{2(\beta+d)} n(n-1) r_n^{\beta+d}}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}} n r_n^{\beta+d/2}} \xrightarrow{\mathcal{D}} N(0,1) \quad \textit{as} \quad n \to \infty.$$

It can be seen from Corollary 2.2 that in part (a) of the previous corollary $L_n(\beta)$ is centred with its expectation, while in (b) the asymptotic expectation is used. In the latter situation, the assumptions on (r_n) are stricter. For the statistics $L_n(\beta)$ with respect to an underlying homogeneous Poisson point process (i.e. the case of complete spatial randomness) central limit theorems are shown in [27, Section 5.1].

Proof of Corollary 2.5: Part (a) is an immediate consequence of Theorem 2.4, (3) and the definition of $\sigma_{\beta,f,n}$. For the proof of (b) recall that $W_{-r_n} = \{x \in W : d(x, \partial W) \geq r_n\}$. It follows from (3) that

$$\frac{d\kappa_d}{2(\beta+d)} \operatorname{Vol}(W_{-r_n}) n(n-1) r_n^{\beta+d} \leq \mathbb{E} L_n(\beta) \leq \frac{d\kappa_d}{2(\beta+d)} \operatorname{Vol}(W) n(n-1) r_n^{\beta+d}.$$

Together with (11), which is valid because we assume (2), and $\operatorname{Vol}(W) = 1$ this yields

$$|\mathbb{E} L_n(\beta) - \frac{d\kappa_d}{2(\beta+d)} n(n-1) r_n^{\beta+d}| \leq \frac{d\kappa_d}{2(\beta+d)} C_W n^2 r_n^{\beta+d+1}$$

so that

$$\lim_{n\to\infty} \frac{|\mathbb{E} L_n(\beta) - \frac{d\kappa_d}{2(\beta+d)} n(n-1) r_n^{\beta+d}|}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}} n r_n^{\beta+d/2}} \leq \lim_{n\to\infty} \frac{\sqrt{d\kappa_d(2\beta+d)}}{\sqrt{2}(\beta+d)} C_W n r_n^{d/2+1} = 0. \tag{12}$$

Hence, the assertion of (b) follows from (a). □

We prepare the proof of Theorem 2.4 by several lemmas, which are formulated for the following more general setting, required later: We assume that the underlying points of $\mathscr{X}_n$ are distributed according to some density $f_n \in L^3(W)$ and that $f_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in W$.

For $n \in \mathbb{N}$ we define

$$W_{f_n} := \{(x,m) \in W \times [0,\infty) : m \leq f_n(x)\}$$

and let $\widehat{X}_1, \dots, \widehat{X}_n$ be independent and uniformly distributed points in W_{f_n}. We denote the collection of these points by $\widehat{\mathscr{X}_n}$. For a point $\hat{x} \in W_{f_n}$ we often use the decomposition $\hat{x} = (x,m)$ with $x \in W$ and $m \in [0, f_n(x)]$. Observe that the first components of $\widehat{X}_1, \dots, \widehat{X}_n$ are distributed according to the density f_n in W. For $\beta \in \mathbb{R}$ we define

$$\widehat{L}_n(\beta) := \frac{1}{2} \sum_{((x_1,m_1),(x_2,m_2)) \in \widehat{\mathscr{X}}^2_{n,\neq}} \mathbf{1}\{\|x_1 - x_2\| \leq r_n\} \, \|x_1 - x_2\|^\beta. \tag{13}$$

If $f_n = f$, $\widehat{L}_n(\beta)$ has the same distribution as $L_n(\beta)$. For $M > 0$ and $a \geq 0$ we define

$$\widehat{L}_{n,M}(\beta) := \frac{1}{2} \sum_{((x_1,m_1),(x_2,m_2)) \in \widehat{\mathscr{X}}^2_{n,\neq}} \mathbf{1}\{m_1, m_2 \leq M\} \, \mathbf{1}\{\|x_1 - x_2\| \leq r_n\} \, \|x_1 - x_2\|^\beta \tag{14}$$

and

$$\widehat{L}_{n,a,M}(\beta) := \frac{1}{2} \sum_{((x_1,m_1),(x_2,m_2)) \in \widehat{\mathscr{X}}^2_{n,\neq}} \mathbf{1}\{m_1, m_2 \leq M\} \, \mathbf{1}\{n^{-2/d} a \leq \|x_1 - x_2\| \leq r_n\} \, \|x_1 - x_2\|^\beta.$$

Moreover, we use the abbreviations $f_{n,M}(x) := \min\{f_n(x), M\}$ and $f_M(x) := \min\{f(x), M\}$ for $x \in W$.

Lemma 2.6 *Let $\beta > -d/2$, $M \geq 1$, $a > 0$ and assume that $n^2 r_n^d \to \infty$ as $n \to \infty$ and that $\lim_{n\to\infty} \operatorname{Var} \widehat{L}_{n,M}(\beta)/\sigma^2_{\beta,f_M,n} = 1$. Then,*

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f_M,n}} - \frac{\widehat{L}_{n,a,M}(\beta) - \mathbb{E}\widehat{L}_{n,a,M}(\beta)}{\sigma_{\beta,f_M,n}} \right)^2 = 0 \tag{15}$$

and

$$\lim_{n\to\infty} \frac{\operatorname{Var} \widehat{L}_{n,a,M}(\beta)}{\sigma^2_{\beta,f_M,n}} = 1. \tag{16}$$

Proof: By definition, we have that

$$\widehat{L}_{n,M}(\beta) - \widehat{L}_{n,a,M}(\beta) = \frac{1}{2} \sum_{((x_1,m_1),(x_2,m_2)) \in \widehat{\mathscr{X}}^2_{n,\neq}} \mathbf{1}\{m_1, m_2 \leq M\} \, \mathbf{1}\{\|x_1 - x_2\| < n^{-2/d} a\} \, \|x_1 - x_2\|^\beta.$$

Now a similar computation as in the proof of Theorem 2.3(a) yields that

$$\operatorname{Var}(\widehat{L}_{n,M}(\beta) - \widehat{L}_{n,a,M}(\beta)) \leq I_1 + I_2$$

with

$$\begin{aligned} I_1 &:= \frac{n^2}{2} \int_{W^2} \mathbf{1}\{\|x - y\| \leq n^{-2/d} a\} \|x - y\|^{2\beta} f_{n,M}(x) f_{n,M}(y) \, \mathrm{d}(x,y) \\ I_2 &:= n^3 \int_W \left(\int_W \mathbf{1}\{\|x - y\| \leq n^{-2/d} a\} \|x - y\|^\beta f_{n,M}(y) \, \mathrm{d}y \right)^2 f_{n,M}(x) \, \mathrm{d}x. \end{aligned}$$

Note that I_1 and I_2 correspond to the first two terms in (7), whereas the third term in (7) was omitted since it is non-positive. Now short computations show that

$$\frac{I_1}{\sigma^2_{\beta,f_M,n}} \leq \frac{d\kappa_d M^2}{2(2\beta + d)} \frac{n^2 n^{-2-4\beta/d} a^{2\beta+d}}{\sigma^{(1)}_{\beta,f_M} n^2 r_n^{2\beta+d}} = \frac{d\kappa_d M^2 a^{2\beta+d}}{2(2\beta + d)\sigma^{(1)}_{\beta,f_M}} \frac{1}{(n^2 r_n^d)^{2\beta/d+1}}$$

and

$$\frac{I_2}{\sigma^2_{\beta,f_M,n}} \le \frac{d^2\kappa_d^2 M^3}{(\beta+d)^2}\frac{n^3 n^{-4-4\beta/d}a^{2\beta+2d}}{\sigma^{(1)}_{\beta,f_M} n^2 r_n^{2\beta+d}} = \frac{d^2\kappa_d^2 M^3 a^{2\beta+2d}}{(\beta+d)^2\sigma^{(1)}_{\beta,f_M}}\frac{1}{n(n^2 r_n^d)^{2\beta/d+1}}.$$

Since $n^2 r_n^d \to \infty$ as $n \to \infty$ and $2\beta/d+1 > 0$, this provides (15). Now (16) follows from combining (15) and $\lim_{n\to\infty} \operatorname{Var} \widehat{L}_{n,M}(\beta)/\sigma^2_{\beta,f_M,n} = 1$. □

Lemma 2.7 *Let $\beta > -d/2$, $M \ge 1$, $a > 0$ and assume that $n^2 r_n^d \to \infty$ as $n \to \infty$ and that $\lim_{n\to\infty} \operatorname{Var} \widehat{L}_{n,M}(\beta)/\sigma^2_{\beta,f_M,n} = 1$. Then,*

$$\frac{\widehat{L}_{n,a,M}(\beta) - \mathbb{E}\widehat{L}_{n,a,M}(\beta)}{\sigma_{\beta,f_M,n}} \xrightarrow{\mathcal{D}} N(0,1) \quad as \quad n \to \infty \tag{17}$$

and

$$\frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f_M,n}} \xrightarrow{\mathcal{D}} N(0,1) \quad as \quad n \to \infty. \tag{18}$$

Proof: From (16) we know that $\lim_{n\to\infty} \operatorname{Var} \widehat{L}_{n,a,M}(\beta)/\sigma^2_{\beta,f_M,n} = 1$. If $\beta \ge 0$, then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\sup_{(x,m_x),(y,m_y)\in W_{f_n}} \mathbf{1}\{m_x, m_y \le M\}\mathbf{1}\{n^{-2/d}a \le \|x-y\| \le r_n\}\|x-y\|^\beta}{n r_n^{\beta+d/2}} \\
&\le \lim_{n\to\infty} \frac{r_n^\beta}{n r_n^{\beta+d/2}} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2 r_n^d}} = 0,
\end{aligned}$$

while for $\beta \in (-d/2, 0)$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\sup_{(x,m_x),(y,m_y)\in W_{f_n}} \mathbf{1}\{m_x, m_y \le M\}\mathbf{1}\{n^{-2/d}a \le \|x-y\| \le r_n\}\|x-y\|^\beta}{n r_n^{\beta+d/2}} \\
&\le \lim_{n\to\infty} \frac{n^{-2\beta/d}a^\beta}{n r_n^{\beta+d/2}} = \lim_{n\to\infty} \frac{a^\beta}{(n^2 r_n^d)^{1/2+\beta/d}} = 0.
\end{aligned}$$

Denoting by $(X_1^{(n)}, m_{X_1^{(n)}})$ an uniformly distributed point in W_{f_n}, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n \sup_{(x,m)\in W_{f_n}} \mathbb{E}\mathbf{1}\{m, m_{X_1^{(n)}} \le M\}\mathbf{1}\{n^{-2/d}a \le \|x-X_1^{(n)}\| \le r_n\}\|x-X_1^{(n)}\|^\beta}{n r_n^{\beta+d/2}} \\
&= \lim_{n\to\infty} \frac{1}{r_n^{\beta+d/2}} \sup_{x\in W} \int_W \mathbf{1}\{n^{-2/d}a \le \|x-y\| \le r_n\}\,\|x-y\|^\beta\, f_{n,M}(y)\,\mathrm{d}y \\
&\le \lim_{n\to\infty} \frac{d\kappa_d M}{\beta+d}\frac{r_n^{\beta+d}}{r_n^{\beta+d/2}} = \lim_{n\to\infty} \frac{d\kappa_d M}{\beta+d} r_n^{d/2} = 0.
\end{aligned}$$

Thus, (17) follows from Theorem B.1. Combining the L^2-covergence in (15) with (17) yields (18). □

In the following we use the abbreviation $\overline{f}_{n,M}(x) := \max\{f_n(x) - M, 0\}$ for $x \in W$ and $M \geq 0$.

Lemma 2.8 *For $n \in \mathbb{N}$, $\beta > -d/2$ and $M \geq 1$,*

$$\begin{aligned}
&\mathbb{E}\bigg(\frac{\widehat{L}_n(\beta) - \mathbb{E}\widehat{L}_n(\beta)}{\sigma_{\beta,f,n}} - \frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f,n}}\bigg)^2 \\
&\leq \frac{d\kappa_d}{(2\beta+d)} \frac{n^2 r_n^{2\beta+d}}{\sigma_{\beta,f,n}^2} \int_W \overline{f}_{n,M}(x)^2 + M\overline{f}_{n,M}(x)\,\mathrm{d}x \\
&\quad + \frac{18d^2\kappa_d^2}{(\beta+d)^2} \frac{n^3 r_n^{2\beta+2d}}{\sigma_{\beta,f,n}^2} \int_W M^2\overline{f}_{n,M}(x) + M\overline{f}_{n,M}(x)^2 + \overline{f}_{n,M}(x)^3\,\mathrm{d}x.
\end{aligned}$$

Proof: By definition we have

$$\widehat{L}_n(\beta) - \widehat{L}_{n,M}(\beta) = \frac{1}{2} \sum_{((x,m_x),(y,m_y)) \in \widehat{\mathscr{X}}_{n,\neq}^2} \mathbf{1}\{m_x > M \text{ or } m_y > M\}\,\mathbf{1}\{\|x-y\| \leq r_n\}\|x-y\|^\beta.$$

From similar arguments as in the proofs of Theorem 2.3(a) and Lemma 2.6, it follows that

$$\mathrm{Var}(\widehat{L}_n(\beta) - \widehat{L}_{n,M}(\beta)) \leq I_1 + I_2$$

with

$$\begin{aligned}
I_1 &:= \frac{n^2}{2} \int_{W_{f_n}^2} \mathbf{1}\{m_1 > M \text{ or } m_2 > M\}\,\mathbf{1}\{\|x_1 - x_2\| \leq r_n\}\,\|x_1 - x_2\|^{2\beta}\,\mathrm{d}((x_1,m_1),(x_2,m_2)) \\
I_2 &:= n^3 \int_{W_{f_n}^3} \mathbf{1}\{m_1 > M \text{ or } m_2 > M\}\,\mathbf{1}\{m_1 > M \text{ or } m_3 > M\}\,\mathbf{1}\{\|x_1 - x_2\| \leq r_n\} \\
&\qquad \mathbf{1}\{\|x_1 - x_3\| \leq r_n\}\,\|x_1 - x_2\|^\beta\,\|x_1 - x_3\|^\beta\,\mathrm{d}((x_1,m_1),(x_2,m_2),(x_3,m_3)).
\end{aligned}$$

For I_1 we obtain the bound

$$\begin{aligned}
I_1 &\leq n^2 \int_{W^2} \mathbf{1}\{\|x-y\| \leq r_n\}\,\|x-y\|^{2\beta}\,\overline{f}_{n,M}(x)\,(\overline{f}_{n,M}(y) + M)\,\mathrm{d}(x,y) \\
&\leq n^2 \int_{W^2} \mathbf{1}\{\|x-y\| \leq r_n\}\,\|x-y\|^{2\beta}\,(\overline{f}_{n,M}(x)^2 + M\overline{f}_{n,M}(x))\,\mathrm{d}(x,y) \\
&\leq \frac{d\kappa_d}{2\beta+d} n^2 r_n^{2\beta+d} \int_W \overline{f}_{n,M}(x)^2 + M\overline{f}_{n,M}(x)\,\mathrm{d}x.
\end{aligned}$$

Because of

$$\mathbf{1}\{m_1 > M \text{ or } m_2 > M\}\,\mathbf{1}\{m_1 > M \text{ or } m_3 > M\} \leq \mathbf{1}\{m_1 > M\} + \mathbf{1}\{m_2 > M, m_3 > M\},$$

we have

$$I_2 \le n^3 \int_{W^3} \mathbf{1}\{\|x_1 - x_2\| \le r_n\}\, \mathbf{1}\{\|x_1 - x_3\| \le r_n\}\, \|x_1 - x_2\|^\beta\, \|x_1 - x_3\|^\beta$$
$$(\overline{f}_{n,M}(x_1)\, f_n(x_2)\, f_n(x_3) + f_n(x_1)\, \overline{f}_{n,M}(x_2)\, \overline{f}_{n,M}(x_3))\, \mathrm{d}(x_1, x_2, x_3).$$

Using that $f_n(x) \le \overline{f}_{n,M}(x) + M$ for $x \in \mathbb{R}^d$, we obtain

$$\overline{f}_{n,M}(x_1)\, f_n(x_2)\, f_n(x_3) + f_n(x_1)\, \overline{f}_{n,M}(x_2)\, \overline{f}_{n,M}(x_3) \le 6 \max_{k,i \in \{1,2,3\}} M^{3-k} \overline{f}_{n,M}(x_i)^k.$$

This implies

$$I_2 \le \frac{18 d^2 \kappa_d^2}{(\beta + d)^2} n^3 r_n^{2\beta + 2d} \int_W M^2 \overline{f}_{n,M}(x) + M \overline{f}_{n,M}(x)^2 + \overline{f}_{n,M}(x)^3 \, \mathrm{d}x,$$

which completes the proof. □

We recall that $f_M(x) := \min\{f(x), M\}$ for $x \in W$ and $M \ge 0$.

Lemma 2.9 *Let $\beta > -d/2$, $M \ge 1$ and $f_n = f$, $n \in \mathbb{N}$. If $f \not\equiv \mathbf{1}_W$ or if $f \equiv \mathbf{1}_W$, W satisfies* (2) *and $n r_n^{d+1} \to 0$ as $n \to \infty$, then*

$$\lim_{n \to \infty} \frac{\operatorname{Var} \widehat{L}_{n,M}(\beta)}{\sigma^2_{\beta, f_M, n}} = 1.$$

Proof: For $M \ge 1$ and $f \equiv \mathbf{1}_W$, $f_M \equiv \mathbf{1}_W$ and the statement is the same as Theorem 2.3(c) because $\widehat{L}_{n,M}(\beta)$ follows the same distribution as $L_n(\beta)$. For $f \not\equiv \mathbf{1}_W$ one can show as in the proof of Theorem 2.3(a) that

$$\begin{aligned}
&\operatorname{Var} \widehat{L}_{n,M}(\beta) \\
&= \frac{n(n-1)}{2} \int_{W^2} \mathbf{1}\{\|x - y\| \le r_n\}\, \|x - y\|^{2\beta}\, f_M(x)\, f_M(y)\, \mathrm{d}(x, y) \\
&\quad + n(n-1)(n-2) \int_W \bigg(\int_W \mathbf{1}\{\|x - y\| \le r_n\}\, \|x - y\|^\beta\, f_M(y)\, \mathrm{d}y \bigg)^2 f_M(x)\, \mathrm{d}x \\
&\quad - n(n-1)(n-3/2) \bigg(\int_{W^2} \mathbf{1}\{\|x - y\| \le r_n\}\, \|x - y\|^\beta\, f_M(x)\, f_M(y)\, \mathrm{d}(x, y) \bigg)^2.
\end{aligned}$$

Now the assertion can be proved as Theorem 2.3(b). □

Proof of Theorem 2.4: We consider the same setting as in the previous lemmas with $f_n = f$ for $n \in \mathbb{N}$ so that $L_n(\beta)$ has the same distribution as $\widehat{L}_n(\beta)$, which we study throughout this proof. For $f \equiv \mathbf{1}_W$ the assertion follows from (18) in Lemma 2.7 because, for $M \ge 1$, $\widehat{L}_n(\beta)$ has the same distribution as $\widehat{L}_{n,M}(\beta)$, $\sigma_{\beta, f_M, n} = \sigma_{\beta, f, n}$ and Lemma 2.9 guarantees that the variance condition in Lemma 2.7 is satisfied. So we assume $f \not\equiv \mathbf{1}_W$ in the sequel.

Let $h : \mathbb{R} \to \mathbb{R}$ a be bounded Lipschitz function whose Lipschitz constant is at most one and let $\varepsilon > 0$. In the following we show

$$\lim_{n\to\infty} \left| \mathbb{E}h\left(\frac{\widehat{L}_n(\beta) - \mathbb{E}\widehat{L}_n(\beta)}{\sigma_{\beta,f,n}} \right) - \mathbb{E}h(N(0,1)) \right| \leq \varepsilon, \tag{19}$$

which yields the assertion.

For $M \geq 1$ the triangle inequality implies

$$\begin{aligned}
&\left| \mathbb{E}h\left(\frac{\widehat{L}_n(\beta) - \mathbb{E}\widehat{L}_n(\beta)}{\sigma_{\beta,f,n}} \right) - \mathbb{E}h(N(0,1)) \right| \\
&\leq \left| \mathbb{E}h\left(\frac{\widehat{L}_n(\beta) - \mathbb{E}\widehat{L}_n(\beta)}{\sigma_{\beta,f,n}} \right) - \mathbb{E}h\left(\frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f,n}} \right) \right| \\
&\quad + \left| \mathbb{E}h\left(\frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f,n}} \right) - \mathbb{E}h\left(\frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f_M,n}} \right) \right| \\
&\quad + \left| \mathbb{E}h\left(\frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f_M,n}} \right) - \mathbb{E}h(N(0,1)) \right| \\
&=: R_{1,n,M} + R_{2,n,M} + R_{3,n,M}.
\end{aligned} \tag{20}$$

It follows from Lemma 2.7 (notice that the variance condition is satisfied because of Lemma 2.9) that $R_{3,n,M}$ vanishes for any $M \geq 1$ as $n \to \infty$. The Lipschitz property of h, the Cauchy-Schwarz inequality and Lemma 2.8 imply that

$$\begin{aligned}
R_{1,n,M}^2 &\leq \mathbb{E}\left(\frac{\widehat{L}_n(\beta) - \mathbb{E}\widehat{L}_n(\beta)}{\sigma_{\beta,f,n}} - \frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f,n}} \right)^2 \\
&\leq \frac{d\kappa_d}{(2\beta+d)} \frac{n^2 r_n^{2\beta+d}}{\sigma_{\beta,f,n}^2} \int_W \overline{f}_M(x)^2 + M\overline{f}_M(x)\,\mathrm{d}x \\
&\quad + \frac{18d^2\kappa_d^2}{(\beta+d)^2} \frac{n^3 r_n^{2\beta+2d}}{\sigma_{\beta,f,n}^2} \int_W M^2\overline{f}_M(x) + M\overline{f}_M(x)^2 + \overline{f}_M(x)^3\,\mathrm{d}x.
\end{aligned}$$

Here the terms depending on n can be bounded by some constants. The dominated convergence theorem with the upper bounds $2f^2$ and $3f^3$ leads to

$$\lim_{M\to\infty} \int_W \overline{f}_M(x)^2 + M\overline{f}_M(x)\,\mathrm{d}x = 0$$

and

$$\lim_{M\to\infty} \int_W M^2\overline{f}_M(x) + M\overline{f}_M(x)^2 + \overline{f}_M(x)^3\,\mathrm{d}x = 0.$$

Hence, there exists an $M_1 \geq 1$ such that $\lim_{n\to\infty} R_{1,n,M} \leq \varepsilon/2$ for $M > M_1$.

A short computation using the Lipschitz continuity of h and the Cauchy-Schwarz inequality shows that

$$R_{2,n,M} \le \left|\frac{\sigma_{\beta,f_M,n}}{\sigma_{\beta,f,n}} - 1\right| \mathbb{E}\left|\frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f_M,n}}\right| \le \left|\frac{\sigma_{\beta,f_M,n}}{\sigma_{\beta,f,n}} - 1\right| \frac{\sqrt{\operatorname{Var}\widehat{L}_{n,M}(\beta)}}{\sigma_{\beta,f_M,n}}.$$

By the monotone convergence theorem and the assumption $f \not\equiv \mathbf{1}_W$, we have $\sigma^{(1)}_{\beta,f_M} \to \sigma^{(1)}_{\beta,f} > 0$ and $\sigma^{(2)}_{\beta,f_M} \to \sigma^{(2)}_{\beta,f} > 0$ as $M \to \infty$. Together with the definitions of $\sigma_{\beta,f_M,n}$ and $\sigma_{\beta,f,n}$ this implies that there exists an $M_2 \ge 1$ such that

$$\lim_{n\to\infty} \left|\frac{\sigma_{\beta,f_M,n}}{\sigma_{\beta,f,n}} - 1\right| \le \frac{\varepsilon}{2}$$

for $M > M_2$. Since, by Lemma 2.9, $\lim_{n\to\infty} \sqrt{\operatorname{Var}\widehat{L}_{n,M}(\beta)}/\sigma_{\beta,f_M,n} = 1$, we obtain $\lim_{n\to\infty} R_{2,n,M} \le \varepsilon/2$ for $M > M_2$. Thus, choosing $M > \max\{M_1, M_2\}$ in (20) and letting $n \to \infty$ yields (19) and completes the proof. $\square$

3 Testing for uniformity

Motivated by Corollary 2.5 we propose testing goodness-of-fit of H_0 in (1) against general alternatives based on the families of statistics

$$T_{e,n}(\beta) = \left(\frac{L_n(\beta) - \frac{1}{2}n(n-1)\int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\}\,\|x-y\|^\beta\,\mathrm{d}(x,y)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}\,n r_n^{\beta+d/2}}\right)^2 \tag{21}$$

and

$$T_{a,n}(\beta) = \left(\frac{L_n(\beta) - \frac{d\kappa_d}{2(\beta+d)}\,n(n-1)r_n^{\beta+d}}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}\,n r_n^{\beta+d/2}}\right)^2, \tag{22}$$

depending on $\beta > -\frac{d}{2}$ and $r_n \in (0,\infty)$. The choice of the sequence (r_n) is discussed in Section 5, where we introduce a parameter k. The indices e and a are abbreviations for 'exact' and 'asymptotic', and they point out that $T_{e,n}(\beta)$ involves $\mathbb{E}L_n(\beta)$, which can be difficult to compute depending on the shape of the observation window W, while $T_{a,n}(\beta)$ uses a simple asymptotic approximation of $\mathbb{E}L_n(\beta)$. Rejection of H_0 will be for large values of $T_{j,n}(\beta)$, $j \in \{a,e\}$. Empirical critical values for $W = [0,1]^d$ can be found in Tables 9 to 12 for dimensions $d = 2,3$ and sample sizes $n \in \{50,100,200,500\}$. Notice that under H_0 and some mild assumptions on (r_n) and W the continuous mapping theorem and Corollary 2.5 yield

$$T_{j,n}(\beta) \xrightarrow{\mathcal{D}} \chi^2_1 \quad \text{as} \quad n \to \infty, \quad j \in \{a,e\}, \beta > -d/2.$$

Here χ_1^2 denotes a random variable having a chi-squared distribution with one degree of freedom. In the following theorem we consider the asymptotic behaviour of $T_{e,n}(\beta)$ and $T_{a,n}(\beta)$ under fixed alternatives. We write $\xrightarrow{\mathbb{P}}$ for convergence in probability.

Theorem 3.1 *Let $\beta > -d/2$ and $f \not\equiv \mathbf{1}_W$. If $n^2 r_n^d \to \infty$ as $n \to \infty$, then*

$$T_{e,n}(\beta) \xrightarrow{\mathbb{P}} \infty \quad \textit{and} \quad T_{a,n}(\beta) \xrightarrow{\mathbb{P}} \infty \quad \textit{as} \quad n \to \infty.$$

Proof: Throughout this proof we denote the terms that are squared in (21) and (22) by $\overline{L}_{e,n}(\beta)$ and $\overline{L}_{a,n}(\beta)$, respectively. In the following we will show that

$$\overline{L}_{j,n}(\beta) \xrightarrow{\mathbb{P}} \infty \quad \text{as} \quad n \to \infty \tag{23}$$

for $j \in \{a, e\}$, which implies the assertion.

Let $M \geq 1$ and $f_n := f$ for $n \in \mathbb{N}$. Recall the definitions of $\widehat{L}_n(\beta)$ and $\widehat{L}_{n,M}(\beta)$ from (13) and (14). Since $\widehat{L}_n(\beta)$ and $L_n(\beta)$ have the same distribution, we can assume without loss of generality that they are identical. All edges that contribute to $\widehat{L}_{n,M}(\beta)$ also contribute to $\widehat{L}_n(\beta)$ so that $\widehat{L}_{n,M}(\beta) \leq \widehat{L}_n(\beta)$. This implies that, for $j \in \{e, a\}$,

$$\overline{L}_{j,n}(\beta) \geq \frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}} n r_n^{\beta+d/2}} + \frac{\mathbb{E}\widehat{L}_{n,M}(\beta) - m_{n,j}(\beta)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}} n r_n^{\beta+d/2}} =: S_{1,n} + S_{2,j,n}$$

with

$$m_{n,e}(\beta) = \frac{1}{2} n(n-1) \int_{W^2} \mathbf{1}\{\|x-y\| \leq r_n\} \|x-y\|^\beta \, \mathrm{d}(x,y)$$

and

$$m_{n,a}(\beta) = \frac{d\kappa_d}{2(\beta+d)} n(n-1) r_n^{\beta+d}.$$

Using the same arguments as in the proof of Theorem 2.1, one can show that

$$\lim_{n\to\infty} \frac{\mathbb{E}\widehat{L}_{n,M}(\beta)}{n^2 r_n^{\beta+d}} = \frac{d\kappa_d}{2(\beta+d)} \int_W f_M(x)^2 \, \mathrm{d}x.$$

By the Cauchy-Schwarz inequality, we have

$$\int_W 1 \, \mathrm{d}x = 1 = \int_W f(x) \, \mathrm{d}x < \sqrt{\int_W f(x)^2 \, \mathrm{d}x} \sqrt{\int_W 1 \, \mathrm{d}x} = \sqrt{\int_W f(x)^2 \, \mathrm{d}x}$$

since $f \not\equiv \mathbf{1}_W$. Together with the monotone convergence theorem this implies that we can choose $M \geq 1$ such that

$$\lim_{n\to\infty} \frac{\mathbb{E}\widehat{L}_{n,M}(\beta)}{n^2 r_n^{\beta+d}} \geq \frac{d\kappa_d}{2(\beta+d)}(1+\varepsilon)$$

for some $\varepsilon \in (0, \infty)$. Since, by Theorem 2.1,

$$\lim_{n\to\infty} \frac{m_{n,j}(\beta)}{n^2 r_n^{\beta+d}} = \frac{d\kappa_d}{2(\beta+d)}$$

for $j \in \{e, a\}$, this shows that $S_{2,e,n}$ and $S_{2,a,n}$ behave at least as $\frac{\sqrt{d\kappa_d(2\beta+d)}}{\sqrt{2}(\beta+d)}\varepsilon n r_n^{d/2}$ as $n \to \infty$. From the Chebyshev inequality and Lemma 2.9 it follows that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(|S_{1,n}| \geq \frac{\sqrt{d\kappa_d(2\beta+d)}}{2\sqrt{2}(\beta+d)}\varepsilon n r_n^{d/2}\right) &\leq \lim_{n\to\infty} \frac{\operatorname{Var} \widehat{L}_{n,M}(\beta)}{\frac{d\kappa_d}{2(2\beta+d)} \frac{d\kappa_d(2\beta+d)}{8(\beta+d)^2} n^2 r_n^{2\beta+d} \varepsilon^2 n^2 r_n^d} \\
&= \frac{16(\beta+d)^2}{(d\kappa_d)^2\varepsilon^2} \lim_{n\to\infty} \frac{\sigma_{\beta,f_M}^{(1)} n^2 r_n^{2\beta+d} + \sigma_{\beta,f_M}^{(2)} n^3 r_n^{2\beta+2d}}{n^4 r_n^{2\beta+2d}} \\
&= \frac{16(\beta+d)^2}{(d\kappa_d)^2\varepsilon^2} \lim_{n\to\infty} \frac{\sigma_{\beta,f_M}^{(1)}}{n^2 r_n^d} + \frac{\sigma_{\beta,f_M}^{(2)}}{n} = 0,
\end{aligned}$$

which implies (23) for $j \in \{a, e\}$. □

Theorem 3.1 yields consistency of $T_{e,n}(\beta)$ and $T_{a,n}(\beta)$ against each fixed alternative $f \not\equiv \mathbf{1}_W$.

4 Behaviour under contiguous alternatives

Let $g \in L^3(W)$ be such that $g \not\equiv 0$ and $\int_W g(x)\,\mathrm{d}x = 0$ and let (a_n) be a positive sequence such that $a_n \to 0$ as $n \to \infty$. In the following we always tacitly assume that $1 + a_n g(x) \geq 0$ for all $x \in W$ and $n \in \mathbb{N}$. This guarantees that $\mathbf{1}_W + a_n g$ is a density. In the sequel we denote by $\widetilde{T}_{e,n}(\beta)$ and $\widetilde{T}_{a,n}(\beta)$ our test statistics in (21) and (22) computed on n i.i.d. points $\widetilde{X}_1, \ldots, \widetilde{X}_n$ distributed according to the density $\mathbf{1}_W + a_n g$ (i.e., we have a triangular scheme).

Theorem 4.1 *Let $\beta > -d/2$ and assume that W satisfies (2), that $n^2 r_n^d \to \infty$, $n r_n^{d+1} \to 0$ and $\min\{n r_n^{d/2+1} a_n, r_n/a_n\} \to 0$ as $n \to \infty$ and that, for $r > 0$,*

$$\int_W \mathbf{1}\{d(x, \partial W) \leq r\} |g(x)|\,\mathrm{d}x \leq C_{W,g}\, r \tag{24}$$

with some constant $C_{W,g} \in (0, \infty)$. Then the following assertions hold:

(a) If $n r_n^{d/2} a_n^2 \to \gamma \in [0, \infty)$ as $n \to \infty$, then

$$\widetilde{T}_{e,n}(\beta) \xrightarrow{\mathcal{D}} \left(Z + \frac{\sqrt{d\kappa_d(2\beta+d)}}{\sqrt{2}(\beta+d)} \int_W g(x)^2\,\mathrm{d}x\ \gamma\right)^2 \quad \text{as} \quad n \to \infty$$

with $Z \sim N(0, 1)$.

(b) *If* $nr_n^{d/2}a_n^2 \to \infty$ *as* $n \to \infty$, *then*

$$\widetilde{T}_{e,n}(\beta) \xrightarrow{\mathbb{P}} \infty \quad as \quad n \to \infty.$$

(c) *If, additionally,* $n^2 r_n^{d+2} \to 0$ *as* $n \to \infty$, *the statements of (a) and (b) also hold for* $\widetilde{T}_{a,n}(\beta)$.

The condition (24) requires that the fluctuations of g in an r-neighbourhood of the boundary of W are at most of order r. Because we assume (2), this is always the case if g is bounded. The limiting random variable in Theorem 4.1(a) follows a non-central chi-squared distribution with one degree of freedom. For $nr_n^{d/2}a_n^2 \to 0$ as $n \to \infty$ Theorem 4.1 implies that $\widetilde{T}_{e,n}(\beta)$ and $\widetilde{T}_{a,n}(\beta)$ behave exactly as $T_{e,n}(\beta)$ and $T_{a,n}(\beta)$ under H_0. As the following result shows one can slightly modify Theorem 4.1 if g vanishes close to the boundary of W. By $\operatorname{supp} g$, we denote the support of g, i.e., the set of all $x \in W$ such that $g(x) \neq 0$. For $A, B \subset \mathbb{R}^d$ let $d(A,B) := \inf_{x\in A, y\in B} \|x-y\|$.

Theorem 4.2 *Let* $\beta > -d/2$ *and assume that* $d(\operatorname{supp} g, \partial W) > 0$, *that* W *satisfies* (2) *and that* $n^2 r_n^d \to \infty$ *and* $nr_n^{d+1} \to 0$ *as* $n \to \infty$. *Then, (a), (b) and (c) of Theorem 4.1 hold.*

We prepare the proofs of Theorem 4.1 and Theorem 4.2 with several lemmas. By $\widetilde{L}_n(\beta)$ we denote the statistic $L_n(\beta)$ with respect to i.i.d. points $\widetilde{X}_1, \dots, \widetilde{X}_n$ distributed according to the density $1 + a_n g$, while $L_n(\beta)$ is with respect to n i.i.d. points uniformly distributed in W.

Lemma 4.3 *Assume that* W *and* g *satisfy* (24) *and let* $n \geq 2$. *Then, for any* $\beta > -d$,

$$\begin{aligned}&\left|\mathbb{E}\widetilde{L}_n(\beta) - \mathbb{E}L_n(\beta) - \frac{n(n-1)a_n^2}{2}\int_{W^2} \mathbf{1}\{\|x-y\| \leq r_n\}\|x-y\|^\beta g(x)g(y)\,\mathrm{d}(x,y)\right| \\ &\leq \frac{d\kappa_d C_{W,g}}{\beta+d} n^2 r_n^{\beta+d+1} a_n.\end{aligned} \tag{25}$$

Moreover, for any $\beta > -d/2$,

$$\left|\operatorname{Var}\widetilde{L}_n(\beta) - \operatorname{Var}L_n(\beta)\right| \leq C\big(n^2 r_n^{2\beta+d} a_n(a_n + r_n) + n^3 r_n^{2\beta+2d} a_n(a_n + r_n + a_n^2 + a_n^3 + a_n^2 r_n)\big) \tag{26}$$

with some constant $C \in (0,\infty)$ *depending on* β, d, $C_{W,g}$ *and* g.

Proof: It follows from (3) in Theorem 2.1 that

$$\begin{aligned}\mathbb{E}\widetilde{L}_n(\beta) - \mathbb{E}L_n(\beta) = &\frac{n(n-1)a_n^2}{2}\int_{W^2} \mathbf{1}\{\|x-y\| \leq r_n\}\|x-y\|^\beta g(x)g(y)\,\mathrm{d}(x,y) \\ &+ n(n-1)a_n \int_{W^2} \mathbf{1}\{\|x-y\| \leq r_n\}\|x-y\|^\beta g(x)\,\mathrm{d}(x,y).\end{aligned} \tag{27}$$

We have

$$\begin{aligned}
&\int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\}\|x-y\|^{\beta} g(x)\,\mathrm{d}(x,y)\\
&= \frac{d\kappa_d r_n^{\beta+d}}{\beta+d} \int_W g(x)\,\mathrm{d}x\\
&\quad + \int_W \mathbf{1}\{d(x,\partial W) \le r_n\}\bigg(\int_W \mathbf{1}\{\|x-y\| \le r_n\}\|x-y\|^2\,\mathrm{d}y - \frac{d\kappa_d r_n^{\beta+d}}{\beta+d}\bigg) g(x)\,\mathrm{d}x.
\end{aligned}$$

Here, the first term is zero since $\int_W g(x)\,\mathrm{d}x = 0$. By (24), the absolute value of the second term can be bounded by

$$\frac{d\kappa_d}{\beta+d} r_n^{\beta+d} \int_W \mathbf{1}\{d(x,\partial W) \le r_n\}|g(x)|\,\mathrm{d}x \le \frac{d\kappa_d C_{W,g}}{\beta+d} r_n^{\beta+d+1},$$

which proves (25).

From Theorem 2.3(a) we can deduce

$$\begin{aligned}
&\operatorname{Var}\widetilde{L}_n(\beta) - \operatorname{Var}L_n(\beta)\\
&= \mathbb{E}\widetilde{L}_n(2\beta) - \mathbb{E}L_n(2\beta)\\
&\quad + n(n-1)(n-2) \int_{W^3} \mathbf{1}\{\|x-y_1\|, \|x-y_2\| \le r_n\}\|x-y_1\|^{\beta}\|x-y_2\|^{\beta}\\
&\qquad\qquad \big(a_n(2g(y_1)+g(x)) + a_n^2(g(y_1)g(y_2) + 2g(y_1)g(x))\\
&\qquad\qquad + a_n^3 g(y_1)g(y_2)g(x)\big)\,\mathrm{d}(y_1,y_2,x)\\
&\quad - \frac{4n-6}{n(n-1)}(\mathbb{E}\widetilde{L}_n(\beta) - \mathbb{E}L_n(\beta))(\mathbb{E}\widetilde{L}_n(\beta) + \mathbb{E}L_n(\beta))\\
&=: \bar{R}_{1,n} + \bar{R}_{2,n} - \bar{R}_{3,n}.
\end{aligned}$$

It follows from

$$\frac{n(n-1)a_n^2}{2}\bigg| \int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\}\|x-y\|^{2\beta} g(x)g(y)\,\mathrm{d}(x,y)\bigg| \le \frac{d\kappa_d}{2(2\beta+d)} \int_W g(x)^2\,\mathrm{d}x\; n^2 r_n^{2\beta+d} a_n^2$$

and (25) that

$$|\bar{R}_{1,n}| \le \frac{d\kappa_d}{2(2\beta+d)} \int_W g(x)^2\,\mathrm{d}x\; n^2 r_n^{2\beta+d} a_n^2 + \frac{d\kappa_d C_{W,g}}{2\beta+d} n^2 r_n^{2\beta+d+1} a_n.$$

From

$$\begin{aligned}
\mathbb{E}\widetilde{L}_n(\beta) + \mathbb{E}L_n(\beta) &\le \frac{d\kappa_d}{2(\beta+d)}\bigg(1 + \int_W (1+a_n g(x))^2\,\mathrm{d}x\bigg) n^2 r_n^{\beta+d}\\
&= \frac{d\kappa_d}{2(\beta+d)}\bigg(2 + a_n^2 \int_W g(x)^2\,\mathrm{d}x\bigg) n^2 r_n^{\beta+d},
\end{aligned}$$

$$\frac{n(n-1)a_n^2}{2}\left|\int_{W^2}\mathbf{1}\{\|x-y\|\le r_n\}\|x-y\|^{\beta}g(x)g(y)\,\mathrm{d}(x,y)\right|\le\frac{d\kappa_d}{2(\beta+d)}\int_W g(x)^2\,\mathrm{d}x\;n^2r_n^{\beta+d}a_n^2$$

and (25) we conclude

$$\begin{aligned}|\bar{R}_{3,n}|&\le C_3\frac{1}{n}(n^2r_n^{\beta+d}a_n^2+n^2r_n^{\beta+d+1}a_n)(1+a_n^2)n^2r_n^{\beta+d}\\&\le C_3n^3r_n^{2\beta+2d}a_n(a_n+r_n+a_n^3+a_n^2r_n)\end{aligned}$$

with some constant $C_3\in(0,\infty)$ depending on β, d, $C_{W,g}$ and g.

By similar arguments as for the second term in (27), one obtains

$$\begin{aligned}&n^3\left|\int_{W^3}\mathbf{1}\{\|x-y_1\|,\|x-y_2\|\le r_n\}\|x-y_1\|^{\beta}\|x-y_2\|^{\beta}a_n\big(2g(y_1)+g(x)\big)\,\mathrm{d}(y_1,y_2,x)\right|\\&\le\frac{6d^2\kappa_d^2C_{W,g}}{(\beta+d)^2}n^3r_n^{2\beta+2d+1}a_n.\end{aligned}$$

Moreover, one can show the inequality

$$\begin{aligned}&n^3\bigg|\int_{W^3}\mathbf{1}\{\|x-y_1\|,\|x-y_2\|\le r_n\}\|x-y_1\|^{\beta}\|x-y_2\|^{\beta}\\&\qquad\qquad\big(a_n^2(g(y_1)g(y_2)+2g(y_1)g(x))+a_n^3g(y_1)g(y_2)g(x)\big)\,\mathrm{d}(y_1,y_2,x)\bigg|\\&\le\frac{3d^2\kappa_d^2}{(\beta+d)^2}\int_W g(x)^2\,\mathrm{d}x\;n^3r_n^{2\beta+2d}a_n^2+\frac{d^2\kappa_d^2}{(\beta+d)^2}\int_W|g(x)|^3\,\mathrm{d}x\;n^3r_n^{2\beta+2d}a_n^3.\end{aligned}$$

Summarising, it follows that

$$|\bar{R}_{2,n}|\le C_2n^3r_n^{2\beta+2d}a_n(a_n+r_n+a_n^2)$$

with some constant $C_2\in(0,\infty)$ depending on β, d, $C_{W,g}$ and g. Combining the estimates for $\bar{R}_{1,n}$, $\bar{R}_{2,n}$ and $\bar{R}_{3,n}$ completes the proof of (26). □

Lemma 4.4 *Let $\beta>-d/2$ and assume that W satisfies (2), that $n^2r_n^d\to\infty$ and $\max\{nr_n^{d+1},nr_n^da_n^3\}\to0$ as $n\to\infty$ and that*

$$\lim_{n\to\infty}\frac{\operatorname{Var}\widetilde{L}_n(\beta)-\operatorname{Var}L_n(\beta)}{n^2r_n^{2\beta+d}}=0. \tag{28}$$

Then,

$$\frac{\widetilde{L}_n(\beta)-\mathbb{E}\widetilde{L}_n(\beta)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}nr_n^{\beta+d/2}}\xrightarrow{\mathcal{D}}N(0,1)\quad\text{as}\quad n\to\infty.$$

We prepare the proof of Lemma 4.4 with the following inequality.

Lemma 4.5 *For $p, q > 0$, $v \in L^{p+q}(W)$ and $a > 0$,*

$$\int_W \max\{v(x) - a, 0\}^p \, dx \leq \frac{1}{a^q} \int_W |v(x)|^{p+q} \, dx.$$

Proof: We have that

$$\int_W |v(x)|^{p+q} \, dx \geq \int_W \mathbf{1}\{v(x) \geq a\}(v(x) - a)^p a^q \, dx = a^q \int_W \max\{v(x) - a, 0\}^p \, dx,$$

which is the desired inequality. □

Proof of Lemma 4.4: In the following we consider the framework from the Lemmas 2.6, 2.7 and 2.8 with $f \equiv \mathbf{1}_W$ and $f_n := \mathbf{1}_W + a_n g$, $n \in \mathbb{N}$. Then, $\widetilde{L}_n(\beta)$ has the same distribution as $\widehat{L}_n(\beta)$. For the latter we will prove convergence to $N(0,1)$ after an appropriate rescaling.

It follows from (28) and Theorem 2.3(c) that

$$\lim_{n\to\infty} \frac{\operatorname{Var} \widehat{L}_n(\beta)}{\sigma_{\beta,f,n}^2} = \lim_{n\to\infty} \frac{\operatorname{Var} \widetilde{L}_n(\beta) - \operatorname{Var} L_n(\beta)}{\sigma_{\beta,f,n}^2} + \lim_{n\to\infty} \frac{\operatorname{Var} L_n(\beta)}{\sigma_{\beta,f,n}^2} = 1. \tag{29}$$

For the rest of this proof we choose $M = 2$. Lemma 2.8 yields

$$\begin{aligned}
&\mathbb{E}\left(\frac{\widehat{L}_n(\beta) - \mathbb{E}\widehat{L}_n(\beta)}{\sigma_{\beta,f,n}} - \frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f,n}}\right)^2 \\
&\leq \frac{d\kappa_d}{(2\beta + d)} \frac{n^2 r_n^{2\beta+d}}{\sigma_{\beta,f,n}^2} \int_W \overline{f}_{n,M}(x)^2 + M\overline{f}_{n,M}(x) \, dx \\
&\quad + \frac{18 d^2 \kappa_d^2}{(\beta + d)^2} \frac{n^3 r_n^{2\beta+2d}}{\sigma_{\beta,f,n}^2} \int_W M^2 \overline{f}_{n,M}(x) + M\overline{f}_{n,M}(x)^2 + \overline{f}_{n,M}(x)^3 \, dx.
\end{aligned} \tag{30}$$

It follows from Lemma 4.5 (with $p = 1$, $q = 2$ and $p = 2$, $q = 1$, respectively) that

$$\int_W \overline{f}_{n,M}(x) \, dx = a_n \int_W \max\{g(x) - 1/a_n, 0\} \, dx \leq a_n^3 \int_W |g(x)|^3 \, dx$$

and

$$\int_W \overline{f}_{n,M}(x)^2 \, dx = a_n^2 \int_W \max\{g(x) - 1/a_n, 0\}^2 \, dx \leq a_n^3 \int_W |g(x)|^3 \, dx.$$

Moreover, we have

$$\int_W \overline{f}_{n,M}(x)^3 \, dx = a_n^3 \int_W \max\{g(x) - 1/a_n, 0\}^3 \, dx \leq a_n^3 \int_W |g(x)|^3 \, dx.$$

Since $\sigma_{\beta,f,n}^2 = \sigma_{\beta,f}^{(1)} n^2 r_n^{2\beta+d}$, the right-hand side of (30) is at most of order

$$\frac{n^2 r_n^{2\beta+d}}{\sigma_{\beta,f,n}^2} a_n^3 + \frac{n^3 r_n^{2\beta+2d}}{\sigma_{\beta,f,n}^2} a_n^3 = \frac{1 + n r_n^d}{\sigma_{\beta,f}^{(1)}} a_n^3,$$

which vanishes as $n \to \infty$. This means that

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{\widehat{L}_n(\beta) - \mathbb{E}\widehat{L}_n(\beta)}{\sigma_{\beta,f,n}} - \frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f,n}}\right)^2 = 0. \tag{31}$$

Together with (29) we see that

$$\lim_{n\to\infty} \frac{\operatorname{Var}\widehat{L}_{n,M}(\beta)}{\sigma^2_{\beta,f,n}} = 1. \tag{32}$$

It follows from Lemma 2.7, where the variance condition is satisfied because of (32) and $\sigma^2_{\beta,f,n} = \sigma^2_{\beta,f_M,n}$, that

$$\frac{\widehat{L}_{n,M}(\beta) - \mathbb{E}\widehat{L}_{n,M}(\beta)}{\sigma_{\beta,f,n}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad n \to \infty.$$

Because of the L^2-convergence in (31) this yields

$$\frac{\widehat{L}_n(\beta) - \mathbb{E}\widehat{L}_n(\beta)}{\sigma_{\beta,f,n}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad n \to \infty,$$

which completes the proof. □

Proof of Theorem 4.1: By Lemma 4.3 we have that

$$\frac{\mathbb{E}\widetilde{L}_n(\beta) - \mathbb{E}L_n(\beta)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}\, n r_n^{\beta+d/2}} = T_n + R_n \tag{33}$$

with

$$T_n := \frac{\sqrt{2\beta+d}}{\sqrt{2d\kappa_d}} \frac{(n-1)a_n^2}{r_n^{\beta+d/2}} \int_{W^2} \mathbf{1}\{\|x-y\| \le r_n\} \|x-y\|^\beta g(x)g(y)\, \mathrm{d}(x,y)$$

and a remainder term R_n satisfying

$$|R_n| \le \frac{C_{W,g}\sqrt{2d\kappa_d(2\beta+d)}}{\beta+d} n r_n^{d/2+1} a_n. \tag{34}$$

As in the proof of Theorem 2.1 one can show that

$$\lim_{n\to\infty} \frac{T_n}{n r_n^{d/2} a_n^2} = \frac{\sqrt{d\kappa_d(2\beta+d)}}{\sqrt{2}(\beta+d)} \int_W g(x)^2\, \mathrm{d}x. \tag{35}$$

For $\gamma = 0$ one obtains $\lim_{n\to\infty} T_n = 0$ and $\lim_{n\to\infty} R_n = 0$. The latter follows from the assumption $\min\{n r_n^{d/2+1} a_n, r_n/a_n\} \to 0$ as $n \to \infty$, whence, by (34), R_n vanishes directly or is of a lower order than T_n and, thus, also vanishes.

For $\gamma > 0$ or $n r_n^{d/2} a_n^2 \to \infty$ as $n \to \infty$, we have that $\lim_{n\to\infty} r_n/a_n = 0$. Indeed, if there was a subsequence (n_m) such that $r_{n_m}/a_{n_m} \ge c$ for some $c > 0$, we would have $n_m r_{n_m}^{d/2+1} a_{n_m} \ge$

$cn_m r_{n_m}^{d/2} a_{n_m}^2$. Then $\min\{n_m r_{n_m}^{d/2+1} a_{n_m}, r_{n_m}/a_{n_m}\}$ would not converge to 0 as $m \to \infty$, which is a contradiction. Because of (34) and (35) it follows from $\lim_{n\to\infty} r_n/a_n = 0$ that $\lim_{n\to\infty} R_n/T_n = 0$, whence T_n is the leading summand in (33).

Assume that $nr_n^{d/2}a_n^2 \to \gamma \in [0,\infty)$ as $n \to \infty$. By (26), we have

$$\lim_{n\to\infty} \frac{|\operatorname{Var}\widetilde{L}_n(\beta) - \operatorname{Var} L_n(\beta)|}{n^2 r_n^{2\beta+d}} \le C \lim_{n\to\infty} a_n(a_n + r_n) + nr_n^d a_n(a_n + r_n + a_n^2 + a_n^3 + a_n^2 r_n) = 0,$$

where we also used that $a_n, r_n, nr_n^{d+1} \to 0$ as $n \to \infty$. Now Lemma 4.4 implies

$$\frac{\widetilde{L}_n(\beta) - \mathbb{E}\widetilde{L}_n(\beta)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}\, nr_n^{\beta+d/2}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad n \to \infty.$$

This together with (33) and the above analysis of the asymptotic behaviour of T_n and R_n yields

$$\frac{\widetilde{L}_n(\beta) - \mathbb{E}L_n(\beta)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}\, nr_n^{\beta+d/2}} \xrightarrow{\mathcal{D}} N\left(\frac{\sqrt{d\kappa_d(2\beta+d)}}{\sqrt{2}(\beta+d)} \int_W g(x)^2 \,\mathrm{d}x\ \gamma, 1\right) \quad \text{as} \quad n \to \infty.$$

Now (a) follows from the continuous mapping theorem.

Next we show part (b). It follows from (26) that

$$\frac{\operatorname{Var}\widetilde{L}_n(\beta)}{(n^2 r_n^{\beta+d} a_n^2)^2} \le C \frac{a_n(a_n + r_n) + nr_n^d a_n(a_n + r_n + a_n^2 + a_n^3 + a_n^2 r_n)}{(nr_n^{d/2} a_n^2)^2} + \frac{\operatorname{Var} L_n(\beta)}{(n^2 r_n^{\beta+d} a_n^2)^2}.$$

The first term on the right-hand side vanishes as $n \to \infty$ since $a_n, r_n, nr_n^{d+1} \to 0$ and $nr_n^{d/2}a_n^2 \to \infty$ as $n \to \infty$. Because $\operatorname{Var} L_n(\beta)$ behaves as $n^2 r_n^{2\beta+d}$, the second term is of order $1/(nr_n^{d/2}a_n^2)^2$ and converges to zero as $n \to \infty$. We thus have

$$\lim_{n\to\infty} \frac{\operatorname{Var}\widetilde{L}_n(\beta)}{(n^2 r_n^{\beta+d} a_n^2)^2} = 0 \quad \text{and} \quad \frac{\widetilde{L}_n(\beta) - \mathbb{E}\widetilde{L}_n(\beta)}{n^2 r_n^{\beta+d} a_n^2} \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty.$$

Together with the fact that T_n is the dominating term in (33) and (35), this means that

$$\frac{\widetilde{L}_n(\beta) - \mathbb{E}L_n(\beta)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}\, n^2 r_n^{\beta+d} a_n^2} \xrightarrow{\mathbb{P}} \frac{\sqrt{d\kappa_d(2\beta+d)}}{\sqrt{2}(\beta+d)} \int_W g(x)^2 \,\mathrm{d}x \quad \text{as} \quad n \to \infty.$$

Because of $nr_n^{d/2}a_n^2 \to \infty$ as $n \to \infty$ this implies

$$\frac{\widetilde{L}_n(\beta) - \mathbb{E}L_n(\beta)}{\sqrt{\frac{d\kappa_d}{2(2\beta+d)}}\, nr_n^{\beta+d/2}} \xrightarrow{\mathbb{P}} \infty \quad \text{as} \quad n \to \infty,$$

which proves part (b).

Part (c) follows from (12) in the proof of Corollary 2.5. □

Proof of Theorem 4.2: Without loss of generality we can assume that $r_n < d(\operatorname{supp} g, \partial W)$ for each n. Consequently, the assumption (24) is satisfied with $C_{W,g} = 0$ for $r = r_n$. Now the proof of Theorem 4.1 works without the additional assumption that $\min\{nr_n^{d/2+1}a_n, r_n/a_n\} \to 0$ as $n \to \infty$ because $R_n = 0$. □

From Theorem 4.1 and Theorem 4.2, we conclude that under the stated assumptions the tests based on $\widetilde{T}_{a,n}(\beta)$ and $\widetilde{T}_{e,n}(\beta)$ are able to detect alternatives which converge to the uniform distribution at rate a_n. Moreover, the theorems could be the foundation of establishing local optimality of the tests by applying the third Le Cam lemma, see Section 5.2 of [21] for a short review of the needed methodology.

5 Simulation

In this section we compare the finite-sample power performance of the test statistics $T_{e,n}(\beta)$ and $T_{a,n}(\beta)$, $\beta > -d/2$, $n \in \mathbb{N}$, with that of some competitors. Since the d-dimensional hypercube $[0,1]^d$ is the mostly used observation window, we restrict our simulation study to this case with $d \in \{2,3\}$. Particular interest will be given to the influence on the finite-sample power of β and r_n in dependence of the chosen alternatives. In each scenario, we consider the sample sizes $n \in \{50, 100, 200, 500\}$ and set the nominal level of significance to 0.05. Since the test statistics depend on the parameter β and the choice of r_n and the empirical finite sample quantile is in some cases far away from the quantile $\chi^2_{1,0.95} \approx 3.8415$ of the limiting distribution, we simulated critical values for $T_{e,n}(\beta)$ and $T_{a,n}(\beta)$ with 100 000 replications, see Tables 9 to 12. Each stated empirical power of the tests in Tables 4 to 8 is based on 10 000 replications and the asterisk $*$ denotes a rejection rate of 100%.

Since there is a vast variety of ways to choose the parameters β and r_n, we chose the parameter configurations to fit the limiting regimes of Corollary 2.5 as well as the following additional property: From (6) we know that the expectation of the average degree $\bar{D}_n$ behaves as $\kappa_d n r_n^d$ for $n \to \infty$ under H_0. This observation motivates the following choices of the radius r_n for $T_{e,n}(\beta)$, namely

$$r_n = \left(\frac{k}{n\kappa_d}\right)^{\frac{1}{d}}, \quad k \in \{1, \dots, 10\},$$

which satisfies $n^2 r_n^d \to \infty$ and $nr_n^{d+1} \to 0$ as $n \to \infty$ and ensures $\mathbb{E}\bar{D}_n \to k$ as $n \to \infty$ under H_0. For the test statistic $T_{a,n}(\beta)$ the additional condition $n^2 r_n^{d+2} \to 0$ as $n \to \infty$ has to be fulfilled, so we choose

$$r_n := \left(\frac{k}{n^{\frac{3}{2}}\kappa_d}\right)^{\frac{1}{d}}, \quad k \in \{1, \dots, 10\},$$

to guarantee this additional assumption for $d \in \{2,3\}$. In this case we have $\mathbb{E}\bar{D}_n \to 0$ as $n \to \infty$, which for $d = 2$ is always the case if $n^2 r_n^{d+2} \to 0$ as $n \to \infty$.

The expected value $\mathbb{E}L_n(\beta)$ depends on the observation window W as well as on the dimension $d \geq 2$. The following lemma provides exact formulae of $\mathbb{E}L_n(\beta)$ for each of the cases simulated.

Lemma 5.1 *Assume $\beta > -d$ and $f \equiv \mathbf{1}_W$.*

(a) If $d = 2$, $W = [0,1]^2$ and $r_n \leq 1$, then

$$\mathbb{E}L_n(\beta) = \frac{n(n-1)}{2}\left(\frac{2\pi}{\beta+2}r_n^{\beta+2} - \frac{8}{\beta+3}r_n^{\beta+3} + \frac{2}{\beta+4}r_n^{\beta+4}\right).$$

(b) If $d = 3$, $W = [0,1]^3$ and $r_n \leq 1$, then

$$\mathbb{E}L_n(\beta) = \frac{n(n-1)}{2}\left(\frac{4\pi}{\beta+3}r_n^{\beta+3} - \frac{6\pi}{\beta+4}r_n^{\beta+4} + \frac{8}{\beta+5}r_n^{\beta+5} - \frac{1}{\beta+6}r_n^{\beta+6}\right).$$

Proof: Let $d \in \{2,3\}$, $W = [0,1]^d$ and $r_n \leq 1$. We apply Corollary 2.2 to obtain

$$\begin{aligned}
\mathbb{E}L_n(\beta) &= \frac{n(n-1)}{2}\int_{W^2}\mathbf{1}\{\|x-y\| \leq r_n\}\|x-y\|^\beta \,\mathrm{d}(x,y) \\
&= \frac{n(n-1)}{2}\int_{B^d(0,r_n)}\|y\|^\beta \int_{\mathbb{R}^d}\mathbf{1}\{x \in W,\, x-y \in W\}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{n(n-1)}{2}\int_{B^d(0,r_n)}\|y\|^\beta \operatorname{Vol}(W \cap (W+y))\,\mathrm{d}y \\
&= \frac{n(n-1)}{2}\int_{B^d(0,r_n)}\|y\|^\beta \prod_{j=1}^{d}(1-|y_j|)\,\mathrm{d}y,
\end{aligned}$$

with $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. The formulae in (a) and (b) follow now from a longer calculation with polar coordinates. □

As competitors to the new test statistics we consider the distance to boundary test (DB-test), see [7], the maximal spacing test (MS-test), see [5, 16], the nearest neighbour type test (NN-test) of [12] as well as the Bickel-Rosenblatt test (BR-test) presented in [31]. We follow the descriptions of the DB- and MS-tests given in [12].

For the NN-test we consider the family of statistics

$$NN_{n,J}^{(\beta)} := \sum_{x \in \mathscr{X}_n}\xi_{n,J}^{(\beta)}(x, \mathscr{X}_n)$$

in dependence of $\beta \in (0,\infty)$, where J is the number of nearest neighbours, with $x^{(k)}$ being the kth nearest neighbour of $x \in \mathscr{X}_n$ and

$$\xi_{n,J}^{(\beta)}(x,\mathscr{X}_n) := \sum_{k=1}^{J}(\kappa_d \|n^{1/d}(x-x^{(k)})\|^d)^\beta.$$

To avoid boundary problems in the computation of the NN-test, we used the same toroid metric in the simulation as in [12]. Since rejection rates depend crucially on the power β and the number of neighbours J taken into account, we chose different values for β and J for the two alternatives where the choice was motivated by Table 2 in [12]. Notice that this test is consistent, but one has to be careful to choose the correct rejection region, which depends on the choice of β.

As a further competitor we consider the fixed bandwidth Bickel-Rosenblatt test (BR-test) on the unit cube, studied in [31]. Using the notation of [31], the corresponding test statistic is

$$I_n^2(h) = -I_n^{2,1}(h) + I_n^{2,2}(h) + W_h(0) + n(W_h \star \overline{U} \star U)(0),$$

with

$$I_n^{2,1}(h) = 2\sum_{i=1}^{n}(W_h \star U)(X_i) \quad \text{and} \quad I_n^{2,2}(h) = \frac{2}{n}\sum_{1\le i<j\le n} W_h(X_i - X_j),$$

where $h > 0$ is a fixed bandwidth. For the sake of completeness we restate the following abbreviations. The convolution product operator is denoted by $\star$, $U = \mathbf{1}_{[0,1]^d}$ is the density of the uniform distribution over the unit hypercube $[0,1]^d$ and for any function g we define $\overline{g}(x) := g(-x)$ and $g_h(x) := g\left(\frac{x}{h}\right)/h^d$ with $h > 0$. Furthermore, we set $W := \overline{K} \star K$, where K is a product kernel on $\mathbb{R}^d$, that is, $K(u) = \prod_{i=1}^d k(u_i)$, $u = (u_1,\dots,u_d) \in \mathbb{R}^d$ with a kernel k on $\mathbb{R}$ (so k is bounded and integrable). Using the arguments and techniques in [31], direct calculations for $d = 2$ and $k(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$, $x \in \mathbb{R}$, being the standard normal density function, give for $h > 0$,

$$\begin{aligned}
I_n^{2,1}(h) &= 2\sum_{i=1}^{n}\left(\Phi\left(\frac{X_{i,1}-1}{\sqrt{2}h}\right) - \Phi\left(\frac{X_{i,1}}{\sqrt{2}h}\right)\right)\left(\Phi\left(\frac{X_{i,2}-1}{\sqrt{2}h}\right) - \Phi\left(\frac{X_{i,2}}{\sqrt{2}h}\right)\right),\\
I_n^{2,2}(h) &= \frac{1}{2\pi n h^4}\sum_{1\le i<j\le n}\exp\left(-\frac{(X_{i,1}-X_{j,1})^2}{4h^2}\right)\exp\left(-\frac{(X_{i,2}-X_{j,2})^2}{4h^2}\right)
\end{aligned}$$

and

$$W_h(0) = \frac{1}{4\pi h^4},$$
$$n(W_h \star \overline{U} \star U)(0) = \frac{4n}{\pi h^2}\left[\sqrt{\pi}\left(\Phi\left(\frac{1}{\sqrt{2}h}\right) - \frac{1}{2}\right) + h\left(\exp\left(-\frac{1}{4h^2}\right) - 1\right)\right]^2,$$

where Φ is the standard normal distribution function and $X_{i,j}$ denotes the jth component of the random vector X_i, with $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$.

The BR-test rejects the null hypothesis for large values of $I_n^2(h)$. Notice that the asymptotic distribution of $I_n^2(h)$ is known, see [31], but not in a closed form. Hence we simulated critical values of $I_n^2(h)$ for $h \in \{0.1, 0.25, 0.5, 1\}$, which can be found in Table 2.

$n\backslash h$	0.1	0.25	0.5	1
50	9113.028	827.3781	72.70593	0.01799183
100	17048.245	1616.5611	144.16370	0.01799641
200	32839.801	3186.1990	286.73272	0.01795072
500	80073.212	7876.7591	713.74621	0.01787482

Table 2: Critical values of the BR-statistic $I_n^2(h)$

Following the studies in [6, 12], we simulated a contamination and a clustering model as alternatives to the uniform distribution. The contamination alternative (CON) is given by the mixture

$$(1 - q_1 - q_2)\mathcal{U}([0,1]^d) + q_1 N_d(c_1, \sigma_1^2 I_d) + q_2 N_d(c_2, \sigma_2^2 I_d),$$

under the condition that all simulated points are located in $[0,1]^d$. Here, $I_d \in \mathbb{R}^{d\times d}$ denotes the identity matrix of order d. The chosen parameters are given in Table 3, where $\Phi^{-1}(p)$, $p \in (0,1)$, denotes the p-quantile of a standard normal distribution. See Figure 1, second row, for a realisation of this model, where the normally distributed contamination points are filled points and filled squares, respectively.

d	q_1	q_2	c_1	c_2	σ_1	σ_2
2	0.135	0.24	$(0.25, 0.25)$	$(0.7, 0.7)$	$0.15 \cdot \Phi^{-1}(\sqrt{0.9})$	$0.2 \cdot \Phi^{-1}(\sqrt{0.9})$
3	0.135	0.24	$(0.25, 0.25, 0.25)$	$(0.7, 0.7, 0.7)$	$0.15 \cdot \Phi^{-1}(\sqrt[3]{0.9})$	$0.2 \cdot \Phi^{-1}(\sqrt[3]{0.9})$

Table 3: Parameter configuration of the CON-alternatives

The clustering alternative (CLU) is motivated by a fixed number of data points version of a Matérn cluster processes, see Section 12.3 in [2], and is designed to destroy the independence.

One first chooses a radius r_{clu} and simulates $\frac{n}{5}$ random points with the uniform distribution $\mathcal{U}([-r_{\text{clu}}, 1 + r_{\text{clu}}]^d)$, that act as centres of clusters. These points will not be part of the final sample. In a second step, one generates 5 points around each centre in a ball with radius r_{clu}. These points are generated independently of each other and follow uniform distributions on the mentioned balls. If a point falls outside $[0,1]^d$, it is replaced by a point that follows a $\mathcal{U}([0,1]^d)$ distribution. In the following we set $r_{\text{clu}} = 0.1$ and a realisation of this model can be found in Figure 1, third row. The clustering alternative is not included in the framework of our theoretical results since the points are, by construction, not independent. Nevertheless it is interesting to see how the test statistics behave for such alternatives, which were also considered in the simulation study in [7].

We now present the simulation results for $d = 2$. Table 4 exhibits the empirical percentage of rejection of the competing procedures under discussion. An asterisk stands for power of 100%, and in each row the best performing procedures have been highlighted using boldface ciphers. Clearly, $I_n^2(0.1)$ and $NN_{n,15}^{(0.5)}$ dominate the other procedures for the CON-alternative, but as noted in [12] the performance of $NN_{n,J}^{(0.5)}$ might even increase for bigger values of J. Comparison with $T_{e,n}(\beta)$ for $\beta = -0.5$ (see Table 5) shows that the presented new methods are for sample sizes of $n = 100, 200, 500$ as good as and for $n = 50$ nearly as good as the best competitor $I_n^2(0.1)$. As one can witness throughout the Tables 5 and 6, $T_{e,n}(\beta)$ dominates $T_{a,n}(\beta)$ for small sample sizes, while the power is similar to the best competitors. In case of the CLU alternative $T_{e,n}(\beta)$ gives the overall highest performance for $\beta = -0.5$ over small sample sizes of $n = 50, 100, 200$, while the only procedure that is better for $n = 500$ is again $NN_{n,15}^{(0.5)}$. Notice that the asymptotic version $T_{a,n}(\beta)$ might even achieve higher performance if one considers bigger radii, since it attains the highest rates for the biggest values of k. A closer look at these tables reveals the dependency of the new tests on the choice of β and k. Interestingly, the highest performance is given for both alternatives and $T_{j,n}(\beta)$, $j \in \{a, e\}$, for the choice of $\beta = -0.5$. The best choice of k obviously depends on the sample size.

Observe that the simulation results for $d = 3$ in Tables 7 and 8 show higher rejection rates for $T_{j,n}(\beta)$ than in the bivariate setting. Since the other methods were too time consuming to implement or to simulate we restrict the comparison to the DB-test. As can be seen in Table 7 the new tests dominate the DB-method for $\beta = -0.5$ and nearly for every value of k.

6 Conclusions and open problems

We have introduced two new families of consistent goodness-of-fit tests of uniformity based on random geometric graphs. As the simulation section shows, the presented methods are serious competitors to existing methods, even dominating them for right choices of the parameters β and r_n (or k). Clearly, a natural question is to find (data dependent) best choices of them. Another obvious extension of the presented methods would be to find tests of uniformity on (lower dimensional) manifolds, including special cases of directional statistics as the circle or the sphere (for existing methods see Chapter 6 of either [21] or [23]). Section 4 invites to further investigate in view of concepts of locally optimal tests. Since the approach is fairly general, an extension would be testing the fit of $X_1, \ldots, X_n$ to some parametric family $\{f(\cdot, \vartheta) : \vartheta \in \Theta\}$ of densities for a specific parameter space Θ (eventually the procedures would use a suitable estimator $\widehat{\vartheta}_n$ of ϑ). In view of the special interest in the case of unknown support of the data, see [5, 6], we indicate that the definition of $T_{a,n}(\beta)$ is not dependent on the shape of the underlying observation window and therefore is applicable in this setting (as long as the observation window has volume one).

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Appendix A A consequence of Lebesgue's differentiation theorem

Lemma A.1 *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a measurable function with $\|g\|_\infty := \sup_{y \in \mathbb{R}^d} |g(y)| < \infty$ and let $\beta > -d$. Then, for almost all $x \in \mathbb{R}^d$,*

$$\lim_{r \to 0} \frac{1}{r^{\beta+d}} \int_{B^d(x,r)} \|x-y\|^\beta \, g(y) \, \mathrm{d}y = \frac{d\kappa_d}{\beta+d} g(x).$$

Proof: We choose $p > 1$ subject to $p\beta > -d$. Then for any $x \in \mathbb{R}^d$ and $r > 0$,

$$\begin{aligned}
&\left| \frac{1}{r^{\beta+d}} \int_{B^d(x,r)} \|x-y\|^\beta g(y) \, \mathrm{d}y - \frac{d\kappa_d}{\beta+d} g(x) \right| \\
&\leq \frac{1}{r^d} \int_{B^d(x,r)} \frac{\|y-x\|^\beta}{r^\beta} |g(y) - g(x)| \, \mathrm{d}y \\
&\leq \left(\frac{1}{r^d} \int_{B^d(x,r)} \frac{\|y-x\|^{p\beta}}{r^{p\beta}} \, \mathrm{d}y \right)^{1/p} \left(\frac{1}{r^d} \int_{B^d(x,r)} |g(y) - g(x)|^{p/(p-1)} \, \mathrm{d}y \right)^{(p-1)/p} \\
&= \left(\frac{d\kappa_d}{p\beta+d} \right)^{1/p} \left(\frac{1}{r^d} \int_{B^d(x,r)} |g(y) - g(x)|^{p/(p-1)} \, \mathrm{d}y \right)^{(p-1)/p},
\end{aligned}$$

where we have used the Hölder inequality in the second last step. By Lebesgue's differentiation theorem (see, for example, [29, Theorem 8.8]), we have

$$\lim_{r \to \infty} \frac{1}{r^d} \int_{B^d(x,r)} |g(y) - g(x)| \, \mathrm{d}y = 0.$$

for almost all $x \in \mathbb{R}^d$. Since $|g(y) - g(x)|^{p/(p-1)} \leq \left(2\|g\|_\infty\right)^{1/(p-1)} |g(y) - g(x)|$, we have

$$\lim_{r \to \infty} \frac{1}{r^d} \int_{B^d(x,r)} |g(y) - g(x)|^{1/(p-1)} \, \mathrm{d}y = 0.$$

for almost all $x \in \mathbb{R}^d$. Together with the above inequalities this proves the statement. □

Appendix B A central limit theorem for a triangular scheme of U-statistics

In the following we provide a central limit theorem for second-order U-statistics of a triangular scheme of random vectors, which is a slight generalisation of [17, Theorem 2.1].

For each $n \in \mathbb{N}$ let $Y_1^{(n)}, \dots, Y_n^{(n)}$ be i.i.d. random vectors in $\mathbb{R}^d$, whose distribution may depend on n. We use the shorthand notation $\mathscr{Y}_n = \{Y_1^{(n)}, \dots, Y_n^{(n)}\}$, $n \in \mathbb{N}$, in the sequel. For $n \in \mathbb{N}$ let $h_n : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a bounded, symmetric and measurable function and let

$$S_n := \frac{1}{2} \sum_{(y_1, y_2) \in \mathscr{Y}_{n,\neq}^2} h_n(y_1, y_2).$$

The random variables S_n, $n \in \mathbb{N}$, are so-called second order U-statistics. The following theorem provides a sufficient criterion for the convergence of (S_n), after rescaling, to a standard Gaussian random variable.

Theorem B.1 *Let S_n, $n \in \mathbb{N}$, be as above. Assume that* $\operatorname{Var} S_n > 0$ *for all* $n \in \mathbb{N}$ *and let* $\sigma_n > 0$, $n \in \mathbb{N}$, *be such that* $\lim_{n\to\infty} \operatorname{Var} S_n / \sigma_n^2 = 1$. *If*

$$\lim_{n\to\infty} \frac{1}{\sigma_n} \sup_{y_1, y_2 \in \mathbb{R}^d} |h_n(y_1, y_2)| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{n}{\sigma_n} \sup_{y \in \mathbb{R}^d} \mathbb{E}|h_n(y, Y_1^{(n)})| = 0,$$

then

$$\frac{S_n - \mathbb{E} S_n}{\sigma_n} \xrightarrow{\mathcal{D}} N(0,1) \quad \textit{as} \quad n \to \infty.$$

Proof: In the special case that $(Y_i^{(n)})_{1 \le i \le n < \infty}$ are identically distributed, this is a slightly rewritten version of [17, Theorem 2.1]. Otherwise, there are measurable maps $T_n : [0,1] \to \mathbb{R}^d$, $n \in \mathbb{N}$, such that $Y_i^{(n)}$, $i \in \{1, \dots, n\}$, has the same distribution as $T_n(U)$, where U is a uniformly distributed random variable on $[0,1]$ (see, for example, the proof of Theorem 29.6 in [9]). For $n \in \mathbb{N}$ define $\tilde{h}_n : [0,1]^2 \ni (u_1, u_2) \mapsto h_n(T_n(u_1), T_n(u_2))$ and let $\mathcal{U}_n := \{U_1, \dots, U_n\}$, where $U_1, \dots, U_n$ are independent and uniformly distributed on $[0,1]$. Then, S_n has the same distribution as

$$\widetilde{S}_n := \frac{1}{2} \sum_{(u_1, u_2) \in \mathcal{U}_{n\neq}^2} \tilde{h}_n(u_1, u_2).$$

Since the assumptions of the theorem are satisfied for the U-statistics (S_n), they must also hold for the U-statistics $(\widetilde{S}_n)$. As the underlying random variables of $(\widetilde{S}_n)$ are identically distributed, we are in the previously discussed special case for which the central limit theorem holds. This completes the proof. $\square$

Alt.	n	$I_n^2(0.1)$	$I_n^2(0.25)$	$I_n^2(0.5)$	$I_n^2(1)$	$NN_{n,1}^{(0.5)}$	$NN_{n,15}^{(0.5)}$	DB	MS
CON	50	**74**	40	33	6	16	66	31	6
	100	**96**	66	56	9	19	90	58	14
	200	*	91	83	14	25	98	89	25
	500	*	*	99	36	41	*	*	41
CLU	50	**80**	34	31	42	78	67	28	36
	100	73	30	27	41	74	**82**	28	48
	200	61	26	24	41	58	**90**	28	52
	500	45	23	22	41	32	**96**	29	47

Table 4: Empirical rejection rates of the different competitors ($d = 2$)

B. Ebner and F. Nestmann,
Karlsruhe Institute of Technology (KIT),
Institute of Stochastics,
Englerstr. 2, D-76131 Karlsruhe, Germany.
E-mail: Bruno.Ebner@kit.edu and Franz.Nestmann2@kit.edu

M. Schulte,
University of Bern,
Institute of Mathematical Statistics and Actuarial Science,
Alpeneggstr. 22, CH-3012 Bern, Switzerland
E-mail: Matthias.Schulte@stat.unibe.ch

Alt.	β	$n\backslash k$	1	2	3	4	5	6	7	8	9	10	15	20	25
CON	−0.5	50	39	54	61	66	69	71	**72**	72	72	72	67	60	51
		100	59	77	85	90	92	94	95	95	**96**	96	96	96	95
		200	82	95	98	99	99	*	*	*	*	*	*	*	*
		500	99	*	*	*	*	*	*	*	*	*	*	*	*
CLU		50	95	**97**	97	96	94	91	88	86	83	80	68	59	53
		100	91	96	**97**	97	97	96	95	94	93	92	82	73	65
		200	81	92	95	**96**	96	96	96	96	96	95	91	86	79
		500	59	77	85	89	91	92	93	**94**	94	94	94	92	90
CON	0	50	41	57	64	68	70	**71**	71	71	71	69	59	45	34
		100	64	80	88	91	93	94	95	**96**	96	96	96	94	92
		200	85	96	99	*	*	*	*	*	*	*	*	*	*
		500	*	*	*	*	*	*	*	*	*	*	*	*	*
CLU		50	95	**96**	95	91	87	81	75	68	64	59	43	35	31
		100	92	**96**	96	96	95	93	92	89	86	82	64	51	43
		200	83	92	94	**96**	95	95	95	94	93	92	84	74	63
		500	61	80	86	89	91	92	**93**	93	93	93	91	88	84
CON	1	50	40	53	59	**63**	64	64	64	63	61	58	40	25	16
		100	60	77	85	89	91	93	**94**	94	94	94	93	89	82
		200	83	96	98	99	99	*	*	*	*	*	*	*	*
		500	*	*	*	*	*	*	*	*	*	*	*	*	*
CLU		50	**91**	91	86	78	67	56	47	41	37	34	29	28	29
		100	88	**92**	92	91	88	85	80	74	68	63	40	32	29
		200	79	88	**91**	91	91	90	89	88	86	83	69	54	42
		500	56	75	81	85	87	88	**89**	89	89	89	85	79	73
CON	5	50	29	38	43	**45**	44	43	41	38	35	31	15	8	8
		100	44	61	71	77	80	81	83	83	**84**	83	77	64	45
		200	65	86	93	96	97	98	**99**	99	99	99	99	99	99
		500	97	*	*	*	*	*	*	*	*	*	*	*	*
CLU		50	**72**	68	56	42	30	24	24	24	25	27	29	30	30
		100	**69**	75	73	69	62	55	48	41	35	30	25	26	27
		200	57	69	73	**74**	72	70	68	65	61	57	39	29	26
		500	36	53	61	65	69	70	**71**	71	71	71	65	57	50

Table 5: Empirical rejection rates for T_e ($d = 2$)

Alt.	β	$n\backslash k$	1	2	3	4	5	6	7	8	9	10	15	20	25
CON	−0.5	50	16	18	22	25	27	30	31	33	34	35	**36**	35	32
		100	17	22	27	32	36	40	43	46	48	50	57	60	**61**
		200	18	26	34	41	45	51	56	60	63	66	77	82	**85**
		500	24	36	47	56	64	71	76	80	83	86	94	97	**99**
CLU		50	65	79	85	89	91	92	**93**	93	93	93	92	88	83
		100	41	55	66	72	77	81	83	85	87	87	89	**90**	88
		200	22	31	38	45	50	55	59	62	65	68	75	78	**80**
		500	11	14	17	20	22	25	26	29	30	32	40	46	**50**
CON	0	50	11	15	23	29	27	32	34	32	34	**36**	36	32	27
		100	15	20	28	33	37	41	43	46	47	49	57	59	**60**
		200	16	27	33	43	50	56	61	64	68	71	78	82	**86**
		500	27	39	51	60	67	75	79	83	85	88	95	98	**99**
CLU		50	59	76	86	91	90	92	**93**	92	92	92	89	81	71
		100	38	53	66	74	78	81	83	84	85	85	**88**	87	84
		200	20	31	37	47	54	59	63	66	68	70	75	77	**78**
		500	12	14	18	21	23	27	28	31	31	34	41	46	**51**
CON	1	50	17	19	21	24	27	29	**30**	30	30	30	29	24	20
		100	17	22	27	32	35	39	41	44	46	48	52	**53**	53
		200	18	26	33	40	46	51	55	59	62	65	75	79	**83**
		500	24	36	47	55	63	70	75	79	82	85	93	97	**98**
CLU		50	63	75	81	84	85	**86**	86	86	85	84	76	63	48
		100	39	54	63	69	73	76	78	80	**81**	81	81	79	74
		200	21	30	37	43	48	52	56	59	61	63	69	**70**	70
		500	11	13	16	19	22	23	25	27	28	30	37	41	**43**
CON	5	50	13	15	17	18	20	**22**	21	22	21	21	19	14	10
		100	15	17	20	22	25	27	29	31	33	34	37	**38**	37
		200	16	19	23	27	32	36	39	41	44	47	56	61	**65**
		500	18	24	31	38	43	49	54	58	63	65	78	86	**90**
CLU		50	45	57	61	64	**65**	65	65	64	62	60	46	30	18
		100	30	38	44	48	52	55	57	58	**60**	60	58	54	48
		200	17	22	25	29	32	35	37	39	41	43	**47**	47	47
		500	10	11	12	14	15	16	17	18	19	19	23	**25**	25

Table 6: Empirical rejection rates for T_a ($d = 2$)

Alt.	β	$n\backslash k$	1	2	3	4	5	6	7	8	9	10	15	20	25	DB
CON	−0.5	50	92	95	**96**	96	96	96	95	95	94	93	88	82	75	59
		100	*	*	*	*	*	*	*	*	*	*	*	*	*	89
		200	*	*	*	*	*	*	*	*	*	*	*	*	*	*
		500	*	*	*	*	*	*	*	*	*	*	*	*	*	*
CLU		50	*	99	98	96	94	92	89	86	83	80	67	58	52	22
		100	*	*	*	*	99	98	97	96	94	92	81	71	62	22
		200	*	*	*	*	*	*	*	99	99	99	93	85	77	22
		500	*	*	*	*	*	*	*	*	*	*	*	98	94	23
CON	0	50	92	95	**96**	95	95	95	94	92	91	90	80	66	54	
		100	*	*	*	*	*	*	*	*	*	*	*	*	99	
		200	*	*	*	*	*	*	*	*	*	*	*	*	*	
		500	*	*	*	*	*	*	*	*	*	*	*	*	*	
CLU		50	**99**	97	92	84	77	70	64	58	54	50	38	32	30	
		100	*	*	99	98	95	90	85	80	75	70	53	43	36	
		200	*	*	*	*	*	99	98	96	94	91	75	61	51	
		500	*	*	*	*	*	*	*	*	*	*	97	89	79	
CON	1	50	91	**94**	94	93	92	90	88	84	81	78	54	35	23	
		100	99	*	*	*	*	*	*	*	*	*	99	97	92	
		200	*	*	*	*	*	*	*	*	*	*	*	*	*	
		500	*	*	*	*	*	*	*	*	*	*	*	*	*	
CLU		50	**98**	83	60	47	39	34	31	29	29	28	26	26	25	
		100	*	99	94	80	68	57	49	43	39	36	28	26	25	
		200	*	*	*	99	97	91	83	75	68	61	41	32	29	
		500	*	*	*	*	*	*	*	99	99	98	82	62	48	
CON	5	50	82	**84**	82	78	72	65	57	49	43	35	14	8	8	
		100	98	99	99	99	*	99	99	98	98	97	86	63	40	
		200	*	*	*	*	*	*	*	*	*	*	*	*	*	
		500	*	*	*	*	*	*	*	*	*	*	*	*	*	
CLU		50	**75**	24	23	24	25	25	25	26	26	27	27	27	27	
		100	**98**	75	39	25	24	24	24	23	24	24	25	25	25	
		200	*	99	91	73	53	38	30	27	24	24	24	24	25	
		500	*	*	*	99	98	96	91	84	75	66	33	25	23	

Table 7: Empirical rejection rates for T_e ($d = 3$)

Alt.	β	$n\backslash k$	1	2	3	4	5	6	7	8	9	10	15	20	25
CON	-0.5	50	58	69	74	77	**78**	78	78	78	77	76	69	61	51
		100	74	86	91	94	95	96	96	**97**	97	97	97	96	95
		200	86	96	99	99	*	*	*	*	*	*	*	*	*
		500	97	*	*	*	*	*	*	*	*	*	*	*	*
CLU		50	99	*	*	99	99	99	99	98	98	97	89	76	58
		100	99	*	*	*	*	*	*	*	*	*	99	98	95
		200	97	*	*	*	*	*	*	*	*	*	*	*	*
		500	74	91	96	98	99	99	99	*	*	*	*	*	*
CON	0	50	62	72	75	76	76	**78**	77	75	74	72	66	54	42
		100	70	88	91	94	95	96	96	**97**	97	96	96	95	94
		200	85	96	99	*	*	*	*	*	*	*	*	*	*
		500	98	*	*	*	*	*	*	*	*	*	*	*	*
CLU		50	99	*	99	99	99	99	98	97	95	93	73	45	26
		100	99	*	*	*	*	*	*	*	*	*	99	94	84
		200	96	*	*	*	*	*	*	*	*	*	*	*	99
		500	76	91	96	98	99	99	99	*	*	*	*	*	*
CON	1	50	57	67	71	**73**	73	73	72	70	69	67	54	40	26
		100	72	85	90	93	94	95	95	**96**	96	95	95	93	91
		200	85	96	98	99	*	*	*	*	*	*	*	*	*
		500	97	*	*	*	*	*	*	*	*	*	*	*	*
CLU		50	**99**	99	99	98	97	96	93	88	81	72	29	12	6
		100	99	*	*	*	*	*	*	99	99	99	94	75	48
		200	96	99	*	*	*	*	*	*	*	*	*	99	96
		500	72	89	94	97	98	99	99	99	99	99	*	99	99
CON	5	50	50	55	59	**60**	60	59	57	54	52	48	30	16	7
		100	61	75	81	85	87	88	**89**	89	89	89	87	82	76
		200	74	90	95	97	98	99	99	99	99	*	*	*	*
		500	90	99	*	*	*	*	*	*	*	*	*	*	*
CLU		50	**96**	96	93	89	79	66	49	34	22	15	9	14	18
		100	95	97	**98**	98	97	96	95	93	90	85	50	18	5
		200	86	95	97	**98**	98	98	98	98	98	97	93	83	66
		500	55	74	82	87	90	92	93	94	94	**95**	94	93	89

Table 8: Empirical rejection rates for T_a ($d = 3$)

β	$n\backslash k$	1	2	3	4	5	6	7	8	9	10	15	20	25
−0.5	50	3.5153	3.5611	3.647	3.7462	3.8737	4.0189	4.1848	4.329	4.507	4.669	5.6829	6.5529	7.326
	100	3.684	3.71	3.8253	3.9398	4.081	4.2427	4.3712	4.5251	4.7462	4.9095	5.9538	7.0291	8.1751
	200	3.7486	3.7786	3.8392	3.9334	4.0411	4.1677	4.3276	4.4797	4.6413	4.8084	5.7686	6.7913	7.9071
	500	3.824	3.8526	3.9011	3.9687	4.0495	4.1732	4.2981	4.3739	4.5089	4.638	5.347	6.1883	7.1429
0	50	3.3375	3.5945	3.8075	3.951	4.0252	4.3326	4.4754	4.5129	4.9256	5.1809	6.3216	7.3246	8.203
	100	3.8339	3.7618	3.7106	4.0433	4.193	4.3351	4.6114	4.8324	5.0195	5.2954	6.641	8.0996	9.3564
	200	3.6694	3.7618	3.9031	3.9109	4.1479	4.2694	4.5334	4.7475	4.9236	5.1882	6.347	7.6525	9.1298
	500	3.7991	3.8698	3.8935	4.0124	4.1025	4.3069	4.4478	4.5799	4.6927	4.8885	5.8603	6.9065	8.198
1	50	3.4684	3.5603	3.6979	3.8496	4.0063	4.2318	4.4481	4.6313	4.8555	5.0735	6.2101	7.1905	7.7949
	100	3.6467	3.7174	3.8564	4.0146	4.1474	4.3562	4.5674	4.8293	5.0666	5.3019	6.7023	7.997	9.3607
	200	3.7272	3.7692	3.8592	3.9879	4.188	4.3389	4.5259	4.7382	4.9674	5.2089	6.4348	7.7738	9.2293
	500	3.7983	3.8088	3.921	4.0282	4.1471	4.3119	4.4313	4.5962	4.7598	4.9432	5.9055	7.0258	8.3193
5	50	3.4067	3.396	3.5065	3.5612	3.6555	3.7396	3.8386	3.9407	4.0372	4.1538	4.7196	5.0936	5.2229
	100	3.5517	3.5891	3.6528	3.7325	3.8056	3.9412	4.058	4.2367	4.3675	4.4953	5.2748	6.0161	6.712
	200	3.6891	3.665	3.7209	3.7833	3.9296	4.0139	4.1062	4.2567	4.3563	4.5128	5.2359	6.0929	6.9251
	500	3.7482	3.7563	3.8228	3.8894	3.8786	4.0446	4.1364	4.2289	4.3353	4.4072	5.0608	5.7461	6.596

Table 9: Empirical critical values for T_e ($d = 2$)

β	$n\backslash k$	1	2	3	4	5	6	7	8	9	10	15	20	25
-0.5	50	3.059	3.4495	3.4895	3.5476	3.6013	3.6136	3.671	3.7402	3.8304	3.8987	4.4778	5.2045	6.1158
	100	3.3636	3.6174	3.6356	3.6774	3.6665	3.6884	3.724	3.773	3.8006	3.8531	4.1379	4.5676	5.1179
	200	3.6098	3.6475	3.6837	3.7082	3.728	3.7441	3.7669	3.7857	3.7967	3.8159	4.0084	4.2671	4.5564
	500	3.6948	3.7277	3.7479	3.7491	3.7536	3.7463	3.7814	3.7909	3.837	3.8487	3.8895	4.0106	4.122
0	50	3.5348	3.6357	3.8551	3.6055	3.9197	3.6414	3.8381	4.0619	3.9305	3.8376	4.8105	5.8258	6.8581
	100	3.2805	3.721	3.4082	3.872	3.8025	3.8163	3.8778	3.844	4.0201	4.205	4.4408	4.84	5.7245
	200	3.5862	3.5359	3.728	3.6373	3.5344	3.8531	3.8196	3.8274	3.8627	3.7838	3.976	4.521	4.7926
	500	3.3917	3.8812	3.9251	3.568	3.913	3.7914	3.7388	3.7294	3.8093	3.7881	3.9171	4.0679	4.1481
1	50	3.0182	3.4862	3.5104	3.5512	3.6001	3.6774	3.7808	3.9017	4.0562	4.1931	5.0547	6.1696	7.4639
	100	3.5107	3.5938	3.6393	3.6181	3.6692	3.7113	3.7541	3.8079	3.8998	3.968	4.4981	5.1274	5.9715
	200	3.6403	3.7021	3.7011	3.7726	3.7307	3.7443	3.8137	3.835	3.8984	3.8807	4.1639	4.5612	4.9969
	500	3.682	3.7521	3.7587	3.781	3.7749	3.7912	3.8095	3.8242	3.8517	3.8635	3.931	4.1156	4.3126
5	50	3.2332	3.167	3.331	3.4236	3.4419	3.5016	3.5726	3.6696	3.8171	3.9049	4.5556	5.4372	6.4344
	100	3.154	3.4374	3.5352	3.5248	3.6007	3.6177	3.6573	3.6886	3.7474	3.8571	4.2155	4.6772	5.3163
	200	3.3493	3.5723	3.6399	3.6894	3.6745	3.6561	3.7296	3.7904	3.7717	3.7667	4.0233	4.2971	4.6231
	500	3.5735	3.6706	3.7452	3.7067	3.7459	3.7472	3.7381	3.7855	3.7764	3.813	3.8827	3.9822	4.1625

Table 10: Empirical critical values for T_a $(d = 2)$

β	$n\backslash k$	1	2	3	4	5	6	7	8	9	10	15	20	25
−0.5	50	3.291	3.4152	3.6171	3.8073	3.9982	4.179	4.3982	4.5908	4.7811	4.9581	5.9169	6.687	7.2964
	100	3.5113	3.6402	3.8774	4.0916	4.3269	4.5864	4.8626	5.1119	5.3263	5.6027	6.8917	8.1342	9.2808
	200	3.6032	3.7398	3.9406	4.1741	4.414	4.715	4.9566	5.2263	5.4921	5.7925	7.3647	8.778	10.19
	500	3.7113	3.8272	4.0474	4.2397	4.4686	4.7166	5.0007	5.2743	5.5108	5.7829	7.2489	8.8147	10.3807
0	50	3.1545	3.3134	3.4272	3.9481	4.0208	4.3308	4.4546	4.5199	4.7144	4.9395	5.8986	6.6639	7.391
	100	3.586	3.7246	3.9569	4.1844	4.2736	4.6567	4.9779	5.1929	5.5419	5.8016	7.1195	8.4343	9.559
	200	3.5915	3.7587	3.9103	4.2356	4.439	4.7096	5.1048	5.3459	5.6768	5.9974	7.6054	9.1797	10.6337
	500	3.6335	3.7929	4.0581	4.2893	4.5389	4.7968	5.1122	5.3755	5.6666	5.944	7.5246	9.2364	10.8763
1	50	3.1978	3.2985	3.4761	3.7027	3.8968	4.0746	4.282	4.4805	4.6267	4.797	5.7106	6.2775	6.6809
	100	3.4313	3.5603	3.7796	4.0027	4.2631	4.5141	4.7796	5.0626	5.314	5.5933	6.8797	8.0612	9.1192
	200	3.5185	3.6853	3.8858	4.1833	4.4413	4.7079	4.9798	5.2699	5.5453	5.8156	7.4537	8.887	10.2664
	500	3.6457	3.7906	4.0316	4.2263	4.4733	4.721	5.0232	5.335	5.5669	5.9183	7.4302	9.0472	10.723
5	50	3.0234	3.0631	3.1429	3.2483	3.3453	3.4346	3.5348	3.6421	3.7032	3.7805	4.2232	4.4042	4.4337
	100	3.2868	3.3212	3.4543	3.6047	3.7154	3.8567	4.0369	4.1956	4.3376	4.5078	5.277	5.9176	6.4958
	200	3.3949	3.4875	3.6135	3.7985	3.9709	4.13	4.2746	4.4308	4.6418	4.8225	5.8187	6.7304	7.6005
	500	3.5393	3.6181	3.7635	3.8794	4.0342	4.2151	4.3972	4.6339	4.7776	4.9825	5.9785	7.0771	8.2448

Table 11: Empirical critical values for T_e ($d = 3$)

β	$n\backslash k$	1	2	3	4	5	6	7	8	9	10	15	20	25
-0.5	50	3.2597	3.4875	3.7513	4.0104	4.3327	4.6857	5.0597	5.4407	5.8605	6.3121	8.7125	11.3174	14.1969
	100	3.2199	3.5981	3.7745	4.0337	4.2606	4.5426	4.8688	5.2008	5.5566	5.9118	7.8884	10.063	12.3647
	200	3.6179	3.6867	3.7927	3.9873	4.1672	4.3922	4.6249	4.8978	5.1767	5.4394	7.0396	8.7485	10.5914
	500	3.6869	3.7585	3.8528	3.9549	4.1134	4.2601	4.4272	4.6144	4.8305	5.0434	6.1219	7.3977	8.7644
0	50	3.3955	3.4367	3.8551	4.3677	4.918	4.5172	5.1173	5.7192	6.3221	6.9259	9.1035	12.138	15.1724
	100	3.2805	3.481	4.1082	3.872	4.6225	4.563	5.3235	5.329	6.0867	6.125	8.5008	10.89	13.2845
	200	3.5862	3.5359	3.9098	4.1676	4.195	4.7623	4.6985	5.282	5.2746	5.8607	7.6772	9.5313	11.4006
	500	3.3917	3.8812	3.9251	4.1553	3.913	4.2818	4.6642	4.5909	4.9962	5.3339	6.2479	7.9489	9.2872
1	50	3.1833	3.4018	3.7233	4.1124	4.4785	4.855	5.31	5.7959	6.2698	6.7732	9.603	12.6136	15.9305
	100	3.2867	3.5733	3.8195	4.1181	4.3869	4.7105	5.0822	5.4852	5.8748	6.327	8.5753	11.1363	13.8202
	200	3.4985	3.6399	3.8368	4.0294	4.2826	4.5365	4.8227	5.1292	5.4417	5.7801	7.6549	9.6059	11.761
	500	3.6849	3.7417	3.8808	3.9683	4.1648	4.3516	4.5512	4.7898	5.0224	5.2462	6.5491	8.0747	9.6426
5	50	2.4588	3.2078	3.4256	3.7305	4.0282	4.3659	4.7462	5.1398	5.5282	5.9654	8.2712	10.6489	13.3085
	100	2.9862	3.3926	3.6124	3.8096	4.0335	4.3441	4.6368	4.9501	5.2748	5.6764	7.5187	9.5585	11.7511
	200	3.3645	3.5133	3.6722	3.8172	4.0221	4.255	4.4754	4.7343	4.9896	5.2368	6.8046	8.4616	10.1664
	500	3.5962	3.6743	3.7579	3.859	4.0147	4.1344	4.3064	4.4773	4.7088	4.8687	5.9855	7.2618	8.4907

Table 12: Empirical critical values for T_a $(d = 3)$