CLOSED-FORM EXPANSIONS FOR OPTION PRICES WITH RESPECT TO THE MIXING SOLUTION

KAUSTAV DAS AND NICOLAS LANGRENÉ

ABSTRACT. We consider closed-form expansions for European put option prices within several stochastic volatility frameworks with time-dependent parameters. Our methodology involves writing the put option price as an expectation of a Black-Scholes formula and performing a second-order Taylor expansion around the mean of its argument. The difficulties then faced are computing a number of expectations induced by the Taylor expansion in a closed-form manner. We establish a fast calibration scheme under the assumption that the parameters are piecewise-constant. Furthermore, we perform a sensitivity analysis to investigate the quality of our approximation and show that the errors are well within the acceptable range for application purposes. Lastly, we derive bounds on the remainder term due to the Taylor expansion.

Keywords: Stochastic volatility, Closed-form expansion, Closed-form approximation, Heston, GARCH, Inverse-Gamma

We consider the European put option pricing problem in the context of several stochastic volatility models with time-dependent parameters, namely the Heston, GARCH[1] and Inverse-Gamma[15, 20, 25, 29]. Our goal is to study how European put option prices can be approximated in each of these frameworks via expansion of the so-called mixing solution, which will be detailed later in this paper. We find that via a second-order Taylor expansion of the mixing solution, we can derive an accurate approximation to European put option prices in the aforementioned models. Furthermore, our method works naturally with time-dependent parameters. This is seen a major positive as compared to other methodologies, for example transform methods, which cannot handle time-dependent parameters well. Our method is similar to that of Drimus [9], in which the Heston model is considered with constant parameters. Additionally, we derive a fast calibration scheme under the assumption of piecewise-constant parameters. A sensitivity analysis is performed in order to assess our approximation numerically. Furthermore, we give mathematical bounds on the error term in terms of third-order mixed moments of the underlying variance process.

It has been well established that volatility is highly dependent on the strike and maturity of European option contracts. This phenomenon is called the volatility smile or skew, an attribute the well known Black-Scholes model fails to take into account [7]. In response, there have been a number of frameworks proposed to explain the volatility smiles and

[1]Generalised AutoRegressive Conditional Heteroskedasticity.

skews observed in the market.[2] In particular, stochastic volatility models have been introduced, where the volatility itself is a stochastic process possibly correlated with the spot. However with this added complexity, often option prices cannot be computed in a closed-form fashion. This is detrimental, as closed-form solutions lead to rapid option pricing, a quality necessary for fast calibration of financial models. Without a closed-form solution, option pricing must be done numerically via Monte Carlo or PDE methods, both methods being computational costly.

If we assume that the characteristic function of the log-spot is known explicitly, then the option price can be computed quasi-explicitly[3], albeit under the restrictive assumption of constant or piecewise-constant parameters [8, 15, 24]. One class of models where this occurs are the affine[4] models such as Heston as well as Schöbel and Zhu. However, for non-affine models, the characteristic function of the log-spot is rarely known explicitly, and such a procedure will not be effective. Non-affine models, although usually intractable compared to their affine counterparts, are often far more realistic. This has been shown in a number of studies, see for example Gander and Stephens [11]. For these reasons, numerical procedures such as PDE and Monte Carlo methods have been substantially developed in the literature [3, 27].

Closed-form approximations are an alternative methodology for option pricing, where the option price is instead approximated by an explicit expression. The main advantages are that the option price can be computed rapidly and since transform methods are not used, time-dependent parameters can usually be handled well. One motivation for quick option pricing formulae is calibration, where the option price must be computed several times.

There have been a plethora of results on closed-form expansions in the literature. For example, Lorig et al. [23] derive a general closed-form expression for the price of an option via a PDE approach, as well as its corresponding implied volatility. Hagan et al. [13] use single perturbation techniques to obtain an explicit approximation for the option price and implied volatility in their SABR model. Alòs [2] show that from the mixing solution, one can approximate the put option price by decomposing it into a sum of two terms, one being completely correlation indepedent and the other dependent on correlation. However, neither terms are explicit. Furthermore, similar to our work, Antonelli and Scarlatti [4], Antonelli et al. [5] show that under the assumption of small correlation, an expansion can be performed with respect to the mixing solution, where the resulting expectations can be computed using Malliavin calculus techniques. Similarly, in the case of the time-dependent Heston model, Benhamou et al. [6] consider the mixing solution and expand around vol-vol, performing a combination of Taylor expansions and computing the

[2]For example, local volatility models, stochastic local volatility models and of course, stochastic volatility models.

[3]Quasi-explicit meaning in terms of at most one-dimensional complex integrals, where the integrands are explicit functions.

[4]Affine stochastic volatility models are such that $\ln \mathbb{E}\left(e^{iu\ln(S_t)}\right)$ is affine in $\ln S_0$, i.e., the log of the characteristic function of the log-spot is an affine function in $\ln S_0$ [1, 10].

resulting terms via Malliavin calculus techniques.

Stochastic volatility models usually either model the volatility directly, or indirectly via the variance process. A critical assumption is that volatility or variance has some sort of mean reversion behaviour, and this is supported by empirical evidence, see for example Gatheral [12]. Specifically, for modelling the variance, this class of stochastic volatility models is given by[5]

$$\begin{aligned} \mathrm{d}S_t &= S_t((r^d_t - r^f_t)\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t), \quad S_0, \\ \mathrm{d}V_t &= \kappa_t(\theta_t V_t^{\hat{\mu}} - V_t^{\tilde{\mu}})\mathrm{d}t + \lambda_t V_t^{\mu}\mathrm{d}B_t, \quad V_0 = v_0, \\ \mathrm{d}\langle W, B\rangle_t &= \rho_t \mathrm{d}t, \end{aligned}$$

whereas for modelling the volatility, this class is of the form

$$\begin{aligned} \mathrm{d}S_t &= S_t((r^d_t - r^f_t)\mathrm{d}t + V_t\mathrm{d}W_t), \quad S_0, \\ \mathrm{d}V_t &= \kappa_t(\theta_t V_t^{\hat{\mu}} - V_t^{\tilde{\mu}})\mathrm{d}t + \lambda_t V_t^{\mu}\mathrm{d}B_t, \quad V_0 = v_0, \\ \mathrm{d}\langle W, B\rangle_t &= \rho_t \mathrm{d}t, \end{aligned}$$

for some $\tilde{\mu}, \hat{\mu}$ and $\mu \in \mathbb{R}$. [6] In this paper, we will focus on this class on stochastic volatility models. Some popular models in the literature include:

Model	Variance/Volatility	Dynamics of V	$\hat{\mu}$	$\tilde{\mu}$	μ
Heston [15]	Variance	$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t\sqrt{V_t}\mathrm{d}B_t$	0	1	1/2
Schöbel and Zhu [26]	Volatility	$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t\mathrm{d}B_t$	0	1	0
GARCH [25, 29]	Variance	$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t\mathrm{d}B_t$	0	1	1
Inverse Gamma [20]	Volatility	$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t\mathrm{d}B_t$	0	1	1
3/2 Model [21]	Variance	$\mathrm{d}V_t = \kappa_t(\theta_t V_t - V_t^2)\mathrm{d}t + \lambda_t V_t^{3/2}\mathrm{d}B_t$	1	2	3/2
XGBM/Logistic [22]	Volatility	$\mathrm{d}V_t = \kappa_t(\theta_t V_t - V_t^2)\mathrm{d}t + \lambda_t V_t\mathrm{d}B_t$	1	2	1

This paper is dedicated to detailing how a second-order expansion of the so called mixing solution for stochastic volatility models with time-dependent parameters can give a closed-form approximation for the price of a European put option, as well as how a fast calibration scheme can be implemented. The tractability of our methodology relies largely on the dynamics of the underlying variance process. Our method extends that of Drimus [9], in which the Heston model is considered with constant parameters. Specifically, we consider a variety of of stochastic volatility models including the Heston, GARCH and Inverse-Gamma and we allow for time-dependent parameters. We include a robust error analysis, design a fast calibration scheme and give extensive numerical results. The sections are organised as follows:

[5]Our model formulation here is for FX market purposes, but can be easily adjusted for equity and fixed income markets purposes.

[6]There exist other classes of stochastic volatility models, for example the exponential Ornstein-Uhlenbeck model Wiggins [28] is not apart of either of these classes.

- Section 2 details some preliminary calculations, where we express the put option price as the mixing solution. Once done, a second-order Taylor expansion is performed, giving the closed-form approximation in terms of a number of expectations of functionals of the underlying variance process
- Section 3 details how to derive more convenient expressions for these expectations derived in Section 2 via change of measure techniques. Specifically, we rewrite the spot S_T as a convenient expression so as to construct a term which is a stochastic exponential, thereby defining a Radon-Nikodym derivative. This term allows us to change measure, allowing for more convenient calculations
- Section 4 introduces specific models. As precise dynamics are assumed, the task is to derive explicit expressions for the expectations from Section 3. In particular, we consider the Heston, GARCH and Inverse-Gamma models
- Section 5 gives an error analysis, bounding the error in the expansion in terms of third-order mixed moments of the underlying variance process
- Section 6 details our fast calibration scheme. In particular, we rewrite the pricing functions found for the Heston and GARCH models in Section 4 in terms of specific integral operators, which can be shown to satisfy some convenient recursive properties when parameters are assumed to be piecewise-constant
- Section 7 is dedicated to a numerical error analysis for the Heston and GARCH models

1. Preliminary calculations

Suppose the spot S with variance σ follows the dynamics

$$\begin{aligned}
\mathrm{d}S_t &= S_t((r_t^d - r_t^f)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t), \quad S_0,\\
\mathrm{d}\sigma_t &= \alpha(t,\sigma_t)\mathrm{d}t + \beta(t,\sigma_t)\mathrm{d}B_t, \quad \sigma_0,\\
\mathrm{d}\langle W, B\rangle_t &= \rho_t \mathrm{d}t,
\end{aligned}$$

where W and B are Brownian motions with instantaneous correlation $(\rho_t)_{0\le t\le T}$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{Q})$. Here T is a finite time horizon, where $(r_t^d)_{0\le t\le T}$ and $(r_t^f)_{0\le t\le T}$ are the time-dependent deterministic and foreign interest rates respectively. Furthermore, $(\mathcal{F}_t)_{0\le t\le T}$ is the filtration generated by (W, B) which satisfies the usual assumptions.[7] In the following, $\mathbb{E}(\cdot)$ denotes the expectation under $\mathbb{Q}$, where $\mathbb{Q}$ is a risk-neutral measure which we assume to be chosen. We assume that the drift and and diffusion coefficients of σ are such that σ has a strong solution.

Definition 1.1 (Geometric Brownian motion process)**.** A process Y is called a Geometric Brownian motion (GBM) process if it solves the SDE

$$\mathrm{d}Y_t = \mu_t Y_t \mathrm{d}t + \nu_t Y_t \mathrm{d}\tilde{B}_t, \quad Y_0 = y_0.$$

[7]Meaning that $(\mathcal{F}_t)_{0\le t\le T}$ is right continuous and augmented by $\mathbb{Q}$ null-sets.

Assuming μ and ν are adapted to the Brownian filtration and satisfy some regularity conditions, Y has the well known strong solution

$$Y_t = y_0 \exp\left\{ \int_0^t \left(\mu_u - \frac{1}{2}\nu_u^2 \right) \mathrm{d}u + \int_0^t \nu_u \mathrm{d}\tilde{B}_u \right\}.$$

We denote the process Y as a $\mathrm{GBM}(y_0; \mu_t, \nu_t)$.

We decompose the Brownian motion W as $W_t = \int_0^t \rho_u \mathrm{d}B_u + \int_0^t \sqrt{1-\rho_u^2}\mathrm{d}Z_u$, where Z is an artificial Brownian motion under $\mathbb{Q}$ such that B and Z are independent. Then, noticing S is a $\mathrm{GBM}(S_0; r_t^d - r_t^f, \sqrt{\sigma_t})$, we obtain the strong solution

$$\begin{aligned} S_T &= S_0\xi_T \exp\left\{ \int_0^T (r_t^d - r_t^f)\mathrm{d}t - \frac{1}{2}\int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t + \int_0^T \sqrt{\sigma_t(1-\rho_t^2)}\mathrm{d}Z_t \right\}, \\ \xi_t &:= \exp\left\{ \int_0^t \rho_u \sqrt{\sigma_u}\mathrm{d}B_u - \frac{1}{2}\int_0^t \rho_u^2 \sigma_u \mathrm{d}u \right\}. \end{aligned}$$

1.1. **Pricing a Put option.** Denote $(\mathcal{F}_t^B)_{0\le t\le T}$ to be the filtration generated by the Brownian motion B, as well as $\mathcal{N}(\cdot)$ and $\phi(\cdot)$ to be the standard Normal distribution and density functions respectively. It can be seen that the price of a European put, Put is given by

$$\begin{aligned} \mathrm{Put} &= e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}(K - S_T)_+ = \mathbb{E}\Big\{ e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}\big[(K-S_T)_+ | \mathcal{F}_T^B\big] \Big\} \\ &= \mathbb{E}\left(\mathrm{Put}_{\mathrm{BS}}\left(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t \right) \right), \end{aligned} \tag{1}$$

where

$$\begin{aligned} \mathrm{Put}_{\mathrm{BS}}(x,y) &:= Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(-d_-) - xe^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(-d_+), \\ d_\pm &:= \frac{\ln(x/K) + \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}. \end{aligned}$$

[8] Notice that in eq. (1), we have been able to write the put option price as an expectation of a Black-Scholes formula with $\mathcal{F}_T^B$-measurable arguments. This result, called the mixing solution, is derived in Appendix A. It was first established by Hull and White for the case of independent W and B, then later extended by Willard for the correlated case [16, 29].

1.2. **Expansion.** To compute this price explicitly, we use a Taylor expansion of the function $\mathrm{Put}_{\mathrm{BS}}$, which is smooth on $(\mathbb{R}_+^2; \mathbb{R}_+)$. We expand around the mean of $\left(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t\right)$. That is, around the vector $(\bar{x}, \bar{y}) := (S_0, \int_0^T (1-\rho_t^2)\mathbb{E}(\sigma_t)\mathrm{d}t)$. We assume that $\xi = (\xi_t)_{0\le t\le T}$

[8]The call option scenario is detailed in Appendix B, however we opt to pricing the put option since the put price is bounded by a fixed constant. The call price can be derived from the Put-Call parity.

is a martingale, and hence $\mathbb{E}(\xi_T) = 1$. [9]

$$
\begin{aligned}
&\mathrm{Put_{BS}}\left(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t\right) \approx \mathrm{Put_{BS}}(\bar{x},\bar{y}) \\
&+ \partial_x \mathrm{Put_{BS}}(\bar{x},\bar{y}) S_0(\xi_T - 1) + \partial_y \mathrm{Put_{BS}}(\bar{x},\bar{y}) \left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right) \\
&+ \frac{1}{2}\partial_{xx}\mathrm{Put_{BS}}(\bar{x},\bar{y}) S_0^2(\xi_T-1)^2 + \frac{1}{2}\partial_{yy}\mathrm{Put_{BS}}(\bar{x},\bar{y}) \left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 \\
&+ \partial_{xy}\mathrm{Put_{BS}}(\bar{x},\bar{y}) S_0(\xi_T-1)\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right).
\end{aligned}
$$

Taking expectation gives a second-order approximation to the price, which we will denote as $\mathrm{Put}^{(2)}$. Notice that $\mathrm{Put_{BS}}(\bar{x},\bar{y})$ is a deterministic quantity, thus the first order terms will vanish.

$$
\begin{aligned}
\mathrm{Put}^{(2)} &= \mathrm{Put_{BS}}(\bar{x},\bar{y}) \\
&+ \frac{1}{2}\partial_{xx}\mathrm{Put_{BS}}(\bar{x},\bar{y}) S_0^2 \mathbb{E}(\xi_T-1)^2 + \frac{1}{2}\partial_{yy}\mathrm{Put_{BS}}(\bar{x},\bar{y}) \mathbb{E}\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 \\
&+ \partial_{xy}\mathrm{Put_{BS}}(\bar{x},\bar{y}) S_0 \mathbb{E}\left\{(\xi_T-1)\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right\}.
\end{aligned}
\tag{2}
$$

The above partial derivatives are the second-order Black-Scholes Greeks, given as

$$
\begin{aligned}
\partial_{xx}\mathrm{Put_{BS}} &= \frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{x\sqrt{y}}, \\
\partial_{yy}\mathrm{Put_{BS}} &= \frac{xe^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{4y^{3/2}}(d_+d_- - 1), \\
\partial_{xy}\mathrm{Put_{BS}} &= (-1)\frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)d_-}{2y},
\end{aligned}
$$

[9]There exists many sufficient conditions for $(\xi_t)_{0\le t\le T}$ to be a martingale, for instance the Novikov condition, $\mathbb{E}\left(e^{\frac{1}{2}\int_0^T \rho_t^2\sigma_t \mathrm{d}t}\right) < \infty$. However these conditions may impose restrictions on the parameters which may be tighter than what is necessary. For this reason we assume that ξ is a martingale. We refer the reader to Kazamaki [18] or Klebaner and Liptser [19] for sufficient conditions on when stochastic exponentials are martingales.

which are Gamma, Volga and Vanna respectively. What remains to be done is the calculation of each of the expectations, which are

$$\mathbb{E}(\xi_T - 1)^2, \tag{3}$$

$$\mathbb{E}\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2, \tag{4}$$

$$\mathbb{E}\left\{(\xi_T - 1)\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right\}. \tag{5}$$

Remark 1.2 (Greeks)**.** The Put Delta is obtained via partial differentiation of the Put price with respect to the underlying S_0.

$$\begin{aligned}
\partial_{S_0}\mathrm{Put}^{(2)} &= \partial_x \mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) \\
&+ \frac{1}{2}\left[2S_0\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) + S_0^2\partial_{xxx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\right]\mathbb{E}(\xi_T-1)^2 \\
&+ \frac{1}{2}\partial_{xyy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\mathbb{E}\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 \\
&+ \left[\partial_{xy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) + S_0\partial_{xxy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\right]\mathbb{E}\left\{(\xi_T-1)\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right\}.
\end{aligned} \tag{6}$$

The Put Gamma is obtained via partial differentiation of the Delta with respect to the underlying S_0.

$$\begin{aligned}
\partial_{S_0S_0}\mathrm{Put}^{(2)} &= \partial_{xx} \mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) \\
&+ \frac{1}{2}\left[2\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) + 2S_0\partial_{xxx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) + S_0^2\partial_{xxxx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\right]\mathbb{E}(\xi_T-1)^2 \\
&+ \frac{1}{2}\partial_{xxyy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\mathbb{E}\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 \\
&+ \left[2\partial_{xxy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) + S_0\partial_{xxxy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\right]\mathbb{E}\left\{(\xi_T-1)\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right\}.
\end{aligned} \tag{7}$$

The third and fourth-order Black-Scholes Greeks can be found in the literature, for example see Haug [14].

2. Calculation of expectations

The following lemmas will be useful in order to calculate eq. (3), eq. (4) and eq. (5).

Lemma 2.1. By Girsanov's theorem, there exists a probability measure $\mathbb{Q}_1 \sim \mathbb{Q}$ defined by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{Q}} := \xi_T = \exp\left\{\int_0^T \rho_u\sqrt{\sigma_u}\mathrm{d}B_u - \frac{1}{2}\int_0^T \rho_u^2\sigma_u\mathrm{d}u\right\},$$

such that $B_t^1 := B_t - \int_0^t \rho_u \sqrt{\sigma_u} \mathrm{d}u$ is a $\mathbb{Q}_1$ Brownian motion. Furthermore, expectations can be calculated under the new measure by the equation $\mathbb{E}(X\xi_T) = \mathbb{E}_{\mathbb{Q}_1}(X)$ or $\mathbb{E}(X) = \mathbb{E}_{\mathbb{Q}_1}(X\frac{1}{\xi_T})$.

We can extend the above idea to a sequence of equivalent measures.

Lemma 2.2. Let $(\mathbb{Q}_n)_{n\geq 0}$ be a sequence of probability measures equivalent to $\mathbb{Q}$, defined by the Radon-Nikodym derivatives

$$\frac{\mathrm{d}\mathbb{Q}_{n+1}}{\mathrm{d}\mathbb{Q}_n} := \xi_T^{(n)} := \exp\left\{\int_0^T \rho_u\sqrt{\sigma_u}\mathrm{d}B_u^n - \frac{1}{2}\int_0^T \rho_u^2\sigma_u \mathrm{d}u\right\}, \quad \xi_T^{(0)} := \xi_T, \quad n \geq 0,$$

where $\mathbb{Q}_0 := \mathbb{Q}$ and $B^0 := B$. Under $\mathbb{Q}_n$, $B_t^n := B_t^{n-1} - \int_0^t \rho_u\sqrt{\sigma_u}\mathrm{d}u$ is a Brownian motion. Furthermore, we have the relationship between densities as

$$\xi_T^{(n)} = \xi_T^{(n-1)} e^{-\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}, \quad n \geq 1. \tag{8}$$

Expectations can also be calculated as

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_n}(X) &= \mathbb{E}_{\mathbb{Q}_{n-1}}(X\xi_T^{(n-1)}), \\ \mathbb{E}_{\mathbb{Q}_{n-1}}(X) &= \mathbb{E}_{\mathbb{Q}_n}\left(X\frac{1}{\xi_T^{(n-1)}}\right). \end{aligned} \tag{9}$$

The two above relationships, eq. (8) and eq. (9), allow for alternative and often more convenient calculations of expectations under $\mathbb{Q}$.

Using these tools, we can now give alternative expressions for the expectations seen in eq. (3), eq. (4) and eq. (5).

2.1. $\mathbb{E}(\xi_T - 1)^2$. First, expanding eq. (3) gives

$$\mathbb{E}(\xi_T - 1)^2 = \mathbb{E}(\xi_T^2) - 1.$$

This second moment can be dealt with a number of changes of measures

$$\begin{aligned} \mathbb{E}(\xi_T^2) &= \mathbb{E}_{\mathbb{Q}_1}(\xi_T) = \mathbb{E}_{\mathbb{Q}_1}(\xi_T^{(1)} e^{\int_0^T \rho_t^2\sigma_t \mathrm{d}t}) \\ &= \mathbb{E}_{\mathbb{Q}_2}(e^{\int_0^T \rho_t^2\sigma_t \mathrm{d}t}). \end{aligned} \tag{10}$$

Under the assumption of constant parameters[10] we may calculate eq. (10) explicitly via the Laplace transform for certain processes σ. To our knowledge however, there exists no explicit solution when parameters are time-dependent, see Hurd and Kuznetsov [17]. Instead, we approximate eq. (10) by expanding the exponential around the mean of $\int_0^T \rho_t^2\sigma_t \mathrm{d}t$

[10] That is, $\alpha(t, \sigma_t) = \alpha(\sigma_t)$, $\beta(t, \sigma_t) = \beta(\sigma_t)$ and $\rho_t = \rho$.

to second-order.

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_2}(e^{\int_0^T \rho_t^2\sigma_t \mathrm{d}t})\\
&\approx \mathbb{E}_{\mathbb{Q}_2}\left\{e^{\int_0^T \rho_t^2\mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t}\left[1+\int_0^T \rho_t^2\left(\sigma_t-\mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\right)\mathrm{d}t+\frac{1}{2}\left(\int_0^T \rho_t^2\left(\sigma_t-\mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t\right)\right)^2\right]\right\}\\
&= e^{\int_0^T \rho_t^2\mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t}\left\{1+\frac{1}{2}\mathbb{E}_{\mathbb{Q}_2}\left[\left(\int_0^T \rho_t^2\left(\sigma_t-\mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t\right)\right)^2\right]\right\}\\
&= e^{\int_0^T \rho_t^2\mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t}\left\{1+\int_0^T \rho_t^2\int_0^t \rho_s^2\mathrm{Cov}_{\mathbb{Q}_2}(\sigma_s,\sigma_t)\mathrm{d}s\mathrm{d}t\right\},
\end{aligned}$$

where we have used the fact that $\left(\int_0^T f(t)\mathrm{d}t\right)^2 = 2\int_0^T f(t)\left(\int_0^t f(s)\mathrm{d}s\right)\mathrm{d}t$.

2.2. $\mathbb{E}\left(\int_0^T(1-\rho_t^2)(\sigma_t-\mathbb{E}(\sigma_t))\mathrm{d}t\right)^2$. To calculate eq. (4), we use the same trick from section 2.1.

$$\mathbb{E}\left(\int_0^T(1-\rho_t^2)(\sigma_t-\mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 = 2\int_0^T(1-\rho_t^2)\left(\int_0^t(1-\rho_s^2)\mathrm{Cov}(\sigma_s,\sigma_t)\mathrm{d}s\right)\mathrm{d}t.$$

2.3. $\mathbb{E}\left\{(\xi_T-1)\left(\int_0^T(1-\rho_t^2)(\sigma_t-\mathbb{E}(\sigma_t))\mathrm{d}t\right)\right\}$. Calculation of the mixed expectation eq. (5) gives

$$\begin{aligned}
\mathbb{E}\left\{(\xi_T-1)\left(\int_0^T(1-\rho_t^2)(\sigma_t-\mathbb{E}(\sigma_t))\mathrm{d}t\right)\right\} &= \int_0^T(1-\rho_t^2)\big(\mathbb{E}(\xi_T\sigma_t)-\mathbb{E}(\sigma_t)\big)\mathrm{d}t\\
&= \int_0^T(1-\rho_t^2)\big(\mathbb{E}_{\mathbb{Q}_1}(\sigma_t)-\mathbb{E}(\sigma_t)\big)\mathrm{d}t.
\end{aligned}$$

3. Pricing equations for specific models

We now introduce specific dynamics for both the spot and its underlying variance process. From Section 2, it is apparent that a closed-form expression for $\mathrm{Put}^{(2)}$ will largely depend on the tractability of the process σ under the original measure $\mathbb{Q}$, as well as the artificial measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$.

3.1. **Heston Model.** Suppose the spot S with variance V follows the Heston dynamics

$$\begin{aligned}
\mathrm{d}S_t &= S_t((r_t^d-r_t^f)\mathrm{d}t+\sqrt{V_t}\mathrm{d}W_t), \quad S_0,\\
\mathrm{d}V_t &= \kappa_t(\theta_t-V_t)\mathrm{d}t+\lambda_t\sqrt{V_t}\mathrm{d}B_t, \quad V_0=v_0,\\
\mathrm{d}\langle W,B\rangle_t &= \rho_t\mathrm{d}t.
\end{aligned}$$

The unobservable parameters mainly stem from the process V:

(1) $(\kappa_t)_{0\le t\le T}$. The time-dependent mean reversion speed of V.

(2) $(\theta_t)_{0\le t\le T}$. The time-dependent mean reversion level of V.

(3) $(\lambda_t)_{0\le t\le T}$. The time-dependent volatility of V (vol-vol).

(4) $(\rho_t)_{0\le t\le T}$. The instantaneous correlation between the spot S and its variance V.

Here we model the variance directly, that is, in the language of the initial sections, $\sigma_t = V_t$. This is convenient for the calculations.

Definition 3.1 (CIR process)**.** A process $\tilde{V}$ is called a CIR process if it solves the SDE

$$\mathrm{d}\tilde{V}_t = \kappa_t(\theta_t - \tilde{V}_t)\mathrm{d}t + \lambda_t\sqrt{\tilde{V}_t}\mathrm{d}\tilde{B}_t, \quad \tilde{V}_0 = \tilde{v}_0,$$

where we assume κ, θ and λ are time-dependent and deterministic and satisfy some regularity conditions. It can be integrated to obtain

$$\tilde{V}_t = \tilde{v}_0 e^{-\int_0^t \kappa_z \mathrm{d}z} + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z}\kappa_u\theta_u \mathrm{d}u + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z}\lambda_u\sqrt{\tilde{V}_u}\mathrm{d}B_u.$$

We denote the process $\tilde{V}$ as a $\mathrm{CIR}(\tilde{v}_0; \kappa_t, \theta_t, \lambda_t)$.

Lemma 3.2 (Moments of the CIR process)**.** Let $\tilde{V}$ be a $\mathrm{CIR}(\tilde{v}_0; \kappa_t, \theta_t, \lambda_t)$. It has the following moments

$$\begin{aligned}
\mathbb{E}(\tilde{V}_t^n) &= e^{-\int_0^t n\kappa_z \mathrm{d}z}\left(v_0^n + \int_0^t e^{\int_0^u n\kappa_z \mathrm{d}z}\left(n\kappa_u\theta_u + \frac{1}{2}n(n-1)\lambda_u^2\right)\mathbb{E}(\tilde{V}_u^{n-1})\mathrm{d}u\right),\\
\mathrm{Var}(\tilde{V}_t) &= \int_0^t \lambda_u^2 e^{-2\int_u^t \kappa_z \mathrm{d}z}\left\{v_0 e^{-\int_0^u \kappa_z \mathrm{d}z} + \int_0^u e^{-\int_p^u \kappa_z \mathrm{d}z}\kappa_p\theta_p \mathrm{d}p\right\}\mathrm{d}u,\\
\mathrm{Cov}(\tilde{V}_s, \tilde{V}_t) &= e^{-\int_s^t \kappa_z \mathrm{d}z}\int_0^s \lambda_u^2 e^{-2\int_u^s \kappa_z \mathrm{d}z}\left\{v_0 e^{-\int_0^u \kappa_z \mathrm{d}z} + \int_0^u e^{-\int_p^u \kappa_z \mathrm{d}z}\kappa_p\theta_p \mathrm{d}p\right\}\mathrm{d}u,\\
\mathbb{E}(\tilde{V}_s^m\tilde{V}_t^n) &= e^{-\int_0^t n\kappa_z \mathrm{d}z}\left(\mathbb{E}(\tilde{V}_s^{m+n}) + \int_s^t e^{\int_0^u n\kappa_z \mathrm{d}z}\left(n\kappa_u\theta_u + \frac{1}{2}n(n-1)\lambda_u^2\right)\mathbb{E}(\tilde{V}_s^m\tilde{V}_u^{n-1})\mathrm{d}u\right),
\end{aligned}$$

$\mathrm{Cov}(\tilde{V}_s^m, \tilde{V}_t^n) = \mathbb{E}(\tilde{V}_s^m\tilde{V}_t^n) - \mathbb{E}(\tilde{V}_s^m)\mathbb{E}(\tilde{V}_t^n)$,

all for $m, n \ge 1$ and $s < t$. The derivation is given in Appendix C.1.

It is clear that the variance process V is a $\mathrm{CIR}(v_0; \kappa_t, \theta_t, \lambda_t)$.

Lemma 3.3. Let $(\mathbb{Q}_n)_{n\ge 0}$ be a sequence of probability measures equivalent to $\mathbb{Q}$, defined by the Radon-Nikodym derivatives

$$\frac{\mathrm{d}\mathbb{Q}_{n+1}}{\mathrm{d}\mathbb{Q}_n} := \xi_T^{(n)} := \exp\left\{\int_0^T \rho_u\sqrt{V_u}\mathrm{d}B_u^n - \frac{1}{2}\int_0^T \rho_u^2 V_u \mathrm{d}u\right\}, \quad \xi_T^{(0)} := \xi_T, \quad n \ge 0,$$

where $\mathbb{Q}_0 := \mathbb{Q}$ and $B^0 := B$. Under $\mathbb{Q}_n$, $B_t^n := B_t^{n-1} - \int_0^t \rho_u\sqrt{V_u}\mathrm{d}u$ is a Brownian motion. For $n \ge 0$, the dynamics of V under the measure $\mathbb{Q}_n$ are

$$\mathrm{d}V_t = (\kappa_t - n\lambda_t\rho_t)\left(\frac{\theta_t\kappa_t}{\kappa_t - n\lambda_t\rho_t} - V_t\right)\mathrm{d}t + \lambda_t\sqrt{V_t}\mathrm{d}B_t^n,$$

which is a $\mathrm{CIR}(v_0; \kappa_t - n\lambda_t\rho_t, \frac{\theta_t\kappa_t}{\kappa_t - n\lambda_t\rho_t}, \lambda_t)$.

Thus, the variance process V is a CIR process under all measures considered, and we have explicit expressions for its moments and covariance. All the terms needed can be calculated explicitly.

3.1.1. *Pricing under the Heston framework.* Adapting Section 1.1 to the Heston framework, the second-order expansion of the option price, denoted $\mathrm{Put}_{\mathrm{H}}^{(2)}$, is given by the following theorem.

Theorem 3.4 (Second-order Heston option price)**.** The second-order Heston price, $\mathrm{Put}_{\mathrm{H}}^{(2)}$ is given by

(11)

$$\begin{aligned}\mathrm{Put}_{\mathrm{H}}^{(2)} &= \mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\\ &+\frac{1}{2}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})S_0^2\mathbb{E}(\xi_T-1)^2+\frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\mathbb{E}\left(\int_0^T(1-\rho_t^2)(V_t-\mathbb{E}(V_t))\mathrm{d}t\right)^2\\ &+\partial_{xy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})S_0\mathbb{E}\left\{(\xi_T-1)\left(\int_0^T(1-\rho_t^2)(V_t-\mathbb{E}(V_t))\mathrm{d}t\right)\right\},\end{aligned}$$

where, referring to Section 2, the three expectations in eq. (11) can be calculated as

$$\begin{aligned}\mathbb{E}(\xi_T-1)^2 &\approx e^{\int_0^T\rho_t^2\mathbb{E}_{\mathbb{Q}_2}(V_t)\mathrm{d}t}\left\{1+\int_0^T\rho_t^2\int_0^t\rho_s^2\mathrm{Cov}_{\mathbb{Q}_2}(V_s,V_t)\mathrm{d}s\mathrm{d}t\right\}-1,\\ \mathbb{E}_{\mathbb{Q}_2}(V_t) &= v_0e^{-\int_0^t\kappa_z-2\lambda_z\rho_z\mathrm{d}z}+\int_0^t e^{-\int_u^t\kappa_z-2\lambda_z\rho_z\mathrm{d}z}\kappa_u\theta_u\mathrm{d}u,\\ \mathrm{Cov}_{\mathbb{Q}_2}(V_s,V_t) &= e^{-\int_s^t\kappa_z-2\lambda_z\rho_z\mathrm{d}z}\int_0^s\lambda_u^2e^{-2\int_u^s\kappa_z-2\lambda_z\rho_z\mathrm{d}z}\left[v_0e^{-\int_0^u\kappa_z-2\lambda_z\rho_z\mathrm{d}z}+\int_0^u e^{-\int_p^u\kappa_z-2\lambda_z\rho_z\mathrm{d}z}\kappa_p\theta_p\mathrm{d}p\right]\mathrm{d}u,\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\left(\int_0^T(1-\rho_t^2)(V_t-\mathbb{E}(V_t))\mathrm{d}t\right)^2\\ &=2\int_0^T(1-\rho_t^2)\left(\int_0^t(1-\rho_s^2)\left[e^{-\int_s^t\kappa_z\mathrm{d}z}\int_0^s\lambda_u^2e^{-2\int_u^s\kappa_z\mathrm{d}z}\left\{v_0e^{-\int_0^u\kappa_z\mathrm{d}z}+\int_0^u e^{-\int_p^u\kappa_z\mathrm{d}z}\kappa_p\theta_p\mathrm{d}p\right\}\mathrm{d}u\right]\mathrm{d}s\right)\mathrm{d}t,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left\{(\xi_T-1)\left(\int_0^T(1-\rho_t^2)(V_t-\mathbb{E}(V_t))\mathrm{d}t\right)\right\}\\ &=\int_0^T(1-\rho_t^2)\left\{v_0\left(e^{-\int_0^t\kappa_z-\lambda_z\rho_z\mathrm{d}z}-e^{-\int_0^t\kappa_z\mathrm{d}z}\right)+\int_0^t\left(e^{-\int_u^t\kappa_z-\lambda_z\rho_z\mathrm{d}z}-e^{-\int_u^t\kappa_z\mathrm{d}z}\right)\kappa_u\theta_u\mathrm{d}u\right\}\mathrm{d}t.\end{aligned}$$

Furthermore, $\bar{x}=S_0$ and $\bar{y}=\int_0^T(1-\rho_t^2)\mathbb{E}(V_t)\mathrm{d}t=\int_0^T(1-\rho_t^2)\left\{v_0e^{-\int_0^t\kappa_z\mathrm{d}z}+\int_0^t e^{-\int_u^t\kappa_z\mathrm{d}z}\kappa_u\theta_u\mathrm{d}u\right\}\mathrm{d}t$.

3.2. **GARCH diffusion model.** Suppose the spot S with variance V follows the GARCH diffusion dynamics

$$\begin{aligned}\mathrm{d}S_t &= S_t((r_t^d-r_t^f)\mathrm{d}t+\sqrt{V_t}\mathrm{d}W_t),\quad S_0,\\ \mathrm{d}V_t &= \kappa_t(\theta_t-V_t)\mathrm{d}t+\lambda_tV_t\mathrm{d}B_t,\quad V_0=v_0,\\ \mathrm{d}\langle W,B\rangle_t &= \rho_t\mathrm{d}t.\end{aligned}$$

Like the Heston model, we model the variance directly. The unobservable parameters mainly stem from the process V:

(1) $(\kappa_t)_{0\le t\le T}$. The time-dependent mean reversion speed of V.

(2) $(\theta_t)_{0\le t\le T}$. The time-dependent mean reversion level of V.

(3) $(\lambda_t)_{0\le t\le T}$. The time-dependent volatility of V (vol-vol).

(4) $(\rho_t)_{0\le t\le T}$. The instantaneous correlation between the spot S and its variance V.

Definition 3.5 (Inverse-Gamma process)**.** A process $\tilde{V}$ is called an Inverse-Gamma (IGa) process if it solves the SDE

$$\mathrm{d}\tilde{V}_t = \kappa_t(\theta_t - \tilde{V}_t)\mathrm{d}t + \lambda_t\tilde{V}_t\mathrm{d}\tilde{B}_t, \quad \tilde{V}_0 = \tilde{v}_0,$$

where we assume κ, θ and λ are time-dependent and deterministic and satisfy some regularity conditions. Denote Y to be a GBM$(1; -\kappa_t, \lambda_t)$. Then the explicit strong solution of of $\tilde{V}$ is

$$\tilde{V}_t = Y_t\left(\tilde{v}_0 + \int_0^t \frac{\kappa_u\theta_u}{Y_u}\mathrm{d}u\right).$$

We denote $\tilde{V}$ to be an IGa$(\tilde{v}_0; \kappa_t, \theta_t, \lambda_t)$.

It is evident that the variance process V is an IGa$(v_0; \kappa_t, \theta_t, \lambda_t)$.

Lemma 3.6 (Moments of IGa process)**.** Let $\tilde{V}$ be an IGa$(\tilde{v}_0; \kappa_t, \theta_t, \lambda_t)$. Then $\tilde{V}$ has the following moments

$$\begin{aligned}
\mathbb{E}(\tilde{V}_t^n) &= e^{\int_0^t \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z\mathrm{d}z}\left(\tilde{v}_0^n + n\int_0^t \kappa_u\theta_u e^{-\int_0^u \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z\mathrm{d}z}\mathbb{E}(\tilde{V}_u^{n-1})\mathrm{d}u\right),\\
\mathrm{Var}(\tilde{V}_t) &= e^{-2\int_0^t \kappa_z\mathrm{d}z}\int_0^t \lambda_u^2\mathbb{E}(\tilde{V}_u^2)e^{2\int_0^u \kappa_z\mathrm{d}z}\mathrm{d}u,\\
\mathrm{Cov}(\tilde{V}_s, \tilde{V}_t) &= \mathrm{Var}(\tilde{V}_s)e^{-\int_s^t \kappa_z\mathrm{d}z},\\
\mathbb{E}(\tilde{V}_s^m\tilde{V}_t^n) &= e^{\int_0^t \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z\mathrm{d}z}\Big(\mathbb{E}(\tilde{V}_s^{m+n})e^{-\int_0^s \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z\mathrm{d}z}\\
&\quad + n\int_s^t \kappa_u\theta_u e^{-\int_0^u \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z\mathrm{d}z}\mathbb{E}(\tilde{V}_s^m\tilde{V}_u^{n-1})\mathrm{d}u\Big),\\
\mathrm{Cov}(\tilde{V}_s^m, \tilde{V}_t^n) &= \mathbb{E}(\tilde{V}_s^m\tilde{V}_t^n) - \mathbb{E}(\tilde{V}_s^m)\mathbb{E}(\tilde{V}_t^n),
\end{aligned}$$

all for $m, n \ge 1$ and $s < t$. The derivation is given in Appendix C.2.

Lemma 3.7. Let $(\mathbb{Q}_n)_{n\ge 0}$ be a sequence of probability measures equivalent to $\mathbb{Q}$, defined by the Radon-Nikodym derivatives

$$\frac{\mathrm{d}\mathbb{Q}_{n+1}}{\mathrm{d}\mathbb{Q}_n} := \xi_T^{(n)} := \exp\left\{\int_0^T \rho_u\sqrt{V_u}\mathrm{d}B_u^n - \frac{1}{2}\int_0^T \rho_u^2 V_u\mathrm{d}u\right\}, \quad \xi_T^{(0)} := \xi_T, \quad n \ge 0,$$

where $\mathbb{Q}_0 := \mathbb{Q}$ and $B^0 := B$. Under $\mathbb{Q}_n$, $B_t^n := B_t^{n-1} - \int_0^t \rho_u\sqrt{V_u}\mathrm{d}u$ is a Brownian motion. For $n \ge 0$, the dynamics of V under the measure $\mathbb{Q}_n$ are

$$\mathrm{d}V_t = \kappa_t\left(\theta_t - V_t + \frac{n\lambda_t\rho_t}{\kappa_t}V_t^{3/2}\right)\mathrm{d}t + \lambda_t V_t\mathrm{d}B_t^n.$$

Proposition 3.8. Let $\mathbb{Q}_n$ be defined as in Lemma 3.7. Under the measures $\mathbb{Q}_n$, $n \geq 1$, V has no known explicit solution, nor known explicit moments.

Proof. The above SDE is a linear diffusion type SDE. From Appendix D, it is known that if an explicit solution exists, it is given by

$$V_t = Y_t/F_t,$$

where F is a GBM$(1; \lambda_t^2, -\lambda_t)$ and Y is the solution to the integral equation (written in differential form)

$$\mathrm{d}Y_t = \left(\kappa_t\theta_t F_t - \kappa_t Y_t + \frac{n\lambda_t\rho_t}{\kappa_t} Y_t^{3/2} F_t^{-1/2}\right)\mathrm{d}t.$$

Define $A_t := \kappa_t\theta_t F_t$ and $C_t := \frac{n\lambda_t\rho_t}{\kappa_t} F_t^{-1/2}$. Then first note that A_t and C_t are both non-differentiable in t. Thus

$$\mathrm{d}Y_t = \left(A_t - \kappa_t Y_t + C_t Y_t^{3/2}\right)\mathrm{d}t.$$

As far as we know, there is no explicit solution to these types of ODEs in the literature, even when A and C are differentiable. □

As for explicit moments, it is unclear how to approach this problem. There seems to be no approach to this problem in the literature, especially in the case of time-dependent parameters.

3.2.1. *Pricing under the GARCH diffusion framework:* $\rho = 0$. The change of measure technique gives an intractable dynamic for V; we cannot appeal to it for calculating expectations. However, in the case of $\rho = 0$ a.e., this implies $\xi_T = 1$ $\mathbb{Q}$ a.s., and one will notice that the terms in the expansion requiring a change of measure will disappear. Of course, the cost is the unrealistic assumption that spot and volatility movements are uncorrelated. We hope to mitigate this issue in future work by combining this approach with small correlation expansion methods, see Antonelli and Scarlatti [4], Antonelli et al. [5].

Theorem 3.9 (Second-order GARCH option price)**.** Assume $\rho = 0$ a.e., then the second-order expansion of the GARCH option price, denoted $\mathrm{Put}^{(2)}_{\mathrm{GARCH}}$ is

$$\mathrm{Put}^{(2)}_{\mathrm{GARCH}} = \mathrm{Put}_{\mathrm{BS}}(\bar{x}, \bar{y}) + \frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\bar{x}, \bar{y})\mathbb{E}\left(\int_0^T (V_t - \mathbb{E}(V_t))\mathrm{d}t\right)^2. \tag{12}$$

Here the expectation is

$$\mathbb{E}\left(\int_0^T (V_t - \mathbb{E}(V_t))\mathrm{d}t\right)^2 = 2\int_0^T \left(\int_0^t \mathrm{Cov}(V_s, V_t)\mathrm{d}s\right)\mathrm{d}t.$$

Furthermore, $\bar{x} = S_0$ and $\bar{y} = \int_0^T \mathbb{E}(V_t)\mathrm{d}t$. Both $\mathrm{Cov}(V_s, V_t)$ and $\mathbb{E}(V_t)$ are given in Lemma 3.6.

3.3. **Inverse-Gamma model.** Suppose the spot S with volatility V follows the Inverse-Gamma dynamics

$$\begin{aligned} \mathrm{d}S_t &= S_t((r_t^d - r_t^f)\mathrm{d}t + V_t \mathrm{d}W_t), \quad S_0, \\ \mathrm{d}V_t &= \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t, \quad V_0 = v_0, \\ \mathrm{d}\langle W, B\rangle_t &= \rho_t \mathrm{d}t, \end{aligned}$$

The unobservable parameters mainly stem from the process V

(1) $(\kappa_t)_{0\le t\le T}$. The time-dependent mean reversion speed of V.

(2) $(\theta_t)_{0\le t\le T}$. The time-dependent mean reversion level of V.

(3) $(\lambda_t)_{0\le t\le T}$. The time-dependent volatility of V (vol-vol).

(4) $(\rho_t)_{0\le t\le T}$. The instantaneous correlation between the spot S and its volatility V.

Note that unlike the Heston model, we are no longer modelling the variance directly. Instead, we model its square root, the volatility. To arrive at the desired framework, one replaces σ_t with V_t^2 from the initial sections. Immediately, it is clear that the calculations are less straight forward, as the process V^2 is not nearly as convenient as V.

Lemma 3.10. Let $(\mathbb{Q}_n)_{n\ge 0}$ be a sequence of probability measures equivalent to $\mathbb{Q}$, defined by the Radon-Nikodym derivatives

$$\frac{\mathrm{d}\mathbb{Q}_{n+1}}{\mathrm{d}\mathbb{Q}_n} := \xi_T^{(n)} := \exp\left\{\int_0^T \rho_u V_u \mathrm{d}B_u^n - \frac{1}{2}\int_0^T \rho_u^2 V_u^2 \mathrm{d}u\right\}, \quad \xi_T^{(0)} := \xi_T, \quad n \ge 0,$$

where $\mathbb{Q}_0 := \mathbb{Q}$ and $B^0 := B$. Under $\mathbb{Q}_n$, $B_t^n := B_t^{n-1} - \int_0^t \rho_u V_u \mathrm{d}u$ is a Brownian motion. For $n \ge 0$, the dynamics of V under the measure $\mathbb{Q}_n$ are

$$\mathrm{d}V_t = \kappa_t\left(\theta_t - V_t + \frac{n\lambda_t\rho_t}{\kappa_t}V_t^2\right)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t^n.$$

Under the measures $\mathbb{Q}_n$, $n \ge 1$, V has no explicit solution, nor explicit moments. This can seen in a similar way of Proposition 3.8; the resulting integral equation needed to be solved has no known explicit solution. Thus, we cannot explicitly calculate some of the terms in the expansion for the IGa model.

3.3.1. *Pricing under the Inverse-Gamma framework: $\rho = 0$.* Again, the dynamics of V are intractable under $\mathbb{Q}_n$. Assuming $\rho = 0$ a.e. will eliminate the terms we cannot calculate.

Theorem 3.11 (Second-order IGa option price)**.** Assume $\rho = 0$ a.e., then the second-order expansion of the option price, denoted $\mathrm{Put}_{\mathrm{IGa}}^{(2)}$, is

$$\mathrm{Put}_{\mathrm{IGa}}^{(2)} = \mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) + \frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\mathbb{E}\left(\int_0^T (V_t^2 - \mathbb{E}(V_t^2))\mathrm{d}t\right)^2. \tag{13}$$

Here the expectation is

$$\mathbb{E}\left(\int_0^T (V_t^2 - \mathbb{E}(V_t^2))\mathrm{d}t\right)^2 = 2\int_0^T \left(\int_0^t \mathrm{Cov}(V_s^2, V_t^2)\mathrm{d}s\right)\mathrm{d}t.$$

Furthermore, $\bar{x} = S_0$ and $\bar{y} = \int_0^T \mathbb{E}(V_t^2)\mathrm{d}t$. Both $\mathrm{Cov}(V_s^2, V_t^2)$ and $\mathbb{E}(V_t^2)$ are given in Lemma 3.6.

4. Error Analysis

We present an explicit bound on the error term in our expansion in terms of third-order mixed moments of the corresponding variance process. Specifically, this means bounding the remainder term in the second-order expansion of the function $\mathrm{Put}_{\mathrm{BS}}$, and for the case when $\rho \neq 0$, the error term associated with the expansion of $e^{\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}$.

Evidently, we will need explicit expressions for the error terms, which are given by Taylor's theorem. We only consider the results up to second-order.

Theorem 4.1 (Taylor's theorem for $f : \mathbb{R} \mapsto \mathbb{R}$)**.** Let $A \subseteq \mathbb{R}, B \subseteq \mathbb{R}$ and $f : A \mapsto B$ be a $C^3(\mathbb{R}; \mathbb{R})$ function in a closed interval about the point $a \in A$. Then Taylor's theorem states that the Taylor series of f around the point a is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + R,$$

where

$$R = \frac{1}{2}\int_a^x (x-u)^2 f'''(u)\mathrm{d}u = \frac{1}{2}(x-a)^3 \int_0^1 (1-u)^2 f'''(a+u(x-a))\mathrm{d}u.$$

Theorem 4.2 (Taylor's theorem for $g : \mathbb{R}^2 \mapsto \mathbb{R}$)**.** Let $A \subseteq \mathbb{R}^2, B \subseteq \mathbb{R}$ and $g : A \mapsto B$ be a $C^3(\mathbb{R}^2; \mathbb{R})$ function in a closed ball about the point $(a, b) \in A$. Then Taylor's theorem states that the Taylor series of g around the point (a, b) is given by

$$\begin{aligned} g(x,y) &= g(a,b) + g_x(a,b)(x-a) + g_y(a,b)(y-b) \\ &\quad + \frac{1}{2}g_{xx}(a,b)(x-a)^2 + \frac{1}{2}g_{yy}(a,b)(y-b)^2 + g_{xy}(a,b)(x-a)(y-b) + R, \end{aligned}$$

where

$$\begin{aligned} R &= \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1!\alpha_2!} E_\alpha(x,y)(x-a)^{\alpha_1}(y-b)^{\alpha_2} \\ E_\alpha(x,y) &= \int_0^1 (1-u)^2 \frac{\partial^3}{\partial x^{\alpha_1}\partial y^{\alpha_2}} g(a+u(x-a), b+u(y-b))\mathrm{d}u, \end{aligned}$$

with $\alpha := (\alpha_1, \alpha_2)$ and $|\alpha| := \alpha_1 + \alpha_2$.

First, we deal with the error associated with the function $\mathrm{Put}_{\mathrm{BS}}$. Recall the expansion of the Black-Scholes formula at the vector $(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t)$ around the point $(\bar{x}, \bar{y}) :=$

$(S_0, \int_0^T \mathbb{E}(\sigma_t)(1-\rho_t^2)\mathrm{d}t)$ and a general variance process σ

$$\begin{aligned}
\mathrm{Put}_{\mathrm{BS}}\left(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t\right) &= \mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) \\
&+ \partial_x \mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) S_0(\xi_T - 1) + \partial_y \mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) \left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right) \\
&+ \frac{1}{2}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) S_0^2(\xi_T-1)^2 + \frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 \\
&+ \partial_{xy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y}) S_0(\xi_T-1)\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right) + \mathcal{E}_{\mathrm{BS}}(\sigma).
\end{aligned}$$

Using Theorem 4.2 for the function $\mathrm{Put}_{\mathrm{BS}}$, this gives the error term as

$$\mathcal{E}_{\mathrm{BS}}(\sigma) = \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1!\alpha_2!} E_\alpha\left(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t\right) S_0^{\alpha_1}(\xi_T-1)^{\alpha_1}\left(\int_0^T (1-\rho_u^2)(\sigma_u - \mathbb{E}(\sigma_u))\mathrm{d}u\right)^{\alpha_2}$$

$$\begin{aligned}
E_\alpha(\cdot,\cdot) &= \int_0^1 (1-u)^2 \\
&\frac{\partial^3}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\mathrm{Put}_{\mathrm{BS}}\left(S_0 + uS_0(\xi_T-1), \int_0^T (1-\rho_t^2)\mathbb{E}(\sigma_t)\mathrm{d}t + u\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right)\mathrm{d}u.
\end{aligned}$$

Next, we look at the error associated with the calculation of $\mathbb{E}\xi_T^2$. Let us look at this term without the expectation.

$$\xi_T^2 = \left(\xi_T^2 e^{-\int_0^T \rho_u^2\sigma_u \mathrm{d}u}\right) e^{\int_0^T \rho_u^2\sigma_u\mathrm{d}u}.$$

We expand $e^{\int_0^T \rho_u^2\sigma_u\mathrm{d}u}$ around the mean of the exponential's argument under $\mathbb{Q}_2$. Note that $\xi_T^2 e^{-\int_0^T \rho_u^2\sigma_u\mathrm{d}u}$ is the Radon-Nikodym derivative which changes measure from $\mathbb{Q}$ to $\mathbb{Q}_2$. Expanding to second-order gives

$$\begin{aligned}
e^{\int_0^T \rho_u^2\sigma_u\mathrm{d}u} &= \Bigg\{ e^{\int_0^T \rho_u^2\mathbb{E}_{\mathbb{Q}_2}(\sigma_u)\mathrm{d}u}\left(1 + \int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))\mathrm{d}u + \frac{1}{2}\left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))\mathrm{d}u\right)^2\right) \\
&+ \frac{1}{2}\left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u)\mathrm{d}u\right)^3 \int_0^1 (1-u)^2 e^{\int_0^T \rho_m^2\mathbb{E}_{\mathbb{Q}_2}(\sigma_m)\mathrm{d}m} e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))\mathrm{d}m}\mathrm{d}u \Bigg\}.
\end{aligned}$$

Finally, the coefficient in front of ξ_T^2 in the pricing formula is $\frac{1}{2}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})S_0^2$. Thus, the error associated with this expansion, denoted by $\tilde{\mathcal{E}}(\sigma)$, is

$$\begin{aligned}
\tilde{\mathcal{E}}(\sigma) &= \frac{1}{4}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})S_0^2\xi_T^2 e^{-\int_0^T \rho_u^2\sigma_u\mathrm{d}u}\left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))\mathrm{d}u\right)^3 \\
&\cdot \int_0^1 (1-u)^2 e^{\int_0^T \rho_m^2\mathbb{E}_{\mathbb{Q}_2}(\sigma_m)\mathrm{d}m} e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))\mathrm{d}m}\mathrm{d}u.
\end{aligned}$$

In the end, the total expansion error can be summarised by the following theorem.

Theorem 4.3 (Total expansion error)**.** The error due to Taylor expansions for a general variance process σ is given by

$$\mathcal{E}(\sigma) = \mathcal{E}_{\mathrm{BS}}(\sigma) + \tilde{\mathcal{E}}(\sigma),$$

where

$$\mathcal{E}_{\mathrm{BS}}(\sigma) = \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1!\alpha_2!} E_\alpha\left(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t\right) S_0^{\alpha_1}(\xi_T-1)^{\alpha_1}\left(\int_0^T (1-\rho_u^2)(\sigma_u - \mathbb{E}(\sigma_u))\mathrm{d}u\right)^{\alpha_2},$$

$$E_\alpha(\cdot,\cdot) = \int_0^1 (1-u)^2$$
$$\frac{\partial^3}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\mathrm{Put}_{\mathrm{BS}}\left(S_0 + uS_0(\xi_T-1), \int_0^T (1-\rho_t^2)\mathbb{E}(\sigma_t)\mathrm{d}t + u\left(\int_0^T (1-\rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right)\mathrm{d}u,$$

and

$$\tilde{\mathcal{E}}(\sigma) = \frac{1}{4}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})S_0^2\xi_T^2 e^{-\int_0^T \rho_u^2\sigma_u \mathrm{d}u}\left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))\mathrm{d}u\right)^3$$
$$\cdot \int_0^1 (1-u)^2 e^{\int_0^T \rho_m^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)\mathrm{d}m} e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))\mathrm{d}m}\mathrm{d}u.$$

Corollary 4.4 (Total expansion error: $\rho = 0$)**.** The error due to Taylor expansions for a general variance process σ and $\rho = 0$ a.e., denoted $\mathcal{E}_0$, is given by

$$\mathcal{E}_0(\sigma) = \frac{1}{2}\left(\int_0^T (\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^3 \int_0^1 (1-u)^2\partial_{yyy}\mathrm{Put}_{\mathrm{BS}}\left(S_0, \int_0^T \mathbb{E}(\sigma_t)\mathrm{d}t + u\left(\int_0^T (\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right)\mathrm{d}u,$$

which is just $\mathcal{E}_{\mathrm{BS}}$ when $\rho = 0$ a.e.

The hope now is to be able to bound $\mathbb{E}(\mathcal{E}(\sigma))$ in terms of the moments of the variance process σ.

Lemma 4.5 (Error bounds for general σ)**.** Suppose that M_{α_1,α_2} are uniform bounds on $\frac{\partial^3\mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$, where $\alpha_1+\alpha_2 = 3$. The error term in the pricing formula is bounded as

$$|\mathbb{E}(\mathcal{E}(\sigma))| \leq \frac{1}{3}\sum_{|\alpha|=3}\frac{|\alpha|}{\alpha_1!\alpha_2!}M_{\alpha_1,\alpha_2}S_0^{\alpha_1}\mathbb{E}\left\{|\xi_T - 1|^{\alpha_1}\left(\int_0^T (1-\rho_u^2)|\sigma_u - \mathbb{E}(\sigma_u)|\mathrm{d}u\right)^{\alpha_2}\right\}$$
$$+ \frac{1}{4}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})S_0^2 e^{\int_0^T \rho_m^2\mathbb{E}(\sigma_m)\mathrm{d}m}\mathbb{E}_{\mathbb{Q}_2}\left\{\left(\int_0^T \rho_u^2|\sigma_u - \mathbb{E}(\sigma_u)|\mathrm{d}u\right)^3\right.$$
$$\left.\cdot \int_0^1 (1-u)^2 e^{u\int_0^T \rho_m^2|\sigma_m - \mathbb{E}(\sigma_m)|\mathrm{d}m}\mathrm{d}u\right\}.$$

Corollary 4.6 (Error bounds for general σ: $\rho = 0$)**.** Suppose that M is a uniform bound on $\partial_{yyy}\mathrm{Put}_{\mathrm{BS}}$. The error term in the pricing formula is bounded as

$$|\mathbb{E}(\mathcal{E}_0(\sigma))| \leq \frac{M}{3!}\mathbb{E}\left(\int_0^T |\sigma_t - \mathbb{E}(\sigma_t)|\mathrm{d}t\right)^3 = M\int_0^T\int_0^t\int_0^s \mathbb{E}|(\sigma_t - \mathbb{E}\sigma_t)(\sigma_s - \mathbb{E}\sigma_s)(\sigma_u - \mathbb{E}\sigma_u)|\mathrm{d}u\mathrm{d}s\mathrm{d}t.$$

5. Fast calibration

In this section, we devise a fast calibration scheme under the assumption of piecewise-constant parameters. To this end, define the integral operator

$$\omega_T^{(k,l)} := \int_0^T l_u e^{\int_0^u k_z \mathrm{d}z} \mathrm{d}u, \tag{14}$$

[11] and recursively, define

$$\omega_T^{(k^{(n)},l^{(n)}),(k^{(n-1)},l^{(n-1)}),\dots,(k^{(1)},l^{(1)})} := \omega_T^{\left(k^{(n)},l^{(n)}w_\cdot^{(k^{(n-1)},l^{(n-1)}),\cdots,(k^{(1)},l^{(1)})}\right)}, \quad n \in \mathbb{N}. \tag{15}$$

[12] It is clear that the pricing equations can be written in terms of the integral operators eq. (14) and eq. (15).

Assume a time discretisation of $[0,T]$ as $\{0 = T_0, T_1, \dots, T_{N-1}, T_N = T\}$, with $\Delta T_i := T_{i+1} - T_i$ and $\Delta T_0 \equiv 1$. When the dummy functions are piecewise constant, i.e., $l_t^{(n)} = l_i^{(n)}$ on $t \in [T_i, T_{i+1})$ and similarly for $k^{(n)}$, then we can recursively calculate the integral operators eq. (14) and eq. (15). Define

$$e_t^{(k^{(n)},\dots,k^{(1)})} := e^{\int_0^t \sum_{j=1}^n k_z^{(j)} \mathrm{d}z},$$
$$\varphi_{T_i,t}^{(k,p)} := \int_{T_i}^t \gamma_i^p(u) e^{\int_{T_i}^u k_z \mathrm{d}z} \mathrm{d}u,$$

where $\gamma_i(u) := (u - T_i)/\Delta T_i$ and $p \in \mathbb{N} \cup \{0\}$. And recursively

$$\varphi_{T_i,t}^{(k^{(n)},p_n),\dots,(k^{(2)},p_2),(k^{(1)},p_1)} := \int_{T_i}^t \gamma_i^{p_n}(u) e^{\int_{T_i}^u k_z^{(n)} \mathrm{d}z} \varphi_{T_i,u}^{(k^{(n-1)},p_{n-1}),\dots,(k^{(2)},p_2),(k^{(1)},p_1)} \mathrm{d}u,$$

[11] Sufficient conditions for when $\omega_T^{(k,l)}$ exists and is finite is when $u \mapsto l_u$ and $u \mapsto e^{\int_0^u \kappa_z \mathrm{d}z}$ are real valued Borel measurable functions that belong to $L_p(\mathbb{R})$ and $L_q(\mathbb{R})$ respectively for some p and q that satisfy $1/p + 1/q = 1$.

[12] For example
$$\omega_T^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} = \int_0^T l_{u_3}^{(3)} e^{\int_0^{u_3} k_z^{(3)} \mathrm{d}z} \left(\int_0^{u_3} l_{u_2}^{(2)} e^{\int_0^{u_2} k_z^{(2)} \mathrm{d}z} \left(\int_0^{u_2} l_{u_1}^{(1)} e^{\int_0^{u_1} k_z^{(1)} \mathrm{d}z} \mathrm{d}u_1 \right) \mathrm{d}u_2 \right) \mathrm{d}u_3.$$

where $p_n \in \mathbb{N} \cup \{0\}$. [13] With the assumption that the dummy functions are piecewise-constant, we can obtain the integral operator at time T_{i+1} expressed by terms at T_i.

$$
\begin{aligned}
\omega_{T_{i+1}}^{(k^{(1)},l^{(1)})} &= \omega_{T_i}^{(k^{(1)},l^{(1)})} + l_i^{(1)} e_{T_i}^{(k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(1)},0)}, \\
\omega_{T_{i+1}}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} &= \omega_{T_i}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(2)} e_{T_i}^{(k^{(2)})} \varphi_{T_i,T_{i+1}}^{(k^{(2)},0)} \omega_{T_i}^{(k^{(1)},l^{(1)})} \\
&+ l_i^{(2)} l_i^{(1)} e_{T_i}^{(k^{(2)},k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(2)},0),(k^{(1)},0)}, \\
\omega_{T_{i+1}}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} &= \omega_{T_i}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(3)} e_{T_i}^{(k^{(3)})} \varphi_{T_i,T_{i+1}}^{(k^{(3)},0)} \omega_{T_i}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(3)} l_i^{(2)} e_{T_i}^{(k^{(3)},k^{(2)})} \varphi_{T_i,T_{i+1}}^{(k^{(3)},0),(k^{(2)},0)} \omega_{T_i}^{(k^{(1)},l^{(1)})} \\
&+ l_i^{(3)} l_i^{(2)} l_i^{(1)} e_{T_i}^{(k^{(3)},k^{(2)},k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(3)},0),(k^{(2)},0),(k^{(1)},0)}, \\
\omega_{T_{i+1}}^{(k^{(4)},l^{(4)}),\ldots,(k^{(1)},l^{(1)})} &= \omega_{T_i}^{(k^{(4)},l^{(4)}),(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(4)} e_{T_i}^{(k^{(4)})} \varphi_{T_i,T_{i+1}}^{(k^{(4)},0)} \omega_{T_i}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(4)} l_i^{(3)} e_{T_i}^{(k^{(4)},k^{(3)})} \varphi_{T_i,T_{i+1}}^{(k^{(4)},0),(k^{(3)},0)} \omega_{T_i}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(4)} l_i^{(3)} l_i^{(2)} e_{T_i}^{(k^{(4)},k^{(3)},k^{(2)})} \varphi_{T_i,T_{i+1}}^{(k^{(4)},0),(k^{(3)},0),(k^{(2)},0)} \omega_{T_i}^{(k^{(1)},l^{(1)})} \\
&+ l_i^{(4)} l_i^{(3)} l_i^{(2)} l_i^{(1)} e_{T_i}^{(k^{(4)},k^{(3)},k^{(2)},k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(4)},0),(k^{(3)},0),(k^{(2)},0),(k^{(1)},0)}, \\
\omega_{T_{i+1}}^{(k^{(5)},l^{(5)}),\ldots,(k^{(1)},l^{(1)})} &= \omega_{T_i}^{(k^{(5)},l^{(5)}),(k^{(4)},l^{(4)}),(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} e_{T_i}^{(k^{(5)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0)} \omega_{T_i}^{(k^{(4)},l^{(4)}),(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} l_i^{(4)} e_{T_i}^{(k^{(5)},k^{(4)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0),(k^{(4)},0)} \omega_{T_i}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} l_i^{(4)} l_i^{(3)} e_{T_i}^{(k^{(5)},k^{(4)},k^{(3)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0),(k^{(4)},0),(k^{(3)},0)} \omega_{T_i}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} l_i^{(4)} l_i^{(3)} l_i^{(2)} e_{T_i}^{(k^{(5)},k^{(4)},k^{(3)},k^{(2)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0),(k^{(4)},0),(k^{(3)},0),(k^{(2)},0)} \omega_{T_i}^{(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} l_i^{(4)} l_i^{(3)} l_i^{(2)} l_i^{(1)} e_{T_i}^{(k^{(5)},k^{(4)},k^{(3)},k^{(2)},k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0),(k^{(4)},0),(k^{(3)},0),(k^{(2)},0),(k^{(1)},0)}.
\end{aligned}
$$

[13] For example

$$
\varphi_{T_i,t}^{(k^{(3)},p_3),(k^{(2)},p_2),(k^{(1)},p_1)} = \int_{T_i}^{t} \gamma_i^{p_3}(u_3) e^{\int_{T_i}^{u_3} k_z^{(3)} \mathrm{d}z} \left(\int_{T_i}^{u_3} \gamma_i^{p_2}(u_2) e^{\int_{T_i}^{u_2} k_z^{(2)} \mathrm{d}z} \left(\int_{T_i}^{u_2} \gamma_i^{p_1}(u_1) e^{\int_{T_i}^{u_1} k_z^{(1)} \mathrm{d}z} \mathrm{d}u_1 \right) \mathrm{d}u_2 \right) \mathrm{d}u_3.
$$

[14] The only terms here that are not explicit are the functions $e_{\cdot}^{(\cdot,\dots,\cdot)}$ and $\varphi_{T_i,\cdot}^{(\cdot,\cdot),\dots,(\cdot,\cdot)}$. For $t \in (T_i, T_{i+1}]$, we can derive the following:

$$e_t^{(k^{(n)},\dots,k^{(1)})} = e_{T_i}^{(k^{(n)},\dots,k^{(1)})} e^{\Delta T_i \gamma_i(t) \sum_{j=1}^n k_i^{(j)}} = e^{\sum_{m=0}^{i-1} \Delta T_m \sum_{j=1}^n k_m^{(j)}} e^{\Delta T_i \gamma_i(t) \sum_{j=1}^n k_i^{(j)}},$$

where $e_0^{(k^{(n)},\dots,k^{(1)})} = 1$. By using integration by parts and basic integration properties, we find that for $n \geq 2$

$$\varphi_{T_i,t}^{(k,p)} = \begin{cases} \frac{1}{k_i}\left(\gamma_i^p(t) e^{k_i \Delta T_i \gamma_i(t)} - \frac{p}{\Delta T_i}\varphi_{T_i,t}^{(k,p-1)}\right), & k_i \neq 0, p \geq 1, \\ \frac{1}{k_i}\left(e^{k_i \Delta T_i \gamma_i(t)} - 1\right), & k_i \neq 0, p = 0, \\ \frac{1}{p+1}\Delta T_i \gamma_i^{p+1}(t), & k_i = 0, p \geq 0, \end{cases}$$

and

$$\varphi_{T_i,t}^{(k^{(n)},p_n),\dots,(k^{(1)},p_1)} = \begin{cases} \frac{1}{k_i^{(n)}}\Big(\gamma_i^{p_n}(t) e^{k_i^{(n)}\Delta T_i \gamma_i(t)} \varphi_{T_i,t}^{(k^{(n-1)},p_{n-1}),\dots,(k^{(1)},p_1)} \\ -\frac{p_n}{\Delta T_i}\varphi_{T_i,t}^{(k^{(n)},p_n-1),(k^{(n-1)},p_{n-1}),\dots,(k^{(1)},p_1)} \\ -\varphi_{T_i,t}^{(k^{(n)}+k^{(n-1)},p_n+p_{n-1}),(k^{(n-2)},p_{n-2}),\dots,(k^{(1)},p_1)}\Big), & k_i^{(n)} \neq 0, p_n \geq 1, \\ \frac{1}{k_i^{(n)}}\Big(e^{k_i^{(n)}\Delta T_i \gamma_i(t)} \varphi_{T_i,t}^{(k^{(n-1)},p_{n-1}),\dots,(k^{(1)},p_1)} \\ -\varphi_{T_i,t}^{(k^{(n)}+k^{(n-1)},p_{n-1}),(k^{(n-2)},p_{n-2}),\dots,(k^{(1)},p_1)}\Big), & k_i^{(n)} \neq 0, p_n = 0, \\ \frac{\Delta T_i}{p_n+1}\Big(\gamma_i^{p_n+1}(t) \varphi_{T_i,t}^{(k^{(n-1)},p_{n-1}),\dots,(k^{(1)},p_1)} \\ -\varphi_{T_i,t}^{(k^{(n-1)},p_n+p_{n-1}+1),(k^{(n-2)},p_{n-2}),\dots,(k^{(1)},p_1)}\Big), & k_i^{(n)} = 0, p_n \geq 0. \end{cases}$$

5.1. **Heston calibration.**

Recall the second-order pricing function $\mathrm{Put}_{\mathrm{H}}^{(2)}$,

$$\begin{aligned} \mathrm{Put}_{\mathrm{H}}^{(2)} &= \mathrm{Put}_{\mathrm{BS}}(\bar{x}, \bar{y}) \\ &+ \frac{1}{2}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\bar{x}, \bar{y}) S_0^2 \mathbb{E}(\xi_T - 1)^2 + \frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\bar{x}, \bar{y}) \mathbb{E}\left(\int_0^T (1-\rho_t^2)(V_t - \mathbb{E}(V_t))\mathrm{d}t\right)^2 \\ &+ \partial_{xy}\mathrm{Put}_{\mathrm{BS}}(\bar{x}, \bar{y}) S_0 \mathbb{E}\left\{(\xi_T - 1)\left(\int_0^T (1-\rho_t^2)(V_t - \mathbb{E}(V_t))\mathrm{d}t\right)\right\}. \end{aligned}$$

[14] In general

$$\omega_{T_{i+1}}^{(k^{(n)},l^{(n)}),\dots,(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} = \sum_{q=1}^{n+1} \omega_{T_i}^{(k^{(n-q+1)},l^{(n-q+1)}),\dots,(k^{(1)},l^{(1)})} \left(\prod_{j=0}^{q-2} l_i^{(n-j)}\right) e_{T_i}^{(k^{(n-q+2)},\dots,k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(n-q+2)},0),\dots,(k^{(1)},0)},$$

where whenever the index goes outside of $\{1,\dots,n\}$, then that term is equal to 1.

The three expectations were calculated in Section 3.1.1. We can write them in terms of the integral operators eq. (14) and eq. (15).

$$\mathbb{E}(\xi_T-1)^2 \approx \exp\left\{v_0\omega_T^{(-(\kappa-2\lambda\rho),\rho^2)}+\omega_T^{(-(\kappa-2\lambda\rho),\rho^2),(\kappa-2\lambda\rho,\kappa\theta)}\right\}\left\{1+v_0\omega_T^{(-(\kappa-2\lambda\rho),\rho^2),(-(\kappa-2\lambda\rho),\rho^2),(\kappa-2\lambda\rho,\lambda^2)}\right.$$
$$\left.+\,\omega_T^{(-(\kappa-2\lambda\rho),\rho^2),(-(\kappa-2\lambda\rho),\rho^2),(\kappa-2\lambda\rho,\lambda^2),(\kappa-2\lambda\rho,\kappa\theta)}\right\}-1.$$

$$\mathbb{E}\left(\int_0^T(1-\rho_t^2)(V_t-\mathbb{E}(V_t))\mathrm{d}t\right)^2=2v_0\omega_T^{(-\kappa,1-\rho^2),(-\kappa,1-\rho^2),(\kappa,\lambda^2)}+2\omega_T^{(-\kappa,1-\rho^2),(-\kappa,1-\rho^2),(\kappa,\lambda^2),(\kappa,\kappa\theta)}.$$

$$\mathbb{E}\left\{(\xi_T-1)\left(\int_0^T(1-\rho_t^2)(V_t-\mathbb{E}(V_t))\mathrm{d}t\right)\right\}=v_0\left(\omega_T^{(-(\kappa-\lambda\rho),1-\rho^2)}-\omega_T^{(-\kappa,1-\rho^2)}\right)+\omega_T^{(-(\kappa-\lambda\rho),1-\rho^2),(\kappa-\lambda\rho,\kappa\theta)}$$
$$-\,\omega_T^{(-\kappa,1-\rho^2),(\kappa,\kappa\theta)}.$$

Furthermore $\bar{x}=S_0$ and $\bar{y}=v_0\omega_T^{(-\kappa,1-\rho^2)}+\omega_T^{(-\kappa,1-\rho^2),(\kappa,\kappa\theta)}$.

Assuming the parameters are all piecewise-constant on $\{0=T_0,T_1,\dots,T_{N-1},T_N=T\}$, i.e.,

$$(\kappa_t,\theta_t,\lambda_t,\rho_t)=(\kappa_i,\theta_i,\lambda_i,\rho_i),\quad t\in[T_i,T_{i+1}),\qquad i=0,\dots N-1,$$

then we can use the scheme presented in Section 5 to calibrate the Heston parameters.

5.2. **GARCH calibration:** $\rho=0$**.** Recall the second-order pricing function $\mathrm{Put}^{(2)}_{\mathrm{GARCH}}$,

$$\mathrm{Put}^{(2)}_{\mathrm{GARCH}}=\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})+\frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\bar{x},\bar{y})\mathbb{E}\left(\int_0^T(V_t-\mathbb{E}(V_t))\mathrm{d}t\right)^2.$$

We can write the expectation in terms of the integral operators eq. (14) and eq. (15).

$$\mathbb{E}\left(\int_0^T(V_t-\mathbb{E}(V_t))\mathrm{d}t\right)^2=2\Big(v_0^2\omega_T^{(-\kappa,1),(-\kappa,1),(\lambda^2,\lambda^2)}+2v_0\omega_T^{(-\kappa,1),(-\kappa,1),(\lambda^2,\lambda^2),(-(\lambda^2-\kappa),\kappa\theta)}$$
$$+\,2\omega_T^{(-\kappa,1),(-\kappa,1),(\lambda^2,\lambda^2),(-(\lambda^2-\kappa),\kappa\theta),(\kappa,\kappa\theta)}\Big).$$

Furthermore $\bar{x}=S_0$ and

$$\bar{y}=\int_0^T\mathbb{E}(V_t)\mathrm{d}t=v_0\omega_T^{(-\kappa,1)}+\omega_T^{(-\kappa,1),(\kappa,\kappa\theta)}.$$

Assuming the parameters are all piecewise-constant on $\{0=T_0,T_1,\dots,T_{N-1},T_N=T\}$, i.e.,

$$(\kappa_t,\theta_t,\lambda_t)=(\kappa_i,\theta_i,\lambda_i)\quad t\in[T_i,T_{i+1}),\qquad i=0,\dots N-1,$$

then we can use the scheme presented in Section 5 to calibrate the GARCH diffusion parameters.

6. Numerical tests and sensitivity analysis

We test our approximation method by considering the sensitivity of our approximation with respect to one parameter at a time. Specifically, for an arbitrary parameter set $(\mu_1, \mu_2, \ldots, \mu_n)$, we vary only one of the μ_i at a time and keep the rest fixed. Then, we compute implied volatilities via our approximation method as well as the Monte Carlo for strikes corresponding to Put 10, 25 and ATM deltas. Specifically,

$$\text{Error}(\mu) = \sigma_{IM-Approx}(\mu, K) - \sigma_{IM-Monte}(\mu, K)$$

for K corresponding to Put 10, Put 25 and ATM.

For all our simulations, we use 2,000,000 Monte Carlo paths, and 24 time steps per day. This is to reduce the Monte Carlo and discretisation errors sufficiently well.

6.1. Heston sensitivity analysis.

We consider maturity times $T \in \{1/12, 3/12, 6/12, 1\}$. We start from a 'safe' parameter, which are parameters calibrated by Bloomberg USD/JPY FX option price data on 9/07/18. The safe parameter set is $(S_0, v_0, r_d, r_f) = (100.00, 0.0036, 0.02, 0)$ with

$$(\kappa, \theta, \lambda, \rho) = \begin{cases} (5.000, 0.019, 0.414, -0.391), & T = 1/12, \\ (5.000, 0.011, 0.414, -0.391), & T = 3/12, \\ (5.000, 0.009, 0.414, -0.391), & T = 6/12, \\ (5.000, 0.009, 0.414, -0.391), & T = 1. \end{cases}$$

In our analysis, we vary one of the $(\kappa, \theta, \lambda, \rho)$ and keep the rest fixed.

Varying κ
We vary κ over the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

κ	1	2	3	4	5	6	7	8
1M	-1.31	0.10	1.12	1.85	2.40	2.82	3.14	3.39
3M	-51.25	-28.87	-16.79	-9.69	-5.25	-2.35	-0.39	0.97
6M	-125.36	-60.43	-31.45	-16.74	-8.64	-3.92	-1.07	0.70
1Y	-198.02	-72.32	-29.82	-12.27	-4.22	-0.32	1.66	2.67

Figure 1. κ: Error for ATM implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	-0.86	0.60	1.61	2.33	2.86	3.25	3.55	3.78
3M	-49.82	-27.87	-16.03	-9.08	-4.74	-1.92	-0.02	1.29
6M	-121.83	-57.64	-29.37	-15.03	-7.19	-2.66	0.05	1.70
1Y	-194.14	-69.80	-27.98	-10.85	-3.11	0.60	2.41	3.31

FIGURE 2. κ: Error for Put 25 implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	-0.79	0.77	1.84	2.59	3.14	3.55	3.86	4.09
3M	-48.42	-26.74	-15.11	-8.30	-4.09	-1.35	0.47	1.72
6M	-118.47	-55.45	-27.71	-13.80	-6.19	-1.81	0.80	2.37
1Y	-191.35	-68.26	-27.22	-10.51	-2.94	0.65	2.41	3.27

FIGURE 3. κ: Error for Put 10 implied volatilities in basis points

Varying θ
We vary θ over the set
$\{7\text{e-}03, 10\text{e-}03, 13\text{e-}03, 16\text{e-}03, 19\text{e-}03, 22\text{e-}03, 25\text{e-}03, 28\text{e-}03\}$.

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	-0.26	0.70	1.42	1.97	2.40	2.75	3.04	3.28
3M	-12.55	-6.34	-2.78	-0.55	0.95	2.02	2.80	3.39
6M	-14.60	-7.29	-3.45	-1.14	0.34	1.35	2.07	2.60
1Y	-8.21	-3.30	-0.81	0.62	1.51	2.10	2.50	2.79

FIGURE 4. θ: Error for ATM implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	-0.20	0.77	1.51	2.07	2.50	2.86	3.15	3.40
3M	-11.62	-5.76	-2.38	-0.25	1.17	2.17	2.91	3.47
6M	-13.50	-6.47	-2.76	-0.56	0.86	1.82	2.50	3.00
1Y	-6.48	-1.69	0.75	2.17	3.06	3.67	4.10	4.41

FIGURE 5. θ: Error for Put 25 implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	-0.09	0.87	1.58	2.12	2.54	2.88	3.16	3.39
3M	-11.88	-6.22	-3.01	-1.01	0.31	1.23	1.90	2.41
6M	-11.89	-4.97	-1.31	0.87	2.29	3.26	3.95	4.48
1Y	-6.33	-1.89	0.29	1.50	2.24	2.71	3.03	3.25

FIGURE 6. θ: Error for Put 10 implied volatilities in basis points

Varying λ
We vary λ over the set $\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$.

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	2.80	2.97	2.24	-0.56	-6.59	-16.77	-31.73	-51.86
3M	2.95	1.50	-4.39	-17.01	-37.79	-67.67	-107.02	-155.90
6M	2.77	0.53	-6.99	-22.03	-45.92	-79.29	-122.64	-176.50
1Y	3.08	1.78	-3.14	-13.33	-29.85	-53.44	-84.54	-123.02

FIGURE 7. λ: Error for ATM implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	3.80	4.11	3.51	0.86	-4.98	-14.88	-29.54	-49.41
3M	3.59	2.42	-3.08	-15.25	-35.61	-64.91	-103.80	-152.11
6M	3.60	1.70	-5.50	-20.04	-43.33	-76.10	-118.78	-171.52
1Y	3.32	2.10	-2.73	-12.74	-29.21	-52.35	-82.97	-121.09

FIGURE 8. λ: Error for Put 25 implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	3.56	3.91	3.40	0.90	-4.73	-14.36	-28.73	-48.09
3M	4.04	3.06	-2.19	-13.96	-33.74	-62.50	-100.87	-148.80
6M	3.84	2.06	-4.91	-19.30	-42.33	-74.85	-117.27	-169.57
1Y	3.71	2.78	-1.70	-11.29	-26.99	-49.50	-79.77	-117.51

FIGURE 9. λ: Error for Put 10 implied volatilities in basis points

Varying ρ
We vary ρ over the set $\{-0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1, 0\}$.

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	52.52	24.09	9.93	2.59	-1.25	-3.20	-4.08	-4.28
3M	53.60	21.85	4.84	-4.89	-10.63	-13.97	-15.71	-16.17
6M	52.02	20.13	2.84	-7.24	-13.33	-17.01	-18.99	-19.55
1Y	48.20	19.17	4.09	-4.33	-9.24	-12.11	-13.60	-13.91

FIGURE 10. ρ: Error for ATM implied volatilities in basis points

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	55.43	26.08	11.36	3.66	-0.42	-2.54	-3.55	-3.85
3M	57.58	24.55	6.73	-3.55	-9.66	-13.29	-15.24	-15.89
6M	54.73	21.62	3.61	-6.90	-13.27	-17.11	-19.19	-19.83
1Y	49.97	19.92	4.28	-4.47	-9.55	-12.50	-14.02	-14.34

FIGURE 11. ρ: Error for Put 25 implied volatilities in basis points

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	56.28	26.32	11.26	3.37	-0.79	-2.92	-3.91	-4.17
3M	60.03	26.03	7.61	-3.05	-9.42	-13.20	-15.28	-16.01
6M	58.74	24.43	5.63	-5.45	-12.23	-16.40	-18.75	-19.61
1Y	54.35	23.14	6.70	-2.62	-8.14	-11.45	-13.27	-13.87

FIGURE 12. ρ: Error for Put 10 implied volatilities in basis points

The above sensitivity analysis is consistent with what we were expecting. For example, for large maturity T, large vol-vol λ or large correlation $|\rho|$, the component-wise variance of the expansion vector increases. Thus, when these parameters are large, we expect the approximation to break down. As we can see, this indeed occurs. For realistic parameter values we see that the error in magnitude is around 10-50bps, which is reasonable for application purposes.

6.2. **GARCH sensitivity analysis.** We start from the same 'safe' parameter set from Section 6.1, albeit with $\rho = 0$ always.
Varying κ
We vary κ over the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

κ	1	2	3	4	5	6	7	8
1M	0.044	0.090	0.122	0.146	0.163	0.174	0.182	0.186
3M	0.003	0.019	0.026	0.028	0.027	0.025	0.022	0.019
6M	0.020	0.027	0.027	0.024	0.021	0.019	0.016	0.015
1Y	-0.047	-0.015	-0.004	0.000	0.002	0.002	0.003	0.003

FIGURE 13. κ: Error for ATM implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	0.044	0.092	0.127	0.152	0.171	0.183	0.192	0.197
3M	0.045	0.056	0.060	0.059	0.056	0.052	0.048	0.043
6M	-0.031	-0.013	-0.006	-0.003	-0.003	-0.002	-0.002	-0.002
1Y	-0.064	-0.029	-0.016	-0.010	-0.006	-0.004	-0.002	-0.002

FIGURE 14. κ: Error for Put 25 implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	0.072	0.118	0.151	0.175	0.192	0.204	0.212	0.216
3M	-0.074	-0.052	-0.040	-0.034	-0.032	-0.031	-0.031	-0.031
6M	-0.004	0.004	0.005	0.003	0.001	0.000	-0.001	-0.002
1Y	-0.044	-0.013	-0.003	0.001	0.003	0.003	0.003	0.003

FIGURE 15. κ: Error for Put 10 implied volatilities in basis points

Varying θ
We vary θ over the set
$\{7\text{e-}03, 10\text{e-}03, 13\text{e-}03, 16\text{e-}03, 19\text{e-}03, 22\text{e-}03, 25\text{e-}03, 28\text{e-}03\}$.

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	0.039	0.077	0.112	0.143	0.171	0.198	0.223	0.246
3M	0.046	0.061	0.073	0.085	0.095	0.104	0.113	0.121
6M	0.012	0.019	0.024	0.029	0.033	0.036	0.040	0.043
1Y	-0.040	-0.045	-0.050	-0.054	-0.058	-0.062	-0.066	-0.069

FIGURE 16. θ: Error for ATM implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	0.051	0.089	0.124	0.156	0.185	0.212	0.237	0.261
3M	0.019	0.031	0.042	0.051	0.059	0.066	0.072	0.079
6M	0.013	0.017	0.020	0.023	0.026	0.029	0.031	0.033
1Y	-0.047	-0.055	-0.063	-0.069	-0.075	-0.081	-0.086	-0.091

Figure 17. θ: Error for Put 25 implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	0.043	0.081	0.116	0.147	0.176	0.203	0.228	0.252
3M	0.024	0.038	0.049	0.059	0.067	0.075	0.082	0.089
6M	-0.010	-0.009	-0.008	-0.008	-0.007	-0.007	-0.007	-0.007
1Y	-0.034	-0.038	-0.042	-0.045	-0.049	-0.052	-0.055	-0.058

Figure 18. θ: Error for Put 10 implied volatilities in basis points

Varying λ
We vary λ over the set $\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$.

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	-0.184	-0.182	-0.179	-0.176	-0.173	-0.168	-0.161	-0.153
3M	-0.030	-0.026	-0.020	-0.012	0.000	0.016	0.040	0.073
6M	-0.015	-0.017	-0.018	-0.015	-0.007	0.010	0.039	0.083
1Y	0.004	0.008	0.013	0.021	0.034	0.055	0.088	0.135

Figure 19. λ: Error for ATM implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	-0.194	-0.197	-0.200	-0.203	-0.204	-0.205	-0.204	-0.201
3M	-0.032	-0.028	-0.023	-0.015	-0.004	0.013	0.036	0.070
6M	-0.019	-0.024	-0.026	-0.025	-0.017	-0.002	0.026	0.069
1Y	0.008	0.014	0.021	0.032	0.048	0.073	0.109	0.160

Figure 20. λ: Error for Put 25 implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	-0.185	-0.184	-0.183	-0.180	-0.177	-0.173	-0.167	-0.159
3M	-0.036	-0.033	-0.029	-0.023	-0.012	0.003	0.026	0.059
6M	-0.010	-0.010	-0.007	-0.001	0.011	0.031	0.063	0.112
1Y	-0.008	-0.011	-0.012	-0.009	-0.002	0.014	0.040	0.082

FIGURE 21. λ: Error for Put 10 implied volatilities in basis points

The GARCH error behaves well, with most errors being less than 1bp in magnitude. In contrast to the Heston analysis, this is most likely due to the case that the correlation ρ is assumed to be 0 always. Otherwise, the approximation behaves as we expect, with errors being larger for large maturity T and vol-vol λ, as the variance of the expansion point will grow with these parameters.

7. CONCLUSION

We have considered a closed-form expansion formula for European put option prices in the context of stochastic volatility models with time-dependent parameters. Our method involves a second-order Taylor expansion of the mixing solution, and then the explicit computation of a number of expectations via use of change of measure techniques. Such a method has been considered by Drimus with respect to the Heston model with constant parameters [9]. We extend this method as we consider time-dependent parameters, which requires an additional approximation of one of the expectations. Under the Heston framework, we find that all expectations induced by the expansion are able to be computed explicitly. Furthermore, we attempt to generalise this method to the GARCH diffusion and IGa models. We show that any sort of explicit solution or explicit moments of the resulting variance process after the change of measure is non-trivial to compute. By assuming $\rho = 0$ a.e. in the GARCH and IGa models, we are able to work around this problem, albeit with the added assumption of uncorrelated spot and volatility movements. We give general error bounds due to the Taylor expansions of the mixing solution in terms of third-order moments and mixed moments of the underlying variance process, We believe this to be a major contribution as it is a first in the literature for this type of methodology. We devise a fast calibration scheme which exploits a recursive property of our integral operators. Lastly, we perform a sensitivity analysis to investigate the quality of our approximation numerically in the Heston and GARCH models, and find that the error is well within the range appropriate for application purposes and behaves as we expect for certain parameter values, such as long maturity, large vol-vol and large correlation. Also, it is worth noting that the explicit expressions for moments and mixed moments of the IGa process may be the first in the literature. The purely probabilistic mixing solution approach, which is the backbone of our expansion method, is very appealing due to its generality and ability to handle time-dependent parameters. Further research would be needed to combine it, with no correlation restriction, with the type of non-affine stochastic volatility models favored by practitioners.

Appendix A. Mixing Solution

Under the risk-neutral measure $\mathbb{Q}$, suppose that the spot S with variance σ follows the dynamics

$$
\begin{aligned}
\mathrm{d}S_t &= S_t((r_t^d - r_t^f)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t), \quad S_0,\\
\mathrm{d}\sigma_t &= \alpha(t,\sigma_t)\mathrm{d}t + \beta(t,\sigma_t)\mathrm{d}B_t, \quad \sigma_0,\\
\mathrm{d}\langle W, B\rangle_t &= \rho_t \mathrm{d}t.
\end{aligned}
$$

We give an outline of the result

$$
\begin{aligned}
\mathrm{Put} = e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}(K - S_T)_+ &= \mathbb{E}\Big\{e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}\big[(K-S_T)_+|\mathcal{F}_T^B\big]\Big\}\\
&= \mathbb{E}\left(\mathrm{Put}_{\mathrm{BS}}\left(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t\right)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Put}_{\mathrm{BS}}(x,y) &:= Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(-d_-) - xe^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(-d_+)\\
d_\pm &:= \frac{\ln(x/K) + \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}.
\end{aligned}
$$

This is referred to as the mixing solution by Hull and White [16].

Proof. By writing the driving Brownian motion of the spot as $W_t = \int_0^t \rho_u \mathrm{d}B_u + \int_0^t \sqrt{1-\rho_u^2}\mathrm{d}Z_u$, where Z is an artificial Brownian motion under $\mathbb{Q}$ which is independent of B, this gives the strong solution of S as

$$
\begin{aligned}
S_T &= S_0\xi_T \exp\left\{\int_0^T (r_t^d - r_t^f)\mathrm{d}t - \frac{1}{2}\int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t + \int_0^T \sqrt{\sigma_t(1-\rho_t^2)}\mathrm{d}Z_t\right\},\\
\xi_t &:= \exp\left\{\int_0^t \rho_u\sqrt{\sigma_u}\mathrm{d}B_u - \frac{1}{2}\int_0^t \rho_u^2\sigma_u \mathrm{d}u\right\}.
\end{aligned}
$$

First, notice that both σ and ξ are adapted to the filtration $(\mathcal{F}_t^B)_{0\le t\le T}$. Thus, it is evident that $S_T|\mathcal{F}_T^B$ will have a log-normal distribution

$$
\begin{aligned}
S_T|\mathcal{F}_T^B &\sim \mathcal{LN}\left(\tilde{\mu}(T), \tilde{\sigma}^2(T)\right),\\
\tilde{\mu}(T) &:= \ln(S_0\xi_T) + \int_0^T (r_t^d - r_t^f)\mathrm{d}t - \frac{1}{2}\int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t,\\
\tilde{\sigma}^2(T) &:= \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t.
\end{aligned}
$$

Hence the calculation of $e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}((K-S_T)_+|\mathcal{F}_T^B)$ will result in a Black-Scholes like formula

$$
\begin{aligned}
&e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}((K-S_T)_+|\mathcal{F}_T^B)\\
&= Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K)-\tilde{\mu}(T)}{\tilde{\sigma}(T)}\right) - e^{-\int_0^T r_t^d \mathrm{d}t}e^{\tilde{\mu}(T)+\frac{1}{2}\tilde{\sigma}^2(T)}\mathcal{N}\left(\frac{\ln(K)-\tilde{\mu}(T)-\tilde{\sigma}^2(T)}{\tilde{\sigma}(T)}\right)\\
&= Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K)-\tilde{\mu}(T)-\frac{1}{2}\tilde{\sigma}^2(T)}{\tilde{\sigma}(T)}+\frac{1}{2}\tilde{\sigma}(T)\right)\\
&- S_0\xi_T e^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K)-\tilde{\mu}(T)-\frac{1}{2}\tilde{\sigma}^2(T)}{\tilde{\sigma}(T)}-\frac{1}{2}\tilde{\sigma}(T)\right)\\
&= Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K/S_0\xi_T)-\int_0^T(r_t^d-r_t^f)\mathrm{d}t}{\tilde{\sigma}(T)}+\frac{1}{2}\tilde{\sigma}(T)\right)\\
&- S_0\xi_T e^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K/S_0\xi_T)-\int_0^T(r_t^d-r_t^f)\mathrm{d}t}{\tilde{\sigma}(T)}-\frac{1}{2}\tilde{\sigma}(T)\right).
\end{aligned}
$$

It is immediate that $e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}((K-S_T)_+|\mathcal{F}_T^B) = \mathrm{Put}_{\mathrm{BS}}\left(S_0\xi_T, \tilde{\sigma}^2(T)\right)$. □

Appendix B. Call options

The calculation of call option prices follows a similar methodology to that of put options. We state the result without the proof.

$$
\begin{aligned}
\mathrm{Call} = e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}(S_T-K)_+ &= \mathbb{E}\Big\{e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}\big[(S_T-K)_+|\mathcal{F}_T^B\big]\Big\}\\
&= \mathbb{E}\left(\mathrm{Call}_{\mathrm{BS}}\left(S_0\xi_T, \int_0^T \sigma_t(1-\rho_t^2)\mathrm{d}t\right)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Call}_{\mathrm{BS}}(x,y) &:= xe^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(d_+) - Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(d_-)\\
d_\pm &:= \frac{\ln(x/K)+\int_0^T(r_t^d-r_t^f)\mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}.
\end{aligned}
$$

The expansion will be the same as the put option case, and it can be seen via the Put-Call parity

$$
\mathrm{Call}_{\mathrm{BS}}(x,y) - \mathrm{Put}_{\mathrm{BS}}(x,y) = xe^{-\int_0^T r_t^f \mathrm{d}t} - Ke^{-\int_0^T r_t^d \mathrm{d}t},
$$

that the second-order Call Greeks will be the same as the Put Greeks. Explicitly

$$\begin{aligned}
\partial_{xx}\text{Call}_{\text{BS}} &= \frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{x\sqrt{y}}, \\
\partial_{yy}\text{Call}_{\text{BS}} &= \frac{xe^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{4y^{3/2}}(d_+d_- - 1), \\
\partial_{xy}\text{Call}_{\text{BS}} &= (-1)\frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)d_-}{2y}.
\end{aligned}$$

Interestingly, the only difference between the Call and Put option price is the zero-th order term.

Appendix C. Calculation of moments

In this appendix we derive expressions for some of the moments, mixed moments and covariances of the CIR and IGa processes used in this paper. Although the results for the CIR process are well known, one could deem the calculation of such terms for the IGa as non-trivial. In fact, according to our knowledge, we have not seen a derivation of IGa moments with time-dependent parameters in the literature.

C.1. **Deriving the CIR moments.** Let V be a $\text{CIR}(v_0; \kappa_t, \theta_t, \lambda_t)$. It satisfies the SDE

$$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t\sqrt{V_t}\mathrm{d}B_t, \quad V_0 = v_0,$$

where we assume κ, θ and λ are time-dependent and deterministic and satisfy some regularity conditions. For $s < t$, it can be integrated to obtain

$$V_t = V_s e^{-\int_s^t \kappa_z \mathrm{d}z} + \int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z}\kappa_u\theta_u \mathrm{d}u + \int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z}\lambda_u\sqrt{V_u}\mathrm{d}B_u. \tag{16}$$

In particular, for $s = 0$

$$V_t = v_0 e^{-\int_0^t \kappa_z \mathrm{d}z} + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z}\kappa_u\theta_u \mathrm{d}u + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z}\lambda_u\sqrt{V_u}\mathrm{d}B_u. \tag{17}$$

It has the moments

$$\begin{aligned}
\mathbb{E}(V_t^n) &= e^{-\int_0^t n\kappa_z \mathrm{d}z}\left(v_0^n + \int_0^t e^{\int_0^u n\kappa_z \mathrm{d}z}\left(n\kappa_u\theta_u + \frac{1}{2}n(n-1)\lambda_u^2\right)\mathbb{E}(V_u^{n-1})\mathrm{d}u\right) \\
\text{Var}(V_t) &= \int_0^t \lambda_u^2 e^{-2\int_u^t \kappa_z \mathrm{d}z}\left\{v_0 e^{-\int_0^u \kappa_z \mathrm{d}z} + \int_0^u e^{-\int_p^u \kappa_z \mathrm{d}z}\kappa_p\theta_p \mathrm{d}p\right\}\mathrm{d}u. \\
\text{Cov}(V_s, V_t) &= e^{-\int_s^t \kappa_z \mathrm{d}z}\int_0^s \lambda_u^2 e^{-2\int_u^s \kappa_z \mathrm{d}z}\left\{v_0 e^{-\int_0^u \kappa_z \mathrm{d}z} + \int_0^u e^{-\int_p^u \kappa_z \mathrm{d}z}\kappa_p\theta_p \mathrm{d}p\right\}\mathrm{d}u \\
\mathbb{E}(V_s^m V_t^n) &= e^{-\int_0^t n\kappa_z \mathrm{d}z}\left(\mathbb{E}(V_s^{m+n}) + \int_s^t e^{\int_0^u n\kappa_z \mathrm{d}z}\left(n\kappa_u\theta_u + \frac{1}{2}n(n-1)\lambda_u^2\right)\mathbb{E}(V_s^m V_u^{n-1})\mathrm{d}u\right)
\end{aligned}$$

$$\text{Cov}(V_s^m, V_t^n) = \mathbb{E}(V_s^m V_t^n) - \mathbb{E}(V_s^m)\mathbb{E}(V_t^n),$$

all for $m, n \geq 1$ and $s < t$.

We give an outline for obtaining $\mathrm{Var}(V_t)$ and $\mathrm{Cov}(V_s, V_t)$. The other terms follow a similar methodology.

Proof. Notice that $\mathrm{Var}(V_t) = \mathbb{E}(V_t - \mathbb{E}(V_t))^2$, thus using eq. (17) and $\mathbb{E}(V_t)$,

$$\mathrm{Var}(V_t) = \mathbb{E}\left(\int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z}\lambda_u\sqrt{V_u}\mathrm{d}B_u\right)^2 = \int_0^t e^{-2\int_u^t \kappa_z \mathrm{d}z}\lambda_u^2\mathbb{E}(V_u)\mathrm{d}u.$$

Assume $s < t$. Using the representation of V_t in terms of V_s eq. (16), we have

$$\begin{aligned}\mathrm{Cov}(V_s, V_t) &= \mathrm{Cov}\left(V_s, V_s e^{-\int_s^t \kappa_z \mathrm{d}z} + \int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z}\kappa_u\theta_u \mathrm{d}u + \int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z}\lambda_u\sqrt{V_u}\mathrm{d}B_u\right)\\ &= e^{-\int_s^t \kappa_u \mathrm{d}u}\mathrm{Var}(V_s),\end{aligned}$$

where we have used that V_s is independent of the Itô integral $\int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z}\lambda_u\sqrt{V_u}\mathrm{d}B_u$. □

C.2. **Deriving the IGa moments.** Let V be an $\mathrm{IGa}(v_0; \kappa_t, \theta_t, \lambda_t)$. It satisfies the SDE

$$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t, \quad V_0 = v_0,$$

where we assume κ, θ and λ are time-dependent and deterministic and satisfy some regularity conditions. Denoting Y to be a $\mathrm{GBM}(1; -\kappa_t, \lambda_t)$, then for $s < t$, V has the explicit strong solution

$$V_t = V_s\frac{Y_t}{Y_s}\left(\frac{v_0 + \int_0^t \kappa_u\theta_u/Y_u \mathrm{d}u}{v_0 + \int_0^s \kappa_u\theta_u/Y_u \mathrm{d}u}\right).$$

In particular, for $s = 0$,

$$V_t = Y_t\left(v_0 + \int_0^t \frac{\kappa_u\theta_u}{Y_u}\mathrm{d}u\right).$$

It has the following moments

$$\begin{aligned}\mathbb{E}(V_t^n) &= e^{\int_0^t \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\left(v_0^n + n\int_0^t \kappa_u\theta_u e^{-\int_0^u \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\mathbb{E}(V_u^{n-1})\mathrm{d}u\right)\\ \mathrm{Var}(V_t) &= e^{-2\int_0^t \kappa_z \mathrm{d}z}\int_0^t \lambda_u^2\mathbb{E}(V_u^2)e^{2\int_0^u \kappa_z \mathrm{d}z}\mathrm{d}u\\ \mathrm{Cov}(V_s, V_t) &= \mathrm{Var}(V_s)e^{-\int_s^t \kappa_z \mathrm{d}z}\\ \mathbb{E}(V_s^m V_t^n) &= e^{\int_0^t \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\Big(\mathbb{E}(V_s^{m+n})e^{-\int_0^s \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\\ &\quad + n\int_s^t \kappa_u\theta_u e^{-\int_0^u \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\mathbb{E}(V_s^m V_u^{n-1})\mathrm{d}u\Big)\\ \mathrm{Cov}(V_s^m, V_t^n) &= \mathbb{E}(V_s^m V_t^n) - \mathbb{E}(V_s^m)\mathbb{E}(V_t^n),\end{aligned}$$

all for $m, n \geq 1$ and $s < t$. We show how to obtain $\mathbb{E}(V_s^n V_t^m)$. The other terms follow a similar methodology.

Proof. We consider the differential of V^n.

$$\mathrm{d}(V_t^n) = \left(n\kappa_t\theta_t V_t^{n-1} + \left(\frac{1}{2}n(n-1)\lambda_t^2 - n\kappa_t\right)V_t^n\right)\mathrm{d}t + n\lambda_t V_t^n \mathrm{d}B_t$$
$$\Rightarrow V_t^n = V_s^n + \int_s^t n\kappa_u\theta_u V_u^{n-1} + \left(\frac{1}{2}n(n-1)\lambda_u^2 - n\kappa_u\right)V_u^n \mathrm{d}u + \int_s^t n\lambda_u V_u^n \mathrm{d}B_u.$$

Multiplying both sides by V_s^m and taking expectation

$$\mathbb{E}(V_s^m V_t^n) = \mathbb{E}(V_s^{n+m}) + \int_s^t n\kappa_u\theta_u \mathbb{E}(V_s^m V_u^{n-1}) + \left(\frac{1}{2}n(n-1)\lambda_u^2 - n\kappa_u\right)\mathbb{E}(V_s^m V_u^n)\mathrm{d}u.$$

Differentiating both sides in t and denoting $M_s^{m,n}(t) := \mathbb{E}(V_s^m V_t^n)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}M_s^{m,n}(t) = n\kappa_t\theta_t M_s^{m,n-1}(t) + \left(\frac{1}{2}n(n-1)\lambda_t^2 - n\kappa_t\right)M_s^{m,n}(t).$$

This is a first order ODE, which can be solved with the integrating factor method by integrating from s to t. □

Appendix D. Solutions to SDEs with linear diffusion

Suppose the diffusion X solves the SDE

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t X_t \mathrm{d}B_t, \quad X_0 = x_0 \tag{18}$$

where $(\sigma_t)_{0\le t\le T}$ is adapted to the Brownian filtration and $f(\cdot, X_\cdot)$ and σ satisfy some regularity conditions so that a strong solution for X exists. Then if an explicit solution exists, it is given by

$$X_t = Y_t/F_t,$$

where F is a GBM$(1; \sigma_t^2, -\sigma_t)$, i.e.,

$$\mathrm{d}F_t = \sigma_t^2 F_t \mathrm{d}t - \sigma_t F_t \mathrm{d}B_t, \quad F_0 = 1$$
$$\Rightarrow F_t = \exp\left\{\int_0^t \frac{1}{2}\sigma_u^2 \mathrm{d}u - \int_0^t \sigma_u \mathrm{d}B_u\right\},$$

and Y solves the integral equation (written in differential form)

$$\mathrm{d}Y_t = F_t f\left(t, \frac{Y_t}{F_t}\right)\mathrm{d}t, \quad Y_0 = x_0. \tag{19}$$

Proof. We essentially verify that this form of X satisfies the SDE eq. (18).

$$\begin{aligned}
\mathrm{d}\left(\frac{Y_t}{F_t}\right) &= \mathrm{d}\left(1/F_t\right)Y_t + \frac{1}{F_t}\mathrm{d}Y_t + \mathrm{d}\left(1/F_t\right)\mathrm{d}Y_t \\
&= \left(\left\{\frac{\sigma_t}{F_t}\mathrm{d}B_t - \frac{\sigma_t^2}{F_t}\mathrm{d}t\right\} + \frac{\sigma_t^2}{F_t}\mathrm{d}t\right)Y_t + f\left(t, \frac{Y_t}{F_t}\right)\mathrm{d}t + 0 \\
&= \frac{Y_t}{F_t}\sigma_t \mathrm{d}B_t + f\left(t, \frac{Y_t}{F_t}\right)\mathrm{d}t \\
&= \sigma_t X_t \mathrm{d}B_t + f(t, X_t)\mathrm{d}t.
\end{aligned}$$

□

References

[1] Aurélien Alfonsi et al. *Affine diffusions and related processes: simulation, theory and applications*, volume 6. Springer, 2015.

[2] Elisa Alòs. A generalization of the Hull and White formula with applications to option pricing approximation. *Finance and Stochastics*, 10(3):353–365, 2006.

[3] Leif BG Andersen. Efficient simulation of the Heston stochastic volatility model. 2007.

[4] F. Antonelli and S. Scarlatti. Pricing options under stochastic volatility: a power series approach. *Finance and Stochastics*, 13(2):269–303, 2009.

[5] F. Antonelli, A. Ramponi, and S. Scarlatti. Exchange option pricing under stochastic volatility: a correlation expansion. *Review of Derivatives Research*, 13(1):45–73, 2010.

[6] Eric Benhamou, Emmanuel Gobet, and Mohammed Miri. Time-dependent Heston model. *SIAM Journal on Financial Mathematics*, 1(1):289–325, 2010.

[7] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, pages 637–654, 1973.

[8] Peter Carr and Dilip Madan. Option valuation using the fast Fourier transform. *Journal of computational finance*, 2(4):61–73, 1999.

[9] Gabriel G Drimus. Closed-form convexity and cross-convexity adjustments for Heston prices. *Quantitative Finance*, 11(8):1137–1149, 2011.

[10] Darrell Duffie, Damir Filipović, Walter Schachermayer, et al. Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3):984–1053, 2003.

[11] Matthew PS Gander and David A Stephens. Stochastic volatility modelling in continuous time with general marginal distributions: Inference, prediction and model selection. *Journal of Statistical Planning and Inference*, 137(10):3068–3081, 2007.

[12] Jim Gatheral. *The volatility surface: a practitioner's guide*, volume 357. John Wiley & Sons, 2011.

[13] Patrick S Hagan, Deep Kumar, Andrew S Lesniewski, and Diana E Woodward. Managing smile risk. *The Best of Wilmott*, page 249, 2002.

[14] Espen Gaarder Haug. *The complete guide to option pricing formulas*, volume 2. McGraw-Hill New York, 2007.

[15] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[16] John Hull and Alan White. The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, 42(2):281–300, 1987.

[17] Tom R Hurd and Alexey Kuznetsov. Explicit formulas for Laplace transforms of stochastic integrals. *Markov Processes and Related Fields*, 14(2):277–290, 2008.

[18] Norihiko Kazamaki. *Continuous Exponential Martingales and BMO*. Springer, 2006.

[19] Fima Klebaner and Robert Liptser. When a stochastic exponential is a true martingale. Extension of the Benes method. *Theory of Probability & Its Applications*, 58(1): 38–62, 2014.

[20] Nicolas Langrené, Geoffrey Lee, and Zili Zhu. Switching to non-affine stochastic volatility: A closed-form expansion for the Inverse-Gamma model. *International Journal of Theoretical and Applied Finance*, 19(05):1650031, 2016.

[21] Alan Lewis. Option valuation under stochastic volatility with mathematica code. *International Review of Economics & Finance*, 11(3):331–333, 2002.

[22] Alan L Lewis. Exact solutions for a GBM-type stochastic volatility model having a stationary distribution. *arXiv preprint arXiv:1809.08635*, 2018.

[23] Matthew Lorig, Stefano Pagliarani, and Andrea Pascucci. Explicit implied volatilities for multi-factor local-stochastic volatility models. *Mathematical Finance*, 2015.

[24] Mikhailov, Sergei and Nögel, Ulrich. *Heston's stochastic volatility model: Implementation, calibration and some extensions*. John Wiley and Sons, 2004.

[25] Daniel B Nelson. ARCH models as diffusion approximations. *Journal of Econometrics*, 45(1-2):7–38, 1990.

[26] Rainer Schöbel and Jianwei Zhu. Stochastic volatility with an Ornstein–Uhlenbeck process: an extension. *Review of Finance*, 3(1):23–46, 1999.

[27] Anthonie W Van der Stoep, Lech A Grzelak, and Cornelis W Oosterlee. The Heston stochastic-local volatility model: Efficient Monte Carlo simulation. *International Journal of Theoretical and Applied Finance*, 17(07):1450045, 2014.

[28] James B Wiggins. Option values under stochastic volatility: Theory and empirical estimates. *Journal of financial economics*, 19(2):351–372, 1987.

[29] Gregory Willard. Calculating prices and sensitivities for path-independent derivative securities in multifactor models. *Journal of Derivatives*, 5(1):45–61, 1997.