

# Geometric Scattering on Manifolds

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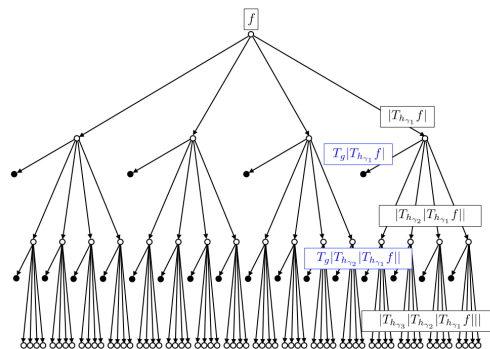
## Abstract

We present a mathematical model for geometric deep learning based upon a scattering transform defined over manifolds, which generalizes the wavelet scattering transform of Mallat. This geometric scattering transform is (locally) invariant to isometry group actions, and we conjecture that it is stable to actions of the diffeomorphism group.

## 1 Introduction

Convolutional neural networks (CNNs) are revolutionizing imaging science for two and three dimensional images over Euclidean domains. However, many images, and more generally data sets, are intrinsically non-Euclidean and are better modeled through other mathematical structures, such as graphs or manifolds. This has led to the development of *geometric deep learning* [1, 2] (and references therein), which refers to a body of research that aims to translate the principles of CNNs to these non-Euclidean structures. In the process, various challenges have arisen, including how to define such networks, how to train them efficiently, and how to analyze their mathematical properties. In this letter we focus primarily on the last question in a relatively general context, akin to several existing methods, which illustrates the fundamental properties of such networks.

We present a geometric version of the scattering transform (Figure 1), which is similar to the one introduced in [3], but here is defined over manifolds instead of Euclidean space. The Euclidean scattering transform can be, on the one hand, thought of as a mathematical model for standard CNNs, but has, on the other hand, obtained state of the art or near state of the art empirical results in computer vision [4, 5, 6], audio signal processing [7, 8, 9], and even quantum chemistry [10, 11]. In Section 3 we define the geometric scattering transform, and provide results showing it encodes localized isometry invariant descriptions of



**Figure 1:** The geometric scattering transform. Black: Equivariant intermediate layers. Blue: Invariant output coefficients at each layer.

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signal data defined on a manifold. These results generalize local translation and rotation invariance on Euclidean domains. The underlying spectral integral operators that are the foundation of the geometric scattering transform are presented in Section 2. In Section 4 we discuss the stability of the geometric scattering transform to the action of diffeomorphisms. We provide a framework in which to quantify how much a diffeomorphism differs from being an isometry, provide results proving that individual spectral integral operators are Lipschitz stable to diffeomorphism actions within this framework, and conjecture that these results can be extended to the geometric scattering transform. Finally, we provide a short conclusion in Section 5.

### 1.1 Notation

Let  $\mathcal{M}$  be a smooth, compact, and connected,  $d$ -dimensional Riemannian manifold without boundary contained in  $\mathbb{R}^n$ . Let  $r : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$  denote the geodesic distance between two points, and let  $\Delta$  be the Laplace-Beltrami operator on  $\mathcal{M}$ . The eigenfunctions and non-unique eigenvalues of  $-\Delta$  are denoted by  $\varphi_k$  and  $\lambda_k$ , respectively. Since  $\mathcal{M}$  is compact, the spectrum of  $-\Delta$  is countable and we may assume that  $\{\varphi_k\}_{k \in \mathbb{N}}$  forms an orthonormal basis for  $L^2(\mathcal{M})$ . The set of unique eigenvalues of  $-\Delta$  is denoted by  $\Lambda$ , and for  $\lambda \in \Lambda$  we let  $m(\lambda)$  and  $E_\lambda$  denote the corresponding multiplicities and eigenspaces. For a diffeomorphism  $\zeta : \mathcal{M} \rightarrow \mathcal{M}$ , we let  $V_\zeta$  be the operator  $V_\zeta f(x) = f(\zeta^{-1}(x))$ , and let  $\|\zeta\|_\infty = \sup_{x \in \mathcal{M}} r(x, \zeta(x))$ .

## 2 Spectral Integral Operators

For a smooth function  $\eta$ , we define a *spectral kernel*  $K_\eta$  by

$$K_\eta(x, y) = \sum_{k \in \mathbb{N}} \eta(\lambda_k) \varphi_k(x) \overline{\varphi_k}(y) \quad (1)$$

and refer to the integral operator  $T_\eta$ , with kernel  $K_\eta$ , as a *spectral integral operator*. It can be verified that

$$T_\eta f(x) = \int_{\mathcal{M}} K_\eta(x, y) f(y) dV(y) = \sum_{k \in \mathbb{N}} \eta(\lambda_k) \langle f, \varphi_k \rangle \varphi_k(x).$$

Since  $\{\varphi_k\}_{k \in \mathbb{N}}$  is an orthonormal basis for  $L^2(\mathcal{M})$ , it follows that

$$\|T_\eta f\|_2^2 = \sum_{k \in \mathbb{N}} |\eta(\lambda_k)|^2 |\langle f, \varphi_k \rangle|^2.$$

Therefore, if  $\|\eta\|_\infty \leq 1$ , then  $T$  is a nonexpansive operator on  $L^2(\mathcal{M})$ . Operators of this form are analogous to convolution operators defined on  $\mathbb{R}^d$  since like the latter they are diagonalized in the Fourier basis. To further emphasize this connection, we note the following theorem which shows that spectral integral operators are equivariant with respect to isometries.

**Theorem 2.1.** *Let  $T_\eta$  be a spectral integral operator. Then, if  $\zeta$  is an isometry,*

$$T_\eta V_\zeta f = V_\zeta T_\eta f$$

*for all  $f \in L^2(\mathcal{M})$ .*

The proof of Theorem 2.1 can be found in Appendix B.1. We will consider frame operators that are constructed using a countable family of spectral integral operators. In particular, we assume that we have a low-pass filter  $g$ , satisfying  $|g(0)| \geq |g(\lambda)|$ , and a family of high-pass filters  $\{h_\gamma\}_{\gamma \in \Gamma}$  with  $h_\gamma(0) = 0$ , which satisfy a Littlewood-Paley type condition

$$A \leq m(\lambda) \left[ |g(\lambda)|^2 + \sum_{\gamma \in \Gamma} |h_\gamma(\lambda)|^2 \right] \leq B, \quad \forall \lambda \in \Lambda \quad (2)$$

for some  $0 < A \leq B$ . A frame analysis operator is then defined by

$$\Phi f = \{T_g f, T_{h_\gamma} f : \gamma \in \Gamma\},$$

where  $T_g$  and  $T_{h_\gamma}$  are the spectral integral operators corresponding to  $\eta = g$  and  $\eta = h_\gamma$ , respectively. Figure 2 illustrates the low pass operator  $T_g$  applied to the Stanford bunny manifold.

**Proposition 2.2.** *Under the Littlewood-Paley condition (2),  $\Phi$  is a bounded operator from  $\mathbf{L}^2(\mathcal{M})$  to  $\ell^2(\mathbf{L}^2(\mathcal{M}))$  and  $A\|f\|_2^2 \leq \|\Phi f\|_{2,2}^2 := \|T_g f\|_2^2 + \sum_{\gamma \in \Gamma} \|T_{h_\gamma} f\|_2^2 \leq B\|f\|_2^2$  for all  $f \in \mathbf{L}^2(\mathcal{M})$ . In particular, if  $A = B = 1$ , then  $\Phi$  is an isometry.*

**Remark 2.3.** In [13], R. Coifman and M. Maggioni used the heat semigroup to construct a class filters such that the high-pass filters  $\{h_j\}_{j \in \mathbb{Z}}$  form a wavelet frame and a low-pass filter  $g_J$  is chosen so that  $\{g_J, h_j : j \geq J\}$  satisfies (2). This construction can be generalized to the manifold setting after making suitable adjustments to account for the multiplicities of the eigenvalues.

### 3 The Geometric Scattering Transform

The *geometric scattering transform* is a non-linear operator constructed through an alternating cascade of spectral integral operators and nonlinearities. Let  $M : \mathbf{L}^2(\mathcal{M}) \rightarrow \mathbf{L}^2(\mathcal{M})$  be the modulus operator,  $Mf(x) = |f(x)|$ , and for each  $\gamma \in \Gamma$ , we let  $U_\gamma f(x) = MT_{h_\gamma} f(x) = |T_{h_\gamma} f(x)|$ . We define an operator  $U : \mathbf{L}^2(\mathcal{M}) \rightarrow \ell^2(\mathbf{L}^2(\mathcal{M}))$ , called the one-step scattering propagator, by

$$Uf = \{T_g f, U_\gamma f : \gamma \in \Gamma\}.$$

The  $m$ -step scattering propagator is constructed by iteratively applying the one-step scattering propagator. For  $m \geq 1$ , let  $\Gamma_m$  be the set of all paths of the form  $\vec{\gamma} = (\gamma_1, \dots, \gamma_m)$ . Let  $\Gamma_0$  denote the empty set, and let  $\Gamma_\infty = \bigcup_{m=1}^\infty \Gamma_m$  denote the set of all finite paths. For  $\vec{\gamma} \in \Gamma_m$ , let

$$U_{\vec{\gamma}} f(x) = U_{\gamma_m} \cdots U_{\gamma_1} f(x), \quad \vec{\gamma} = (\gamma_1, \dots, \gamma_m),$$

and for  $\mathcal{P} \subset \Gamma_\infty$ , we define  $U[\mathcal{P}]f$  as the collection of all path propagators with paths in  $\mathcal{P}$ ,

$$U[\mathcal{P}]f = \{U_{\vec{\gamma}} f : \vec{\gamma} \in \mathcal{P}\}.$$

The scattering transform  $S_{\vec{\gamma}}$  over a path  $\vec{\gamma} \in \Gamma_\infty$  is defined as the integration of  $U_{\vec{\gamma}}$  against the low-pass integral operator  $T_g$ , i.e.

$$S_{\vec{\gamma}} f(x) = T_g U_{\vec{\gamma}} f(x).$$

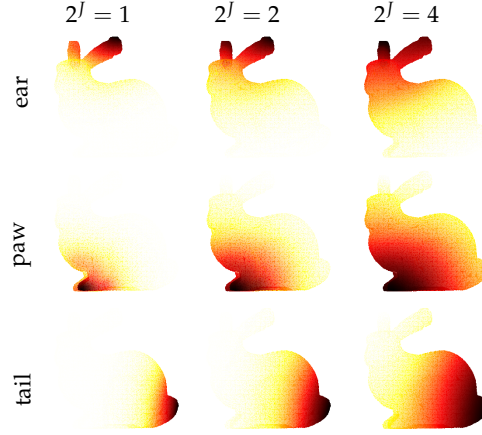
Analogously to  $U[\mathcal{P}]$ , we define

$$S[\mathcal{P}]f = \{S_{\vec{\gamma}} f : \vec{\gamma} \in \mathcal{P}\}.$$

The operator  $S[\Gamma_\infty] : \mathbf{L}^2(\mathcal{M}) \rightarrow \ell^2(\mathbf{L}^2(\mathcal{M}))$  is referred to as the scattering transform. The following proposition shows that  $S[\Gamma_\infty]$  is nonexpansive. The proof is nearly identical to [3, Proposition 2.5], and is thus omitted.

**Proposition 3.1.** *If the Littlewood-Paley condition (2) holds, then*

$$\|S[\Gamma_\infty]f_1 - S[\Gamma_\infty]f_2\|_{2,2} \leq \|f_1 - f_2\|, \quad \forall f_1, f_2 \in \mathbf{L}^2(\mathcal{M}).$$



**Figure 2:** Illustration of  $T_{g_J} \delta_x$  over the Stanford bunny manifold [12] with Gaussian  $g_J$ , three  $J$  scales, and three Diracs  $\delta_x$  centered at different regions of the manifold.

The scattering transform is invariant to the action of the isometry group on the inputted signal  $f$  up to a factor that depends upon the decay of the low-pass spectral function  $g$ . If the low-pass spectral function  $g$  is rapidly decaying and satisfies  $|g(\lambda)| \leq Ce^{-t\lambda}$  for some constant  $C$  and  $t > 0$  (e.g., the heat kernel), then the following theorem establishes isometric invariance up to the scale  $t^d$ .

**Theorem 3.2.** *Let  $\zeta$  be an isometry. If the Littlewood-Paley condition (2) holds and  $|g(\lambda)| \leq Ce^{-t\lambda}$  for some constant  $C$  and  $t > 0$ , then there exists a constant  $C(\mathcal{M}) < \infty$ , such that*

$$\|S[\Gamma_\infty]f - S[\Gamma_\infty]V_\zeta f\|_{2,2} \leq C(\mathcal{M})t^{-d}\|\zeta\|_\infty\|U[\Gamma_\infty]f\|_{2,2} \quad \forall f \in L^2(\mathcal{M}).$$

See Appendix B.2 for the proof.

## 4 Stability to Diffeomorphisms

As stated in Theorem 2.1, spectral integral operators are equivariant to the action of isometries. This fact is crucial to proving Theorem 3.2 because it allows us to estimate

$$\|S[\Gamma_\infty]f - V_\zeta S[\Gamma_\infty]f\|_{2,2} \quad (3)$$

instead of

$$\|S[\Gamma_\infty]f - S[\Gamma_\infty]V_\zeta f\|_{2,2}. \quad (4)$$

In [3], it is shown that the Euclidean scattering transform  $S_{Euc}$  is stable to the action of certain diffeomorphisms which are close to being translations. A key step in the proof is a bound on the commutator norm  $\|[S_{Euc}[\Gamma_\infty], V_\zeta]\|$ , which then allows the author to bound a quantity analogous to (3) instead of bounding (4) directly. This motivates us to study the commutator of spectral integral operators with  $V_\zeta$  for diffeomorphisms which are close to being isometries.

For technical reasons, we will assume that  $\mathcal{M}$  is two-point homogeneous, that is, for any two pairs of points,  $(x_1, x_2), (y_1, y_2)$  such that  $r(x_1, x_2) = r(y_1, y_2)$ , there exists an isometry  $\zeta : \mathcal{M} \rightarrow \mathcal{M}$  such that  $\zeta(x_1) = y_1$  and  $\zeta(x_2) = y_2$ . In order to quantify how far a diffeomorphism  $\zeta$  differs from being an isometry we will consider two quantities:

$$A_1(\zeta) = \sup_{\substack{x, y \in \mathcal{M} \\ x \neq y}} \left| \frac{r(\zeta(x), \zeta(y)) - r(x, y)}{r(x, y)} \right|, \quad (5)$$

and

$$A_2(\zeta) = \left( \sup_{x \in \mathcal{M}} \left| |\det[D\zeta(x)]| - 1 \right| \right) \left( \sup_{x \in \mathcal{M}} \left| \det[D\zeta^{-1}(y)] \right| \right). \quad (6)$$

We let  $A(\zeta) = \max\{A_1(\zeta), A_2(\zeta)\}$  and note that if  $\zeta$  is an isometry, then  $A(\zeta) = 0$ . The following theorem, which is proved in Appendix B.3, bounds the operator norm of  $[T_\eta, V_\zeta]$  in terms of  $A(\zeta)$  and a quantity depending upon  $\eta$ .

**Theorem 4.1.** *Assume that  $\mathcal{M}$  is two-point homogeneous, and let  $T_\eta$  be a spectral integral operator. Then there exists a constant  $C(\mathcal{M}) > 0$  such that for any diffeomorphism  $\zeta : \mathcal{M} \rightarrow \mathcal{M}$ ,*

$$\|[T_\eta, V_\zeta]\| \leq C(\mathcal{M})A(\zeta)B(\eta) \quad (7)$$

where

$$B(\eta) = \max \left\{ \sum_{k \in \mathbb{N}} \eta(\lambda_k) \lambda_k^{(d+1)/4}, \left( \sum_{k \in \mathbb{N}} \eta(\lambda_k)^2 \right)^{\frac{1}{2}} \right\}.$$

**Remark 4.2.** We conjecture that when  $\Phi$  is constructed to be a wavelet frame as in Remark 2.3, we can use (7) to prove a bound on  $\|[S[\Gamma_\infty], V_\zeta]\|$  in terms of  $A(\zeta)$ . If true, this result would allow us to show that the scattering transform is stable to diffeomorphisms using methods analogous to the ones in [3].

## 5 Conclusion

We have presented a path towards understanding the mathematical properties of geometric deep networks through the notion of the geometric scattering transform. Recently, related analyses for graphs have been presented in [14, 15, 16]. We remark that the analysis proposed here applies to compact Riemannian manifolds of arbitrary dimension, thus providing a road map that goes beyond 2D surface or 3D shape matching. Looking ahead, such an approach naturally lends itself to research avenues that synthesize geometric deep learning and geometric data analysis (e.g., manifold learning [17, 18, 19]), which in turn has the potential to bridge the graph and manifold theories for geometric deep networks.

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# Appendices

## A Auxiliary Results

In the following subsections, we will state and prove some auxiliary results that will be needed to prove our main theorems.

### A.1 Stability of Spectral Integral Operators to Left Composition

**Theorem A.1.** *For every spectral integral operator  $T = T_\eta$ , there exists a constant  $C(\mathcal{M}, \eta) > 0$  such that for any diffeomorphism  $\zeta : \mathcal{M} \rightarrow \mathcal{M}$ ,*

$$\|T - V_\zeta T\| \leq C(\mathcal{M}, \eta) \|\zeta\|_\infty,$$

where

$$C(\mathcal{M}, \eta) = C'(\mathcal{M}) \sum_{k \in \mathbb{N}} \eta(\lambda_k) \lambda_k^{d/2}.$$

*Proof.* Let  $K = K_\eta$  be the kernel of  $T$ . By the definition of  $V_\zeta$  and the Cauchy-Schwartz inequality,

$$\begin{aligned} |Tf(x) - V_\zeta Tf(x)| &= \left| \int_{\mathcal{M}} [K(x, y) - K(\zeta^{-1}(x), y)] f(y) dV(y) \right| \\ &\leq \|f\|_2 \left( \int_{\mathcal{M}} |K(x, y) - K(\zeta^{-1}(x), y)|^2 dV(y) \right)^{1/2} \\ &\leq \|f\|_2 \|\nabla K\|_\infty \left( \int_{\mathcal{M}} |r(x, \zeta^{-1}(x))|^2 dV(y) \right)^{1/2} \\ &\leq \|f\|_2 \sqrt{\text{vol}(\mathcal{M})} \|\nabla K\|_\infty \|\zeta\|_\infty, \end{aligned} \tag{8}$$

where line (8) follows from the fact that for all  $F \in \mathbf{C}^1(\mathcal{M})$ ,

$$|F(x) - F(x')| \leq \|\nabla F\|_\infty r(x, x').$$

It follows that  $\|Tf - V_\zeta Tf\|_2 \leq \|f\|_2 \text{vol}(\mathcal{M}) \|\nabla K\|_\infty \|\zeta\|_\infty$ , and so

$$\|T - V_\zeta T\| \leq \text{vol}(\mathcal{M}) \|\nabla K\|_\infty \|\zeta\|_\infty.$$

Lemma A.3 shows

$$\|\nabla K\|_\infty \leq C(\mathcal{M}) \sum_{k \in \mathbb{N}} \eta(\lambda_k) \lambda_k^{d/2},$$

and therefore

$$\|T - V_\zeta T\| \leq C(\mathcal{M}) \left( \sum_{k \in \mathbb{N}} \eta(\lambda_k) \lambda_k^{d/2} \right) \|\zeta\|_\infty.$$

□

**Corollary A.2.** *Let  $T = T_\eta$  be a spectral integral operator. If  $|\eta(\lambda)| \leq Ce^{-t\lambda}$  for some constant  $C$  and  $t > 0$ , then there exists a constant  $C(\mathcal{M}) > 0$  such that for any diffeomorphism  $\zeta : \mathcal{M} \rightarrow \mathcal{M}$ ,*

$$\|T - V_\zeta T\| \leq C(\mathcal{M}) t^{-d} \|\zeta\|_\infty.$$

*Proof.* [20, Theorem 2.4] proves that for any  $x \in \mathcal{M}$ ,  $\alpha \geq 0$ , and  $t > 0$ ,

$$\sum_{k \geq 1} \lambda_k^\alpha e^{-t\lambda_k} |\varphi_k(x)|^2 \leq C(\mathcal{M}) (\alpha + 1) t^{-(d+2\alpha)/2}.$$

Integrating both sides over  $\mathcal{M}$  yields:

$$\sum_{k \geq 1} \lambda_k^\alpha e^{-t\lambda_k} \leq C(\mathcal{M}) (\alpha + 1) t^{-(d+2\alpha)/2}. \quad (9)$$

Using Theorem A.1 and (9) with  $\alpha = d/2$  gives

$$\begin{aligned} \|T - V_\zeta T\| &\leq C(\mathcal{M}) \left( \sum_{k \geq 1} \lambda_k^{d/2} e^{-t\lambda_k} \right) \|\zeta\|_\infty \\ &\leq C(\mathcal{M}) t^{-d} \|\zeta\|_\infty. \end{aligned}$$

□

**Lemma A.3.** *Let*

$$K_\lambda = \sum_{k: \lambda_k = \lambda} \varphi_k(x) \bar{\varphi}_k(y),$$

*so that if  $K_\eta$  is the kernel of a spectral integral operator defined as in (1), we may write*

$$K_\eta(x, y) = \sum_\lambda \eta(\lambda) K_\lambda(x, y).$$

*Then, there exists a constant  $C(\mathcal{M}) > 0$  such that*

$$\|\nabla K_\lambda\|_\infty \leq C(\mathcal{M}) m(\lambda) \min \left\{ \lambda^{d/2}, \lambda^{(d+1)/4} \right\}.$$

*As a consequence,*

$$\|\nabla K\|_\infty \leq C(\mathcal{M}) \sum_{\lambda \in \Lambda} \eta(\lambda) m(\lambda) \min \left\{ \lambda^{d/2}, \lambda^{(d+1)/4} \right\} = C(\mathcal{M}) \sum_{k \in \mathbb{N}} \eta(\lambda_k) \min \left\{ \lambda_k^{d/2}, \lambda_k^{(d+1)/4} \right\}.$$

*Proof.* For any  $k$  such that  $\lambda_k = \lambda$ , it is a consequence of Hörmander's local Weyl law [21] (see also [22]) that

$$\|\varphi_k\|_\infty \leq C(\mathcal{M}) \lambda^{(d-1)/4}. \quad (10)$$

Theorem 1 of [22] shows that

$$\|\nabla \varphi_k\|_\infty \leq C(\mathcal{M}) \sqrt{\lambda} \|\varphi_k\|_\infty. \quad (11)$$

It follows that

$$\begin{aligned}
|\nabla K_\lambda(x, y)|^2 &= \left| \sum_{k:\lambda_k=\lambda} \nabla \varphi_k(x) \bar{\varphi}_k(y) \right|^2 \\
&\leq \left( \sum_{k:\lambda_k=\lambda} |\nabla \varphi_k(x)|^2 \right) \left( \sum_{k:\lambda_k=\lambda} |\varphi_k(y)|^2 \right) \\
&\leq C(\mathcal{M}) m(\lambda) \lambda^{(d-1)/2} \sum_{k:\lambda_k=\lambda} |\nabla \varphi_k(x)|^2 \\
&\leq C(\mathcal{M}) m(\lambda) \lambda^{(d+1)/2} \sum_{k:\lambda_k=\lambda} \|\varphi_k\|_\infty^2 \\
&\leq C(\mathcal{M}) m(\lambda)^2 \lambda^d.
\end{aligned}$$

Alternatively, Theorem 3.2 of [23] shows that

$$\sum_{k:\lambda_k=\lambda} |\varphi_k(y)|^2 = C(\mathcal{M}) m(\lambda).$$

Substituting this into the above string of inequalities yields

$$|\nabla K_\lambda(x, y)|^2 \leq C(\mathcal{M}) m(\lambda)^2 \lambda^{(d+1)/2}.$$

□

## A.2 Commutator Estimate for Radial Kernels

We will say that a kernel operator

$$Tf(x) = \int_{\mathcal{M}} K(x, y) f(y) dV(y)$$

is radial if

$$K(x, y) = \kappa(r(x, y))$$

for some  $\kappa : [0, \infty) \rightarrow \mathbb{C}$ . The following theorem establishes a commutator estimate for operators with radial kernels.

**Theorem A.4.** *Let  $T$  be a kernel integral operator with a radial kernel  $K(x, y) = \kappa(r(x, y))$  for some  $\kappa \in \mathbf{C}^1(\mathbb{R})$ . Then there exists constants  $C_1(\mathcal{M}, K)$  and  $C_2(\mathcal{M}, K)$  such that*

$$\|[T, V_\zeta]f\|_2 \leq \|f\|_2 [C_1(\mathcal{M}, K) A_1(\zeta) + C_2(\mathcal{M}, K) A_2(\zeta)].$$

Here  $A_1(\zeta)$  and  $A_2(\zeta)$  are defined as in (5) and (6) respectively,

$$C_1(\mathcal{M}, K) = \|\nabla K\|_\infty \text{diam}(\mathcal{M}) \text{vol}(\mathcal{M}),$$

and

$$C_2(\mathcal{M}, K) = \|K\|_{\mathbf{L}^2(\mathcal{M} \times \mathcal{M})}.$$

*Proof.* We first compute

$$\begin{aligned}
|[T, V_\zeta]f(x)| &= \left| \int_{\mathcal{M}} K(x, y) f(\zeta^{-1}(y)) dV(y) - \int_{\mathcal{M}} K(\zeta^{-1}(x), y) f(y) dV(y) \right| \\
&= \left| \int_{\mathcal{M}} K(x, \zeta(y)) f(y) |\det[D\zeta(y)]| dV(y) - \int_{\mathcal{M}} K(\zeta^{-1}(x), y) f(y) dV(y) \right| \\
&= \left| \int_{\mathcal{M}} f(y) [K(x, \zeta(y)) |\det[D\zeta(y)]| - K(\zeta^{-1}(x), y)] dV(y) \right| \\
&\leq \left| \int_{\mathcal{M}} f(y) [K(x, \zeta(y)) (|\det[D\zeta(y)]| - 1)] dV(y) \right| + \left| \int_{\mathcal{M}} f(y) [K(x, \zeta(y)) - K(\zeta^{-1}(x), y)] dV(y) \right| \\
&\leq \| |\det[D\zeta(y)]| - 1 \|_\infty \left| \int_{\mathcal{M}} f(y) K(x, \zeta(y)) dV(y) \right| + \left| \int_{\mathcal{M}} f(y) [K(x, \zeta(y)) - K(\zeta^{-1}(x), y)] dV(y) \right|.
\end{aligned}$$



Therefore, by the Cauchy-Schwartz inequality,

$$\begin{aligned} \| [T, V_\zeta] f \|_2 &\leq \| f \|_2 \left[ \| |\det[D\zeta(y)]| - 1 \|_\infty \left( \int_{\mathcal{M}} \int_{\mathcal{M}} |K(x, \zeta(y))|^2 dV(y) dV(x) \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left( \int_{\mathcal{M}} \int_{\mathcal{M}} |K(x, \zeta(y)) - K(\zeta^{-1}(x), y)|^2 dV(y) dV(x) \right)^{\frac{1}{2}} \right]. \end{aligned}$$

We may bound the first integral by observing

$$\int_{\mathcal{M}} \int_{\mathcal{M}} |K(x, \zeta(y))|^2 dV(y) dV(x) \leq \| \det[D\zeta^{-1}(y)] \|_\infty^2 \int_{\mathcal{M}} \int_{\mathcal{M}} |K(x, y)|^2 dV(y) dV(x).$$

To bound the second integral observe, that by the mean value theorem and the assumption that  $K$  is radial, we have

$$\begin{aligned} \int_{\mathcal{M}} \int_{\mathcal{M}} |K(x, \zeta(y)) - K(\zeta^{-1}(x), y)|^2 dV(y) dV(x) &= \\ &= \int_{\mathcal{M}} \int_{\mathcal{M}} \left| \kappa(r(x, \zeta(y))) - \kappa(r(\zeta^{-1}(x), y)) \right|^2 dV(y) dV(x) \\ &\leq \| \kappa' \|_\infty^2 \int_{\mathcal{M}} \int_{\mathcal{M}} \left| r(x, \zeta(y)) - r(\zeta^{-1}(x), y) \right|^2 dV(y) dV(x) \\ &\leq [\| \kappa' \|_\infty A_1(\zeta)]^2 \int_{\mathcal{M}} \int_{\mathcal{M}} |r(x, \zeta(y))|^2 dV(y) dV(x) \\ &\leq [\| \kappa' \|_\infty A_1(\zeta) \text{diam}(\mathcal{M}) \text{vol}(\mathcal{M})]^2. \end{aligned}$$

Lastly, since  $K(x, y) = \kappa(r(x, y))$ , we see that

$$\| \nabla K \|_\infty = \| \kappa' \|_\infty,$$

which completes the proof.  $\square$

## B The Proof of Theorems

In this section, we will give the proofs of Theorems 2.1, 3.2, and 4.1. The proofs of Propositions 2.2 and 3.1 are similar to the proofs of Propositions 2.1 and 2.5 in [3] respectively.

### B.1 The Proof of Theorem 2.1

*Proof.* Write  $T = T_\eta$  and  $K = K_\eta$ , and recall that

$$K(x, y) = \sum_{k \in \mathbb{N}} \eta(\lambda_k) \varphi_k(x) \bar{\varphi}_k(y) = \sum_{\lambda \in \Lambda} \eta(\lambda) K_\lambda(x, y), \quad (12)$$

where

$$K_\lambda(x, y) = \sum_{k: \lambda_k = \lambda} \varphi_k(x) \bar{\varphi}_k(y).$$

Let  $\zeta : \mathcal{M} \rightarrow \mathcal{M}$  be an isometry and set  $\psi_k(x) = \varphi_k(\zeta(x))$ . For  $\lambda \in \Lambda$ , define the vectors  $\vec{\varphi}_\lambda(x) \in \mathbb{C}^{m(\lambda)}$  and  $\vec{\psi}_\lambda(x) \in \mathbb{C}^{m(\lambda)}$  as

$$\vec{\varphi}_\lambda(x) = (\varphi_k(x))_{k: \lambda_k = \lambda} \quad \text{and} \quad \vec{\psi}_\lambda(x) = (\psi_k(x))_{k: \lambda_k = \lambda}.$$

Since  $\zeta$  is an isometry,  $\{\varphi_k\}_{k: \lambda_k = \lambda}$  and  $\{\psi_k\}_{k: \lambda_k = \lambda}$  are both orthonormal bases for  $E_\lambda$ . Therefore, there exists an  $m(\lambda) \times m(\lambda)$  unitary matrix  $A_\lambda$  (that does not depend upon  $x$ ) such that

$$\vec{\psi}_\lambda(x) = A_\lambda \vec{\varphi}_\lambda(x).$$

Using this fact, we see that

$$\begin{aligned}
K_\lambda(\zeta(x), \zeta(y)) &= \sum_{k: \lambda_k = \lambda} \varphi_k(\zeta(x)) \bar{\varphi}_k(\zeta(y)) \\
&= \sum_{k: \lambda_k = \lambda} \psi_k(x) \bar{\psi}_k(y) \\
&= \langle \vec{\psi}_\lambda(x), \vec{\psi}_\lambda(y) \rangle \\
&= \langle A_\lambda \vec{\varphi}_\lambda(x), A_\lambda \vec{\varphi}_\lambda(y) \rangle = \langle \vec{\varphi}_\lambda(x), \vec{\varphi}_\lambda(y) \rangle = K_\lambda(x, y).
\end{aligned}$$

It follows from (12) that  $K(\zeta(x), \zeta(y)) = K(x, y)$  for all  $(x, y) \in \mathcal{M} \times \mathcal{M}$ . Now writing  $x = \zeta(\tilde{x})$ , we have

$$\begin{aligned}
TV_\zeta f(x) &= \int_{\mathcal{M}} K(x, y) f(\zeta^{-1}(y)) dV(y) \\
&= \int_{\mathcal{M}} K(x, \zeta(z)) f(z) dV(z) \\
&= \int_{\mathcal{M}} K(\zeta(\tilde{x}), \zeta(z)) f(z) dV(z) \\
&= \int_{\mathcal{M}} K(\tilde{x}, z) f(z) dV(z) \\
&= Tf(\tilde{x}) = Tf(\zeta^{-1}(x)) = V_\zeta Tf(x).
\end{aligned}$$

□

## B.2 The Proof of Theorem 3.2

*Proof.* We rewrite  $S[\mathcal{P}]f - S[\mathcal{P}]V_\zeta f$  as

$$S[\mathcal{P}]f - S[\mathcal{P}]V_\zeta f = [V_\zeta, S[\mathcal{P}]]f + S[\mathcal{P}]f - V_\zeta S[\mathcal{P}]f.$$

Theorem 2.1 proves that spectral integral operators commute with isometries. Since the modulus operator does as well, it follows that  $[V_\zeta, S[\mathcal{P}]]f = 0$  and

$$\|S[\mathcal{P}]f - S[\mathcal{P}]V_\zeta f\|_{2,2} = \|S[\mathcal{P}]f - V_\zeta S[\mathcal{P}]f\|_{2,2}.$$

Since  $S[\mathcal{P}] = T_g U[\mathcal{P}]$ , we see that

$$\begin{aligned}
\|S[\mathcal{P}]f - V_\zeta S[\mathcal{P}]f\|_{2,2} &= \|T_g U[\mathcal{P}]f - V_\zeta T_g U[\mathcal{P}]f\|_{2,2} \\
&\leq \|T_g - V_\zeta T_g\| \|U[\mathcal{P}]f\|_{2,2},
\end{aligned} \tag{13}$$

and since  $|g(\lambda)| \leq Ce^{-t\lambda}$ , Corollary A.2 shows that

$$\|T_g - V_\zeta T_g\| \leq C(\mathcal{M})t^{-d}\|\zeta\|_\infty,$$

which completes the proof. □

## B.3 The Proof of Theorem 4.1

*Proof.* We write  $T = T_\eta$  and  $K = K_\eta$ . If  $\mathcal{M}$  is two-point homogeneous and  $r(x, y) = r(x', y')$ , then by the definition of two-point homogeneity there exists an isometry  $\tilde{\zeta}$  mapping  $x \mapsto x'$  and  $y \mapsto y'$ . Therefore, we may use the proof of Theorem 2.1 to see that  $K(x', y') = K(x, y)$ . It follows that  $K(x, y)$  is radial and so we may write  $K(x, y) = \kappa(r(x, y))$  for some  $\kappa \in \mathbf{C}^1$ .

Applying Theorem A.4, we see that

$$\|[T, V_\zeta]\| \leq C(\mathcal{M}) \left[ \|\nabla K\|_\infty A_1(\zeta) + \|K\|_{L^2(\mathcal{M} \times \mathcal{M})} A_2(\zeta) \right].$$

Lemma A.3 implies that

$$\|\nabla K_\lambda\|_\infty \leq C(\mathcal{M}) \sum_{k \in \mathbb{N}} \eta(\lambda_k) \lambda_k^{(d+1)/4},$$

and since  $\{\varphi_k\}_{k=0}^{\infty}$  forms an orthonormal basis for  $\mathbf{L}^2(\mathcal{M})$ , it can be checked that

$$\|K\|_{\mathbf{L}^2(\mathcal{M} \times \mathcal{M})} = \left( \sum_{k=0}^{\infty} |\eta(\lambda_k)|^2 \right)^{1/2}.$$

Therefore, the proof is complete since

$$A(\zeta) = \max\{A_1(\zeta), A_2(\zeta)\} \text{ and } B(\eta) = \max \left\{ \sum_{k \in \mathbb{N}} \eta(\lambda_k) \lambda_k^{(d+1)/4}, \left( \sum_{k \in \mathbb{N}} \eta(\lambda_k)^2 \right)^{\frac{1}{2}} \right\}.$$

□