UNITARY INVARIANTS FOR COMMUTING TUPLES OF HYPERCONTRACTIONS

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ABSTRACT. In this paper, we introduce the notion of characteristic functions for commuting tuples of m-hypercontractions on Hilbert spaces and investigate some properties. We prove that the characteristic function is a complete unitary invariant. We also offer some factorization properties of characteristic functions.

1. Introduction

We denote the Euclidean unit ball by $\mathbb{B}^n = \{z \in \mathbb{C}^n : ||z|| < 1\}$. For any positive integer $\ell \geq 0$ and a complex Hilbert space \mathcal{E} , we denote by $\mathbb{H}_{\ell}(\mathbb{B}^n, \mathcal{E})$ the \mathcal{E} -valued weighted Bergman space with domain \mathbb{B}^n , that is

$$\mathbb{H}_{\ell}(\mathbb{B}^n, \mathcal{E}) := \Big\{ f = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} z^{\alpha} \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : \|f\|^2 = \sum_{\alpha \in \mathbb{N}^n} \frac{\|f_{\alpha}\|^2}{\rho_{\ell}(\alpha)} < \infty \Big\},\,$$

where $\rho_{\ell}(\alpha) = \frac{(\ell+|\alpha|-1)!}{\alpha!(\ell-1)!}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. It is also a reproducing kernel Hilbert space with kernel

$$K_{\ell}: \mathbb{B}^n \times \mathbb{B}^n \to \mathcal{B}(\mathcal{E}), \quad K_{\ell}(z, w) = \frac{1_{\mathcal{E}}}{(1 - \langle z, w \rangle)^{\ell}}.$$

In particular, if $\ell = 1$, $\mathbb{H}_1(\mathbb{B}^n, \mathcal{E})$ is known as the Drury-Arveson space and we use $H_n^2(\mathcal{E})$ to denote it. Corresponding to an arbitrary sequence of positive numbers $\beta = \{\beta_n\}_{n=0}^{\infty}$ with $\beta_0 = 1$ and $\liminf \beta_j^{\frac{1}{j}} \geq 1$, one also defines an \mathcal{E} -valued weighted reproducing kernel Hilbert space $H_n^2(\beta, \mathcal{E})$ on \mathbb{B}^n by

$$H_n^2(\beta, \mathcal{E}) := \left\{ f = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} z^{\alpha} : \|f\|_{H_n^2(\beta, \mathcal{E})}^2 = \sum_{k=0}^{\infty} \beta_k \sum_{|\alpha| = k} \frac{\alpha!}{|\alpha|!} \|f_{\alpha}\|_{\mathcal{E}}^2 < \infty \right\}$$

with kernel

$$K_{\beta}(z,w) = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \langle z, w \rangle^n I_{\mathcal{E}}.$$

For the particular choice $\beta_j = 1$ for all j, $H_n^2(\beta, \mathcal{E})$ becomes the Drury-Arveson space $H_n^2(\mathcal{E})$. Also for $\beta_j = \frac{j!(n-1)!}{(n+j-1)!}$ and $\beta_j = \frac{j!n!}{(n+j)!}$, $H_n^2(\beta, \mathcal{E})$ corresponds to the Hardy space and the Bergman space over the open unit Ball, respectively. We refer the article of Shields ([13]) for detailed information about these spaces and weighted shift operators.

For an *n*-tuple of commuting bounded linear operators $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n$, we say T is a row contraction if the operator

$$\mathcal{H}^n \to \mathcal{H}, \ (h_1, \dots, h_n) \mapsto \sum_{i=1}^n T_i h_i$$

is a contraction. That is, the operator viewed as a row operator $T:\mathcal{H}^n\to\mathcal{H}$ is a contraction and therefore its adjoint $T^*:\mathcal{H}\to\mathcal{H}^n$ is also a contraction where $T^*=\begin{bmatrix}T_1^*\\\vdots\\T_n^*\end{bmatrix}:\mathcal{H}\to\mathcal{H}^n$ is viewed as a column operator. Consider the associated completely positive map

$$\sigma_T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), X \mapsto \sum_{i=1}^n T_i X T_i^*.$$

Using the map σ_T , for each $k \in \mathbb{N}$, the defect operators of different orders of T are defined as follows. First consider the operator

$$\Delta_T^{(k)} = (1 - \sigma_T)^k (1_{\mathcal{H}}) = \sum_{j=0}^k (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \sum_{|\alpha|=j} \gamma_\alpha T^\alpha T^{*\alpha} \quad (k \in \mathbb{N}),$$

where $\gamma_{\alpha} = \frac{|\alpha|!}{\alpha!}$ for $\alpha \in \mathbb{N}^n$. Note that T is a row contraction if and only if $\Delta_T^{(1)} \geq 0$. Using the above notation, T is said to be a m-hypercontraction if $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(m)} \geq 0$. For such a m-hypercontraction, the defect operator of order m is $D_{m,T^*} = (\Delta_T^{(m)})^{\frac{1}{2}}$ and the corresponding defect space is $\mathcal{D}_{m,T^*} = \overline{\operatorname{ran}}(\Delta_T^{(m)})^{\frac{1}{2}}$. This is also well-known that for a m-hypercontraction T, $\Delta_T^{(k)} \geq 0$ for all $1 \leq k \leq m$. An m-hypercontraction $T \in \mathcal{B}(\mathcal{H})^n$ is said to be pure if

$$SOT - \lim_{k \to \infty} \sigma_T^k(I_{\mathcal{H}}) = 0.$$

In [8], it has shown that a pure m-hypercontraction can be dilated to weighted-shift operators on $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. More precisely, given a pure m-hypercontraction $T \in \mathcal{B}(\mathcal{H})^n$, there is an isometric map $\pi_m : \mathcal{H} \to \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ defined by

$$\pi_m h(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha) (D_{m,T^*} T^{*\alpha} h) z^{\alpha} = D_{m,T^*} (1 - ZT^*)^{-m} h, \quad (h \in \mathcal{H})$$

such that $\pi_m T_i^* = M_{z_i}^* \pi_m$ for all i = 1, ..., n, where $Z : \mathcal{H}^n \to \mathcal{H}$ defined by $Z(h_1, ..., h_n) = \sum_{i=1}^n z_i h_i$ and M_{z_i} is the shift on $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ for all i = 1, ..., n. The dilation map π_m is called the canonical dilation map of T. Thus $\mathcal{Q} = \operatorname{ran} \pi_m$ is a co-invariant subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ and $(T_1, ..., T_n) \cong (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, ..., P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}})$. In other words, given a m-hypercontraction we can associate a co-invariant subspace \mathcal{Q} of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$, which we call as the model space of T, such that $T \cong (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, ..., P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}})$. On the other hand, a Beurling-Lax-Halmos type theorem (see [12, Theorem 4.4]) for invariant subspaces of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E}_*)$ says that a closed subspace \mathcal{S} of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E}_*)$ is an invariant subspace if and only if

there exists a Hilbert space \mathcal{E} and a partial isometric multiplier $\Phi \in \operatorname{Mul}(H_n^2(\mathcal{E}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{E}_*))$ such that

$$\mathcal{S} = \Phi H_n^2(\mathcal{E}).$$

Combining above two results we have that given a m-hypercontraction T there exists a Hilbert space \mathcal{E} and a partial isometric multiplier $\Phi \in \operatorname{Mul}(H_n^2(\mathcal{E}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{E}_*))$ such that the model space of T is $(\Phi H_n^2(\mathcal{E}))^{\perp}$. But the partial isometric multiplier Φ is not unique and description of these partial isometric multipliers are not known explicitly in general. Recently, in [7], the author find a method to construct such a partial isometric multiplier. In this article, we find a general, explicit and simpler recipe to construct such partial isometric multipliers and study them by considering their factorization and by finding relations with classical characteristic functions.

The plan of the paper is as follows. In Section 2, we find a general recipe to construct characteristic functions for m-hypercontractions. In Section 3, we have considered factorization of multipliers from vector valued Durry-Arveson spaces to weighted reproducing kernel Hilbert spaces on \mathbb{B}^n and find some uniqueness result when the multipliers are K-inner. In last section, we apply results from the previous sections and obtain factorization of characteristic functions of m-hypercontractions. We also find some relation with the classical characteristic function.

2. Characteristic Functions for Hypercontrctions

In this section, we describe an explicit general recipe to construct characteristic functions of commuting m-hypercontractions. To begin with, we define some useful operators and study their properties which will be used to obtain the characteristic functions explicitly. Recall that for a m-hypercontraction T, D_{m,T^*} and \mathcal{D}_{m,T^*} are the defect operator and defect space of T of order m respectively. Consider the operator $C_{m,T}: \mathcal{H} \to l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ defined by $h \mapsto (\rho_{m-1}(\alpha)^{\frac{1}{2}}D_{m,T^*}T^{*\alpha}h)_{\alpha\in\mathbb{N}^n}$ for all $h \in \mathcal{H}$, where $\rho_m(\alpha) = \frac{(m+|\alpha|-1)!}{\alpha!(m-1)!}$. In other words, $C_{m,T}$ has the following column matrix representation

$$C_{m,T} = \begin{bmatrix} \vdots \\ \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{m,T^*} T^{*\alpha} \\ \vdots \end{bmatrix}_{\alpha \in \mathbb{N}^n}.$$

Let $\pi_m: \mathcal{H} \to \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ be the canonical dilation map of T defined by

$$\pi_m h(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha) (D_{m,T^*} T^{*\alpha} h) z^{\alpha} \quad (z \in \mathbb{B}^n).$$

Since the dilation map π_m is an isometry, we have

$$\sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha) T^{\alpha} D_{m,T^*}^2 T^{*\alpha} = I_{\mathcal{H}}.$$

Also, by an elementary calculation, we have the following relation among multi-variable coefficients $\rho_{\ell}(\alpha)$ for $\alpha \in \mathbb{N}^n$ and $\ell > 1$:

$$\rho_{\ell}(\alpha) = \rho_{\ell-1}(\alpha) + \sum_{i=1,\alpha,i\geq 1}^{n} \rho_{\ell}(\alpha - e_i),$$

where for $1 \le i \le n$, $\alpha - e_i = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_n)$. Then note that,

$$TT^* = T\left(\sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha) T^{\alpha} D_{m,T^*}^2 T^{*\alpha}\right) T^*$$

$$= \sum_{i=1}^n \sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha) T^{\alpha + e_i} D_{m,T^*}^2 T^{*(\alpha + e_i)}$$

$$= \sum_{\alpha} \left(\sum_{i=1,\alpha_i \ge 1}^n \rho_m(\alpha - e_i)\right) T^{\alpha} D_{m,T^*}^2 T^{*\alpha}$$

$$= \sum_{\alpha} \left(\rho_m(\alpha) - \rho_{m-1}(\alpha)\right) T^{\alpha} D_{m,T^*}^2 T^{*\alpha}$$

$$= I_{\mathcal{H}} - C_{m,T}^* C_{m,T}.$$

Therefore, $\begin{bmatrix} T^* \\ C_{m,T} \end{bmatrix} : \mathcal{H} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ is an isometry. Then by adding a suitable Hilbert space \mathcal{E} , we can find $B \in \mathcal{B}(\mathcal{E}, \mathcal{H}^n)$, $D \in \mathcal{B}(\mathcal{E}, l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*}))$ such that

$$U = \begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$$

is a co-isometry. Such a co-isometry plays a crucial role in what follows and we make the following definition to refer such a co-isometry.

DEFINITION 2.1. For a m-hypercontraction T on \mathcal{H} , a triple (\mathcal{E}, B, D) consists of a Hilbert space \mathcal{E} and operators $B \in \mathcal{B}(\mathcal{E}, \mathcal{H}^n)$, $D \in \mathcal{B}(\mathcal{E}, l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*}))$ is a characteristic triple if the corresponding block operator matrix

$$\begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$$

is a co-isometry.

For a characteristic triple (\mathcal{E}, B, D) of T, let

$$D = \begin{bmatrix} \vdots \\ D_{\alpha} \\ \vdots \end{bmatrix}_{\alpha \in \mathbb{N}^n},$$

where $D_{\alpha} \in \mathcal{B}(\mathcal{E}, D_{m,T^*})$ for all $\alpha \in \mathbb{N}^n$. Since the corresponding block operator matrix is a co-isometry, we have the following relations:

(a)
$$T^*T + BB^* = I_{\mathcal{H}}$$

(b)
$$T^*C_{m,T}^* + BD^* = 0$$
, that is, $\rho_{m-1}(\alpha)^{\frac{1}{2}}T^*T^{\alpha}D_{T^*} + BD_{\alpha}^* = 0$ for all $\alpha \in \mathbb{N}^n$

(c)
$$C_{m,T}T + DB^* = 0$$
, that is, $\rho_{m-1}(\alpha)^{\frac{1}{2}}D_{T^*}T^{*\alpha}T + D_{\alpha}B^* = 0$ for all $\alpha \in \mathbb{N}^n$

(d) $C_{m,T}C_{m,T}^* + DD^* = I_{l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})}$, that is,

(i)
$$\rho_{m-1}(\alpha)D_{m,T^*}T^{*\alpha}T^{\alpha}D_{m,T^*} + D_{\alpha}D_{\alpha}^* = I_{\mathcal{D}_{m,T^*}}$$
 for all $\alpha \in \mathbb{N}^n$ and

(ii)
$$\rho_{m-1}(\alpha)^{\frac{1}{2}}\rho_{m-1}(\beta)^{\frac{1}{2}}D_{m,T^*}T^{*\alpha}T^{\beta}D_{m,T^*} + D_{\alpha}D_{\beta}^* = 0, \alpha \neq \beta \text{ for all } \alpha, \beta \in \mathbb{N}^n.$$

Now we define an operator-valued analytic function $\Phi: \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$ by

(2.1)
$$\Phi(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + D_{m,T^*} (1 - ZT^*)^{-m} ZB \qquad (z \in \mathbb{B}^n).$$

Therefore,

(2.2)
$$\Phi(w)^* = \sum_{\beta \in \mathbb{N}^n} \rho_{m-1}(\beta)^{\frac{1}{2}} D_{\beta}^* \bar{w}^{\beta} + B^* W^* (1 - TW^*)^{-m} D_{m,T^*} \qquad (w \in \mathbb{B}^n).$$

Using the above relations (a),(b),(c) and (d) we calculate:

$$\Phi(z)\Phi(w)^* = \sum_{\alpha,\beta\in\mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} \rho_{m-1}(\beta)^{\frac{1}{2}} D_{\alpha} D_{\beta}^* z^{\alpha} \bar{w}^{\beta}$$

$$+ \sum_{\alpha\in\mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} B^* W^* (1 - TW^*)^{-m} D_{m,T^*}$$

$$+ D_{m,T^*} (1 - ZT^*)^{-m} ZB \sum_{\beta\in\mathbb{N}^n} \rho_{m-1}(\beta)^{\frac{1}{2}} D_{\beta}^* \bar{w}^{\beta}$$

$$+ D_{m,T^*} (1 - ZT^*)^{-m} ZBB^* W^* (1 - TW^*)^{-m} D_{m,T^*}$$

$$= \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha) \underbrace{D_{\alpha} D_{\alpha}^{*}}_{\alpha} z^{\alpha} \bar{w}^{\alpha} + \sum_{\alpha,\beta \in \mathbb{N}^{n},\alpha \neq \beta} \rho_{m-1}(\alpha)^{\frac{1}{2}} \rho_{m-1}(\beta)^{\frac{1}{2}} \underbrace{D_{\alpha} D_{\beta}^{*}}_{\beta} z^{\alpha} \bar{w}^{\beta}$$

$$+ \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha)^{\frac{1}{2}} z^{\alpha} \underbrace{D_{\alpha} B^{*}}_{\alpha} W^{*} (1 - TW^{*})^{-m} D_{m,T^{*}}$$

$$+ D_{m,T^{*}} (1 - ZT^{*})^{-m} Z \sum_{\beta \in \mathbb{N}^{n}} \rho_{m-1}(\beta)^{\frac{1}{2}} \underbrace{BD_{\beta}^{*}}_{\beta} \bar{w}^{\beta}$$

$$+ D_{m,T^{*}} (1 - ZT^{*})^{-m} Z \underbrace{BB^{*}}_{\beta} W^{*} (1 - TW^{*})^{-m} D_{m,T^{*}}$$

$$= \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha) (1 - \rho_{m-1}(\alpha) D_{m,T^{*}} T^{*\alpha} T^{\alpha} D_{m,T^{*}}) z^{\alpha} \bar{w}^{\alpha}$$

$$- \sum_{\alpha,\beta \in \mathbb{N}^{n}, \alpha \neq \beta} \rho_{m-1}(\alpha)^{\frac{1}{2}} \rho_{m-1}(\beta)^{\frac{1}{2}} \left[\rho_{m-1}(\alpha)^{\frac{1}{2}} \rho_{m-1}(\beta)^{\frac{1}{2}} D_{m,T^{*}} T^{*\alpha} T^{\beta} D_{m,T^{*}} \right] z^{\alpha} \bar{w}^{\beta}$$

$$- \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha)^{\frac{1}{2}} z^{\alpha} \left[\rho_{m-1}(\alpha)^{\frac{1}{2}} D_{m,T^{*}} T^{*\alpha} T \right] W^{*} (1 - TW^{*})^{-m} D_{m,T^{*}}$$

$$- D_{m,T^{*}} (1 - ZT^{*})^{-m} Z \sum_{\beta \in \mathbb{N}^{n}} \rho_{m-1}(\beta)^{\frac{1}{2}} \left[\rho_{m-1}(\beta)^{\frac{1}{2}} T^{*} T^{\beta} D_{m,T^{*}} \right] \bar{w}^{\beta}$$

$$+ D_{m,T^{*}} (1 - ZT^{*})^{-m} Z B B^{*} W^{*} (1 - TW^{*})^{-m} D_{m,T^{*}}$$

$$= \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha) (z\bar{w})^{\alpha} - \sum_{\alpha,\beta \in \mathbb{N}^{n}} \rho_{m-1}(\alpha) \rho_{m-1}(\beta) D_{m,T^{*}} T^{*\alpha} T^{\beta} D_{m,T^{*}} z^{\alpha} \bar{w}^{\beta}$$

$$- D_{m,T^{*}} \Big[\sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha) (zT^{*})^{\alpha} \Big] TW^{*} (1 - TW^{*})^{-m} D_{m,T^{*}}$$

$$- D_{m,T^{*}} (1 - ZT^{*})^{-m} ZT^{*} \Big[\sum_{\beta \in \mathbb{N}^{n}} \rho_{m-1}(\beta) (\bar{w}T)^{\beta} \Big] D_{m,T^{*}}$$

$$+ D_{m,T^{*}} (1 - ZT^{*})^{-m} Z (1 - T^{*}T) W^{*} (1 - TW^{*})^{-m} D_{m,T^{*}}$$

$$= (1 - \langle z, w \rangle)^{-(m-1)} I_{\mathcal{D}_{m,T^*}} - D_{m,T^*} \Big[\sum_{\alpha \in \mathbb{N}^n} \rho_{m-1}(\alpha) T^{*\alpha} z^{\alpha} \Big] \Big[\sum_{\beta \in \mathbb{N}^n} \rho_{m-1}(\beta) T^{\beta} \bar{w}^{\beta} \Big] D_{m,T^*}$$

$$- D_{m,T^*} (1 - ZT^*)^{-(m-1)} TW^* (1 - TW^*)^{-m} D_{m,T^*}$$

$$- D_{m,T^*} (1 - ZT^*)^{-m} ZT^* (1 - TW^*)^{-(m-1)} D_{m,T^*}$$

$$+ D_{m,T^*} (1 - ZT^*)^{-m} Z (1 - T^*T) W^* (1 - TW^*)^{-m} D_{m,T^*}$$

$$= (1 - \langle z, w \rangle)^{-(m-1)} I_{\mathcal{D}_{m,T^*}} - D_{m,T^*} (1 - ZT^*)^{-(m-1)} (1 - TW^*)^{-(m-1)} D_{m,T^*}$$

$$- D_{m,T^*} (1 - ZT^*)^{-(m-1)} TW^* (1 - TW^*)^{-m} D_{m,T^*}$$

$$- D_{m,T^*} (1 - ZT^*)^{-m} ZT^* (1 - TW^*)^{-(m-1)} D_{m,T^*}$$

$$+ D_{m,T^*} (1 - ZT^*)^{-m} Z (1 - T^*T) W^* (1 - TW^*)^{-m} D_{m,T^*}$$

Hence,

$$\frac{1}{(1-\langle z,w\rangle)^{m-1}}I_{\mathcal{D}_{m,T^*}} - \Phi(z)\Phi(w)^*
= D_{m,T^*}(1-ZT^*)^{-(m-1)}(1-TW^*)^{-(m-1)}D_{m,T^*}
+ D_{m,T^*}(1-ZT^*)^{-(m-1)}TW^*(1-TW^*)^{-m}D_{m,T^*}
+ D_{m,T^*}(1-ZT^*)^{-m}ZT^*(1-TW^*)^{-(m-1)}D_{m,T^*}
- D_{m,T^*}(1-ZT^*)^{-m}Z(1-T^*T)W^*(1-TW^*)^{-m}D_{m,T^*}
= (1-\langle z,w\rangle)D_{m,T^*}(1-ZT^*)^{-m}(1-TW^*)^{-m}D_{m,T^*}.$$

And therefore,

(2.3)
$$\frac{1}{(1-\langle z,w\rangle)^m} I_{\mathcal{D}_{m,T^*}} - \frac{\Phi(z)\Phi(w)^*}{1-\langle z,w\rangle} = D_{m,T^*} (1-ZT^*)^{-m} (1-TW^*)^{-m} D_{m,T^*}.$$

Thus we have find a general recipe of constructing characteristic functions of m-hypercontractions as follows.

THEOREM 2.2. Let T be a pure m-hypercontraction on \mathcal{H} , and let $\mathcal{Q} \subseteq \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ be the model space for T. Suppose (\mathcal{E}, B, D) is a characteristic triple of T. Then

$$\Phi(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + D_{m,T^*} (1 - ZT^*)^{-m} ZB \qquad (z \in \mathbb{B}^n),$$

defines a partial isometric multiplier from $H_n^2(\mathcal{E})$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ such that

$$Q^{\perp} = \Phi H_n^2(\mathcal{E}).$$

Proof: Since the kernel of \mathcal{Q}^{\perp} is $D_{m,T^*}(1-ZT^*)^{-m}(1-TW^*)^{-m}D_{m,T^*}$, the proof follows from (2.3).

REMARK 2.3. By [6, Theorem 6.5], if Θ_T is any other characteristic function of T then Θ_T and Φ , as in the above theorem, are related via a partial isometry. More precisely, if Θ_T is a multiplier from $H_n^2(\mathcal{E}')$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ then there exists a partial isometry $V \in \mathcal{B}(\mathcal{E}'; \mathcal{E})$ such that for all $z \in \mathbb{B}^n$

$$\Theta_T(z) = \Phi(z)V.$$

Two row contractions $T = (T_1, \dots, T_n)$ and $R = (R_1, \dots, R_n)$ on \mathcal{H} are said to be unitary equivalent if there exist a unitary U on \mathcal{H} such that $T_i = UR_iU^*$, for all $i = 1, \dots, n$.

DEFINITION 2.4. The characteristic functions Φ_T and Φ_R of two pure m-hypercontractions T and R are said to coincide if there exists two unitary $\Gamma: (Ker M_{\Phi_R})^{\perp} \to (Ker M_{\Phi_T})^{\perp}$ and $\tau: \mathcal{D}_{m,T^*} \to \mathcal{D}_{m,R^*}$ such that

$$M_{\Phi_R}|_{(Ker M_{\Phi_R})^{\perp}} = (I \otimes \tau) M_{\Phi_T} \Gamma.$$

Theorem 2.5. Two pure m-hypercontractions are unitarily equivalent if and only if their characteristic functions coincide.

Proof. Suppose T and R are two pure m-hypercontractions on \mathcal{H} and they are unitary equivalent via a unitary U on \mathcal{H} , that is, $T_i = UR_iU^*$, for all $i = 1, \ldots, n$. Then it is easy to see that

$$\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) = (I \otimes U)\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,R^*}) \text{ and } \mathcal{Q}_T = (I \otimes U)\mathcal{Q}_R,$$

where \mathcal{Q}_T and \mathcal{Q}_R are the model space of T are R respectively. Therefore, we also have $\mathcal{Q}_T^{\perp} = (I \otimes U) \mathcal{Q}_R^{\perp}$. Now if Φ_T and Φ_R are two characteristic function of T and R corresponding to the characteristic triples $(\mathcal{E}_T, B_T, D_T)$ and $(\mathcal{E}_R, B_R, D_R)$ respectively, then

$$\Phi_T H_n^2(\mathcal{E}_T) = (I \otimes U) \Phi_R H_n^2(\mathcal{E}_R).$$

By applying Douglas lemma, there exist a unitary $\Gamma: (\operatorname{Ker} M_{\Phi_R})^{\perp} \to (\operatorname{Ker} M_{\Phi_T})^{\perp}$ such that

$$M_{\Phi_R}|_{(\operatorname{Ker} M_{\Phi_R})^{\perp}} = (I \otimes U^*) M_{\Phi_T} \Gamma.$$

For the only if part by definition we have two unitary, $\tau: \mathcal{D}_{m,T^*} \to \mathcal{D}_{m,R^*}$ and $\Gamma: (\operatorname{Ker} M_{\Phi_R})^{\perp} \to (\operatorname{Ker} M_{\Phi_T})^{\perp}$, such that $M_{\Phi_R}|_{(\operatorname{Ker} M_{\Phi_R})^{\perp}} = (I \otimes \tau) M_{\Phi_T} \Gamma$.

Therefore,

$$\mathcal{Q}_T^{\perp} = M_{\Phi_T} H_n^2(\mathcal{E}_T) = (I \otimes \tau^*) M_{\Phi_R} (\mathrm{Ker} M_{\Phi_R})^{\perp} = (I \otimes \tau^*) \mathcal{Q}_R^{\perp}.$$

So,

$$Q_{T} = \mathbb{H}_{m}(\mathbb{B}^{n}, \mathcal{D}_{m,T^{*}}) \ominus M_{\Phi_{T}}H_{n}^{2}(\mathcal{E}_{T})$$

$$= (I \otimes \tau^{*})\mathbb{H}_{m}(\mathbb{B}^{n}, \mathcal{D}_{m,R^{*}}) \ominus (I \otimes \tau^{*})M_{\Phi_{R}}H_{n}^{2}(\mathcal{E}_{R})$$

$$= (I \otimes \tau^{*})\big(\mathbb{H}_{m}(\mathbb{B}^{n}, \mathcal{D}_{m,R^{*}}) \ominus M_{\Phi_{R}}H_{n}^{2}(\mathcal{E}_{R})\big)$$

$$= (I \otimes \tau^{*})\mathcal{Q}_{R}.$$

Combining these we have

$$P_{\mathcal{Q}_T}(M_z \otimes I_{\mathcal{D}_{m,T^*}})|_{\mathcal{Q}_T} \cong P_{\mathcal{Q}_R}(M_z \otimes I_{\mathcal{D}_{m,R^*}})|_{\mathcal{Q}_R},$$

and therefore $T \cong R$.

Next we find factorization of a characteristic function of a pure *m*-hypercontraction corresponding to an invariant subspace of the *m*-hypercontraction. This is analogous to the Sz.-Nagy-Foias factorization result for a single contraction obtained in [9]. We use the following factorization result to obtain the factorization in our context. The result is true for more general class of reproducing kernel Hilbert space but we sate it only for Drury-Arveson space as the present situation demands. For a proof, we refer the interested reader to [1].

THEOREM 2.6. (Agler, McCarthy) Let K be the kernel of the Drury-Arveson space H_n^2 on \mathbb{B}^n and $\mathcal{L}_1, \mathcal{L}_2$ and \mathcal{L}_3 be Hilbert spaces. Suppose $\Phi : \mathbb{B}^n \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ and $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{L}_3, \mathcal{L}_2)$ are given functions. Then the following are equivalent:

(i) The function

$$[\Phi(z)\Phi(w)^* - \Theta(z)\Theta(w)^*]K(z,w)$$

is a $\mathcal{B}(\mathcal{L}_2)$ -valued kernel on \mathbb{B}^n .

(ii) There exists $\Psi \in S(\mathcal{L}_3, \mathcal{L}_1)$ of norm at most one such that for all $z \in \mathbb{B}^n$,

$$\Phi(z)\Psi(z) = \Theta(z).$$

The factorization result in the present context is as follows.

THEOREM 2.7. Let $T = (T_1, ..., T_n)$ be a pure m-hypercontraction on \mathcal{H} and let (\mathcal{E}_T, B, D) be a characteristic triple of T. If $\mathcal{H}_1 \subseteq \mathcal{H}$ is a joint T-invariant subspace, then there exist a Hilbert space \mathcal{E} , and two multipliers $\Phi_1 \in Mul(H_n^2(\mathcal{E}_T), H_n^2(\mathcal{E}))$ and $\Phi_2 \in Mul(H_n^2(\mathcal{E}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}))$ such that

$$\Phi_T = \Phi_2 \Phi_1$$
.

where Φ_T is the characteristic function of T corresponding to (\mathcal{E}_T, B, D) .

Proof. Let $\pi_m: \mathcal{H} \to \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ be the canonical dilation of the pure m-hyper contraction T, and let $\mathcal{Q} = \pi_m \mathcal{H}$ is the model space of T. Then we know that

$$Q = \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \Phi_T H_n^2(\mathcal{E}_T).$$

Now if $\mathcal{H}_1 \subseteq \mathcal{H}$ is a T-invariant subspace, then $\mathcal{H}_2 := \mathcal{H} \ominus \mathcal{H}_1$ is invariant for T^* and therefore $\pi_m \mathcal{H}_2$ is a co-invariant subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. Since $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \pi_m \mathcal{H}_2$ is a M_z -invariant subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$, by invoking Theorem 4.4 of ([12]), we have a Hilbert space \mathcal{E} and a partial isometric multiplier $\Phi_2 \in \mathrm{Mul}(H_n^2(\mathcal{E}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}))$ such that

$$\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \pi_m \mathcal{H}_2 = \Phi_2 H_n^2(\mathcal{E}).$$

By construction,

$$\mathcal{H}_{1} \cong \pi_{m} \mathcal{H}_{1} = \pi_{m} \mathcal{H} \ominus \pi_{m} \mathcal{H}_{2}$$

$$= \left\{ \mathbb{H}_{m}(\mathbb{B}^{n}, \mathcal{D}_{m,T^{*}}) \ominus \Phi_{T} H_{n}^{2}(\mathcal{E}_{T}) \right\} \ominus \left\{ \mathbb{H}_{m}(\mathbb{B}^{n}, \mathcal{D}_{m,T^{*}}) \ominus \Phi_{2} H_{n}^{2}(\mathcal{E}) \right\}$$

$$= \Phi_{2} H_{n}^{2}(\mathcal{E}) \ominus \Phi_{T} H_{n}^{2}(\mathcal{E}_{T}).$$

Here, $\pi_m \mathcal{H}_1$ is a RKHS with the kernel function

$$K(z, w) = \frac{\Phi_2(z)\Phi_2(w)^* - \Phi_T(z)\Phi_T(w)^*}{1 - \langle z, w \rangle}.$$

Therefore, from the above Theorem 2.6, we have a multiplier $\Phi_1 \in \text{Mul}(H_n^2(\mathcal{E}_T), H_n^2(\mathcal{E}))$ such that

$$\Phi_T(z) = \Phi_2(z)\Phi_1(z)$$
, for all $z \in \mathbb{B}^n$.

Hence, it completes the proof.

3. Factorization of Multipliers in $S_{\beta}(\mathcal{H},\mathcal{K})$

Multipliers from a Drury-Arveson space to a weighted reproducing kernel Hilbert space always factors through contractive multipliers between Drury-Arveson spaces. We establish such factorization in this section and also consider analogous factorization in the case of K-inner functions.

For this section, we fix a weight $\beta = \{\beta_j\}_{j=0}^{\infty}$ such that β is a decreasing sequence of positive numbers and

(3.1)
$$K_{\beta}(z,w) = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \langle z, w \rangle^n I_{\mathcal{E}}$$

defines a kernel on \mathbb{B}^n , for some Hilbert space \mathcal{E} . For a fixed such weight and Hilbert spaces \mathcal{H} and \mathcal{K} , we denote by $S_{\beta}(\mathcal{H},\mathcal{K})$ the space of Schur multipliers from $H_n^2(\mathcal{H})$ to $H_n^2(\beta,\mathcal{K})$. We also denote by $S(\mathcal{H},\mathcal{K})$ the space of Schur multipliers from $H_n^2(\mathcal{H})$ to $H_n^2(\mathcal{K})$. We show that any element $S \in S_{\beta}(\mathcal{H},\mathcal{K})$ can be factorized as $S = \Psi_{\beta}\tilde{S}$ for some contractive multiplier $\tilde{S} \in S(\mathcal{H}, l^2(\mathbb{N}^n, \mathcal{K}))$ and a fixed multiplier $\Psi_{\beta} \in S_{\beta}(l^2(\mathbb{N}^n, \mathcal{K}), \mathcal{K})$. The multiplier Ψ_{β} appears in the above factorization only depends on the weight β . To begin with, we describe this fixed factor. Define, a sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ such that $\gamma_0 = 1$ and $\gamma_n = (\beta_n^{-1} - \beta_{n-1}^{-1})^{-1}$ for all $n \geq 1$ and an operator-valued function $\Psi_{\beta} : \mathbb{B}^n \to \mathcal{B}(l^2(\mathbb{N}^n, \mathcal{K}), \mathcal{K})$ by

$$\Psi_{\beta}(z)(\{k_{\alpha}\}_{\alpha \in \mathbb{N}^n}) = \sum_{j=0}^{\infty} \frac{1}{\sqrt{\gamma_j}} \left(\sum_{|\alpha|=j} \left(\frac{|\alpha|!}{\alpha!} \right)^{\frac{1}{2}} z^{\alpha} k_{\alpha} \right)$$

where $z \in \mathbb{B}^n$ and $\{k_{\alpha}\}_{{\alpha} \in \mathbb{N}^n} \in l^2(\mathbb{N}^n, \mathcal{K})$. The following simple calculation shows that Ψ_{β} is well-defined:

$$\begin{split} \left\| \sum_{j=0}^{\infty} \frac{1}{\sqrt{\gamma_{j}}} \Big(\sum_{|\alpha|=j} \left(\frac{|\alpha|!}{\alpha!} \right)^{\frac{1}{2}} z^{\alpha} k_{\alpha} \Big) \right\|_{\mathcal{K}} &\leq \sum_{j=0}^{\infty} \frac{1}{\sqrt{\gamma_{j}}} \sum_{|\alpha|=j} \left(\frac{|\alpha|!}{\alpha!} \right)^{\frac{1}{2}} |z|^{\alpha} |k_{\alpha}||_{\mathcal{K}} \\ &\leq \sum_{j=0}^{\infty} \frac{1}{\sqrt{\gamma_{j}}} \Big[\sum_{|\alpha|=j} \left(\frac{|\alpha|!}{\alpha!} \right) |z|^{2\alpha} \Big]^{\frac{1}{2}} \Big[\sum_{|\alpha|=j} |k_{\alpha}||_{\mathcal{K}}^{2} \Big]^{\frac{1}{2}} \\ &\leq \Big[\sum_{j=0}^{\infty} \frac{1}{\gamma_{j}} \langle z, z \rangle^{j} \Big]^{\frac{1}{2}} \Big[\sum_{j=0}^{\infty} \sum_{|\alpha|=j} |k_{\alpha}||_{\mathcal{K}}^{2} \Big]^{\frac{1}{2}} \\ &= K_{\gamma}(z, z)^{\frac{1}{2}} ||k||_{l^{2}(\mathbb{N}^{n}, \mathcal{Y})}, \end{split}$$

where K_{γ} is a kernel on \mathbb{B}^n with weight γ . For each $z \in \mathbb{B}^n$, one can represent $\Psi_{\beta}(z)$ as the row operator

$$\Psi_{\beta}(z) = \left[\cdots, \frac{1}{\sqrt{\gamma_{|\alpha|}}} \left(\frac{|\alpha|!}{\alpha!}\right)^{\frac{1}{2}} z^{\alpha} I_{\mathcal{K}}, \cdots\right]_{\alpha \in \mathbb{N}^n}.$$

Also it follows from the definition of Ψ_{β} that

$$M_{\Psi_{\beta}}|_{l^2(\mathbb{N}^n,\mathcal{K})}: l^2(\mathbb{N}^n,\mathcal{K}) \to H_n^2(\gamma,\mathcal{K}), \quad \{k_{\alpha}\} \mapsto \Psi_{\beta}(.)(\{k_{\alpha}\})$$

is an isometry. We collect further property of Ψ_{β} in the next lemma.

LEMMA 3.1. The map $M_{\Psi_{\beta}}: H_n^2(l^2(\mathbb{N}^n, \mathcal{K})) \to H_n^2(\beta, \mathcal{K})$ is a co-isometry.

Proof: Note that, for $z, w \in \mathbb{B}^n$,

$$(1 - \langle z, w \rangle) K_{\beta}(z, w) = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \langle z, w \rangle^n - \sum_{n=0}^{\infty} \frac{1}{\beta_n} \langle z, w \rangle^{n+1}$$

$$= \frac{1}{\beta_0} + \sum_{n=0}^{\infty} \left[\frac{1}{\beta_{n+1}} \langle z, w \rangle^{n+1} - \frac{1}{\beta_n} \langle z, w \rangle^{n+1} \right]$$

$$= \frac{1}{\beta_0} + \sum_{n=0}^{\infty} \left[\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n} \right] \langle z, w \rangle^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\gamma_n} \langle z, w \rangle^n$$

$$= K_{\gamma}(z, w),$$

and therefore,

$$\Psi_{\beta}(z)\Psi_{\beta}(w)^* = \sum_{j=0}^{\infty} \frac{1}{\gamma_j} \sum_{|\alpha|=j} \frac{j!}{\alpha!} z^{\alpha} \bar{w}^{\alpha} I_{\mathcal{K}}$$

$$= \sum_{j=0}^{\infty} \frac{1}{\gamma_j} \langle z, w \rangle^j I_{\mathcal{K}}$$

$$= K_{\gamma}(z, w) I_{\mathcal{K}}$$

$$= (1 - \langle z, w \rangle) K_{\beta}(z, w) I_{\mathcal{K}}.$$

Hence

(3.3)
$$\frac{\Psi_{\beta}(z)\Psi_{\beta}(w)^*}{(1-\langle z,w\rangle)} = K_{\beta}(z,w)I_{\mathcal{K}},$$

and the proof follows.

The above introduced factorization Theorem 2.6 need to be invoked to obtain the following factorization result. We now state the main theorem of this section.

THEOREM 3.2. Let β be as in (3.1). Then $S \in S_{\beta}(\mathcal{H}, \mathcal{K})$ if and only if there is a contractive multiplier $\tilde{S} \in S(\mathcal{H}, l^2(\mathbb{N}^n, \mathcal{K}))$ such that

$$S(z) = \Psi_{\beta}(z)\tilde{S}(z).$$

Proof: First assume that $S(z) = \Psi_{\beta}(z)\tilde{S}(z)$ for all $z \in \mathbb{B}^n$, where $\tilde{S} \in S(\mathcal{H}, l^2(\mathbb{N}^n, \mathcal{K}))$ is a contractive multiplier. Then, using (3.3), we have for $z, w \in \mathbb{B}^n$,

(3.4)
$$K_{\beta}(z,w)I_{\mathcal{K}} - \frac{S(z)S(w)^{*}}{(1-\langle z,w\rangle)} = \frac{\Psi_{\beta}(z)\Psi_{\beta}(w)^{*}}{1-\langle z,w\rangle} - \frac{\Psi_{\beta}(z)\tilde{S}(z)\tilde{S}(w)^{*}\Psi_{\beta}(w)^{*}}{1-\langle z,w\rangle}$$
$$= \Psi_{\beta}(z) \left[\frac{I_{l^{2}(\mathbb{N}^{n},\mathcal{K})} - \tilde{S}(z)\tilde{S}(w)^{*}}{1-\langle z,w\rangle}\right]\Psi_{\beta}(w)^{*}.$$

Thus $K_{\beta}(z,w)I_{\mathcal{K}} - \frac{S(z)S(w)^*}{(1-\langle z,w\rangle)}$ defines a kernel on \mathbb{B}^n , and therefore $S \in S_{\beta}(\mathcal{H},\mathcal{K})$.

Conversely, suppose that $S \in S_{\beta}(\mathcal{H}, \mathcal{K})$, that is, $M_S : H_n^2(\mathcal{H}) \to H_n^2(\beta, \mathcal{K})$ is bounded operator. Then, using (3.3) again, we have

$$K_{\beta}(z,w)I_{\mathcal{K}} - \frac{S(z)S(w)^*}{1 - \langle z, w \rangle} = \frac{\Psi_{\beta}(z)\Psi_{\beta}(w)^* - S(z)S(w)^*}{1 - \langle z, w \rangle}.$$

The proof now follows from Theorem 2.6.

Remark 3.3. For n = 1, the above factorization results recovers the result of Ball and Bolotnikov [5].

Next we restrict ourselves to the subclass of K-inner functions (defined below) in $S_{\beta}(\mathcal{H}, \mathcal{K})$ and show that, in this case, the above factorization becomes unique. An element $S \in S_{\beta}(\mathcal{H}, \mathcal{K})$, is K-inner if $||S(h)||_{H_{n}^{2}(\beta, \mathcal{K})} = ||h||$ for all $h \in \mathcal{H}$ and

$$S(\mathcal{H}) \perp \bigvee_{\alpha \in \mathbb{N}^n, \alpha \neq 0} M_z^{\alpha} S(\mathcal{H}).$$

The notion of K-inner function has been introduced recently in [6]. The multiplication of a K-inner function $\tilde{S} \in S(\mathcal{H}, l^2(\mathbb{N}^n, \mathcal{K}))$ with the above defined operator Ψ_{β} is a K-inner function. The unique factorization of K-inner functions in $S_{\beta}(\mathcal{H}, \mathcal{K})$ is given next.

THEOREM 3.4. Let β be as in (3.1). For a K-inner function $S \in S_{\beta}(\mathcal{H}, \mathcal{K})$ there exists a unique contractive multiplier $\tilde{S} \in S(\mathcal{H}, l^2(\mathbb{N}^n, \mathcal{K}))$ such that \tilde{S} is a K-inner function and

$$S(z) = \Psi_{\beta}(z)\tilde{S}(z) \quad (z \in \mathbb{B}^n).$$

Proof. Suppose $S \in \mathcal{S}_{\beta}(\mathcal{H}, \mathcal{K})$ is a K-inner function. By Theorem 3.2, let

$$S = \Psi_{\beta} S'$$

be a factorization of S for some contractive multiplier $S' \in S(\mathcal{H}, l^2(\mathbb{N}^n, \mathcal{K}))$. First, we show that S' is a K-inner function. To this end, for $h \in \mathcal{H}$, let S'(h) = f + g where $f \in \text{Ker} M_{\Psi_{\beta}}$ and $g \in (\text{Ker} M_{\Psi_{\beta}})^{\perp}$. Since $M_{\Psi_{\beta}}$ is a co-isometry, we have

$$\|h\|_{\mathcal{H}} = \|M_{\Psi_{\beta}} M_{S'} h\|_{H_n^2(\beta,\mathcal{K})} = \|M_{\Psi_{\beta}} g\|_{H_n^2(\beta,\mathcal{K})} = \|g\|_{H_n^2(l^2(\mathbb{N}^n,\mathcal{K}))} < \|S' h\|_{H_n^2(l^2(\mathbb{N}^n,\mathcal{K}))} \le \|h\|.$$

This shows that $||S'h||_{H_n^2(l^2(\mathbb{N}^n,\mathcal{K}))} = ||h||_{\mathcal{H}}$ and $S'h \perp \operatorname{Ker} M_{\Psi_{\beta}}$ for all $h \in \mathcal{H}$. Now the orthogonality

$$S'(\mathcal{H}) \perp \bigvee_{\alpha \in \mathbb{N}^n, \alpha \neq 0} M_z^{\alpha} S'(\mathcal{H}),$$

follows from that of S and the fact that $M_{\Psi_{\beta}}$ is an isometry on $\operatorname{Ran}S'(\mathcal{H})$. Hence S' is a K-inner function and $S'(\mathcal{H}) \perp \operatorname{Ker} M_{\Psi_{\beta}}$.

For the uniqueness part, suppose $S = \Psi_{\beta}S''$ is an another factorization of S for some contractive multiplier $S'' \in S(\mathcal{H}, l^2(\mathbb{N}^n, \mathcal{K}))$. Then by the above argument S'' is a K-inner function and $S''(\mathcal{H}) \perp \operatorname{Ker} M_{\Psi_{\beta}}$. Then for all $h \in \mathcal{H}$,

$$M_{S'}(h) = M_{\Psi_{\beta}}^* M_S(h) = M_{S''}(h),$$

and the uniqueness follows.

4. Factorization of Characteristic Functions

We combine results obtained in the previous sections and find explicit factorization of characteristic functions of m-hypercontractions. Recall that a characteristic function for an m-hypercontraction T corresponds to a triple (\mathcal{E}, D, B) as in Definition 2.1. In such a case, the characteristic function of T is given by

$$S(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + D_{m,T^*} (1 - ZT^*)^{-m} ZB$$

for all $z \in \mathbb{B}^n$. Since S is a multiplier from $H_n^2(\mathcal{E})$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ and $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) = H_n^2(\beta, \mathcal{D}_{m,T^*})$ with

$$\beta_j = \binom{m+j-1}{j}^{-1} = \frac{j!(m-1)!}{(m+j-1)!} \quad (j \in \mathbb{N}),$$

then $S \in S_{\beta}(\mathcal{E}, \mathcal{D}_{m,T^*})$ and by Theorem 3.2 $S = \Psi_{\beta}\tilde{S}$ for some contractive multiplier $\tilde{S} \in S(\mathcal{E}, l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*}))$. Also note that, in this case, the sequence $\gamma = \{\gamma_j\}_{j=0}^{\infty}$ has the following form:

$$\gamma_j^{-1} = \beta_j^{-1} - \beta_{j-1}^{-1} = \frac{(m+j-1)!}{j!(m-1)!} - \frac{(m+j-2)!}{(j-1)!(m-1)!}$$
$$= \frac{(m+j-2)!}{j!(m-1)!} (m-1)$$
$$= \frac{(m+j-2)!}{j!(m-2)!} = {m+j-2 \choose j},$$

and therefore,

$$\Psi_{\beta}(z) = \left[\cdots, \frac{1}{\sqrt{\gamma_{|\alpha|}}} \left(\frac{|\alpha|!}{\alpha!}\right)^{\frac{1}{2}} z^{\alpha} I_{\mathcal{D}_{m,T^*}}, \cdots\right]_{\alpha \in \mathbb{N}^n}.$$

The next theorem describe the factor \tilde{S} explicitly as a transfer function of some associated co-isometry.

Theorem 4.1. Let T be a m-hypercontraction, and let

$$S(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + D_{m,T^*} (1 - ZT^*)^{-m} ZB$$

be a characteristic function of T corresponding to a triple (\mathcal{E}, B, D) . Then $S(z) = \Psi_{\beta}(z)\tilde{S}(z)$, where $\tilde{S}(z) = D + C_{m,T}(1 - ZT^*)^{-1}ZB$ is the transfer function of the canonical co-isometry

$$\begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$$

associated to the triple (\mathcal{E}, B, D) .

Proof. The factor $\tilde{S}(z)$ can be expanded as

$$\tilde{S}(z) = D + C_{m,T} (1 - ZT^*)^{-1} ZB$$

$$= D + C_{m,T} \sum_{i=1}^{n} \left(\sum_{\alpha \in \mathbb{N}^n} z^{\alpha} T^{*\alpha} \right) z_i B_i$$

$$= D + \sum_{i=1}^{n} \sum_{\alpha \in \mathbb{N}^n} \left(C_{m,T} T^{*\alpha} B_i \right) z^{\alpha + e_i},$$

where e_i is the *n*-tuple in \mathbb{N}^n with 1 in the *i*-th place and 0 elsewhere. Again using the definition of the operator $C_{m,T}$ as defined earlier, we can further simplify and view $\tilde{S}(z)$ as a column operator as follows:

$$\tilde{S}(z) = \begin{bmatrix} \vdots \\ D_{\alpha} + \sum_{i=1}^{n} \sum_{\alpha' \in \mathbb{N}^{n}} \left(\rho_{m-1}(\alpha)^{\frac{1}{2}} D_{m,T^{*}} T^{*(\alpha+\alpha')} B_{i} \right) z^{\alpha'+e_{i}} \\ \vdots \end{bmatrix}_{\alpha \in \mathbb{N}^{n}}.$$

Also recall that Ψ_{β} has the representation as a row operator

$$\Psi_{\beta}(z) = \left[\cdots \frac{1}{\sqrt{\gamma_{j}}} \left(\frac{j!}{\alpha!}\right)^{\frac{1}{2}} z^{\alpha} I_{\mathcal{D}_{m,T^{*}}} \cdots \right]_{\alpha \in \mathbb{N}^{n}}.$$

Now for $z \in \mathbb{B}^n$, we compute the product

$$\begin{split} \Psi_{\beta}(z)\tilde{S}(z) &= \sum_{\alpha \in \mathbb{N}^{n}} \frac{1}{\sqrt{\gamma_{j}}} (\frac{j!}{\alpha!})^{\frac{1}{2}} z^{\alpha} \Big(D_{\alpha} + \sum_{i=1}^{n} \sum_{\alpha' \in \mathbb{N}^{n}} \Big(\rho_{m-1}(\alpha)^{\frac{1}{2}} D_{m,T^{*}} T^{*(\alpha+\alpha')} B_{i} \Big) z^{\alpha'+e_{i}} \Big) \\ &= \sum_{\alpha \in \mathbb{N}^{n}} \frac{1}{\sqrt{\gamma_{j}}} (\frac{j!}{\alpha!})^{\frac{1}{2}} D_{\alpha} z^{\alpha} + \sum_{\alpha,\alpha' \in \mathbb{N}^{n}} \sum_{i=1}^{n} \Big(\frac{1}{\sqrt{\gamma_{j}}} (\frac{j!}{\alpha!})^{\frac{1}{2}} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{m,T^{*}} T^{*(\alpha+\alpha')} B_{i} \Big) z^{\alpha+\alpha'+e_{i}} \\ &= \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + \sum_{\alpha \in \mathbb{N}^{n}} \sum_{\alpha' \in \mathbb{N}^{n}} \sum_{i=1}^{n} \Big(\rho_{m-1}(\alpha) D_{m,T^{*}} T^{*(\alpha+\alpha')} B_{i} \Big) z^{\alpha+\alpha'+e_{i}} \\ &= \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + D_{m,T^{*}} \sum_{i=1}^{n} z_{i} \Big(\sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha) T^{*\alpha} z^{\alpha} \Big) \Big(\sum_{\alpha' \in \mathbb{N}^{n}} T^{*\alpha'} z^{\alpha'} \Big) B_{i} \\ &= \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + D_{m,T^{*}} \Big(\sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha) T^{*\alpha} z^{\alpha} \Big) \Big(\sum_{\alpha' \in \mathbb{N}^{n}} T^{*\alpha'} z^{\alpha'} \Big) ZB \\ &= \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + D_{m,T^{*}} (1 - ZT^{*})^{-(m-1)} (1 - ZT^{*})^{-1} ZB \\ &= \sum_{\alpha \in \mathbb{N}^{n}} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_{\alpha} z^{\alpha} + D_{m,T^{*}} (1 - ZT^{*})^{-m} ZB \\ &= S(z). \end{split}$$

This completes the proof

Now let T be a pure m_2 -hypercontraction. Then for $1 \leq m_1 \leq m_2$, T is also a m_1 -hypercontraction. Suppose that $(\mathcal{E}_1, B_1, D_1)$ and $(\mathcal{E}_2, B_2, D_2)$ are characteristic triples of T considered as a m_1 -hypercontraction and m_2 -hypercontraction, respectively. Then by the previous theorem the corresponding characteristic functions have factors \tilde{S}_1 and \tilde{S}_2 which are the transfer function of the co-isometries

$$\begin{bmatrix} T^* & B_1 \\ C_{m_1,T} & D_1 \end{bmatrix} : \mathcal{H} \oplus \mathcal{E}_1 \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m_1,T^*})$$

and

$$\begin{bmatrix} T^* & B_2 \\ C_{m_2,T} & D_2 \end{bmatrix} : \mathcal{H} \oplus \mathcal{E}_2 \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m_2,T^*}),$$

respectively. We find a relationship between \tilde{S}_1 and \tilde{S}_2 next.

To this end, we first note the following identities which follows from the co-isometric property of the above two block-matrices. For all i = 1, 2,

- (i) $C_{m_i,T}C_{m_i,T}^* + D_iD_i^* = I_{l^2(\mathbb{N}^n,\mathcal{D}_{m_i,T^*})};$
- (ii) $T^*T + B_i B_i^* = I_{\mathcal{H}};$
- (iii) $T^*C^*_{m_i,T} + B_iD^*_i = 0.$

Since, $\begin{bmatrix} T^* \\ C_{m_i,T} \end{bmatrix} : \mathcal{H} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m_i,T^*})$ is an isometry for i = 1, 2, we also have

$$TT^* + C_{m_i,T}^* C_{m_i,T} = I_{\mathcal{H}} \qquad (i = 1, 2).$$

Therefore,

$$C_{m_1,T}^*C_{m_1,T} = C_{m_2,T}^*C_{m_2,T}$$
 and $B_1B_1^* = B_2B_2^*$,

where the second identity follows from (ii). In both the cases, Douglas lemma guarantees the existence of two isometries $Y \in \mathcal{B}(\overline{\operatorname{ran}} C_{m_2,T}; l^2(\mathbb{N}^n, \mathcal{D}_{m_1,T^*}))$ and $X \in \mathcal{B}(\overline{\operatorname{ran}} B_1^*; \mathcal{E}_2)$ such that

$$(4.1) YC_{m_2,T} = C_{m_1,T} \text{ and } XB_1^* = B_2^*.$$

Now note the chain of identity, using (iii) and (4.1).

$$D_1B_1^* = -C_{m_1,T}T = -YC_{m_2,T}T = YD_2B_2^* = YD_2XB_1^*.$$

This implies that $D_1|_{(\text{Ker}B_1)^{\perp}} = YD_2X$. Finally, using the above identities we have the following relation between the corresponding transfer functions:

(4.2)
$$\tilde{S}_{1}(z)|_{(\operatorname{Ker}B_{1})^{\perp}} = \left[D_{1} + C_{m_{1},T}(1 - ZT^{*})^{-1}ZB_{1}\right]_{(\operatorname{Ker}B_{1})^{\perp}} \\
= YD_{2}X + YC_{m_{2},T}(1 - ZT^{*})^{-1}ZB_{2}X \\
= Y\tilde{S}_{2}(z)X.$$

Thus we have proved the following result.

PROPOSITION 4.2. Let T be a pure m_2 -hypercontraction. Suppose that $(\mathcal{E}_1, B_1, D_1)$ and $(\mathcal{E}_2, B_2, D_2)$ are characteristic triples of T considered as a m_1 -hypercontraction and m_2 -hypercontraction

 $(1 \leq m_1 < m_2)$, with \tilde{S}_1 and \tilde{S}_2 are the factor of the respective characteristic functions. Then there exist isometries $Y \in \mathcal{B}(\overline{ran} \ C_{m_2,T}; l^2(\mathbb{N}^n, \mathcal{D}_{m_1,T^*}))$ and $X \in \mathcal{B}(\overline{ran} \ B_1^*; \mathcal{E}_2)$ such that

$$\tilde{S}_1(z)|_{(KerB_1)^{\perp}} = Y\tilde{S}_2(z)X$$

foe all $z \in \mathbb{B}^n$.

As a special case of the above result, we exhibit relations between Sz.-Nagy Foias characteristic function and transfer function corresponding to a characteristic triple of a m-hyper contraction. Let T be a pure m-hypercontraction ($m \ge 2$). Then T is also a row contraction, and by Sz.-Nagy Foias we know $\begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix} : \mathcal{H} \oplus \mathcal{D}_T \to \mathcal{H}^n \oplus \mathcal{D}_{T^*}$ is a unitary. The Sz.-Nagy Foias characteristic function of T is denoted by Θ_T and defined as the transfer of the above unitary block matrix, that is

$$\Theta_T(z) = -T + D_{T^*}(1 - ZT^*)^{-1}ZD_T.$$

THEOREM 4.3. Let T be a pure m-hypercontraction $(m \geq 2)$, and let $(\mathcal{E}_m, B_m, D_m)$ be a characteristic triple of the m-hypercontraction T. If T is also a strict contraction then there exist two isometries $Y \in \mathcal{B}(\overline{ran} \ C_{m,T}; l^2(\mathbb{N}^n, \mathcal{D}_{T^*}))$ and $X \in \mathcal{B}(\mathcal{D}_T; \mathcal{E}_m)$ such that

$$\Theta_T = Y\tilde{S}X$$

where Θ_T is the Sz.-Nagy Foias characteristic function and \tilde{S} is the transfer function of the characteristic triple $(\mathcal{E}_m, B_m, D_m)$.

Proof: Using the canonical unitary operator $\begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix}$: $\mathcal{H} \oplus \mathcal{D}_T \to \mathcal{H}^n \oplus \mathcal{D}_{T^*}$, we first find a characteristic triple of the row contraction T as follows. Let $\iota_0 : \mathcal{D}_{T^*} \to l^2(\mathbb{N}^n, \mathcal{D}_{T^*})$ be the inclusion map defined by

$$\iota_0(D_{T^*}h)(\alpha) = \begin{cases} D_{T^*}h, & \text{if } \alpha = 0\\ 0 & \text{otherwise} \end{cases}.$$

Set $\mathcal{E} := l^2(\mathbb{N}^n \setminus \{0\}, \mathcal{D}_{T^*})$. Let $B_1 = [D_T, 0] : \mathcal{D}_T \oplus \mathcal{E} \to \mathcal{H}^n$ be the row operator defined by $B_1(h, e) = D_T h$ for all $h \in \mathcal{D}_T$ and $e \in \mathcal{E}$. Finally, let $D_1 : \mathcal{D} \oplus \mathcal{E} \to l^2(\mathbb{N}^n, \mathcal{D}_{T^*})$ defined by

$$D_1(h, e)(\alpha) = \begin{cases} -Th & \text{if } \alpha = 0\\ e(\alpha) & \text{otherwise} \end{cases}.$$

Then a simple calculation shows that

$$\begin{bmatrix} T^* & B_1 \\ \iota_0 D_{T^*} & D_1 \end{bmatrix} : \mathcal{H} \oplus (\mathcal{D}_T \oplus \mathcal{E}) \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{T^*})$$

is a unitary, and therefore $(\mathcal{D}_T \oplus \mathcal{E}, B_1, D_1)$ is a characteristic triple for T when it is considered as a row contraction. Also note that if \tilde{S}_1 is the transfer function corresponding to the triple then for all $z \in \mathbb{B}^n$,

$$\Theta_T(z) = \tilde{S}_1(z)|_{\mathcal{D}_T}.$$

By applying theorem 4.2, for \tilde{S}_1 and S, we get isometries $Y \in \mathcal{B}(\overline{\operatorname{ran}} C_{m,T}; l^2(\mathbb{N}^n, \mathcal{D}_{T^*}))$ and $X \in \mathcal{B}(\mathcal{D}_T; \mathcal{E}_m)$ such that

$$\tilde{S}_1(z)|_{(\operatorname{Ker} B_1)^{\perp}} = Y \tilde{S}(z) X,$$

for all $z \in \mathbb{B}^n$. Since T is a strict row contraction, $(\operatorname{Ker} B_1)^{\perp} = \mathcal{D}_T$. This completes the proof.

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