On q-tensor product of Cuntz algebras

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To 75-th birthday of our teacher Yurii S. Samoilenko

Abstract

We consider C^*-algebra $\mathcal{E}^q_{n,m}$, which is a q-twist of two Cuntz-Toeplitz algebras. For the case $|q| < 1$ we give an explicit formula, which untwists the q-deformation, thus showing that the isomorphism class of $\mathcal{E}^q_{n,m}$ does not depend of q. For the case $|q| = 1$ we give an explicit description of all ideals in $\mathcal{E}^q_{n,m}$. In particular $\mathcal{E}^q_{n,m}$ contains unique largest ideal $\mathcal{M}_q$. Then we identify $\mathcal{E}^q_{n,m}/\mathcal{M}_q$ with the Rieffel deformation of $\mathcal{O}_n \otimes \mathcal{O}_m$ and use a K-theoretical argument to show that the isomorphism class does not depend on q.

Key Words: Cuntz-Toeplitz algebra, Rieffel's deformation, q-deformation, Fock representation, K-theory.

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1 Introduction

In this paper we study a class of C^*-algebras generated by isometries satisfying certain twisted commutation relations. Since the objects we are considering belong to the class of quadratic $*$-algebras allowing Wick ordering, we recall the basic theory of Wick algebras and fit the algebras we are interested in into the general framework.

Let $\{T_{ij}^{kl},\ i,j,k,l=\overline{1,d}\}\subset\mathbb{C}$, $T_{ij}^{kl}=\overline{T}_{ji}^{lk}$. Denote by $W(T)$, see [14], the $*$-algebra generated by elements a_j, a_j^*, $j=\overline{1,d}$ subject to the relations

$$a_i^*a_j=\delta_{ij}\mathbf{1}+\sum_{k,l=1}^{d}T_{ij}^{kl}a_la_k^*.$$

It was shown in [14] that properties of $W(T)$ depend on a self-adjoint operator T called the operator of coefficients of $W(T)$. Namely, let $\mathcal{H}=\mathbb{C}^d$ and $e_1,\ldots,$ e_d be the standard orthonormal basis of $\mathcal{H}$. Construct

$$T\colon\mathcal{H}^{\otimes 2}\to\mathcal{H}^{\otimes 2},\quad Te_k\otimes e_l=\sum_{i,j=1}^{d}T_{ik}^{lj}e_i\otimes e_j.$$

Notice, that the subalgebra of $W(T)$ generated by $\{a_j\}_{j=1}^d$ is free and can be identified with the tensor algebra $\mathcal{F}=\bigoplus_{n=0}^{\infty}\mathcal{H}^{\otimes n}$.

Definition 1. The Fock representation of $W(T)$ is the unique irreducible representation $\pi_{F,T}$ determined by a cyclic vector Ω, $||\Omega||=1$, such that

$$\pi_{F,T}(a_j^*)\Omega=0,\ j=\overline{1,d}.$$

The problem of existence of $\pi_{F,T}$ is non-trivial and is one of the central problems in representation theory of Wick algebras. Some sufficient conditions are collected in the following theorem, see [11, 14, 12].

Theorem 1. *The Fock representation $\pi_{F,T}$ of $W(T)$ exists if one of the conditions below is satisfied*

- *The operator of coefficients $T\geq 0$;*
- $||T||<\sqrt{2}-1$

- *If T is braided, i.e.* $(\mathbf{1}\otimes T)(T\otimes\mathbf{1})(\mathbf{1}\otimes T)=(T\otimes\mathbf{1})(\mathbf{1}\otimes T)(\mathbf{1}\otimes T)$ *on* $\mathcal{H}^{\otimes 3}$ *and* $||T||\le 1$. *Moreover, if* $||T||<1$ *then* $\pi_{F,T}$ *is a faithful representation of* $W(T)$ *and* $||\pi_{F,T}(a_j)||<(1-||T||)^{-\frac{1}{2}}$. *If* $||T||=1$, *one can not guarantee boundedness of* $\pi_{F,T}$ *and in this case* $\ker\pi_{F,T}$ *is a* $*$*-ideal* $\mathcal{I}_2$ *generated as a* $*$*-ideal by* $\ker(\mathbf{1}+T)$. *Hence* $\pi_{F,T}$ *is a faithful representation of* $W(T)/\mathcal{I}_2$.

Another important question in the theory of Wick algebras is the question of stability of isomorphism class of $\mathcal{W}(T)=C^*(W(T))$ for the case $||T||<1$. The following problem was posed in [15].

Conjecture 1. Let $T:\mathcal{H}^{\otimes 2}\to\mathcal{H}^{\otimes 2}$ be a self-adjoint braided operator and $||T||<1$. Then $\mathcal{W}(T)\simeq\mathcal{W}(0)$.

In particular, the authors of [15] have shown that the conjecture holds for the case $||T||<\sqrt{2}-1$, for more results on the subject see [30, 31].

Consider in a few more details the case $T=0$. If $d=\dim\mathcal{H}=1$, then $W(0)$ is generated by single isometry s, $s^*s=1$. In this case the universal C^*-algebra $\mathcal{E}$ of $W(0)$ exists and is isomorphic to C^*-algebra generated by unilateral shift S on $l_2(\mathbb{Z}_0)$. Notice also that $\pi_{F,0}(s)=S$, so the Fock representation of C^*-algebra $\mathcal{E}$ is faithful. The ideal $\mathcal{I}$ in $\mathcal{E}$, generated by $\mathbf{1}-ss^*$ is isomorphic to the algebra of compact operators and $\mathcal{E}/\mathcal{I}\simeq C(S^1)$, see [28]. When $d\ge 2$, $W(0)$ is generated by s_j, s_j^*, such that

$$s_i^*s_j=\delta_{ij}\mathbf{1},\quad i,j=\overline{1,d}.$$

The Fock representation $\pi_{F,d}$ acts on $\mathcal{F}:=\mathcal{F}_d$ as follows

$$\pi_{F,d}(s_j)\Omega=e_j,\quad \pi_{F,d}(s_j)e_{i_1}\otimes\cdots\otimes e_{i_k}=e_j\otimes e_{i_1}\otimes\cdots\otimes e_{i_k},\ k\ge 1,$$
$$\pi_{F,d}(s_j^*)\Omega=0,\quad \pi_{F,d}(s_j^*)e_{i_1}\otimes\cdots\otimes e_{i_k}=\delta_{ji_1}e_{i_2}\otimes\cdots\otimes e_{i_k},\ k\ge 1.$$

The universal C^*-algebra generated by $W(0)$ with $d\ge 2$ exists and is called the Cuntz-Toeplitz agebra $\mathcal{O}_d^{(0)}$. It is isomorphic to $C^*(\pi_{F,d}(W(0)))$, so the Fock representation of $\mathcal{O}_d^{(0)}$ is faithful, see [27]. Further, the ideal $\mathcal{I}$ generated by $1-\sum_{j=1}^d s_js_j^*$ is the unique largest ideal in $\mathcal{O}_d^{(0)}$. It is isomorphic to the algebra of compact operators on $\mathcal{F}_d$. The quotient $\mathcal{O}_d^{(0)}/\mathcal{I}$ is called the Cuntz algebra $\mathcal{O}_d$. It is nuclear (as well as $\mathcal{O}_d^{(0)}$), simple and purely infinite, see [27] for more details.

Now let $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^m$, $|q| \le 1$ and the operator T is defined as follows

$$Tu_1 \otimes u_2 = 0, \; Tv_1 \otimes v_2 = 0, \; u_1, u_2 \in \mathbb{C}^n, \; v_1, v_2 \in \mathbb{C}^m,$$
$$Tu \otimes v = qv \otimes u, \; Tv \otimes u = \overline{q} u \otimes v, \; u \in \mathbb{C}^n, \; v \in \mathbb{C}^m.$$

We denote the corresponding Wick algebra by $WE_{n,m}^q$ and by $\mathcal{E}_{n,m}^q$ its universal C^*-algebra. Notice that T satisfies the braid relation and $||T|| = |q| \le 1$ for any $n, m \in \mathbb{N}$. In particular, the Fock representation $\pi_{F,q}$ exists for $|q| \le 1$ and is faithful on $WE_{n,m}^q$ if $|q| < 1$.

The case $n = 1$, $m = 1$ was studied by various authors. Namely, $WE_{1,1}^q$ is generated by isometries s_1, s_2 subject to the relations

$$s_1^* s_2 = q s_2 s_1^*.$$

It is easy to see that the corresponding universal C^*-algebra $\mathcal{E}_{1,1}^q$ exists for any $|q| \le 1$.

If $|q| < 1$ the main result of [13] states that $\mathcal{E}_{1,1}^q \simeq \mathcal{E}_{1,1}^{(0)} = \mathcal{O}_2^{(0)}$ for any $|q| < 1$. In particular the Fock representation of $\mathcal{E}_{1,1}^q$ is faithful.

The case $|q| = 1$ was considered in [19, 20, 21]. In this situation the additional relation

$$s_2 s_1 = q s_1 s_2$$

holds in $\mathcal{E}_{1,1}^q$. It was shown that $\mathcal{E}_{1,1}^q$ is nuclear for any $|q| = 1$. Let $\mathcal{M}_q$ be ideal generated by the projections $1 - s_1 s_1^*$ and $1 - s_2 s_2^*$. Then $\mathcal{E}_{1,1}/\mathcal{M}_q \simeq \mathcal{A}_q$, where $\mathcal{A}_q$ is the non-commutative torus, see [22],

$$\mathcal{A}_q = C^*(u_1, u_2 \,|\, u_1^* u_1 = u_1 u_1^* = \mathbf{1}, \; u_2^* u_2 = u_2 u_2^* = \mathbf{1}, \; u_2^* u_1 = q u_1 u_2^*).$$

If q is not a root of unity then the corresponding non-commutative torus $\mathcal{A}_q$ is simple and $\mathcal{M}_q$ is the unique largest ideal in $\mathcal{E}_{1,1}^q$. Let us stress that unlike the case $|q| < 1$, the C^*-isomorphism class of $\mathcal{E}_{1,1}^q$ is "unstable" with respect to q. Namely, $\mathcal{E}_{1,1}^{q_1} \simeq \mathcal{E}_{1,1}^{q_2}$ iff $\mathcal{A}_{q_1} \simeq \mathcal{A}_{q_2}$, see [19, 20, 21].

One can consider another higher-dimensional analog of $\mathcal{E}_{1,1}^q$. For a set $\{q_{ij}\}_{i,j=1}^d$ of complex numbers such that $|q_{ij}| \le 1$, $q_{ij} = \overline{q}_{ji}$ and $d > 2$, one can consider the C^*-algebra $\mathcal{E}_{\{q_{ij}\}}$, generated by s_j, s_j^*, $j = \overline{1,d}$ subject to relations

$$s_j^* s_j = 1, \quad s_i^* s_j = q_{ij} s_j s_i^*.$$

The case $|q_{ij}| < 1$ was considered in [24], where it was proved that $\mathcal{E}_{\{q_{ij}\}}$ is nuclear and the Fock representation is faithful. It turned out that the fixed point C^*-subalgebra of $\mathcal{E}_{\{q_{ij}\}}$ w.r.t. the canonical action of $\mathbb{T}^d$ is an AF-algebra and is independent of $\{q_{ij}\}$. However the conjecture that $\mathcal{E}_{\{q_{ij}\}} \simeq \mathcal{E}_{\{0\}}$ remains open.

The case $|q_{ij}| = 1$ was studied in [19, 20, 23]. It was shown that $\mathcal{E}_{\{q_{ij}\}}$ is nuclear for any such family $\{q_{ij}\}$ and the Fock representation is faithful. For more details on ideal structure and representation theory see [20, 23].

In this paper we focus on the study of $\mathcal{E}_{n,m}^q$ with $n, m \geq 2$ (the case $n = 1$, $m \geq 2$ will be considered separately, see [25]). It is generated by isometries $\{s_j\}_{j=1}^n$, and $\{t_r\}_{r=1}^m$, satisfying commutation relations of the following form

$$s_i^* s_j = 0,\ t_r^* t_s = 0,\ i \neq j,\ r \neq s,\ s_j^* t_r = q t_r s_j^*. \tag{1}$$

The analysis is separated into two conceptually different cases $|q| < 1$ and $|q| = 1$.

Namely, we show that if $|q| < 1$, then $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^0 = \mathcal{O}_{n+m}^{(0)}$, where the lattest is the Cuntz-Toeplitz algebra on $n + m$ generators.

For the case $|q| = 1$ we prove that $\mathcal{E}_{n,m}^q$ is nuclear, contains unique largest ideal $\mathcal{M}_q$ and the quotient $\mathcal{O}_n \otimes_q \mathcal{O}_m = \mathcal{E}_{n,m}^q/\mathcal{M}_q$ is simple and purely infinite for any q specified above. Then we use Kirchberg-Philips classification Theorem, see [1, 2], to get one of our main results. Namely we show that

$$\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$$

for any $q \in \mathbb{C}$, $|q| = 1$. Further we prove, that Fock representation of $\mathcal{E}_{n,m}^q$ is faithful for any $|q| = 1$ and use this fact to prove that $\mathcal{E}_{n,m}^q$ is isomorphic to Rieffel deformation of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$. Next we show that the isomorphism class of $\mathcal{M}_q$ is independent on q and consider $\mathcal{E}_{n,m}^q$ as an (essential) extension of $\mathcal{O}_n \otimes \mathcal{O}_m$ by $\mathcal{M}_q$ and compute the corresponding Ext group. In particular if $\gcd(n-1, m-1) = 1$ this group is zero. Thus in this case, $\mathcal{E}_{n,m}^q$ and $\mathcal{E}_{n,m}^1$ both determine the zero class in $\mathsf{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q)$. We stress that unlike the case of extensions by compacts, one can not immediately deduce that two trivial essential extensions are isomorphic. So the problem of isomorphism $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^1$ remains open for further investigations.

2 The case $|q| < 1$

We start with some lemmas. Let Λ_n denote the set of all words in alphabet $\{\overline{1,n}\}$. For any non-empty $\mu = (\mu_1, \ldots, \mu_k)$ and family of elements $b_1, \ldots b_n$, we denote by b_μ the product $b_{\mu_1}\cdots b_{\mu_k}$, we also put $b_\emptyset = \mathbf{1}$. In this section we assume that any word μ belongs to Λ_n.

Lemma 1. *Let $Q = \sum_{i=1}^n s_i s_i^*$, then*

$$\sum_{|\mu|=k} s_\mu Q s_\mu^* = \sum_{|\nu|=k+1} s_\nu s_\nu^*$$

Proof. Evident. □

Lemma 2. *For any $x \in \mathcal{E}_{n,m}^q$ one has*

$$||\sum_{|\mu|=k} s_\mu x s_\mu^*|| \leq ||x||.$$

Proof.
1. First prove the claim for positive x. In this case one has $0 \leq x \leq ||x||\mathbf{1}$. Hence $0 \leq s_\mu x s_\mu^* \leq ||x|| s_\mu s_\mu^*$ and

$$||\sum_{|\mu|=k} s_\mu x s_\mu^*|| \leq ||x|| \cdot ||\sum_{|\mu|=k} s_\mu s_\mu^*||.$$

Note that $s_\mu^* s_\lambda = \delta_{\mu\,\lambda}$, $\mu, \lambda \in \Lambda_n$, $|\mu| = |\lambda| = k$, implying that $\{s_\mu s_\mu^* \mid |\mu| = k\}$ form the family of pairwise orthogonal projections. Hence $||\sum_{|\mu|=k} s_\mu s_\mu^*|| = 1$ and the statement for positive x is proved.

2. For any $x \in \mathcal{E}_{n,m}^q$ denote by $A = \sum_{|\mu|=k} s_\mu x s_\mu^*$, then $A^* = \sum_{|\mu|=k} s_\mu x s_\mu^*$ and

$$A^* A = \sum_{|\mu|=k} s_\mu x^* x s_\mu^*.$$

Then by proved above

$$||A||^2 = ||A^* A|| \leq ||x^* x|| = ||x||^2.$$

□

Construct $\widetilde{t}_l = (\mathbf{1} - Q)t_l$, $r = \overline{1,m}$.

Lemma 3. *The following commutation relations hold*

$$s_i^* \widetilde{t}_l = 0,\ i = \overline{1,n},\ l = \overline{1,m}, \quad \widetilde{t}_r^* \widetilde{t}_l = 0,\ l \neq r, \quad \widetilde{t}_r^* \widetilde{t}_r = \mathbf{1} - |q|^2 Q > 0.$$

Proof.
1. Note that $s_i^*(\mathbf{1}-Q)=0$, implying $s_i^*\widetilde{t}_l=0$ for any $i=\overline{1,n}$ and $l=\overline{1,m}$.
2. Further

$$\begin{aligned}\widetilde{t}_r^*\widetilde{t}_l &= t_r^*(\mathbf{1}-Q)t_l = t_r^*t_l-\sum_{i=1}^n t_r^*s_is_i^*t_l=\delta_{rl}\mathbf{1}-\sum_{i=1}^n|q|^2s_it_r^*t_ls_i^* = \\ &= \delta_{rl}(\mathbf{1}-|q|^2Q).\end{aligned}$$

□

Proposition 1. *For any $r=\overline{1,m}$ one has*

$$t_r=\sum_{k=0}^\infty\sum_{|\mu|=k}q^ks_\mu\widetilde{t}_rs_\mu^*.$$

In particular the family $\{s_i,\widetilde{t}_r,\ i=\overline{1,n},\ r=\overline{1,m}\}$ generates $\mathcal{E}_{n,m}^q$.

Proof. Denote by $M_k^r=\sum_{|\mu|=k}q^ks_\mu\widetilde{t}_rs_\mu^*$, $k\in\mathbb{Z}_+$. Then

$$M_0^r=t_r-Qt_r=t_r-\sum_{|\mu|=1}s_\mu s_\mu^*t_r$$

and

$$\begin{aligned}M_k^r &= \sum_{|\mu|=k}q^ks_\mu(\mathbf{1}-Q)t_rs_\mu^*=\sum_{|\mu|=k}s_\mu(\mathbf{1}-Q)s_\mu^*t_r= \\ &= \sum_{|\mu|=k}s_\mu s_\mu^*t_r-\sum_{|\mu|=k+1}s_\mu s_\mu^*t_r.\end{aligned}$$

Then

$$S_N^r=\sum_{k=0}^N M_k^r=t_r-\sum_{|\mu|=N+1}s_\mu s_\mu^*t_r=t_r-q^{N+1}\sum_{|\mu|=N+1}s_\mu t_rs_\mu^*.$$

Since $||\sum_{|\mu|=N+1}s_\mu t_rs_\mu^*||\le||t_r||=1$ one has that $S_N^r\to t_r$ in $\mathcal{E}_{n,m}^q$ when $N\to\infty$. □

Suppose that $\mathcal{E}_{n,m}^q$ is realised by Hilbert space operators. Consider the left polar decomposition $\widetilde{t}_r=\widehat{t}_r\cdot c_r$, where $c_r^2=\widetilde{t}_r^*\widetilde{t}_r=\mathbf{1}-|q|^2Q>0$ implying that $\widehat{t}_r$ is an isometry and

$$\widehat{t}_r=\widetilde{t}_rc_r^{-1}\in\mathcal{E}_{n,m}^q,\quad r=\overline{1,m}.$$

Lemma 4. *The following commuation relations hold*

$$s_i^*\widehat{t}_r = 0,\ i = \overline{1,n},\ r = \overline{1,m}, \quad \widehat{t}_r^*\widehat{t}_l = \delta_{rl}\mathbf{1},\ r,l = \overline{1,m}.$$

Proof. Indeed for any $i = \overline{1,n}$ and $r = \overline{1,m}$ one has

$$s_i^*\widehat{t}_r = s_i^*\widetilde{t}_r\, c_r^{-1} = 0,$$

and

$$\widehat{t}_r^*\widehat{t}_l = c_r^{-1}\widetilde{t}_r^*\widetilde{t}_l c_r^{-1} = 0,\ \text{if } r \neq l.$$

□

Summing up the results stated above we get the following

Theorem 2. *Let* $\widetilde{t}_r = (\mathbf{1} - Q)t_r$ *and* $\widehat{t}_r = \widetilde{t}_r \cdot (1 - |q|^2 Q)^{-\frac{1}{2}}$, $r = \overline{1,m}$. *Then the family* $\{s_i, \widehat{t}_r,\ i = \overline{1,n},\ r = \overline{1,m}\}$ *generates* $\mathcal{E}_{n,m}^q$ *and*

$$s_i^* s_j = \delta_{ij}\mathbf{1},\ t_r^* t_l = \delta_{rl}\mathbf{1},\ s_i^* t_r = 0,\ i,j = \overline{1,n},\ r,l = \overline{1,m}.$$

Proof. It remains to note that $\widetilde{t}_r = \widehat{t}_r(1 - |q|^2 Q)^{\frac{1}{2}}$, so $\widetilde{t}_r \in C^*(\widehat{t}_r, Q)$, so by Proposition 1 the elements s_i, $\widehat{t}_r$, $i = \overline{1,n}$, $r = \overline{1,m}$, generate $\mathcal{E}_{n,m}^q$. □

Corollary 1. *Denote by* v_i, $i = \overline{1,n}{+}m$ *isometries generating* $\mathcal{E}_{n,m}^0 = \mathcal{O}_{n+m}^0$. *Then Theorem 2 implies that the correspondence*

$$v_i \mapsto s_i,\ i = \overline{1,n}, \quad v_{n+r} \mapsto \widehat{t}_r,\ r = \overline{1,m}$$

extends uniquely to surjective homomorphism $\varphi\colon \mathcal{E}_{n,m}^0 \to \mathcal{E}_{n,m}^q$.

Our next aim is to construct the inverse homomorphism $\psi\colon \mathcal{E}_{n,m}^q \to \mathcal{E}_{n,m}^0$. To do it, denote by

$$\widetilde{Q} = \sum_{i=1}^{n} v_i v_i^*$$

and put $\widetilde{w}_r = v_{n+r}(1 - |q|^2\widetilde{Q})^{\frac{1}{2}}$, $r = \overline{1,m}$. Then $\widetilde{w}_r^*\widetilde{w}_r = 1 - |q|^2\widetilde{Q}$ and $\widetilde{w}_r^*\widetilde{w}_l = 0$ if $r \neq l$, $r,l = \overline{1,m}$. Construct

$$w_r = \sum_{k=0}^{\infty}\sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^*,\ r = \overline{1,m},$$

where μ runs over Λ_n, and as above $v_\mu = v_{\mu_1}\cdots v_{\mu_k}$. Note that the series above converges w.r.t norm in $\mathcal{E}_{n,m}^0$.

Lemma 5. *The following commutation relations hold*

$$w_r^* w_l = \delta_{rl}\mathbf{1}, \quad v_i^* w_r = q w_r v_i^*, \quad i = \overline{1,n}, \ r,l = \overline{1,m}.$$

Proof. First we note that $v_i^* \widetilde{w}_r = 0$ and $\widetilde{w}_r^* v_i = 0$ for any $i = \overline{1,n}$ and $j = \overline{1,m}$ implying that

$$v_\delta^* \widetilde{w}_r = 0, \ \widetilde{w}_r^* v_\delta = 0, \ \text{for any nonempty } \delta \in \Lambda_n, \ r = \overline{1,m}.$$

Let $|\lambda| \neq |\mu|$, $\lambda, \mu \in \Lambda_n$. If $|\lambda| > |\mu|$. then $\lambda = \widehat{\lambda}\gamma$ with $|\lambda| = |mu|$ and

$$v_\lambda^* v_\mu = \delta_{\widehat{\lambda}\mu} v_\gamma^*.$$

Otherwise $\mu = \widehat{\mu}\beta$, $|\widehat{\mu}| = |\lambda|$ and

$$v_\lambda^* v_\mu = \delta_{\lambda\widehat{\mu}} v_\beta.$$

So, if $|\lambda| > |\mu|$ one has

$$v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* = \delta_{\widehat{\lambda}\mu} v_\lambda \widetilde{w}_r^* v_\gamma^* \widetilde{w}_r v_\mu = 0$$

and if $|\mu| > |\lambda|$, then

$$v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* = \delta_{\lambda\widehat{\mu}} v_\lambda \widetilde{w}_r^* v_\beta \widetilde{w}_r v_\mu = 0$$

Since $v_\mu^* v_\lambda = \delta_{\mu\lambda}\mathbf{1}$, if $|\mu| = |\lambda|$, one has

$$w_r^* w_r = \lim_{N\to\infty} \Big(\sum_{k=0}^{N} \sum_{|\lambda|=k} |q|^k v_\lambda \widetilde{w}_r^* v_\lambda^*\Big) \cdot \Big(\sum_{l=0}^{N} \sum_{|\mu|=l} |q|^l v_\mu \widetilde{w}_r v_\mu^*\Big) =$$

$$= \lim_{N\to\infty} \sum_{k,l=0}^{N} \sum_{|\lambda|=k, |\mu|=l} |q|^{k+l} v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* = \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{|\lambda|,|\mu|=k,} |q|^{2k} v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* =$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{|\mu|=k} |q|^{2k} v_\mu \widetilde{w}_r^* \widetilde{w}_r v_\mu^* = \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{|\mu|=k} |q|^{2k} v_\mu (\mathbf{1} - |q|^2 \widetilde{Q}^2) v_\mu^* =$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} \Big(\sum_{|\mu|=k} |q|^{2k} v_\mu v_\mu^* - \sum_{|\mu|=k+1} |q|^{2k+2} v_\mu v_\mu^* \Big)$$

$$= \lim_{N\to\infty} (\mathbf{1} - |q|^{2N+2} \sum_{|\mu|=N+1} v_\mu v_\mu^*) = \mathbf{1}.$$

Since $\widetilde{w}_r^*\widetilde{w}_l = 0$, $r \neq l$, the same arguments as above imply that $w_r^* w_l = 0$, $r \neq l$.

For any non-empty $\mu \in \Lambda_n$ denote by $\sigma(\mu) = \emptyset$, if $|\mu| = 1$ and $\sigma(\mu) = (\mu_2, \ldots, \mu_k)$, if $|\mu| = k > 1$. Further, for any $i = \overline{1,n}$, $r = \overline{1,m}$ one has

$$v_i^* w_r = \sum_{k=0}^{\infty} \sum_{|\mu|=k} q^k s_i^* v_\mu \widetilde{w}_r v_\mu^* = v_i^* \widetilde{w}_r + \sum_{k=1}^{\infty} \sum_{|\mu|=k} q^k \delta_{i\mu_1} v_{\sigma(\mu)} \widetilde{w}_r v_{\sigma(\mu)}^* v_i^* =$$
$$= q \sum_{k=0}^{\infty} \sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^* v_i^* = q w_r v_i^*.$$

□

Lemma 6. *For any $r = \overline{1,m}$ one has $\widetilde{w}_r = (\mathbf{1} - \widetilde{Q}) w_r$.*

Proof. First note that $(\mathbf{1} - \widetilde{Q}) v_i = 0$, $i = \overline{1,n}$, implies that

$$(\mathbf{1} - \widetilde{Q}) v_\mu = 0, \ |\mu| \in \Lambda_n, \ \mu \neq \emptyset.$$

Then

$$(\mathbf{1} - \widetilde{Q}) w_r = (\mathbf{1} - \widetilde{Q})(\sum_{k=0} \sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^*) =$$
$$= (\mathbf{1} - \widetilde{Q}) \widetilde{w}_r + \sum_{k=1} \sum_{|\mu|=k} q^k (\mathbf{1} - \widetilde{Q}) v_\mu \widetilde{w}_r v_\mu^* = (\mathbf{1} - \widetilde{Q}) \widetilde{w}_r.$$

To complete the proof it remains to note that $\widetilde{Q} v_{n+r} = 0$, $r = \overline{1,m}$. So, $\widetilde{Q}\widetilde{w}_r = \widetilde{Q} v_{n+r} (\mathbf{1} - |q|^2 \widetilde{Q})^{\frac{1}{2}} = 0$. □

Theorem 3. *Let v_i, $i = \overline{1, n+m}$, be isometries generating $\mathcal{E}_{n,m}^0$, construct $\widetilde{Q} = \sum_{i=1}^n v_i v_i^*$. Put*

$$\widetilde{w}_r = v_{n+r} (\mathbf{1} - |q|^2 \widetilde{Q})^{\frac{1}{2}} \text{ and } w_r = \sum_{k=0} \sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^*.$$

Then

$$v_i^* v_j = \delta_{ij} \mathbf{1}, \quad w_r^* w_l = \delta_{rl} \mathbf{1}, \quad v_i^* w_r = q w_r v_i^*, \quad i,j = \overline{1,n}, \ r,l = \overline{1,m}.$$

Moreover, the family $\{v_i, w_r, \ i = \overline{1,n}, \ r = \overline{1,m}\}$ generates $\mathcal{E}_{n,m}^0$.

Proof. We need to prove only last statement of the theorem. To do so let us note that

$$v_{n+r} = \widetilde{w}_r(\mathbf{1} - |q|^2\widetilde{Q})^{-\frac{1}{2}} = (\mathbf{1} - \widetilde{Q})w_r(\mathbf{1} - |q|^2\widetilde{Q})^{-\frac{1}{2}} \in C^*(w_r, v_i,\ i = \overline{1,n}).$$

Hence v_i, w_r, $i = \overline{1,n}$, $r = \overline{1,m}$, generate $\mathcal{E}^0_{n,m}$. □

Corollary 2. *The statement of Theorem 3 and universal property of $\mathcal{E}^q_{n,m}$ imply the existence of surjective homomorphism $\psi\colon \mathcal{E}^q_{n,m} \to \mathcal{E}^0_{n,m}$ defined by*

$$\psi(s_i) = v_i, \quad \psi(t_r) = w_r, \quad i = \overline{1,n},\ r = \overline{1,m}.$$

Now we are ready to formulate the main result of this section.

Theorem 4. *For any $q \in \mathbb{C}$, $|q| < 1$ one has an isomorphism $\mathcal{E}^q_{n,m} \simeq \mathcal{E}^0_{n,m}$.*

Proof. By Theorem 2 we have the surjective homomorphism $\varphi\colon \mathcal{E}^0_{n,m} \to \mathcal{E}^q_{n,m}$ defined by

$$\varphi(v_i) = s_i, \quad \varphi(v_{n+r}) = \widehat{t}_r, \quad i = \overline{1,n},\ r = \overline{1,m}.$$

Show that $\psi\colon \mathcal{E}^q_{n,m} \to \mathcal{E}^0_{n,m}$ constructed in Corollary 2 is inverse for φ. Indeed $\psi(s_i) = v_i$, $i = \overline{1,n}$, imply that

$$\psi(\mathbf{1} - Q) = \mathbf{1} - \widetilde{Q}.$$

Then as $\psi(t_r) = w_r$ we get

$$\psi(\widetilde{t}_r) = \psi((\mathbf{1} - Q)t_r) = (\mathbf{1} - \widetilde{Q})w_r = \widetilde{w}_r, \quad r = \overline{1,m}$$

and

$$\psi(\widehat{t}_r) = \psi(\widetilde{t}_r(\mathbf{1} - |q|^2Q)^{-\frac{1}{2}}) = \widetilde{w}_r(\mathbf{1} - |q|^2\widetilde{Q})^{-\frac{1}{2}} = v_{n+r}, \quad r = \overline{1,m}.$$

So, $\psi\varphi(v_i) = \psi(s_i) = v_i$, $\psi\varphi(v_{n+r}) = \psi(\widehat{t}_r) = v_{n+r}$, $i = \overline{1,n}$, $r = \overline{1,m}$ and

$$\psi\varphi = id_{\mathcal{E}^0_{n,m}}.$$

Show that $\varphi\psi = id_{\mathcal{E}^q_{n,m}}$. Indeed

$$\varphi(\widetilde{w}_r) = \varphi(v_{n+r}(\mathbf{1} - |q|^2\widetilde{Q})^{\frac{1}{2}}) = \widehat{t}_r(\mathbf{1} - |q|^2Q)^{\frac{1}{2}} = \widetilde{t}_r, \quad r = \overline{1,d}.$$

Then for any $r = \overline{1,m}$ one has

$$\varphi(w_r) = \sum_{k=0}\sum_{|\mu|=k} q^k\varphi(v_\mu)\varphi(\widetilde{w}_r)\varphi v^*_\mu = \sum_{k=0}\sum_{|\mu|=k} q^k s_\mu\widetilde{t}_r s^*_\mu = t_r.$$

So, $\varphi\psi(s_i) = \varphi(v_i) = s_i$, $\varphi\psi(t_r) = \varphi(w_r) = t_r$, $i = \overline{1,n},$, $r = \overline{1,m}$. □

3 The case $|q| = 1$

In this section, we discuss the case $|q| = 1$. Notice that for $|q| = 1$, the relations in $\mathcal{E}^q_{n,m}$ imply that $t_j s_i = q s_i t_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Indeed, for $B_{ij} = t_j s_i - q s_i t_j$ we have directly $B^*_{ij} B_{ij} = 0$.

3.1 Preliminaries

In this part we collect some general facts about C^*-dynamical systems, crossed products and Rieffel deformations which we will need in our considerations.

3.1.1 Fixed point subalgebras

First we recall how properties of a fixed point subalgebra of a C^*-algebra with an action of a compact group are related to properties of the whole algebra.

Definition 2. Let A be a C^*-algebra with an action γ of a compact group G. A fixed point subalgebra A^γ is a subset of all $a \in A$ such that $\gamma_g(a) = a$ for all $g \in G$.

Notice that for every action of a compact group G on a C^*-algebra A one can construct a faithful conditional expectation $E_\gamma : A \to A^\gamma$ onto the fixed point subalgebra, given by

$$E_\gamma(a) = \int_G \gamma_g(a) d\lambda,$$

where λ is the Haar measure on G.

A homomorphism $\varphi : A \to B$ between C^*-algebras with actions α and β of a compact group G is called equivariant when

$$\varphi \circ \alpha_g = \beta_g \circ \varphi \text{ for any } g \in G.$$

Proposition 2. *([17], Section 4.5, Theorem 1, 2)*

1. *Let γ be an action of a compact group G on a C^*-algebra A. Then A is nuclear if and only if A^γ is nuclear.*

2. *Let $\varphi : A \to B$ be an equivariant $*$-homomorphism. Then φ is injective on A if and only if φ is injective on A^α.*

3.1.2 Crossed products

Recall that given a locally compact group G and a C^*-algebra A with a G-action α, one has two natural inclusions

$$i_A : A \to M(A \rtimes_\alpha G),\ i_G : G \to M(A \rtimes_\alpha G),$$

$$(i_A(a)f)(s) = af(s), \quad (i_G(t)f)(s) = \alpha_t(f(t^{-1}s)), \quad t, s \in G,\ a \in A,$$

for $f \in C_c(G, A)$.

Remark 1. Obviously, $i_G(s)$ is a unitary element of $M(A \rtimes_\alpha G)$ for any $s \in G$. Recall that i_G determines the homomorphism denoted also by i_G

$$i_G \colon C^*(G) \to M(A \rtimes_\alpha G)$$

defined by

$$i_G(f) = \int_G f(s) i_G(s) d\lambda(s),$$

where λ is the left Haar measure on G.

Notice that for any $g \in C_c(G, A)$ one has

$$(i_G(f)g)(t) = f \cdot_\alpha g,$$

where $\cdot_\alpha$ denotes the product in $A \rtimes_\alpha G$. In particular, when A is unital we can identify $i_G(f)$ with $f \cdot_\alpha \mathbf{1}_A$ and in fact i_G maps $C^*(G)$ into $A \rtimes_\alpha G$. Also notice that

$$i_G(t) i_A(a) i_G(t)^{-1} = i_A(\alpha_t(a)) \in M(A \rtimes_\alpha G).$$

If φ is an equivariant homomorphism between C^*-algebras A with a G-action α and B with a G-action β, then one can define the homomorphism

$$\varphi \rtimes G : A \rtimes_\alpha G \to B \rtimes_\beta G,\ (\varphi \rtimes G)(f)(t) = \varphi(f(t)), \quad f \in C_c(G, A).$$

Let A be unital C^*-algebra with G-action α. Then $\iota_A \colon \mathbb{C} \to A$,

$$\iota_A(\lambda) = \lambda \mathbf{1}_A,$$

is an equivariant homomorphism, where G acts trivially on $\mathbb{C}$. As $\mathbb{C} \rtimes G = C^*(G)$ one has that

$$\iota_A \rtimes G \colon C^*(G) \to A \rtimes_\alpha G.$$

In fact in this case

$$\iota_A \rtimes G = i_G, \tag{2}$$

where $i_G\colon C^*(G) \to A \rtimes_\alpha G$ is described in Remark 1 . Indeed, for any $g \in G_c(G, A)$ one has

$$\begin{aligned}(i_G(f) \cdot_\alpha g)(s) &= \int_G f(t)\alpha_t(g(t^{-1}s))dt = \\ &= \int_G f(t)1_A\alpha_t(g(t^{-1}s))dt = \\ &= ((f(\cdot)1_A) \cdot_\alpha g)(s) = ((\iota_A \rtimes G)(f) \cdot_\alpha g)(s)\end{aligned}$$

implying $i_G(f) = (\iota_A \rtimes G)(f)$ for any $f \in C^*(G)$.

3.1.3 Rieffel's deformation

Below, we recall some basic facts on Rieffel's deformations (see [6] for the details). Given a C^*-algebra A equipped with an action α of $\mathbb{R}^n$, fix a skew symmetric matrix $\Theta \in M_n(\mathbb{R})$ and consider dense subset A_∞ of elements $a \in A$ such that $x \mapsto \alpha_x(a) \in C^\infty(\mathbb{R}^n, A)$, see [18]. For any $a, b \in A_\infty$, define their deformed product as the following oscillatory integral (see [6]):

$$a \cdot_\Theta b := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha_{\Theta(x)}(a)\alpha_y(b)e^{2\pi i\langle x,y\rangle}dxdy$$

(here, $\langle\cdot,\cdot\rangle$ is a scalar product in $\mathbb{R}^n$). For $a \in A_\infty$ define the new involution which coincides with the old one. Then take a closure with respect to special C^*-norm, see [18] for details and obtain the C^*-algebra A_Θ called the Rieffel deformation of A.

Here we will be interested in periodic actions of $\mathbb{R}^n$, i.e. α is an action of $\mathbb{T}^n$. Given a character $\chi \in \widehat{\mathbb{T}}^n \simeq \mathbb{Z}^n$, consider

$$A_\chi = \{a \in A : \alpha_z(a) = \chi(z)a \text{ for every } z \in \mathbb{T}^n\}.$$

Then

$$A = \overline{\bigoplus_{\chi\in\mathbb{Z}^n} A_\chi},$$

where some terms could be equal to zero and $A_{\chi_1} \cdot A_{\chi_2} \subset A_{\chi_1+\chi_2}$, $A_\chi^* = A_{-\chi}$. So, A_χ, $\chi \in \mathbb{Z}^n$, can be treated as homogeneous components of $\mathbb{Z}^n$-grading on

A. Moreover to define an action of $\mathbb{T}^n$ on A is equivalent to equipping it with a $\mathbb{Z}^n$-grading so that $A_pA_q \subset A_{p+q}$ and $A_p^* \subset A_{-p}$ for $p, q \in \mathbb{Z}^n$. For $a \in A_p$ we have $\alpha_t(a) = e^{2\pi i \langle t,p\rangle} a$ (see, e.g., [26]). In particular all homogeneous elements belongs to A_∞.

For the action of $\mathbb{T}^n$, one has an explicit formula for the deformed product of homogeneous elements [8, 3].

Theorem 5. *Suppose A is a C^*-algebra with a $\mathbb{T}^n$-action. Assume that $a \in A_p$, $b \in A_q$. Then*

$$a \cdot_\Theta b = e^{2\pi i \langle \Theta(p), q\rangle} a \cdot b.$$

Proof.

$$\begin{aligned} a \cdot_\Theta b &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i \langle \Theta(x), p\rangle} a e^{2\pi i \langle y,q\rangle} b e^{2\pi i \langle x,y\rangle} dxdy = \\ &= a \cdot b \int_{\mathbb{R}^n} e^{2\pi i \langle y,q\rangle} \int_{\mathbb{R}^n} e^{2\pi i \langle x, -\Theta(p)\rangle} e^{2\pi i \langle x,y\rangle} dxdy = \\ &= a \cdot b \int_{\mathbb{R}^n} e^{2\pi i \langle y,q\rangle} \delta_{y-\Theta(p)} dy = \\ &= e^{2\pi i \langle \Theta(p), q\rangle} a \cdot b. \end{aligned}$$

□

Remark 2. Notice that A_Θ also possesses $\mathbb{Z}^n$-grading such that $(A_\Theta)_p = A_p$ for every $p \in \mathbb{Z}^n$. Notice that due to the result above $a \cdot_\Theta b = a \cdot b$ for any $a, b \in A_{\pm p}$, $p \in \mathbb{Z}^n$. Indeed for any skew symmetric $\Theta \in M_n(\mathbb{R}^n)$ and $p \in \mathbb{Z}^n$ one has $\langle \Theta\, p\,, \pm p\rangle = 0$. The involution on $(A_\Theta)_p$ coincides with the involution on A_p.

Consider a C^* dynamical system $(A, \mathbb{T}^n, \alpha)$ and its covariant representation (π, U) on a Hilbert space $\mathcal{H}$. For any $p \in \mathbb{Z}^n \simeq \widehat{\mathbb{T}}^n$ put

$$\mathcal{H}_p = \{h \in \mathcal{H} \mid U_t h = e^{2\pi i \langle t,p\rangle} h\}.$$

Then $\mathcal{H} = \bigoplus_{p\in\mathbb{Z}^n} \mathcal{H}_p$ (see [26]).

Proposition 3 ([9], Theorem 2.8)**.** *Let (π, U) be a covariant representation of $(A, \mathbb{T}^n, \alpha)$ on a Hilbert space $\mathcal{H}$. Then one can define a representation π_Θ of A_Θ as follows:*

$$\pi_\Theta(a)\xi = e^{2\pi i \langle \Theta(p), q\rangle} \pi(a)\xi,$$

for every $\xi \in \mathcal{H}_q$, $a \in A_p$, $p, q \in \mathbb{Z}^n$. Moreover, π_Θ is faithful if and only if π is faithful.

It is known that Rieffel's deformation can be embedded into $M(A \rtimes_\alpha \mathbb{R}^n)$, but for the periodic actions we have an explicit description of this embedding.

Proposition 4 ([3], Lemma 3.1.1)**.** *The following mapping defines an embedding*

$$i_{A_\Theta} : A_\Theta \to M(A \rtimes_\alpha \mathbb{R}^n),\ i_{A_\Theta}(a_p) = i_A(a_p) i_{\mathbb{R}^n}(-\Theta(p)),$$

where $p \in \mathbb{Z}^n$ and a_p is homogeneous of degree p.

Proposition 5 ([8], Proposition 3.2 and [3], Section 3.1)**.** *Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system with periodic α and unital A. Put A_Θ to be the Rieffel deformation of A. There exist a periodic action α^Θ of $\mathbb{R}^n$ on A_Θ and an isomorphism $\Psi : A_\Theta \rtimes_{\alpha^\Theta} \mathbb{R}^n \to A \rtimes_\alpha \mathbb{R}^n$ such that the following diagram commutes*

$$\begin{tikzcd}
 & C^*(\mathbb{R}^n) \simeq C_0(\mathbb{R}^n) \arrow[dl, "i_{\mathbb{R}^n}"'] \arrow[dr, "i_{\mathbb{R}^n}"] & \\
A_\Theta \rtimes_{\alpha^\Theta} \mathbb{R}^n \arrow[rr, "\Psi"] & & A \rtimes_\alpha \mathbb{R}^n
\end{tikzcd}$$

Namely, $\alpha^\Theta(a) = \alpha(a)$ for any $a \in A_p$, $p \in \mathbb{Z}^n$. It is easy to verify that $i_{A_\Theta} \colon A_\Theta \to M(A \rtimes_\alpha \mathbb{R}^n)$ with $i_{\mathbb{R}^n} \colon \mathbb{R}^n \to M(A \rtimes_\alpha \mathbb{R}^n)$ determine a covariant representation of $(A_\Theta, \mathbb{R}^n, \alpha^\Theta)$ in $M(A \rtimes_\alpha \mathbb{R}^n)$. Hence, by the universal property of crossed product we get the corresponding homomorphism

$$\Psi \colon A_\Theta \rtimes_{\alpha^\Theta} \mathbb{R}^n \to M(A \rtimes_\alpha \mathbb{R}^n).$$

In fact, the range of Ψ coincides with $A \rtimes_\alpha \mathbb{R}^n$ and Ψ defines an isomorphism

$$\Psi \colon A_\Theta \rtimes_{\alpha^\Theta} \mathbb{R}^n \to A \rtimes_\alpha \mathbb{R}^n, \tag{3}$$

see [8], [3] for more detailed considerations.

The following propositions shows that Rieffel's deformation inherits properties of the non-deformed counterpart.

Proposition 6 ([8], Theorem 3.10)**.** *A C^*-algebra A_Θ is nuclear if and only if A is nuclear.*

Proposition 7 ([8], Theorem 3.13)**.** *For a C^*-algebra A one has*

$$K_0(A_\Theta) = K_0(A) \text{ and } K_1(A_\Theta) = K_1(A).$$

3.1.4 Rieffel's deformation of tensor product

In this part we apply Rieffel's deformation procedure to a tensor product of two nuclear unital C^*-algebras equipped with an action of $\mathbb{T}$.

Let A, B be C^*-algebras with actions α and β of $\mathbb{T}$. Then there is a natural action $\alpha \otimes \beta$ of $\mathbb{T}^2$ on $A \otimes B$ defined as

$$(\alpha \otimes \beta)_{\varphi_1,\varphi_2}(a \otimes b) = \alpha_{\varphi_1}(a) \otimes \beta_{\varphi_2}(b).$$

Consider induced gradings on A and B:

$$A = \bigoplus_{p_1 \in \mathbb{Z}} A_{p_1}, \ B = \bigoplus_{p_2 \in \mathbb{Z}} B_{p_2}.$$

Then the corresponding grading on $A \otimes B$ is

$$A \otimes B := \bigoplus_{(p_1,p_2)^t \in \mathbb{Z}^2} A_{p_1} \otimes B_{p_2}.$$

In particular, $a \otimes 1 \in (A \otimes B)_{(p_1,0)^t}$ and $1 \otimes b \in (A \otimes B)_{(0,p_2)^t}$, where $a \in A_{p_1}$ and $b \in B_{p_2}$.

Given $q = e^{2\pi i \varphi_0}$ consider

$$\Theta_q = \begin{pmatrix} 0 & \frac{\varphi_0}{2} \\ -\frac{\varphi_0}{2} & 0 \end{pmatrix}. \tag{4}$$

Thus we can construct the Rieffel's deformation $(A \otimes B)_{\Theta_q}$.

Proposition 8. *One has the following homomorphisms*

$$\eta_A : A \to (A \otimes B)_{\Theta_q}, \ \eta_A(a) = a \otimes 1,$$

$$\eta_B : B \to (A \otimes B)_{\Theta_q}, \ \eta_B(b) = 1 \otimes b,$$

such that for $a \in A_{p_1}$ and $b \in B_{p_2}$ it holds

$$\eta_B(b) \cdot_{\Theta_q} \eta_A(a) = e^{2\pi i p_1 p_2 \varphi_0} \eta_A(a) \cdot_{\Theta_q} \eta_B(b).$$

Proof. Recall that $\mathbb{Z}^2$-homogeneous components of $A \otimes B$ and $(A \otimes B)_{\Theta_q}$ coincide and will be consider the same. Let $e_1 = (1,0)^t$, $e_2 = (0,1)^t$.

Given $a \in A_p$ we have

$$\eta_A(a) = a \otimes 1 \in ((A \otimes B)_{\Theta_q})_{p\,e_1}$$

implying that

$$\eta_A(a)^* = a^* \otimes 1 \in ((A \otimes B)_{\Theta_q})_{-p\,e_1}.$$

Let $a_1 \in A_{p_1}$ and $a_2 \in A_{p_2}$. Then

$$\eta_A(a_1) \cdot_{\Theta_q} \eta_A(a_2) = e^{2\pi i \langle p_1 \Theta_q(e_1), p_2 e_1\rangle}(a_1 \otimes 1)(a_2 \otimes 1) = a_1 a_2 \otimes 1 = \eta_A(a_1 a_2).$$

Thus η_A is a homomorphism. The arguments for η_B are the same.

Given $a \in A_{p_1}$ and $b \in B_{p_2}$ one has

$$\eta_A(a) \cdot_{\Theta_q} \eta_B(b) = e^{2\pi i \langle \Theta_q(p_1 e_1), p_2 e_2\rangle}(a \otimes 1)(1 \otimes b) = e^{-\pi i p_1 p_2 \varphi_0} a \otimes b,$$

$$\eta_B(b) \cdot_{\Theta_q} \eta_A(a) = e^{2\pi i \langle \Theta_q(p_2 e_2), p_1 e_1\rangle}(a \otimes 1)(1 \otimes b) = e^{\pi i p_1 p_2 \varphi_0} a \otimes b,$$

implying

$$\eta_B(b) \cdot_{\Theta_q} \eta_A(a) = e^{2\pi i p_1 p_2 \varphi_0} \eta_A(a) \cdot_{\Theta_q} \eta_B(b).$$

□

3.2 Fock representation of $\mathcal{E}_{n,m}^q$

In this part we show that the Fock representation of $\mathcal{E}_{n,m}^q$ is faithful and apply this result to state that $\mathcal{E}_{n,m}^q$ is isomorphic to the Rieffel deformation $(\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}$, where Θ_q is specified in (4).

Definition 3. The Fock representation of $\mathcal{E}_{n,m}^q$ is the unique up to unitary equivalence irreducible $*$-representation π_F^q determined by the action on vacuum vector Ω, $||\Omega|| = 1$

$$\pi_F^q(s_j^*)\Omega = 0,\ \pi_F^q(t_r^*)\Omega = 0, \quad j = \overline{1,n}, \quad r = \overline{1,m}.$$

Denote by $\pi_{F,n}$ the Fock representation of $\mathcal{O}_n^{(0)} \subset \mathcal{E}_{n,m}^q$ acting on the space

$$\mathcal{F}_n = \mathcal{T}(\mathcal{H}_n) = \mathbb{C}\Omega \oplus \bigoplus_{d=1}^{\infty} \mathcal{H}_n^{\otimes d}, \quad \mathcal{H}_n = \mathbb{C}^n$$

described by formulas

$$\begin{aligned}
&\pi_{F,n}(s_j)\Omega = e_j, \quad \pi_{F,n}(s_j) e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d} = e_j \otimes e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d},\\
&\pi_{F,n}(s_j^*)\Omega = 0, \quad \pi_{F,n}(s_j^*) e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} = \delta_{j i_1} e_{i_2} \otimes \cdots \otimes e_{i_d}, \quad d \in \mathbb{N},
\end{aligned}$$

where $e_1, \ldots, e_n$ is the standard orthonormal basis of $\mathcal{H}_n$. Notice that $\pi_{F,n}$ is the unique irreducible faithful representations of $\mathcal{O}_n^{(0)}$, see for example [15].

Recall that the Fock representation of $\mathcal{E}_{n,m}^q$ exists for any $q \in \mathbb{C}$, $|q| \leq 1$. For $|q| = 1$ one has $||T|| = 1$ and the kernel of the Fock representation of the Wick algebra $WE_{n,m}^q$ coincides with the $*$-ideal $\mathcal{I}_2$ generated by $\ker(\mathbf{1} + T)$, see Introduction. In our case

$$\mathcal{I}_2 = \langle t_r s_j - q s_j t_r,\ j = \overline{1,n},\ r = \overline{1,n} \rangle.$$

Denote by $E_{n,m}^q$ the quotient $WE_{n,m}^q / \mathcal{I}_2$. Obviously, $\mathcal{E}_{n,m}^q = C^*(E_{n,m}^q)$. So one has the following corollary of Theorem 1

Proposition 9. *The Fock representation of $\mathcal{E}_{n,m}^q$ exists and is faithful on the $*$-subalgebra $E_{n,m}^q \subset \mathcal{E}_{n,m}^q$.*

Below we give an explicit formula for $\pi_F(s_j)$, $\pi_F(t_r)$. Consider the Fock representations $\pi_{F,n}$ and $\pi_{F,m}$ of $*$-subalgebras $C^*(\{s_1, \ldots, s_n\}) = \mathcal{O}_n^{(0)} \subset \mathcal{E}_{n,m}^q$ and $C^*(\{t_1, \ldots, t_m\}) = \mathcal{O}_m^{(0)} \subset E_{n,m}^q$ resp. Denote by $\Omega_n \in \mathcal{F}_n$ and $\Omega_m \in \mathcal{F}_m$ the corresponding vacuum vectors.

Theorem 6. *The Fock representation π_F^q of $\mathcal{E}_{n,m}^q$ acts on the space $\mathcal{F} = \mathcal{F}_n \otimes \mathcal{F}_m$ as follows*

$$\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes d_m(q^{-\frac{1}{2}}),\ j = \overline{1,n},$$
$$\pi_F^q(t_r) = d_n(q^{\frac{1}{2}}) \otimes \pi_{F,m}(t_r),\ r = \overline{1,m},$$

where $d_k(\lambda)$ acts on $\mathcal{F}_k$, $k = n, m$ by

$$d_k(\lambda)\Omega_k = \Omega_k, \quad d_k(\lambda)X = \lambda^l X,\ X \in \mathcal{H}_k^{\otimes l},\ l \in \mathbb{N}.$$

Proof. It is a direct calculation to verify that the operators defined above satisfy the relations of $\mathcal{E}_{n,m}^q$. Since $\pi_{F,k}$ is irreducible on $\mathcal{F}_k$, $k = m, n$, the representation π_F^q is irreducible on $\mathcal{F}_n \otimes \mathcal{F}_m$. Finally put $\Omega = \Omega_n \otimes \Omega_m$, then obviously

$$\pi_F^q(s_j^*)\Omega = 0, \text{ and } \pi_F^q(t_r^*)\Omega = 0, \quad j = \overline{1,n}, r = \overline{1,m}$$

Thus π_F^q is the Fock representation of $\mathcal{E}_{n,m}^q$. $\square$

Remark 3. In some cases it will be more convenient to present operators of Fock representation of $\mathcal{E}_{n,m}^q$ in the form

$$\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{F}_m},\ j = \overline{1,n},$$
$$\pi_F^q(t_r) = d_n(q) \otimes \pi_{F,m}(t_r),\ r = \overline{1,m},$$

or

$$\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes d_m(q^{-1}),\ j = \overline{1,n},$$
$$\pi_F^q(t_r) = \mathbf{1}_{\mathcal{F}_n} \otimes \pi_{F,m}(t_r),\ r = \overline{1,m},$$

which are obviously unitary equivalent to the one presented in the statement above.

Consider the action α of $\mathbb{T}^2$ on $\mathcal{E}_{n,m}^q$

$$\alpha_{\varphi_1,\varphi_2}(s_i) = e^{2\pi i\varphi_1}s_i,\ \alpha_{\varphi_1,\varphi_2}(t_r) = e^{2\pi i\varphi_2}t_r.$$

Recall, see Section 3.1.1, that the conditional expectation, associated to α is denoted by E_α.

Proposition 10. *The fixed point C^*-subalgebra $(\mathcal{E}_{n,m}^q)^\alpha$ is an AF-algebra.*

Proof. The family $\{s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*,\ \mu_1,\nu_1 \in \Lambda_n,\ \mu_2,\nu_2 \in \Lambda_m\}$ is dense in $\mathcal{E}_{n,m}^q$, thus the family $\{E_\alpha(s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*),\ \mu_1,\nu_1 \in \Lambda_n,\ \mu_2,\nu_2 \in \Lambda_m\}$ is dense in $(\mathcal{E}_{n,m}^q)^\alpha$. Further,

$$E_\alpha(s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*) = 0,\ \text{if } |\mu_1| \neq |\nu_1| \text{ or } |\mu_2| \neq |\nu_2|$$

and $E_\alpha(s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*) = s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*$ otherwise. Hence

$$(\mathcal{E}_{n,m}^q)^\alpha = c.l.s.\{s_{\mu_1}s_{\nu_1}^*t_{\mu_2}t_{\nu_2}^*,\ |\mu_1| = |\nu_1|, |\mu_2| = |\nu_2|,\ \mu_1,\nu_1 \in \Lambda_n,\ \mu_2,\nu_2 \in \Lambda_m\}$$

Put $\mathcal{A}_{1,0}^0 = \mathbb{C}$,

$$\mathcal{A}_{1,0}^{k_1} = c.l.s.\{s_{\mu_1}s_{\nu_1}^*, |\mu_1| = |\nu_1| = k_1,\ \mu_1,\nu_1 \in \Lambda_n\},\ k_1 \in \mathbb{N},$$

and $\mathcal{A}_{2,0}^0 = \mathbb{C}$,

$$\mathcal{A}_{2,0}^{k_2} = c.l.s.\{t_{\mu_2}t_{\nu_2}^*, |\mu_2| = |\nu_2| = k_2,\ \mu_1,\nu_1 \in \Lambda_n\},\ k_2 \in \mathbb{N}.$$

It is easy to see that $xy = yx$, $x \in \mathcal{A}_{1,0}^{k_1}$, $y \in \mathcal{A}_{2,0}^{k_2}$. Let

$$\mathcal{A}_k^{\alpha} = \sum_{k_1+k_2=k} \mathcal{A}_{1,0}^{k_1} \cdot \mathcal{A}_{2,0}^{k_2}.$$

Evidently $\mathcal{A}_k^{\alpha}$ is finite-dimensional for any $k \in \mathbb{Z}_+$ and

$$(\mathcal{E}_{n,m}^q)^{\alpha} = \overline{\bigcup_{k\in\mathbb{Z}_+} \mathcal{A}_k^{\alpha}}. \qquad \square$$

Remark 4. Define unitary operators U_{φ_1,φ_2}, $(\varphi_1,\varphi_2) \in \mathbb{T}^2$ on $\mathcal{F}_n \otimes \mathcal{F}_m$ as follows:

$$U_{\varphi_1,\varphi_2} = d_n(e^{2\pi i\varphi_1}) \otimes d_m(e^{2\pi i\varphi_2}).$$

Then $(\pi_F^q, U_{\varphi_1,\varphi_2})$ is a covariant representation of $(\mathcal{E}_{n,m}^q, \mathbb{T}^2, \alpha)$.

Theorem 7. *The Fock representation π_F^q of $\mathcal{E}_{m,n}^q$ is faithful.*

Proof. Consider the action α^{π} of $\mathbb{T}^2$ on $\pi_F^q(\mathcal{E}_{n,m}^q)$ induced the action α on $\mathcal{E}_{n,m}^q$

$$\alpha_{\varphi_1,\varphi_2}^{\pi} \pi_F^q(s_j) = e^{2\pi i\varphi_1}\pi_F^q(s_j) := S_{j,\varphi_1,\varphi_2},$$
$$\alpha_{\varphi_1,\varphi_2}^{\pi} \pi_F^q(t_r) = e^{2\pi i\varphi_2}\pi_F^q(t_r) := T_{r,\varphi_1,\varphi_2}.$$

To see that $\alpha_{\varphi_1,\varphi_2}^q$ is an automorphism of $\pi_F^q(\mathcal{E}_{n,m}^q)$ we notice that the operators $S_{j,\varphi_1,\varphi_2}$, $T_{r,\varphi_1,\varphi_2}$, satisfy the defining relations in $\mathcal{E}_{n,m}^q$ and

$$S_{j,\varphi_1,\varphi_2}^*\Omega = T_{r,\varphi_1,\varphi_2}^*\Omega = 0.$$

Evidently the family $\{S_{j,\varphi_1,\varphi_2}, S_{j,\varphi_1,\varphi_2}^*, T_{r,\varphi_1,\varphi_2}, T_{r,\varphi_1,\varphi_2}^*\}_{j=\overline{1,n}}^{r=\overline{1,m}}$ is irreducible and therefore defines the Fock representation of $\mathcal{E}_{n,m}^q$. Thus, by the uniqueness of the Fock representation, there exists a unitary V_{φ_1,φ_2} on $\mathcal{F}$ such that for any $j = \overline{1,n}$, $r = \overline{1,m}$ one has

$$S_{j,\varphi_1,\varphi_2} = \mathsf{Ad}(V_{\varphi_1,\varphi_2}) \circ \pi_F^q(S_j), \quad T_{r,\varphi_1,\varphi_2} = \mathsf{Ad}(V_{\varphi_1,\varphi_2}) \circ \pi_F^q(T_r)$$

implying that $\alpha_{\varphi_1,\varphi_2}^{\pi}$ is an automorphism of $\pi_F^q(\mathcal{E}_{n,m}^q)$ for any $(\varphi_1,\varphi_2) \in \mathbb{T}^2$. Evidently,

$$\pi_F^q \colon \mathcal{E}_{n,m}^q \to \pi_F^q(\mathcal{E}_{n,m}^q)$$

is equivariant w.r.t. α and α^{π}.

By Proposition 2 the representation π_F^q is faithful on $\mathcal{E}_{n,m}^q$ if and only if it is faithful on $(\mathcal{E}_{n,m}^q)^\alpha$. Further, by Proposition 10,

$$(\mathcal{E}_{n,m}^q)^\alpha = \overline{\bigcup_{k\in\mathbb{Z}_+} \mathcal{A}_k^\alpha}.$$

Evidently $\mathcal{A}_k^\alpha \subset E_{n,m}^q$, $k \in \mathbb{Z}_+$. Hence by Proposition 9, π_F^q is faithful on $\mathcal{A}_k^\alpha$ for any $k \in \mathbb{Z}_+$. Then the statement follows from the fact that a representation of an AF-algebra is injective if and only if it is injective on the finite-dimensional subalgebras. □

The next step is to construct a representation of $(\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}$ corresponding to the Fock representation $\pi_{F,n} \otimes \pi_{F,m}$ of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$.

Notice that $(\pi_{F,n}\otimes\pi_{F,m},\ U_{\varphi_1,\varphi_2})$ determines a covariant representation of $(\mathcal{O}_n^0 \otimes \mathcal{O}_m^0, \alpha, \mathbb{T}^2)$, where as above

$$\alpha_{\varphi_1,\varphi_2}(s_j \otimes \mathbf{1}) = e^{2\pi i\varphi_1}(s_j \otimes \mathbf{1}), \quad \alpha_{\varphi_1,\varphi_2}(\mathbf{1} \otimes t_r) = e^{2\pi i\varphi_2}(\mathbf{1} \otimes t_r).$$

Notice that for $p = (p_1, p_2)^t \in \mathbb{Z}_+^2$ the subspace $\mathcal{H}_n^{\otimes p_1} \otimes \mathcal{H}_m^{\otimes p_2}$ is the $(p_1, p_2)^t$-homogeneous component of $\mathcal{F}$ related to the action of U_{φ_1,φ_2} and $(\mathcal{F})_p = \{0\}$ for any $p \in \mathbb{Z}^2 \setminus \mathbb{Z}_+^2$.

Recall also that $\widehat{s}_j = s_j \otimes \mathbf{1}$ is contained in $e_1 = (1, 0)^t$-homogeneous component and $\widehat{t}_r = \mathbf{1} \otimes t_r$ is in $e_2 = (0, 1)^t$-homogeneous component w.r.t. α. Now one can apply Proposition 3. Namely, given $\xi = \xi_1 \otimes \xi_2 \in \mathcal{H}_n^{\otimes p_1} \otimes \mathcal{H}_m^{\otimes p_2}$ one gets

$$\begin{aligned}(\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q}(\widehat{s}_j)\,\xi &= e^{2\pi i\langle\Theta_q\, e_1,\, p\rangle}\,\pi_{F,n} \otimes \pi_{F,m}(\widehat{s}_j)\,\xi = \\ &= \pi_{F,n}(s_j)\xi_1 \otimes e^{-\pi i\, p_2\, \varphi_0}\,\xi_2 = (\pi_{F,n}(s_j) \otimes d_m(q^{-\frac{1}{2}}))\,\xi,\end{aligned}$$

and

$$\begin{aligned}(\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q}(\widehat{t}_r)\,\xi &= e^{2\pi i\langle\Theta_q\, e_2,\, p\rangle}\,\pi_{F,n} \otimes \pi_{F,m}(\widehat{t}_r)\xi = \\ &= e^{\pi i\, p_1\, \varphi_0}\,\xi_1 \otimes \pi_{F,m}(t_r)\,\xi_2 = (d_n(q^{\frac{1}{2}}) \otimes \pi_{F,m}(t_r))\,\xi.\end{aligned}$$

Notice that for any $j = \overline{1, n}$ and $r = \overline{1, m}$

$$(\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q}(\widehat{s}_j^*)\Omega = 0,\ (\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q}(\widehat{t}_r^*)\Omega = 0.$$

Theorem 8. *For any $q \in \mathbb{C}$, $|q| = 1$, the C^*-algebra $\mathcal{E}_{n,m}^q$ is isomorphic to $(\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}$.*

Proof. Proposition 8 implies that elements $(\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q} \ni \widehat{s}_j = s_j \otimes \mathbf{1}$ and $(\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q} \ni \widehat{t}_r = \mathbf{1} \otimes t_r$ satisfy

$$\widehat{s}_j^* \widehat{s}_i = \delta_{ij} \mathbf{1} \otimes \mathbf{1}, \ \widehat{t}_r^* \widehat{t}_s = \delta_{rs} \mathbf{1} \otimes \mathbf{1}, \ \widehat{t}_r^* \widehat{s}_j = q \widehat{s}_j \widehat{t}_r^*$$

Hence, by universal property one can construct a surjective homomorphism $\Phi \colon \mathcal{E}_{n,m}^q \to (\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}$ defined by

$$\Phi(s_j) = \widehat{s}_j, \quad \Phi(t_r) = \widehat{t}_r, \quad j = \overline{1,n}, \ r = \overline{1,m}.$$

Notice that due to considerations above $\pi_F^q = (\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q} \circ \Phi$. Since π_F is faithful representation of $\mathcal{E}_{n,m}^q$ we deduce that Φ is injective. $\square$

Nuclearity of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ and Proposition 6 immediately imply the following

Corollary 3. *The C^*-algebra $\mathcal{E}_{n,m}^q$ is nuclear for any $q \in \mathbb{C}$, $|q| = 1$.*

The nuclearity of $\mathcal{E}_{n,m}^q$ can also be shown using more explicit arguments. One can use the standard trick of untwisting the q-deformation in the crossed product, which clarifies informally the nature of isomorphism (3). Namely, consider the action α_q of $\mathbb{Z}$ on $\mathcal{E}_{n,m}^q$ defined on the generators as

$$\alpha_q^k(s_j) = e^{\pi i k \varphi_0} s_j, \quad \alpha_q^k(t_r) = e^{-\pi i k \varphi_0} t_r, \quad j = 1, \ldots, n, \ r = 1, \ldots, m, \ k \in \mathbb{Z}.$$

Denote by the same symbol the similar action on $\mathcal{E}_{n,m}^1 \simeq \mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$. Here we denote by $\widetilde{s}_j$ and $\widetilde{t}_r$ the generators of $\mathcal{E}_{n,m}^1$.

Proposition 11. *For any value of $\varphi_0 \in [0,1)$ one has the isomorphism $\mathcal{E}_{n,m}^q \rtimes_{\alpha_q} \mathbb{Z} \simeq \mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$.*

Proof. Recall that $\mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$ is generated as a C^*-algebra by elements $\widetilde{s}_j$, $\widetilde{t}_r$ and a unitary u, such that the following relations satisfied

$$u \widetilde{s}_j u^* = e^{i\pi\varphi_0} \, \widetilde{s}_j, \quad u \widetilde{t}_r u^* = e^{-i\pi\varphi_0} \, \widetilde{t}_r, \quad j = \overline{1,n}, \ r = \overline{1,m}.$$

Put $\widehat{s}_j = \widetilde{s}_j \, u$ and $\widehat{t}_r = \widetilde{t}_r \, u$. Obviously, $\widehat{s}_j$, $\widehat{t}_r$ and u generate $\mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$. Further,

$$\widehat{s}_j^* \widehat{s}_k = \delta_{jk} \mathbf{1}, \quad \widehat{t}_r^* \widehat{t}_l = \delta_{rl} \mathbf{1}$$

and

$$\widehat{s}_j \widehat{t}_r = \widetilde{s}_j u \widetilde{t}_r u = e^{-i\pi\varphi_0} \widetilde{s}_j \widetilde{t}_r u^2 = e^{-i\pi\varphi_0} \widetilde{t}_r \widetilde{s}_j u^2 = e^{-2\pi i \varphi_0} \widetilde{t}_r u \widetilde{s}_j u = \overline{q} \, \widehat{s}_j \widehat{t}_r.$$

In a similar way we get $\widehat{s}_j^*\widehat{t}_r = q\widehat{t}_r\widehat{s}_j^*$, $j=\overline{1,n}$, $r=\overline{1,m}$. Finally

$$u\widehat{s}_j u^* = e^{i\pi\varphi_0}\widehat{s}_j, \quad u\widehat{t}_r u^* = e^{-i\pi\varphi_0}\widehat{t}_r.$$

Hence the correspondence

$$s_j \mapsto \widehat{s}_j, \quad t_j \mapsto \widehat{t}_j, \quad u \mapsto u$$

determines a homomorphism $\Phi_q \colon \mathcal{E}_{n,m}^q \rtimes_{\alpha_q} \mathbb{Z} \to \mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$. The inverse is constructed evidently.

$\square$

Let us show nuclearity of $\mathcal{E}_{n,m}^q$ again. Indeed $\mathcal{E}_{n,m}^1 = \mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ is nuclear. Then so is the crossed product $\mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$. Then due to the isomorphism above $\mathcal{E}_{n,m}^q \rtimes_{\alpha_q} \mathbb{Z}$ is nuclear, implying nuclearity of $\mathcal{E}_{n,m}^q$, see [10].

We finish this part by analog of well-known Wold decomposition theorem for a single isometry.

Theorem 9 (Generalised Wold decomposition). *Let $\pi \colon \mathcal{E}_{n,m}^q \to \mathbb{B}(\mathcal{H})$ be a $*$-representation. Then*

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4,$$

where each $\mathcal{H}_j$, $j=1,2,3,4$, is invariant w.r.t. π, and for $\pi_j = \pi_{|\mathcal{H}_j}$ one has

- *$\mathcal{H}_1 = \mathcal{F} \otimes \mathcal{K}$ for some Hilbert space $\mathcal{K}$ and $\pi_1 = \pi_F^q \otimes \mathbf{1}_{\mathcal{K}}$;*
- *$\pi_2(\mathbf{1}-Q) = 0$, $\pi_2(\mathbf{1}-P) \neq 0$;*
- *$\pi_3(\mathbf{1}-P) = 0$, $\pi_3(\mathbf{1}-Q) \neq 0$;*
- *$\pi_4(\mathbf{1}-Q) = 0$, $\pi_4(\mathbf{1}-P) = 0$;*

where any of $\mathcal{H}_j$, $j=2,3,4$, could be zero.

Proof. We will use the fact that any representation of $\mathcal{O}_n^{(0)}$ is a direct sum of multiple of the Fock representation and a representation of $\mathcal{O}_n$.

So, restrict π to $\mathcal{O}_n^{(0)} \subset \mathcal{E}_{n,m}^q$ and decompose $\mathcal{H} = \mathcal{H}_F \oplus \mathcal{H}_F^\perp$, where

$$\pi(\mathbf{1}-Q)_{|\mathcal{H}_F^\perp} = 0$$

and $\pi(\mathcal{O}_n^0)_{|\mathcal{H}_F}$ is multiple of the Fock representation. Denote

$$S_j := \pi(s_j)_{|\mathcal{H}_F},\ Q := \pi(Q)_{|\mathcal{H}_F}.$$

Since

$$\mathcal{H}_F = \oplus_{\lambda\in\Lambda_n} S_\lambda(\ker Q),$$

it is invariant w.r.t. $\pi(t_r)$, $\pi(t_r^*)$, $r = \overline{1,m}$. Indeed, $t_r Q = Q t_r$ in $\mathcal{E}_{n,m}^q$ implying the invariance of $\ker Q$ w.r.t. $\pi(t_r)$ and $\pi(t_r^*)$. Denote $\ker Q$ by $\mathcal{G}$ and $T_r := \pi(t_r)_{|\mathcal{G}}$. Then

$$\pi(t_r) S_\lambda \xi = q^{|\lambda|} S_\lambda \pi(t_r)\xi = q^{|\lambda|} S_\lambda T_r \xi,\ \xi \in \mathcal{G}.$$

Thus $\mathcal{H}_F \simeq \mathcal{F}_n \otimes \mathcal{G}$ with

$$\pi(s_j)_{|\mathcal{H}_F} = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{G}},\ \pi(t_r)_{|\mathcal{H}_F} = d_n(q) \otimes T_r,\ j = \overline{1,n},\ r = \overline{1,m},$$

where the family $\{T_r\}$ determines $*$-representation $\widetilde{\pi}$ of $\mathcal{O}_m^{(0)}$ on $\mathcal{G}$.

Further decompose $\mathcal{G}$ as $\mathcal{G} = \mathcal{G}_F \oplus \mathcal{G}_F^\perp$ into orthogonal sum of subspaces invariant w.r.t $\widetilde{\pi}$, where $\mathcal{G}_F = \mathcal{F}_m \otimes \mathcal{K}$,

$$\widetilde{\pi}_{\mathcal{G}_F}(t_r) = \pi_{F,m}(t_r) \otimes \mathbf{1}_{\mathcal{K}},\ r = \overline{1,m},\ \text{and } \widetilde{\pi}_{|\mathcal{G}_F^\perp}(\mathbf{1} - P) = 0.$$

Thus $\mathcal{H}_F = (\mathcal{F}_n \otimes \mathcal{F}_m \otimes \mathcal{K}) \oplus \left(\mathcal{F}_n \otimes \mathcal{G}_F^\perp\right)$ and

$$\pi_{\mathcal{H}_F}(s_j) = (\pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{F}_m} \otimes \mathbf{1}_K) \oplus (\pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{G}_F^\perp}),\ j = \overline{1,n},$$
$$\pi_{\mathcal{H}_F}(t_r) = (d_n(q) \otimes \pi_{F,m}(t_r) \otimes \mathbf{1}_K) \oplus \Big(d_n(q) \otimes \widetilde{\pi}_{|\mathcal{G}_F^\perp}(t_r)\Big),\ r = \overline{1,m}.$$

Put $\mathcal{H}_1 = \mathcal{F}_n \otimes \mathcal{F}_m \otimes \mathcal{K} = \mathcal{F} \otimes \mathcal{K}$ and notice that that $\pi_{|\mathcal{H}_1} = \pi_F^q \otimes \mathbf{1}_{\mathcal{K}}$, see 3. Put $\mathcal{H}_3 = \mathcal{F}_n \otimes \mathcal{G}_F^\perp$ and $\pi_3 = \pi_{|\mathcal{H}_3}$ i.e.

$$\pi_3(s_j) = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{G}_F^\perp},\quad \pi_3(t_r) = d_n(q) \otimes \widetilde{\pi}_{|\mathcal{G}_F^\perp}(t_r),\ j = \overline{1,n},\ r = \overline{1,m}.$$

Evidently, $\pi_3(\mathbf{1} - P) = 0$ and $\pi_3(\mathbf{1} - Q) \neq 0$.

Finally, applying considerations above to the invariant subspace $\mathcal{H}_F^\perp$ one can show that there exists a decomposition

$$\mathcal{H}_F^\perp = \mathcal{H}_2 \oplus \mathcal{H}_4$$

into the orthogonal sum of invariant subspaces, where

- $\mathcal{H}_2 = \mathcal{F}_m \otimes \mathcal{L}$ and

  $$\pi_2(s_j) := \pi_{|\mathcal{H}_2}(s_j) = d_m(\overline{q}) \otimes \widehat{\pi}(s_j), \quad \pi_2(t_r) := \pi_{|\mathcal{H}_2}(t_r) = \pi_{F,m}(t_r) \otimes \mathbf{1}_{\mathcal{L}},$$

  for a representation $\widehat{\pi}$ of $\mathcal{O}_n$. Evidently, $\pi_2(\mathbf{1} - Q) = 0$, $\pi_2(\mathbf{1} - P) \neq 0$.

- For $\pi_4 := \pi_{|\mathcal{H}_4}$ one has

  $$\pi_4(\mathbf{1} - Q) = 0, \quad \pi_4(\mathbf{1} - P) = 0.$$

$\square$

3.3 Ideals in $\mathcal{E}_{n,m}^q$

In this part we give a complete description of ideals in $\mathcal{E}_{n,m}^q$ and prove their independence on the deformation parameter q.

For

$$Q = \sum_{j=1}^{n} s_j s_j^*, \quad P = \sum_{r=1}^{m} t_r t_r^*.$$

we consider two-sided ideals $\mathcal{M}_q$ generated by $1 - P$ and $1 - Q$, $\mathcal{J}_1^q$ generated by $1 - Q$, $\mathcal{J}_2^q$ generated by $\mathbf{1} - P$, and $\mathcal{J}_q$ by $(\mathbf{1} - Q)(\mathbf{1} - P)$. Evidently,

$$\mathcal{J}_q = \mathcal{J}_1^q \cap \mathcal{J}_2^q = \mathcal{J}_1^q \cdot \mathcal{J}_2^q.$$

Below we will that any ideal in $\mathcal{E}_{n,m}^q$ coincides with the one listed above.

To clarify the structure of $\mathcal{J}_1^q$, $\mathcal{J}_2^q$ and $\mathcal{J}_q$ we use the construction of twisted tensor product of a certain C^*-algebra with the algebra of compact operators $\mathbb{K}$, see [21]. Let us give a brief review of the construction, adapted to our situation.

Recall that the C^*-algebra $\mathbb{K}$ can be considered as a universal C^*-algebra generated by a closed linear span of elements $e_{\mu\nu}$, $\mu, \nu \in \Lambda_m$ subject to the relations

$$e_{\mu_1\nu_1} e_{\mu_2\nu_2} = \delta_{\mu_2\nu_1} e_{\mu_1\nu_2}, \quad e_{\mu_1\nu_1}^* = e_{\nu_1\mu_1}, \quad \nu_i, \mu_i \in \Lambda_m,$$

here $e_\emptyset := e_{\emptyset\emptyset}$ is the minimal projection. Let A be a C^*-algebra, and

$$\alpha = \{\alpha_\mu,\ \mu \in \Lambda_m\} \subset \mathsf{Aut}(A), \quad \text{where } \alpha_\emptyset = \mathrm{id}_A.$$

Definition 4. Let $e_{\mu\nu}$, $\mu, \nu \in \Lambda_m$ be generators of $\mathbb{K}$ specified above. Construct C^*-algebra

$$B_\alpha = C^*(a \in A, e_{\mu\nu} \in \mathbb{K} \mid \ ae_{\mu\nu} = e_{\mu\nu}\alpha_\nu^{-1}(\alpha_\mu(a)).$$

We define

$$A \otimes_\alpha \mathbb{K} := C^*(ax \mid \ a \in A, \ x \in \mathbb{K}) \subset B_\alpha.$$

Notice that B_α exists for any C^*-algebra A and family $\alpha \subset \mathsf{Aut}(A)$.

Remark 5.
1. Let $x_\mu = e_{\mu\emptyset}$, then $ax_\mu = x_\mu\alpha_\mu(a)$, $ax_\mu^* = x_\mu^*\alpha_\mu^{-1}(a)$, $a \in A$, compare with [21].
2. For any $a \in A$ one has $e_{\mu\nu}a = \alpha_\mu^{-1}(\alpha_\nu(a))e_{\mu\nu}$ implying that

$$(ae_{\mu\nu})^* = \alpha_\mu^{-1}(\alpha_\nu(a))e_{\nu\mu}.$$

3. For any $a_1, a_2 \in A$ one has $(a_1e_{\mu_1\nu_1})(a_2e_{\mu_2\nu_2}) = \delta_{\nu_1\mu_2}a_1\alpha_{\mu_1}^{-1}(\alpha_{\mu_2}(a_2))e_{\mu_1\nu_2}$.

Proposition 12 ([21])**.** *Let A be a C^*-algebra and*

$$\alpha = \{\alpha_\mu, \ \mu \in \Lambda_m\} \subset \mathsf{Aut}(A) \ \textit{with} \ \alpha_\emptyset = \mathrm{id}_A.$$

Then the correspondence

$$ae_{\mu\nu} \mapsto \alpha_\mu(a) \otimes e_{\mu\nu}, \quad a \in A, \ \mu, \nu \in \Lambda_m$$

extends by linearity and continuity to isomorphism

$$\Delta_\alpha \colon A \otimes_\alpha \mathbb{K} \to A \otimes \mathbb{K},$$

where Δ_α^{-1} is constructed via the correspondence

$$a \otimes e_{\mu\nu} \mapsto \alpha_\mu^{-1}(a)e_{\mu\nu}, \quad a \in A, \ \mu, \nu \in \Lambda_m.$$

Remark 6. For $x_\mu = e_{\mu\emptyset}$, $\mu \in \Lambda_m$ one has, compare with [21]

$$\Delta_\alpha(ax_\mu) = \alpha_\mu(a) \otimes x_\mu, \quad \Delta_\alpha(ax_\mu^*) = a \otimes x_\mu^*.$$

Due to universal property of $A\otimes\mathbb{K}$ the isomorphism constructed above allows one to consider $A \otimes_\alpha \mathbb{K}$ as the C^*-algebra having the following universal property. For any C^*-algebra D_α generated by A and elements $g_{\mu\nu}$, $\mu,\nu \in \Lambda_m$ with

$$g_{\mu_1\nu_1}g_{\mu_2\nu_2} = \delta_{\mu_1\nu_2}g_{\mu_1\nu_2}, \quad g^*_{\mu_1\nu_1} = g_{\nu_1\mu_1}, \quad \nu_i,\mu_i \in \Lambda_m,$$

and

$$ag_{\mu\nu} = g_{\mu\nu}\alpha_\nu^{-1}(\alpha_\mu(a)$$

consider a C^*-subalgebra $D\mathbb{K}_\alpha$ generated by pairs $ag_{\mu\nu}$, $a \in A$, $\mu,\nu \in \Lambda_m$. Then there exists a surjective homomorphism

$$\Delta_D\colon A \otimes_\alpha \mathbb{K} \to D\mathbb{K}_\alpha, \quad \text{such that } \Delta_D(ae_{\mu\nu}) = ag_{\mu\nu},\ a \in A,\ \mu,\nu \in \Lambda_m.$$

The following functorial property of $\otimes_\alpha\mathbb{K}$ can be derived easily. Consider

$$\alpha = (\alpha_\mu)_{\mu\in\Lambda_m} \subset \mathsf{Aut}(A),\ \beta = (\beta_\mu)_{\mu\in\Lambda_m} \subset \mathsf{Aut}(B).$$

Suppose $\varphi : A \to B$ is equivariant, i.e. $\varphi(\alpha_\mu(a)) = \beta_\mu(\varphi(a))$ for any $a \in A$ and $\mu \in \Lambda_m$. Then one can define homomorphism

$$\varphi\otimes_\alpha^\beta : A \otimes_\alpha \mathbb{K} \to B \otimes_\beta \mathbb{K},\ \varphi \otimes_\alpha^\beta (ak) = \varphi(a)k,\ a \in A,\ k \in \mathbb{K},$$

making the following diagram commutative

$$\begin{array}{ccc} A \otimes_\alpha \mathbb{K} & \xrightarrow{\varphi\otimes_\alpha^\beta} & B \otimes_\beta \mathbb{K} \\ \downarrow{\scriptstyle\Delta_\alpha} & & \downarrow{\scriptstyle\Delta_\beta} \\ A \otimes \mathbb{K} & \xrightarrow{\varphi\otimes\mathrm{id}_\mathbb{K}} & B \otimes \mathbb{K} \end{array} \tag{5}$$

Namely, it is easy to verify that

$$(\Delta_\beta^{-1} \circ (\varphi \otimes \mathrm{id}_\mathbb{K}) \circ \Delta_\alpha)(ae_{\mu\nu}) = \varphi(a)e_{\mu\nu} = \varphi \otimes_\alpha^\beta (ae_{\mu\nu}), \quad a \in A,\ \mu,\nu \in \Lambda_m.$$

An important consequence of commutativity of the diagram above is exactness of the functor $\otimes_\alpha\mathbb{K}$. Let

$$\beta = (\beta_\mu)_{\mu\in\Lambda_m} \subset \mathsf{Aut}(B),\ \alpha = (\alpha_\mu)_{\mu\in\Lambda_m} \subset \mathsf{Aut}(A),\ \gamma = (\gamma_\mu)_{\mu\in\Lambda_m} \subset \mathsf{Aut}(C)$$

and consider a short exact sequence

$$0 \longrightarrow B \xrightarrow{\varphi_1} A \xrightarrow{\varphi_2} C \longrightarrow 0$$

where φ_1, φ_2 are equivariant homomorphisms. Then the induced sequence

$$0 \longrightarrow B \otimes_\beta \mathbb{K} \xrightarrow{\varphi_1 \otimes_\beta^\alpha} A \otimes_\alpha \mathbb{K} \xrightarrow{\varphi_2 \otimes_\alpha^\gamma} C \otimes_\gamma \mathbb{K} \longrightarrow 0$$

is also short exact.

Now we are ready to study the structure of ideals $\mathcal{J}_1^q, \mathcal{J}_2^q, \mathcal{J}_q \subset \mathcal{E}_{n,m}^q$. We start with $\mathcal{J}_1^q$. Notice that

$$\mathcal{J}_1^q = c.l.s.\{\, t_{\mu_2} t_{\nu_2}^* s_{\mu_1}(\mathbf{1} - Q) s_{\nu_1}^*,\ \mu_1, \nu_1 \in \Lambda_n,\ \mu_2, \nu_2 \in \Lambda_m\}$$

Put $E_{\mu_1\nu_1} = s_{\mu_1}(\mathbf{1} - Q) s_{\nu_1}^*,\ \mu_1, \nu_1 \in \Lambda_n$. Then $E_{\mu_1\nu_1}$ satisfy relations for matrix units generating $\mathbb{K}$. Moreover, $c.l.s.\{E_{\mu\nu},\ \mu, \nu \in \Lambda_n\}$ is an ideal in $\mathcal{O}_n^{(0)}$ isomorphic to $\mathbb{K}$.

Consider the family $\alpha^q = (\alpha_\mu)_{\mu \in \Lambda_n} \subset \mathsf{Aut}(\mathcal{O}_m^{(0)})$ defined as

$$\alpha_\mu(t_r) = q^{|\mu|} t_r, \quad \alpha_\mu(t_r^*) = q^{-|\mu|} t_r^*, \quad \mu \in \Lambda_n,\ r = \overline{1, m}.$$

Proposition 13. *The correspondence* $a e_{\mu\nu} \mapsto a E_{\mu\nu},\ a \in \mathcal{O}_m^{(0)},\ \mu, \nu \in \Lambda_n$, *extends to an isomorphism*

$$\Delta_{q,1} \colon \mathcal{O}_m^{(0)} \otimes_{\alpha^q} \mathbb{K} \to \mathcal{J}_1^q.$$

Proof. Let us stress that for any $\mu_1, \nu_1 \in \Lambda_n$ and $\mu_2, \nu_2 \in \Lambda_m$ one has

$$t_{\mu_2} t_{\nu_2}^* E_{\mu_1\nu_1} = q^{(|\nu_1| - |\mu_1|)(|\mu_2| - |\nu_2|)} E_{\mu_1\nu_1} t_{\mu_2} t_{\nu_2}^* = E_{\mu_1\nu_1} \alpha_{\nu_1}^{-1}(\alpha_{\mu_1}(t_{\mu_2} t_{\nu_2}^*)).$$

Thus, due to the universal property of $\mathcal{O}_m^{(0)} \otimes_{\alpha^q} \mathbb{K}$ the correspondence

$$a e_{\mu\nu} \mapsto a E_{\mu\nu}$$

determines a surjective homomorphism $\Delta_{q,1} \colon \mathcal{O}_m^{(0)} \otimes_{\alpha^q} \mathbb{K} \to \mathcal{J}_1^q$.

It remains to show that $\Delta_{q,1}$ is injective. Since the Fock representation of $\mathcal{E}_{n,m}^q$ is faithful, we can identify $\mathcal{J}_1^q$ with $\pi_F^q(\mathcal{J}_1^q)$. It will be convenient for us to use the following form of the Fock representation, see Remark 3

$$\begin{aligned}
\pi_F^q(s_j) &= \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{F}_m} := S_j \otimes \mathbf{1}_{\mathcal{F}_m},\ j = \overline{1, n}, \\
\pi_F^q(t_r) &= d_n(q) \otimes \pi_{F,m}(t_r) := d_n(q) \otimes T_r,\ r = \overline{1, m}.
\end{aligned}$$

In particular, for any $\mu_1, \nu_1 \in \Lambda_n,\ \mu_2, \nu_2 \in \Lambda_m$

$$\pi_F^q(t_{\mu_2} t_{\nu_2}^* E_{\mu_1\nu_1}) = d_n(q^{|\mu_2| - |\nu_2|}) S_{\mu_1}(\mathbf{1} - Q) S_{\nu_1} \otimes T_{\mu_2} T_{\nu_2}^*.$$

Consider $\Delta_{q,1}\circ\Delta_{\alpha^q}^{-1}\colon \mathcal{O}_m^{(0)}\otimes\mathbb{K}\to\pi_F^q(\mathcal{J}_1^q)$. We intend to show that

$$\Delta_{q,1}\circ\Delta_{\alpha^q}^{-1}=\pi_F^1,$$

where π_F^1 is restriction of the Fock representation of $\mathcal{O}_n^{(0)}\otimes\mathcal{O}_m^{(0)}$ to $\mathbb{K}\otimes\mathcal{O}_m^{(0)}$, where $\mathbb{K}$ is generated by $E_{\mu\nu}$ specified above. Notice that the family

$$\{t_{\mu_2}t_{\nu_2}^*\otimes E_{\mu_1\nu_1},\ \mu_1,\nu_1\in\Lambda_n,\ \mu_2,\nu_2\in\Lambda_m\}$$

generates $\mathcal{O}_m^{(0)}\otimes\mathbb{K}$. Then

$$\Delta_{\alpha^q}^{-1}(t_{\mu_2}t_{\nu_2}^*\otimes E_{\mu_1\nu_1})=\alpha_{\mu_1}^{-1}(t_{\mu_2}t_{\nu_2}^*)e_{\mu_1\nu_1}=q^{-|\mu_1|(|\mu_2|-|\nu_2|)}t_{\mu_2}t_{\nu_2}^*e_{\mu_1\nu_1},$$

and

$$\begin{aligned}\Delta_{q,1}\circ\Delta_{\alpha^q}^{-1}(t_{\mu_2}t_{\nu_2}^*\otimes E_{\mu_1\nu_1})&=q^{-|\mu_1|(|\mu_2|-|\nu_2|)}\pi_F^q(t_{\mu_2}t_{\nu_2}^*E_{\mu_1\nu_1})=\\&=q^{-|\mu_1|(|\mu_2|-|\nu_2|)}d_n(q^{|\mu_2|-|\nu_2|})S_{\mu_1}(\mathbf{1}-Q)S_{\nu_1}^*\otimes T_{\mu_2}T_{\nu_2}^*=\\&=q^{-|\mu_1|(|\mu_2|-|\nu_2|)}q^{|\mu_1|(|\mu_2|-|\nu_2|)}S_{\mu_1}d_n(q^{|\mu_2|-|\nu_2|})(\mathbf{1}-Q)S_{\nu_1}^*\otimes T_{\mu_2}T_{\nu_2}^*=\\&=S_{\mu_1}(\mathbf{1}-Q)S_{\nu_1}^*\otimes T_{\mu_2}T_{\nu_2}^*=\pi_F^1(E_{\mu_1\nu_1}\otimes t_{\mu_2}t_{\nu_2}^*),\end{aligned}$$

where we used relations $d_n(\lambda)S_j=\lambda S_jd_n(\lambda)$ $j=\overline{1,n}$, $\lambda\in\mathbb{C}$ and the obvious fact that

$$d_n(\lambda)(\mathbf{1}-Q)=\mathbf{1}-Q.$$

To complete the proof we recall that π_F^0 is faithful representation of $\mathcal{O}_n^{(0)}\otimes\mathcal{O}_m^0$, so its restriction to $\mathbb{K}\otimes\mathcal{O}_m^{(0)}$ is also faithful, implying the injectivity of Δ_q. ☐

Remark 7. Evidently $\mathcal{J}_q$ is a closed linear span of the family

$$\{\,t_{\mu_2}(1-P)t_{\nu_2}^*s_{\mu_1}(\mathbf{1}-Q)s_{\nu_1}^*,\ \mu_1,\nu_1\in\Lambda_n,\ \mu_2,\nu_2\in\Lambda_m\}\subset\mathcal{J}_1^q$$

Moreover *c.l.s.*$\{t_{\mu_2}(1-P)t_{\nu_2}^*,\ \mu_2,\nu_2\in\Lambda_m\}=\mathbb{K}\subset\mathcal{O}_m^{(0)}$. It is easy to see that

$$\alpha_\mu(t_{\mu_2}(1-P)t_{\nu_2}^*)=q^{|\mu|(|\mu_2|-|\nu_2|)}t_{\mu_2}(1-P)t_{\nu_2}^*,$$

so for every $\alpha_\mu\in\alpha^q$ can be regarded as element of $\mathsf{Aut}(\mathbb{K})$.

The moment reflection and Proposition 13 give the following corollary

Proposition 14. *Restriction of $\Delta_{q,1}$ to $\mathbb{K}\otimes_{\alpha^q}\mathbb{K}\subset\mathcal{O}_m^{(0)}\otimes_{\alpha^q}\mathbb{K}$ gives an isomorphism*

$$\Delta_{q,1}\colon\mathbb{K}\otimes_{\alpha^q}\mathbb{K}\to\mathcal{J}_q.$$

To deal with $\mathcal{J}_2^q$ we define the family $\beta^q = \{\beta_\mu,\ \mu \in \Lambda_m\} \subset \mathsf{Aut}(\mathcal{O}_n^{(0)})$ defined as

$$\beta_\mu(s_j) = q^{-|\mu|}s_j,\ \beta_\mu(s_j^*) = q^{|\mu|}s_j^*,\ j = \overline{1,n}.$$

Proposition 15. *One has an isomorphism* $\Delta_{q,2}\colon \mathcal{O}_n^{(0)} \otimes_{\beta^q} \mathbb{K} \to \mathcal{J}_2^q$.

Obviously, $\Delta_{q,2}$ induces the isomorphism $\mathbb{K} \otimes_{\beta^q} \mathbb{K} \simeq \mathcal{J}_q$, where the first term is ideal in $\mathcal{O}_n^{(0)}$ and the second in $\mathcal{O}_m^{(0)}$ resp.

Denote

$$\varepsilon_n\colon \mathbb{K} \to \mathcal{O}_n^{(0)},\ \varepsilon_m\colon \mathbb{K} \to \mathcal{O}_m^{(0)}$$

to be the canonical embeddings and

$$q_n\colon \mathcal{O}_n^{(0)} \to \mathcal{O}_n,\ q_m\colon \mathcal{O}_m^{(0)} \to \mathcal{O}_m$$

to be the quotient maps. Let also

$$\varepsilon_{q,j}\colon \mathcal{J}_q \to \mathcal{J}_j^q,\ j = 1,2,$$

be the embeddings and

$$\pi_{q,j}\colon \mathcal{J}_j^q \to \mathcal{J}_j^q/\mathcal{J}_q,\ j = 1,2,$$

the quotient maps. Notice also that families $\alpha^q \subset \mathsf{Aut}(\mathcal{O}_m^{(0)})$, $\beta^q \subset \mathsf{Aut}(\mathcal{O}_n^{(0)})$ determine corresponding families of automorphisms of $\mathcal{O}_m$ and $\mathcal{O}_n$ resp. also denoted by α^q and β^q.

Theorem 10. *One has the following isomorphism of extensions*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}_q \arrow[r, "\varepsilon_{q,1}"] \arrow[d, "\Delta_{\alpha^q}\circ\Delta_{q,1}^{-1}"] & \mathcal{J}_1^q \arrow[r, "\pi_{q,1}"] \arrow[d, "\Delta_{\alpha^q}\circ\Delta_{q,1}^{-1}"] & \mathcal{J}_1^q/\mathcal{J}_q \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathbb{K}\otimes\mathbb{K} \arrow[r, "\varepsilon_m\otimes\mathrm{id}_{\mathbb{K}}"] & \mathcal{O}_m^0\otimes\mathbb{K} \arrow[r, "q_m\otimes\mathrm{id}_{\mathbb{K}}"] & \mathcal{O}_m\otimes\mathbb{K} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}_q \arrow[r, "\varepsilon_{q,2}"] \arrow[d, "\Delta_{\beta^q}\circ\Delta_{q,2}^{-1}"] & \mathcal{J}_2^q \arrow[r, "\pi_{q,2}"] \arrow[d, "\Delta_{\beta^q}\circ\Delta_{q,2}^{-1}"] & \mathcal{J}_2^q/\mathcal{J}_q \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathbb{K}\otimes\mathbb{K} \arrow[r, "\varepsilon_n\otimes\mathrm{id}_{\mathbb{K}}"] & \mathcal{O}_n^0\otimes\mathbb{K} \arrow[r, "q_n\otimes\mathrm{id}_{\mathbb{K}}"] & \mathcal{O}_n\otimes\mathbb{K} \arrow[r] & 0
\end{tikzcd}$$

Proof. Indeed, each row in diagram (6) below is exact and every non-dashed vertical arrow is an isomorphism. Bottom left and bottom right squares are commutative due to (5). Top left square is commutative due to consideration in the proof of Proposition 13 combined with Remark 7. Hence there exists unique isomomorphism

$$\Phi_{q,1} \colon \mathcal{I}_1^q/\mathcal{I}_q \to \mathcal{O}_m \otimes_{\alpha^q} \mathbb{K}$$

making the diagram (6) commutative

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_q \arrow[r, "\varepsilon_{q,1}"] \arrow[d, "\Delta_{q,1}^{-1}"] & \mathcal{I}_1^q \arrow[r, "\pi_{q,1}"] \arrow[d, "\Delta_{q,1}^{-1}"] & \mathcal{I}_1^q/\mathcal{I}_q \arrow[r] \arrow[d, dashed, "\Phi_{q,1}"] & 0 \\
0 \arrow[r] & \mathbb{K} \otimes_{\alpha^q} \mathbb{K} \arrow[r, "\varepsilon_m \otimes_{\alpha^q}^{\alpha^q}"] \arrow[d, "\Delta_{\alpha^q}"] & \mathcal{O}_m^0 \otimes_{\alpha^q} \mathbb{K} \arrow[r, "q_m \otimes_{\alpha^q}^{\alpha^q}"] \arrow[d, "\Delta_{\alpha^q}"] & \mathcal{O}_m \otimes_{\alpha^q} \mathbb{K} \arrow[r] \arrow[d, "\Delta_{\alpha^q}"] & 0 \\
0 \arrow[r] & \mathbb{K} \otimes \mathbb{K} \arrow[r, "\varepsilon_m \otimes \mathrm{id}_{\mathbb{K}}"] & \mathcal{O}_m^0 \otimes \mathbb{K} \arrow[r, "q_m \otimes \mathrm{id}_{\mathbb{K}}"] & \mathcal{O}_m \otimes \mathbb{K} \arrow[r] & 0
\end{tikzcd} \tag{6}$$

The proof for $\mathcal{I}_2^q$ is the same. ☐

The following Lemma follows from the fact that $\mathcal{M}_q = \mathcal{I}_1^q + \mathcal{I}_2^q$

Lemma 7.

$$\mathcal{M}_q/\mathcal{I}_q \simeq \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q \simeq \mathcal{O}_m \otimes \mathbb{K} \oplus \mathcal{O}_n \otimes \mathbb{K}.$$

Theorem 10 implies that $\mathcal{I}_q, \mathcal{I}_1^q, \mathcal{I}_2^q$ are stable C^*-algebra. It follows from [29], Proposition 6.12, that extension of a stable C^*-algebra by $\mathbb{K}$ is also stable. Thus the Lemma 7 implies immediately the following important corollary.

Corollary 4. *For any $q \in \mathbb{C}$, $|q| = 1$, the C^*-algebra $\mathcal{M}_q$ is stable.*

Denote the Calkin algebra by Q. Recall that for C^*-algebras A and B the isomorphism

$$\mathsf{Ext}(A \oplus B, \mathbb{K}) \simeq \mathsf{Ext}(A, \mathbb{K}) \oplus \mathsf{Ext}(B, \mathbb{K})$$

is given as follows. Let

$$\iota_1 : A \to A \oplus B,\ \iota_1(a) = (a, 0), \quad \iota_2 : B \to A \oplus B,\ \iota_2(b) = (0, b).$$

For a Busby invariant $\tau : A \oplus B \to Q$ define

$$\mathsf{F} : \mathsf{Ext}(A \oplus B, \mathbb{K}) \to \mathsf{Ext}(A, \mathbb{K}) \oplus \mathsf{Ext}(B, \mathbb{K}), \quad \mathsf{F}(\tau) = (\tau \circ \iota_1, \tau \circ \iota_2).$$

It can be shown, see [32], that F determines the group isomorphism.

Remark 8. Consider extension

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0 \tag{7}$$

Define β to be the unique map such that

$$\beta(e)i(b) = i(eb), \quad \text{for every } b \in B,\ e \in E.$$

Then Busby invariant τ is the unique map which makes the diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "i"] & M(B) \arrow[r] & M(B)/B \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] & E \arrow[r] \arrow[u, "\beta"] & A \arrow[r] \arrow[u, "\tau"] & 0
\end{tikzcd}$$

We will use both notations $[E]$ and $[\tau]$ in order to denote class of the extension (7) in $\mathsf{Ext}(A, B)$.

Let $[\mathcal{M}_q] \in \mathsf{Ext}(\mathcal{J}_1^q/\mathcal{J}_q \oplus \mathcal{J}_2^q/\mathcal{J}_q, \mathcal{J}_q)$, $[\mathcal{J}_1^q] \in \mathsf{Ext}(\mathcal{J}_1^q/\mathcal{J}_q, \mathcal{J}_q)$, $[\mathcal{J}_2^q] \in \mathsf{Ext}(\mathcal{J}_2^q/\mathcal{J}_q, \mathcal{J}_q)$ respectively be the classes of the following extensions

$$0 \to \mathcal{J}_q \to \mathcal{M}_q \to \mathcal{J}_1^q/\mathcal{J}_q \oplus \mathcal{J}_2^q/\mathcal{J}_q \to 0,$$
$$0 \to \mathcal{J}_q \to \mathcal{J}_1^q \to \mathcal{J}_1^q/\mathcal{J}_q \to 0,$$
$$0 \to \mathcal{J}_q \to \mathcal{J}_2^q \to \mathcal{J}_2^q/\mathcal{J}_q \to 0.$$

Lemma 8.

$$[\mathcal{M}_q] = ([\mathcal{J}_1^q], [\mathcal{J}_2^q]) \in \mathsf{Ext}(\mathcal{J}_1^q/\mathcal{J}_q, \mathcal{J}_q) \oplus \mathsf{Ext}(\mathcal{J}_2^q/\mathcal{J}_q, \mathcal{J}_q) \simeq \mathsf{Ext}(\mathcal{J}_1^q/\mathcal{J}_q \oplus \mathcal{J}_2^q/\mathcal{J}_q, \mathcal{J}_q),$$

Proof. Consider the following morphism of extensions:

$$\begin{tikzcd}
& \mathcal{J}_q \arrow[rr, "i"] \arrow[dd, equal] & & M(\mathcal{J}_q) \arrow[rr] \arrow[dd, equal] & & M(\mathcal{J}_q)/\mathcal{J}_q \arrow[dd, equal] \\
\mathcal{J}_q \arrow[ur, equal] \arrow[rr] \arrow[dd, equal] & & \mathcal{J}_1^q \arrow[ur, "\beta_1"] \arrow[rr] \arrow[dd, hook] & & \mathcal{J}_1^q/\mathcal{J}_q \arrow[ur, "\tau_{\mathcal{J}_1^q}"] \arrow[dd] & \\
& \mathcal{J}_q \arrow[rr] & & M(\mathcal{J}_q) \arrow[rr] & & M(\mathcal{J}_q)/\mathcal{J}_q \\
\mathcal{J}_q \arrow[ur, equal] \arrow[rr] & & \mathcal{M}_q \arrow[ur, "\beta_2"] \arrow[rr] & & \mathcal{J}_1^q/\mathcal{J}_q \oplus \mathcal{J}_2^q/\mathcal{J}_q \arrow[ur, "\tau_{\mathcal{M}_q}"] &
\end{tikzcd}$$

Here

$$\beta_1 \colon \mathcal{I}_1^q \to M(\mathcal{I}_q), \quad \beta_2 \colon \mathcal{M}_q \to M(\mathcal{I}_q)$$

are homomorphisms introduced in Remark 8, the arrow

$$j_1 \colon \mathcal{I}_1^q \hookrightarrow \mathcal{M}_q$$

is the inclusion, and the arrow

$$\iota_1 : \mathcal{I}_1^q/\mathcal{I}_q \to \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q$$

has the form $\iota_1(x) = (x, 0)$.

Notice that for every $b \in \mathcal{I}_q$ and $x \in \mathcal{I}_1^q$ one has

$$(\beta_2 \circ j_1)(x) i(b) = i(j_1(x) b) = i(xb) = \beta_1(x) i(b).$$

By uniqueness of β_1 we get $\beta_2 \circ j_1 = \beta_1$. Thus the following diagram commutes

$$\begin{tikzcd} \mathcal{I}_1^q \arrow[r, "\beta_1"] \arrow[d, hook] & M(\mathcal{I}_q) \arrow[d, equal] \\ \mathcal{M}_q \arrow[r, "\beta_2"] & M(\mathcal{I}_q) \end{tikzcd}$$

Further, Remark 8 implies that for Busby invariants $\tau_{\mathcal{I}_1^q}$ and $\tau_{\mathcal{M}_q}$ the squares below are commutative

$$\begin{tikzcd} M(\mathcal{I}_q) \arrow[r] & M(\mathcal{I}_q)/\mathcal{I}_q \\ \mathcal{I}_1^q \arrow[r] \arrow[u, "\beta_1"] & \mathcal{I}_1^q/\mathcal{I}_q \arrow[u, "\tau_{\mathcal{I}_1^q}"'] \end{tikzcd}, \quad \begin{tikzcd} M(\mathcal{I}_q) \arrow[r] & M(\mathcal{I}_q)/\mathcal{I}_q \\ \mathcal{M}_q \arrow[r] \arrow[u, "\beta_2"] & \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q \arrow[u, "\tau_{\mathcal{M}_q}"'] \end{tikzcd}.$$

Hence the square

$$\begin{tikzcd} \mathcal{I}_1^q/\mathcal{I}_q \arrow[r, "\tau_{\mathcal{I}_1^q}"] \arrow[d, "\iota_1"] & M(\mathcal{I}_q)/\mathcal{I}_q \arrow[d, equal] \\ \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q \arrow[r, "\tau_{\mathcal{M}_q}"] & M(\mathcal{I}_q)/\mathcal{I}_q \end{tikzcd}.$$

is also commutative. Thus, $\tau_{\mathcal{I}_1^q} = \tau_{\mathcal{M}_q} \circ \iota_1$. By the same arguments we get $\tau_{\mathcal{I}_2^q} = \tau_{\mathcal{M}_q} \circ \iota_2$, where

$$\iota_2 \colon \mathcal{I}_2^q/\mathcal{I}_q \to \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q, \quad \iota_2(y) = (0, y).$$

Thus
$$[\tau_{\mathcal{M}_q}] = ([\tau_{\mathcal{M}_q} \circ \iota_1], [\tau_{\mathcal{M}_q} \circ \iota_2]) = ([\tau_{\mathcal{I}_1^q}], [\tau_{\mathcal{I}_2^q}]).$$

$\square$

In the next theorem we give a description of all ideals in $\mathcal{E}_{n,m}^q$.

Theorem 11. *Any ideal $J \subset \mathcal{E}_{n,m}^q$ coincides with one of $\mathcal{I}_q$, $\mathcal{I}_1^q$, $\mathcal{I}_2^q$, $\mathcal{M}_q$.*

Proof. First we notice that $\mathcal{I}_1^q/\mathcal{I}_q \simeq \mathcal{O}_m \otimes \mathbb{K}$, $\mathcal{I}_2^q/\mathcal{I}_q \simeq \mathcal{O}_n \otimes \mathbb{K}$ are simple. Hence for any ideal $\mathcal{J}$ such that $\mathcal{I}_q \subseteq \mathcal{J} \subseteq \mathcal{I}_1^q$ or $\mathcal{I}_q \subseteq \mathcal{J} \subseteq \mathcal{I}_2^q$, one has either $\mathcal{J} = \mathcal{I}_q$ or $\mathcal{J} = \mathcal{I}_1^q$ or $\mathcal{J} = \mathcal{I}_2^q$.

Further, using the fact that $\mathcal{M}_q = \mathcal{I}_1^q + \mathcal{I}_2^q$ and $\mathcal{I}_q = \mathcal{I}_1^q \cap \mathcal{I}_2^q$ we get
$$\mathcal{M}_q/\mathcal{I}_1^q \simeq \mathcal{I}_2^q/\mathcal{I}_q \simeq \mathcal{O}_n \otimes \mathbb{K}.$$
So if $\mathcal{I}^q \subseteq \mathcal{J} \subseteq \mathcal{M}_q$, then again either $\mathcal{J} = \mathcal{I}_1^q$ or $\mathcal{J} = \mathcal{M}_q$.

Below, Theorem 14, we show that $\mathcal{E}_{n,m}^q/\mathcal{M}_q$ is simple ad purely infinite. In particular $\mathcal{M}_q$ contains any ideal in $\mathcal{E}_{n,m}^q$, see Corollary 7.

Let $\mathcal{J} \subset \mathcal{E}_{n,m}^q$ be an ideal and π be a faithful representation of $\mathcal{E}_{n,m}^q/\mathcal{J}$. Notice that the Fock component π_1 in the Wold decomposition of π is zero. Thus, see Theorem 9,
$$\pi = \pi_2 \oplus \pi_3 \oplus \pi_4 \tag{8}$$
and $\mathcal{J} = \ker \pi = \ker \pi_2 \cap \ker \pi_3 \cap \ker \pi_4$. Let us describe these kernels. Suppose that the component π_2 is non-zero. Since $\pi_2(\mathbf{1} - Q) = 0$ and $\pi_2(\mathbf{1} - P) \neq 0$ we have
$$\mathcal{I}_1^q \subseteq \ker \pi_2 \subsetneq \mathcal{M}_q$$
implying $\ker \pi_2 = \mathcal{I}_1^q$. Using the same considerations one can deduce that if the component π_3 is non-zero, then $\ker \pi_3 = \mathcal{I}_2^q$, and if π_4 is non-zero, then $\ker \pi_4 = \mathcal{M}_q$.

Finally, if in (8) π_2 and π_3 are non-zero then $\mathcal{J} = \ker \pi = \mathcal{I}_q$. If either $\pi_2 \neq 0$ and $\pi_3 = 0$ or $\pi_3 \neq 0$ and $\pi_2 = 0$, then either $\mathcal{J} = \mathcal{I}_1^q$ or $\mathcal{J} = \mathcal{I}_2^q$. In the case $\pi_2 = 0$ and $\pi_3 = 0$ one has $\mathcal{J} = \ker \pi_4 = \mathcal{M}_q$. $\square$

Corollary 5. *All ideals in $\mathcal{E}_{n,m}^q$ are essential. The ideal $\mathcal{I}^q$ is the unique minimal ideal.*

In particular, the extension
$$0 \to \mathcal{I}_q \to \mathcal{M}_q \to \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q \to 0$$

is essential. Indeed the ideal $\mathbb{K} = \mathcal{I}_q \subset \mathcal{E}^q_{n,m}$ is the unique minimal ideal. Since the ideal of ideal in C^*-algebra is the ideal in the whole algebra, $\mathcal{I}_q$ is the unique minimal ideal in $\mathcal{M}_q$, thus it is essential in $\mathcal{M}_q$.

The following proposition is a corollary of Voiculescu's Theorem, see Theorem 15.12.3 of [10]

Proposition 16. *Let E_1, E_2 be two essential extensions of a nuclear C^*-algebra A by $\mathbb{K}$. If $[E_1] = [E_2] \in \mathsf{Ext}(A, \mathbb{K})$ then $E_1 \simeq E_2$.*

Theorem 12. *For any $q \in \mathbb{C}$, $|q| = 1$, one has $\mathcal{M}_q \simeq \mathcal{M}_1$.*

Proof. By Theorem 10, $[\mathcal{I}^q_1] \in \mathsf{Ext}(\mathcal{O}_m \otimes \mathbb{K}, \mathbb{K})$, and $[\mathcal{I}^q_2] \in \mathsf{Ext}(\mathcal{O}_n \otimes \mathbb{K}, \mathbb{K})$ do not depend on q. By Lemma 8, $[\mathcal{M}_q]$ does not depend on q. Thus by Corollary 5 and Proposition 16, $\mathcal{M}_q \simeq \mathcal{M}_1$. $\square$

3.4 Simplicity and pure infiniteness of $\mathcal{O}_n \otimes_q \mathcal{O}_m$

The next step is to show that the quotient $\mathcal{O}_n \otimes_q \mathcal{O}_m = \mathcal{E}^q_{n,m}/\mathcal{M}_q$, being nuclear, is also simple and purely infinite.

It is easy to see that

$$\mathcal{M}_q = c.l.s.\{s_{\mu_1} t_{\nu_1} (\mathbf{1} - P)^{\varepsilon_1} (\mathbf{1} - Q)^{\varepsilon_2} t^*_{\nu_2} s^*_{\mu_2}\},$$

where $\mu_j \in \Lambda_n$, $\nu_j \in \Lambda_m$, $j = 1, 2$, and $\varepsilon_j \in \{0, 1\}$, $\varepsilon_1^2 + \varepsilon_2^2 \neq 0$.

We denote the generators of $\mathcal{O}_n \otimes_q \mathcal{O}_m$ in the same way as generators of $\mathcal{E}^q_{n,m}$. Notice, that for any $k \in \mathbb{N}$ the following relations hold in $\mathcal{O}_n \otimes_q \mathcal{O}_m$

$$\sum_{\lambda \in \Lambda_n, |\lambda| = k} s_\lambda s^*_\lambda = \mathbf{1}, \qquad \sum_{\nu \in \Lambda_m, |\nu| = k} t_\nu t^*_\nu = \mathbf{1},$$

and

$$\mathcal{O}_n \otimes_q \mathcal{O}_m = c.l.s.\{s_{\mu_1} s^*_{\mu_2} t_{\nu_1} t^*_{\nu_2},\ \mu_i \in \Lambda_n,\ \nu_j \in \Lambda_m\}.$$

Consider the action α of $\mathbb{T}^2$ on $\mathcal{O}_n \otimes_q \mathcal{O}_m$

$$\alpha_{\varphi_1,\varphi_2}(s_j) = e^{2\pi i \varphi_1} s_j, \quad \alpha_{\varphi_1,\varphi_2}(t_r) = e^{2\pi i \varphi_2} t_r,\ j = \overline{1,n},\ r = \overline{1,m}.$$

Construct the corresponding faithful conditional expectation E_α and denote by $\mathcal{A}_q$ the fixed point C^*-algebra of $(\mathcal{O}_n \otimes_q \mathcal{O}_m)^\alpha$, see Section 3.1.1. Similarly to the case of $\mathcal{E}^q_{n,m}$, one has

$$E_\alpha(s_{\mu_1} s^*_{\mu_2} t_{\nu_1} t^*_{\nu_2}) = 0, \quad \text{if either } |\mu_1| \neq |\mu_2| \ \text{ or } |\nu_1| \neq |\nu_2|,$$
$$E_\alpha(s_{\mu_1} s^*_{\mu_2} t_{\nu_1} t^*_{\nu_2}) = s_{\mu_1} s^*_{\mu_2} t_{\nu_1} t^*_{\nu_2} \ \text{ if } |\mu_1| = |\mu_2| \ \text{ and } |\nu_1| = |\nu_2|.$$

Lemma 9. *If $\nu_1, \nu_2 \in \Lambda_m$, then*

$$s_j t_{\nu_1} t_{\nu_2}^* = \overline{q}^{|\nu_1|-|\nu_2|} t_{\nu_1} t_{\nu_2}^* s_j,\ \ s_j^* t_{\nu_1} t_{\nu_2}^* = q^{|\nu_1|-|\nu_2|} t_{\nu_1} t_{\nu_2}^* s_j^*,\ \ j = \overline{1,n}$$

If $\mu_1, \mu_2 \in \Lambda_n$, then

$$t_i s_{\mu_1} s_{\mu_2}^* = q^{|\mu_1|-|\mu_2|} s_{\mu_1} s_{\mu_2}^* t_i,\ \ t_i^* s_{\mu_1} s_{\mu_2}^* = \overline{q}^{|\mu_1|-|\mu_2|} s_{\mu_1} s_{\mu_2}^* t_i^*,\ \ i = \overline{1,m}$$

As well as in the proof of Proposition 10, denote

$$\mathcal{A}_1^0 = \mathbb{C}, \quad \mathcal{A}_1^k = \operatorname{span}\{s_{\mu_1} s_{\mu_2}^*,\ |\mu_1| = |\mu_2| = k,\ \mu_i \in \Lambda_n\},\ k \in \mathbb{N},$$
$$\mathcal{A}_2^0 = \mathbb{C}, \quad \mathcal{A}_2^k = \operatorname{span}\{t_{\nu_1} t_{\nu_2}^*,\ |\nu_1| = |\nu_2| = k,\ \nu_i \in \Lambda_m\},\ k \in \mathbb{N}.$$

Recall also that $\mathcal{A}_1^k \simeq M_{n^k}(\mathbb{C})$ and $\mathcal{A}_2^k \simeq M_{m^k}(\mathbb{C})$, see [27].

Put $\mathcal{A}_q^0 := \mathbb{C}$ and

$$\mathcal{A}_q^k := \sum_{k_1+k_2=k} \mathcal{A}_1^{k_1} \cdot \mathcal{A}_2^{k_2}$$

and

$$\mathcal{A}_q = \overline{\bigcup_{k \in \mathbb{Z}_+} \mathcal{A}_q^k}.$$

By Lemma 9, for any $x \in \mathcal{A}_1^k$ and $y \in \mathcal{A}_2^l$ one has $xy = yx$. Thus $\mathcal{A}_q$ is an AF-subalgebra in $\mathcal{O}_n \otimes_q \mathcal{O}_m$ and $\mathcal{A}_{q_1} \simeq \mathcal{A}_{q_2}$ for any $q_1, q_2 \in \mathbb{C}$, $|q_1| = |q_2| = 1$.

To prove pure infiniteness of $\mathcal{O}_n \otimes_q \mathcal{O}_m$ we essentially follow Chapter V.4 of [4].

Denote by Fin_q^k the span of monomials $s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^*$ such that

$$\max\{|\mu_1|, |\mu_2|\} + \max\{|\nu_1|, |\nu_2|\} \le k.$$

Proposition 17. *For any $k \in \mathbb{N}$ there exists an isometry $w_k \in \mathcal{O}_n \otimes_q \mathcal{O}_m$ such that*

$$E_\alpha(x) = w_k^* x w_k,\ \textit{for any } x \in Fin_q^k$$

and $w_k^* y w_k = y$ for any $y \in \mathcal{A}_q^k$.

Proof. Let $s_\gamma = s_1^{2k} s_2$ and $t_\gamma = t_1^{2k} t_2$. Consider isometries

$$w_{k,1} = \sum_{|\delta|=k, \delta \in \Lambda_n} s_\delta s_\gamma s_\delta^*$$

and

$$w_{k,2} = \sum_{|\lambda|=k, \lambda\in\Lambda_m} t_\lambda t_\gamma t_\lambda^*.$$

Then, see Lemma V.4.5 of [4],

$$w_{k,1}^* s_{\mu_1} s_{\mu_2}^* w_{k,1} = 0, \text{ if } |\mu_1| \neq |\mu_2|, \ |\mu_i| \leq k, \mu_i \in \Lambda_n$$

and

$$w_{k,1}^* s_{\mu_1} s_{\mu_2}^* w_{k,1} = s_{\mu_1} s_{\mu_2}^*, \text{ if } |\mu_1| = |\mu_2|, \ |\mu_i| \leq k, \mu_i \in \Lambda_n.$$

Analogously,

$$w_{k,2}^* t_{\nu_1} t_{\nu_2}^* w_{k,2} = 0, \text{ if } |\nu_1| \neq |\nu_2|, \ |\nu_i| \leq k, \nu_i \in \Lambda_m$$

and

$$w_{k,2}^* t_{\nu_1} t_{\nu_2}^* w_{k,2} = t_{\nu_1} t_{\nu_2}^*, \text{ if } |\nu_1| = |\nu_2|, \ |\nu_i| \leq k, \nu_i \in \Lambda_m.$$

By Lemma 9 we get

$$\begin{aligned} w_{k,1} t_{\nu_1} t_{\nu_2}^* &= \overline{q}^{(|\nu_1|-|\nu_2|)(2k+1)} t_{\nu_1} t_{\nu_2}^* w_{k,1}, \\ w_{k,1}^* t_{\nu_1} t_{\nu_2}^* &= q^{(|\nu_1|-|\nu_2|)(2k+1)} t_{\nu_1} t_{\nu_2}^* w_{k,1}^*, \end{aligned}$$

$$\begin{aligned} w_{k,2} s_{\mu_1} s_{\mu_2}^* &= q^{(|\mu_1|-|\mu_2|)(2k+1)} s_{\mu_1} s_{\mu_2}^* w_{k,2}, \\ w_{k,2}^* s_{\mu_1} s_{\mu_2}^* &= \overline{q}^{(|\mu_1|-|\mu_2|)(2k+1)} s_{\mu_1} s_{\mu_2}^* w_{k,2}^*. \end{aligned}$$

$$w_{k,2} w_{k,1} = q^{(2k+1)^2} w_{k,1} w_{k,2}, \ w_{k,2}^* w_{k,1} = \overline{q}^{(2k+1)^2} w_{k,1} w_{k,2}^*.$$

Let $w_k = w_{k,2} w_{k,1}$. Evidently w_k is the isometry. Then for any $|\mu_i| \leq k$ and $|\nu_i| \leq k$ one has

$$w_k^* s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^* w_k = q^{((|v_1|-|v_2|)-(|\mu_1|-|\mu_2|))(2k+1)} w_{k,1}^* s_{\mu_1} s_{\mu_2}^* w_{k,1} w_{k,2}^* t_{\nu_1} t_{\nu_2}^* w_{k,2},$$

implying that for any $|\mu_i| \leq k$ and $|\nu_i| \leq k$

$$w_k^* s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^* w_k = 0, \text{ if } |\mu_1| \neq |\mu_2| \text{ or } |\nu_1| \neq |\nu_2|$$

and

$$w_k^* s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^* w_k = s_{\mu_1} s_{\mu_2}^* t_{\nu_1} t_{\nu_2}^*, \text{ if } |\mu_1| = |\mu_2| \text{ and } |\nu_1| = |\nu_2|.$$

Hence for any $x \in Fin_q^k$ one has $w_k^* x w_k = E_\alpha(x)$ and $w_k^* y w_k = y$ for $y \in \mathcal{A}_q^k$. $\square$

Remark 9. Since $\mathcal{A}_q^k$ is finite-dimensional, it is a direct sum of full matrix algebras, where matrix units are represented by the elements $s_{\mu_1} t_{\nu_1} t_{\nu_2}^* s_{\mu_2}^*$, $|\mu_1| = |\mu_2|$, $|\nu_1| = |\nu_2|$ and $|\mu_1| + |\nu_1| = k$. In particular any minimal projection in $\mathcal{A}_q^k$ is unitary equivalent in $\mathcal{A}_q^k$ to a "matrix-unit projection" having form $s_{\mu_1} t_{\nu_1} t_{\nu_1}^* s_{\mu_1}^*$ with $|\mu_1| + |\nu_1| = k$. So any minimal projection in $\mathcal{A}_q^k$ has the form $u\,u^*$ for some isometry $u \in \mathcal{E}_{n,m}^q$.

The following statement is the main result of this Subsection.

Theorem 13. *For any non-zero $x \in \mathcal{O}_n \otimes_q \mathcal{O}_m$ with $|q| = 1$ there exist $a, b \in \mathcal{O}_n \otimes_q \mathcal{O}_m$ such that $axb = 1$.*

Proof. The proof coincides with the proof of Theorem V.4.6 in [4]. We present it here for the reader's convenience.

Let $\mathcal{O}_n^q \otimes \mathcal{O}_m^q \ni x \neq 0$. Then $x^*x > 0$ and $E_\alpha(x^*x) > 0$. After normalisation of x we can suppose that $||E_\alpha(x^*x)|| = 1$. Find $k \in \mathbb{N}$ and $y = y^* \in Fin_q^k$ such that $||x^*x - y|| < \frac{1}{4}$. Since E_α is a contraction one has

$$||E_\alpha(x^*x) - E_\alpha(y)|| < \frac{1}{4} \text{ and } ||E_\alpha(y)|| > \frac{3}{4}.$$

Further, $w_k^* y w_k = E_\alpha(y)$. Since $E_\alpha(y) = E_\alpha(y)^* \in \mathcal{A}_q^k$ by the spectral theorem for a self-adjoint operator on a finite-dimensional Hilbert space there exists a minimal projection $p \in \mathcal{A}_q^k$ such that

$$pE_\alpha(y) = E_\alpha(y)p = ||E_\alpha(y)|| \cdot p.$$

As noted above, $p = u\,u^*$ for an isometry $u \in \mathcal{O}_n \otimes_q \mathcal{O}_m$. Put

$$z = ||E_\alpha(y)||^{-\frac{1}{2}} u^* p w_k^*.$$

Then $||z|| < \frac{2}{\sqrt{3}}$ and

$$\begin{aligned} zyz^* &= ||E_\alpha(y)||^{-1} u^* p w_k^* y w_k p u = ||E_\alpha(y)||^{-1} u^* p E_\alpha(y) p u = \\ &= ||E_\alpha(y)||^{-1} ||E_\alpha(y)|| u^* p u = u^*\,u\,u^*\,u = 1. \end{aligned}$$

Then

$$||1 - zx^*xz^*|| = ||zyz^* - zx^*xz^*|| \leq ||z||^2 \cdot ||y - x^*x|| < \frac{4}{3}\frac{1}{4} = \frac{1}{3}.$$

Hence zx^*xz^* is invertible in $\mathcal{O}_n \otimes_q \mathcal{O}_m$. Let $c \in \mathcal{O}_n \otimes_q \mathcal{O}_m$ satisfies $czx^*xz^* = 1$, then for $a = czx^*$ and $b = z^*$ one has $axb = 1$. $\square$

The following corollary is immediate.

Theorem 14. *The C^*-algebra $\mathcal{O}_n \otimes_q \mathcal{O}_m$ is nuclear, simple and purely infinite.*

Given $q = e^{2\pi i \varphi_0}$ consider

$$\Theta_q = \begin{pmatrix} 0 & \frac{\varphi_0}{2} \\ -\frac{\varphi_0}{2} & 0 \end{pmatrix}. \tag{9}$$

and construct the Rieffel deformation $(\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q}$.

Corollary 6. *The following isomorphism holds:*

$$\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq (\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q}.$$

Proof. As in the proof of Theorem 8, the universal property of $\mathcal{O}_n \otimes_q \mathcal{O}_m$ implies that the correspondence

$$s_j \mapsto s_j \otimes 1, \quad t_r \mapsto 1 \otimes t_r, \quad j = \overline{1,n},\ r = \overline{1,m}$$

extends to a surjective homomorphism $\Phi\colon \mathcal{O}_n \otimes_q \mathcal{O}_m \to (\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q}$. Finally, the simplicity of $\mathcal{O}_n \otimes_q \mathcal{O}_m$ one has that Φ is an isomorphism. $\square$

Remark 10. The isomorphism established in Corollary 6 is equivariant with respect to the introduced above actions of $\mathbb{T}^2$ on $\mathcal{O} \otimes_q \mathcal{O}_m$ and $(\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q}$ respectively.

The simplicity of $\mathcal{O}_n \otimes_q \mathcal{O}_m$ implies that $\mathcal{M}_q \subset \mathcal{E}_{n,m}^q$ is the largest ideal.

Corollary 7. *The ideal $\mathcal{M}_q \subset \mathcal{E}_{n,m}^q$ is the unique largest ideal.*

Proof. Let $\eta\colon \mathcal{E}_{n,m}^q \to \mathcal{O}_n \otimes_q \mathcal{O}_m$ be the quotient homomorphism. Suppose that $\mathcal{J} \subset \mathcal{E}_{n,m}^q$ is a two-sided $*$-ideal. Due to the simplicity of $\mathcal{O}_n \otimes_q \mathcal{O}_m$ we have that either $\eta(\mathcal{J}) = \{0\}$ and $\mathcal{J} \subset \mathcal{M}_q$ or $\eta(\mathcal{J}) = \mathcal{O}_n \otimes_q \mathcal{O}_m$. In the latter case $1 + x \in \mathcal{J}$ for a certain $x \in \mathcal{M}_q$. For any $0 < \varepsilon < 1$ choose $N_\varepsilon \in \mathbb{N}$, such that for

$$x_\varepsilon = \sum_{\substack{\varepsilon_1,\varepsilon_2 \in \{0,1\},\\ \varepsilon_1^2+\varepsilon_2^2 \neq 0}} \sum_{\substack{\mu_1,\mu_2 \in \Lambda_n,\\ |\mu_j| \leq N_\varepsilon}} \sum_{\substack{\nu_1,\nu_2 \in \Lambda_m,\\ |\nu_j| \leq N_\varepsilon}} \Psi^{(\varepsilon_1,\varepsilon_2)}_{\mu_1,\mu_2\nu_1\nu_2} s_{\mu_1} t_{\nu_1} (\mathbf{1} - P)^{\varepsilon_1} (\mathbf{1} - Q)^{\varepsilon_2} t_{\nu_2}^* s_{\mu_2^*} \in \mathcal{M}_q$$

one has $||x - x_\varepsilon|| < \varepsilon$. Notice that for any $\mu \in \Lambda_n$, $\nu \in \Lambda_m$ with $|\mu|, |\nu| > N_\varepsilon$ one has $s_\mu^* t_\nu^* x_\varepsilon = 0$.

Fix $\mu \in \Lambda_n$ and $\nu \in \Lambda_m$, $|\mu| = |\nu| > N_\varepsilon$, then

$$y_\varepsilon = s_\mu^* t_{\nu^*} (\mathbf{1} - x) t_\nu s_\mu = \mathbf{1} - s_\mu^* t_\nu^* (x - x_\varepsilon) t_\nu s_\mu \in \mathcal{J}.$$

Thus $||s_\mu^* t_\nu^* (x - x_\varepsilon) t_\nu s_\mu|| < \varepsilon$ implies that y_ε is invertible, so $\mathbf{1} \in \mathcal{J}$. $\square$

3.5 The isomorphism $\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$

In this section we prove the main result of Section 3. Namely, we show that

$$\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m.$$

In [5], the authors have shown that for every C^*-algebra A with an action α of $\mathbb{R}$, there exists a KK-isomorphism $t_\alpha \in KK_1(A, A \rtimes_\alpha \mathbb{R})$. This t_α is a generalization of the Connes-Thom isomorphisms for K-theory. Below we will denote $\circ : KK(A,B) \times KK(B,C) \to KK(A,C)$ to be the Kasparov product, $\boxtimes : KK(A,B) \times KK(C,D) \to KK(A \otimes C, B \otimes D)$ to be the exterior tensor product. Given a homomorphism $\phi : A \to B$ we denote $[\phi] \in KK(A,B)$ to be the induced KK-morphism. For more details see [33, 10].

List some properties of t_α we will use below:

1. Inverse of t_α is given by $t_{\widehat{\alpha}}$, where $\widehat{\alpha}$ is the dual action.
2. If $A = \mathbb{C}$ with the trivial action of $\mathbb{R}$ then the corresponding element $t_1 \in KK_1(\mathbb{C}, C_0(\mathbb{R})) \simeq \mathbb{Z}$ is the generator of the group.
3. Let $\phi : (A, \alpha) \to (B, \beta)$ be an equivariant homomorphism. Then the following diagram commutes in KK-theory

$$\begin{array}{ccc} A & \xrightarrow{t_\alpha} & A \rtimes_\alpha \mathbb{R} \\ \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\phi\rtimes\mathbb{R}} \\ B & \xrightarrow{t_\beta} & B \rtimes_\beta \mathbb{R} \end{array}$$

4. Let β be an action of $\mathbb{R}$ on B. If the action $\gamma = \mathrm{id}_A \otimes \beta$ on $A \otimes B$ then

$$t_\gamma = 1_A \boxtimes t_\beta.$$

We will need the classification result by Kirchberg and Philips:

Theorem 15 ([1], Corollary 4.2.2)**.** *Let A and B be separable nuclear unital purely infinite simple C^*-algebras, and suppose that there exists an invertible element $\eta \in KK(A,B)$ such that $[\iota_A] \circ \eta = [\iota_B]$, where $\iota_A : \mathbb{C} \to A$ is $\lambda \mapsto \lambda 1_A$, and similarly for ι_B. Then A and B are isomorphic.*

Theorem 16. *C^*-algebras $\mathcal{O}_n \otimes_q \mathcal{O}_m$ and $\mathcal{O}_n \otimes \mathcal{O}_m$ are isomorphic.*

Proof. Throughout the proof we will distinguish between the actions of $\mathbb{T}^2$ on $\mathcal{O}_n \otimes \mathcal{O}_m$ and on $\mathcal{O}_n \otimes_q \mathcal{O}_m$, denoting the latter by α^q. Due to Theorem 14, the both algebras are separable nuclear unital simple and purely infinite.

Further, Corollary 6, Proposition 5 and Remark 10 yield the isomorphism

$$\Psi : (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_\alpha \mathbb{R}^2 \to (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2,$$

Decompose the crossed products as follows:

$$(\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_\alpha \mathbb{R}^2 \simeq (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_{\alpha_1} \mathbb{R} \rtimes_{\alpha_2} \mathbb{R},$$

$$(\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2 \simeq (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha_1^q} \mathbb{R} \rtimes_{\alpha_2^q} \mathbb{R}.$$

Define

$$t_\alpha = t_{\alpha_1} \circ (1_{C_0(\mathbb{R})} \boxtimes t_{\alpha_2}) \in KK(\mathcal{O}_n \otimes \mathcal{O}_m, (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_\alpha \mathbb{R}^2),$$

$$t_{\alpha^q} = t_{\alpha_1^q} \circ (1_{C_0(\mathbb{R})} \boxtimes t_{\alpha_2^q}) \in KK(\mathcal{O}_n \otimes_q \mathcal{O}_m, (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2),$$

Then

$$\eta = t_{\alpha^q} \circ [\Psi] \circ t_\alpha^{-1} \in KK(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{O}_n \otimes \mathcal{O}_m)$$

is a KK-isomorphism. The property $[\iota_{\mathcal{O}_n \otimes_q \mathcal{O}_m}] \circ \eta = [\iota_{\mathcal{O}_n \otimes \mathcal{O}_m}]$ follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "t_1 \circ (1_{C_0(\mathbb{R})} \boxtimes t_1)"] \arrow[d, "\iota_{\mathcal{O}_n \otimes_q \mathcal{O}_m}"] & & C_0(\mathbb{R}^2) \arrow[rr, "(1_{C_0(\mathbb{R})} \boxtimes t_1)^{-1} \circ t_1^{-1}"] \arrow[dl, "\iota_{\mathcal{O}_n \otimes_q \mathcal{O}_m} \rtimes \mathbb{R}^2"'] \arrow[dr, "\iota_{\mathcal{O}_n \otimes \mathcal{O}_m} \rtimes \mathbb{R}^2"] & & \mathbb{C} \arrow[d, "\iota_{\mathcal{O}_n \otimes \mathcal{O}_m}"'] \\
\mathcal{O}_n \otimes_q \mathcal{O}_m \arrow[r, "t_{\alpha^q}"] & (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2 \arrow[rr, "\Psi"] & & (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_\alpha \mathbb{R}^2 \arrow[r, "t_\alpha^{-1}"] & \mathcal{O}_n \otimes \mathcal{O}_m
\end{tikzcd}$$

□

3.6 Computation of Ext for $\mathcal{E}_{n,m}^q$

In this section we compute $\mathsf{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q)$. We use the isomorphism $\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$, $|q| = 1$.

Recall the notion of UCT property for KK-theory, see [10]:

Definition 5. Suppose A and B are separable nuclear C^*-algebras. We say that (A, B) satisfies the Universal Coefficient Theorem (UCT) if the following sequence is exact

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to KK_{*+1}(A, B) \to Hom(K_{*+1}(A), K_{*+1}(B)) \to 0.$$

We say that A satisfies UCT if (A, B) satisfies UCT for every B.

It is known that $\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$ satisfies UCT.

The following statement is an easy consequence of the Kunneth formula.

Theorem 17. *Let* $d = \gcd(n-1, m-1)$. *Then*

$$K_0(\mathcal{O}_n \otimes_q \mathcal{O}_m) \simeq \mathbb{Z}/d\mathbb{Z},\ K_1(\mathcal{O}_n \otimes_q \mathcal{O}_m) \simeq \mathbb{Z}/d\mathbb{Z},$$

Proof. The Kunneth formula, see [10] Theorem 23.1.3, for K-theory gives the following short exact sequences

$$0 \to K_*(\mathcal{O}_n) \otimes_{\mathbb{Z}} K_*(\mathcal{O}_m) \to K_*(\mathcal{O}_n \otimes \mathcal{O}_m) \to Tor_1^{\mathbb{Z}}(K_{*+1}(\mathcal{O}_n), K_{*+1}(\mathcal{O}_m)) \to 0,$$

It is a well known fact in homological algebra that for an abelian group A

$$Tor_1^{\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) \simeq Ann_A(d) = \{a \in A \mid da = 0\}.$$

In particular,

$$Tor_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/\gcd(n,m)\mathbb{Z}.$$

Recall that, see [34]

$$K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z},\ K_1(\mathcal{O}_n) = 0.$$

Hence for $\mathcal{O}_n \otimes \mathcal{O}_m$ one has the following short exact sequences:

$$0 \to \mathbb{Z}/(n-1)\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(m-1)\mathbb{Z} \to K_0(\mathcal{O}_n \otimes \mathcal{O}_m) \to 0 \to 0,$$

$$0 \to 0 \to K_1(\mathcal{O}_n \otimes \mathcal{O}_m) \to \mathbb{Z}/d\mathbb{Z} \to 0.$$ □

Next step is to compute K-theory of $\mathcal{M}_q$.

Theorem 18. *Let* $d = \gcd(n-1, m-1)$. *Then*

$$K_0(\mathcal{M}_q) \simeq \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z},\ K_1(\mathcal{M}_q) \simeq 0,$$

Proof. By Theorem 8, Proposition 7 and [34] Proposition 3.9

$$K_0(\mathcal{E}_{n,m}^q) = K_0((\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}) = K_0(\mathcal{O}_n^0 \otimes \mathcal{O}_m^0) = \mathbb{Z},$$

$$K_1(\mathcal{E}_{n,m}^q) = K_1((\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}) = K_1(\mathcal{O}_n^0 \otimes \mathcal{O}_m^0) = 0.$$

Applying the 6-term exact sequence for

$$0 \to \mathbb{K} \to \mathcal{M}_q \to \mathcal{O}_n \otimes \mathbb{K} \oplus \mathcal{O}_m \otimes \mathbb{K} \to 0$$

we get

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & K_0(\mathcal{M}_q) & \longrightarrow & \mathbb{Z}/(n-1)\mathbb{Z} \oplus \mathbb{Z}/(m-1)\mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & K_1(\mathcal{M}_q) & \longleftarrow & 0
\end{array}$$

Then

$$K_1(\mathcal{M}_q) = 0$$

and elementary properties of finitely generated abelian groups imply that

$$K_0(\mathcal{M}_q) = \mathbb{Z} \oplus \mathsf{Tors},$$

where Tors is a direct sum of finite cyclic groups.
Further, the following exact sequence

$$0 \longrightarrow \mathcal{M}_q \longrightarrow \mathcal{E}^q_{n,m} \to \mathcal{O}_n \otimes_q \mathcal{O}_m \longrightarrow 0$$

gives

$$\begin{array}{ccccc}
K_0(\mathcal{M}_q) & \xrightarrow{p} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \\
{\scriptstyle i}\uparrow & & & & \downarrow \\
\mathbb{Z}/d\mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

The map $p : K_0(\mathcal{M}_q) \simeq \mathbb{Z} \oplus \mathsf{Tors} \to \mathbb{Z}$ has form $p = (p_1, p_2)$, where

$$p_1 \colon \mathbb{Z} \to \mathbb{Z}, \quad p_2 \colon \mathsf{Tors} \to \mathbb{Z}.$$

Evidently $p_2 = 0$ and $p \neq 0$ implies that $\ker p_1 = \{0\}$. Thus,

$$\ker p = \mathsf{Tors} = \mathsf{Im}(i) \simeq \mathbb{Z}/d\mathbb{Z}.$$

□

Theorem 19. *Let $d = \gcd(n-1, m-1)$. The following isomorphism holds*

$$\mathsf{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q) \simeq \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}.$$

Proof. Recall that for nuclear C^*-algebras $\mathsf{Ext}(A, B) \simeq KK_1(A, B)$. Since $\mathsf{Hom}(K_1(\mathcal{O}_n \otimes_q \mathcal{O}_m), K_1(\mathcal{M}_q)) = 0$, one has, see [37],

$$KK_1(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q) \simeq Ext^1_{\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}.$$

□

As a corollary of Theorem 19, for the case of $\gcd(n-1, m-1) = 1$ one can immediately deduce that extension classes of

$$0 \to \mathcal{M}_q \to \mathcal{E}^q_{n,m} \to \mathcal{O}_n \otimes_q \mathcal{O}_m \to 0$$

and

$$0 \to \mathcal{M}_1 \to \mathcal{E}^1_{n,m} \to \mathcal{O}_n \otimes \mathcal{O}_m \to 0$$

coincide in $\mathsf{Ext}(\mathcal{O}_n \otimes \mathcal{O}_m, \mathcal{M}_1)$ and are trivial. These extensions are essential, however in general case one does not have an immediate generalization of Proposition 16. Thus the study of the problem whether $\mathcal{E}^q_{n,m} \simeq \mathcal{E}^1_{n,m}$ would require further investigations, see [36, 35].

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