

J -FUSION FRAME OPERATOR FOR KREIN SPACES

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ABSTRACT. In this article we find a necessary and sufficient condition under which a given collection of subspace is a J -fusion frame for a Krein space \mathbb{K} . We also approximate J -fusion frame bounds of a J -fusion frame by the upper and lower bounds of the synthesis operator. Then, we obtain the J -fusion frame bounds of the canonical J -dual fusion frame. Finally, we address the problem of characterizing those bounded linear operators in \mathbb{K} for which the image of J -fusion frame is also a J -fusion frame.

1. INTRODUCTION

The study of frames for Krein spaces was originally initiated by Giribet et al. [10] in 2012. However, apart from the work by Giribet et al. recently, in 2015 an independent work in this direction by Esmeral et al. [8] has been proposed. The idea to extend the notion of frame theory from definite inner product spaces to indefinite inner product spaces is certainly an interesting research area, and this area is vastly under-developed. Krein spaces has rich application in modern mathematics [5, 2]. So studying frame theory for Krein spaces is important for solving problems in Krein spaces. In [12] Karmakar et al. discovered a genuine flaw in the definition of [8] by providing an example to establish the claim. So to study frame theory for Krein spaces we will use the definition of Giribet et al. [10] as the basic definition since this definition is motivated purely from the geometric intuition.

Fusion frame in Hilbert spaces has many important and fascinating applications such as distributed processing in sensor networks, Filter Bank theory, communications in packet based system etc. [6, 4], so motivated by this we studied J -fusion frames in [13].

This article is organized as follows. At first we give a very brief overview of the basic notations and terminologies and then in the main result section we introduce some operators corresponding to J -fusion frames. In the first subsection we prove that if $\{(W_i, v_i) : i \in I\}$ is a J -fusion frame for a Krein space \mathbb{K} , then $\{(J(W_i), v_i) : i \in I\}$ is also a J -fusion frame for \mathbb{K} . The following subsection consists an example to point out an error in [13] and we introduce J -fusion frame equations and also the necessary and sufficient condition under which a fusion frame for \mathbb{K} is a J -fusion frame for the Krein space \mathbb{K} . In the next subsection we approximate

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J -fusion frame bounds of a J -fusion frame by the upper and lower bounds of the fusion frame operator. J -frame operator and J -fusion frame operator discussed in the subsequent subsection and also in this section we calculate the J -fusion frame bounds of the canonical J -dual fusion frame. In the last subsection we address the problem of characterizing those bounded linear operators in \mathbb{K} for which the image of J -fusion frame is again a J -fusion frame by providing another necessary and sufficient condition.

In the following paragraphs we briefly recapitulate the basic notations and terminologies.

1.1. Backgrounds and terminologies. Let M be a closed subspace of a Krein space \mathbb{K} and π_M be an orthogonal projection from \mathbb{K} onto M . So, we have $\pi_M^2 = \pi_M$ and $\pi_M^* = \pi_M$. Here the range of the projection, $R(\pi_M) = M$ and Null space of π_M , $N(\pi_M) = M^\perp$. Now if M is a projectively complete subspace of \mathbb{K} then the J -orthogonal projection from \mathbb{K} onto M exists. Let Q_M be the J -orthogonal projection from \mathbb{K} onto M . Here range of the J -projection Q_M , $R(Q_M) = M$ and Null space, $N(Q_M) = M^{\perp J}$.

Let $\pi_M^\#$ be the J -adjoint of π_M . Then we have $\pi_M^\# = J\pi_M J$ and since $\pi_{JM} = J\pi_M J$, thus we have $\pi_{JM} = \pi_M^\#$.

Let W be a subspace of a Krein space \mathbb{K} . Also, let us assume that \mathbb{P}^{++} denotes the set of all J -positive subspaces of \mathbb{K} while \mathbb{P}^+ denotes the set of all J -non-negative subspaces of \mathbb{K} . Similarly, let \mathbb{P}^{--} and \mathbb{P}^- respectively denote the set of all J -negative and J -non-positive subspaces of \mathbb{K} . Then, $W \in \mathbb{P}^+ \cup \mathbb{P}^- \cup \tilde{\mathbb{P}}$. Throughout in our work we consider either $W \in \mathbb{P}^+ \cup \mathbb{P}^{--}$ or $W \in \mathbb{P}^{++} \cup \mathbb{P}^-$. Without any loss of generality, we assume $W \in \mathbb{P}^+ \cup \mathbb{P}^{--}$ to establish our results.

Let $\{W_i : i \in I\}$ be a collection of subspaces of the Krein space \mathbb{K} such that $W_i \in \mathbb{P}^+ \cup \mathbb{P}^{--}$, $\forall i \in I$. We consider the space $(\sum_{i \in I} \oplus W_i)$. If $f \in (\sum_{i \in I} \oplus W_i)$ then $f = \{f_i\}_{i \in I}$, where $f_i \in W_i$ for each $i \in I$. Let $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. We define $[f, g] = \sum_{i \in I} [f_i, g_i]$, where $f, g \in (\sum_{i \in I} \oplus W_i)$. If the series is unconditionally convergent then $[\cdot, \cdot]$ defines an inner product on $(\sum_{i \in I} \oplus W_i)$. Now consider the space $(\sum_{i \in I} \oplus W_i)_{\ell_2} = \left\{ f : (\sum_{i \in I} \oplus W_i) : \sum_{i \in I} \|f_i\|_J^2 < \infty \right\}$. We will use this space frequently in our work.

The definition of J -fusion frame is already given in [13] but we have observed that the Theorem 2.4 in [13] is not always true. So, in this article we deduce J -fusion frame equations (correcting from our earlier results) for Krein spaces to obtain more important results. For the sake of completeness of this article the definition of J -fusion frame is given below.

Let $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ be a Bessel family of closed subspaces of a Krein space \mathbb{K} with synthesis operator $T_{W,v} \in L\left((\sum_{i \in I} \oplus W_i)_{\ell_2}, \mathbb{K}\right)$ such that $W_i \in \mathbb{P}^+ \cup \mathbb{P}^{--}$, $\forall i \in I$. Let $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. Now consider the orthogonal decomposition of

$(\sum_{i \in I} \oplus W_i)_{\ell_2}$ given by

$$(\sum_{i \in I} \oplus W_i)_{\ell_2} = (\sum_{i \in I_+} \oplus W_i)_{\ell_2} \oplus (\sum_{i \in I_-} \oplus W_i)_{\ell_2}$$

and denote by P_{\pm} the orthogonal projection onto $(\sum_{i \in I_{\pm}} \oplus W_i)_{\ell_2}$. Also, let $T_{W,v_{\pm}} = T_{W,v} P_{\pm}$. If $M_{\pm} = \overline{\sum_{i \in I_{\pm}} W_i}$, notice that $\sum_{i \in I_{\pm}} W_i \subseteq R(T_{W,v_{\pm}}) \subseteq M_{\pm}$ and $R(T_{W,v}) = R(T_{W,v_+}) + R(T_{W,v_-})$.

Definition 1.1. The Bessel family of closed subspaces $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ is a *J-fusion frame* for \mathbb{K} if $R(T_{W,v_+})$ is a maximal uniformly *J*-positive subspace of \mathbb{K} and $R(T_{W,v_-})$ is a maximal uniformly *J*-negative subspace of \mathbb{K} .

Let $\{(W_i, v_i) : i \in I\}$ be a *J*-fusion frame for \mathbb{K} then $(\sum_{i \in I} \oplus W_i, [\cdot, \cdot])$ is a Krein space. The fundamental symmetry, denoted by J_2 , is defined as $J_2(f) = \{f_i : i \in I_+\} \cup \{-f_i : i \in I_-\}$ for all f . Also $[f, g]_{J_2} = \sum_{i \in I_+} [f_i, g_i] - \sum_{i \in I_-} [f_i, g_i]$.

2. MAIN RESULTS

2.1. Operators corresponding J-fusion frame. Let $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ be a *J*-fusion frame for the Krein space \mathbb{K} . Then $\{W_i : i \in I_+\}$ is a collection of uniformly *J*-positive subspaces of \mathbb{K} and $\{W_i : i \in I_-\}$ is a collection of uniformly *J*-negative subspaces of \mathbb{K} . Let $T_{W,v}^{\#}$ be the *J*-adjoint operator of the synthesis operator $T_{W,v}$ which is called the analysis operator of the *J*-fusion frame \mathbb{F} . Now, $T_{W,v}^{\#} = (T_{W,v_+}^{\#} + T_{W,v_-}^{\#})$ and $N(T_{W,v_+}^{\#})^{[\perp]} = R(T_{W,v_+}) = M_+$. We want to calculate $T_{W,v_+}^{\#}$.

We know that $T_{W,v_+}^{\#} = T_{W,v_+}^* J = \{v_i \pi_{W_i} J(f)\}_{i \in I_+}$ for all $f \in \mathbb{K}$. Similarly as above we have $T_{W,v_-}^{\#}(f) = -\{v_i \pi_{W_i} J(f)\}_{i \in I_-}$ for all $f \in \mathbb{K}$. So, $T_{W,v}^{\#}(f) = \{\sigma_i v_i \pi_{JW_i}(f)\}_{i \in I}$ for all $f \in \mathbb{K}$. Here $\sigma_i = 1$ if $i \in I_+$ and $\sigma_i = -1$ if $i \in I_-$.

Lemma 2.1. Let $\{(W_i, v_i) : i \in I\}$ be a *J*-fusion frame for the Krein space \mathbb{K} . Then $\{(J(W_i), v_i) : i \in I\}$ is also a *J*-fusion frame for \mathbb{K} .

Proof. Let $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. Then $W_i \subset M_+$ for all $i \in I_+$ and $W_i \subset M_-$ for all $i \in I_-$. Then we have $J(W_i) \subset J(M_+)$ for all $i \in I_+$ and $J(W_i) \subset J(M_-)$ for all $i \in I_-$. Since $\{(W_i, v_i) : i \in I\}$ be a *J*-fusion frame for \mathbb{K} , so $M_+ = \overline{\sum_{i \in I_+} W_i}$ and $M_- = \overline{\sum_{i \in I_-} W_i}$. From these it readily follows that $J(M_+) = \overline{\sum_{i \in I_+} J(W_i)}$ and $J(M_-) = \overline{\sum_{i \in I_-} J(W_i)}$. Now let $f \in M_+$, then $J(f) \in J(M_+)$. So,

$$\begin{aligned} [\pi_{JW_i} J(f), J(f)] &= [\pi_{JW_i} f, J(f)] = [J \pi_{W_i} J(f), J(f)] \\ &= [\pi_{W_i} J(f), f] \end{aligned}$$

In terms of the inequality (2.1) we have

$$A_{\pm}[J(f), J(f)] \leq \sum_{i \in I_{\pm}} v_i^2 |[\pi_{J(W_i)} J(f), J(f)]| \leq B_{\pm}[J(f), J(f)], \quad \forall J(f) \in J(M_{\pm})$$

So, we can say that $\{(J(W_i), v_i) : i \in I\}$ is a J -fusion frame for \mathbb{K} with the same J -fusion frame bounds. \square

Let $\{(W_i, v_i)\}_{i \in I}$ be a J -fusion frame for the Krein space \mathbb{K} . Then $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. The linear operator $S_{W,v} : \mathbb{K} \rightarrow \mathbb{K}$ defined by $S_{W,v}(f) = \sum_{i \in I} \sigma_i v_i^2 \pi_{W_i} J(f)$ is said to be the J -fusion frame operator for the J -fusion frame $\{(W_i, v_i)\}_{i \in I}$. Here $\sigma_i = 1$ if $i \in I_+$ and $\sigma_i = -1$ if $i \in I_-$. From the above it readily follows that $S_{W,v} = T_{W,v} T_{W,v}^\#$. Also $S_{W,v}$ is the sum of two J -positive operators. Let $S_{W,v_+} : \mathbb{K} \rightarrow \mathbb{K}$ is defined by $S_{W,v_+}(f) = \sum_{i \in I_+} v_i^2 \pi_{W_i} J(f)$. Then $[S_{W,v_+}(f), f] = [\sum_{i \in I_+} v_i^2 \pi_{W_i} J(f), f] = \sum_{i \in I_+} v_i^2 [\pi_{W_i} J(f), J(f)]_J$. So, S_{W,v_+} is a J -positive operator and also $S_{W,v_+} = T_{W,v_+} T_{W,v_+}^\#$. Similarly, let $S_{W,v_-} = -T_{W,v_-} T_{W,v_-}^\#$. Then S_{W,v_-} is also a J -positive operator. We have $S_{W,v} = S_{W,v_+} - S_{W,v_-}$. If $\{(W_i, v_i)\}_{i \in I}$ is a J -fusion frame for the Krein space \mathbb{K} with synthesis operator $T_{W,v} \in L\left(\left(\sum_{i \in I} \oplus W_i\right)_{\ell_2}, \mathbb{K}\right)$ then the J -fusion frame operator $S_{W,v}$ is bijective and J -selfadjoint.

Given a closed subspace M of \mathbb{K} , the Gramian operator G_M is defined as $G_M := \pi_M J|_M$. If M is J -non-negative then the Gramian operator is J -selfadjoint, bounded and positive.

2.2. J -fusion frame equation. In [13] Karmakar et al. assumed that $\pi_{W_i}|_{M_\pm} = Q_{W_i}|_{M_\pm}$, but it is wrong as we can see from the following example.

Example 2.2. Let

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

in \mathbb{C}^3 and let $M = \{(x, y, z) : z = \epsilon(x + y)\}$, where $\epsilon \in (0, \frac{1}{\sqrt{2}})$. Then M is uniformly J -positive and so is its subspace $W = \text{span}\{w\}$, where $w = (0, 1, \epsilon)$. Now let $w_0 = \frac{w}{\|w\|_J} = \frac{w}{\sqrt{1+\epsilon^2}}$ and $w_1 = \frac{w}{\sqrt{[w,w]}} = \frac{w}{\sqrt{1-\epsilon^2}}$. Then $\pi_W(x) = \frac{1}{1+\epsilon^2} \langle x, w \rangle w$ and $Q_W(x) = \frac{1}{1-\epsilon^2} [x, w] w$, for $x \in \mathbb{C}^3$. Hence for $x = (1, 1, 2\epsilon) \in M$ we get,

$$\pi_W(x) = \frac{1+2\epsilon^2}{1+\epsilon^2} w \quad \text{and} \quad Q_W(x) = \frac{1-2\epsilon^2}{1-\epsilon^2} w$$

. So, $\pi_{W_i}|_M \neq Q_{W_i}|_M$.

Now, according to the definition of J -fusion frame the collection $\{(W_i, v_i) : i \in I_\pm\}$ must be a fusion frame for $(M_\pm, \pm[\cdot, \cdot])$.

Let us consider the operator $\pi_{W_i} J$, for $i \in I_+$. For the purpose of our work let us assume that W_i is a subspace of M_+ , where M_+ is uniformly J -positive, hence M_+ is projectively complete. Hence $\mathbb{K} = M_+ [\oplus] M_+^\perp$ and also $\mathbb{K} = W_i [\oplus] W_i^\perp$. It is easy to see that $N(\pi_{W_i} J) = W_i^\perp$ and $R(\pi_{W_i} J) = W_i$. Also, we have $(\pi_{W_i} J)^\# = J \pi_{W_i}^\# = J \pi_{JW_i} = \pi_{W_i} J$ i.e. $\pi_{W_i} J$ is J -selfadjoint. Also, $(\pi_{W_i} J)^2 = \pi_{W_i} \pi_{JW_i} = \pi_{W_i} \pi_{W_i} J = \pi_{W_i} J$.

With respect to the above observations we have the following result.

Theorem 2.3. Let $\{(W_i, v_i) : i \in I\}$ be a J -fusion frame for the Krein space \mathbb{K} . Then $\{(W_i, v_i) : i \in I_\pm\}$ is a fusion frame for $(M_\pm, \pm[\cdot, \cdot])$ i.e.

$$A_\pm[f, f] \leq \sum_{i \in I_\pm} v_i^2 |[\pi_{W_i} J(f), f]| \leq B_\pm[f, f] \text{ for every } f \in M_\pm, \quad (2.1)$$

where $-\infty < B_- \leq A_- < 0 < A_+ \leq B_+ < \infty$ are constants.

The main result of this section is the converse problem which we will prove in the following theorem.

Theorem 2.4. Let $\mathbb{F} = \{f_i\}_{i \in I}$ be a frame for \mathbb{K} . If $I_\pm = \{i \in I : \pm[f_i, f_i] \geq 0\}$ and $M_\pm = \overline{\text{span}\{f_i : i \in I_\pm\}}$ then \mathbb{F} is a J -frame if $M_\pm \cap M_\pm^{[\perp]} = \{0\}$ and there exist constants $B_- \leq A_- < 0 < A_+ \leq B_+$ such that

$$A_\pm[f, f] \leq \sum_{i \in I_\pm} v_i^2 |[\pi_{W_i} J(f), f]| \leq B_\pm[f, f], \quad \forall f \in M_\pm \quad (2.2)$$

Proof. Let M_+ be non-degenerated subspace of \mathbb{K} and there exist constants $0 < A_+ \leq B_+$ such that

$$A_+[f, f] \leq \sum_{i \in I_+} v_i^2 |[\pi_{W_i} J(f), f]| \leq B_+[f, f], \quad \forall f \in M_+$$

then, by Theorem 3.15 of [13], M_+ is uniformly J -positive. So, \exists a real number $\alpha > 0$ such that

$$\alpha \|\pi_{M_+}(f)\|^2 \leq [\pi_{M_+}(f), \pi_{M_+}(f)] \leq \|\pi_{M_+}(f)\|^2, \quad \forall f \in \mathbb{K}.$$

Therefore we can say that there exist constants $0 < A \leq B$ such that

$$A \|\pi_{M_+}(f)\|^2 \leq \|T_{W, v_+}^\# \pi_{M_+}(f)\|^2 \leq B \|\pi_{M_+}(f)\|^2, \quad \forall f \in \mathbb{K}.$$

The above equation can be written as

$$A\pi_{M_+} \leq \pi_{M_+} J T_{W, v_+} T_{W, v_+}^* J \pi_{M_+} \leq B\pi_{M_+}.$$

Then by Douglas theorem [7] we have $R(\pi_{M_+} J T_{W, v_+}) = R(\pi_{M_+}) = M_+$. Furthermore, $\pi_{J(M_+)}(R(T_{W, v_+})) = R(\pi_{J(M_+)} T_{W, v_+}) = R((J\pi_{M_+} J) T_{W, v_+}) = J(R(\pi_{M_+} J T_{W, v_+})) = J(M_+)$. So, we have $\pi_{J(M_+)}(R(T_{W, v_+})) = J(M_+)$. Therefore, taking the pre-image of $\pi_{J(M_+)}(R(T_{W, v_+}))$ by $\pi_{J(M_+)}$, we have

$$\mathbb{K} = R(T_{W, v_+}) \oplus J(M_+)^{\perp} \subseteq M_+ \oplus M_+^{[\perp]} = \mathbb{K}.$$

Thus, $R(T_{W, v_+}) = M_+$ and \mathbb{F}_+ is a frame for M_+ . Analogously, $\mathbb{F}_- = \{f_i\}_{i \in I_-}$ is a frame for M_- . Finally since \mathbb{F} is a frame for \mathbb{K} ,

$$\mathbb{K} = R(T_{W, v}) = R(T_{W, v_+}) + R(T_{W, v_-})$$

, which proves the maximality of $R(T_{W, v_\pm})$. Hence, \mathbb{F} is a J -frame for \mathbb{K} . \square

2.3. Bounds of J -fusion frame.

Definition 2.5. Let $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ be a J -fusion frame for \mathbb{K} , then there exist constants B_- , A_- , A_+ and B_+ such that $-\infty < B_- \leq A_- < 0 < A_+ \leq B_+ < \infty$. These constants are the J -fusion frame bounds of the J -fusion frame \mathbb{F} in \mathbb{K} . If these bounds are optimal then they are called optimal J -fusion frame bound.

Definition 2.6. The reduced minimum modulus $\gamma(T)$ of an operator $T \in L(\mathbb{H}, \mathbb{K})$ is defined by

$$\gamma(T) = \inf\{\|Tx\| : x \in N(T)^\perp, \|x\| = 1\}.$$

It is well known that $\gamma(T) = \gamma(T^*) = \gamma(TT^*)^{\frac{1}{2}}$.

We want to calculate J -fusion frame bounds of a J -fusion frame in a Krein space. Let $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ be a J -frame of subspaces for the Krein space \mathbb{K} . Then for all $f \in M_+$ we have

$$\begin{aligned} \sum_{i \in I_+} v_i^2 [\pi_{W_i} J(f), f] &= \sum_{i \in I_+} v_i^2 [\pi_{W_i} J(f), \pi_{W_i} J(f)] \\ &= \|T_{W, v_+}^\#(f)\|_{J_2}^2 \leq \|T_{W, v_+}^\#\|_{J_2}^2 \|f\|_J^2 \\ &\leq \frac{1}{\gamma(G_{M_+})} \|T_{W, v_+}\|_J^2 [f, f] \end{aligned}$$

Comparing with the inequality (2.1), we have $B_+ = \frac{1}{\gamma(G_{M_+})} \|T_{W, v_+}\|_J^2$. Again for all $f \in M_+$, we have

$$\sum_{i \in I_+} v_i^2 [\pi_{W_i} J(f), f] = \sum_{i \in I_+} v_i^2 [\pi_{W_i} J(f), \pi_{W_i} J(f)] = \|T_{W, v_+}^\#(f)\|_{J_2}^2$$

Now, if $f \in \mathbb{K}$, then $T_{W, v_+}^\#(f) = \{v_i \pi_{W_i} J(f)\}_{i \in I_+}$. So we have,

$$\|T_{W, v_+}^\#(f)\|_{J_2}^2 = \sum_{i \in I_+} v_i^2 [\pi_{W_i} J(f), \pi_{W_i} J(f)].$$

Also, $\pi_{M_+} J(f) \in M_+$, so $T_{W, v_+}^\#(\pi_{M_+} J(f)) = \{v_i \pi_{W_i} \pi_{M_+} J(f)\}_{i \in I_+}$. Hence, we have $T_{W, v_+}^\#(\pi_{M_+} J(f)) = \{v_i \pi_{W_i} J(f)\}$. Hence,

$$\begin{aligned} \|T_{W, v_+}^\#(f)\|_{J_2}^2 &\geq \|T_{W, v_+}^\# \pi_{M_+} J(f)\|_{J_2}^2 \\ &\geq \gamma(T_{W, v_+}^\#)^2 \|\pi_{M_+} J(f)\|_J^2 \\ &= \gamma(T_{W, v_+})^2 \|\pi_{M_+} J(f)\|_J^2 \\ &= \gamma(T_{W, v_+})^2 \|G_{M_+}(f)\|_J^2 \\ &\geq \gamma(T_{W, v_+})^2 \gamma(G_{M_+})^2 \|f\|_J^2 \\ &\geq \gamma(T_{W, v_+})^2 \gamma(G_{M_+})^2 [f, f] \end{aligned}$$

Comparing with the inequality (2.1) we have $A_+ = \gamma(T_{W, v_+})^2 \gamma(G_{M_+})^2$. Similarly we have $B_- = -\frac{1}{\gamma(G_{M_-})} \|T_{W, v_-}\|_J^2$ and $A_- = -\gamma(T_{W, v_-})^2 \gamma(G_{M_-})^2$.

Here of course the J -fusion frame bounds calculated above are not optimal.

The above discussion can be summerized as follows

Theorem 2.7. *Let $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ be a J -frame of subspaces for the Krein space \mathbb{K} with optimal J -fusion frame bounds B_- , A_- , A_+ and B_+ such that $-\infty < B_- \leq A_- < 0 < A_+ \leq B_+ < \infty$. Then we have the following inequality:*

$$-\frac{1}{\gamma(G_{M_-})} \|T_{W, v_-}\|_J^2 \leq B_- \leq A_- \leq -\gamma(T_{W, v_-})^2 \gamma(G_{M_-})^2 < 0 \\ < \gamma(T_{W, v_+})^2 \gamma(G_{M_+})^2 \leq A_+ \leq B_+ \leq \frac{1}{\gamma(G_{M_+})} \|T_{W, v_+}\|_J^2.$$

Let $\{(W_i, v_i) : i \in I\}$ be a J -fusion frame for the Krein space \mathbb{K} . Then according to our definition $M_+ = \overline{\sum_{i \in I_+} W_i}$ and $M_- = \overline{\sum_{i \in I_-} W_i}$. So, M_+ and M_- are closed uniformly J -positive and J -negative subspaces respectively. Now, since $(\mathbb{K}, [\cdot, \cdot], J)$ be a Krein space, so let $\mathbb{K} = \mathbb{K}^+ [\dot{+}] \mathbb{K}^-$ be the canonical decomposition of \mathbb{K} . Let K be the angular operator of M_+ with respect to \mathbb{K}^+ . Then $\|K\| = \frac{1-\gamma(G_{M_+})}{1+\gamma(G_{M_+})}$, and also the domain of definition of K is \mathbb{K}^+ . Similarly, let Q be the angular operator of M_- with respect to \mathbb{K}^- . Then $\|Q\| = \frac{1-\gamma(G_{M_-})}{1+\gamma(G_{M_-})}$, and also the domain of definition of Q is \mathbb{K}^- .

2.4. J -frame operator and J -fusion frame operator. Let $(\mathbb{K}, [\cdot, \cdot], J)$ be a Krein space and $\{f_i : i \in I\}$ be a J -frame in \mathbb{K} . Then a careful investigation reveals that the family of vectors is not arbitrarily scattered in the Krein space. In fact the set of all positive elements form a maximal uniformly J -positive subspace $M_+ = \overline{\text{span}}\{f_i : i \in I_+\}$ and the set of all negative elements form a maximal uniformly J -negative subspace $M_- = \overline{\text{span}}\{f_i : i \in I_-\}$. Now, if we apply the J -frame operator S^{-1} on the J -frame vectors then we know that the corresponding image set also decomposes the Krein space into two parts namely $M_+^{[\perp]}$ and $M_-^{[\perp]}$. So, we have a nice distribution for the family $\{S^{-1}f_i : i \in I\}$. In a rough sense we can say that the inverse of the J -frame operator *i.e.* S^{-1} rotates any uniformly J -definite subspace onto a uniformly J -definite subspace preserving the *definiteness* 2.13 (given below). Now, let $-\infty < B_- \leq A_- < 0 < A_+ \leq B_+ < \infty$ be the optimal J -frame bounds for the J -frame $\{f_i : i \in I\}$. Let $\{S^{-1}f_i : i \in I\}$ be the canonical J -dual frame for $\{f_i : i \in I\}$ in \mathbb{K} . Therefore, the optimal frame bounds of this frame also exists. The next theorem provide us with a relation between the optimal bounds of a given J -frame and the corresponding canonical J -dual frame.

Theorem 2.8. *Let $\{f_i : i \in I\}$ be a J -frame for the Krein space \mathbb{K} with optimal frame bounds $-\infty < B_- \leq A_- < 0 < A_+ \leq B_+ < \infty$. Then the canonical J -dual frame has optimal frame bounds $-\infty < \frac{1}{A_-} \leq \frac{1}{B_-} < 0 < \frac{1}{B_+} \leq \frac{1}{A_+} < \infty$.*

Proof. Let S be the J -frame operator for the J -frame $\{f_i : i \in I\}$. Now, consider the operator $S_+|_{M_+}$, it is a bijective, J -positive and J -selfadjoint. Also, it is a frame operator for $\{f_i : i \in I_+\}$ in the Hilbert space $(M_+, [\cdot, \cdot])$. So, $A_+I|_{M_+} \leq S_+|_{M_+} \leq B_+I|_{M_+}$. Hence, $\frac{1}{B_+}I|_{M_+} \leq (S_+|_{M_+})^{-1} \leq \frac{1}{A_+}I|_{M_+}$. But we know that

$(S_+|_{M_+})^{-1} = S_+^{-1}|_{M_+^\perp}$. Hence, from the definition of J -frame it easily follows that $\frac{1}{B_+}$ and $\frac{1}{A_+}$ are the optimal frame bounds of the frame $\{S^{-1}f_i : i \in I_+\}$. Similarly, we can show that $\frac{1}{A_-}$ and $\frac{1}{B_-}$ are the optimal frame bounds of the frame $\{S^{-1}f_i : i \in I_-\}$. Hence, we establish the result. \square

The following theorem is the generalization of the fundamental identity for frames in Hilbert spaces [4].

Theorem 2.9. *Let $\{f_i : i \in I\}$ be a J -frame for the Krein space \mathbb{K} with canonical J -dual frame $\{S^{-1}f_i : i \in I\}$. Then for all $I_1 \subset I$ and for all $f \in \mathbb{K}$ we have*

$$\sum_{i \in I_1} \sigma_i |[f, f_i]|^2 - \sum_{i \in I} \sigma_i |[S_{I_1} f, S^{-1}f_i]|^2 = \sum_{i \in I_1^c} \sigma_i |[f, f_i]|^2 - \sum_{i \in I} \sigma_i |[S_{I_1^c} f, S^{-1}f_i]|^2,$$

where $\sigma_i = 1$ if $i \in I_+$ and $\sigma_i = -1$ if $i \in I_-$.

Proof. Let S denotes the frame operator for $\{f_i : i \in I\}$. Then we have $S(f) = \sum_{i \in I} \sigma_i [f, f_i] f_i$. Also since $S = S_{I_1} + S_{I_1^c}$, then $I = S^{-1}S_{I_1} + S^{-1}S_{I_1^c}$. From the operator theory we have $S^{-1}S_{I_1} - S^{-1}S_{I_1^c} = S^{-1}S_{I_1}S^{-1}S_{I_1} - S^{-1}S_{I_1^c}S^{-1}S_{I_1^c}$. Then for every $f, g \in \mathbb{K}$, we have

$$[S^{-1}S_{I_1}(f), g] - [S^{-1}S_{I_1}S^{-1}S_{I_1}(f), g] = [S_{I_1}(f), S^{-1}g] - [S^{-1}S_{I_1}(f), S_{I_1}S^{-1}g]$$

Now if we choose $g = S(f)$, then the above equation reduces to

$$= [S_{I_1}(f), f] - [S^{-1}S_{I_1}(f), S_{I_1}(f)] = \sum_{i \in I_1} \sigma_i |[f, f_i]|^2 - \sum_{i \in I} \sigma_i |[S_{I_1} f, S^{-1}f_i]|^2$$

Now replacing I_1 by I_1^c we can have the other part of the equality. Combining we finally get

$$\sum_{i \in I_1} \sigma_i |[f, f_i]|^2 - \sum_{i \in I} \sigma_i |[S_{I_1} f, S^{-1}f_i]|^2 = \sum_{i \in I_1^c} \sigma_i |[f, f_i]|^2 - \sum_{i \in I} \sigma_i |[S_{I_1^c} f, S^{-1}f_i]|^2$$

\square

Theorem 2.8 can easily be generalized in the setting for J -fusion frame. We only state the result in the following theorem. We note that Casazza et al. [6] calculated the canonical fusion frame bounds for Hilbert spaces in a more general setting, however, an error was pointed out by Gavrutu [9]. But in the current work we calculated the canonical J -fusion frame bounds in the following theorem different from their approaches due to the nice structure of J -fusion frame for Krein spaces.

Theorem 2.10. *Let $\{(W_i, v_i) : i \in I\}$ be a J -fusion frame for the Krein space \mathbb{K} with optimal frame bounds $-\infty < B_- \leq A_- < 0 < A_+ \leq B_+ < \infty$. Then the canonical J -dual fusion frame has optimal frame bounds $-\infty < \frac{1}{A_-} \leq \frac{1}{B_-} < 0 < \frac{1}{B_+} \leq \frac{1}{A_+} < \infty$.*

2.5. Bounded linear operators acting on J -fusion frames. In this section we want to address the problem of characterizing those bounded operators $T : \mathbb{K} \rightarrow \mathbb{K}$, such that $\{(T(W_i), v_i) : i \in I\}$ is a J -fusion frame for \mathbb{K} , if $\{(W_i, v_i) : i \in I\}$ is a J -fusion frame for \mathbb{K} . Now to form J -fusion frame, the subspaces $T(W_i)$ must be uniformly definite. The image of a closed, uniformly definite subspace under a bounded invertible linear operator may be a neutral subspace.

The following lemma is the Krein space version of a theorem in [9]. We need this result in the next part of our work.

Lemma 2.11. *Let $T : \mathbb{K} \rightarrow \mathbb{K}$ be any bounded linear operator and V be any closed regular subspace of \mathbb{K} . Then $Q_V T^\# = Q_V T^\# Q_{\overline{T(V)}}$, where Q_V is the J -orthogonal projection onto V .*

Example 2.12. We define an inner product $[\cdot, \cdot]$ on the sequence space ℓ^2 in the following way. Let $\{e_n\}_{n \in \mathbb{N}}$ be the countable orthonormal basis where $[e_{2n}, e_{2n}] = -1$, $[e_{2n-1}, e_{2n-1}] = 1$ for all $n \in \mathbb{N}$, and also $[e_i, e_j] = 0$ for $i \neq j$. The fundamental symmetry $J : \ell^2 \rightarrow \ell^2$ is defined by $J(\sum_{n \in \mathbb{N}} c_n e_n) = (\sum_{n \in \mathbb{N}} \sigma_n c_n e_n)$, where $\sum_{n \in \mathbb{N}} c_n e_n \in \ell^2$ and $\sigma_n = 1$, if n is odd, $\sigma_n = -1$, if n is even. Then the triple $(\ell^2, [\cdot, \cdot], J)$ forms a Krein space. Consider the invertible linear operator $T : \ell^2 \rightarrow \ell^2$ defined by $T(\{c_n\}_{n \in \mathbb{N}}) = (c_1 + c_2, c_1 + 2c_2, c_3, \dots)$. Now if $M = \text{span}\{e_1\}$, then M is a uniformly J -positive definite subspace. But $T(M) = \text{span}\{(1, 1, 0, \dots)\}$ is a neutral subspace of ℓ^2 .

Now we consider some restrictions on the linear operator T , so that $\{(T(W_i), v_i) : i \in I\}$ is also a J -fusion frame for \mathbb{K} . Before we proceed any further we need the following notations.

The set of all neutral vectors in \mathbb{K} is called the neutral part of \mathbb{K} and will be denoted by β^0 . The symbol β^{00} will stand for

$$\beta^{00} = \{x \in \mathbb{K} : [x, x] = 0, x \neq 0\}$$

We denote by β^{++} (respectively, β^{--}) the set consisting of the zero element together with all positive (negative) elements of \mathbb{K} and by β^+ (respectively, β^-) the set of all non-negative (non-positive) elements of \mathbb{K} . Thus e.g.

$$\beta^{++} = \{x \in \mathbb{K} : [x, x] > 0 \text{ or } x = 0\},$$

$$\beta^+ = \{x \in \mathbb{K} : [x, x] \geq 0\}.$$

Also, let μ^+ (respectively, μ^-) be the set of all uniformly J -positive (J -negative) subspaces for \mathbb{K} . The set of all maximal subspaces of \mathbb{K} is denoted by $\Psi(\mathbb{K})$ and the set of all regular subspaces of \mathbb{K} is denoted by $\Omega(\mathbb{K})$. Here we would like to mention that $\mu^+ \cup \mu^- \subseteq \Omega(\mathbb{K})$.

We need the following definitions.

Definition 2.13. Let T be a bounded linear operator on a Krein space \mathbb{K} . We say that T *preserves definiteness* if $T(V) \in \mu^+ \cup \mu^-$ whenever $V \in \mu^+ \cup \mu^-$, where V is a subspace of \mathbb{K} . We also say that T *preserves definiteness with sign* if the linear operator preserves definiteness and also the sign of the subspaces V and $T(V)$ remains same i.e. either $V, T(V) \in \mu^+$ or $V, T(V) \in \mu^-$.

Definition 2.14. Let T be a bounded linear operator on a Krein space \mathbb{K} . We say that T *preserves maximality* if $T(V) \in \Psi(\mathbb{K})$ whenever $V \in \Psi(\mathbb{K})$.

Definition 2.15. Let T be a bounded linear operator on a Krein space \mathbb{K} . We say that T *preserves regularity* if $T(V) \in \Omega(\mathbb{K})$ if $V \in \Omega(\mathbb{K})$.

Theorem 2.16. Let T be a bounded surjective linear operator on a Krein space \mathbb{K} . Also, let

(i) T preserves definiteness with sign.

(ii) T preserves maximality.

(iii) T preserves regularity.

Then $\{(T(W_i), v_i) : i \in I\}$ is a J -fusion frame for the Krein space \mathbb{K} , if $\{(W_i, v_i) : i \in I\}$ be a J -fusion frame for \mathbb{K} .

Proof. Let $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. For $i \in I_+$ we choose W_i where each W_i is a closed, definite subspace of \mathbb{K} . Since, T preserves definiteness with sign, hence, $T(W_i)$ is also positive definite for $i \in I_+$. Further, $T(W_i)$ is also closed since the image of closed subspace is also closed as T is bounded and linear. Now $M_+ = \overline{\sum_{i \in I_+} W_i}$ is a maximal uniformly J -positive subspace of \mathbb{K} . Also, we have $T(M_+) \subset \overline{\sum_{i \in I_+} T(W_i)}$. By virtue of our assumptions, $\overline{\sum_{i \in I_+} T(W_i)}$ is a positive subspace of \mathbb{K} . But, since T preserves maximality, hence, $T(M_+) = \overline{\sum_{i \in I_+} T(W_i)}$. Similarly, for $i \in I_-$ we can show that $\overline{\sum_{i \in I_-} T(W_i)} = T(M_-) \subset \mathbb{K}$ is a maximal negative subspace of \mathbb{K} . Now, we use our regularity assumption. Since, T preserves regularity, hence, $T(M_+)$ and $T(M_-)$ are also regular. Using the corollary 7.17 of [?] we have $T(M_+)$ and $T(M_-)$ are maximal uniformly J -positive and J -negative subspaces respectively. So, we have a decomposition of \mathbb{K} , i.e. $\mathbb{K} = T(M_+) \oplus T(M_-)$. Now, let θ be the synthesis operator for the Bessel sequence of subspaces $\{W_i, v_i) : i \in I\}$. Hence, θ is a surjective bounded linear operator. Then the mapping $T\theta$ is well defined and surjective. Now, from the definition of J -fusion frame, it easily follows that $\{(T(W_i), v_i) : i \in I\}$ is also a J -fusion frame for the Krein space \mathbb{K} . \square

Remark 2.17. Let the linear operator T considered above is also injective. Then from we know that T is a scaler multiple of J -isometry. Therefore, the class of operators are just J -unitary operators modulo multiplication by non-zero scalars.

Remark 2.18. The conditions of the above theorem are sufficient but not necessary. In fact we can get necessary conditions on T which we thought worth mentioning.

Theorem 2.19. Let $\{(W_i, v_i) : i \in I\}$ be a J -fusion frame for a Krein space \mathbb{K} and T be a bounded surjective linear operator on \mathbb{K} such that $\{(T(W_i), v_i) : i \in I\}$ is also a J -fusion frame for \mathbb{K} . Then there exists a index set I^0 such that $I^0 = I$ and $\mathbb{K} = \overline{\sum_{i \in I_+^0} T(W_i)} \oplus \overline{\sum_{i \in I_-^0} T(W_i)}$ where $I_+^0 \cup I_-^0 = I^0$. Also $\overline{\sum_{i \in I_+^0} T(W_i)}$ and $\overline{\sum_{i \in I_-^0} T(W_i)}$ are maximal uniformly J -definite subspaces but of course with opposite signs.

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