

Likelihood Ratio Test in Multivariate Linear Regression: from Low to High Dimension

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Abstract

Multivariate linear regressions are widely used statistical tools in many applications to model the associations between multiple related responses and a set of predictors. To infer such associations, it is often of interest to test the structure of the regression coefficients matrix, and the likelihood ratio test (LRT) is one of the most popular approaches in practice. Despite its popularity, it is known that the classical χ^2 approximations for LRTs often fail in high-dimensional settings, where the dimensions of responses and predictors (m, p) are allowed to grow with the sample size n . Though various corrected LRTs and other test statistics have been proposed in the literature, the fundamental question of when the classic LRT starts to fail is less studied, an answer to which would provide insights for practitioners, especially when analyzing data with m/n and p/n small but not negligible. Moreover, the power performance of the LRT in high-dimensional data analysis remains underexplored. To address these issues, the first part of this work gives the asymptotic boundary where the classical LRT fails and develops the corrected limiting distribution of the LRT for a general asymptotic regime. The second part of this work further studies the test power of the LRT in the high-dimensional setting. The result not only advances the current understanding of asymptotic behavior of the LRT under alternative hypothesis, but also motivates the development of a power-enhanced LRT. The third part of this work considers the setting with $p > n$, where the LRT is not well-defined. We propose a two-step testing procedure by first performing dimension reduction and then applying the proposed LRT. Theoretical properties are developed to ensure the validity of the proposed method. Numerical studies are also presented to demonstrate its good performance.

1 Introduction

Multivariate linear regressions are widely used in econometrics, financial engineering, psychometrics and many other areas of applications to model the relationships between multiple related responses and a set of predictors. Suppose we have n observations of m -dimensional responses $\mathbf{y}_i = (y_{i,1}, \dots, y_{i,m})^\top$ and p -dimensional predictors $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,p})^\top$, for $i = 1, \dots, n$. Let $Y = (\mathbf{y}_1, \dots, \mathbf{y}_n)^\top$ be the $n \times m$ response matrix and $X = (\mathbf{x}_1, \dots, \mathbf{x}_n)^\top$ be the $n \times p$ design matrix. The multivariate linear regression model assumes $Y = XB + E$, where B is a $p \times m$ matrix of unknown regression parameters and $E = (\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_n)^\top$ is an $n \times m$ matrix of regression errors, with $\boldsymbol{\epsilon}_i$'s independently sampled from an m -dimensional Gaussian distribution $\mathcal{N}(\mathbf{0}, \Sigma)$.

Under the multivariate linear regression model, we are interested in testing the null hypothesis $H_0 : CB = \mathbf{0}_{r \times m}$, where C is an $r \times p$ matrix with rank $r \leq p$ and $\mathbf{0}_{r \times m}$ is an all-zero matrix of size $r \times m$. This is often called general linear hypothesis in multivariate analysis and has been popularly used in multivariate analysis of variance (see, e.g., Muirhead, 2009). Different choices of the testing matrix C are of interest in various applications. For instance, if B is partitioned as $B^\top = [B_1^\top, B_2^\top]$, where B_1 is a $r \times m$ matrix; then the null hypothesis of $B_1 = \mathbf{0}$ is equivalent to taking $C = [I_r, \mathbf{0}]$, which can be used to test the significance of the first r predictors of X . Another example is to test the equivalence of the effects of a set of $r + 1$ predictors (such as different levels of some categorical variables), where C may take a form of

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & -1 \end{pmatrix}_{r \times p}. \quad (1)$$

To test $H_0 : CB = \mathbf{0}_{r \times m}$, a popularly used approach in the literature is the likelihood ratio test (LRT) (Anderson, 2003; Muirhead, 2009). Specifically, when $n > m + p$, Σ is positive definite and X has rank p , the likelihood ratio test statistic is

$$L_n = \frac{\det(S_E)^{n/2}}{\det(S_E + S_X)^{n/2}},$$

where $S_E = Y^\top [I - X(X^\top X)^{-1}X^\top]Y$ and $S_X = (C\hat{B})^\top [C(X^\top X)^{-1}C^\top]^{-1}C\hat{B}$ are the residual sum of squares and the regression sum of squares matrices, and $\hat{B} = (X^\top X)^{-1}X^\top Y$ is the least squares estimator. Assuming m and p are fixed, it is well known that $-2 \log L_n$ converges weakly to a χ^2 distribution as $n \rightarrow \infty$ under the null hypothesis (Anderson, 2003).

However, in the high-dimensional settings where the dimension parameters (p, m, r) are allowed to increase with n , the likelihood ratio test suffers from several issues. First, under the null hypothesis, the limiting distribution of $-2 \log L_n$ may not be a χ^2 distribution any more. The failure of the χ^2 approximations of LRT distributions under high dimensions has been studied by researchers under various model settings. For instance, Bai et al. (2009) examined two LRTs on testing covariance matrices, showed that their χ^2 approximations perform poorly, and proposed the corrected normal limiting distributions. Jiang and Yang (2013) and Jiang and Qi (2015) studied classical LRTs on testing sample means and covariance matrices, showed that the χ^2 approximations also fail as the dimensions increase. Moreover, Bai et al. (2013) considered the LRT on testing linear hypotheses in high-dimensional multivariate linear regressions, demonstrated the failure of χ^2 approximation and derived the corrected LRT. Note that Bai et al. (2009) only consider the high-dimensional settings where $m, r, n - p$ are proportional to each other with $m \leq r$. Despite these existing works, it is still unclear under which asymptotic regimes the χ^2 approximation of LRT starts to fail. An answer to this question would provide insights for practitioners especially when analyzing data with m/n and p/n small but not negligible.

The second issue of the likelihood ratio test in the high-dimensional settings concerns with its power performance under alternative hypotheses. When $n > p + m$, the statistic $-2 \log L_n = n \sum_{i=1}^{\min\{m,r\}} \log(1 + \lambda_i)$, where λ 's are the eigenvalues of $S_X^{1/2} S_E^{-1} S_X^{1/2}$, therefore it is expected that the asymptotic power behavior of LRT would depend on an averaged effect of all the eigenvalues. Such an study of the eigenvalues of the random matrix $S_X^{1/2} S_E^{-1} S_X^{1/2}$ under alternative hypotheses remains still underexplored in the literature.

The third issue of the likelihood ratio test arises when the dimension parameters p and m are large such that $n < p + m$. In this situation, the likelihood ratio test is not well defined due to the singularity of the matrix S_E . This excludes the likelihood ratio test for many high-dimensional applications with $p > n$ or $m > n$ (e.g., Donoho, 2000; Clarke et al., 2008; Fan et al., 2014). When $m > n$, the linear hypothesis testing problem has been studied in depth for specific submodels such as one-way MANOVA (Srivastava and Fujikoshi, 2006; Hu et al., 2017; Zhou et al., 2017; Cai and Xia, 2014, etc.). More generally, Li et al. (2018) recently proposed a modified LRT for the general linear hypothesis tests via spectral shrinkage. However, these works assume that p is fixed.

This paper aims to address the above problems. First, under the null hypothesis, we derive the asymptotic boundary when the χ^2 approximation fails as the dimension parameters (p, m, r) increase with the sample size n . Moreover, we develop the corrected limiting distribution of $\log L_n$ for a general asymptotic regime of (p, m, r, n) . Second, under alternative hypotheses, we characterize the statistical power of $\log L_n$ in the high-dimensional setting. Using a technique of analyzing partial differential equations induced by the test statistic, we show that the likelihood ratio test is powerful when the trace of the signal matrix $(CB)\Sigma^{-1}(CB)^\top$ is large while it would lose power under a low-rank signal matrix. With unknown alternatives in practice, we propose an enhanced likelihood ratio test such that it is also powerful against low-rank alternative signal matrices. The power-enhanced test statistic combines the LRT statistic and the largest eigenvalue (Johnstone, 2008, 2009)

to further improve the test power against low-rank alternatives. Third, when $n < p$ and the likelihood ratio test is not well defined, we propose to use a two-step testing procedure by first performing dimension reduction to covariates and responses and then using the proposed (enhanced) likelihood ratio test. To control the estimation error induced by the dimension reduction in the first step, we employ a *repeated* data-splitting approach and show the asymptotic type I error is well controlled under the null hypothesis. Simulation results further confirm the good performance of the proposed approach.

The rest of the paper is organized as follows. In Section 2, we examine when the classic LRT fails under the null hypothesis and propose a corrected limiting distribution of $\log L_n$. We further analyze the power of $\log L_n$ and propose the power-enhanced test statistic in Section 3. In Section 4, we discuss the multi-split LRT procedure when $n < p$. Simulation studies and a real dataset analysis on breast cancer are reported in Sections 5 and 6.

2 When likelihood ratio test starts to fail?

In traditional multivariate regression analysis where the dimension parameters p, m, r are considered as fixed numbers, the χ^2 approximation of the likelihood ratio test,

$$-2\log L_n \xrightarrow{D} \chi_{mr}^2, \quad \text{as } n \rightarrow \infty, \quad (2)$$

is used for $H_0 : CB = \mathbf{0}_{r \times m}$ (Muirhead, 2009; Anderson, 2003), where \xrightarrow{D} denotes the convergence in distribution. However, it has been noted that the χ^2 approximation of the distribution of the LRT often performs poorly in high-dimensional applications (see, e.g., Bai and Saranadasa, 1996; Jiang et al., 2012; Bai et al., 2009, 2013; Jiang and Yang, 2013).

As the three dimension parameters m, p, r are allowed to grow with n , it is of interest to examine the phase transition boundary where the χ^2 approximation fails. This is described

in the following theorem.

Theorem 1 *Consider $n > p + m$, $p \geq r$, $mr \rightarrow \infty$ and $\max\{p, m, r\}/n \rightarrow 0$ as $n \rightarrow \infty$. Let $\chi_{mr}^2(\alpha)$ denote the upper α -quantile of χ_{mr}^2 distribution. Then as $n \rightarrow \infty$,*

$$P\{-2 \log L_n > \chi_{mr}^2(\alpha)\} \rightarrow \alpha,$$

for any significance level α , if and only if

$$\lim_{n \rightarrow \infty} \sqrt{mr}(p + m/2 - r/2)n^{-1} = 0. \quad (3)$$

Theorem 1 gives the necessary and sufficient condition on (m, p, r, n) such that the χ^2 approximation (2) fails. To further examine the implications of (3), we consider two special cases. Specifically, let $m = \lfloor n^\eta \rfloor$ and $p = \lfloor n^\epsilon \rfloor$ with $\eta, \epsilon \in (0, 1)$, where $\lfloor \cdot \rfloor$ denotes the floor of a number. When r is fixed, (3) implies $\sqrt{m}(p + m/2) = o(n)$, that is, $\max\{\epsilon, \eta\} + \eta/2 < 1$. When $r = p = \lfloor n^\epsilon \rfloor$, (3) implies $\sqrt{mp}(p + m) = o(n)$, that is, $\max\{\eta, \epsilon\} + (\eta + \epsilon)/2 < 1$. For these two cases, we correspondingly give two (η, ϵ) -regions in Figure 1 satisfying the constraint (3). In these two regions, when ϵ becomes close to 0, the largest η approaches $2/3$. This implies that when p is small, the largest m such that (3) holds is of order $n^{2/3}$. It is the same for both fixed r and $r = p$ cases as p is small and $r \leq p$. In addition, when η goes to 0, the largest ϵ values for fixed r and $r = p$ converge to 1 and $2/3$ respectively. This indicates that when m is small, the largest p values satisfying (3) are of order n and $n^{2/3}$ for the two cases respectively. Moreover, when $m = p$, the largest orders of m and p for the two cases are $n^{2/3}$ and $n^{1/2}$ respectively.

To illustrate this phase transition phenomenon, we present a simple simulation experiment. We set $\Sigma = I_m$, and estimate the type I error of the χ^2 approximation (2) with 10^4

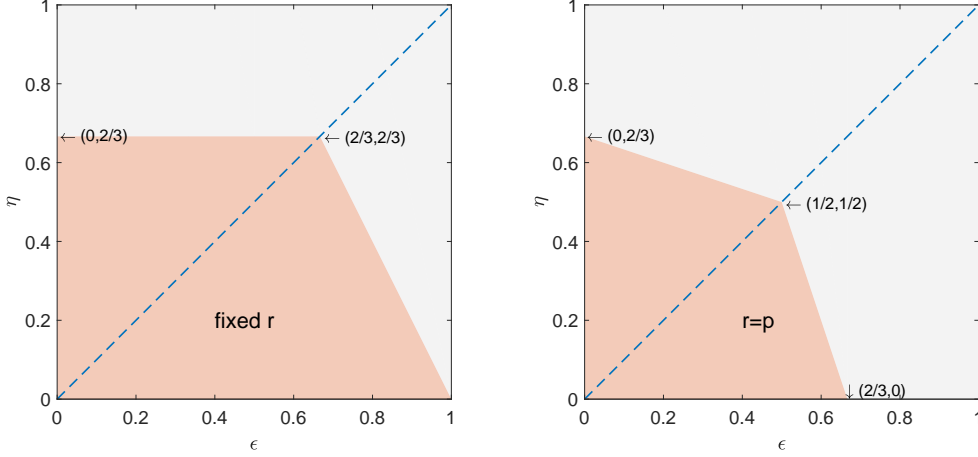


Figure 1: η versus ϵ when r is fixed and $r = p$

repetitions under the following four cases:

- (a) fixed $m = r = 2$, $p = \lfloor n^\eta \rfloor$; (b) fixed $p = r = 2$, $m = \lfloor n^\eta \rfloor$;
- (c) fixed $m = 2$, $p = r = \lfloor n^\eta \rfloor$; (d) $p = m = r = \lfloor n^\eta \rfloor$,

where $\eta \in \{1/24, \dots, 23/24\}$. In Figure 2, we plot the estimated type I errors against η values for $n = 100$ and 300 respectively. The plots show consistent patterns with the theoretical results. In particular, when $p = m = r = \lfloor n^\eta \rfloor$, the χ^2 approximation begins to fail for η around $1/2$. When p and r are fixed and $m = \lfloor n^\eta \rfloor$, or when m is fixed and $p = r = \lfloor n^\eta \rfloor$, the χ^2 approximation begins to fail for η around $2/3$. When m and r are fixed and $p = \lfloor n^\eta \rfloor$, the χ^2 approximation begins to fail later than the other three curves, which is consistent with the theoretical results.

It is worthy to mention that the sufficient and necessary constraint in (3) also characterizes the bias of χ^2 approximation. Specifically, under the conditions of Theorem 1, we have $E(-2 \log L_n - \chi^2_{mr}) / \sqrt{\text{var}(\chi^2_{mr})} = \sqrt{mr}(p + m/2 - r/2 + 1/2)n^{-1}\{1 + o(1)\}$. Thus when p, m, r are large such that (3) is violated and the χ^2 approximation fails, the bias of χ^2 approximation increases as $\sqrt{mr}(p + m/2 - r/2 + 1/2)n^{-1}$ becomes larger. This can be

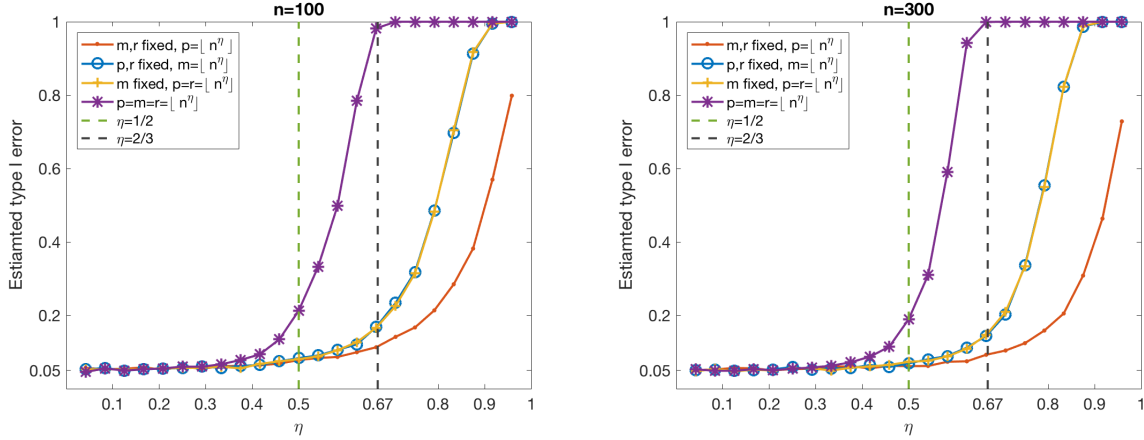


Figure 2: Estimated type I error using χ^2 approximation (2)

seen from Figure 2 and is further illustrated by simulation studies in Section 5.

In the classic regime with fixed m and p , researchers have also proposed the Bartlett correction of the LRT that $-2\rho \log L_n \xrightarrow{D} \chi_{mr}^2$, where $\rho = 1 - (p - r/2 + m/2 + 1/2)/n$. In particular, for any $z \in \mathbb{R}$, this corrected approximation gets rid of the first order approximation error $O(n^{-1})$; that is, for any z , $P(-2\rho \log L_n < z) - P(\chi_{mr}^2 < z) = O(n^{-2})$ when m and p are fixed. Similarly to Theorem 1, the χ^2 approximation with Bartlett correction also fails as m and p increase with n . The phase transition boundary is characterized in the following result.

Theorem 2 Consider $n > p + m$, $p \geq r$, $mr \rightarrow \infty$ and $\max\{p, m, r\}/n \rightarrow 0$ as $n \rightarrow \infty$. Let $\chi_{mr}^2(\alpha)$ denote the upper α -quantile of χ_{mr}^2 distribution. Then as $n \rightarrow \infty$,

$$P\{-2\rho \log L_n > \chi_{mr}^2(\alpha)\} \rightarrow \alpha,$$

for any significance level α , if and only if $\lim_{n \rightarrow \infty} \sqrt{mr}(r^2 + m^2)n^{-2} = 0$.

Interestingly, the phase transition threshold in Proposition 2 only involves m and r , and does not depend on p . It implies that when m and r are fixed, the error of the corrected approximation does not increase with p under the asymptotic regime in Proposition 2; when

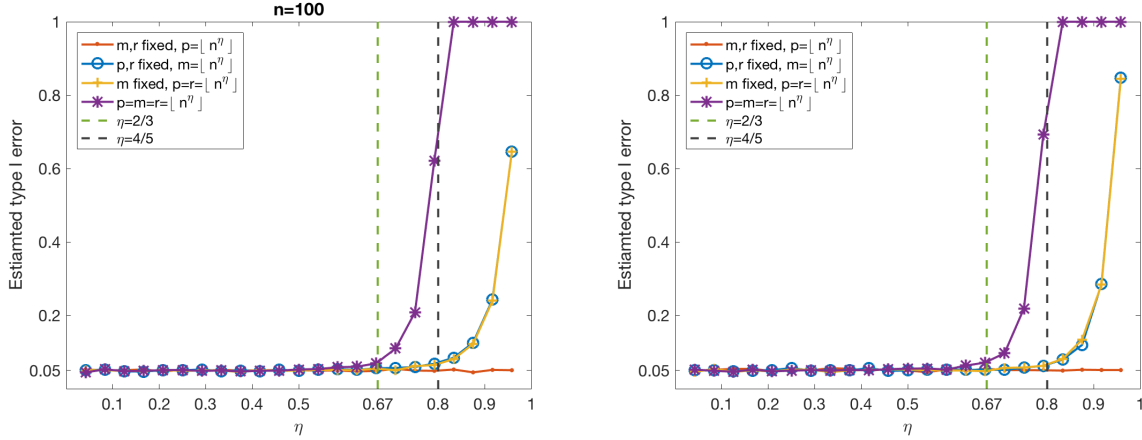


Figure 3: Estimated type I error using χ^2 approximation with Bartlett correction

r is fixed and $m = \lfloor n^\eta \rfloor$, or when m is fixed, and $p = \lfloor n^\eta \rfloor$, the χ^2 approximation with Bartlett correction fails when $\eta \geq 4/5$; when $m = r = \lfloor n^\eta \rfloor$, the corrected approximation fails when $\eta \geq 2/3$.

To illustrate the phenomenon, we present a numerical experiment on the χ^2 approximation with Bartlett correction in Figure 3 under the same set-up as in Figure 2. It shows that when m and r are fixed and $p = \lfloor n^\eta \rfloor$, the estimated type I errors do not increase with η . In addition, when p and r are fixed and $m = \lfloor n^\eta \rfloor$, or when m is fixed and $p = r = \lfloor n^\eta \rfloor$, the corrected χ^2 approximation begins to fail around $\eta = 4/5$. When $p = m = r = \lfloor n^\eta \rfloor$, the corrected χ^2 approximation begins to fail around $\eta = 2/3$. These numerical results are also consistent with the theoretical results.

To have a unified limiting distribution for analyzing high-dimensional data as m, p, r diverge, we derive a corrected normal limiting distribution for the likelihood ratio test statistic.

Theorem 3 *When $n > p + m$, $p \geq r$, $mr \rightarrow \infty$, and $n - p - \max\{m - r, 0\} \rightarrow \infty$ as $n \rightarrow \infty$, the likelihood ratio test statistic L_n has corrected form T_1 satisfying*

$$T_1 := \frac{-2 \log L_n + \mu_n}{n \sigma_n} \xrightarrow{D} \mathcal{N}(0, 1), \quad (4)$$

where

$$\begin{aligned}\mu_n &= n(n - m - p - 1/2) \log \frac{(n + r - p - m)(n - p)}{(n - p - m)(n + r - p)} + nr \log \frac{(n + r - p - m)}{(n + r - p)} \\ &\quad + nm \log \frac{(n - p)}{(n + r - p)}, \\ \sigma_n^2 &= 2 \log \frac{(n + r - p - m)(n - p)}{(n - p - m)(n + r - p)}.\end{aligned}\tag{5}$$

The theorem covers the asymptotic regime where $mr \rightarrow \infty$, $\max\{p, m, r\}/n \rightarrow 0$ and the constraint (3) holds. We can show that under this case, $\mu_n \rightarrow -mr$ and $(n\sigma_n)^2 \rightarrow 2mr$, which are consistent with the mean and variance of χ_{mr}^2 approximation. Even when m or r is small, the normal approximation (4) could still perform well as long as mr is large enough. The simulations in Section 5 show that the performance of the χ^2 and normal approximations could be similar in low dimensions.

Alternatively, under some high dimensional settings, we can check that no χ^2 or even noncentral χ^2 distribution could match the asymptotic mean and variance of $-2\log L_n$ in Theorem 3. Specifically, if the distribution of $-2\log L_n$ could be approximated by some χ^2 distribution, then we should have $-(n\sigma_n)^2/\mu_n \rightarrow 2$, which is, however, not satisfied as $p/n, m/n, r/n$ increase; or if the distribution of $-2\log L_n$ could be approximated by some noncentral χ^2 distribution with degrees of freedom k_n , then we should have $k_n = -2\mu_n - n^2\sigma_n^2/2$, which, however, will become negative as $p/n, m/n, r/n$ increase. Thus it implies that the χ^2 -type approximation for $-2\log L_n$ could fail fundamentally under high dimensions.

Remark 1 *A similar result on the asymptotic normality of $\log L_n$ in Theorem 3 was proved in Bai et al. (2013). However, there are several differences between our result and Bai et al. (2013). First, our asymptotic regime is more general than Bai et al. (2013). Specifically, Bai et al. (2013) requires that $m < r$, $\min\{m, r\} \rightarrow \infty$, and $m/(n - p)$ converges to a constant*

in $(0, 1)$, while we only need $mr \rightarrow \infty$ and $n - p - \max\{m - r, 0\} \rightarrow \infty$. Our analysis covers the case when $m/(n - p) \rightarrow 0$ and even when the limit does not exist. Second, the proof of Bai et al. (2013) is based on the random matrix theory, while we prove Theorem 3 by a moment generating function technique motivated by Jiang and Yang (2013).

3 Power analysis and an enhanced likelihood ratio test

Although the limiting behaviors of LRTs for high dimensional data have been recently explored under different testing problems, the power of LRTs is less studied and remains a challenging problem, as discussed in Jiang and Yang (2013). In this section, we focus on the high-dimensional multivariate linear regression and analyze the power of the LRT statistic. Moreover, based on the theoretical result, we propose a power-enhanced testing approach to further improve the power of the LRT.

To examine the power of LRT statistic, we introduce the classic canonical form of the LRT problem, which writes $H_0 : CB = \mathbf{0}_{r \times m}$ to an equivalent form as follows (Muirhead, 2009). Specifically, consider the matrix decomposition $X = O[I_p, \mathbf{0}_{p \times (n-p)}]^\top D$, where O is an $n \times n$ orthogonal matrix and D is a $p \times p$ nonsingular real matrix. Given D , we have similar decomposition $CD^{-1} = E[I_r, \mathbf{0}_{r \times (p-r)}]V$, where E is an $r \times r$ nonsingular matrix and V is a $p \times p$ orthogonal matrix. This gives that $CB = CD^{-1}DB = E[I_r, \mathbf{0}_{r \times (p-r)}]VDB$, and therefore $H_0 : CB = \mathbf{0}_{r \times m}$ is equivalent to $M_1 = \mathbf{0}_{r \times m}$, where we define $M_1 = [I_r, \mathbf{0}_{r \times (p-r)}]VDB = E^{-1}CB$.

We next describe the relationship between M_1 and the LRT statistic through a linear transformation of Y . Let V_1 denote the first r rows of V . Define $Y_1^* = [V_1, \mathbf{0}_{r \times (n-p)}]O^\top Y$ and $Y_3^* = [\mathbf{0}_{p \times p}, I_{n-p}]O^\top Y$. We then know that $Y_1^{*\top} Y_1^* = S_X$ and $Y_3^{*\top} Y_3^* = S_E$. We further define $\tilde{S}_X = \Sigma^{-1/2} S_X \Sigma^{-1/2}$, $\tilde{S}_E = \Sigma^{-1/2} S_E \Sigma^{-1/2}$, and $\Omega = \Sigma^{-1/2} M_1^\top M_1 \Sigma^{-1/2}$. Then we can write the LRT statistic $-2 \log L_n = n \sum_{i=1}^{\min\{m, r\}} \log(1 + \lambda_i)$, where λ 's are the eigenvalues of

$\tilde{S}_E^{-1}\tilde{S}_X$. Given the fact that $E(\tilde{S}_E^{-1}\tilde{S}_X) = (rI_m + \Omega)/(n - p)$, it is then expected that the power of LRT depends on an averaged effect of all the eigenvalues of Ω .

We focus on the alternatives where the signal matrix Ω is of low rank and p, m, r increase proportionally with n . In particular, we assume Ω has a fixed rank m_0 , and write $\Omega = n\Delta$, where Δ has fixed nonzero eigenvalues $\delta_1, \dots, \delta_{m_0}$. Note that this is reasonable when the entries in $M_1\Sigma^{-1/2}$ are $O(1)$, as the entries in Ω could be $O(n)$ with r proportional to n . The following proposition specifies how the power of the LRT statistic T_1 depends on the eigenvalues of Ω .

Theorem 4 *Consider the setting where p, m, r increase proportionally with n , and $p/n = \rho_p$, $m/n = \rho_m$, and $r/n = \rho_r$, where $\rho_p, \rho_m, \rho_r \in (0, 1)$ are fixed constants and $\rho_p + \rho_m < 1$. Given $\Delta = \Omega/n$ with fixed nonzero eigenvalues $\delta_1, \dots, \delta_{m_0}$, define $W_\Delta = \sum_{j=1}^{m_0} \log[1 + \delta_j(1 + \rho_r - \rho_p)^{-1}]$. There exists a constant $A_1 > 0$ such that $P(T_1 > z_\alpha) \rightarrow 1 - \Phi(z_\alpha - A_1 W_\Delta)$, where $\Phi(\cdot)$ and z_α denote the cumulative distribution function and the upper α -quantile of $\mathcal{N}(0, 1)$, respectively.*

Theorem 4 establishes the relationship between the eigenvalues of Ω and the power of T_1 under the high-dimensional and low-rank signals. It implies that when W_Δ is large, T_1 has high power. On the other hand, the LRT could be highly underpowered when W_Δ is small. As in real applications, the underlying truth is usually unknown, it is desired to have a testing procedure with high statistical power against a variety of alternatives.

To enhance the power of LRT, we propose to combine it with the Roy's test statistic that is based on the largest eigenvalue of $S_E^{-1}S_X$ (Roy, 1953). In particular, Johnstone (2008, 2009) extended Roy's test to high dimensional settings and proposed the largest eigenvalue test statistic

$$T_2 = [\log\{\theta_{n,1}/(1 - \theta_{n,1})\} - \tilde{\mu}_n]/\tilde{\sigma}_n, \quad (6)$$

where $\theta_{n,1} = \lambda_{\max}\{(S_E + S_X)^{-1}S_E\}$ with $\lambda_{\max}(\cdot)$ denoting the largest eigenvalue, and $\tilde{\mu}_n = 2 \log \tan\{(\phi + \gamma/2)\}$ and $\tilde{\sigma}_n^3 = 16(n - p + r - 1)^{-2}\{\sin^2(\phi + \gamma) \sin \phi \sin \gamma\}^{-1}$ with $\sin^2(\gamma/2) = \{\min(m, r) - 1/2\}/(n - p + r - 1)$ and $\sin^2(\phi/2) = \{\max(m, r) - 1/2\}/(n - p + r - 1)$. Moreover, Johnstone (2008) proved that under high-dimensional null hypothesis,

$$T_2 \xrightarrow{D} \mathcal{TW}, \quad (7)$$

where \mathcal{TW} denotes a Tracy-Widom distribution. Under alternative hypothesis, Dharmawansa et al. (2018) further studied the spiked alternative with $\Omega = rUHU^\top$, where U is an $m \times m_0$ matrix with orthonormal columns and fixed m_0 , and $H = \text{diag}(h_1, \dots, h_{m_0})$ with $h_1 > \dots > h_{m_0}$. They showed that the phase transition threshold for h 's is a constant that depends on the limit of $p/n, m/n, r/n$. Note that with fixed r/n , there exists a constant $c_2 > 0$ such that $\delta_1 = c_2 h_1$. This implies that when δ_1 is a constant large enough, the power of T_2 can converge to 1, while the LRT statistic T_1 may only have power smaller than 1 by Theorem 4. On the other hand, when δ_1 is below the phase transition threshold, T_1 could be more powerful than T_2 .

We then propose the combined test statistic as

$$T_3 = T_1 + T_2 * I(T_2 \geq F_n). \quad (8)$$

where F_n is a positive constant depending on n . We next show that with properly chosen F_n , the proposed test statistic T_3 could enhance the power of T_1 without sacrificing the size asymptotically. First, under the null hypothesis, the type I error rate of T_3 is controlled if $P\{T_2 \geq F_n\} \rightarrow 0$. From the limiting distribution of T_2 in (7), we then need $F_n \rightarrow \infty$ as the sample size increases. Second, under alternative hypotheses, we have $P(T_3 > z_\alpha) \geq P(T_1 > z_\alpha)$ since $T_2 * I\{T_2 > F_n\} \geq 0$ for positive F_n . This guarantees that the power of T_3

is at least as large as that of the LRT statistic T_1 . Moreover, consider the case when W_Δ is relatively small but δ_1 is significantly above the phase transition threshold, where T_2 is more powerful than T_1 ; then if F_n does not grow too quickly, T_3 would be also powerful. In summary, we can choose F_n to be a slow varying function and the combined test statistic T_3 could improve the power of T_1 with little size distortion. Through extensive simulation studies, we find $F(n) = \max\{\log \log n, 2\}$ exhibits good performance.

4 Likelihood ratio test when $p > n$

When the number of predictors is large such that $p > n$, S_E becomes singular and the test statistics T_1 , T_2 and T_3 can not be directly applied. To deal with this issue, we propose a multiple data splitting procedure, which repeatedly splits the data into two random subsets. We use the first subset to perform dimension reduction and obtain a manageable size of predictors, and then apply the proposed likelihood ratio test on the second subset. The test statistics from different data splittings are aggregated together to provide the final test statistic. The random splits of data ensure correct size control of the test type I error. Similar ideas are used in other high-dimensional problems (Thall et al., 1997; Wasserman and Roeder, 2009; Berk et al., 2013, etc.). We next describe the proposed procedure.

Consider the setting when $p > n$ and $m < n$. Denote $B = [\mathbf{b}_1, \dots, \mathbf{b}_p]^\top$ and $\mathcal{M}_* = \{k : \mathbf{b}_k \neq \mathbf{0}, 1 \leq k \leq p\}$. We assume the “sparsity” structure that the responses only depend on a subset of the predictors (or transformed predictors) such that $n > m + |\mathcal{M}_*|$. Let $X_{\mathcal{M}_*}$ be the $n \times |\mathcal{M}_*|$ submatrix of X with columns indexed by \mathcal{M}_* and $B_{\mathcal{M}_*}$ be the $|\mathcal{M}_*| \times m$ submatrix of B with rows indexed by \mathcal{M}_* . The underlying model then satisfies $Y = X_{\mathcal{M}_*} B_{\mathcal{M}_*} + E$. Under this model, for any subset $\mathcal{M} \subseteq \{1, \dots, p\}$ such that $\mathcal{M} \supseteq \mathcal{M}_*$ and $n > m + |\mathcal{M}|$, testing $CB = \mathbf{0}$ is equivalent to $C_{\mathcal{M}} B_{\mathcal{M}} = \mathbf{0}$ and the likelihood ratio test is then applicable. Here $C_{\mathcal{M}}$ denotes the $r \times |\mathcal{M}|$ submatrix of C with columns indexed by \mathcal{M} , and $B_{\mathcal{M}}$ denotes

the $|\mathcal{M}| \times m$ submatrix of B with rows indexed by \mathcal{M} .

To obtain such a set $\mathcal{M} \supseteq \mathcal{M}_*$, we propose a screening method for the multivariate linear regression. The seminal work of Fan and Lv (2008) first introduced a sure independence screening procedure that can significantly reduce the number of predictors while preserving the true linear model with an overwhelming probability. This procedure has been further developed in various settings, including generalized linear model (Fan and Song, 2010), additive model (Fan et al., 2011), and quantile regression (He et al., 2013). As SIS builds on marginal correlations between the response and the features, various extensions of correlation have been proposed to deal with more general cases (Hall and Miller, 2009; Zhu et al., 2011; Li et al., 2012a,b; Shao and Zhang, 2014; Wang and Leng, 2016). More recently, people also propose some methods that accommodate conditional correlation (Barut et al., 2016). However, many of the existing works target at the settings with univariate response variable, and then can only consider the information from each response variable separately.

To utilize the joint information from multivariate response variables, we propose a screening method that selects the columns of X by their canonical correlations with Y . The canonical correlation is a widely used dimension reduction criterion inferring information from cross-covariance matrices in multivariate analysis (see, e.g., Hardoon et al., 2004; Härdle and Simar, 2015; Bao et al., 2017). Specifically, for each column vector $\mathbf{x}^j = (x_{1,j}, \dots, x_{n,j})^\top$, $j = 1, \dots, p$, we first compute its canonical correlation with Y denoted by

$$\omega_j = \max_{\mathbf{a} \in \mathbb{R}^m} \frac{\mathbf{a}^\top (Y - \mathbf{1}_n \bar{Y})^\top (\mathbf{x}^j - \bar{x}^j \mathbf{1}_n)}{\sqrt{\{\mathbf{a}^\top (Y - \mathbf{1}_n \bar{Y})^\top (Y - \mathbf{1}_n \bar{Y}) \mathbf{a}\} \times \{(\mathbf{x}^j - \bar{x}^j \mathbf{1}_n)^\top (\mathbf{x}^j - \bar{x}^j \mathbf{1}_n)\}}},$$

where $\bar{x}^j = \sum_{i=1}^n x_{i,j}/n$, \bar{Y} is the row mean vector of Y , and $\mathbf{1}_n$ is an all 1 column vector of length n . Then for $0 < \delta < 1$, we select $\lfloor \delta p \rfloor$ columns of X with the highest canonical correlations with Y , and define the selected column set as $\mathcal{M}_\delta = \{j : |\omega_j| \text{ is among the largest } \lfloor \delta p \rfloor \text{ of all } 1 \leq j \leq p\}$. In practice, we choose an integer $\lfloor \delta p \rfloor$ such that $n_T > \lfloor \delta p \rfloor + m$

to apply the likelihood ratio test. On the other hand, we keep $\lfloor \delta p \rfloor$ large to increase the probability of $\mathcal{M}_\delta \supseteq \mathcal{M}_*$. The following theoretical result provides the desired screening property that $P(\mathcal{M}_* \subseteq \mathcal{M}_\delta) \rightarrow 1$ for properly chosen δ .

To derive Theorem 5, we need some regularity conditions. We use $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ to denote the largest and smallest eigenvalues of a matrix respectively; $\text{diag}(\cdot)$ denotes the vector of diagonal elements of a matrix; $\max \text{diag}(\cdot)$ and $\min \text{diag}(\cdot)$ represent the maximum and minimum value of the diagonal elements of a matrix respectively; $\|\cdot\|$ denotes the ℓ_2 -norm of a vector; and $e_i = (0, \dots, 0, 1, 0, \dots, 0)^\top$ denotes the indicator vector with 1 on the i th entry.

Condition 1 *Rows of X and E independently follow multivariate Gaussian distribution with covariance matrices Σ_x and Σ respectively. There exist constants $\tau, t \geq 0$ and $c_1, c_2, c_3, c_4, c_5 > 0$ such that $\lambda_{\max}(\Sigma_x) \leq c_1 n^\tau$, $\lambda_{\min}(\Sigma) \geq c_2 n^{-t}$, $\min \text{diag}(\Sigma_x) \geq c_3$, $\max \text{diag}(\Sigma) \leq c_4$ and $\max \text{diag}(B^\top \Sigma_x B) \leq c_5$.*

Condition 2 *For some constants $\kappa, u, c_6 > 0$ and $c_7 > 0$, and fixed $i \in \mathcal{M}_*$, there exists $\mathbf{a}_{0,i} \in \mathbb{R}^m$ with $\|\mathbf{a}_{0,i}\| = 1$ such that $\max\{\|\Sigma_x^{1/2} B \mathbf{a}_{0,i}\|, \|\Sigma^{1/2} \mathbf{a}_{0,i}\|\} \leq c_6 n^u$ and $|e_i^\top \Sigma_x B \mathbf{a}_{0,i}| \sigma_{x,i}^{-1} \geq c_7 n^{-\kappa}$, where $\sigma_{x,i}^2$ is the i -th diagonal element of Σ_x .*

Condition 3 *Assume $m = O(n^s)$ with $0 \leq s < 1$; $2u + 2\kappa + \tau + t + s < 1$; $p > c_9 n$ for some constant $c_9 > 1$, and $\log p = O(n^\pi)$ for some constant $\pi \in (0, 1 - 2\kappa - 2u - t - s)$.*

Remark 2 *In Condition 1, we assume that X and E follow the Gaussian distribution for the ease of theoretical developments. We allow the eigenvalues of Σ_x and Σ to diverge or degenerate as n grows, which is similarly assumed in Fan and Lv (2008) and Wang and Leng (2016) etc. in studying the linear regression with univariate response. The boundedness of the diagonal elements of Σ and $B^\top \Sigma_x B$ is satisfied when the variances of response variables are $O(1)$. Condition 2 implies that there exists a combination of the response variables*

whose absolute covariance with the i -th predictor is sufficiently large. In particular, suppose for each $i \in \mathcal{M}_*$, there exists $k_i \in \{1, \dots, m\}$ such that $\text{cov}(x_{1,i}, y_{1,k_i})\sigma_{x,i}^{-1} \geq c_7 n^{-\kappa}$. Then Condition 2 is satisfied under Condition 1. Condition 3 allows the number of predictors p grow exponentially with n . The requirement $2u + 2\kappa + \tau + t + s < 1$ is satisfied when the eigenvalues of Σ_x , $B^\top \Sigma_x B$ and Σ do not diverge or degenerate too fast with n , and the covariance between $x_{1,i}$ and $\mathbf{y}_1^\top \mathbf{a}_{0,i}$ is sufficiently large.

Theorem 5 Under conditions 1-3, if $\delta n^{1-2u-2\kappa-\tau-t-s} \rightarrow \infty$ as $n \rightarrow \infty$, then we have, for some constant $c_0 > 0$,

$$P(\mathcal{M}_* \subseteq \mathcal{M}_\delta) = 1 - O[\exp\{-c_0 n^{1-2\kappa-2u-t-s} / \log n\}], \quad (9)$$

which converges to 1 as $n \rightarrow \infty$.

Remark 3 When testing the coefficients of the first r predictors of X , such as $C = [I_r, \mathbf{0}_{r \times (p-r)}]$, we can keep the first r predictors, denoted by X_1 , in the model, while screen the remaining predictors, denoted by X_2 . In particular, we can apply the screening procedure on the residuals \tilde{R} , from the regression of Y on X_1 , and X_2 . More generally when C is a matrix of rank r , we can use this conditional screening procedure through a linear transformation of data. In particular, given the singular value decomposition $C = UVD^\top$, we can transform X and B into $\tilde{X} = XD$ and $\tilde{B} = D^\top B$. Then testing $H_0 : CB = \mathbf{0}_{r \times m}$ is equivalent to testing $H_0 : [I_r, \mathbf{0}_{r \times (p-r)}]\tilde{B} = \mathbf{0}_{r \times m}$ under the model of the transformed data $Y = \tilde{X}\tilde{B} + E$. The theoretical result similar to Theorem 5 can be obtained under properly adjusted assumptions.

Remark 4 Theorem 5 implies that when the tuning parameters are chosen appropriately, with an overwhelming probability, the submodel \mathcal{M}_δ will contain the true model \mathcal{M}_* . The proposed screening procedure uses the canonical correlation, which is an extension of the marginal correlation in Fan and Lv (2008). Note that the proposed method aggregates the

joint information of the response variables, and therefore could be better than simply applying the marginal screening with respect to each response variable. Computationally, calculating the canonical correlations has low cost and is pre-implemented in many existing softwares. Alternatively, we may generalize the other proposed screening methods in the literature to the multivariate regression setting, such as the high dimensional ordinary least squares projection method in Wang and Leng (2016). Similar screening properties can be obtained and the following testing procedure can still apply.

Given a proper screening approach, we propose a data splitting procedure to apply the likelihood ratio test. We randomly split n observations into two independent sets – the screening data $\{X_S, Y_S\}$ of size n_S and the testing data $\{X_T, Y_T\}$ of size n_T . We use $\{X_S, Y_S\}$ to select \mathcal{M} and apply the proposed likelihood ratio test to the testing data $\{X_T, Y_T\}$ with the selected predictors in \mathcal{M} . Data splitting avoids the influence of the screening step on the inference step and provides valid inference as widely recognized in the literature (Cox, 1975; Meinshausen et al., 2009; Berk et al., 2013; Taylor and Tibshirani, 2015; Wager et al., 2016; Robins et al., 2017; Chernozhukov et al., 2018; Wager and Athey, 2018). We also demonstrate that the type I error rate can not be controlled without splitting the data by the simulation studies in Section 5.

It is known that the testing result from a single random split is sensitive to the arbitrary split choice and would be difficult to reproduce the result (Breiman, 1996; Meinshausen et al., 2009; Yu, 2013; Efron, 2014; Meinshausen and Bühlmann, 2010). Therefore, we further propose to use multiple splits and aggregate the obtained results. Note that computing test statistics by splitting data can be viewed as a resampling method. Since resampling methods usually do not perform well for approximating statistics depending on the eigenvalues of high-dimensional random matrices (Karoui and Purdom, 2016), and the test statistics computed after splitting data are correlated, it is challenging to combine the statistics in a valid and

efficient method.

In this paper, we adopt the general p -value combination method proposed by Meinshausen et al. (2009), which is motivated from multiple hypothesis testing. Specifically, we randomly split data J times, and compute all the J p -values for different splits. For each $j = 1, \dots, J$, we compute p -value $p^{(j)}$ with data splitting. Then for $\gamma \in (0, 1)$, define $Q(\gamma) = \min\{1, q_\gamma(\{p^{(j)}/\gamma; j = 1, \dots, J\})\}$, where q_γ denotes empirical γ quantile function. As a proper selection of γ may be difficult, we use an adaptive version below. Let $\gamma_{\min} \in (0, 1)$ be a lower bound for γ , and define the adjusted p -value p_t as

$$p_t = \min \left\{ 1, (1 - \log \gamma_{\min}) \inf_{\gamma \in (\gamma_{\min}, 1)} Q(\gamma) \right\}. \quad (10)$$

The extra correction factor $1 - \log \gamma_{\min}$ ensures the type I error controlled at level α despite the adaptive search for the best quantile. For the adaptive multi-split adjusted p -value (10), the null hypothesis is rejected when $p_t < \alpha$, where α is the prespecified threshold. Following the proof of Theorem 3.2 in Meinshausen et al. (2009), we obtain the proposition below.

Proposition 6 *Under H_0 , for any J random sample splits, if (9) holds for each split, then $\limsup_{n \rightarrow \infty} P(p_t \leq \alpha) \leq \alpha$.*

Proposition 6 shows that the multi-split and aggregation procedure can control the type I error of the test. To apply the multi-split procedure, we need to choose two parameters J and γ_{\min} . In practice, we choose J that is slightly large and of the same order of n . We next discuss the choice of γ_{\min} . In order to improve the test power, we want to choose γ_{\min} such that $\limsup_{n \rightarrow \infty} P(p_t \leq \alpha)$ in Proposition 6 is maximized to be close to α under H_0 . From the proof of Proposition 6, it is sufficient to make $\operatorname{argmax}_{\gamma \in (0, 1)} P\{Q(\gamma) \leq \alpha\} \in (\gamma_{\min}, 1)$, as the adaptive search of γ in p_t is adjusted by the correction factor $1 - \log \gamma_{\min}$. Note that $\{Q(\gamma) \leq \alpha\}$ is equivalent to $\{\psi(\alpha\gamma) \geq \gamma\}$ with $\psi(u) = J^{-1} \sum_{j=1}^J 1\{p^{(j)} \leq u\}$. It is

then equivalent to examine the γ values where $P\{\psi(\alpha\gamma) \geq \gamma\}$ is closest to the upper bound α . To evaluate this, we consider two extreme cases with a given J . When $p^{(j)}$'s are highly dependent, $P\{\psi(\alpha\gamma) \geq \gamma\} \simeq P\{p^{(1)} \leq \alpha\gamma\} = \alpha\gamma$, which approaches to α when γ is close to 1. When $p^{(j)}$'s are nearly independent, $J\gamma \leq 1$ and $\alpha\gamma$ is small, $P\{\psi(\alpha\gamma) \geq \gamma\} \simeq P\{\min_j p^{(j)} \leq \alpha\gamma\} \simeq 1 - (1 - \alpha\gamma)^J \simeq J\alpha\gamma$; then $J\alpha\gamma \rightarrow \alpha$ if $\gamma \rightarrow J^{-1}$. When the dependence among $p^{(j)}$'s is between these two extreme cases, we expect that the maximum of $P\{\psi(\alpha\gamma) \geq \gamma\}$ would be achieved at some $\gamma \in [J^{-1}, 1)$. Since the true correlation is unknown in practice, we recommend to take γ_{\min} slightly smaller than J^{-1} in the simulations so that the candidate γ range contains $[J^{-1}, 1)$.

We conduct a simulation study to illustrate how the value of $P\{\psi(\alpha\gamma) \geq \gamma\}$ depends on the correlations of the p -values. We consider an “ideal” case with equal correlated p -values. Specifically we generate $p^{(j)} = 1 - \Phi(V_{J,j})$ for $j = 1, \dots, J$, where $V_J = (V_{J,1}, \dots, V_{J,J})^\top \sim \mathcal{N}(\mathbf{0}, \Sigma_J)$ with $\Sigma_J = (1 - \rho)I_{J,J} + \rho \mathbf{1}_J \mathbf{1}_J^\top$. Note that larger ρ value implies larger correlations between $p^{(j)}$'s. We take $J = 200$ and use 10^6 Monte Carlo repetitions to estimate $P\{\psi(\alpha\gamma) \geq \gamma\}$. Figure 4 gives the simulation results for $\rho \in \{0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 1\}$, and $\gamma \in (0, 1)$ and $(0, 0.01)$ respectively. When ρ is small, the largest value of $P\{\psi(\alpha\gamma) \geq \gamma\}$ is attained at $5 \times 10^{-3} = J^{-1}$; when $\rho = 1$, the largest value is attained at $\gamma = 1$. These observations are consistent with the above theoretical argument.

We give a summary of the whole testing procedure considering the case when X dimension p is large.

Procedure: For $j = 1, \dots, J$,

1. Randomly split data into screening dataset $\{X_S, Y_S\}$ and testing dataset $\{X_T, Y_T\}$, which are of sizes n_S and n_T respectively.
2. On screening dataset $\{X_S, Y_S\}$: compute the canonical correlations between Y_S and each column of matrix X_S , then select the columns whose corresponding correlations

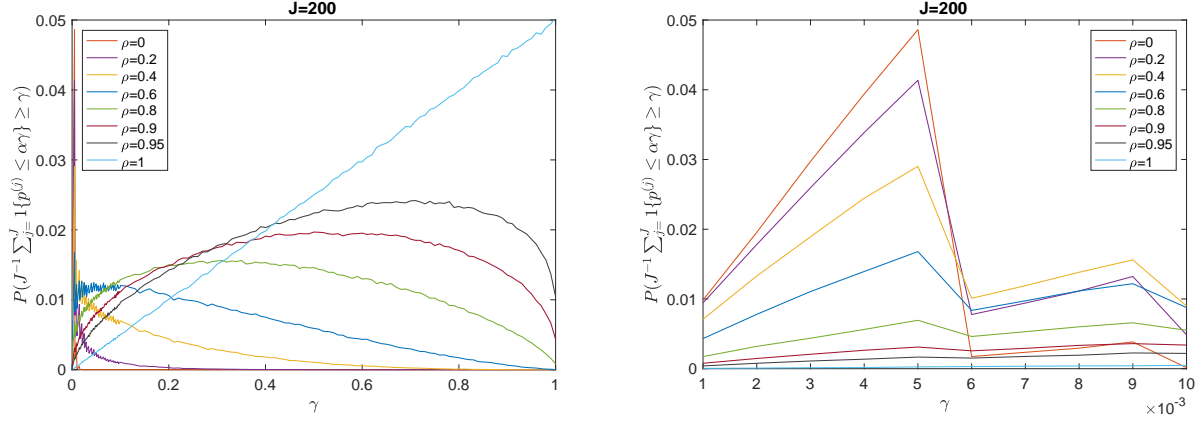


Figure 4: Estimated $P\{\psi(\alpha\gamma) \geq \gamma\}$ versus γ under different correlation levels

are the largest $\lfloor \delta p \rfloor$ ones. The column indexes we select form a set $\mathcal{S}_C \subseteq \{1, \dots, p\}$.

3. On testing dataset $\{X_T, Y_T\}$: choose the columns of X_T indexed by \mathcal{S}_C to obtain $X_{\mathcal{S}_C}$, and use $\{X_{\mathcal{S}_C}, Y_T\}$ to compute the proposed test statistic T_3 in (8) and obtain the p -value $p^{(j)}$.

After obtaining the set of p -values, $\{p^{(j)} : j = 1, \dots, J\}$, compute p_t as defined in (10). Reject the null hypothesis if $p_t \leq \alpha$. ■

Remark 5 When the dimension of response Y is large ($m > n$), we also need to reduce the dimension of response vectors to apply the likelihood ratio test. We can use a principal component analysis (PCA) or factor analysis method to perform the dimension reduction. In the simulation studies, we select the first m_0 principal components of Y_S as the columns of a matrix \hat{W} , where m_0 satisfies $m_0 + p < n_T$ and can be chosen by parallel analysis (Buja and Eyuboglu, 1992; Dobriban and Owen, 2017; Dobriban, 2017). Then we transform the responses Y_T in testing data to obtain $\hat{\mu}_T = Y_T \hat{W}$, which only has m_0 columns. We then use the transformed data $\{X_T, \hat{\mu}_T\}$ to examine $CB\hat{W} = \mathbf{0}$. By the independence between the screening and testing datasets, the test is valid. Under the sparse model setting, the signal matrix $X_{M_*} B_{M_*}$ has a low rank decomposition and we expect that the dimension reduction

procedure still maintains high power. This is verified by simulation studies in Section 5, which shows that applying dimension reduction on responses may even boost power for certain sparse models. Alternatively, other dimension reduction techniques can be applied (e.g., Yuan et al., 2007; Shen and Huang, 2008; Ma, 2013; Vu and Lei, 2013; Lei et al., 2015). When both m and p are large, we can simultaneously apply dimension reduction on Y and X to reduce both m and p .

5 Simulations

In this section, we report some simulation studies to evaluate the theoretical results and proposed methods in this paper for $n < p + m$ and $n > p + m$ respectively.

5.1 When $n < p + m$

When $n < p + m$, we conduct simulations under null and alternative hypotheses respectively to examine the type I error and power of our proposed test statistics.

In the first simulation setting, we sample the test statistics by simulating data following the canonical form introduced in Section 3. To be specific, we generate random matrices Y_1^* of size $r \times m$ and Y_3^* of size $(n - p) \times m$, where the rows of Y_1^* and Y_3^* are independent m -variate Gaussian with covariance I_m , and $E(Y_1^*) = M_1$ and $E(Y_3^*) = \mathbf{0}$. Under the canonical form, we know H_0 is equivalent to $M_1 = \mathbf{0}_{r \times m}$ as discussed. In the following, each simulation is based on 10,000 replications with significance level 0.05.

Under the null hypothesis, we compare the traditional χ^2 approximation (2) with the normal approximations for T_1 in (4) and T_3 in (8). In particular, we study how the dimension parameters p, m, r influence these approximations by varying only one parameter each time. Figure 5 gives estimated type I errors as p increases. It shows that as p, m, r become larger, the χ^2 approximation (2) performs poorly, while the normal approximations for T_1 and T_3

still control the type I error well. Other simulation results with varying m or r are given in the supplement material and similar patterns are observed.

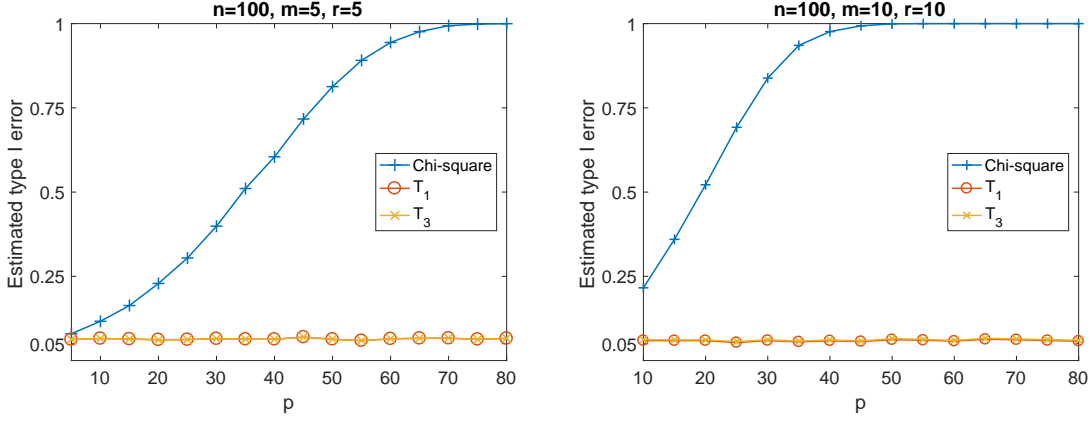


Figure 5: Estimated type I error versus p

Under alternative hypotheses, we compare the power of the test statistics T_1 , T_2 , T_3 and show the power improvement of T_3 over T_1 and T_2 . Specifically, under the canonical form, we simulate data with $M_1 = \text{diag}(\delta_1, \dots, \delta_{r_k}, 0, \dots, 0)$, that is, a diagonal matrix with r_k nonzero elements. It follows that $\Omega = \text{diag}(\delta_1^2, \dots, \delta_{r_k}^2, 0, \dots, 0)$ has rank r_k . Under this setup, we test four different cases:

$$(a) \ r_k = 1; \ (b) \ r_k = 2, \delta_1 = \delta_2; \ (c) \ r_k = 2, \delta_1 = 10\delta_2; \ (d) \ r_k = 3, \delta_1 = \delta_2 = \delta_3$$

when $n = 100, m = 20, p = 50, r = 30$. For each case, we plot estimated powers versus $\text{tr}(\Omega)/m$ in Figure 6. Results in Figure 6 show that when the rank of Ω , r_k , is small or the significant entries in Ω has low rank, T_2 is more powerful than T_1 ; and when r_k or the rank of significant entries in Ω increases, T_1 becomes more powerful. Moreover, in both sparse and non-sparse cases, the combined statistic T_3 has power close to the better one among T_1 and T_2 , with the type I error well controlled. These patterns are consistent with our theoretical power analysis in Section 3.

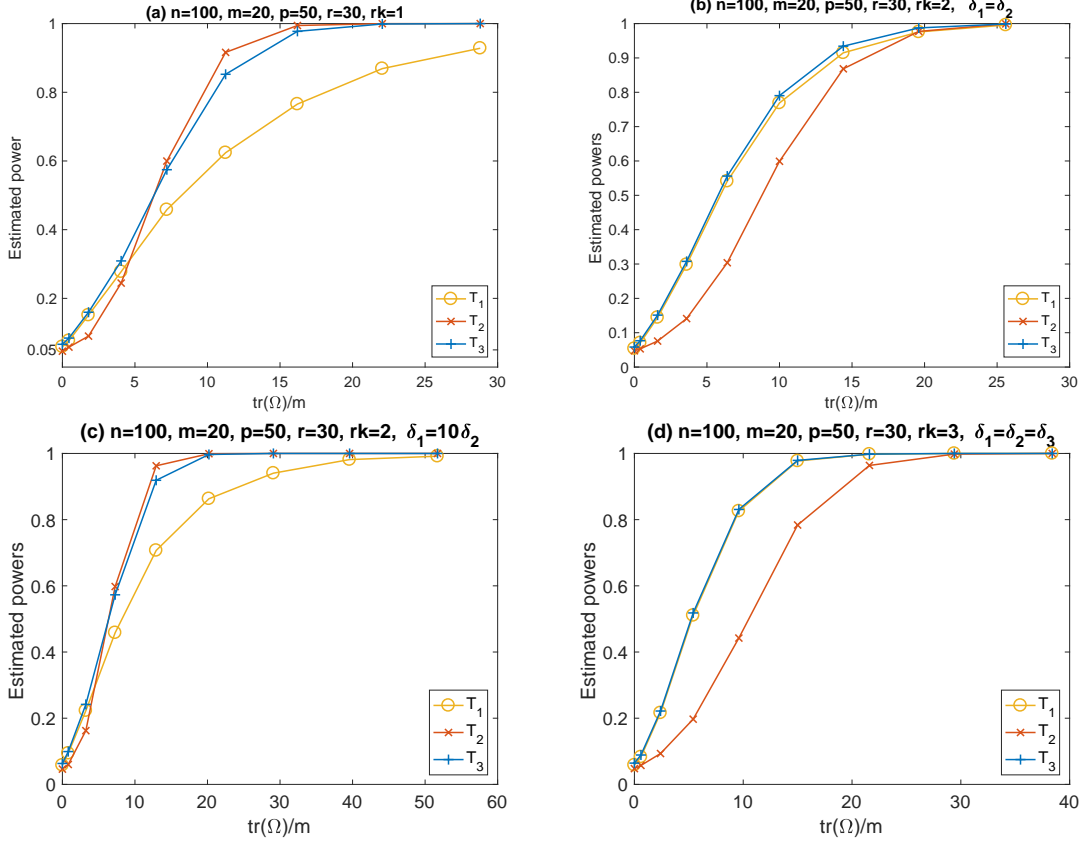


Figure 6: Powers of T_1 , T_2 and T_3 versus $\text{tr}(\Omega)/m$ when $n=100$, $m=20$, $p=50$, $r=30$

In the second simulation setting, we generate data from the multivariate regression model $Y = XB + E$, where the rows of X and E are independent multivariate Gaussian with covariance matrices $\Sigma_x = (\rho^{|i-j|})_{p \times p}$ and $\Sigma = (\rho^{|i-j|})_{m \times m}$ respectively. We consider a sparse scenario when only the $(1,1)$ -entry of B is nonzero with a value v_d . We also consider a dense scenario when all the entries of B are independently generated from $\mathcal{N}(0, \sigma_d^2)$. For each scenario, we estimate the test powers for different v_d or σ_d^2 values, which are referred to as the signal sizes. We take $n = 100$, $m = 20$, $p = 50$, $r = 30$ and conduct 10,000 simulations for two different C matrices. In the first case, we take $C = [I_r, \mathbf{0}_{r \times (p-r)}]$, where I_r is an identity matrix of dimension $r \times r$, $\mathbf{0}_{r \times (p-r)}$ is an all zero matrix of dimension $r \times (p-r)$. Then $H_0 : CB = \mathbf{0}_{r \times m}$ examines the relationship between Y and the first r predictors of

X . In the second case, we take C to be of the form (1). Then $H_0 : CB = \mathbf{0}_{r \times m}$ tests the equivalence of effects of the first r predictors and the last predictor. For two types of B and two types of C matrices, we plot the estimated powers of T_1 , T_2 , T_3 versus signal sizes in Figure 7 with $\rho = 0.7$. We provide the simulation results with respect to other ρ values in the supplementary material, where similar results can be observed.

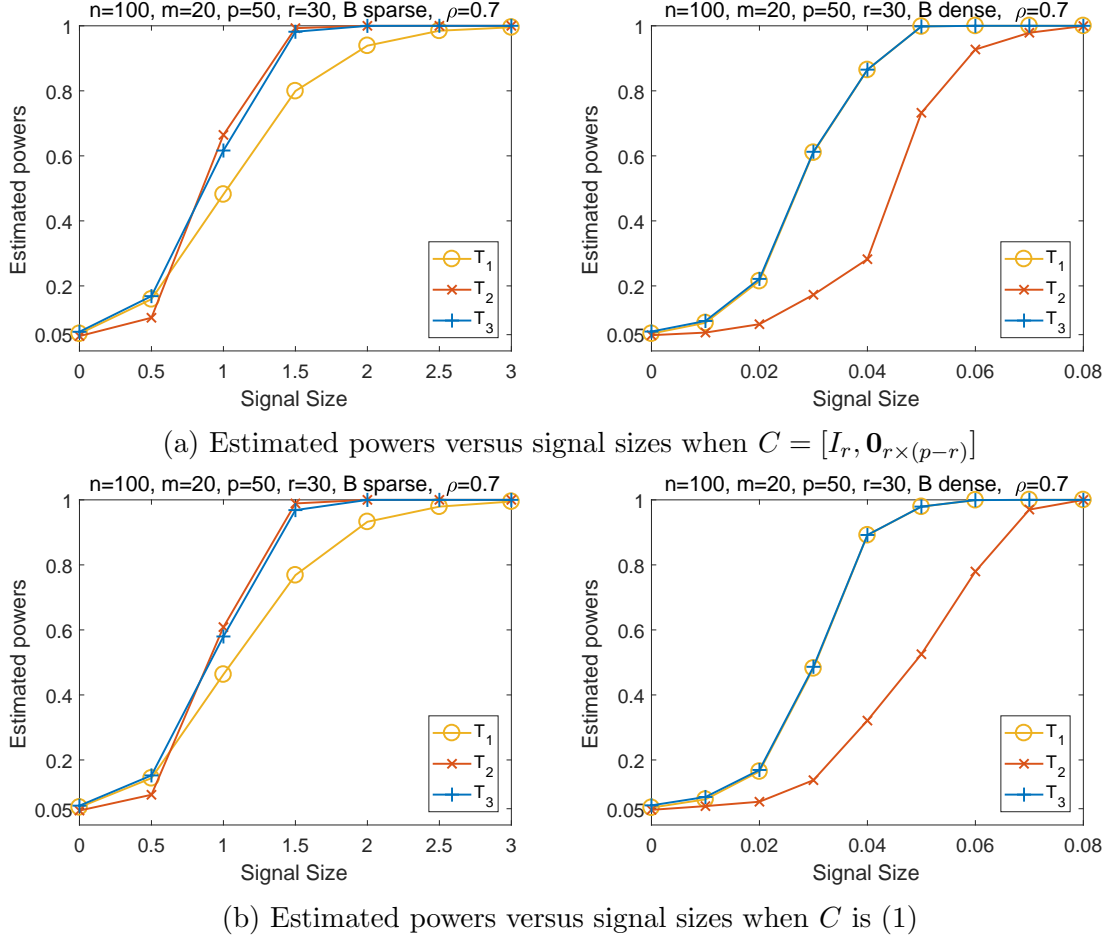


Figure 7: Estimated powers versus signal sizes

Figure 7 shows that under the dense B scenario, T_1 is more powerful than T_2 ; but under the sparse B scenario, T_2 is more powerful than T_1 . In addition, the combined statistic T_3 still maintains high power under both scenarios. These results demonstrate the good performance of the proposed statistic T_3 . Note that the pattern we observe in Figure 7 is

similar to that in Figure 6, which indicates that the conclusion we obtain under the canonical form can be instructive when considering the linear form.

5.2 When $n < p + m$

This section studies the simulation setting when $n < p + m$ and evaluates the performance of our proposed procedure in Section 4. Specifically, we take $C = [I_r, \mathbf{0}_{r \times (p-r)}]$ and B to be a $p \times m$ diagonal matrix with σ_s in the first r_k diagonal entries, where σ_s represents the signal size that varies in simulation. The rows of X and E are independent multivariate Gaussian with covariance matrices $\Sigma_x = (\rho^{|i-j|})_{p \times p}$ and $\Sigma = (\rho^{|i-j|})_{m \times m}$ respectively. We set $n = 100$, $p = 120$, $r = 120$, and test the cases when $m \in \{20, 120\}$, $r_k \in \{5, 10\}$, $\rho \in \{0.3, 0.7\}$. We conduct each simulation with 200 replications, and split the data into screening and testing datasets with ratio 3:7 (the performances of 2:8 and 4:6 are similar in our simulations). Figure 8 reports the simulation results when $r_k = 10$ while the other results are presented in the supplementary material. In Figure 8, “screening” represents the proposed screening procedure on X (with 20% features selected); “PCA” represents the principle component analysis on Y as in Remark 5; J represents the number of splits and $J = 0$ represents testing on the same data without splitting.

Figure 8 shows that when we do not split the data ($J = 0$), the type I errors can not be controlled under all cases. If we split the data once ($J = 1$), the type I errors become closer to the significance level, but can still be unstable. If we use the multi-split method with 200 splits ($J = 200$), the type I errors become well controlled. The results imply the necessity of data splitting for the proposed two-stage testing procedure, and show that the multiple splits help us to obtain stable results. In addition, in the four cases, the multi-split method ($J = 200$) also achieves higher power than the single split ($J = 1$) as the signal size increases. Moreover, for cases (a) and (b) in Figure 8 with the single split of data ($J = 1$),

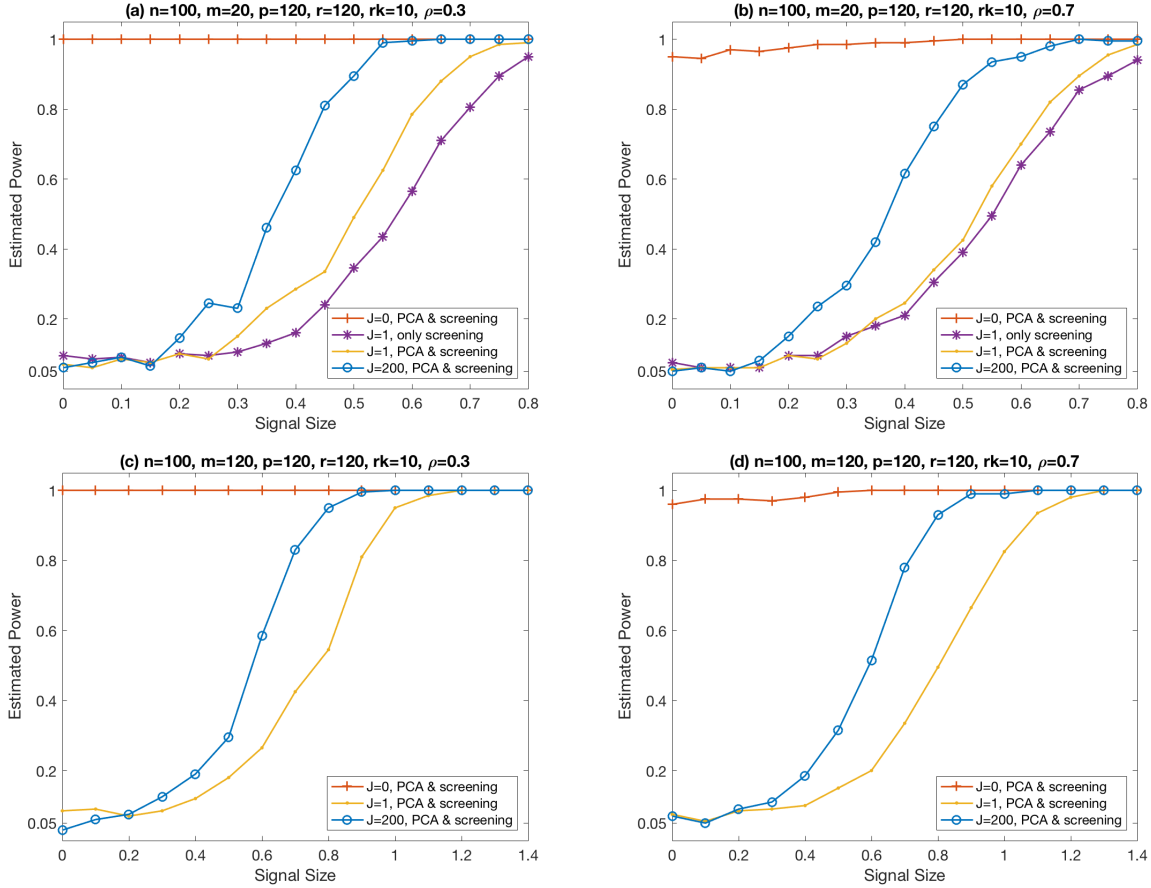


Figure 8: Estimated powers versus signal sizes when $n < m + p$

we also compare the test power when only screening on X with the power when performing dimension reduction for both X and Y . The results are given by the curves “ $J = 1$, only screening” and “ $J = 1$, PCA & screening”, respectively. We can observe that the test power is slightly enhanced by performing dimension reduction for both X and Y .

For the simulations presented in Sections 5.1 and 5.2, we also conduct similar studies when the parameters take different values, and X and Y take discrete values or the statistical error follows a heavy-tail t distribution, which are provided in the supplementary material. We observe similar patterns to those in Figures 7 and 8, which show that the proposed methods are robust to the normal assumption of the statistical error.

6 Real Data Analysis

We demonstrate the application of our proposed method by analyzing a breast cancer dataset from Chin et al. (2006), which was also studied by Chen et al. (2013) and Molstad and Rothman (2016). The dataset is available in the R package **PMA**, and consists of measured gene expression profiles (GEPs) and DNA copy-number variations (CNVs) for $n = 89$ subjects. Prior studies have demonstrated a link between DNA copy-number variations and cancer risk (see, e.g., Peng et al., 2010). It is of interest to further examine the relationship between CNVs and GEPs, where multivariate regression methods are useful.

In this study, we regress CNVs on GEPs with respect to three chromosomes 8, 17, and 22 and test whether they are associated (i.e., $C = I_p$). The m -variate response is the CNVs data and the p -variate predictor is the GEPs data, where the dimension parameters are $(p, m) = (673, 138), (1161, 87), (516, 18)$ for the three chromosomes correspondingly. As the parameters p and m are either comparable or larger than sample size $n = 89$, we use the proposed testing procedure in Section 4 to perform the test. In particular, we choose the screening data size $n_S = 26$ and the testing data size $n_T = 63$, which is approximately 3:7. We reduce the column dimension of response CNVs data matrix by parallel analysis and select the columns of GEPs data matrix by the screening method in Section 4. To include as much information of predictors as possible, we select the number of predictors between 40-50 in the screening. For each chromosome, we split the data $J = 2000$ times and obtain the corresponding p -values, $p^{(j)}$ for $j = 1, \dots, J$, from the limiting distribution of the statistic T_3 . We then compute the final p -value p_t following (10) and reject the null hypothesis if $p_t < \alpha$.

We summarize the testing results in Table 1. The column “ p_0 ” indicates the number of selected predictors, and the columns “ $k_1 \rightarrow k_2$ ” under “Chromosome pair” means that we use GEPs from k_1 th chromosome to predict CNVs from k_2 th chromosome. For each setting,

the symbols “x” and “✓” indicate that we reject and accept the null hypothesis respectively. The test results show that the null hypotheses get rejected when CNVs and GEPs are from the same chromosome, which makes biological sense. On the other hand, if we use GEPs from 8th chromosome to predict CNVs from 22nd chromosome, the null hypothesis is accepted; if we use GEPs from 8th chromosome to predict CNVs from 17th chromosome or GEPs from 17th chromosome to predict CNVs from 22nd chromosome, the null hypotheses are rejected. These conclusions indicate different relationships between CNVs and GEPs of different chromosomes, which might deserve further investigation by scientists.

To further illustrate the test results, we report the histograms of $\{p^{(j)} : j = 1, \dots, J\}$ in Figure 9, which are obtained when selecting 45 predictors under each case. It shows that the median of p -values obtained from the same chromosome is very small, which is consistent with the rejection decisions by our procedure. Besides, for $8 \rightarrow 17$ and $17 \rightarrow 22$, although the histograms of the p -values become less concentrated around 0, most of the p -values are still close to 0, and the two medians of p -values are smaller than 0.05. On the other hand, the histogram of the p -values for $8 \rightarrow 22$ is flatter. It thus shows consistency with our decision results that we only accept null hypothesis when using GEPs from 8th chromosome to predict CNVs from 22nd chromosome.

p_0	Chromosome pair					
	$8 \rightarrow 8$	$17 \rightarrow 17$	$22 \rightarrow 22$	$8 \rightarrow 17$	$17 \rightarrow 22$	$8 \rightarrow 22$
40	x	x	x	x	x	✓
45	x	x	x	x	x	✓
50	x	x	x	x	x	✓

Table 1: Decision results

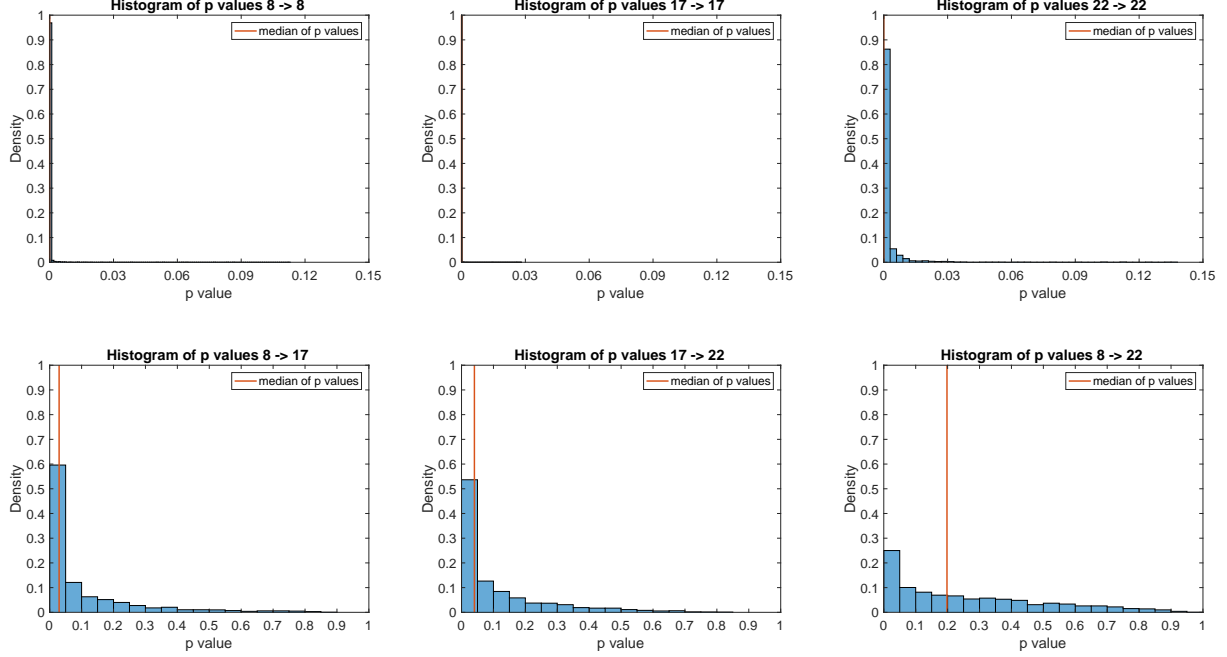


Figure 9: Histogram of p -values

7 Conclusions and Discussions

In this paper, we study the likelihood ratio test for $H_0 : CB = \mathbf{0}_{r \times m}$ in high-dimensional multivariate linear regression, where p and m are allowed to increase with n . Under the null hypothesis, we derive the asymptotic boundary where the classical χ^2 approximation fails, and further develop the corrected limiting distribution of $\log L_n$ for a general asymptotic regime of (p, m, r, n) . Under alternative hypotheses, we characterize the statistical power of $\log L_n$ in the high-dimensional setting, and propose a power-enhanced test statistic. In addition, when $n < p + m$ and then LRT is not well-defined, we propose to use a two-step testing procedure with repeated data-splitting.

The study on LRT of multivariate linear regression can also be extended to vector non-parametric regression models. In specific, for $k = 1, \dots, m$, suppose the k -th response variable depends on the p -dimensional predictor vector \mathbf{x} through the regression equation $y_k = m_k(\mathbf{x}) + e_k$, where the m_k 's are unknown smooth functions and e_k 's are the error

terms. We begin with the case when the predictor is univariate. Then we can model $m_k(x)$ by using regression splines: $m_k(x) = \sum_{j=1}^M b_{k,j} \phi_j(x)$, where $\Phi = (\phi_j : k = 1, \dots, M)^\top$ are some basis functions. Write $\mathbf{y} = (y_1, \dots, y_m)^\top$, $\mathbf{e} = (e_1, \dots, e_m)^\top$ and $B = (b_{k,j})_{M \times m}$, then we have $\mathbf{y} = B^\top \Phi + \mathbf{e}$, which is in the form of the multivariate linear regression. To test the coefficients B , we can apply the method in this paper. More generally, when the predictors are multivariate, additive models (Hastie and Tibshirani, 1986) are commonly used to finesse the “curse of dimensionality”. The multivariate functions m_k are written as $m_k(\mathbf{x}) = m_{k,1}(x_1) + \dots + m_{k,p}(x_p)$, for $k = 1, \dots, m$, and here $m_{k,1}, \dots, m_{k,p}$ are univariate functions. Suppose Φ_1, \dots, Φ_p are the basis functions for $m_{k,1}, \dots, m_{k,p}$ respectively. Then we have $\mathbf{y} = \tilde{B}^\top \tilde{\Phi} + \mathbf{e}$, where $\tilde{B} = (B_1^\top, \dots, B_p^\top)^\top$ and $\tilde{\Phi} = (\Phi_1^\top, \dots, \Phi_p^\top)^\top$. Therefore, we can apply the LRT method in this paper to test the structure of the coefficients matrix \tilde{B} .

This work establishes the theoretical results under the assumption that the error term E follows Gaussian distribution, but we expect that the conclusion holds over a larger range of distributions. Numerically, we conduct simulations for discrete distribution and heavy-tail t distribution, which are provided in the supplementary material. The simulation results show similar patterns and imply that the theoretical results may be still valid. Theoretically, Bai et al. (2013) showed that the linear spectral of F -matrix $S_1 S_2^{-1}$ also has asymptotic normal distribution, without specifying the distribution of entries of S_1 and S_2 to be normal. But they assumed that entries of S_1 and S_2 are i.i.d., which is usually not satisfied in general multivariate regression analysis. Recently, Li et al. (2018) proposed a modified LRT via non-linear spectral shrinkage, and established its asymptotic normality without normal assumption on E when m is proportional to n . However, they assumed that p , the number of predictors, is fixed. Thus the asymptotic distribution of $\log L_n$ for general high-dimensional non-Gaussian cases remains an open question in the literature.

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Supplement to “Likelihood Ratio Test in Multivariate Linear Regression: from Low to High Dimension”

We give proofs of main results and additional simulations in the supplementary material. Specifically, in Section A–E, we prove Theorem 1–5, respectively. We present the proof of Proposition 6 in Section F and provide additional simulations in Section G.

A Theorem 1

We give the proof of Theorem 1 in Section A.1, and a lemma used in the proof is proved in Section A.2.

A.1 Proof of Theorem 1

We prove Theorem 1 based on the result of Theorem 3, which is proved in Section C.

When p, m, r are all fixed, we know that $-2 \log L_n \xrightarrow{D} \chi_{mr}^2$ as $n \rightarrow \infty$. Note that $E(\chi_{mr}^2) = mr$, $\text{var}(\chi_{mr}^2) = 2mr$, and when $mr \rightarrow \infty$, $(\chi_{mr}^2 - mr)/\sqrt{2mr} \xrightarrow{D} \mathcal{N}(0, 1)$. It follows that $P\{\chi_{mr}^2 > \sqrt{2mr}z_\alpha + mr\} \rightarrow \alpha$ and

$$\chi_{mr}^2(\alpha) = \sqrt{2mr} \times \{z_\alpha + o(1)\} + mr, \quad (11)$$

where z_α denotes the upper α -quantile of $\mathcal{N}(0, 1)$.

We define the asymptotic regime $\mathcal{R}_A = \{(p, m, r, n) : n > p + m, p \geq r, mr \rightarrow \infty, \text{ and } \max\{p, m, r\}/n \rightarrow 0 \text{ as } n \rightarrow \infty\}$. Under the asymptotic regime \mathcal{R}_A , Theorem 3 shows that $(-2 \log L_n + \mu_n)/(n\sigma_n) \xrightarrow{D} \mathcal{N}(0, 1)$. Note that

$$P\{-2 \log L_n > \chi_{mr}^2(\alpha)\} = P\left\{\frac{-2 \log L_n + \mu_n}{n\sigma_n} > \frac{\chi_{mr}^2(\alpha) + \mu_n}{n\sigma_n}\right\}.$$

Then when $n \rightarrow \infty$, $P\{-2\log L_n > \chi_{mr}^2(\alpha)\} \rightarrow \alpha$ is equivalent to

$$\frac{\chi_{mr}^2(\alpha) + \mu_n}{n\sigma_n} \rightarrow z_\alpha, \quad \text{as } n \rightarrow \infty. \quad (12)$$

When $mr \rightarrow \infty$, by (11), we know (12) is equivalent to

$$\frac{\sqrt{2mr} \times \{z_\alpha + o(1)\} + mr + \mu_n}{n\sigma_n} \rightarrow z_\alpha, \quad \text{as } n \rightarrow \infty. \quad (13)$$

(13) holds for any significance level α if and only if $n\sigma_n = \sqrt{2mr}\{1 + o(1)\}$ and $(\mu_n + mr)/\sqrt{2mr} = o(1)$.

In the following, we will prove that under \mathcal{R}_A , $n\sigma_n = \sqrt{2mr}\{1 + o(1)\}$ in the first step, derive the form of μ_n in the second step, and obtain the conclusion in the third step.

Step 1. Note that

$$\sigma_n^2 = 2 \log \frac{(n+r-p-m)(n-p)}{(n-p-m)(n+r-p)}.$$

By Taylor expansion, we know that $\log(1-a) = -a - a^2/2 - a^3/3 + O(a^4)$ for $a = o(1)$.

Under \mathcal{R}_A , we know that $p/n, m/n, r/n \rightarrow 0$ and $r/(n-p-m) \rightarrow 0$. Then we have

$$\begin{aligned} \log \frac{n+r-p-m}{n-p-m} &= \log \left(1 + \frac{r}{n-p-m} \right) \\ &= \frac{r}{n-p-m} - \frac{1}{2} \frac{r^2}{(n-p-m)^2} + \frac{1}{3} \frac{r^3}{(n-p-m)^3} + O\left(\frac{r^4}{n^4}\right), \end{aligned} \quad (14)$$

and similarly,

$$\begin{aligned} -\log \frac{n-p+r}{n-p} &= -\log \left(1 + \frac{r}{n-p} \right) \\ &= -\frac{r}{n-p} + \frac{1}{2} \frac{r^2}{(n-p)^2} - \frac{1}{3} \frac{r^3}{(n-p)^3} + O\left(\frac{r^4}{n^4}\right). \end{aligned} \quad (15)$$

Since for any numbers a and b , $a^2 - b^2 = (a - b)(a + b)$ and $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$, we then know

$$\begin{aligned}
& (14) + (15) \\
&= \frac{r}{n-p-m} - \frac{r}{n-p} - \frac{1}{2} \left\{ \frac{r^2}{(n-p-m)^2} - \frac{r^2}{(n-p)^2} \right\} \\
& \quad + \frac{1}{3} \left\{ \frac{r^3}{(n-p-m)^3} - \frac{r^3}{(n-p)^3} \right\} + O\left(\frac{r^4}{n^4}\right) \\
&= \frac{rm}{(n-p-m)(n-p)} - \frac{1}{2} r^2 \times \frac{m(2n-2p-m)}{(n-p-m)^2(n-p)^2} + O\left\{\frac{r^3(m+r)}{n^4}\right\}. \quad (16)
\end{aligned}$$

We next examine the first two terms in (16). Note that for $a = o(1)$ and $b = o(1)$, $1/(1-a) = 1+a+O(a^2)$ and $1/\{(1-a)(1-b)\} = 1+a+b+O(a^2+b^2)$. Then for the first term in (16), we have

$$\begin{aligned}
\frac{rm}{(n-p-m)(n-p)} &= \frac{rm}{n^2 \{1 - (p+m)/n\} (1-p/n)} \\
&= \frac{rm}{n^2} \left\{ 1 + \frac{2p+m}{n} + O\left(\frac{p^2+m^2}{n^2}\right) \right\}. \quad (17)
\end{aligned}$$

In addition, note that for $a = o(1)$ and $b = o(1)$, $1/\{(1-a)^2(1-b)^2\} = 1+2a+2b+O(a^2+b^2)$.

Then for the second term in (16), we have

$$\begin{aligned}
& -\frac{1}{2} r^2 \times \frac{m(2n-2p-m)}{(n-p-m)^2(n-p)^2} \\
&= -nmr^2 \left\{ 1 - \frac{p+m/2}{n} \right\} \frac{1}{n^4} \left\{ 1 + \frac{2(p+m)}{n} + \frac{2p}{n} + O\left(\frac{p^2+m^2}{n^2}\right) \right\} \\
&= -\frac{mr^2}{n^3} \left\{ 1 + \frac{3p+3m/2}{n} + O\left(\frac{p^2+m^2}{n^2}\right) \right\}. \quad (18)
\end{aligned}$$

Combining (17) and (18), we obtain

$$\begin{aligned}
(16) &= (17) + (18) + O\left\{\frac{r^3(m+r)}{n^4}\right\} \\
&= \frac{rm}{n^2} + \frac{rm}{n^2}\left\{\frac{2p+m-r}{n}\right\} + O\left\{\frac{mr(m^2+r^2+p^2)}{n^4}\right\}.
\end{aligned} \tag{19}$$

We then know that $\sigma_n^2 = 2 \times (16) = (2mr/n^2) \times \{1 + o(1)\}$, and thus $n\sigma_n = \sqrt{2mr}\{1 + o(1)\}$.

Step 2. In this step, we derive the form of μ_n . Under the asymptotic region \mathcal{R}_A , we know that by Lemma 7 and Taylor expansion,

$$\begin{aligned}
\mu_n &= -mr\left\{1 + \frac{p-r}{n} + O\left(\frac{p^2+r^2}{n^2}\right)\right\} \\
&\quad - \frac{1}{2} \times \frac{mr(m+r)}{n}\left\{1 + O\left(\frac{p+r}{n}\right)\right\} + o(1)mr \times \frac{p+m+r}{n} \\
&= -mr - mr\frac{p+m/2-r/2}{n} + o(1)mr \times \frac{p+m+r}{n}.
\end{aligned}$$

Step 3. As discussed, under \mathcal{R}_A , (13) holds for any level α , if and only if $n\sigma_n = \sqrt{2mr}\{1 + o(1)\}$ and $(\mu_n + mr)/\sqrt{2mr} = o(1)$. In the first step, we have shown that $n\sigma_n = \sqrt{2mr}\{1 + o(1)\}$ under \mathcal{R}_A . In the second step, we obtain the form of μ_n . Thus we have

$$\frac{\mu_n + mr}{\sqrt{2mr}} = -\frac{\sqrt{mr}}{\sqrt{2}}\left(\frac{p+m/2-r/2}{n}\right) \times \{1 + o(1)\},$$

which converges to 0, if and only if $\lim_{n \rightarrow \infty} \sqrt{mr}(p+m/2-r/2)n^{-1} = 0$.

A.2 Lemma in the proof of Theorem 1

Lemma 7 *Under the asymptotic regime \mathcal{R}_A ,*

$$\begin{aligned}\mu_n &= -\frac{nmr}{n+r-p} - \frac{1}{2} \frac{nmr(m+r)}{(n+r-p)^2} - \frac{nmr(m^2/3 + mr/2 + r^2/3)}{(n+r-p)^3} \\ &\quad + O(1) \frac{mr(m^3 + r^3)}{n^3} + O(mr/n).\end{aligned}$$

Proof. By the definition of μ_n in Theorem 3,

$$\begin{aligned}\mu_n &= n(n-m-p-1/2) \log \frac{(n+r-p-m)(n-p)}{(n-p-m)(n+r-p)} + nr \log \frac{(n+r-p-m)}{(n+r-p)} \\ &\quad + nm \log \frac{(n-p)}{(n+r-p)}.\end{aligned}$$

Note that

$$\begin{aligned}&\log \frac{(n+r-p-m)(n-p)}{(n-p-m)(n+r-p)} \\ &= \log \frac{n+r-p-m}{n+r-p} + \log \frac{n-p}{n+r-p} - \log \frac{n-p-m}{n+r-p} \\ &= \log \left(1 - \frac{m}{n+r-p}\right) + \log \left(1 - \frac{r}{n+r-p}\right) - \log \left(1 - \frac{m+r}{n+r-p}\right).\end{aligned}$$

It follows that

$$\mu_n = n(n-m-p+r-1/2) \log \left(1 - \frac{m}{n+r-p}\right) \tag{20}$$

$$+ n(n-p-1/2) \log \left(1 - \frac{r}{n+r-p}\right) \tag{21}$$

$$- n(n-m-p-1/2) \log \left(1 - \frac{m+r}{n+r-p}\right), \tag{22}$$

where $\mu_n = (20) + (21) + (22)$. For simplicity, we will then analyze (20)/ n , (21)/ n and (22)/ n separately.

By Taylor expansion, we have

$$\begin{aligned}\log\left(1 - \frac{m}{n+r-p}\right) &= -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{m}{n+r-p}\right)^k, \\ \log\left(1 - \frac{r}{n+r-p}\right) &= -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{r}{n+r-p}\right)^k, \\ \log\left(1 - \frac{m+r}{n+r-p}\right) &= -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{m+r}{n+r-p}\right)^k.\end{aligned}$$

Then

$$\begin{aligned}(20)/n &= -(n+r-p) \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{m}{n+r-p}\right)^k + (m+1/2) \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{m}{n+r-p}\right)^k \\ &= -\sum_{k=1}^{\infty} \frac{1}{k} \frac{m^k}{(n+r-p)^{k-1}} + \sum_{k=1}^{\infty} \frac{1}{k} \frac{m^{k+1}}{(n+r-p)^k} + \sum_{k=1}^{\infty} \frac{1}{k} \frac{m^k/2}{(n+r-p)^k}.\end{aligned}$$

$$\begin{aligned}(21)/n &= -(n+r-p) \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{r}{n+r-p}\right)^k + (r+1/2) \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{r}{n+r-p}\right)^k \\ &= -\sum_{k=1}^{\infty} \frac{1}{k} \frac{r^k}{(n+r-p)^{k-1}} + \sum_{k=1}^{\infty} \frac{1}{k} \frac{r^{k+1}}{(n+r-p)^k} + \sum_{k=1}^{\infty} \frac{1}{k} \frac{r^k/2}{(n+r-p)^k}.\end{aligned}$$

$$\begin{aligned}(22)/n &= (n+r-p) \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{m+r}{n+r-p}\right)^k - (m+r+1/2) \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{m+r}{n+r-p}\right)^k \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \frac{(m+r)^k}{(n+r-p)^{k-1}} - \sum_{k=1}^{\infty} \frac{1}{k} \frac{(m+r)^{k+1}}{(n+r-p)^k} - \sum_{k=1}^{\infty} \frac{1}{k} \frac{(m+r)^k/2}{(n+r-p)^k}.\end{aligned}$$

We then have

$$\begin{aligned} & \{(20) + (21) + (22)\}/n \\ = & \sum_{k=1}^{\infty} \frac{1}{k} \frac{(m+r)^k - m^k - r^k}{(n+r-p)^{k-1}} - \sum_{k=1}^{\infty} \frac{1}{k} \frac{(m+r)^{k+1} - m^{k+1} - r^{k+1}}{(n+r-p)^k} \end{aligned} \quad (23)$$

$$- \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{(m+r)^k - m^k - r^k}{(n+r-p)^k}, \quad (24)$$

where $\{(20) + (21) + (22)\}/n = (23) + (24)$. As $(m+r)^1 - m^1 - r^1 = 0$, we know

$$\begin{aligned} (23) &= \sum_{k=2}^{\infty} \frac{1}{k} \frac{(m+r)^k - m^k - r^k}{(n+r-p)^{k-1}} - \sum_{k=1}^{\infty} \frac{1}{k} \frac{(m+r)^{k+1} - m^{k+1} - r^{k+1}}{(n+r-p)^k} \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k} \right) \frac{(m+r)^{k+1} - m^{k+1} - r^{k+1}}{(n+r-p)^k} \\ &= -\frac{1}{2} \frac{2mr}{n+r-p} \end{aligned} \quad (25)$$

$$- \frac{1}{6} \frac{(m+r)^3 - m^3 - r^3}{(n+r-p)^2} \quad (26)$$

$$- \frac{1}{12} \frac{(m+r)^4 - m^4 - r^4}{(n+r-p)^3} \quad (27)$$

$$- \sum_{k=4}^{\infty} \frac{1}{k(k+1)} \frac{(m+r)^{k+1} - m^{k+1} - r^{k+1}}{(n+r-p)^k}, \quad (28)$$

where $(23) = (25) + (26) + (27) + (28)$. We then have $n \times (25) = -nmr(n+r-p)^{-1}$, $n \times (26) = -2^{-1}nmr(m+r)(n+r-p)^{-2}$, and $n \times (27) = -nmr(m^2/3 + mr/2 + r^2/3) \times (n+r-p)^{-3}$.

In addition

$$\begin{aligned} |(28)| &= \sum_{k=4}^{\infty} \frac{1}{k(k+1)} \frac{\sum_{q=1}^k \binom{k+1}{q} m^q r^{k+1-q}}{(n+r-p)^k} \\ &\leq \frac{mr}{n+r-p} \sum_{k=4}^{\infty} \frac{1}{k} \times \frac{2^{k+1}(\max\{m, r\})^{k-1}}{(n+r-p)^{k-1}} \\ &= \frac{mr}{n+r-p} O\left\{\left(\frac{\max\{m, r\}}{n+r-p}\right)^3\right\} = O(1) \frac{mr(m^3 + r^3)}{n^4}, \end{aligned}$$

where the in two last equations, we use the property of remainder term in Taylor expansion and the condition that $\max\{p, m, r\} = o(n)$. Thus we know $n \times (28) = mr \times O\{(m^3 + r^3)/n^3\}$. Moreover,

$$\begin{aligned}
(24) &= \sum_{k=1}^{\infty} \frac{1}{k} \frac{(m+r)^k - m^k - r^k}{(n+r-p)^k} \\
&= \sum_{k=2}^{\infty} \frac{1}{k} \frac{\sum_{q=1}^{k-1} \binom{k}{q} m^q r^{k-q}}{(n+r-p)^k} \\
&\leq \frac{mr}{(n+r-p)^2} \sum_{k=2}^{\infty} \frac{1}{k} \times \frac{2^k (\max\{m, r\})^{k-2}}{(n+r-p)^{k-2}} \\
&= O(mr/n^2),
\end{aligned}$$

where in the last equation we use the fact that

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{1}{k} \times \frac{2^k (\max\{m, r\})^{k-2}}{(n+r-p)^{k-2}} \\
&\leq 2 + \sum_{k=3}^{\infty} \frac{1}{k-2} \times \frac{2^k (\max\{m, r\})^{k-2}}{(n+r-p)^{k-2}} \\
&= 2 + 4 \sum_{k=1}^{\infty} \frac{1}{k} \times \frac{2^k (\max\{m, r\})^k}{(n+r-p)^k} \\
&= 2 + 4 \log[1 - \{2 \max\{m, r\}/(n+r-p)\}].
\end{aligned}$$

In summary,

$$\begin{aligned}
\mu_n &= (20) + (21) + (22) \\
&= n \times \{(23) + (24)\} \\
&= -\frac{nmr}{n+r-p} - \frac{1}{2} \frac{nmr(m+r)}{(n+r-p)^2} - \frac{nmr(m^2/3 + mr/2 + r^2/3)}{(n+r-p)^3} \\
&\quad + O(1) \frac{mr(m^3 + r^3)}{n^3} + O(mr/n).
\end{aligned}$$

■

B Theorem 2

The proof is similar to the proof of Theorem 1. When p, m, r are all fixed, by Bartlett correction, we know that with $\rho = 1 - (p - r/2 + m/2 + 1/2)/n$, $-2\rho \log L_n \xrightarrow{D} \chi_{mr}^2$ as $n \rightarrow \infty$. Note that under \mathcal{R}_A , $\rho = 1 + o(1)$. Then similarly to the proof of Theorem 1, we know that under \mathcal{R}_A , $P\{-2\rho \log L_n > \chi_{mr}^2(\alpha)\} \rightarrow \alpha$ holds for any given significance level α if and only if $n\sigma_n = \sqrt{2mr}\{1 + o(1)\}$ and $(\mu_n + mr/\rho)/\sqrt{2mr} = o(1)$.

Following the same argument as in the proof of Theorem 1, we know that under \mathcal{R}_A , $n\sigma_n = \sqrt{2mr}\{1 + o(1)\}$. In addition, by Taylor expansion,

$$\begin{aligned}
\frac{mr}{\rho} &= \frac{mr}{1 - (p + m/2 - r/2 + 1/2)/n} \\
&= \frac{mr}{1 - (p + m/2 - r/2)/n} + mrO\left(\frac{1}{n}\right) \\
&= \frac{nmr}{n - p + r - (m + r)/2} + mrO\left(\frac{1}{n}\right) \\
&= \frac{nmr}{n - p + r} \sum_{k=0}^{\infty} \left\{ \frac{m + r}{2(n - p + r)} \right\}^k + mrO\left(\frac{1}{n}\right) \\
&= \frac{nmr}{n - p + r} + \frac{nmr(m + r)}{2(n + r - p)^2} + \frac{nmr(m + r)^2}{4(n - p + r)^3} \\
&\quad + mr \times O\left(\frac{m^3 + r^3}{n^3}\right) + mrO\left(\frac{1}{n}\right),
\end{aligned}$$

where in the last equation, we use the fact that $\sum_{k=3}^{\infty} [(m + r)/\{2(n - p + r)\}]^k = O\{(m^3 + r^3)/n^3\}$.

$r^3)/n^3\}$ as $\max\{p, m, r\} = o(n)$. Then under \mathcal{R}_A , by Lemma 7,

$$\begin{aligned}
& \frac{\mu_n - (-mr/\rho)}{\sqrt{2mr}} \tag{29} \\
&= \frac{1}{\sqrt{2mr}} \left\{ -\frac{nmr}{n+r-p} - \frac{nmr(m+r)}{2(n+r-p)^2} - \frac{nmr(m^2/3 + mr/2 + r^2/3)}{(n+r-p)^3} \right. \\
&\quad \left. + \frac{nmr}{n-p+r} + \frac{nmr(m+r)}{2(n+r-p)^2} + \frac{nmr(m+r)^2}{4(n-p+r)^3} \right\} \\
&\quad + \sqrt{mr} \times O\left(\frac{m^3 + r^3}{n^3}\right) + O(\sqrt{mr}/n) \\
&= -\frac{\sqrt{mr}}{\sqrt{2}} \frac{1}{12} \frac{n(m^2 + r^2)}{(n+r-p)^3} + \sqrt{mr} \times O\left(\frac{m^3 + r^3}{n^3}\right) + O(\sqrt{mr}/n) \\
&= -\frac{\sqrt{mr}(m^2 + r^2)}{12\sqrt{2}n^2} + o(1) \times \frac{\sqrt{mr}(m^2 + r^2)}{n^2} + O(\sqrt{mr}/n),
\end{aligned}$$

where in the last equation, we use the fact that $\max\{p, m, r\} = o(n)$. Then we know that (29) = 0 if and only if $\sqrt{mr}(m^2 + r^2)/n^2 \rightarrow 0$.

C Theorem 3

In this section, we give the proof of Theorem 3, where the main proof is in Section C.1 and some lemmas used are provided and proved in Section C.2.

C.1 Proof of Theorem 3

Proof. To prove the central limit theorem that $H_n := \{-2 \log L_n + \mu_n\}/(n\sigma_n) \xrightarrow{D} \mathcal{N}(0, 1)$, it is sufficient to show

$$E \exp \left\{ \frac{\log L_n - \mu_n/2}{n\sigma_n/2} s \right\} \rightarrow \exp\{s^2/2\}, \tag{30}$$

as $n \rightarrow \infty$ and $|s| < 1$, where σ_n^2 and μ_n are defined in Theorem 3. Equivalently, it suffices to show that for any subsequence $\{n_k\}$, there is a further subsequence $\{n_{k_j}\}$ such that $H_{n_{k_j}}$

converges to $\mathcal{N}(0, 1)$ in distribution as $j \rightarrow \infty$. In the following, the further subsequence is selected in a way such that the subsequential limits of some bounded quantities (to be specified in the proof below) exist, which is guaranteed by Bolzano-Weierstrass Theorem. Therefore, we only need to verify the theorems by assuming that the limits for these bounded quantities exist. In the following, we give the proof by discussing two settings $r \geq m$ and $m \geq r$ separately.

Case 1. When $r \geq m$ and $r \rightarrow \infty$. By Lemma 9, under the null hypothesis, the distribution of L_n can be reexpressed as the distribution of a product of independent beta random variables. Let $h = 2s/(n\sigma_n)$, by Lemma 8, then under the null hypothesis, L_n 's h th moment can be written as

$$E \exp \left\{ \frac{\log L_n}{n\sigma_n/2} s \right\} = E(L_n^h) = \frac{\Gamma_m\{\frac{1}{2}n(1+h) - \frac{1}{2}p\} \Gamma_m\{\frac{1}{2}(n+r-p)\}}{\Gamma_m\{\frac{1}{2}(n-p)\} \Gamma_m\{\frac{1}{2}n(1+h) + \frac{1}{2}(r-p)\}}, \quad (31)$$

where $\Gamma_m(a)$, $a \in \mathbb{C}$ and $\text{Re}(a) > (m-1)/2$, is the multivariate Gamma function defined to be

$$\Gamma_m(a) = \int_{A>0} e^{-\text{tr}(A)} \det A^{a-(m+1)/2} (dA). \quad (32)$$

The above integration is taken over the space of positive definite $m \times m$ matrices, i.e., $\{A_{m \times m} : A \succ 0\}$; and $\text{tr}(A)$ is the trace of A . Note that when $m = 1$, $\Gamma_m(a)$ becomes the usual definition of Gamma function. By Lemma 10, $\Gamma_m(a)$ can be written as a product of ordinary Gamma functions as

$$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{j=1}^m \Gamma\{a - (j-1)/2\}.$$

Note that $n > m + p$ and $r \geq 1$. Thus the limits of $m/(n+r-p)$ and $m/(n-p)$ are

in $[0, 1]$ for all n . Applying the subsequence argument above, for any subsequence $\{n_k\}$, we take a further subsequence n_{k_j} such that $m_{k_j}/(n_{k_j} + r_{k_j} - p_{k_j})$ and $m_{k_j}/(n_{k_j} - p_{k_j})$ converge to some constants in $[0, 1]$. Thus without loss of generality, we consider the cases when $m/(n + r - p)$ and $m/(n - p)$ converge to some constants in $[0, 1]$. Next we give the proof by discussing different cases below.

Case 1.1 If $m/(n + r - p) \rightarrow \gamma > 0$, this implies that $m \rightarrow \infty$ as $n \rightarrow \infty$. And as $r \geq m$ and $n > p + m$, we know $m/(n + r - p) \leq 1/2$, then $\gamma \in (0, 1/2]$. Since $1 \geq m/(n - p) \geq m/(n + r - p)$, then $m/(n - p) \rightarrow \gamma' \in (0, 1]$.

If $\gamma' \in (0, 1)$, $nh \times [-\log\{1 - m/(n - p)\}]^{1/2} = O(1)$, which satisfies the assumption of Lemma 5.4 in Jiang and Yang (2013). If $\gamma' = 1$, as

$$\sigma_n^2 = 2 \log \left(1 - \frac{m}{n + r - p} \right) - 2 \log \left(1 - \frac{m}{n - p} \right), \quad (33)$$

and $m/(n + r - p) \rightarrow \gamma \in (0, 1/2]$, we know σ_n^2 has leading order $\log\{1 - m/(n - p)\}$. Then as $nh\sigma_n = O(1)$ by definition, we also know $nh \times [-\log\{1 - m/(n - p)\}]^{1/2} = O(1)$, which satisfies the assumption of Lemma 5.4 in Jiang and Yang (2013). Following the lemma, we have

$$\begin{aligned} \log \frac{\Gamma_m\{\frac{1}{2}n(1 + h) - \frac{1}{2}p\}}{\Gamma_m\{\frac{1}{2}(n - p)\}} &= \log \frac{\Gamma_m\{\frac{1}{2}(n - p) + \frac{1}{2}nh\}}{\Gamma_m\{\frac{1}{2}(n - p)\}} \\ &= - \left\{ \frac{n^2 h^2}{4} + \frac{nh}{2} \left(n - m - p - \frac{1}{2} \right) \right\} \log \left(1 - \frac{m}{n - p} \right) \\ &\quad + \frac{mnh}{2} \{ \log(n - p) - \log 2e \} + o(1), \end{aligned} \quad (34)$$

and similarly, we can obtain

$$\begin{aligned}
\log \frac{\Gamma_m\{\frac{1}{2}(n+r-p)\}}{\Gamma_m\{\frac{1}{2}n(1+h) + \frac{1}{2}(r-p)\}} &= \log \frac{\Gamma_m\{\frac{1}{2}(n+r-p)\}}{\Gamma_m\{\frac{1}{2}(n+r-p) + \frac{1}{2}nh\}} \\
&= \left\{ \frac{n^2h^2}{4} + \frac{nh}{2} \left(n+r-m-p-\frac{1}{2} \right) \right\} \log \left(1 - \frac{m}{n+r-p} \right) \\
&\quad - \frac{mnh}{2} \{ \log(n+r-p) - \log 2e \} + o(1). \tag{35}
\end{aligned}$$

Combining (31), (34) and (35), we have

$$\begin{aligned}
\log E \exp \left\{ \frac{\log L_n}{n\sigma_n/2} s \right\} &= \frac{n^2h^2}{4} \log \frac{(n+r-p-m)(n-p)}{(n-p-m)(n+r-p)} + \frac{h\mu_n}{2} + o(1) \\
&= \frac{s^2}{2} + \frac{h\mu_n}{2} + o(1),
\end{aligned}$$

where

$$\begin{aligned}
\mu_n &= n(n-m-p-1/2) \log \frac{(n+r-p-m)(n-p)}{(n-p-m)(n+r-p)} + nr \log \frac{(n+r-p-m)}{(n+r-p)} \\
&\quad + nm \log \frac{(n-p)}{(n+r-p)}.
\end{aligned}$$

Therefore, $\log E \exp \left\{ \frac{\log L_n - \mu_n/2}{n\sigma_n/2} s \right\} = s^2/2 + o(1)$ is proved.

Case 1.2 We discuss the case when $m/(n+r-p) \rightarrow 0$ and $m/(n-p) \rightarrow 0$ below. By Lemma 13, we know that when $n-p \rightarrow \infty$ and $r \rightarrow \infty$,

$$\begin{aligned}
& \log \frac{\Gamma_m\{\frac{1}{2}n(1+h) - \frac{1}{2}p\}}{\Gamma_m\{\frac{1}{2}(n-p)\}} \\
&= - \left\{ 2m + \left(n-p-m - \frac{1}{2} \right) \log \left(1 - \frac{m}{n-p} \right) \right\} \frac{nh}{2} \\
& \quad - \left\{ \frac{m}{n-p} + \log \left(1 - \frac{m}{n-p} \right) \right\} \frac{n^2 h^2}{4} \\
& \quad + m \left\{ \frac{(n-p+nh)}{2} \log \frac{(n-p+nh)}{2} - \frac{(n-p)}{2} \log \frac{(n-p)}{2} \right\} + o(1), \tag{36}
\end{aligned}$$

and

$$\begin{aligned}
& \log \frac{\Gamma_m\{\frac{1}{2}(n+r-p)\}}{\Gamma_m\{\frac{1}{2}n(1+h) + \frac{1}{2}(r-p)\}} \\
&= \left\{ 2m + \left(n+r-p-m - \frac{1}{2} \right) \log \left(1 - \frac{m}{n+r-p} \right) \right\} \frac{nh}{2} \\
& \quad - m \left\{ \frac{(n+r-p+nh)}{2} \log \frac{(n+r-p+nh)}{2} - \frac{(n+r-p)}{2} \log \frac{(n+r-p)}{2} \right\} \\
& \quad + \left\{ \frac{m}{n+r-p} + \log \left(1 - \frac{m}{n+r-p} \right) \right\} \frac{n^2 h^2}{4} + o(1). \tag{37}
\end{aligned}$$

By Taylor expansion of the log function, we have

$$\begin{aligned}
\sigma_n^2 &= 2 \log \left(1 - \frac{m}{n+r-p} \right) - 2 \log \left(1 - \frac{m}{n-p} \right) \\
&= \frac{2mr}{(n-p)(n+r-p)} \{1 + o(1)\}, \tag{38}
\end{aligned}$$

where the second order terms of Taylor expansion of the log functions is ignorable as $m = o(n-p)$. Also, as $r \rightarrow \infty$,

$$h = \frac{s}{n\sigma_n/2} = \frac{s\sqrt{2(n-p)(n+r-p)}}{n\sqrt{mr}} \{1 + o(1)\} \rightarrow 0. \tag{39}$$

Therefore, combining (31), (36) and (37), we obtain

$$\begin{aligned}
& \log E \exp \left\{ \frac{\log L_n}{n\sigma_n/2} s \right\} \\
= & \frac{n^2 h^2}{4} \log \frac{(n+r-p-m)(n-p)}{(n-p-m)(n+r-p)} + \frac{n^2 h^2}{4} \left(\frac{m}{n+r-p} - \frac{m}{n-p} \right) \\
& + \frac{nh}{2} (n-m-p-1/2) \log \frac{(n+r-p-m)(n-p)}{(n-p-m)(n+r-p)} \\
& + \frac{nh}{2} r \log \frac{(n+r-p-m)}{(n+r-p)} + \frac{nh}{2} m \log \frac{n-p+nh}{n+r-p+nh} \\
& + \frac{m(n+r-p)}{2} \log \frac{n+r-p}{n+r-p+nh} \\
& + \frac{m(n-p)}{2} \log \frac{n-p+nh}{n-p} + o(1).
\end{aligned} \tag{40}$$

We then analyze the terms in (40) separately. By (39),

$$\begin{aligned}
& \frac{n^2 h^2}{4} \left(\frac{m}{n+r-p} - \frac{m}{n-p} \right) \\
= & - \frac{s^2(n-p)(n+r-p)}{2mr} \times \frac{mr}{(n-p)(n+r-p)} \{1 + o(1)\} \\
= & - \frac{s^2}{2} + o(1).
\end{aligned} \tag{41}$$

In addition, as $nh/(n-p) \rightarrow 0$ and $nh/(n+r-p) \rightarrow 0$, we have

$$\begin{aligned}
& \frac{m(n+r-p)}{2} \log \frac{n+r-p}{n+r-p+nh} \\
= & - \frac{m(n+r-p)}{2} \left\{ \frac{nh}{n+r-p} - \frac{n^2 h^2}{2(n+r-p)^2} + R_{n,1} \right\},
\end{aligned} \tag{42}$$

and

$$\begin{aligned} & \frac{m(n-p)}{2} \log \frac{n-p+nh}{n-p} \\ &= \frac{m(n-p)}{2} \left\{ \frac{nh}{n-p} - \frac{n^2 h^2}{2(n-p)^2} + R_{n,2} \right\}, \end{aligned} \quad (43)$$

where the remainder terms

$$R_{n,1} = \sum_{k=3}^{\infty} \frac{1}{k} (-1)^{k+1} \frac{(nh)^k}{(n+r-p)^k}, \quad R_{n,2} = \sum_{k=3}^{\infty} \frac{1}{k} (-1)^{k+1} \frac{(nh)^k}{(n-p)^k}. \quad (44)$$

Then we have

$$\begin{aligned} & (42) + (43) \\ &= \frac{mn^2 h^2}{4(n+r-p)} - \frac{mn^2 h^2}{4(n-p)} - \frac{m(n+r-p)}{2} R_{n,1} + \frac{m(n-p)}{2} R_{n,2} \\ &= -\frac{s^2}{2} + o(1), \end{aligned} \quad (45)$$

where in the last equation, we use (41) and Lemma 14. Furthermore, by $nh/(n+r-p) \rightarrow 0$ and (39), we have

$$\begin{aligned} & \frac{nh}{2} m \log \frac{n-p+nh}{n+r-p+nh} \\ &= \frac{nh}{2} m \log \left\{ \frac{n-p}{n+r-p} + \frac{nh r}{(n+r-p)(n+r-p+nh)} \right\} \\ &= \frac{nh}{2} m \log \frac{n-p}{n+r-p} + \frac{nmh}{2} \frac{n+r-p}{n-p} \frac{nh r}{(n+r-p)(n+r-p+nh)} + o(1) \\ &= \frac{nh}{2} m \log \frac{n-p}{n+r-p} + s^2 + o(1). \end{aligned} \quad (46)$$

Combining (40), (41), (45) and (46), we obtain $\log E \exp \left\{ \frac{\log L_n - \mu_n/2}{n\sigma_n/2} s \right\} = s^2/2 + o(1)$.

Case 1.3 When $m/(n+r-p) \rightarrow 0$ and $m/(n-p) \rightarrow \gamma \in (0, 1]$, we know (34) still holds following similar analysis to Case 1.1. And (37) also holds following similar analysis to Case 1.2. To establish (30), we next show that under this case, the difference between the result of (35) and (37) is ignorable.

$$\begin{aligned}
& (37) - (35) \\
&= 2m \frac{nh}{2} + \frac{mnh}{2} \{ \log(n+r-p) - \log 2e \} + \frac{n^2 h^2}{4} \times \frac{m}{n+r-p} \\
&\quad - \frac{m(n+r-p)}{2} \log \frac{n+r-p+nh}{n+r-p} - \frac{mnh}{2} \log \frac{n+r-p+nh}{2} + o(1) \\
&= mnh + \frac{mnh}{2} \left\{ \log \left(\frac{n+r-p}{2} \right) - 1 \right\} + \frac{n^2 h^2}{4} \times \frac{m}{n+r-p} \\
&\quad - \frac{m(n+r-p)}{2} \log \left(1 + \frac{nh}{n+r-p} \right) - \frac{mnh}{2} \log \left(\frac{n+r-p}{2} + \frac{nh}{2} \right) + o(1). \tag{47}
\end{aligned}$$

We then analyze the terms in (47) separately.

Since $m/(n-p) \rightarrow \gamma \in (0, 1]$, similarly to (33), we know that $nh = 2s/\sigma_n = O(s)$. As $m/(n+r-p) \rightarrow 0$, it follows that $n^2 h^2 m/(n+r-p) \rightarrow 0$. Applying Taylor expansion, we then have

$$\begin{aligned}
& \frac{mnh}{2} \log \left(\frac{n+r-p}{2} + \frac{nh}{2} \right) \\
&= \frac{mnh}{2} \left\{ \log \left(\frac{n+r-p}{2} \right) + O \left(\frac{nh}{n+r-p} \right) \right\} \\
&= \frac{mnh}{2} \log \left(\frac{n+r-p}{2} \right) + O \left(\frac{mn^2 h^2}{n+r-p} \right) \\
&= \frac{mnh}{2} \log \left(\frac{n+r-p}{2} \right) + o(1). \tag{48}
\end{aligned}$$

Similarly, by $nh = O(s)$, $m/(n+r-p) \rightarrow 0$, and Taylor expansion, we have

$$\begin{aligned}
& \frac{m(n+r-p)}{2} \log \left(1 + \frac{nh}{n+r-p} \right) \\
&= \frac{m(n+r-p)}{2} \left\{ \frac{nh}{n+r-p} + O\left(\frac{n^2 h^2}{(n+r-p)^2} \right) \right\} \\
&= \frac{mnh}{2} + o(1).
\end{aligned} \tag{49}$$

In summary, combining (48) and (49), we have $(47) = (37) - (35) = o(1)$. Then by the results in Case 1.1, we get the same conclusion as in Case 1.1.

Case 2. When $m > r$, $m \rightarrow \infty$. According to Lemma 9, we can make the following substitution

$$m \rightarrow r, \quad r \rightarrow m, \quad n-p \rightarrow n+r-p-m.$$

Then the substituted mean and variance are

$$\begin{aligned}
\mu_n &= n(n-p-m-1/2) \log \frac{(n-p)(n-p+r-m)}{(n-p-m)(n+r-p)} + nm \log \frac{(n-p)}{(n+r-p)} \\
&\quad + nr \log \frac{(n-p+r-m)}{(n+r-p)},
\end{aligned}$$

and

$$\sigma_n^2 = 2 \log \frac{(n-p)(n-p+r-m)}{(n-p-m)(n+r-p)},$$

which take the same forms as those in the setting when $r \geq m$. And the theorem can be proved following similar analysis when $m \rightarrow \infty$, $n-p+r-m \rightarrow \infty$. ■

C.2 Lemmas in the proof of Theorem 3

Lemma 8 (Corollary 10.5.2 in Muirhead (2009)) *Under the null hypothesis, L_n 's h -th moment can be written as*

$$E(L_n^h) = \frac{\Gamma_m\{\frac{1}{2}n(1+h) - \frac{1}{2}p\}\Gamma_m\{\frac{1}{2}(n+r-p)\}}{\Gamma_m\{\frac{1}{2}(n-p)\}\Gamma_m\{\frac{1}{2}n(1+h) + \frac{1}{2}(r-p)\}}.$$

Lemma 9 (Theorem 10.5.3 in Muirhead (2009)) *Under the null hypothesis, when $n-p \geq m$ and $r \geq m$, $\frac{2}{n} \log L_n$ has the same distribution as $\sum_{i=1}^m \log V_i$, where V_i 's are independent random variables and $V_i \sim \text{beta}(\frac{1}{2}(n-p-i+1), \frac{1}{2}r)$; when $n-p \geq m \geq r$, $\frac{2}{n} \log L_n$ has the same distribution as $\sum_{i=1}^r \log V_i$, where V_i 's are independent and $V_i \sim \text{beta}(\frac{1}{2}(n+r-p-m-i+1), \frac{1}{2}m)$.*

Lemma 10 (Theorem 2.1.12 in Muirhead (2009)) *The multivariate Gamma function defined in (32) can be written as*

$$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{j=1}^m \Gamma(a - (j-1)/2).$$

Lemma 11 *Consider m is fixed and $a \rightarrow \infty$. We have*

$$\frac{1}{a-1} \sum_{i=1}^m \frac{i-1}{a-i} = \left\{ \frac{1}{2} \left(\sigma_a^2 - \frac{m}{(a-1)^2} \right) \right\} \{1 + O(1/a)\}, \quad (50)$$

$$\sum_{i=1}^m \{\log(a-1) - \log(a-i)\} = -\mu_a + O\left(\frac{m^2}{a^2}\right), \quad (51)$$

where $\mu_a = -(m-a+3/2) \log\{1-m/(a-1)\} + (a-1)m/a$ and $\sigma_a^2 = -2[m/(a-1) + \log\{1-m/(a-1)\}]$.

Proof. We first prove (50). As m is fixed and $a \rightarrow \infty$, we have

$$\sigma_a^2 = -2 \left[\frac{m}{a-1} + \log \left(1 - \frac{m}{a-1} \right) \right] = \left(\frac{m}{a-1} \right)^2 \{1 + O(m/a)\},$$

and

$$\frac{1}{a-1} \sum_{i=1}^m \frac{i-1}{a-i} = \frac{1}{a-1} \sum_{i=1}^m \frac{i-1}{a-1} + \epsilon_a = \frac{m(m-1)}{2(a-1)^2} + \epsilon_a,$$

where $|\epsilon_a| \leq 2(a-1)^{-3} \sum_{i=1}^m (i-1)^2 \leq 3(m/a)^3$. Therefore,

$$\begin{aligned} \frac{1}{a-1} \sum_{i=1}^m \frac{i-1}{a-i} &= \frac{m(m-1)}{2(a-1)^2} \left\{ 1 + O\left(\frac{m}{a}\right) \right\} \\ &= \left[\frac{1}{2} \left\{ \sigma_a^2 - \frac{m}{(a-1)^2} \right\} \right] \{1 + O(1/a)\}, \end{aligned}$$

where in the last equation, we use the fact that $O(m/a) = O(1/a)$ as m is fixed. Then (50) is proved.

We then prove (51). Recall Stirling formula, (see, e.g., p. 368 ?)

$$\log \Gamma(x) = (x - 1/2) \log x - x + \log \sqrt{2\pi} + \frac{1}{12x} + O(x^{-3})$$

as $x \rightarrow \infty$. Therefore,

$$\begin{aligned} &\log \Gamma(a-1) - \log \Gamma(a-m-1) \\ &= (a-3/2) \log(a-1) - (a-m-3/2) \log(a-m-1) - m \\ &\quad + \frac{1}{12} \left(\frac{1}{a-1} - \frac{1}{a-m-1} \right) + O(a^{-3}) \\ &= (a-3/2) \log(a-1) - (a-m-3/2) \log(a-m-1) - m + O(ma^{-2}). \end{aligned}$$

Since for integers $k \geq 1$, $\Gamma(k) = (k-1)! = \prod_{i=1}^{k-1} i$. Then we have

$$\begin{aligned}
& \sum_{i=1}^m \{\log(a-1) - \log(a-i)\} \\
&= m \log(a-1) - \{\log \Gamma(a-1) - \log \Gamma(a-m-1)\} + \log \left(1 - \frac{m}{a-1}\right) \\
&= m \log(a-1) - (a-3/2) \log(a-1) + (a-m-3/2) \log(a-m-1) \\
&\quad + m + \log \left(1 - \frac{m}{a-1}\right) + O(ma^{-2}) \\
&= -(m-a+3/2) \log \left(1 - \frac{m}{a-1}\right) + m - \frac{m}{a-1} + O\left(\frac{m^2}{a^2}\right) \\
&= -(m-a+3/2) \log \left(1 - \frac{m}{a-1}\right) + \frac{a-1}{a}m + O\left(\frac{m^2}{a^2}\right) \\
&= -\mu_a + O\left(\frac{m^2}{a^2}\right),
\end{aligned}$$

where in the last two equations, we use the fact that $\frac{a-2}{a-1}m = \frac{a-1}{a}m + O(ma^{-2})$. ■

Lemma 12 *Consider m is fixed and $a \rightarrow \infty$. Define*

$$g_i(x) = \left(\frac{a-i}{2} + x\right) \log \left(\frac{a-i}{2} + x\right) - \left(\frac{a-1}{2} + x\right) \log \left(\frac{a-1}{2} + x\right)$$

for $1 \leq i \leq m$ and $x > -(a-m)/2$. Let μ_a and σ_a be as in Lemma 11. If $t = o(a)$ and $mt^2/a^2 = o(1)$, we have that as $a \rightarrow \infty$,

$$\sum_{i=1}^m \{g_i(t) - g_i(0)\} = \mu_a t + \frac{\sigma_a^2}{2} t^2 + o(1).$$

Proof. We know for $1 \leq i \leq m$,

$$\begin{aligned}
g_i'(x) &= \log\left(\frac{a-i}{2} + x\right) - \log\left(\frac{a-1}{2} + x\right), \\
g_i''(x) &= \frac{1}{\frac{a-i}{2} + x} - \frac{1}{\frac{a-1}{2} + x} = \frac{\frac{i-1}{2}}{\left(\frac{a-i}{2} + x\right)\left(\frac{a-1}{2} + x\right)}, \\
g_i^{(3)}(x) &= -\frac{1}{\left(\frac{a-i}{2} + x\right)^2} + \frac{1}{\left(\frac{a-1}{2} + x\right)^2} \\
&= -\frac{\frac{i-1}{2} \cdot \frac{2a-i-1}{2} + (i-1)x}{\left(\frac{a-i}{2} + x\right)^2 \left(\frac{a-1}{2} + x\right)^2}.
\end{aligned}$$

By Taylor expansion,

$$\begin{aligned}
g_i(t) - g_i(0) &= g_i'(0)t + \frac{t^2}{2}g_i''(0) + \frac{t^3}{6}g_i^{(3)}(\xi_i) \\
&= \{\log(a-i) - \log(a-1)\}t + \frac{i-1}{(a-1)(a-i)}t^2 + \frac{t^3}{6}g_i^{(3)}(\xi_i).
\end{aligned}$$

For $1 \leq i \leq m$, fixed m and $0 \leq \xi_i \leq t = o(a)$, we have $\sup_{|\xi_i| \leq |t|, 1 \leq i \leq m} |g_i^{(3)}(\xi_i)| \leq ca^{-3}$, where c denotes an universal constant. Therefore, as $t = o(a)$, $|t^3 g_i^{(3)}(\xi_i)| \leq ct^3 a^{-3} = o(1)$. In addition, by Lemma 11, and the fact that $mt^2/(a-1)^2 = o(1)$, we have as $a \rightarrow \infty$,

$$\begin{aligned}
\sum_{i=1}^m \{g_i(t) - g_i(0)\} &= \mu_a t + \left[\frac{1}{2} \left(\sigma_a^2 - \frac{m}{(a-1)^2} \right) \right] t^2 + o(1) \\
&= \mu_a t + \frac{\sigma_a^2}{2} t^2 + o(1).
\end{aligned}$$

■

Lemma 13 Consider $n - p \rightarrow \infty$, $r \rightarrow \infty$, $m/(n - p) \rightarrow 0$ and $m/(n - p + r) \rightarrow 0$. For $t = nh/2$, $a = n - p + r$ or $a = n - p$, we have

$$\log \frac{\Gamma_m\left(\frac{a-1}{2} + t\right)}{\Gamma_m\left(\frac{a-1}{2}\right)} = v_a t + \vartheta_a t^2 + \gamma_a(t) + o(1),$$

where

$$v_a = -[2m + (a - m - 3/2) \log\{1 - m/(a - 1)\}]; \quad \vartheta_a = -[m/(a - 1) + \log\{1 - m/(a - 1)\}];$$

$$\gamma_a(t) = m \left\{ \left(\frac{a-1}{2} + t \right) \log \left(\frac{a-1}{2} + t \right) - \frac{a-1}{2} \log \left(\frac{a-1}{2} \right) \right\}.$$

Proof. By Lemma 10, we know

$$\log \frac{\Gamma_m(\frac{a-1}{2} + t)}{\Gamma_m(\frac{a-1}{2})} = \sum_{i=1}^m \log \frac{\Gamma(\frac{a-i}{2} + t)}{\Gamma(\frac{a-i}{2})}. \quad (52)$$

To prove the lemma, we expand each summed term in (52), $\log\{\Gamma(\frac{a-i}{2} + t)/\Gamma(\frac{a-i}{2})\}$, by Lemma A.1. in Jiang and Qi (2015). To apply the lemma, we first need to check the condition that for each $1 \leq i \leq m$, $t \in [-\delta(a-i)/2, \delta(a-i)/2]$ for any given $\delta \in (0, 1)$.

Recall that we previously define $nh = 2s/\sigma_n$ in Section C.1. Then $t = nh/2 = s\sigma_n^{-1}$. Note that when $m/(n-p)$ and $m/(n-p+r) \rightarrow 0$,

$$\begin{aligned} \sigma_n^2 &= \frac{1}{2} \log \left(1 - \frac{m}{n+r-p} \right) - \frac{1}{2} \log \left(1 - \frac{m}{n-p} \right) \\ &= \frac{mr}{2(n-p)(n+r-p)} \{1 + o(1)\}. \end{aligned}$$

Thus we have

$$t = O(s) \sqrt{\frac{(n-p)(n-p+r)}{mr}}. \quad (53)$$

For $a = n - p + r$ or $a = n - p$, and $1 \leq i \leq m$, by (53), we then have

$$\begin{aligned}
\frac{t}{a-i} &\leq \frac{t}{n-p-m} = O(s) \sqrt{\frac{(n-p)(n-p+r)}{mr(n-p-m)^2}} \\
&= O(s) \sqrt{\frac{1+r/(n-p)}{mr\{1-m/(n-p)\}^2}} \\
&= O(s) \sqrt{\left\{\frac{1}{mr} + \frac{1}{m(n-p)}\right\}\{1+o(1)\}} = o(1),
\end{aligned}$$

where the last two equations follow from the condition that $m/(n-p) \rightarrow 0, r \rightarrow \infty$ and $n-p \rightarrow \infty$. Then we know that for each $1 \leq i \leq m$, $t \in [-\delta(a-i)/2, \delta(a-i)/2]$ for any given $\delta \in (0, 1)$.

Therefore, the condition of Lemma A.1. in Jiang and Qi (2015) is satisfied. By that lemma, we know when $a \rightarrow \infty$, for uniformly $1 \leq i \leq m$,

$$\begin{aligned}
\log \frac{\Gamma(\frac{a-i}{2} + t)}{\Gamma(\frac{a-i}{2})} &= \left(\frac{a-i}{2} + t\right) \log \left(\frac{a-i}{2} + t\right) - \frac{a-i}{2} \log \frac{a-i}{2} \\
&\quad - t - \frac{t}{a-i} + O\left(\frac{t^2}{a^2}\right).
\end{aligned}$$

Write $\frac{t}{a-i} = \frac{t}{a} + \frac{t}{a} \times \frac{i}{a-i}$. Then similarly to Lemma 11, we have

$$\sum_{i=1}^m \frac{t}{a-i} = \frac{mt}{a} + \frac{tm(m+1)}{2a(a-1)} + O\left\{\frac{t}{a} \times \left(\frac{m}{a}\right)^3\right\}. \quad (54)$$

For $a = n - p$, by (53), $m/(n-p) \rightarrow 0$ and $m \leq r$,

$$\begin{aligned}
\frac{tm(m+1)}{a(a-1)} &= O(s) \sqrt{\frac{(n-p)(n-p+r)}{mr}} \frac{m^2}{(n-p)^2} \\
&= O(s) \sqrt{\frac{m(n-p+r)}{r(n-p)}} \frac{m}{n-p} \\
&= O(s) \sqrt{\frac{m}{\min\{n-p, r\}}} \frac{m}{n-p} = o(1).
\end{aligned}$$

For $a = n - p + r$, similar conclusion, $tm(m+1)/\{a(a-1)\} = o(1)$, holds by substituting $n - p$ with $n - p + r$. In addition, for $a = n - p$ or $a = n - p + r$, by (53),

$$\begin{aligned} \frac{t}{a} \leq \frac{t}{n-p} &= O(s) \sqrt{\frac{\max\{n-p, r\}}{mr(n-p)}} \\ &= O(s) \sqrt{\frac{1}{m \times \min\{n-p, r\}}} = o(1). \end{aligned} \quad (55)$$

Then based on (54) and (55), we obtain

$$\sum_{i=1}^m \left\{ -t - \frac{t}{a-i} + O(t^2/a^2) \right\} = -mt - \frac{mt}{a} + o(1).$$

Therefore, from (52), we have

$$\begin{aligned} &\log \frac{\Gamma_m(\frac{a-1}{2} + t)}{\Gamma_m(\frac{a-1}{2})} \\ &= -\frac{(a+1)mt}{a} + \sum_{i=1}^m \left\{ \left(\frac{a-i}{2} + t \right) \log \left(\frac{a-i}{2} + t \right) - \frac{a-i}{2} \log \frac{a-i}{2} \right\} + o(1). \end{aligned} \quad (56)$$

For $1 \leq i \leq m$, define the function

$$g_i(x) = \left(\frac{a-i}{2} + x \right) \log \left(\frac{a-i}{2} + x \right) - \left(\frac{a-1}{2} + x \right) \log \left(\frac{a-1}{2} + x \right),$$

and $x > -(a-m)/2$. We then know that the summation term “ \sum ” in (56) equals to

$$m \left[\left(\frac{a-1}{2} + t \right) \log \left(\frac{a-1}{2} + t \right) - \frac{a-1}{2} \log \frac{a-1}{2} \right] + \sum_{i=1}^m \{g_i(t) - g_i(0)\}. \quad (57)$$

We then examine the function $\sum_{i=1}^m \{g_i(t) - g_i(0)\}$ in (57). Note that by (55), we know $t = o(a)$, $mt^2/a^2 = o(1)$ and $mt/a = O(1)$ as $m < n - p$ and $m \leq r$. Thus the conditions of Lemma 12 and Lemma A.3. in Jiang and Qi (2015) are satisfied when m is fixed and $m \rightarrow \infty$

respectively. When m is fixed, we apply Lemma 12; when $m \rightarrow \infty$, we apply Lemma A.3. in Jiang and Qi (2015). Then we obtain

$$\sum_{i=1}^m \{g_i(t) - g_i(0)\} = \mu_a t + \frac{1}{2} \sigma_a^2 t^2 + o(1),$$

where

$$\begin{aligned} \mu_a &= (m - a + 3/2) \log \left(1 - \frac{m}{a-1} \right) - m \frac{a-1}{a}, \\ \sigma_a^2 &= -2 \left[\frac{m}{a-1} + \log \left(1 - \frac{m}{a-1} \right) \right]. \end{aligned}$$

Therefore, the proposition can be proved by noticing

$$\begin{aligned} v_a &= -\frac{(a+1)m}{a} + \mu_a; \quad \vartheta_a = \sigma_a^2/2; \\ \gamma_a(t) &= m \left[\left(\frac{a-1}{2} + t \right) \log \left(\frac{a-1}{2} + t \right) - \frac{a-1}{2} \log \frac{a-1}{2} \right]. \end{aligned}$$

■

Lemma 14 *Under Case 1 in Section C.1, $R_{n,1}$ and $R_{n,2}$ defined in (44) satisfy*

$$-\frac{m(n+r-p)}{2} R_{n,1} + \frac{m(n-p)}{2} R_{n,2} = o(1).$$

Proof. Note that

$$\begin{aligned}
& -\frac{m(n+r-p)}{2}R_{n,1} + \frac{m(n-p)}{2}R_{n,2} \\
&= \frac{m}{2} \left[\sum_{k=3}^{\infty} \frac{1}{k} (-nh)^k \left\{ \frac{1}{(n+r-p)^{k-1}} - \frac{1}{(n-p)^{k-1}} \right\} \right] \\
&= \frac{m}{2} \left\{ \sum_{k=3}^{\infty} \frac{1}{k} (-nh)^k \frac{-\sum_{q=1}^{k-1} \binom{k-1}{q} r^q (n-p)^{k-1-q}}{(n+r-p)^{k-1} (n-p)^{k-1}} \right\} \\
&= \frac{mnh}{2} \sum_{k=3}^{\infty} \frac{1}{k} \left(\frac{-nh}{n+r-p} \right)^{k-1} \sum_{q=1}^{k-1} \binom{k-1}{q} \left(\frac{r}{n-p} \right)^q. \tag{58}
\end{aligned}$$

If $r/(n-p) = 1$,

$$|(58)| \leq mnh \sum_{k=3}^{\infty} \left(\frac{2nh}{n+r-p} \right)^{k-1} = O\left\{ mnh \frac{n^2 h^2}{(n+r-p)^2} \right\},$$

where

$$\begin{aligned}
\frac{mnh \times n^2 h^2}{(n+r-p)^2} &= O\left\{ \frac{m\sqrt{(n-p)(n+r-p)}}{\sqrt{mr}} \times \frac{(n-p)(n+r-p)}{mr(n+r-p)^2} \right\} \\
&= O\left\{ \frac{m\sqrt{r^2}}{\sqrt{mr}} \times \frac{r^2}{mr \times r^2} \right\} = o(1),
\end{aligned}$$

as $r \rightarrow \infty$.

If $r/(n-p) > 1$, as $\{nh/(n+r-p)\} \times \{r/(n-p)\} = O\{\sqrt{r}/\sqrt{m(n-p)(n+r-p)}\} = o(1)$,

$$|(58)| \leq mnh \sum_{k=3}^{\infty} \left(\frac{2nh}{n+r-p} \times \frac{r}{n-p} \right)^{k-1} = O\left\{ mnh \left(\frac{2nh}{n+r-p} \right)^2 \left(\frac{r}{n-p} \right)^2 \right\},$$

where

$$\begin{aligned}
& mn h \left(\frac{nh}{n+r-p} \right)^2 \left(\frac{r}{n-p} \right)^2 \\
&= O \left\{ \frac{m \sqrt{(n-p)(n+r-p)}}{\sqrt{mr}} \times \frac{(n-p)(n+r-p)}{mr(n+r-p)^2} \times \frac{r^2}{(n-p)^2} \right\} \\
&= O \left\{ \frac{\sqrt{(n-p)(n+r-p)}}{\sqrt{mr}} \frac{r}{(n-p)(n+r-p)} \right\} \\
&= O \left\{ \frac{r}{\sqrt{mr(n+r-p)(n-p)}} \right\} = o(1),
\end{aligned}$$

as $n+r-p \geq r$ and $n-p \rightarrow \infty$.

If $r/(n-p) < 1$,

$$|(58)| \leq mn h \sum_{k=3}^{\infty} \left(\frac{nh}{n+r-p} \right)^{k-1} \frac{r}{(n-p)} = O \left\{ mn h \frac{(nh)^2}{(n+r-p)^2} \times \frac{r}{(n-p)} \right\},$$

where

$$\begin{aligned}
& mn h \frac{(nh)^2}{(n+r-p)^2} \times \frac{r}{(n-p)} \\
&= O \left\{ \frac{m \sqrt{(n-p)(n+r-p)}}{\sqrt{mr}} \times \frac{(n-p)(n+r-p)}{mr(n+r-p)^2} \times \frac{r}{(n-p)} \right\} \\
&= O \left\{ \frac{\sqrt{n-p}}{\sqrt{mr(n+r-p)}} \right\} = o(1).
\end{aligned}$$

■

D Theorem 4

We give the main proof of Theorem 4 in Section D.1, where we use some concepts of hypergeometric function, which is introduced in Section D.2, and the lemmas we use are given and proved in Section D.3.

D.1 Proof of Theorem 4

As $p/n = \rho_p$, $r/n = \rho_r$, $m/n = \rho_m$ with $\rho_p, \rho_r, \rho_m \in (0, 1)$ and $\rho_p + \rho_m < 1$, we know that σ_n^2 in Theorem 3 satisfies

$$\sigma_n^2 = 2 \log \left\{ \left(1 - \frac{\rho_m}{1 + \rho_r - \rho_p} \right) \left(1 - \frac{\rho_m}{1 - \rho_p} \right)^{-1} \right\},$$

which is a positive constant, and we write the constant as σ^2 . Then $T_1 = \{-2 \log L_n + \mu_n\}/(n\sigma)$, and we examine the moment generating function $E\{2s \log L_n/(n\sigma)\}$. Let $h = 2s/(n\sigma)$. By Lemma 15, we have

$$\begin{aligned} E\{2s \log L_n/(n\sigma)\} &= E\{\exp(h \log L_n)\} \\ &= EL_n^h = E_0 L_n^h \times {}_1F_1\left(\frac{nh}{2}; \frac{1}{2}(n+r-p) + \frac{nh}{2}; -\frac{1}{2}\Omega\right), \end{aligned} \quad (59)$$

where $E_0 L_n^h$ is the moment generating function of $\log L_n$ under H_0 , and ${}_1F_1$ is the hypergeometric function, which depends on Ω only through its eigenvalues symmetrically.

As ${}_1F_1$ only depends on Ω via its eigenvalues symmetrically, without loss of generality, we consider the alternative with $\Omega = \text{diag}(w_1, \dots, w_m)$ and $w_1 \geq \dots \geq w_m \geq 0$. Let $\xi_a = nh/2$, $\xi_b = (n+r-p+nh)/2$ and $Q = -\Omega/2 = \text{diag}(-w_1/2, \dots, -w_m/2)$, then we write ${}_1F_1(nh/2, (n+r-p+nh)/2; -\Omega/2)$ as ${}_1F_1(\xi_a; \xi_b; Q)$. Note that we assume that Ω has fixed rank m_0 in Theorem 4, then $\omega_1 \geq \dots \geq \omega_{m_0} > 0$ are m_0 nonzero eigenvalues of Ω . Further

define $\tilde{Q} = \text{diag}(-\omega_1/2, \dots, -\omega_{m_0}/2)$. By Lemma 17, we know ${}_1F_1(\xi_a; \xi_b; Q) = {}_1F_1(\xi_a; \xi_b; \tilde{Q})$. Then to evaluate ${}_1F_1(\xi_a; \xi_b; Q)$ when Q has fixed rank, without loss of generality, we consider ${}_1F_1(\xi_a; \xi_b; \tilde{Q})$.

Let $W = \log {}_1F_1(\xi_a; \xi_b; \tilde{Q})$ and $\tilde{Q} = -n\tilde{\Delta}/2$ with $\tilde{\Delta} = \text{diag}(\delta_1, \dots, \delta_{m_0})$. From Lemma 18, we know that $W(\tilde{\Delta})$ is the unique solution of each of the m_0 partial differential equations

$$\begin{aligned} & \left[\frac{1}{2}(n+r-p-m_0+1) + \frac{nh}{2} + \frac{1}{2}n\delta_j + \frac{1}{2} \sum_{i \neq j}^{m_0} \frac{\delta_j}{\delta_j - \delta_i} \right] \frac{\partial W}{\partial \delta_j} \\ & + \delta_j \left[\frac{\partial^2 W}{\partial \delta_j^2} + \left(\frac{\partial W}{\partial \delta_j} \right)^2 \right] - \frac{1}{2} \sum_{i \neq j}^{m_0} \frac{\delta_i}{\delta_j - \delta_i} \frac{\partial W}{\partial \delta_i} = -\frac{nh}{2} \times n, \end{aligned} \quad (60)$$

for $j = 1, \dots, m_0$, subject to the conditions that $W(\tilde{\Delta})$ is (a) a symmetric function of $\delta_1, \dots, \delta_{m_0}$, and (b) analytic at $\tilde{\Delta} = \mathbf{0}_{m_0 \times m_0}$ with $W(\mathbf{0}_{m_0 \times m_0}) = 0$. As $r/n = \rho_r$, $p/n = \rho_p$, m_0 is a fixed number and $nh = 2s/\sigma$, we can write (60) into

$$\begin{aligned} & \left[\frac{1}{2}(1 + \rho_r - \rho_p)n + \frac{1}{2}(2s/\sigma - m_0 + 1) + \frac{1}{2}n\delta_j + \frac{1}{2} \sum_{i \neq j}^{m_0} \frac{\delta_j}{\delta_j - \delta_i} \right] \frac{\partial W}{\partial \delta_j} \\ & + \delta_j \left[\frac{\partial^2 W}{\partial \delta_j^2} + \left(\frac{\partial W}{\partial \delta_j} \right)^2 \right] - \frac{1}{2} \sum_{i \neq j}^{m_0} \frac{\delta_i}{\delta_j - \delta_i} \frac{\partial W}{\partial \delta_i} = -\frac{s}{\sigma} \times n. \end{aligned} \quad (61)$$

Similarly to Theorem 10.5.6 in Muirhead (2009), we write $W(\tilde{\Delta}) = P_0(\tilde{\Delta}) + P_1(\tilde{\Delta})/n + \dots$. Note that $nh = 2s/\sigma$. Matching n on both sides of (61), we obtain

$$\left[\frac{1}{2}(1 + \rho_r - \rho_p)n + \frac{1}{2}n\delta_j \right] \frac{\partial P_0}{\partial \delta_j} = -\frac{sn}{\sigma}.$$

Solving this subject to conditions (a) and (b), we obtain

$$P_0(\tilde{\Delta}) = -\frac{2s}{\sigma} \sum_{j=1}^{m_0} \log \left(1 + \frac{\delta_j}{1 + \rho_r - \rho_p} \right).$$

Then we have $W(\tilde{\Delta}) = P_0(\tilde{\Delta}) + O(n^{-1})$. From (59), we know

$$EL_n^h = E_0 L_n^h \times e^{\log_1 F_1} = E_0 e^{\frac{s}{n\sigma/2} \log(L_n) + W}. \quad (62)$$

Write $W_\Delta = \sum_{j=1}^{m_0} \log[1 + \delta_j(1 + \rho_r - \rho_p)^{-1}]$ and $A_1 = 2/\sigma$. (30) and (62) show that $\{\log L_n - \mu_n/2\}/(n\sigma_n/2) \xrightarrow{D} \mathcal{N}(-A_1 W_\Delta, 1)$, and thus $\{-2 \log L_n + \mu_n\}/(n\sigma) \xrightarrow{D} \mathcal{N}(A_1 W_\Delta, 1)$. Then the power $P(T_1 > z_\alpha) \rightarrow \bar{\Phi}(z_\alpha - A_1 W_\Delta)$.

D.2 Brief review of hypergeometric function

We rephrase some related definitions and results about hypergeometric function, where the details can be found in Chapter 7 in Muirhead (2009).

Let k be a positive integer; a partition κ of k is written as $\kappa = (k_1, k_2, \dots)$, where $\sum_t k_t = k$ and $k_1 \geq k_2 \geq \dots$ are non-negative integers. In addition, let M be an $m \times m$ symmetric matrix with eigenvalues l_1, \dots, l_m , and let $\kappa = (k_1, k_2, \dots)$ be a partition of k into no more than m nonzero parts. We write the zonal polynomial of M corresponding to κ as $C_\kappa(M)$. Then by the definition, we know the hypergeometric function ${}_1F_1(\xi_a; \xi_b; Q)$ satisfies

$${}_1F_1(\xi_a; \xi_b; Q) = \sum_{k=0}^{\infty} \sum_{\kappa:k} \frac{(\xi_a)_\kappa}{(\xi_b)_\kappa} \frac{C_\kappa(Q)}{k!}, \quad (63)$$

where $\sum_{\kappa:k}$ represents the summation over the partitions $\kappa = (k_1, \dots, k_m)$, $k_1 \geq \dots \geq k_m \geq 0$, of k , $C_\kappa(Q)$ is the zonal polynomial of Q corresponding to κ , and the generalized hypergeometric coefficient $(\xi)_\kappa$ is given by $(\xi)_\kappa = \prod_{i=1}^t (\xi - (i-1)/2)_{k_i}$ with $(a)_{k_i} = a(a+1) \dots (a+k_i-1)$ and $(a)_0 = 1$.

We then characterize the zonal polynomials $C_\kappa(M)$. For given partition $\kappa = (k_1, k_2, \dots)$ of k , define the monomial symmetric functions $N_\kappa(M) = \sum_{\{i_1, \dots, i_t\}} l_{i_1}^{k_1} \dots l_{i_t}^{k_t}$, where t is the number of nonzero parts in the partition κ , and the summation is over the distinct

permutations (i_1, \dots, i_t) of t different integers from $1, \dots, m$. For another partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we write $\kappa > \lambda$ if $k_i > \lambda_i$ for the first index i for which the parts in κ and λ are unequal. Then we have $C_\kappa(M) = \sum_{\lambda \leq \kappa} c_{\kappa, \lambda} N_\lambda(M)$, where $c_{\kappa, \lambda}$ are constants.

D.3 Lemmas in the proof of Theorem 4

Lemma 15 $EL_n^h = E_0 L_n^h \times {}_1F_1(nh/2; (n+r-p+nh)/2; -\Omega/2)$.

Proof. The result follows from Theorem 10.5.1 in Muirhead (2009). ■

Lemma 16 *Suppose matrix M of size $m \times m$ has m eigenvalues l_1, \dots, l_m , but only has m_0 positive eigenvalues l_1, \dots, l_{m_0} and $\tilde{M} = \text{diag}(l_1, \dots, l_{m_0})$. Then for given partition κ , the zonal polynomial functions satisfy $N_\kappa(M) = N_\kappa(\tilde{M})$.*

Proof. By the definition of monomial function $N_\lambda(M)$, we note that $\sum_{\{i_1, \dots, i_t\}} l_{i_1}^{k_1} \dots l_{i_t}^{k_t} = \sum_{\{\tilde{i}_1, \dots, \tilde{i}_t\}} l_{\tilde{i}_1}^{k_1} \dots l_{\tilde{i}_t}^{k_t}$, where $\sum_{\{\tilde{i}_1, \dots, \tilde{i}_t\}}$ represents the summation over the distinct permutations $(\tilde{i}_1, \dots, \tilde{i}_t)$ of t different integers from $1, \dots, m_0$. It follows that $N_\lambda(M) = N_\lambda(\tilde{M})$, where $\tilde{M} = \text{diag}(l_1, \dots, l_{m_0})$. ■

Lemma 17 *Suppose Q has fixed rank m_0 , then ${}_1F_1(\xi_a; \xi_b; Q) = {}_1F_1(\xi_a; \xi_b; \tilde{Q})$.*

Proof. As Q has rank m_0 , it only has m_0 nonzero eigenvalues. To prove the lemma, we note that the hypergeometric function can be expressed as the linear combination of the zonal polynomials of a matrix. We then state two properties of the zonal polynomial functions $C_\kappa(Q)$. First, by Corollary 7.2.4 in Muirhead (2009), we know that when κ is a partition of k into more than m_0 nonzero parts, $C_\kappa(Q) = 0$. Second, when κ is a partition of k into fewer than m_0 nonzero parts, $C_\kappa(Q) = C_\kappa(\tilde{Q})$. To see this, we note that $C_\kappa(Q) = \sum_{\lambda \leq \kappa} c_{\lambda, \kappa} N_\lambda(Q)$ and the constants $c_{\lambda, \kappa}$ do not depend on the eigenvalues of Q . Then by Lemma 16, we know that $C_\kappa(M) = C_\kappa(\tilde{M})$. Finally, by the definition in (63), we have ${}_1F_1(\xi_a; \xi_b; Q) = {}_1F_1(\xi_a; \xi_b; \tilde{Q})$. ■

Lemma 18 $W = \log {}_1F_1(\xi_a; \xi_b; \tilde{Q})$ with $\tilde{Q} = -n\tilde{\Delta}/2$ discussed in Section D.1 is the unique solution of each of the m_0 partial differential equations

$$\begin{aligned} & \left[\frac{1}{2}(n+r-p-m_0+1) + \frac{nh}{2} + \frac{1}{2}n\delta_j + \frac{1}{2} \sum_{i \neq j}^{m_0} \frac{\delta_j}{\delta_j - \delta_i} \right] \frac{\partial W}{\partial \delta_j} \\ & + \delta_j \left[\frac{\partial^2 W}{\partial \delta_j^2} + \left(\frac{\partial W}{\partial \delta_j} \right)^2 \right] - \frac{1}{2} \sum_{i \neq j}^{m_0} \frac{\delta_i}{\delta_j - \delta_i} \frac{\partial W}{\partial \delta_i} = -\frac{nh}{2} \times n, \end{aligned}$$

for $j = 1, \dots, m_0$, subject to the conditions that $W(\tilde{\Delta})$ is (a) a symmetric function of $\delta_1, \dots, \delta_{m_0}$, and (b) analytic at $\tilde{\Delta} = \mathbf{0}_{m_0 \times m_0}$ with $W(\mathbf{0}_{m_0 \times m_0}) = 0$.

Proof. As m_0 is fixed, the result follows from Theorem 7.5.6 in Muirhead (2009) by changing of variables. ■

E Theorem 5

We give the main proof of Theorem 5 in Section E.1, and the lemmas we use in the proof are given and proved in Section E.2.

E.1 Proof of Theorem 5

Before proceeding to the proof, we define some notations and provide some preliminary results. Note that by the form of ω_j , we could assume $E(X) = \mathbf{0}$ with loss of generality. Let $Z = X\Sigma_x^{-1/2}$. We know that the entries in Z are i.i.d. $\mathcal{N}(0, 1)$ by Condition 1, and then with probability 1, the $n \times p$ matrix Z has full rank n . Let $\mu_1^{1/2}, \dots, \mu_n^{1/2}$ be the n singular values of Z . Then $Z^\top Z$ has the eigendecomposition

$$Z^\top Z = U^\top \text{diag}(\mu_1, \dots, \mu_n, 0, \dots, 0)U, \quad (64)$$

where U belongs to the orthogonal group $\mathcal{O}(p)$. We write $U^\top = (\mathbf{u}_1, \dots, \mathbf{u}_p)$. It follows that the Moore-Penrose generalized inverse of (64) is

$$(Z^\top Z)^+ = \sum_{i=1}^n \frac{1}{\mu_i} \mathbf{u}_i \mathbf{u}_i^\top.$$

Moreover, we have the decomposition

$$S := (Z^\top Z)^+ Z^\top Z = U^\top \text{diag}(I_n, 0) U = \tilde{U}^\top \tilde{U}, \quad (65)$$

where $\tilde{U} = (I_n, \mathbf{0})_{n \times p} U$ and $(I_n, \mathbf{0})_{n \times p}$ represents an $n \times p$ matrix with first n columns being I_n and 0 in the remaining columns. Since $X = Z \Sigma_x^{1/2}$, by (64), we know that

$$X^\top X = \Sigma_x^{1/2} \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} \Sigma_x^{1/2}. \quad (66)$$

In addition, define $P = I_n - \mathbf{1}_n \mathbf{1}_n^\top / n$. We can then write ω_i equivalently as

$$\omega_i = \max_{\mathbf{a}: \|\mathbf{a}\|=1} \frac{\mathbf{a}^\top Y^\top P^\top P \mathbf{x}^i}{\sqrt{(\mathbf{a}^\top Y^\top P^\top P Y \mathbf{a}) \{(\mathbf{x}^i)^\top P^\top P \mathbf{x}^i\}}}.$$

By the property of ω_i , we assume without loss of generality that X and E have mean zero. Furthermore, suppose $\text{diag}(\Sigma_x) = \text{diag}(\sigma_{x,1}, \dots, \sigma_{x,p})$ and let

$$\zeta_i = \max_{\mathbf{a}: \|\mathbf{a}\|=1} \frac{\mathbf{a}^\top Y^\top P^\top P \mathbf{x}^i}{\sigma_{x,i} \sqrt{n \times \mathbf{a}^\top Y^\top P^\top P Y \mathbf{a}}}. \quad (67)$$

Then by Lemma 24, we know $\omega_i = \zeta_i \{1 + o(1)\}$ with probability $1 - O\{\exp(-c_0 n / \log n)\}$ for some constant $c_0 > 0$. As $Y = XB + E$, we have

$$\zeta_i = \max_{\mathbf{a}: \|\mathbf{a}\|=1} \frac{\mathbf{a}^\top B^\top X^\top P^\top P \mathbf{x}^i + \mathbf{a}^\top E^\top P^\top P \mathbf{x}^i}{\sigma_{x,i} \sqrt{n \times \mathbf{a}^\top (Y^\top P^\top P Y) \mathbf{a}}} = \xi_i + \eta_i, \quad (68)$$

where

$$\xi_i = \max_{\mathbf{a}: \|\mathbf{a}\|=1} \frac{\mathbf{a}^\top B^\top X^\top P^\top P \mathbf{x}^i}{\sigma_{x,i} \sqrt{n \mathbf{a}^\top (Y^\top P^\top P Y) \mathbf{a}}}, \quad \eta_i = \max_{\mathbf{a}: \|\mathbf{a}\|=1} \frac{\mathbf{a}^\top E^\top P^\top P \mathbf{x}^i}{\sigma_{x,i} \sqrt{n \mathbf{a}^\top (Y^\top P^\top P Y) \mathbf{a}}}.$$

Moreover, we write $B = [\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_m]$, where $\boldsymbol{\beta}_j$ represents the j -th column of B . We then study ξ_i and η_i separately.

Step 1: We first examine $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)^\top$.

Step 1.1 (bounding $\|\boldsymbol{\xi}\|$ from above) For $i = 1, \dots, p$,

$$|\xi_i| \leq \{n \lambda_{\min}(Y^\top P^\top P Y)\}^{-1/2} \sigma_{x,i}^{-1} \|B^\top X^\top P^\top P \mathbf{x}^i\|,$$

where $\|\cdot\|$ represents the ℓ_2 -norm of a vector. Then we know

$$\|\boldsymbol{\xi}\|^2 = \sum_{i=1}^p \xi_i^2 \leq \{n \lambda_{\min}(Y^\top P^\top P Y)\}^{-1} \sum_{i=1}^p \sigma_{x,i}^{-2} \|B^\top X^\top P^\top P \mathbf{x}^i\|^2 \quad (69)$$

By Lemma 25, we know that there exist constants c_1 and c_0 such that $\lambda_{\min}(Y^\top P^\top P Y) \geq c_1 n^{1-t}$ with probability $1 - O\{\exp(-c_0 n)\}$. To bound $\|\boldsymbol{\xi}\|$ from above, we then examine $\sum_{i=1}^p \sigma_{x,i}^{-2} \|B^\top X^\top P^\top P \mathbf{x}^i\|^2$. Since $\min_{1 \leq i \leq p} \sigma_{x,i}^2 \geq c_3$ by Condition 1,

$$\begin{aligned} \sum_{i=1}^p \sigma_{x,i}^{-2} \|B^\top X^\top P^\top P \mathbf{x}^i\|^2 &\leq c_3^{-1} \sum_{i=1}^p \sum_{k=1}^m (\boldsymbol{\beta}_k^\top X^\top P^\top P \mathbf{x}^i)^2 \\ &= c_3^{-1} \sum_{k=1}^m \boldsymbol{\beta}_k^\top X^\top P^\top P \sum_{i=1}^p \mathbf{x}^i (\mathbf{x}^i)^\top P^\top P X \boldsymbol{\beta}_k. \end{aligned} \quad (70)$$

As $\sum_{i=1}^p \mathbf{x}^i (\mathbf{x}^i)^\top = XX^\top$ and $P^\top P = I_n - \mathbf{1}_n \mathbf{1}_n^\top / n$, we have

$$\begin{aligned} (70) &= c_3^{-1} \sum_{k=1}^m \|\boldsymbol{\beta}_k^\top X^\top (I_n - \mathbf{1}_n \mathbf{1}_n^\top / n) X\|^2 \\ &\leq 2c_3^{-1} \times (A_{\boldsymbol{\xi},1} + A_{\boldsymbol{\xi},2}), \end{aligned} \quad (71)$$

where $A_{\boldsymbol{\xi},1} = \sum_{k=1}^m \|\boldsymbol{\beta}_k^\top X^\top X\|^2$ and $A_{\boldsymbol{\xi},2} = \sum_{k=1}^m \|\boldsymbol{\beta}_k^\top X^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) X\|^2$.

We next examine $A_{\boldsymbol{\xi},1}$ and $A_{\boldsymbol{\xi},2}$ separately. By (66),

$$\begin{aligned} A_{\boldsymbol{\xi},1} &= \sum_{k=1}^m \boldsymbol{\beta}_k^\top \Sigma_x^{1/2} \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} \Sigma_x^{1/2} \Sigma_x^{1/2} \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} \Sigma_x^{1/2} \boldsymbol{\beta}_k \\ &\leq p^2 \{\lambda_{\max}(p^{-1} Z Z^\top)\}^2 \lambda_{\max}(\Sigma_x) \sum_{k=1}^m \boldsymbol{\beta}_k^\top \Sigma_x^{1/2} \tilde{U}^\top \tilde{U} \Sigma_x^{1/2} \boldsymbol{\beta}_k, \end{aligned} \quad (72)$$

where in the last inequality, we use the fact that $\tilde{U} \Sigma_x \tilde{U}^\top \preceq \lambda_{\max}(\Sigma_x) I_n$, and $\text{diag}(\mu_1, \dots, \mu_n) \preceq p \lambda_{\max}(p^{-1} Z Z^\top) I_n$, as $\mu_1^{1/2}, \dots, \mu_n^{1/2}$ are the singular values of Z . We then bound (72) from above by examining $\boldsymbol{\beta}_k^\top \Sigma_x^{1/2} \tilde{U}^\top \tilde{U} \Sigma_x^{1/2} \boldsymbol{\beta}_k$. For fixed $k = 1, \dots, m$, let $Q \in \mathcal{O}(p)$ such that $\Sigma_x^{1/2} \boldsymbol{\beta}_k = \|\Sigma_x^{1/2} \boldsymbol{\beta}_k\|_2 Q e_1$. By (65) and Lemma 20, we know

$$\boldsymbol{\beta}_k^\top \Sigma_x^{1/2} \tilde{U}^\top \tilde{U} \Sigma_x^{1/2} \boldsymbol{\beta}_k = \|\Sigma_x^{1/2} \boldsymbol{\beta}_k\|^2 \langle Q^\top S Q e_1, e_1 \rangle \stackrel{(d)}{=} \|\Sigma_x^{1/2} \boldsymbol{\beta}_k\|^2 \langle S e_1, e_1 \rangle. \quad (73)$$

By Condition 1, $\|\Sigma_x^{1/2} \boldsymbol{\beta}_k\|^2 = \boldsymbol{\beta}_k^\top \Sigma_x \boldsymbol{\beta}_k \leq c_5$ for some constant $c_5 > 0$. Then by (73) and Lemma 21, we know for some positive constants c_0 and c_1 ,

$$P(\boldsymbol{\beta}_k^\top \Sigma_x^{1/2} \tilde{U}^\top \tilde{U} \Sigma_x^{1/2} \boldsymbol{\beta}_k > c_1 n / p) \leq O\{\exp(-c_0 n)\}. \quad (74)$$

Combining (72), Lemma 22, Condition 1 and (74), we then know for some positive constants c_0 and c , with probability $1 - O\{m \exp(-c_0 n)\}$, $A_{\boldsymbol{\xi},1} \leq c m p^2 n^\tau n / p = c m p n^{1+\tau}$.

For $A_{\xi,2}$, note that

$$A_{\xi,2} = \sum_{k=1}^m \beta_k^\top \Sigma_x^{1/2} Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z \Sigma_x Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z \Sigma_x^{1/2} \beta_k. \quad (75)$$

Similarly, considering fixed $k = 1, \dots, m$, we let $Q \in \mathcal{O}(p)$ such that $\Sigma_x^{1/2} \beta_k = \|\Sigma_x^{1/2} \beta_k\| Q e_1$.

Then

$$\begin{aligned} & \beta_k^\top \Sigma_x^{1/2} Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z \Sigma_x Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z \Sigma_x^{1/2} \beta_k \\ &= \|\Sigma_x^{1/2} \beta_k\|^2 e_1^\top Q^\top Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z \Sigma_x Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z Q e_1 \\ &\leq \lambda_{\max}(\Sigma_x) \|\Sigma_x^{1/2} \beta_k\|^2 e_1^\top Q^\top Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z Q Q^\top Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z Q e_1 \\ &\stackrel{(d)}{=} \lambda_{\max}(\Sigma_x) \|\Sigma_x^{1/2} \beta_k\|^2 \|Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z e_1\|^2, \end{aligned} \quad (76)$$

where in the last equality, we use the fact that $ZQ \stackrel{(d)}{=} Z$. Since the entries in Z are i.i.d. $\mathcal{N}(0, 1)$, we have $L = (L_1, \dots, L_p)^\top = Z^\top \mathbf{1}_n / \sqrt{n} \sim \mathcal{N}(\mathbf{0}_{p \times 1}, I_p)$ with $L_1 = \mathbf{1}_n^\top Z e_1 / \sqrt{n}$. It follows that $\|Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z e_1\|^2 = \sum_{j=1}^p L_1^2 L_j^2$. Since $L_1^2 \sim \chi_1^2$, there exist constants c_0 and c_1 such that $P(|L_1^2 - 1| > c_1 n) \leq O\{\exp(-c_0 n)\}$. Moreover, note that L_j^2 's are i.i.d. χ_1^2 -distributed random variables. By Lemma 19, for some positive constants c_0 and c_1 , when $p \geq n$,

$$P\left(\sum_{j=2}^p L_j^2 / (p-1) > 1 + c_1\right) \leq O\{\exp(-c_0 p)\} \leq O\{\exp(-c_0 n)\}. \quad (77)$$

Thus there exist constants c and c_0 such that with probability $1 - O\{\exp(-c_0 n)\}$,

$$\|Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z e_1\|^2 = \sum_{j=1}^p L_1^2 L_j^2 = L_1^4 + L_1^2 \sum_{j=2}^p L_j^2 \leq c p n.$$

By Condition 1, $\lambda_{\max}(\Sigma_x) \leq c_1 n^\tau$ and $\|\Sigma_x^{1/2} \beta_k\|^2 \leq c_2$ for some constant c_1 and c_2 . From

(75) and (76), we know that $A_{\xi,2} \leq cmpn^{\tau+1}$ with probability $1 - O\{m \exp(-c_0 n)\}$.

In summary, we obtain that for some constants c and c_0 , $A_{\xi,1}$ and $A_{\xi,2} \leq cmpn^{\tau+1}$ with probability $1 - O\{\exp(-c_0 n)\}$. Then by (69), (71) and Lemma 25, we have for some positive constants c_1 and c_0 ,

$$P\{\|\xi\|^2 > c_1 n^{-(1+1-t)} p m n^{1+\tau}\} \leq O\{m \exp(-c_0 n)\} = O\{\exp(-c_0 n)\}, \quad (78)$$

where the last equality is from Condition 3.

Step 1.2 (bounding $|\xi_i|$ for $i \in \mathcal{M}_*$ from below) Without loss of generality, we consider $B \neq \mathbf{0}_{p \times m}$. For fixed $i \in \mathcal{M}_*$,

$$\begin{aligned} \xi_i &= \max_{\mathbf{a}: \|\mathbf{a}\|_2=1} \frac{\mathbf{a}^\top B^\top X^\top P^\top P \mathbf{x}^i}{\sigma_{x,i} \sqrt{n \times \mathbf{a}^\top (Y^\top P^\top P Y) \mathbf{a}}} \\ &\geq \{n \times \mathbf{a}_{0,i}^\top Y^\top P^\top P Y \mathbf{a}_{0,i}\}^{-1/2} \sigma_{x,i}^{-1} |\mathbf{a}_{0,i}^\top B^\top X^\top P^\top P X e_i|, \end{aligned} \quad (79)$$

where $\mathbf{a}_{0,i}$ in the last inequality is specified in Condition 2. To bound $|\xi_i|$ from below, we then examine (79). By Lemma 25, there exist constants c_0 and c_2 such that with probability $1 - O\{\exp(-c_0 n)\}$,

$$\mathbf{a}_{0,i}^\top Y^\top P^\top P Y \mathbf{a}_{0,i} \leq c_2 n^{2u+1}. \quad (80)$$

Moreover, as $P^\top P = I_n - \mathbf{1}_n \mathbf{1}_n^\top / n$, $\sigma_{x,i}^{-1} \mathbf{a}_{0,i}^\top B^\top X^\top P^\top P X e_i = \tilde{A}_{\xi,i,1} - \tilde{A}_{\xi,i,2}$, where $\tilde{A}_{\xi,i,1} = \sigma_{x,i}^{-1} \mathbf{a}_{0,i}^\top B^\top X^\top X e_i$ and $\tilde{A}_{\xi,i,2} = \sigma_{x,i}^{-1} \mathbf{a}_{0,i}^\top B^\top X^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) X e_i$.

We first consider $\tilde{A}_{\xi,i,1}$. From (66),

$$\sigma_{x,i}^{-1} \mathbf{a}_{0,i}^\top B^\top X^\top X e_i = \sigma_{x,i}^{-1} \mathbf{a}_{0,i}^\top B^\top \Sigma_x^{1/2} \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} \Sigma_x^{1/2} e_i. \quad (81)$$

Note that for fixed $i = 1, \dots, p$, $\|\Sigma_x^{1/2} e_i\|^2 \sigma_{x,i}^{-2} = 1$. Then there exists $\tilde{Q} \in \mathcal{O}(p)$ such that $\Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} = \tilde{Q} e_1$, and

$$\begin{aligned} & \Sigma_x^{1/2} B \mathbf{a}_{0,i} - \langle \Sigma_x^{1/2} B \mathbf{a}_{0,i}, \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} \rangle \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} \\ &= (\|\Sigma_x^{1/2} B \mathbf{a}_{0,i}\|^2 - \langle \Sigma_x^{1/2} B \mathbf{a}_{0,i}, \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} \rangle^2)^{1/2} \tilde{Q} e_2. \end{aligned} \quad (82)$$

By Condition 2, there exists constant c such that $\|\Sigma_x^{1/2} B \mathbf{a}_{0,i}\| \leq cn^u$. Thus

$$\Sigma_x^{1/2} B \mathbf{a}_{0,i} = \langle \Sigma_x^{1/2} B \mathbf{a}_{0,i}, \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} \rangle \tilde{Q} e_1 + O(n^u) \tilde{Q} e_2. \quad (83)$$

Let $T_{\eta,1} = \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} \tilde{Q} e_1$. As $\tilde{Q} e_1 = \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1}$, it follows that

$$(81) = \langle \Sigma_x^{1/2} B \mathbf{a}_{0,i}, \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} \rangle e_1^\top \tilde{Q}^\top T_{\eta,1} + O(n^u) e_2^\top \tilde{Q}^\top T_{\eta,1}. \quad (84)$$

Since the uniform distribution on the orthogonal group $\mathcal{O}(p)$ is invariant under itself, $\tilde{U} \tilde{Q} \stackrel{(d)}{=} \tilde{U}$. Then as $(\mu_1, \dots, \mu_n)^\top$ is independent of \tilde{U} by Lemma 20, we know that $\tilde{Q}^\top T_{\eta,1} \stackrel{(d)}{=} \mathbf{R}$, where $\mathbf{R} = (R_1, \dots, R_p)^\top = \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} e_1$. By (84), we then have

$$(81) \stackrel{(d)}{=} \xi_{i,1} + \xi_{i,2}, \quad (85)$$

where $\xi_{i,1} = \langle \Sigma_x^{1/2} B \mathbf{a}_{0,i}, \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} \rangle R_1$ and $\xi_{i,2} = O(n^u) R_2$.

We next examine $\xi_{i,1}$ and $\xi_{i,2}$ separately. For $\xi_{i,1}$, as $\mu_1, \dots, \mu_n \geq p \lambda_{\min}(p^{-1} Z Z^\top)$, and by (65), we have

$$R_1 \geq p e_1^\top \tilde{U}^\top \lambda_{\min}(p^{-1} Z Z^\top) I_n \tilde{U} e_1 = p \lambda_{\min}(p^{-1} Z Z^\top) \langle S e_1, e_1 \rangle.$$

Thus, by Condition 1, Lemmas 21 and 22, and Bonferroni inequality, we have for some

positive constants c_1 and c_0 ,

$$P(R_1 < c_1 p \times n/p) \leq O\{\exp(-c_0 n)\}. \quad (86)$$

This, along with Condition 2, show that for some positive constants c_1 and c_0 ,

$$P(|\xi_{i,1}| < c_1 n^{1-\kappa}) \leq O\{\exp(-c_0 n)\}. \quad (87)$$

We then consider $\xi_{i,2} = O(n^u)R_2$. By Lemma 23, we know that there exist positive constants c_1 and c_0 such that $P(|R_2| > c_1 n^{1/2}|W_1|) \leq O\{\exp(-c_0 n)\}$, where W_1 is an independent $\mathcal{N}(0, 1)$ -distributed random variable. It follows that for some positive constants c_1 and c_0 , we have

$$P(|\xi_{i,2}| > c_1 n^{u+1/2}|W_1|) \leq O\{\exp(-c_0 n)\}. \quad (88)$$

For some constant $c_2 > 0$, let $x_n = \sqrt{2c_2}n^{1-\kappa-u}/\sqrt{\log n}$. Then by the classical Gaussian tail bound, we have

$$P(n^{1/2}|W| > x_n) \leq \sqrt{2/\pi} \frac{\exp\{-c_2 n^{1-2\kappa-2u}/\log n\}}{\sqrt{2c_2}n^{1/2-\kappa-u}/\sqrt{\log n}} \leq O\{\exp(-c_2 n^{1-2\kappa-2u}/\log n)\},$$

which, along with inequality (88), show that for some positive constants c_1 and c_0 ,

$$P(|\xi_{i,2}| > c_1 n^u x_n) \leq O[\exp\{-c_0 n^{1-2\kappa-2u}/\log n\}], \quad (89)$$

where $n^u x_n = \sqrt{2c_2}n^{1-\kappa}/\sqrt{\log n}$.

We then consider $\tilde{A}_{\xi,i,2}$. Similarly, we take $\tilde{Q} \in \mathcal{O}(p)$ satisfying $\Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} = \tilde{Q} e_1$ and

(82). As $Z\tilde{Q} \stackrel{(d)}{=} Z$ and by (83), similarly to (85), we have

$$\tilde{A}_{\xi,i,2} = \mathbf{a}_{0,i}^\top B^\top \Sigma_x^{1/2} Z^\top (\mathbf{1}_n \mathbf{1}_n^\top / n) Z \tilde{Q} e_1 \stackrel{(d)}{=} \tilde{\xi}_{i,1} + \tilde{\xi}_{i,2},$$

where $\tilde{\xi}_{i,1} = \langle \Sigma_x^{1/2} B \mathbf{a}_{0,i}, \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} \rangle e_1^\top Z^\top \mathbf{1}_n \mathbf{1}_n^\top Z e_1 / n$ and $\tilde{\xi}_{i,2} = O(n^u) e_2^\top Z^\top \mathbf{1}_n \mathbf{1}_n^\top Z e_1 / n$. Note that $\mathbf{1}_n^\top Z e_1 / \sqrt{n} \sim \mathcal{N}(0, 1)$ and $\mathbf{1}_n^\top Z e_2 / \sqrt{n} \sim \mathcal{N}(0, 1)$ independently. Then for some positive constants c_1 and c_0 ,

$$\begin{aligned} P(|e_1^\top Z^\top \mathbf{1}_n \mathbf{1}_n^\top Z e_1 / n| > c_1 n^{1-\kappa-u} / \log n) &\leq O[\exp\{-c_0 n^{1-\kappa-u} / \log n\}], \\ P(|e_2^\top Z^\top \mathbf{1}_n \mathbf{1}_n^\top Z e_1 / n| > c_1 n^{1-\kappa-u} / \log n) &\leq O[\exp\{-c_0 n^{1-\kappa-u} / \log n\}]. \end{aligned}$$

These, combined with (86), show that there exist some constants c_1 and c_0 such that

$$\begin{aligned} P(|\tilde{\xi}_{i,1}| > c_1 |\xi_{i,1}| n^{-u-\kappa} / \log n) &\leq P(|\tilde{\xi}_{i,1}| > c_1 \langle \Sigma_x^{1/2} B \mathbf{a}_{0,i}, \Sigma_x^{1/2} e_i \sigma_{x,i}^{-1} \rangle n^{1-\kappa-u} / \log n) \\ &\quad + P(R_1 < c_1 n) \\ &\leq O\{\exp(-c_0 n^{1-\kappa-u} / \log n)\}, \\ P(|\tilde{\xi}_{i,2}| > c_1 n^{1-\kappa} / \log n) &\leq O\{\exp(-c_0 n^{1-\kappa-u} / \log n)\}. \end{aligned} \tag{90}$$

In summary, by Bonferroni's inequality, combining (79), (80), (85), (87), (89) and (90), we have for some positive constants c_1 and c_0 ,

$$P(|\xi_i| < c_1 (n \times n^{2u+1})^{-1/2} n^{1-\kappa}) \leq O[\exp\{-c_0 n^{1-2\kappa-2u} / \log n\}], \quad i \in \mathcal{M}_*. \tag{91}$$

Step 2 We next examine $\boldsymbol{\eta} = (\eta_1, \dots, \eta_p)^\top$ defined in (68).

Step 2.1 (bounding $\|\boldsymbol{\eta}\|_2$ from above) By Condition 1,

$$\eta_i = \max_{\mathbf{a}: \|\mathbf{a}\|=1} \frac{\mathbf{a}^\top E^\top P^\top P \mathbf{x}^i}{\sigma_{x,i} \sqrt{n \times \mathbf{a}^\top Y^\top P^\top P Y \mathbf{a}}} \leq \{n\lambda_{\min}(Y^\top P^\top P Y)\}^{-1/2} c_3^{-1} \|E^\top P^\top P \mathbf{x}^i\|. \quad (92)$$

Let $\boldsymbol{\epsilon}^j$ denote the j -th column of E , then $E = [\boldsymbol{\epsilon}^1, \dots, \boldsymbol{\epsilon}^m]$. As $P^\top P = I_n - \mathbf{1}_n \mathbf{1}_n^\top / n$, we have $\|E^\top P^\top P \mathbf{x}^i\|^2 = \sum_{j=1}^m \{(\boldsymbol{\epsilon}^j)^\top \mathbf{x}^i - (\boldsymbol{\epsilon}^j)^\top \mathbf{1}_n \mathbf{1}_n^\top \mathbf{x}^i / n\}^2$. Note that $\sum_{i=1}^p \mathbf{x}^i (\mathbf{x}^i)^\top = X X^\top$. Then by (92),

$$\begin{aligned} \sum_{i=1}^p \eta_i^2 &\leq c_3^{-2} \{n\lambda_{\min}(Y^\top P^\top P Y)\}^{-1} \sum_{i=1}^p \sum_{j=1}^m 2 \times [\{(\boldsymbol{\epsilon}^j)^\top \mathbf{x}^i\}^2 + \{(\boldsymbol{\epsilon}^j)^\top \mathbf{1}_n \mathbf{1}_n^\top \mathbf{x}^i / n\}^2] \\ &= 2c_3^{-2} \{n\lambda_{\min}(Y^\top P^\top P Y)\}^{-1} \\ &\quad \times \sum_{j=1}^m \{(\boldsymbol{\epsilon}^j)^\top \sum_{i=1}^p \mathbf{x}^i (\mathbf{x}^i)^\top \boldsymbol{\epsilon}^j + (\boldsymbol{\epsilon}^j)^\top \mathbf{1}_n \mathbf{1}_n^\top \sum_{i=1}^p \mathbf{x}^i (\mathbf{x}^i)^\top \mathbf{1}_n \mathbf{1}_n^\top \boldsymbol{\epsilon}^j / n^2\} \\ &= 2c_3^{-2} \{n\lambda_{\min}(Y^\top P^\top P Y)\}^{-1} \sum_{j=1}^m \{(\boldsymbol{\epsilon}^j)^\top X X^\top \boldsymbol{\epsilon}^j + (\boldsymbol{\epsilon}^j)^\top \mathbf{1}_n \mathbf{1}_n^\top X X^\top \mathbf{1}_n \mathbf{1}_n^\top \boldsymbol{\epsilon}^j / n^2\} \\ &\leq 2c_3^{-2} \{n\lambda_{\min}(Y^\top P^\top P Y)\}^{-1} \lambda_{\max}(\Sigma_x) \sum_{j=1}^m (A_{\eta,j,1} + A_{\eta,j,2}), \end{aligned} \quad (93)$$

where $A_{\eta,j,1} = (\boldsymbol{\epsilon}^j)^\top Z Z^\top \boldsymbol{\epsilon}^j$ and $A_{\eta,j,2} = (\boldsymbol{\epsilon}^j)^\top \mathbf{1}_n \mathbf{1}_n^\top Z Z^\top \mathbf{1}_n \mathbf{1}_n^\top \boldsymbol{\epsilon}^j / n^2$.

Note that $A_{\eta,j,1} \leq p\lambda_{\max}(p^{-1} Z Z^\top) \|\boldsymbol{\epsilon}^j\|^2$. Suppose $\text{diag}(\Sigma) = (\sigma_{\epsilon,1}^2, \dots, \sigma_{\epsilon,m}^2)^\top$. Then by Condition 1 and Lemma 19, we know for some positive constants c and c_0 ,

$$P(\|\boldsymbol{\epsilon}^j\|_2^2 > cn\sigma_{\epsilon,j}^2 / \log n) \leq \exp(-c_0 n / \log n). \quad (94)$$

In addition, $A_{\eta,j,2} \leq p\lambda_{\max}(p^{-1} Z Z^\top) \times (\mathbf{1}_n^\top \boldsymbol{\epsilon}^j)^2 / n$. Similarly to (94), by Condition 1 and the tail bound of the Chi-squared distribution, there exist some positive constants c and c_0 ,

$$P\{(\mathbf{1}_n^\top \boldsymbol{\epsilon}^j)^2 / n > cn\sigma_{\epsilon,j}^2 / \log n\} \leq O\{\exp(-c_0 n / \log n)\}. \quad (95)$$

Combining (94) and (95), we know that for some constants c_1, c_2 and c_0 , with probability $1 - O\{m \exp(-c_0 n / \log n)\}$,

$$A_{\eta,j,1} + A_{\eta,j,2} \leq c_1 p n \sum_{j=1}^m \sigma_{\epsilon,j}^2 / \log n \leq c_2 p n m / \log n, \quad (96)$$

where the last inequality is from $\text{diag}(\Sigma) \leq c_4$ for some constant $c_4 > 0$ by Condition 1.

Combining (93), (96), Lemma 25 and Conditions 1 and 3, we know for some positive constants c_1, c_2 and c_0 ,

$$\begin{aligned} P(\|\boldsymbol{\eta}\|^2 > c_1 \{n \times n^{1-t}\}^{-1} p n^{1+\tau} m / \log n) &\leq O\{m \exp(-c_2 n / \log n)\} \\ &= O\{\exp(-c_0 n / \log n)\}. \end{aligned} \quad (97)$$

Step 2.2 (bounding $|\eta_i|$ from above) From Step 2.1, we know

$$\eta_i^2 \leq \{n \lambda_{\min}(Y^\top P^\top P Y)\}^{-1} \sigma_{x,i}^{-2} \sum_{j=1}^m (\boldsymbol{\epsilon}_j^\top P^\top P \mathbf{x}^i)^2. \quad (98)$$

Then conditioning on X , $\sigma_{x,i}^{-1} \boldsymbol{\epsilon}_j^\top P^\top P \mathbf{x}^i \sim \mathcal{N}(0, \sigma_{\epsilon,j}^2 (\mathbf{x}^i)^\top P^\top P \mathbf{x}^i \sigma_{x,i}^{-2})$. Let \mathcal{E}_1 be the event $\{\text{var}(\sigma_{x,i}^{-1} \boldsymbol{\epsilon}_j^\top P^\top P \mathbf{x}^i | X) \leq c_1 n\}$ for some constant $c_1 > 0$. Note that

$$\text{var}(\boldsymbol{\epsilon}_{\epsilon,j}^\top P^\top P \mathbf{x}^i \sigma_{x,i}^{-1} | X) = \sigma_{\epsilon,j}^2 \{(\mathbf{x}^i)^\top \mathbf{x}^i - (\mathbf{x}^i)^\top \mathbf{1}_n \mathbf{1}_n^\top \mathbf{x}^i / n\} \sigma_{x,i}^{-2} \leq \sigma_{\epsilon,j}^2 (\mathbf{x}^i)^\top \mathbf{x}^i \sigma_{x,i}^{-2}.$$

Using the same argument as in Step 1.1, we can show that, there exist some positive constants c_1 and c_0 ,

$$P(\mathcal{E}_1^c) \leq P\{\sigma_{\epsilon,j}^2 (\mathbf{x}^i)^\top \mathbf{x}^i \sigma_{x,i}^{-2} > c_1 n\} \leq O\{\exp(-c_0 n)\}, \quad (99)$$

where \mathcal{E}_1^c represents the complement of the event \mathcal{E}_1 . On the event \mathcal{E}_1 , for any $a > 0$, by Condition 1, we have

$$P(|\boldsymbol{\epsilon}_j^\top P^\top P \mathbf{x}^i| \sigma_{x,i}^{-1} > a | X, \mathcal{E}_1) \leq P\{\sqrt{c_1 n} |W| > a\}, \quad (100)$$

where W is an independent $\mathcal{N}(0, 1)$ -distributed random variable. Thus, combining (99) and (100), we have

$$P(|\boldsymbol{\epsilon}_j^\top P^\top P \mathbf{x}^i| \sigma_{x,i}^{-1} > a) \leq O\{\exp(-c_0 n)\} + P\{\sqrt{c_1 n} |W| > a\}. \quad (101)$$

Let $x'_n = \sqrt{2c_0 c_1} n^{1-\kappa-t/2-s/2-u} / \sqrt{\log n}$. Invoking the classical Gaussian tail bound, we have

$$P\{\sqrt{c_1 n} |W| > x'_n\} = O\{\exp(-c_0 n^{1-2\kappa-t-s-2u} / \log n)\}.$$

By (98) and Lemma 25, we then have

$$\begin{aligned} P(|\eta_i| > (n^{1+1-t})^{-1/2} x'_n \sqrt{m}) &\leq \sum_{j=1}^m P(|\boldsymbol{\epsilon}_j^\top P^\top P \mathbf{x}^i| > x'_n) \\ &\quad + P\{\lambda_{\min}(Y^\top P^\top P Y) < c_1 n^{1-t}\}, \end{aligned}$$

where $(n^{1+1-t})^{-1/2} x'_n \sqrt{m} = \sqrt{2c_0 c_1 m} n^{-\kappa-s/2-u} / \sqrt{\log n} \leq c_2 n^{-\kappa-u} / \sqrt{\log n}$ for some constant $c_2 > 0$ by Condition 3. In summary, we have

$$\begin{aligned} &P[\max_{1 \leq i \leq p} |\eta_i| > c_2 n^{-\kappa-u} / \sqrt{\log n}] \\ &\leq O[p \exp\{-c_2 n^{1-2\kappa-t-s-2u} / \log n\}] \\ &= O[\exp\{-c_0 n^{1-2\kappa-t-s-2u} / \log n\}], \end{aligned} \quad (102)$$

where the last equality is from Condition 3.

Step 3. We combine the results in Steps 1 and 2. By Bonferroni's inequality, it follows from (78), (91), (97) and (102) that, for some positive constants \tilde{c}_1, \tilde{c}_2 and \tilde{c} ,

$$\begin{aligned} & P\left\{\min_{i \in \mathcal{M}_*} |\zeta_i| < \tilde{c}_1(n \times n^{2u+1})^{-1/2} n^{1-\kappa} \text{ or } \|\boldsymbol{\zeta}\|^2 > \tilde{c}_2(n \times n^{1-t})^{-1} n^{1+\tau} pm\right\} \\ & \leq O[|\mathcal{M}_*| \exp\{-\tilde{c} n^{1-2\kappa-2u-t-s} / \log n\}]. \end{aligned} \quad (103)$$

By Lemma 24 and (103), we know that there exist some positive constants c_1, c_2 and c ,

$$\begin{aligned} & P\left\{\min_{i \in \mathcal{M}_*} |\omega_i| < c_1(n \times n^{2u+1})^{-1/2} n^{1-\kappa} \text{ or } \|\boldsymbol{\omega}\|^2 > c_2(n \times n^{-t})^{-1} n^{1+\tau} pm\right\} \\ & \leq O[|\mathcal{M}_*| \exp\{-c n^{1-2\kappa-2u-t-s} / \log n\}], \end{aligned}$$

which is $O[\exp\{-c_0 n^{1-2\kappa-2u-t-s} / \log n\}]$ for some constant $c_0 > 0$ by Condition 3 and $|\mathcal{M}_*| \leq p$. This shows that with overwhelming probability $1 - O[\exp\{-c_0 n^{1-2\kappa-2u-t-s} / \log n\}]$, the magnitudes of $\omega_i, i \in \mathcal{M}_*$ are uniformly at least of order $c_1(n \times n^{2u+1})^{-1/2} n^{1-\kappa}$, and for some positive constants c_1 and c_2 ,

$$\#\{1 \leq k \leq p : |\omega_k| \geq \min_{i \in \mathcal{M}_*} |\omega_i|\} \leq c_1 \frac{(n \times n^{2u+1}) \times pm n^{1+\tau}}{n \times n^{1-t} \times (n^{1-\kappa})^2} \leq c_2 p n^{s+2u+2\kappa+\tau+t-1},$$

where the last inequality is from Condition 3. Thus, if the proportion δ of features selected satisfies $\delta n^{1-2\kappa-2u-\tau-t-s} \rightarrow \infty$, then $\delta p \geq c_2 p n^{s+\tau+t+2\kappa+2u-1}$ when δ is sufficiently large, and we know with probability $1 - O[\exp\{-c_0 n^{1-2\kappa-2u-\tau-t-s} / \log n\}]$, $\mathcal{M}_* \subseteq \mathcal{M}_\delta$.

E.2 Lemmas in the proof of Theorem 5

Lemma 19 (Lemma 3 in Fan and Lv (2008)) *Let $\vartheta_i, i = 1, 2, \dots, n$ be i.i.d. χ_1^2 -distributed random variables. Then for any $\epsilon > 0$, we have $P(n^{-1} \sum_{i=1}^n \vartheta_i > 1 + \epsilon) \leq \exp(-A_\epsilon n)$, where $A_\epsilon = [\epsilon - \log(1 + \epsilon)]/2 > 0$; for any $\epsilon \in (0, 1)$, $P(n^{-1} \sum_{i=1}^n \vartheta_i < 1 - \epsilon) \leq \exp(-B_\epsilon n)$, where*

$$B_\epsilon = [-\epsilon - \log(1 - \epsilon)]/2 > 0.$$

Lemma 20 (Lemma 1 in Fan and Lv (2008)) *For U and $(\mu_1, \dots, \mu_n)^\top$ in (64), and \tilde{O} uniformly distributed on the orthogonal group $\mathcal{O}(p)$, we know that*

$$(I_n, \mathbf{0})_{n \times p} U \stackrel{(d)}{=} (I_n, \mathbf{0})_{n \times p} \tilde{O}, \quad (104)$$

and $(\mu_1, \dots, \mu_n)^\top$ is independent of $(I_n, \mathbf{0})_{n \times p} U$.

Proof. As $\mu_1^{1/2}, \dots, \mu_n^{1/2}$ are n singular values of Z , we know that Z has the singular value decomposition $Z = V_1 D_1 U$, where $V_1 \in \mathcal{O}(n)$, $U \in \mathcal{O}(p)$ is given in (64), and D_1 is an $n \times p$ diagonal matrix whose diagonal elements are $\mu_1^{1/2}, \dots, \mu_n^{1/2}$. Since the entries in Z are i.i.d. $\mathcal{N}(0, 1)$, for any $\tilde{O} \in \mathcal{O}(p)$, $Z\tilde{O} \stackrel{(d)}{=} Z$. Thus, conditional on V_1 and $(\mu_1, \dots, \mu_n)^\top$, the conditional distribution of $(I_n, \mathbf{0})_{n \times p} U$ is invariant under $\mathcal{O}(p)$. This shows that (104) holds for \tilde{O} uniformly distributed on the orthogonal group $\mathcal{O}(p)$, and $(\mu_1, \dots, \mu_n)^\top$ is independent of $(I_n, \mathbf{0})_{n \times p} U$. ■

Lemma 21 (Lemma 4 in Fan and Lv (2008)) *S defined in (65) is uniformly distributed on the Grassmann manifold $\mathcal{G}_{p,n}$. For any constant $c_0 > 0$, there are constants c_1 and c_2 with $0 < c_1 < 1 < c_2$ such that*

$$P(\langle Se_1, e_1 \rangle < c_1 n/p \text{ or } > c_2 n/p) \leq 4 \exp(-c_0 n).$$

Lemma 22 *The matrix Z is of size $n \times p$ and the matrix \tilde{Z} is of size $n \times m$ with Condition 3 satisfied. The entries in Z and \tilde{Z} are i.i.d. $\mathcal{N}(0, 1)$. For some constants $c_1, c_0 > 0$,*

$$P\{\lambda_{\max}(n^{-1} Z Z^\top) > c_1 \text{ or } \lambda_{\min}(n^{-1} Z^\top Z) < 1/c_1\} \leq \exp(-c_0 n). \quad (105)$$

There exist some constants $c_1 > 1$, $c > 0$ and $c_0 > 0$, when $n > c$,

$$P[\lambda_{\max}\{n^{-1}(P\tilde{Z})^\top P\tilde{Z}\} > c_1 \text{ or } \lambda_{\min}\{n^{-1}(P\tilde{Z})^\top P\tilde{Z}\} < 1/c_1] \leq \exp(-c_0 n / \log n). \quad (106)$$

Proof. As the entries in Z are i.i.d. $\mathcal{N}(0, 1)$, by Appendix A.7 in Fan and Lv (2008), we know that (105) holds. For \tilde{Z} , since its entries are also i.i.d. $\mathcal{N}(0, 1)$ and $n > c_7 m$ for some $c_7 > 1$, symmetrically, we know there exist constants $\tilde{c}_1 > 1$ and $\tilde{c}_0 > 0$ such that

$$P\{\lambda_{\max}(n^{-1}\tilde{Z}^\top \tilde{Z}) > \tilde{c}_1 \text{ or } \lambda_{\min}(n^{-1}\tilde{Z}^\top \tilde{Z}) < 1/\tilde{c}_1\} \leq \exp(-\tilde{c}_0 n). \quad (107)$$

Since $(P\tilde{Z})^\top P\tilde{Z} = \tilde{Z}^\top P\tilde{Z} = \tilde{Z}^\top \tilde{Z} - \tilde{Z}^\top \mathbf{1}_n \mathbf{1}_n^\top \tilde{Z}/n$, by Weyl's inequality, we have

$$\begin{aligned} \lambda_{\max}\{(P\tilde{Z})^\top P\tilde{Z}\} &\leq \lambda_{\max}(\tilde{Z}^\top \tilde{Z}) + \lambda_{\max}(-\tilde{Z}^\top \mathbf{1}_n \mathbf{1}_n^\top \tilde{Z}/n), \\ \lambda_{\min}\{(P\tilde{Z})^\top P\tilde{Z}\} &\geq \lambda_{\min}(\tilde{Z}^\top \tilde{Z}) + \lambda_{\min}(-\tilde{Z}^\top \mathbf{1}_n \mathbf{1}_n^\top \tilde{Z}/n). \end{aligned} \quad (108)$$

Let $A_Z = \tilde{Z}^\top \mathbf{1}_n \mathbf{1}_n^\top \tilde{Z}/n$. As $\text{rank}(A_Z) = 1$ and $\text{tr}(A_Z) \geq 0$, we know $\lambda_{\max}(-A_Z) = -\lambda_{\min}(A_Z) = 0$ and $\lambda_{\min}(-A_Z) = -\lambda_{\max}(A_Z) = -\text{tr}(A_Z) = -\mathbf{1}_n^\top \tilde{Z} \tilde{Z}^\top \mathbf{1}_n/n$.

We then examine $\mathbf{1}_n^\top \tilde{Z} \tilde{Z}^\top \mathbf{1}_n/n$. For $z_{ij} \sim \mathcal{N}(0, 1)$ independently,

$$\mathbf{1}_n^\top \tilde{Z} \tilde{Z}^\top \mathbf{1}_n/n = \sum_{j=1}^m \left(\sum_{i=1}^n z_{ij}/\sqrt{n} \right)^2 \sim \chi_m^2.$$

By Lemma 19, we know for the random variable $\tilde{W} \sim \chi_m^2$ and any constant $c_2 > 0$, there exists constant $c_3 > 0$ such that

$$P\{|\tilde{W}/m - 1| > c_2 n/(m \log n)\} \leq \exp\{-c_3 m \times n/(m \log n)\}, \quad (109)$$

This implies that with probability $1 - O\{\exp(-c_3 n / \log n)\}$, $\lambda_{\max}(A_Z/n) = \mathbf{1}_n^\top \tilde{Z} \tilde{Z}^\top \mathbf{1}_n/n^2 \leq$

$c_2/\log n$ for some constant $c_2 > 0$, as $m = O(n^s)$ with $s \in [0, 1)$.

When n is sufficiently large, there exists constant c_1 such that $1 > c_1 > \tilde{c}_1$ and $1/c_1 + c_2/\log n < 1/\tilde{c}_1$. Thus by (108) and (109), we know there exists constant $c_0 > 0$ such that with probability $1 - \exp(-c_0 n/\log n)$,

$$\begin{aligned} \{\lambda_{\min}(n^{-1}(P\tilde{Z})^\top P\tilde{Z}) < 1/c_1\} &\subseteq \{\lambda_{\min}(n^{-1}\tilde{Z}^\top \tilde{Z}) < 1/c_1 + c_2/\log n\} \\ &\subseteq \{\lambda_{\min}(n^{-1}\tilde{Z}^\top \tilde{Z}) < 1/\tilde{c}_1\}, \\ \{\lambda_{\max}(n^{-1}(P\tilde{Z})^\top P\tilde{Z}) > c_1\} &\subseteq \{\lambda_{\max}(n^{-1}\tilde{Z}^\top \tilde{Z}) > \tilde{c}_1\}. \end{aligned}$$

By (107), (106) is then proved.

■

Lemma 23 *For $\mathbf{R} = (R_1, \dots, R_p)^\top = \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} e_1$, there exist positive constants c_1 and c_0 such that $P(|R_2| > c_1 n^{1/2} |W_1|) \leq O\{\exp(-c_0 n)\}$, where $|W_1|$ is an independent $\mathcal{N}(0, 1)$ -distributed random variable.*

Proof. Let $\tilde{\mathbf{R}} = (R_2, \dots, R_p)^\top$. We first show that $\tilde{\mathbf{R}}$ is invariant under the orthogonal group $\mathcal{O}(p-1)$. For any $Q \in \mathcal{O}(p-1)$, let $\tilde{Q} = \text{diag}(1, Q) \in \mathcal{O}(p)$. By Lemma 20, we know that \tilde{U} is independent of $\text{diag}(\mu_1, \dots, \mu_n)$ and $\tilde{Q}\tilde{U} \stackrel{(d)}{=} \tilde{U}$. Thus

$$\tilde{Q}^\top \mathbf{R} = \tilde{Q}^\top \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} \tilde{Q} \tilde{Q}^\top e_1 \stackrel{(d)}{=} \tilde{U}^\top \text{diag}(\mu_1, \dots, \mu_n) \tilde{U} e_1,$$

where we use the fact that $\tilde{Q}^\top e_1 = e_1$. This implies that $\tilde{\mathbf{R}}$ is invariant under the orthogonal group $\mathcal{O}(p-1)$. It follows that $\tilde{\mathbf{R}} \stackrel{(d)}{=} \|\tilde{\mathbf{R}}\| \mathbf{W} / \|\mathbf{W}\|_2$, where $\mathbf{W} = (W_1, \dots, W_{p-1})^\top \sim \mathcal{N}(0, I_{p-1})$, independent of $\|\tilde{\mathbf{R}}\|$.

In particular, we have $R_2 \stackrel{(d)}{=} \|\tilde{\mathbf{R}}\| W_1 / \|\mathbf{W}\|_2$. Note that $\|\tilde{\mathbf{R}}\| \leq \|\mathbf{R}\|$ and $\|\mathbf{R}\|^2 =$

$e_1^\top \tilde{U}^\top \text{diag}(\mu_1^2, \dots, \mu_n^2) \tilde{U} e_1$. Since $\mu_1, \dots, \mu_n \leq p \lambda_{\max}(p^{-1} Z Z^\top)$,

$$\|\mathbf{R}\|^2 \leq \{\lambda_{\max}(p^{-1} Z Z^\top)\}^2 p^2 e_1^\top \tilde{U}^\top \tilde{U} e_1 = \{\lambda_{\max}(p^{-1} Z Z^\top)\}^2 p^2 \langle S e_1, e_1 \rangle.$$

By Lemmas 20 and 22, we then know for some positive constants c_1 and c_0 , $P(\|\mathbf{R}\| > c_1 \sqrt{pn}) \leq O\{\exp(-c_0 n)\}$. Moreover, by Lemma 19, we know for some constants c_1 and c_0 , $P\{\|\mathbf{W}\|^2 < c_1(p-1)\} \leq \exp(-c_0 n)$ when $p > n$. Thus we obtain that for some constants c_1 and c_0 , $P(|R_2| > c_1 |W_1| n^{1/2}) \leq O\{\exp(-c_0 n)\}$, where W_1 is an independent $\mathcal{N}(0, 1)$ -distributed random variable. ■

Lemma 24 *For some constant $c_0 > 0$, $\omega_i = \zeta_i \{1 + o(1)\}$ with probability $1 - O\{\exp(-c_0 n / \log n)\}$.*

Proof. By the definitions of ω_i and ζ_i and $P^\top P = P$, we know

$$\begin{aligned} \omega_i - \zeta_i &\leq \max_{\mathbf{a}} \frac{\mathbf{a}^\top Y^\top P \mathbf{x}^i (\sqrt{n} \sigma_{x,i} - \sqrt{(\mathbf{x}^i)^\top P \mathbf{x}^i})}{\sigma_{x,i} \sqrt{n((\mathbf{x}^i)^\top P \mathbf{x}^i)(\mathbf{a}^\top Y^\top P Y \mathbf{a})}} = \zeta_i \frac{\sqrt{n} \sigma_{x,i} - \sqrt{(\mathbf{x}^i)^\top P \mathbf{x}^i}}{\sqrt{(\mathbf{x}^i)^\top P \mathbf{x}^i}}, \\ \zeta_i - \omega_i &\leq \max_{\mathbf{a}} \frac{\mathbf{a}^\top Y^\top P^\top \mathbf{x}^i (\sqrt{(\mathbf{x}^i)^\top P \mathbf{x}^i} - \sqrt{n} \sigma_{x,i})}{\sigma_{x,i} \sqrt{n((\mathbf{x}^i)^\top P \mathbf{x}^i)(\mathbf{a}^\top Y^\top P Y \mathbf{a})}} = \zeta_i \frac{\sqrt{(\mathbf{x}^i)^\top P \mathbf{x}^i} - \sqrt{n} \sigma_{x,i}}{\sqrt{(\mathbf{x}^i)^\top P \mathbf{x}^i}}. \end{aligned}$$

Thus

$$|\omega_i - \zeta_i| \leq \zeta_i \left| \frac{\sqrt{(\mathbf{x}^i)^\top P \mathbf{x}^i} - \sqrt{n} \sigma_{x,i}}{\sqrt{(\mathbf{x}^i)^\top P \mathbf{x}^i}} \right| = \zeta_i \left| 1 - \sigma_{x,i} \sqrt{n / \{(\mathbf{x}^i)^\top P \mathbf{x}^i\}} \right|.$$

Let $\bar{x}^i = \sum_{k=1}^n x_{k,i} / n$, which is the mean of the entries in \mathbf{x}^i . It follows that $(\mathbf{x}^i)^\top P \mathbf{x}^i = \sum_{k=1}^n (x_{k,i} - \bar{x}^i)^2 = \sum_{k=1}^n x_{k,i}^2 - n(\bar{x}^i)^2$. Then with $\tilde{c}_1 = c_1/2$

$$\begin{aligned} &P(|(\mathbf{x}^i)^\top P \mathbf{x}^i / (n \sigma_{x,i}^2) - 1| > c_1 / \log n) \\ &\leq P\{(\bar{x}^i)^2 / \sigma_{x,i}^2 > \tilde{c}_1 / \log n\} + P\left\{\left| \sum_{k=1}^n x_{k,i}^2 / (n \sigma_{x,i}^2) - 1 \right| > \tilde{c}_1 / \log n\right\}. \end{aligned} \tag{110}$$

By Condition 1, $(\sqrt{n} \bar{x}^i)^2 / \sigma_{x,i}^2 \sim \chi_1^2$. Then by the tail of χ_1^2 distribution, for some constant

$c_0 > 0$, $P\{(\bar{x}^i)^2/\sigma_{x,i}^2 > \tilde{c}_1/\log n\} \leq O\{\exp(-c_0 n/\log n)\}$. In addition, $x_{k,i}^2/\sigma_{x,i}^2$, $k = 1, \dots, n$ are i.i.d. χ_1^2 -distributed random variables. By Lemma 19, there exists some constant $c_0 > 0$,

$$P\left\{\left|\sum_{k=1}^n x_{k,i}^2/(n\sigma_{x,i}^2) - 1\right| > \tilde{c}_1/\log n\right\} \leq O\{\exp(c_0 n/\log n)\}.$$

In summary, we know for any constant $c_1 > 0$, there exists constant $c_0 > 0$ such that (110) $\leq O\{\exp(-c_0 n/\log n)\}$. Thus, for $i = 1, \dots, p$, $\omega_i = \zeta_i\{1 + O(1/\sqrt{\log n})\} = \zeta_i(1 + o(1))$ with probability $1 - O\{p \exp(-c_0 n/\log n)\} = 1 - O\{\exp(-c_0 n/\log n)\}$, where the last equality is from Condition 3. ■

Lemma 25 *Consider $n > c$ for c in Lemma 22. There exist constants c_1, c_2 and c_0 , with probability $1 - O\{\exp(-c_0 n/\log n)\}$,*

$$\lambda_{\min}(Y^\top P^\top P Y) \geq c_1 n^{1-t},$$

and for $\mathbf{a}_{0,i}$ in Condition 2,

$$\mathbf{a}_{0,i}^\top Y^\top P^\top P Y \mathbf{a}_{0,i} \leq c_2 n^{2u+1}. \quad (111)$$

Proof. Since X and E follow independent Gaussian distributions by Condition 1, the rows of Y are independent multivariate Gaussian with mean zero and covariance $\Sigma_y = B^\top \Sigma_x B + \Sigma$. Define $\tilde{Z} = Y \Sigma_y^{-1/2}$. Then \tilde{Z} is of size $n \times m$ and the entries in \tilde{Z} are i.i.d. $\mathcal{N}(0, 1)$. Thus the concentration inequality (106) in Lemma 22 holds. It follows that there exist constants

c_1 and c_0 , with probability $1 - O\{\exp(-c_0 n)\}$,

$$\begin{aligned}
Y^\top P^\top P Y &= \Sigma_y^{1/2} \tilde{Z}^\top P^\top P \tilde{Z} \Sigma_y^{1/2} \\
&\succeq n \Sigma_y^{1/2} \lambda_{\min}(n^{-1} \tilde{Z}^\top P^\top P \tilde{Z}) I_m \Sigma_y^{1/2} \\
&\succeq c_1 n \Sigma_y.
\end{aligned}$$

By Weyl's inequality and Condition 1, we then know

$$\lambda_{\min}(Y^\top P^\top P Y) \geq c_1 n \lambda_{\min}(\Sigma_y) \geq c_1 n \lambda_{\min}(\Sigma) \geq c_1 n^{1-t}. \quad (112)$$

Similarly we know for some constant c_2 , with probability $1 - O\{\exp(-c_0 n)\}$,

$$\begin{aligned}
\mathbf{a}_{0,i}^\top Y^\top P^\top P Y \mathbf{a}_{0,i} &= \mathbf{a}_{0,i}^\top \Sigma_y^{1/2} \tilde{Z}^\top P^\top P \tilde{Z} \Sigma_y^{1/2} \mathbf{a}_{0,i} \\
&\leq c_2 n \mathbf{a}_{0,i}^\top \Sigma_y \mathbf{a}_{0,i} \\
&= c_2 n \mathbf{a}_{0,i}^\top (B^\top \Sigma_x B + \Sigma) \mathbf{a}_{0,i} \\
&\leq c_2 n^{2u+1},
\end{aligned}$$

where the last inequality is from Condition 2. ■

F Proof of Proposition 6 (Meinshausen et al., 2009, Theorem 3.2)

The proof of Proposition 6 directly follows the proof in Meinshausen et al. (2009). For $z \in (0, 1)$, define

$$\psi(z) = \frac{1}{J} \sum_{j=1}^J 1\{p^{(j)} \leq z\}. \quad (113)$$

Note that $\{Q(\gamma) \leq \alpha\}$ and $\{\psi(\alpha\gamma) \geq \gamma\}$ are equivalent. For a random variable U taking values in $[0, 1]$,

$$\sup_{\gamma \in (\gamma_{\min}, 1)} \frac{1\{U \leq \alpha\gamma\}}{\gamma} = \begin{cases} 0 & U \geq \alpha \\ \alpha/U & \alpha\gamma_{\min} \leq U < \alpha \\ 1/\gamma_{\min} & U < \alpha\gamma_{\min}. \end{cases}$$

Thus when U has a uniform distribution on $[0, 1]$,

$$\mathbb{E} \left[\sup_{\gamma \in (\gamma_{\min}, 1)} \frac{1\{U \leq \alpha\gamma\}}{\gamma} \right] = \int_0^{\alpha\gamma_{\min}} \gamma_{\min}^{-1} dx + \int_{\alpha\gamma_{\min}}^{\alpha} \alpha x^{-1} dx = \alpha(1 - \log \gamma_{\min}).$$

Hence, define the event $\mathcal{B}_{(j)}$ as $\mathcal{M}_* \subseteq \mathcal{M}_\delta$ for the j th split, then

$$\begin{aligned} \mathbb{E} \left[\sup_{\gamma \in (\gamma_{\min}, 1)} 1\{p^{(j)} \leq \alpha\gamma\}/\gamma \right] &\leq \mathbb{E} \left\{ \mathbb{E} \left[\sup_{\gamma \in (\gamma_{\min}, 1)} 1\{p^{(j)} \leq \alpha\gamma\}/\gamma \middle| \mathcal{B}_{(j)} \right] \right\} + \frac{1}{\gamma_{\min}} P\{\mathcal{B}_{(j)}\} \\ &\leq \alpha(1 - \log \gamma_{\min}) + O[\exp\{-c_0 n^{1-2\kappa-2u-t-s}/\log n\}], \end{aligned}$$

where the constants c_0, κ, u, t and s are given in Theorem 5. Averaging over J splits yields

$$\mathbb{E} \left[\sup_{\gamma \in (\gamma_{\min}, 1)} \frac{1}{J} \sum_{j=1}^J 1\{p^{(j)}/\gamma \leq \alpha\} \right] \leq \alpha(1 - \log \gamma_{\min}) + O[\exp\{-c_0 n^{1-2\kappa-2u-t-s}/\log n\}].$$

From Markov inequality and (113),

$$\mathbb{E}\left[\sup_{\gamma \in (\gamma_{\min}, 1)} 1\{\psi(\alpha\gamma) \geq \gamma\}\right] \leq \alpha(1 - \log \gamma_{\min}) + O[\exp\{-c_0 n^{1-2\kappa-2u-t-s} / \log n\}].$$

Since $\{Q(\gamma) \leq \alpha\}$ and $\{\psi(\alpha\gamma) \geq \gamma\}$ are equivalent, it follows that

$$P\left[\inf_{\gamma \in (\gamma_{\min}, 1)} Q(\gamma) \leq \alpha\right] \leq \alpha(1 - \log \gamma_{\min}) + O[\exp\{-c_0 n^{1-2\kappa-2u-t-s} / \log n\}],$$

which implies that

$$P\left[\inf_{\gamma \in (\gamma_{\min}, 1)} Q(\gamma)(1 - \log \gamma_{\min}) \leq \alpha\right] \leq \alpha + O[\exp\{-c_0 n^{1-2\kappa-2u-t-s} / \log n\}].$$

By definition of p_t , $\limsup_{n \rightarrow \infty} P[p_t \leq \alpha] \leq \alpha$ is obtained.

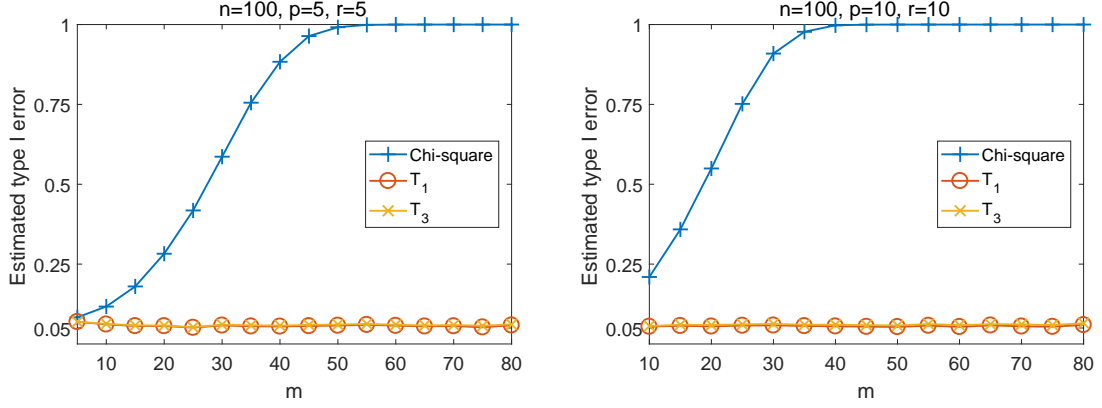
G Supplementary Simulations

G.1 Estimated type I errors

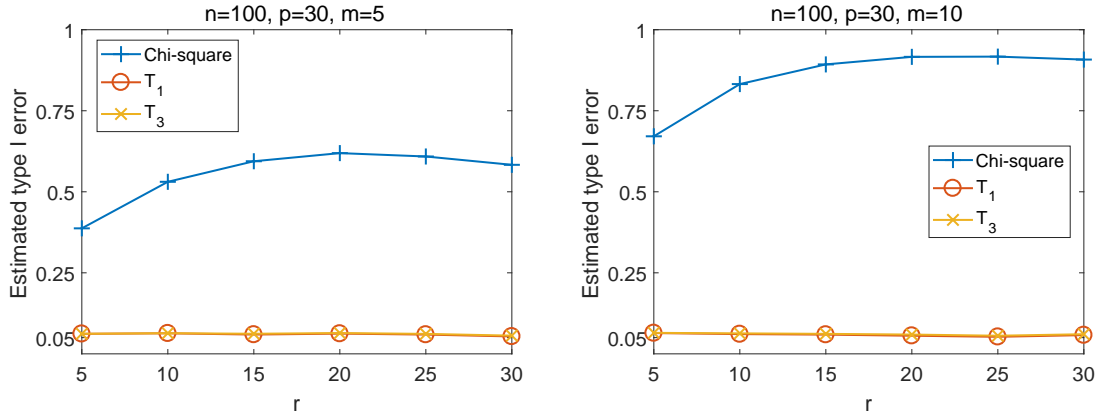
We provide additional simulations under H_0 following the same set-up as in Figure 5. In Figure 10, we present the estimated type I errors of the χ^2 approximation and the normal approximations of T_1 and T_3 with varying m and r respectively. It exhibits similar pattern as in Figure 5, which shows that as p, m, r become larger, the χ^2 approximation performs poorly, while the normal approximations for T_1 and T_3 still control the type I error well.

G.2 Robustness with other distributions

We further conduct some simulations when E does not follow normal distribution, which exhibit similar patterns as in Figure 7 and imply the robustness of the proposed methods.



(a) Estimated type I error versus m

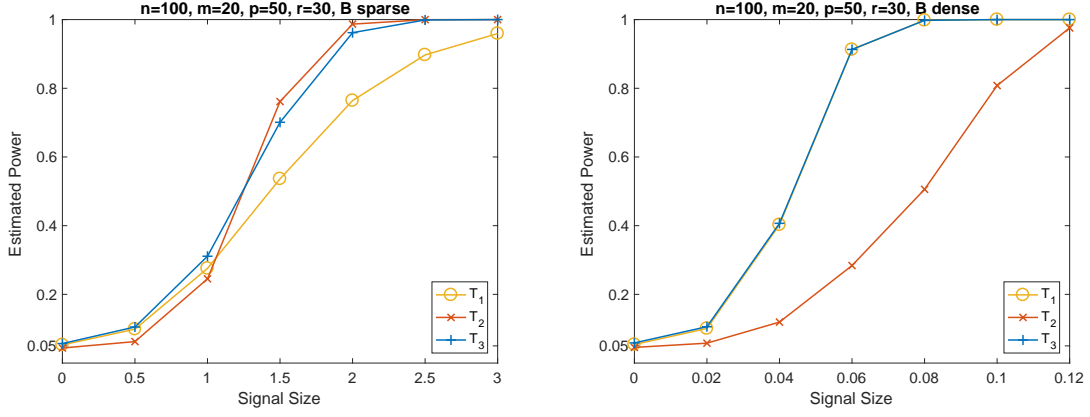


(b) Estimated type I error versus r

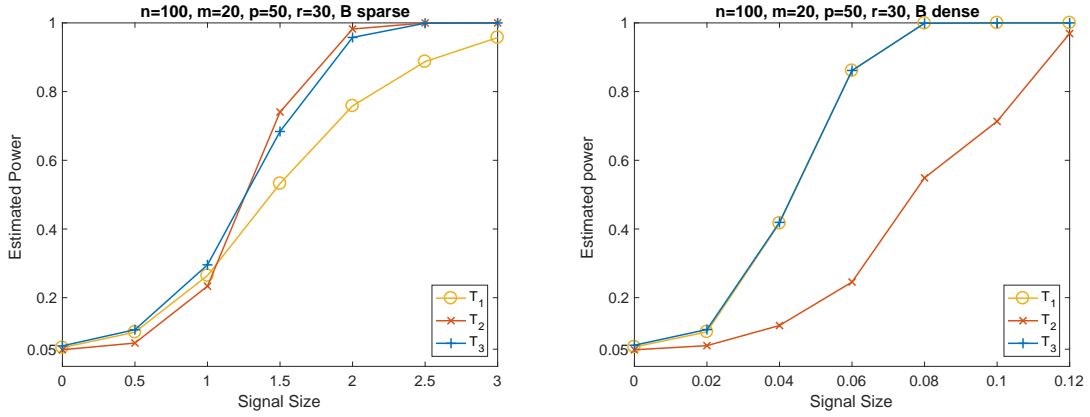
Figure 10: Estimated type I error

G.2.1 X and Y follow multinomial distributions

For $i = 1, \dots, n$ and $j = 1, \dots, p$, we generate the entry $x_{i,j}$ in X independently and identically in the following way. In particular, we first generate $z_{i,j} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$, and set the value



(a) Estimated powers versus signal sizes when $C = [I_r, \mathbf{0}_{r \times (p-r)}]$



(b) Estimated powers versus signal sizes when C is (1)

Figure 11: Estimated powers versus signal sizes

of $x_{i,j}$ as below:

$$x_{i,j} = \begin{cases} -3 & z_{i,j} < -1, \\ -2 & z_{i,j} \in [-1, -0.4), \\ -1 & z_{i,j} \in [-0.4, 0), \\ 1 & z_{i,j} \in [0, 0.4), \\ 2 & z_{i,j} \in [0.4, 1), \\ 3 & z_{i,j} > 1. \end{cases}$$

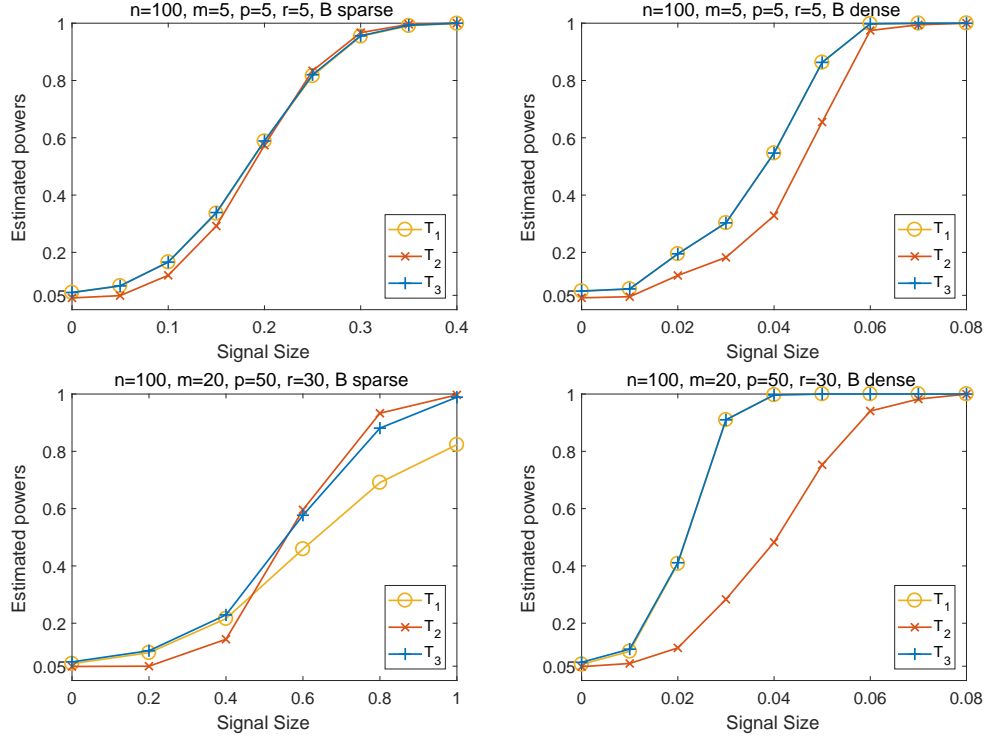
Given B and X , we generate $W = XB + E$, where the entries of E are i.i.d. $\mathcal{N}(0, 1)$. For $i = 1, \dots, n$ and $j = 1, \dots, p$, let $w_{i,j}$ and $y_{i,j}$ denote the entries of W and Y respectively. We then set

$$y_{i,j} = \begin{cases} -3 & w_{i,j} < -1, \\ -2 & w_{i,j} \in [-1, -0.4), \\ -1 & w_{i,j} \in [-0.4, 0), \\ 1 & w_{i,j} \in [0, 0.4), \\ 2 & w_{i,j} \in [0.4, 1), \\ 3 & w_{i,j} > 1. \end{cases}$$

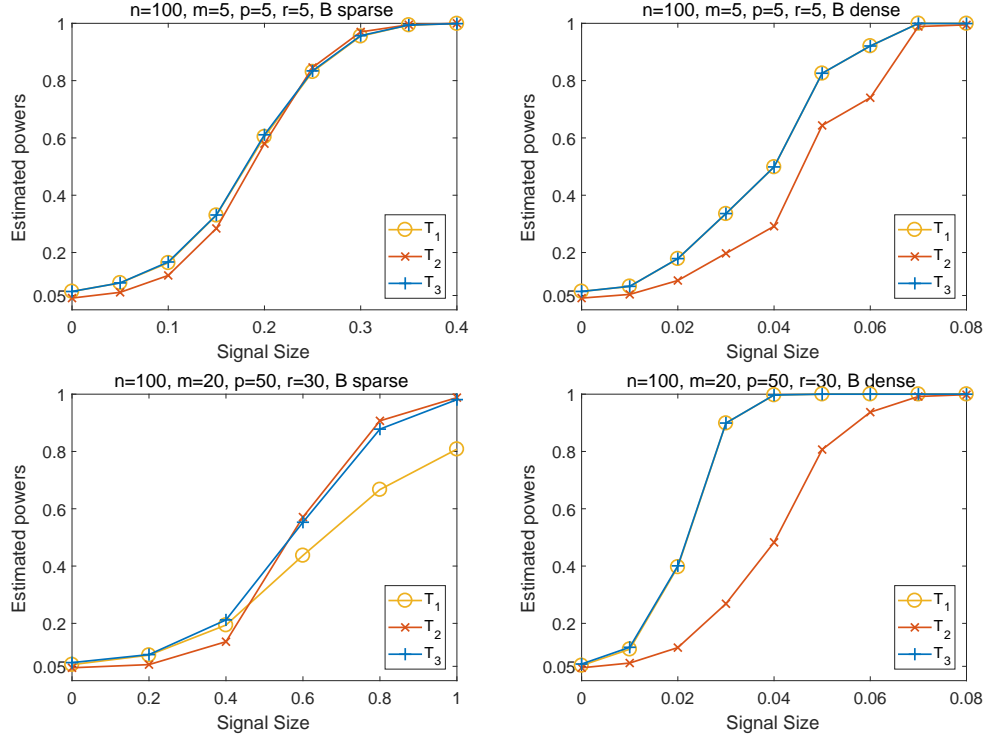
We present the results in Figure 12, where “ B sparse” and “ B dense” represent two different types of B matrix, which are generated following the same method as in Figure 7. Similarly, we take $C = [I_r, \mathbf{0}_{r \times (p-r)}]$ and C of the form (1) in Section 5 respectively. We can observe similar patterns to that in Figure 7.

G.2.2 Errors follow t distribution

In this part, we examine the case when the errors in matrix E independently and identically follows t distribution. In particular, we first generate the entries in X as i.i.d. $\mathcal{N}(0, 1)$. Then we generate the entries in E as i.i.d. t_{df} with $df \in \{3, 5\}$. The results are summarized in Figure 13, where similar patterns are observed as in Figure 7.

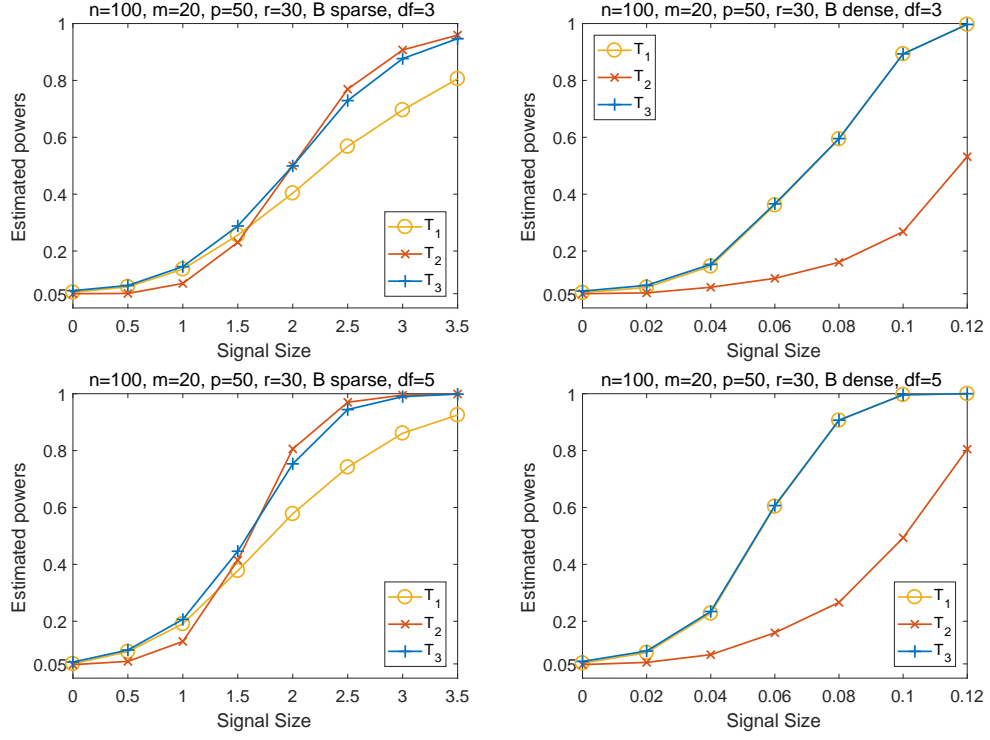


(a) When $C = [I_r, \mathbf{0}_{r \times (p-r)}]$

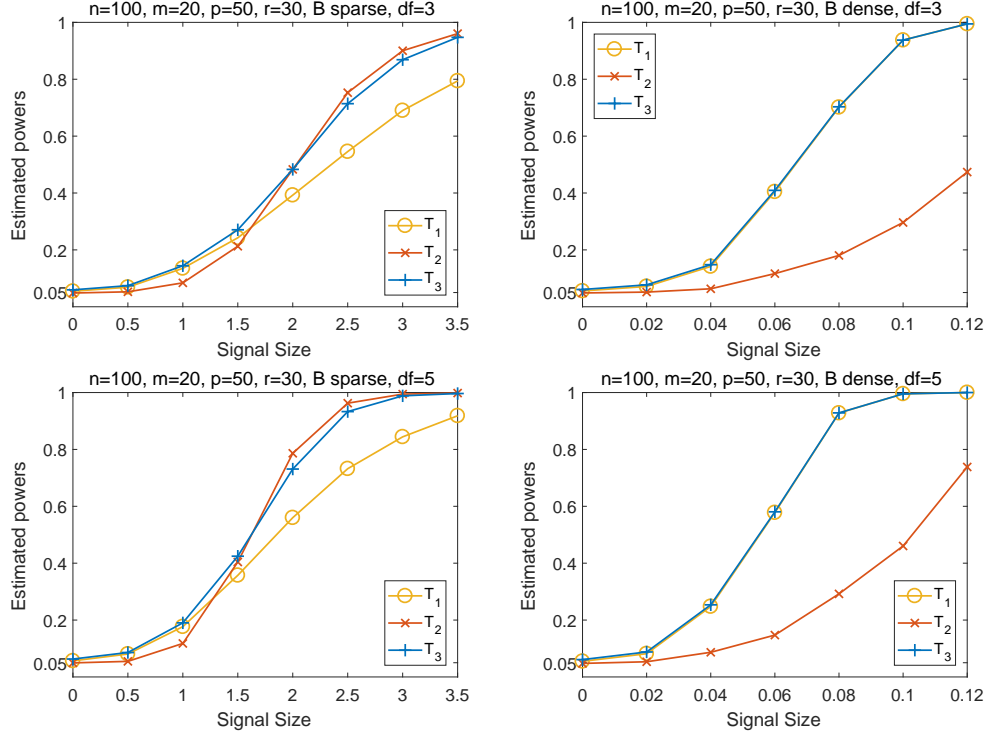


(b) When C is (1)

Figure 12: Power comparison when X and Y follow multinomial distribution



(a) When $C = [I_r, \mathbf{0}_{r \times (p-r)}]$



(b) When C is (1)

Figure 13: Power comparison when entries in E follow t distribution

G.3 Supplementary simulations when $n < p + m$

G.3.1 Supplementary simulations with normal distribution

Under the similar set-up to that of Figure 8, we present additional results with $rk = 5$ in Figure 14, where similar patterns are observed as in Figure 8.

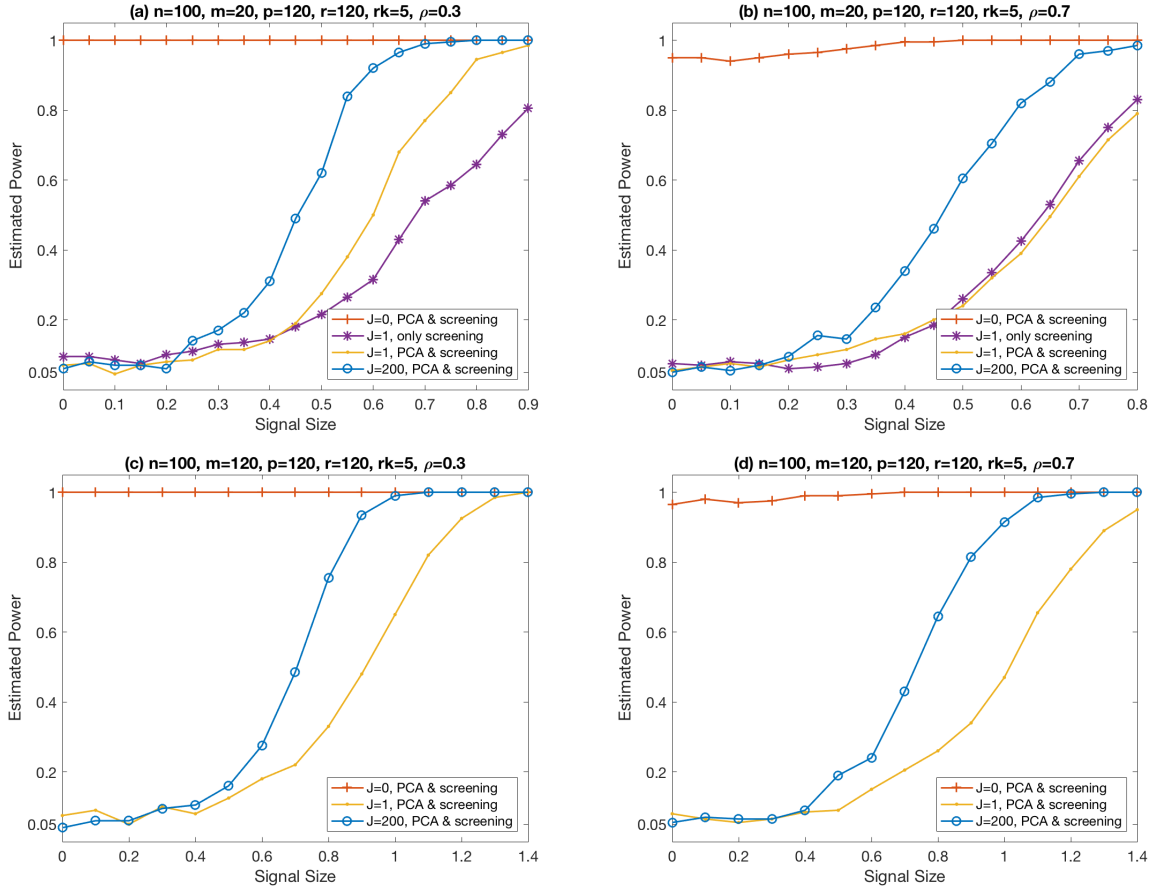


Figure 14: Estimated powers versus signal sizes when $n < m + p$

In addition, under the similar set-up to that of Figure 8, we conduct simulations when $\rho = 0$ and $r_k \in \{1, 5\}$. The results are presented in Figure 15, where similar patterns are observed as in Figure 8.

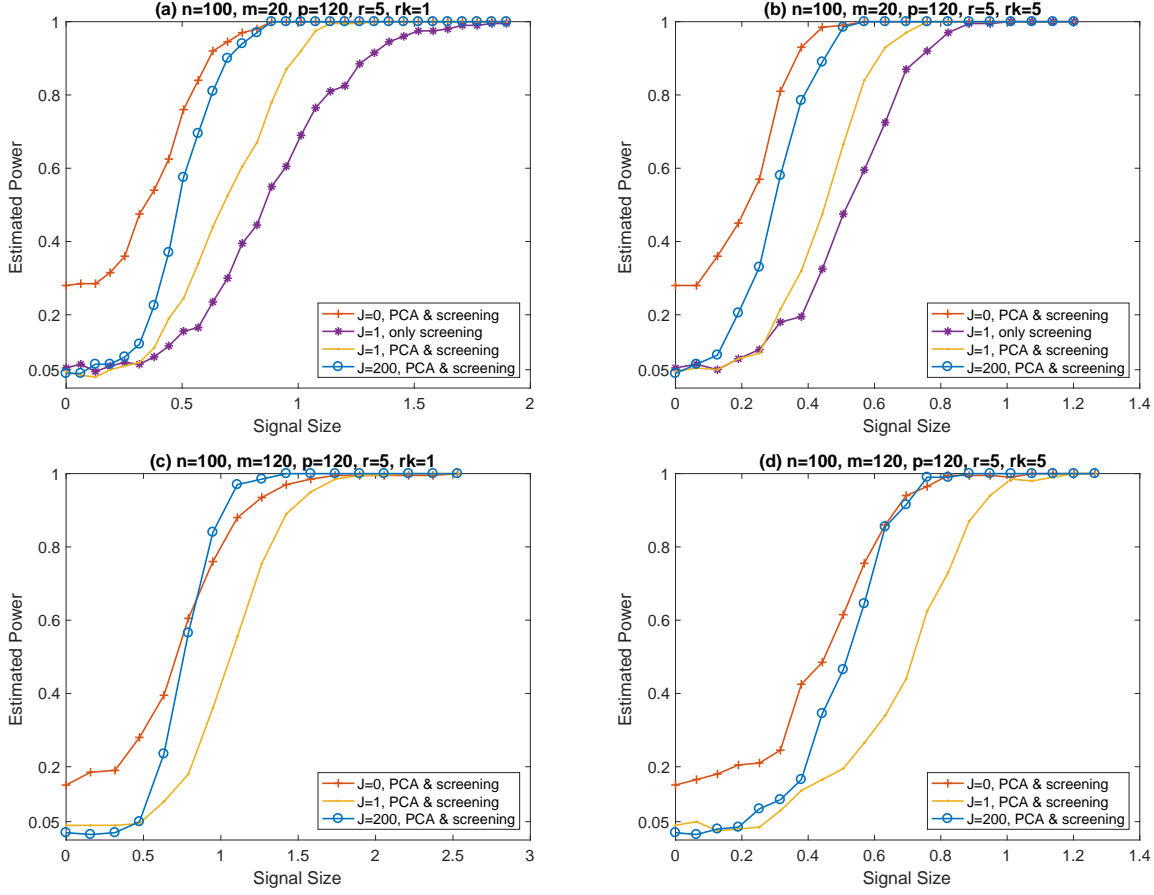
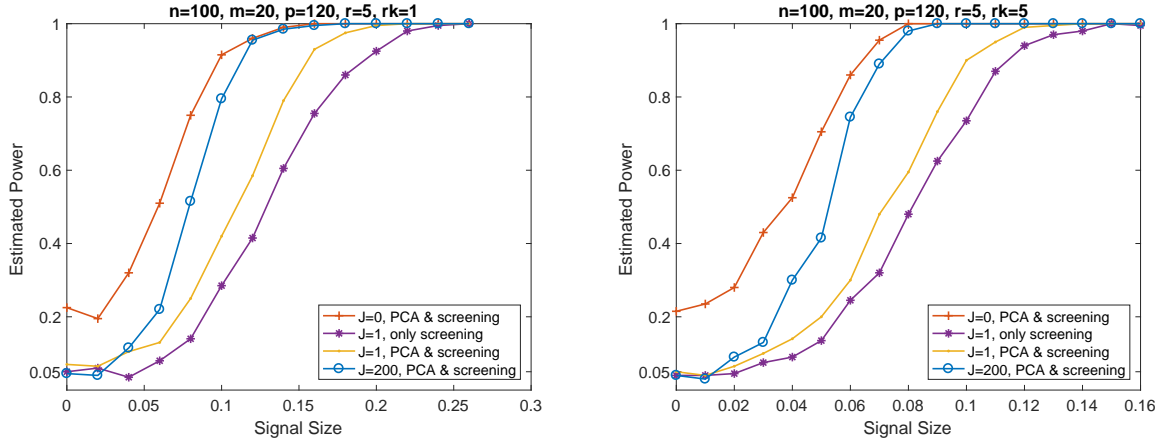


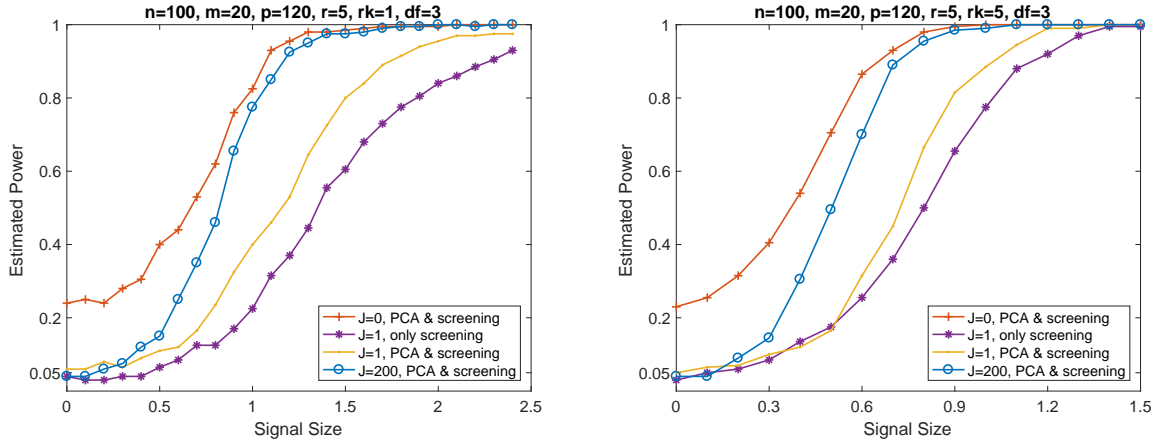
Figure 15: Estimated powers versus signal sizes when $n < m + p$

G.4 Robustness with other distributions

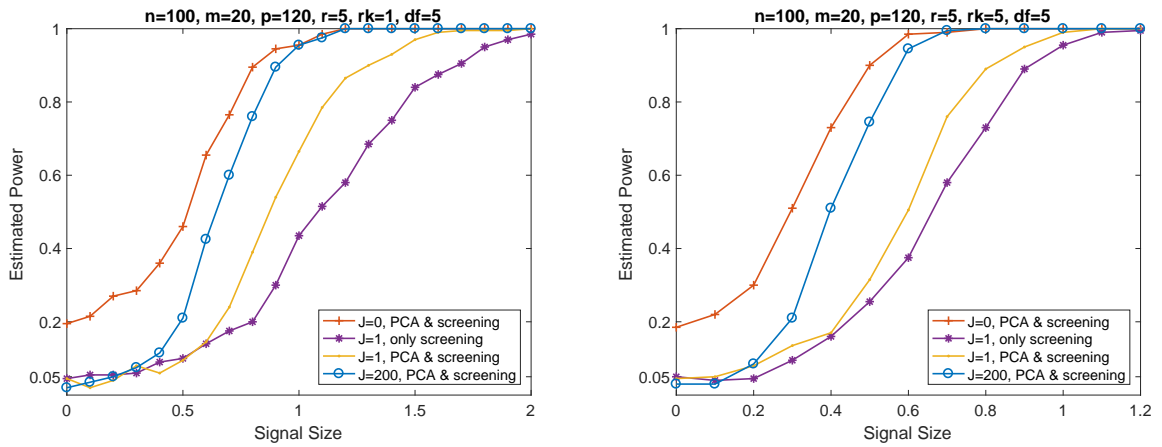
To examine the robustness of the two-step procedure, we generate X and Y following Section G.2 with $n = 100$, $m = 20$, $p = 120$. We then generate B and apply the testing procedure similarly as in Section 5.2 with $r_k \in \{1, 5\}$. The results are presented in Figure 16, where part (a) gives the results when X and Y follow multinomial distribution, and parts (b) and (c) give the results when the error terms in E are i.i.d. t_3 or t_5 . We note that similar patterns are observed as in Figure 8. This shows that the proposed two-step procedure is robust to the normal assumption.



(a) X and Y follow multinomial distributions



(b) Entries in E follow t_3 distribution



(c) Entries in E follow t_5 distribution

Figure 16: Estimated powers of two-step procedure with other distributions

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