

# Weak\* solutions I: A new perspective on solutions to systems of conservation laws

Alexey Miroshnikov\*      Robin Young<sup>†</sup>

## Abstract

We introduce a new notion of solution, which we call weak\* solutions, for systems of conservation laws. These solutions can be used to handle singular situations that standard weak solutions cannot, such as vacuums in Lagrangian gas dynamics or cavities in elasticity. Our framework allows us to treat the systems as ODEs in Banach space. Starting with the observation that solutions act linearly on test functions  $\alpha \in X$ , we require solutions to take values in the dual space  $X^*$  of  $X$ . Moreover, we weaken the usual requirement of measurability of solutions. In order to do this, we develop the calculus of the Gelfand integral, which is appropriate for weak\* measurable functions. We then use the Gelfand calculus to define weak\* solutions, and show that they are stronger than the usual notion of weak solution, although for  $BV$  solutions the notions are equivalent. It is expected that these solutions will also shed light on vexing issues of ill-posedness for multi-dimensional systems.

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## 1 Introduction

Hyperbolic systems of conservation laws are fundamental in the study of fluids and continuum dynamics, expressing basic physical properties such as conservation of mass and momentum. In one dimension, these take the form

$$u_t + F(u)_x = 0, \quad u(x, 0) = u^0(x), \quad (1.1)$$

where  $u \in \mathcal{U} \subset \mathbb{R}^n$ ,  $F : \mathcal{U} \rightarrow \mathbb{R}^n$ , being derived from the integral form

$$\frac{d}{dt} \int_{(a,b)} u \, dx = - \int_{\partial(a,b)} F \cdot n \, ds = -F(u) \Big|_a^b.$$

Here  $u$  is the vector of conserved quantities, and  $F$  is the corresponding flux. Mathematically, these equations present particular challenges because of the ubiquity of shock waves, at which continuum fields such as pressure are discontinuous, so that global classical solutions generally do not exist.

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\*Department of Biostatistics and Epidemiology, University of Massachusetts Amherst, amiroshn@gmail.com

<sup>†</sup>Department of Mathematics and Statistics, University of Massachusetts Amherst, young@math.umass.edu

The study of weak solutions for general systems was initiated by Peter Lax in the 1950's [11], and followed in the 1960's by Glimm's celebrated proof of the global existence of weak solutions, provided the initial data has sufficiently small total variation [9]. A weak solution is a locally bounded, locally integrable function  $u$  satisfying

$$\int_0^T \int_{\mathbb{R}} \left( u(x, t)^\top \partial_t \varphi(x, t) + F(u(x, t))^\top \partial_x \varphi(x, t) \right) dx + \int_{\mathbb{R}} u^0(x)^\top \varphi(0, x) dx = 0, \quad (1.2)$$

for all smooth test functions  $\varphi$  [11, 5]. There is now a mature and complete theory of weak solutions in one space dimension, culminating in Bressan and collaborators' proofs of uniqueness and continuous dependence, both via approximation schemes and via vanishing viscosity [2, 1]. All of these results depend on Glimm's global  $BV$  estimates, which in turn require the initial data to have small amplitude and bounded variation [20].

If the data is large, even for physical systems, the class of weak solutions is not always sufficient to cope with the variety of solutions. In particular, the use of a Lagrangian frame, which is often computationally convenient, presents problems near vacuums, cavities and fractures in the medium, where they are represented by Dirac masses [21, 22]. There have been several varied attempts to extend the class of weak solutions in these and similar contexts [8, 6, 10]. This problem is worse in higher dimensions, for which there are several instances of solutions exhibiting instability and nonuniqueness, independent of entropy considerations [15, 19, 17, 12].

Our initial motivation for this work was to allow Dirac masses, representing vacuums (which are discontinuities in a continuous medium), to appear in solutions of the Euler equations of gas dynamics in a Lagrangian frame. In particular, if the use of Dirac masses can be rigorously justified, their use simplifies the manipulation of solutions in calculations. We would also like to be able to describe solutions in some sense as the solutions of an ODE in a Banach space, thus correctly regarding the conservation law as an evolution equation.

The pairing in (1.2) between the solution  $u(x, t)$  and smooth test functions  $\varphi \in C_c^\infty(\mathbb{R} \times (0, T))$  and the idea of studying the evolution  $t \rightarrow u(t) = u(\cdot, t)$  motivate the following setup:

- $u(t)$  (the value of a solution  $u$ ) must act linearly on spatial test functions  $x \rightarrow \alpha(x)$  that live in a certain Banach space  $X$ . Thus,  $u(t)$  should take values in its dual,  $u(t) \in X^*$ .
- Rather than requiring  $t \rightarrow u(t) \in X^*$  to be Bochner integrable, it is sufficient to require only that  $t \rightarrow \langle u(t), \alpha \rangle$  be integrable for all  $\alpha \in X$ .
- To make sense of the nonlinear flux  $F(U)$  and its derivative in the space  $X^*$  one needs to properly define the flux as the mapping  $F : X^* \rightarrow X^*$  and require that  $C_c^\infty(\Omega)$  for some  $\Omega \subseteq \mathbb{R}$  are dense in  $X$ .

In our present work, regarding  $u = u(t) \in X^*$ , we develop a point of view that enables us to say that  $u(t)$  solves a differential equation in the abstract space

$$u'(t) + \mathbb{D}_x F(u(t)) = 0 \quad \text{in } X^* \quad (1.3)$$

for appropriately interpreted time derivative  $u'$  and flux  $F(u)$ . That is, we view the problem as an evolution without mixing space and time, and our test functions (characterizing the solution spaces) are spatial functions, while for time we use the usual  $C_c^\infty(0, T)$ . We note that weak solutions, defined by (1.2), make no distinction between time and space.

In light of our second observation above, we regard the assumption of strong measurability, in which  $u : [0, T] \rightarrow X^*$  is measurable, as being too strong for our purposes. Instead, we prefer to use the concept of weak\* measurability, so that the scalar function  $\langle u, \alpha \rangle : [0, t] \rightarrow \mathbb{R}$  is measurable for each  $\alpha \in X$ . The appropriate notion of integration in this case is the Gelfand integral, denoted  $\star \int_E u(t) dt$ , with  $E \in \mathfrak{B}([0, T])$  measurable, which is essentially defined by the relations

$$\star \int_E u(t) dt \in X^* \quad \text{and} \quad \left\langle \star \int_E u(t) dt, \alpha \right\rangle = \int_E \langle u(t), \alpha \rangle dt \quad \text{for all } \alpha \in X. \quad (1.4)$$

Although the Gelfand integral goes back to the 1930's, to the authors' knowledge it has not been extensively studied. In this paper we recall the Gelfand integral and develop the calculus thereof, including appropriate versions of the Fundamental Theorem of Calculus and integration by parts formula. We then develop a related notion of Gelfand weak (G-weak) derivative, and introduce the space  $W_{w*}^{1,q}(0, T; X^*)$  as the analogue of the usual space  $W^{1,q}$  of weakly differentiable functions; in particular, our analysis shows that functions from  $W_{w*}^{1,q}(0, T; X^*)$  have absolutely continuous representatives with weak derivatives in the space of weak\* measurable maps  $L_{w*}^q(0, T; X^*)$ , the dual of  $L^{q'}(0, T; X)$ .

Having developed the calculus of the Gelfand integral, we define the notion of weak\* solution. Here weak\* refers to the target space rather than any type of convergence.

Suppose that the initial data for (1.1) is  $u^0 \in X^*$ . We say that the function

$$u \in W_{w*}^{1,q}(0, T; X^*)$$

is a *weak\* solution* to the system (1.1) if

$$\bar{u}(t) - \bar{u}(s) = \star \int_s^t \mathbb{D}_x F(u(\tau)) d\tau \quad \text{in } X^* \quad \text{for } t, s \in [0, T] \quad (1.5)$$

and  $u(0) = u^0$  in  $X^*$ . Here  $\bar{u}$  is the absolutely continuous representative of  $u$  with values in  $X^*$ ,  $F : X^* \rightarrow X^*$  is the flux map, and  $\mathbb{D}_x F(\cdot) : X^* \rightarrow X^*$  is the distributional derivative operator. The formulation (1.5) is equivalent to that of (1.3), with  $u'$  denoting the G-weak derivative of  $u$ .

We note that, unlike the usual definition of weak solution, our definition implicitly imposes a certain regularity on the solution  $u$ , in that  $t \rightarrow u(t)$  must be absolutely continuous in  $X^*$ . Also, although we require  $\mathbb{D}_x F(u(t)) \in X^*$ , this need only be weak\* measurable, rather than strongly measurable as a function of  $t$ . More precisely, the mapping

$$t \rightarrow \mathbb{D}_x F(u(t)) \quad \text{belongs to} \quad L_{w*}^q(0, T; X^*) = L^{q'}(0, T; X^*).$$

Note also that we allow any  $1 \leq q \leq \infty$ , so we allow different rates of growth of solutions.

In principle, by using  $X = C_0$  as the space of spatial test functions, we could admit time varying measures as solutions of (4.1) in  $X^* = M^n$ , the space of Radon measures. This point of view is taken in [13], where the notion of weak\* solution allows for vacuums in gas dynamics and fractures in elasticity. These solutions are not weak solutions as vacuums and fractures are represented by delta measures; the flux mapping  $F$  in that case is extended to measures.

In our present work, we primarily focus on weak\* solutions  $u \in X^* \cap BV^n$ , with  $X = C_0$  and  $X^* = M^n$ . We call these solutions *BV weak\* solutions*. Having defined this class of solutions, we investigate its relation to the familiar definition of entropy weak solutions, and in particular, *BV weak solutions*. We prove that weak\* solutions are distributional solutions, essentially because

the test functions  $\alpha(x)\beta(t)$  are dense in the set  $C_c^\infty(\mathbb{R}^2)$ . In particular, a locally bounded weak\* solution is a weak solution.

Next, we prove that a weak solution with appropriate a.e. bounds on the total variation is a weak\* solution, and in particular, any global solution obtained via Glimm's method or vanishing viscosity is a global weak\* solution with  $q = \infty$ , so the celebrated well-posedness theory developed by Bressan's school applies without change.

As an illustration of weak\* solutions, we quickly derive the Rankine-Hugoniot jump conditions, by explicitly differentiating jumps to get Dirac masses in the derivative, and similarly express the entropy conditions. We also show that the quasilinear form of the equation is satisfied at any Lebesgue point of a weak\* solution. We note that our derivation of the Rankine-Hugoniot conditions is both easier and more general than the usual derivation [5]. By way of example, we express the solution of the Riemann problem as a Bochner integral (which is also the Gelfand integral) in  $W^{1,\infty}(0, T; M^n) \subset W_{w*}^{1,\infty}(0, T; M^n)$ .

Finally, in an appendix, we consider the conditions that imply that an abstract function be G-weak differentiable. Indeed, we show that for  $1 < q \leq \infty$ , the space  $W_{w*}^{1,q}(0, T; X^*)$  is the same as the space  $\widetilde{W}^{1,q}(0, T; X^*)$  introduced by Brezis in [3].

Our goal in this paper has been to develop the calculus of the Gelfand integral and to define the notion of weak\* solution, showing that it is consistent with the usual well-known class of  $BV$  weak solutions. In the upcoming papers [13, 14], we extend the results of this paper, as follows. In [13], we extend the definition in a natural way to rigorously allow the use of Dirac masses to deal with vacuums and other discontinuities in the medium. In [14], we extend these ideas to an abstract and general framework in which we define weak\* solutions for multi-dimensional systems of balance laws. It is hoped that the availability of a wide variety of test function spaces  $X$  will shed light on the problems of instability and nonuniqueness of (strongly measurable) weak solutions.

The paper is arranged as follows. In Section 2, we recall the basic facts we need and establish notation for the rest of the paper. In Section 3 we recall the Gelfand integral and develop the calculus thereof: we do this for an abstract target space  $X^*$ , so we can use these results unchanged in more general contexts. In Section 4 we use the Gelfand integral to define weak\* solutions, and prove that weak\* solutions are distributional solutions, and weak solutions which satisfy appropriate growth conditions are weak\* solutions. Finally, in the appendix, we give a characterization of G-weak differentiable functions and show that  $W_{w*}^q(0, T; X^*)$  is isometrically isomorphic to Brezis' space of absolutely continuous functions.

## 2 Preliminaries

We begin by setting notation and recalling various facts about functions, measures and derivatives that will be useful throughout the paper. For general references, we refer the reader to the books of Royden, Cembranos & Mendoza, Diestel & Uhl, etc. [16, 4, 7].

### 2.1 Measures and Distributions

Let  $\Omega \subset \mathbb{R}$  be an open set. Denote the Borel  $\sigma$ -algebra on  $\Omega$  by  $\mathcal{B}(\Omega)$ , and Lebesgue measure on the measure space  $(\Omega, \mathcal{B}(\Omega))$ , by  $\lambda$ ; we also write  $dx = d\lambda = \lambda(dx)$ .

For any distribution  $T \in \mathcal{D}'(\Omega)$ , we use the standard pairing  $\langle T, \varphi \rangle$  to denote the action of  $T$

on  $\varphi \in \mathcal{D}(\Omega) = C_c^\infty(\Omega)$ . Any integrable function  $f \in L^1_{loc}(\Omega)$  acts as distribution by

$$\langle f, \varphi \rangle := \int f \varphi \, dx, \quad \varphi \in \mathcal{D}(\Omega).$$

Let  $M_{loc}(\Omega)$  denote the set of all sigma-finite Radon measures on the measure space  $(\Omega, \mathcal{B}(\Omega))$ . Any measure  $\nu \in M_{loc}(\Omega)$  is also a distribution via the action

$$\langle \nu, \varphi \rangle := \int \varphi(x) \, \nu(dx) = \int \varphi(x) \, d\nu, \quad \varphi \in \mathcal{D}(\Omega).$$

We embed the integrable functions  $L^1(\Omega)$  in the space of Radon measures by means of the natural mapping  $T : L^1(\Omega) \rightarrow M(\Omega)$ ,  $T(f) =: T_f$ , defined by

$$T_f(E) = \int_E f(x) \, dx, \quad E \in \mathcal{B}(\Omega).$$

If  $f \in L^1(\Omega)$ , it is easy to see that  $T_f$  is a Radon measure satisfying  $\|T_f\|_{M(\Omega)} = \|f\|_{L^1(\Omega)}$ , so that  $T$  is norm preserving, and we can regard

$$L^1_{loc}(\Omega) \subset M_{loc}(\Omega) \subset \mathcal{D}'(\Omega), \quad L^1(\Omega) \subset M(\Omega) \subset \mathcal{D}'(\Omega). \quad (2.1)$$

Next, suppose that  $F : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}$ , with  $\Omega \subset \mathbb{R}$  open. We denote the classical derivative of  $F$  at  $x \in \Omega$  by  $\frac{dF}{dx}$ , wherever this exists. We adopt the convention that if  $\frac{dF}{dx}$  exists  $\lambda$ -almost surely, then we redefine  $\frac{dF}{dx}$  to be zero at all points  $x$  at which the classical derivative is not defined. If  $F \in L^1_{loc}(\Omega)$  has a weak derivative, we denote the weak derivative by  $F'(x)$  or  $DF \in L^1_{loc}(\Omega)$ . Finally, we denote the distributional derivative of  $F$  by  $\mathbb{D}F \in \mathcal{D}'(\Omega)$ .

## 2.2 BV functions

We recall a few basic facts about  $BV$  functions. The total variation of a function  $F : (a, b) \rightarrow \mathbb{R}$  is

$$\mathbb{V}(F, (a, b)) := \sup_{\mathcal{P}} \sum_{k=1}^N |F(x_{k+1}) - F(x_k)|,$$

where  $\mathcal{P} := \{x_k\}_{k=1}^N$  ranges over the set of all finite increasing sequences in  $(a, b)$ , and we say that  $F$  is of function of bounded variation on  $(a, b)$  if  $\mathbb{V}(F, (a, b)) < \infty$ . We denote the space of functions of bounded variation by

$$BV((a, b)) := \{F : (a, b) \rightarrow \mathbb{R} : \mathbb{V}(F, (a, b)) < \infty\}.$$

We note that the usual convention is to treat  $BV$  functions on a closed interval  $[a, b]$ , but we find it convenient to consider an open interval when studying their distributional derivatives. Clearly,  $F$  can be extended to the closed interval  $[a, b]$  by continuity. The classical derivative  $\frac{dF}{dx}$  exists Lebesgue almost surely, the left and right limits  $F(x\pm) := \lim_{z \rightarrow x\pm} F(z)$  exist for all  $x \in (a, b)$ , and the set of all jump discontinuities

$$\mathcal{J} = \{x \in (a, b) : F(x+) - F(x-) \neq 0\}$$

is at most countable. The right-continuous modification  $F^*$  of  $F$  satisfies  $F^* = F_c + F_j + F_s$ , where the functions

$$\begin{aligned} F_c(x) &:= F(a) + \int_a^x \frac{dF}{dx} ds, \\ F_j(x) &:= \sum_{\bar{x} \in \mathcal{J}} H(x - \bar{x}) (F(\bar{x}+) - F(\bar{x}-)), \quad H(x) := \mathcal{X}_{x \geq 0}, \\ F_s(x) &:= F^*(x) - F_c(x) - F_j(x), \end{aligned} \tag{2.2}$$

are the absolutely continuous, jump and singular continuous parts of  $F$ , respectively.

Each  $BV$  function gives rise to a finite signed Borel measure via

$$\nu(y, x] := F(x+) - F(y+), \quad \forall x, y \in (a, b), \tag{2.3}$$

and  $\nu(a, b) := F(b) - F(a)$ , with variation  $\|\nu\|_{M(a,b)} = |\nu|(a, b) = \mathbb{V}(F, (a, b))$ . Applying (2.3) to the decomposition (2.2) of  $F$  yields  $\nu = \nu_c + \nu_j + \nu_s$ , where  $\nu_c$  is absolutely continuous,  $\nu_c \ll \lambda$ , with  $\lambda$ -a.e. Radon-Nikodym derivative  $\frac{d\nu_c}{d\lambda} = \frac{dF}{dx}$ ;  $\nu_j$  is a purely atomic singular measure  $\nu_j \perp \lambda$ ; and  $\nu_s$  is a singular continuous measure,  $\nu_s \perp \lambda$ . Moreover, we have the identities

$$\nu_c(A) = \int_A \frac{dF_c}{dx} dx \quad \text{and} \quad \nu_j = \sum_{\bar{x} \in \mathcal{J}} (F(\bar{x}+) - F(\bar{x}-)) \delta_{\bar{x}}.$$

These decompositions fully describe the distributional derivative of  $F \in BV(a, b)$ , as follows.

**Lemma 2.1.** *Using the above notation,  $F \in BV(a, b)$  distributional derivative given by*

$$\mathbb{D}F = \mathbb{D}F^* = \nu, \quad \mathbb{D}F_c = \nu_c, \quad \mathbb{D}F_j = \nu_j, \quad \mathbb{D}F_s = \nu_s,$$

so that

$$\begin{aligned} \langle \mathbb{D}F, \varphi \rangle &= \int \varphi \nu_c(dx) + \int \varphi \nu_j(dx) + \int \varphi \nu_s(dx) \\ &= \int \varphi \frac{dF}{dx} dx + \sum_{\bar{x} \in \mathcal{J}} (F(\bar{x}+) - F(\bar{x}-)) \varphi(\bar{x}) + \int \varphi \nu_s(dx). \end{aligned}$$

### 2.3 Products of Banach spaces

Let  $X$  be a normed space, with the norm denoted by  $\|\cdot\|_X$ . Recall that the dual  $X^*$  of  $X$  has norm

$$\|\phi\|_{X^*} = \sup_{x \in X, x \neq 0} \frac{|\phi(x)|}{\|x\|_X} = \sup_{x \in X, x \neq 0} \frac{\langle \phi, x \rangle}{\|x\|_X},$$

where  $\langle \phi, x \rangle$  denotes the pairing between  $\phi$  and  $x$ , and  $X^*$  is a Banach space. We recall the well-known property that this pairing distinguishes elements in  $X$  or  $X^*$ : that is, if  $\langle \phi, x \rangle = 0$  for all  $x \in X$ , then  $\phi = 0$ , and if  $\langle \phi, x \rangle = 0$  for all  $\phi \in X^*$ , then  $x = 0$ .

Because we are interested in systems, our solutions will consist of vectors with values in  $(X^*)^n$ . For any  $X$ , we equip the space  $X^n = \{(x_1, x_2, \dots, x_n) : x_i \in X\}$  with the ‘‘Euclidean’’ norm

$$\|x\|_{X^n} := \left( \sum_{i=1}^n \|x_i\|_X^2 \right)^{1/2}, \quad x = (x_1, \dots, x_n) \in X^n. \tag{2.4}$$

We show that there is no ambiguity between  $(X^*)^n$  and  $(X^n)^*$ .

**Lemma 2.2.** For any  $n \geq 1$ , the mapping  $I : (X^*)^n \rightarrow (X^n)^*$ , given by

$$[I(F)](x) = \sum_{i=1}^n F_i(x_i) \quad \text{for } F = (F_1, \dots, F_n) \in (X^*)^n, \quad x = (x_1, \dots, x_n) \in X^n,$$

is an isometric isomorphism,  $\|I(F)\|_{(X^n)^*} = \|F\|_{(X^*)^n}$ .

*Proof.* The map  $I$  is trivially injective and onto; to show it is an isometry, first note that

$$|[I(F)](x)| = \left| \sum_{i=1}^n F_i(x_i) \right| \leq \sum_{i=1}^n \|F_i\|_{X^*} \|x_i\|_X \leq \|F\|_{(X^*)^n} \|x\|_{X^n},$$

by Cauchy-Schwarz in  $\mathbb{R}^n$ , so that  $\|I(F)\|_{(X^n)^*} \leq \|F\|_{(X^*)^n}$ . Next, for  $F \in (X^*)^n$  and  $\varepsilon > 0$ , we can find  $\bar{x}_i \in X$ ,  $i = 1, \dots, n$  such that each  $\|\bar{x}_i\| = 1$  and

$$\left( \sum_{i=1}^n |F_i(\bar{x}_i) - \|F_i\|_{X^*}|^2 \right)^{1/2} \leq \sqrt{n} \varepsilon.$$

Now set

$$\bar{y} = \|F\|_{(X^*)^n}^{-1} (\bar{x}_1 \|F_1\|_{X^*}, \dots, \bar{x}_n \|F_n\|_{X^*}),$$

so that  $\|\bar{y}\|_{X^n} = 1$ , and

$$[I(F)](\bar{y}) - \|F\|_{(X^*)^n} = \|F\|_{(X^*)^n}^{-1} \sum_{i=1}^n \|F_i\|_{X^*} (F_i(\bar{x}_i) - \|F_i\|_{X^*}) \geq -\sqrt{n} \varepsilon,$$

again using Cauchy-Schwarz. Since  $\varepsilon$  is arbitrary, the result follows.  $\square$

**Corollary 2.3.** Let  $\Omega \subset \mathbb{R}^d$  be an open set, and  $C_0(\Omega)$  denote the closure of  $C_c(\Omega)$  under the sup-norm. For  $n \geq 1$ , the mapping  $T : M(\Omega)^n \rightarrow (C_0(\Omega)^n)^*$ , given by

$$\langle T(\mu), \varphi \rangle = \int_{\Omega} \varphi(x) \cdot \mu(dx) := \sum_{i=1}^n \int \varphi_i(x) \mu_i(dx),$$

is an isometric isomorphism,

$$\|T(\mu)\|_{(C_0(\Omega)^n)^*} = \|\mu\|_{M^n(\Omega)} = \left( \sum_{i=1}^n (|\mu_i|(\Omega))^2 \right)^{1/2}.$$

*Proof.* Since  $\Omega \subset \mathbb{R}^d$  is locally compact, the proof for  $n = 1$  follows from the Riesz-Markov and Riesz Representation theorems [16]; the general case follows directly.  $\square$

## 2.4 Measurability and equivalence of functions

Because we want to integrate in Banach spaces, we recall several notions of measurability and note the relationships between them. We use the conventions used in the book of Diestel and Uhl [7]. Unless explicitly stated otherwise,  $f$  is a map of a single variable with values in a Banach space,  $f : [0, T] \rightarrow X$ , and the Borel  $\sigma$ -algebra is always assumed.

The map  $f$  is *measurable*, or  $\mathfrak{B}$ -*measurable*, if  $f^{-1}(A) \in \mathfrak{B} := \mathcal{B}([0, T])$  for each  $A \in \mathcal{B}(X)$ .

The map  $f(t)$  is called  $\lambda$ -*essentially separably valued* if there exists  $N \in \mathfrak{B}$  with  $\lambda(N) = 0$  and a countable set  $H \subset X$  such that  $f([0, T] \setminus N) \subset \overline{H}$ ;  $f$  is *separably valued* if  $f([0, T]) \subset \overline{H}$ .

The map  $f$  is *simple* if there exist  $u_1, u_2, \dots, u_n \in X$  and  $E_1, E_2, \dots, E_n \in \mathfrak{B}$  such that

$$f(t) = \sum_{i=1}^n u_i \chi_{E_i}(t)$$

where  $\chi_{E_i}$  is the indicator function of  $E_i$ .

The map  $f$  is  $\lambda$ -*measurable*, or *strongly measurable*, if there is a sequence of simple functions  $f_n : [0, T] \rightarrow X$  such that

$$\lim_{n \rightarrow \infty} \|f_n(t) - f(t)\| = 0 \quad \lambda\text{-almost surely on } [0, T].$$

It is not hard to show that a map is  $\lambda$ -measurable if and only if it is both  $\mathfrak{B}$ -measurable and  $\lambda$ -essentially separably valued. In particular, if the space  $X$  is separable, then  $\lambda$ -measurability and  $\mathfrak{B}$ -measurability are equivalent.

We say the map  $f : [0, T] \rightarrow X$  is *norm-measurable* if its norm  $\|f(t)\| : [0, T] \rightarrow \mathbb{R}$  is  $\lambda$ -measurable (i.e. Lebesgue measurable).

If  $f : [0, T] \rightarrow X$  is  $\lambda$ -measurable, it is norm-measurable. Moreover, if  $f_n : [0, T] \rightarrow X$  is a sequence of  $\lambda$ -measurable functions which almost surely converge in norm,  $\|f_n(t) - f(t)\| \rightarrow 0$  as  $n \rightarrow \infty$   $\lambda$ -almost surely on  $[0, T]$ , then the limit  $f$  is  $\lambda$ -measurable.

We now define a map  $f : [0, T] \rightarrow X$  to be *weakly measurable*, or *scalar measurable*, if for each  $\phi \in X^*$ , the function

$$\phi(f(\cdot)) = \langle \phi, f(\cdot) \rangle : [0, T] \rightarrow \mathbb{R}$$

is  $\lambda$ -measurable, or equivalently  $\mathfrak{B}$ -measurable.

A measurable map is weakly measurable, and the converse holds if  $f$  is  $\lambda$ -essentially separably valued; this is the statement of Pettis' Measurability Theorem; see [7]. It follows immediately that if  $X$  is separable, then  $f$  is  $\lambda$ -measurable if and only if it is weakly measurable. Pettis's theorem also yields the fact that  $f : [0, T] \rightarrow X$  is  $\lambda$ -measurable if and only if it is the  $\lambda$ -almost everywhere uniform limit of a sequence of countably valued  $\lambda$ -measurable functions.

The map  $f : [0, T] \rightarrow X^*$  is *weak\*-measurable* if for each  $x \in X$ , the function

$$[f(\cdot)](x) = \langle f(\cdot), x \rangle : [0, T] \rightarrow \mathbb{R}$$

is  $\lambda$ -measurable.

In view of the standard embedding  $X \subset X^{**}$ , it follows that an  $X^*$ -valued weak measurable map  $f : [0, T] \rightarrow X^*$  is weak\* measurable.

We now introduce notions of equivalence of functions. Suppose that  $f, g : [0, T] \rightarrow X$  and  $f^*, g^* : [0, T] \rightarrow X^*$ .

The maps  $f$  and  $g$  are  $\lambda$ -*equivalent* if  $f(t) = g(t)$   $\lambda$ -a.e.

The maps  $f$  and  $g$  are *weakly equivalent* if, for all  $\phi \in X^*$ ,  $\langle \phi, f(t) \rangle = \langle \phi, g(t) \rangle$   $\lambda$ -a.e..

The maps  $f^*$  and  $g^*$  are *weak\* equivalent* if, for all  $x \in X$ ,  $\langle f^*(t), x \rangle = \langle g^*(t), x \rangle$   $\lambda$ -a.e..

It is clear that  $\lambda$ -equivalent functions are weakly equivalent, and that weakly equivalent functions with values in  $X^*$  are weak\* equivalent. In fact, the converse of this is true, provided the functions are strongly measurable; see [7].



**Lemma 2.4.** *Suppose that  $f, g : [0, T] \rightarrow X$  and  $f^*, g^* : [0, T] \rightarrow X^*$  are  $\lambda$ -measurable. If  $f$  and  $g$  are weakly equivalent, then they are  $\lambda$ -equivalent, and if  $f^*$  and  $g^*$  are weak\* equivalent, then they are  $\lambda$ -equivalent.*

In this lemma, the requirement that the functions be strongly measurable is essential. We will later consider non-measurable functions that are weak or weak\* measurable, in which cases the corresponding weaker notions of equivalence are appropriate.

## 2.5 The Bochner integral

The integral of a simple function  $h(t) = \sum_{i=1}^N u_i \chi_{E_i} : [0, T] \rightarrow X$  is defined in the obvious way,

$$\int_0^T h(t) \lambda(dt) = \int_0^T h(t) dt := \sum_{i=1}^N u_i \lambda(E_i) \in X.$$

We say that a  $\lambda$ -measurable function  $f : [0, T] \rightarrow X$  is *Bochner integrable*, or *summable*, if there exists a sequence of simple functions  $\{h_n\}_{n \geq 1}$  such that the Lebesgue integral  $\int_0^T \|h_n - f\| dt \rightarrow 0$  as  $n \rightarrow \infty$ . It follows that if  $f$  is summable, its integral over any  $E \in \mathfrak{B}$  exists in the space  $X$ ,

$$\int_E f(t) dt := \lim_{n \rightarrow \infty} \int_E h_n(t) dt.$$

The calculus of the Bochner integral is well known, and many of the usual theorems of the Lebesgue integral translate directly to statements on the Bochner integral. These include a Bochner dominated convergence theorem and a Bochner-Lebesgue differentiation theorem, which states that for  $\lambda$ -almost all  $s \in [0, T]$ ,

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_s^{s+h} \|f(t) - f(s)\| dt = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_s^{s+h} f(t) dt = f(s). \quad (2.5)$$

Hille's theorem states that Bochner integration commutes with closed linear operators, and in particular implies that if  $f : [0, T] \rightarrow X$  and  $f^* : [0, T] \rightarrow X^*$  are Bochner integrable, then for each  $E \in \mathfrak{B}$ ,  $x \in X$  and  $\phi \in X^*$ ,

$$\left\langle \phi, \int_E f(t) dt \right\rangle = \int_E \langle \phi, f(t) \rangle dt \quad \text{and} \quad \left\langle \int_E f^*(t) dt, x \right\rangle = \int_E \langle f^*(t), x \rangle dt.$$

The Bochner integral allows us to introduce  $L^p$  spaces as a natural generalization of the usual space  $L^p(0, T)$  of real valued functions. For each  $1 \leq p \leq \infty$ , the Banach space  $L^p(0, T; X)$  consists of all  $\lambda$ -equivalence classes of Bochner integrable functions  $f : [0, T] \rightarrow X$  such that  $\|f\|_{L^p(0, T; X)} < \infty$ , where

$$\|f\|_{L^p(0, T; X)} := \begin{cases} \left( \int_0^T \|f(t)\|^p dt \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{t \in [0, T]} \|f(t)\|, & p = \infty. \end{cases}$$

Now let  $X, Y$  be Banach spaces with  $Z = X \cap Y$  nonempty, and suppose  $f \in L^1(0, T; X)$ . We say that  $g \in L^1(0, T; Y)$  is the *Bochner weak derivative* or *B-weak derivative* of  $f$ , written  $f'(t) = g(t)$ , provided that  $f, g$  are Bochner integrable as functions from  $[0, T] \rightarrow Z$  and

$$\int_0^T f(t) \varphi'(t) dt = - \int_0^T g(t) \varphi(t) dt$$

for all scalar functions  $\varphi \in C_c^\infty(0, T)$ . For functions with values in  $X^*$ , B-weak derivatives are fully determined by their actions as functionals: for  $f, g \in L^1(0, T; X^*)$ ,  $f' = g$  if and only if for every  $x \in X$  and  $\varphi \in C_c^\infty(0, T)$ ,

$$\int_0^T \langle f(t), x \rangle \varphi'(t) dt = - \int_0^T \langle g(t), x \rangle \varphi(t) dt.$$

For  $1 \leq p \leq \infty$ , the Sobolev space  $W^{1,p}(0, T; X)$  is the set of all functions  $f \in L^p(0, T; X)$  with  $f' \in L^p(0, T; X)$ , with norm

$$\|f\|_{W^{1,p}(0,T;X)} := \begin{cases} (\int_0^T \|f(t)\|^p + \|f'(t)\|^p dt)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{t \in [0,T]} (\|f(t)\| + \|f'(t)\|), & p = \infty. \end{cases}$$

For our purposes, the most important feature of functions in  $W^{1,p}(0, T; X)$  is the Fundamental Theorem of Calculus, which follows from the fact that if  $f \in L^p(0, T; X)$ , the integral  $\int_0^t f(s) ds$  is absolutely continuous as a function of  $t$ .

**Theorem 2.5.** *Let  $f \in W^{1,p}(0, T; X)$  for some  $1 \leq p \leq \infty$ , and define  $\bar{f}(t) = f(0) + \int_0^t f'(s) ds$ . Then  $\bar{f} \in C([0, T]; X)$  is a.e. differentiable and  $\bar{f}' = f$  a.e.. The function  $g : [0, T] \rightarrow X$  given by*

$$g(s) := \begin{cases} \lim_{h \rightarrow 0} \frac{1}{h} (\bar{f}(s+h) - \bar{f}(s)), & \text{if the limit exists,} \\ 0, & \text{otherwise,} \end{cases}$$

*is strongly measurable and satisfies  $g(t) = f'(t)$   $\lambda$ -a.e. on  $[0, T]$ . If  $p = \infty$ , then  $\bar{f}$  is Lipschitz continuous with*

$$\|\bar{f}(t) - \bar{f}(s)\| \leq L |t - s| \quad \text{with} \quad L := \text{ess sup}_{t \in [0,T]} \|f'(t)\|.$$

The following form of Hölder's inequality holds: if  $f \in L^p(0, T; X)$  and  $g^* \in L^q(0, T; X^*)$ ,  $p^{-1} + q^{-1} = 1$ , then  $\langle g^*, f \rangle : [0, T] \rightarrow \mathbb{R}$  is integrable and satisfies

$$\int_0^T |\langle g^*(s), f(s) \rangle| ds \leq \|f\|_{L^p(0,T;X)} \|g^*\|_{L^q(0,T;X^*)}.$$

As a consequence, we can integrate by parts: if  $f \in W^{1,p}(0, T; X)$  and  $g^* \in W^{1,q}(0, T; X^*)$ , then  $\langle g^*, f \rangle \in W^{1,1}(0, T)$ , and

$$\int_0^T \langle g^{*'}(t), f(t) \rangle dt = \langle \bar{g}^*(s), \bar{f}(s) \rangle \Big|_0^T - \int_0^T \langle g^*(t), f'(t) \rangle dt \quad (2.6)$$

where  $\bar{f}, \bar{g}$  are the continuous representatives of  $f$  and  $g$ , respectively.

### 3 The Gelfand integral

It is sometimes too restrictive to consider only the Bochner integral of strongly measurable functions with values in a Banach space. Here we recall the Gelfand integral, which is better suited to our purposes. Throughout this section we fix a Banach space  $X$  and let  $X^*$  denote its dual. The

Gelfand integral is similar to the Dunford integral, but takes its values in the space  $X^*$ . For a detailed discussion see Tulcea and Tulcea [18], Diestel and Uhl [7], or Cembranos and Mendoza [4].

Suppose that we are given a weak\*-measurable function  $\Phi : [0, T] \rightarrow X^*$ , and suppose also that

$$\langle \Phi(\cdot), x \rangle \in L^1(0, T) \quad \text{for all } x \in X.$$

For a given  $E \in \mathfrak{B} = \mathcal{B}([0, T])$ , we define the map  $T_E : X \rightarrow L^1(0, T)$  by

$$T_E(x) = \langle \Phi(\cdot), x \rangle \chi_E(\cdot) \in L^1(0, T).$$

It is clear that  $T_E$  is linear, and if  $x_n \rightarrow x$  and  $T_E(x_n) \rightarrow y$  in  $L^1$ , then by the Riesz-Fischer theorem, a subsequence  $T_E(x_{n_k})(s) \rightarrow y(s)$  a.e., while also  $T_E(x_n)(s) \rightarrow \langle \Phi(s), x \rangle \chi_E(s)$  for all  $s \in [0, T]$ . It follows that  $y \in L^1(0, T)$ , so  $T_E$  is closed, and further, by the closed graph theorem, it is bounded, so we can write  $\|T_E(x)\|_{L^1} \leq \|T_E\| \|x\|$  for all  $x \in X$ . Since integration is a bounded linear operator of  $L^1$  into  $\mathbb{R}$ , it follows that the map

$$x \mapsto \int_0^T T_E(x)(s) ds = \int_E \langle \Phi(s), x \rangle ds$$

is a bounded linear functional on  $X$ , so defines an element of the dual  $X^*$ .

**Definition 3.1.** Let  $\Phi : [0, T] \rightarrow X^*$  be a weak\* measurable function such that  $\langle \Phi(\cdot), x \rangle \in L^1(0, T)$  for every  $x \in X$ . The Gelfand integral of  $\Phi$  over measurable set  $E \subset (0, T)$ , denoted by  $\star \int_E \Phi(s) ds$ , is an element of  $X^*$  defined by

$$\left\langle \star \int_E \Phi(s) ds, x \right\rangle = \int_E \langle \Phi(s), x \rangle ds \quad \text{for all } x \in X. \quad (3.1)$$

### 3.1 The Spaces $L_{w*}^q(0, T; X^*)$

It is clear from (3.1) that if two weak\*-measurable functions are weak\*-equivalent, then their Gelfand integrals coincide. We are thus led to consider equivalence classes,

$$[\Psi] = \{ \tilde{\Psi} : [0, T] \rightarrow X^* : \Psi \text{ and } \tilde{\Psi} \text{ are weakly*-equivalent} \}.$$

We now wish to describe the sets of (equivalence classes of) weak\*-measurable functions which are  $L^q$  Gelfand integrable, and to give an appropriate norm. Note that even if  $\Phi : [0, T] \rightarrow X^*$  is weak\*-measurable, in general its norm  $\|\Phi\| : [0, T] \rightarrow \mathbb{R}$  is not measurable. However, the theory of liftings provides the following lemma, [4, 18].

**Lemma 3.2.** *Each Gelfand integrable function  $\Psi : [0, T] \rightarrow X^*$  is weak\*-equivalent to a map  $\Psi_0 : [0, T] \rightarrow X^*$  whose norm  $\|\Psi_0\| : [0, T] \rightarrow \mathbb{R}$  is measurable. Moreover, if  $\Psi_1$  and  $\Psi_2$  are weak\*-equivalent and norm-measurable, then  $\|\Psi_1\| = \|\Psi_2\|$  almost everywhere.*

It is easy to show that if  $X$  is separable, all weak\*-measurable functions are norm-measurable.

We are now in a position to describe the  $X^*$  valued Gelfand  $L^q$  spaces, for  $1 \leq q \leq \infty$ . Given an equivalence class  $[\Psi]$  of Gelfand integrable functions, set

$$\|[\Psi]\|_q := \inf \{ \|g\|_{L^q(0, T)} : \|\Psi_0(t)\| \leq g(t) \text{ } \lambda\text{-a.e.} \},$$

where  $\Psi_0 \in [\Psi]$  is a norm-measurable element of the equivalence class. It follows that  $\|\cdot\|_q$  is a norm, and we let  $L_{w*}^q(0, T; X^*)$  be the space of equivalence classes  $[\Psi]$  of finite norm,

$$L_{w*}^q(0, T; X^*) := \{[\Psi] : \|\Psi\|_q < \infty\}.$$

It is not difficult to show that  $L_{w*}^q(0, T; X^*)$  is a Banach space and that the trivial inclusion

$$L^q(0, T; X^*) \subset L_{w*}^q(0, T; X^*) \quad \text{via} \quad f \mapsto [f],$$

is a norm-preserving isomorphism. In particular, a Bochner integrable  $f : [0, T] \rightarrow X^*$  is Gelfand integrable, and when they both exist, these integrals coincide. Also, if  $\Psi_0 \in [\Psi] \in L_{w*}^q(0, T; X^*)$  is norm-measurable, then  $\|\Psi_0\| \in L^q(0, T)$  and

$$\|\Psi\|_q = \|\Psi_0(\cdot)\|_{L^q(0, T)}.$$

Similarly,

$$\int_0^T |\langle \Psi_0(t), x \rangle|^q dt \leq \int_0^T \|\Psi_0(t)\|^q \|x\|^q dt = \|\Psi\|_q^q \|x\|^q,$$

and we conclude that for any  $x \in X$  and  $\tilde{\Psi} \in [\Psi] \in L_{w*}^q(0, T; X^*)$ , we have

$$\langle \tilde{\Psi}(\cdot), x \rangle \in L^q(0, T) \quad \text{with} \quad \|\langle \tilde{\Psi}(\cdot), x \rangle\|_{L^q(0, T)} \leq \|\Psi\|_q \|x\|. \quad (3.2)$$

It turns out that  $L_{w*}^q(0, T; X^*)$  is the dual space of  $L^p(0, T; X)$ , provided  $1 \leq p < \infty$  and  $p^{-1} + q^{-1} = 1$ ; see [4] for details.

**Theorem 3.3.** *For each  $\tilde{\Psi} \in [\Psi] \in L_{w*}^q(0, T; X^*)$  and  $f \in L^p(0, T; X)$ , the function*

$$\langle \tilde{\Psi}(\cdot), f(\cdot) \rangle \in L^1(0, T),$$

*and is independent of representative  $\tilde{\Psi}$ . Moreover, the linear map  $[\Psi] \rightarrow I([\Psi]) \in L^p(0, T; X)^*$ , given by*

$$\langle I[\Psi], f \rangle := \int_0^T \langle \tilde{\Psi}(s), f(s) \rangle ds, \quad f \in L^p(0, T; X),$$

*is an isometric isomorphism of  $L_{w*}^q(0, T; X^*)$  onto  $L^p(0, T; X)^*$ .*

### 3.2 Absolute continuity

Although the spaces  $L_{w*}^q(0, T; X^*)$  can be regarded as well known, to the authors' knowledge the calculus of Gelfand integrable functions has not previously been developed. Here we develop this calculus in parallel with the well-known calculus of the Bochner integral. We begin by proving absolute continuity of the Gelfand integral.

**Theorem 3.4.** *Fix  $1 \leq q \leq \infty$ , and suppose  $[\Phi] \in L_{w*}^q(0, T; X^*)$  with  $\tilde{\Phi} \in [\Phi]$ . The linear map  $\Psi(t) : [0, T] \rightarrow X^*$  defined by*

$$\Psi(t) = \star \int_0^t \tilde{\Phi}(s) ds, \quad \text{that is} \quad \langle \Psi(t), x \rangle = \int_0^t \langle \tilde{\Phi}(s), x \rangle ds, \quad x \in X,$$

is absolutely continuous. Also, the total variation function  $\mathbb{V}_\Psi : [0, T] \rightarrow \mathbb{R}$ , defined by

$$\mathbb{V}_\Psi(t) := \mathbb{V}(\Psi, [0, t]) = \sup_{\mathcal{P}_{[0, t]}} \sum \|\Psi(t_n) - \Psi(t_{n-1})\|,$$

the supremum being over partitions  $\mathcal{P}_{[0, t]} = \{0 = t_0 < t_1 < \dots < t_n = t\}$ , is absolutely continuous, with pointwise derivative  $\frac{d\mathbb{V}_\Psi}{dt} \in L^q(0, T)$ , and we have

$$\|\Psi(t) - \Psi(s)\| \leq \mathbb{V}_\Psi(t) - \mathbb{V}_\Psi(s) = \int_s^t \frac{d\mathbb{V}_\Psi}{dt}(\tau) d\tau \leq \int_s^t \|\Phi_0(\tau)\| d\tau \quad (3.3)$$

for all  $0 \leq s \leq t \leq T$ , for  $\Phi_0 \in [\Phi]$  norm-measurable.

*Proof.* Suppose that  $\Phi_0 \in [\Phi]$  is norm-measurable, and let  $E = \bigcup_{n=1}^N (a_n, b_n) \subset [0, T]$  be a union of disjoint intervals. Then  $\|\Phi_0\| \in L^1(0, T)$ , and we have

$$\begin{aligned} \sum_{n=1}^N \|\Psi(b_n) - \Psi(a_n)\| &= \sum_{n=1}^N \sup_{\|x\|=1} \langle \Psi(b_n) - \Psi(a_n), x \rangle \\ &= \sum_{n=1}^N \sup_{\|x\|=1} \int_{a_n}^{b_n} \langle \Phi_0(s), x \rangle ds \\ &\leq \sum_{n=1}^N \int_{a_n}^{b_n} \|\Phi_0(s)\| ds = \int_E \|\Phi_0(s)\| ds, \end{aligned}$$

and absolute continuity follows from that of the Lebesgue integral. Similarly, for any  $0 \leq s \leq t \leq T$ ,

$$\mathbb{V}_\Psi(t) - \mathbb{V}_\Psi(s) \leq \int_s^t \|\Phi_0(\tau)\| d\tau,$$

so  $\mathbb{V}_\Psi(t)$  is absolutely continuous. Thus the derivative  $\frac{d\mathbb{V}_\Psi}{dt}$  is defined  $\lambda$ -a.e., and

$$\mathbb{V}_\Psi(t) - \mathbb{V}_\Psi(s) = \int_s^t \frac{d\mathbb{V}_\Psi}{dt}(\tau) d\tau \quad \text{for all } 0 \leq s \leq t \leq T.$$

It follows that  $0 \leq \frac{d\mathbb{V}_\Psi}{dt} \leq \|\Phi_0(t)\|$   $\lambda$ -a.e. on  $[0, T]$ , and thus  $\frac{d\mathbb{V}_\Psi}{dt} \in L^q(0, T)$ .  $\square$

The Lebesgue Differentiation Theorem implies that for all  $x \in X$ ,

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), x \rangle ds = \langle \tilde{\Phi}(t), x \rangle \quad \lambda\text{-a.e.}$$

Our next task is to extend this to get a Gelfand-Lebesgue Differentiation Theorem. Fix  $1 \leq q \leq \infty$  and  $p^{-1} + q^{-1} = 1$ .

**Theorem 3.5.** *Suppose that  $[\Phi] \in L_{w*}^q(0, T; X^*)$  and  $\tilde{\Phi} \in [\Phi]$ . Then for each  $f \in L^p(0, T; X)$ ,*

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), f(t) \rangle ds = \langle \tilde{\Phi}(t), f(t) \rangle \quad \lambda\text{-a.e.}, \quad \text{and} \quad (3.4)$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), f(s) \rangle ds = \langle \tilde{\Phi}(t), f(t) \rangle \quad \lambda\text{-a.e.} \quad (3.5)$$

If  $X$  is separable then for almost all  $t \in (0, T)$

$$\frac{1}{h} \star \int_t^{t+h} \tilde{\Phi}(s) ds \xrightarrow{w*} \tilde{\Phi}(t) \quad \text{as } h \rightarrow 0.$$

*Proof.* Since  $f \in L^p(0, T; X)$ , it is strongly measurable, and without loss of generality we may assume that it is separably valued. Let  $\{x_n\}$  be a countable dense subset of  $f([0, T])$  and let  $\Phi_0 \in [\Phi]$  be norm-measurable. Define the set

$$E := \left\{ t : \frac{1}{h} \int_t^{t+h} \|\Phi_0(s)\| ds \rightarrow \|\Phi_0(t)\| \right\} \cap \bigcap_{n=1}^{\infty} \left\{ t : \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), x_n \rangle ds \rightarrow \langle \tilde{\Phi}(t), x_n \rangle \right\}$$

where the limits are taken as  $h \rightarrow 0$ . By the Lebesgue Differentiation Theorem,  $E$  has full measure,  $\lambda([0, T] \setminus E) = 0$ . For  $t \in E$  and any  $n$ , we have

$$\begin{aligned} \left| \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), f(t) \rangle ds - \langle \tilde{\Phi}(t), f(t) \rangle \right| &\leq \left| \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), f(t) - x_n \rangle ds \right| \\ &\quad + \left| \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), x_n \rangle ds - \langle \tilde{\Phi}(t), x_n \rangle \right| + \left| \langle \tilde{\Phi}(t), x_n - f(t) \rangle \right| \\ &\leq \left| \frac{1}{h} \int_t^{t+h} \|\Phi_0(s)\| ds \right| \|f(t) - x_n\| \\ &\quad + \left| \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), x_n \rangle ds - \langle \tilde{\Phi}(t), x_n \rangle \right| + \|\tilde{\Phi}(t)\| \|x_n - f(t)\|. \end{aligned}$$

Since  $t \in E$  and  $\{x_n\}$  is dense in  $f([0, T])$ , the right hand side can be made arbitrarily small, and (3.4) follows.

Now set  $B_n = \{t : \|\Phi_0(t)\| < n\}$ . Clearly  $B_n$  is nonempty for all  $n$  beyond some  $N \geq 1$ , and  $\bigcup_n B_n$  has full measure. For any  $n \geq N$  and  $t \in E \cap B_n$ , using (2.5), we get

$$\begin{aligned} \limsup_{h \rightarrow 0} \left| \frac{1}{h} \int_t^{t+h} \langle \tilde{\Phi}(s), f(s) - f(t) \rangle ds \right| &\leq \limsup_{h \rightarrow 0} \left| \frac{1}{h} \int_t^{t+h} \|\Phi_0(s)\| \|f(s) - f(t)\| ds \right| \\ &\leq n \limsup_{h \rightarrow 0} \left| \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\| ds \right| = 0, \end{aligned}$$

and thus (3.5) follows from (3.4).

Now suppose that  $X$  is separable and let  $\{x_n\}_{n \geq 1}$  be dense in  $X$ . Define  $E$  as above and fix  $t \in E$ . Then there is some  $\delta > 0$  such that for all  $0 < |h| < \delta$ ,

$$\frac{1}{h} \int_t^{t+h} \|\Phi_0(s)\| ds \leq \|\Phi_0(t)\| + 1.$$

For each such  $h$ , define  $\phi_t(h) \in X^*$  by

$$\phi_t(h) := \frac{1}{h} \star \int_t^{t+h} \tilde{\Phi}(s) ds,$$

so that, for every  $x \in X$ ,

$$|\langle \phi_t(h), x \rangle| \leq \frac{1}{h} \int_t^{t+h} \|\Phi_0(s)\| ds \|x\| \leq (\|\Phi_0(t)\| + 1) \|x\|,$$

which yields  $\|\phi_t(h)\|_{X^*} \leq \|\Phi_0(t)\| + 1$  for each  $0 < |h| < \delta$ . Now for any fixed  $x \in X$  and each  $n$ , we have

$$\begin{aligned} |\langle \phi_t(h) - \tilde{\Phi}(t), x \rangle| &\leq |\langle \phi_t(h), x - x_n \rangle| + |\langle \phi_t(h) - \tilde{\Phi}(t), x_n \rangle| + |\langle \tilde{\Phi}(t), x_n - x \rangle| \\ &\leq |\langle \phi_t(h) - \tilde{\Phi}(t), x_n \rangle| + (\|\Phi_0(t)\| + 1 + \|\tilde{\Phi}(t)\|) \|x - x_n\|, \end{aligned}$$

and since  $t \in E$ , for every  $n \geq 1$  we have

$$\limsup_{h \rightarrow 0} |\langle \phi_t(h) - \tilde{\Phi}(t), x \rangle| \leq (\|\Phi_0(t)\| + 1 + \|\tilde{\Phi}(t)\|) \|x - x_n\|.$$

Since  $\{x_n\}$  is dense, we have  $\langle \phi_t(h), x \rangle \rightarrow \langle \tilde{\Phi}(t), x \rangle$ , and since  $x$  is arbitrary, the result follows.  $\square$

### 3.3 The Spaces $W_{w*}^{1,q}(0, T; X^*)$

Suppose that  $\Psi, \Phi : [0, T] \rightarrow X^*$  are weakly\* measurable, and  $[\Psi], [\Phi] \in L_{w*}^1(0, T; X^*)$ . We say that  $\Phi$  is the *Gelfand weak derivative* or *G-weak derivative* of  $\Psi$ , written  $\Psi'(t) = \Phi(t)$  or  $[\Phi] = [\Psi']$ , if

$$\begin{aligned} \star \int_0^T \Psi(t) \varphi'(t) dt &= - \star \int_0^T \Phi(t) \varphi(t) dt, \quad \text{that is} \\ \int_0^T \langle \Psi(t), x \rangle \varphi'(t) dt &= - \int_0^T \langle \Phi(t), x \rangle \varphi(t) dt, \quad \text{for all } x \in X, \end{aligned} \tag{3.6}$$

for all scalar functions  $\varphi \in C_c^\infty(0, T)$ .

We now define the space  $W_{w*}^{1,q}(0, T; X^*)$ , for  $1 \leq q \leq \infty$ , to be the set of weak\* equivalence classes  $[\Psi] \in L_{w*}^q(0, T; X^*)$  having G-weak derivative  $[\Psi'] \in L_{w*}^q(0, T; X)$ , with norm

$$\|[\Psi]\|_{W_{w*}^{1,q}(0, T; X)} := \begin{cases} \left( \int_0^T (\|\Psi_0(t)\|^q + \|\Psi'_0(t)\|^q) dt \right)^{1/q}, & 1 \leq q < \infty \\ \text{ess sup}_{t \in [0, T]} (\|\Psi_0(t)\| + \|\Psi'_0(t)\|), & q = \infty, \end{cases}$$

for  $\Psi_0 \in [\Psi]$ ,  $\Psi'_0 \in [\Psi']$  norm-measurable.

We are now in a position to state the Gelfand Fundamental Theorem of Calculus.

**Theorem 3.6.** *Let  $\Psi : [0, T] \rightarrow X^*$  be weakly\* measurable and  $1 \leq q \leq \infty$ .*

- (i)  *$[\Psi] \in W_{w*}^{1,q}(0, T; X^*)$  if and only if there exist  $[\Phi] \in L_{w*}^q(0, T; X^*)$  and  $\psi_0 \in X^*$  such that the mapping*

$$\tilde{\Psi} : [0, T] \rightarrow X^* \quad \text{given by} \quad \tilde{\Psi}(t) := \psi_0 + \star \int_0^t \Phi(s) ds \tag{3.7}$$

*satisfies  $\tilde{\Psi} \in [\Psi]$ .*

- (ii) *If  $[\Psi] \in W_{w*}^{1,q}(0, T; X^*)$ , then there exists an absolutely continuous representative  $\bar{\Psi} \in [\Psi]$  such that*

$$\bar{\Psi}(t) = \bar{\Psi}(0) + \star \int_0^t \Psi'(s) ds. \tag{3.8}$$

*The map  $\bar{\Psi}$  belongs to  $L^q(0, T; X^*)$  and has variation  $\mathbb{V}_{\bar{\Psi}}$  which satisfies (3.3).*

*Proof.* Suppose that  $\tilde{\Psi} \in [\Psi]$  satisfies (3.7). Then for every  $x \in X$ ,  $t \in [0, T]$ , we have

$$\langle \tilde{\Psi}(t), x \rangle = \langle \psi_0, x \rangle + \int_0^t \langle \Phi(s), x \rangle ds \in \mathbb{R},$$

so the function  $t \mapsto \langle \tilde{\Psi}(t), x \rangle$  is in  $W^{1,q}(0, T)$ , with weak derivative  $t \mapsto \langle \Phi(t), x \rangle$ , and (3.6) holds for any  $\varphi \in C_c^\infty(0, T)$ , so  $[\tilde{\Psi}] \in W_{w*}^{1,q}(0, T; X^*)$  with  $[\Phi] = [\tilde{\Psi}']$ .

Now suppose that  $[\Psi] \in W_{w*}^{1,q}(0, T; X^*)$  and let  $\Psi_0 \in [\Psi]$  and  $\Psi'_0 \in [\Psi']$  be norm-measurable. For each  $x \in X$  and  $\varphi \in C_c^\infty(0, T)$ , we have

$$\int_0^T \langle \Psi_0(t), x \rangle \varphi'(t) dt = - \int_0^T \langle \Psi'_0(t), x \rangle \varphi(t) dt,$$

and (3.2) implies  $\langle \Psi_0(\cdot), x \rangle \in W^{1,q}(0, T)$  with weak derivative  $\langle \Psi'_0(\cdot), x \rangle$ . This implies that for every  $x \in X$  there exists a unique absolutely continuous real-valued function  $z_x : [0, T] \rightarrow \mathbb{R}$  such that

$$\begin{aligned} z_x(t) &= \langle \Psi_0(t), x \rangle \quad \lambda\text{-a.e. } t \in [0, T] \\ \text{and } z_x(t) &= z_x(0) + \int_0^t \langle \Psi'_0(s), x \rangle ds \quad \text{for all } t \in [0, T], \end{aligned} \tag{3.9}$$

where  $z_x(0)$  is the trace of the map  $t \rightarrow \langle \Psi_0(t), x \rangle$  at  $t = 0$ .

Note that  $z_x(0)$  does not in general equal to  $\langle \Psi_0(0), x \rangle$ . Therefore our next task is to show that there exists  $\psi_0 \in X^*$  such that  $z_x(0) = \langle \psi_0, x \rangle$  for all  $x \in X$ .

From the linearity of  $\Psi_0(\cdot)$ , the continuity of  $z_x(\cdot)$  and (3.9) it follows that

$$z_{c_1x_1+c_2x_2}(t) = c_1z_{x_1}(t) + c_2z_{x_2}(t) \quad \text{for all } t \in [0, T]$$

for any  $c_1, c_2 \in \mathbb{R}$  and  $x_1, x_2 \in X$ . hence the map  $X \ni x \rightarrow z_x(0)$  is a linear functional.

By (3.9)<sub>1</sub> for each  $x \in X$

$$\int_0^T \langle \Psi_0(t), x \rangle ds = \int_0^T z_x(t) dt = z_x(0)T + \int_0^T \left( \int_0^t \langle \Psi'_0(s), x \rangle ds \right) dt.$$

Since  $t \rightarrow \|\Psi_0\|$  and  $t \rightarrow \|\Psi'_0(t)\|$  are measurable and integrable, we conclude that

$$|z_x(0)| \leq \frac{1}{T} \left( \int_0^T \|\Psi_0(t)\| dt + T \int_0^T \|\Psi'_0(t)\| dt \right) \|x\|.$$

Thus,  $X \ni x \rightarrow z_x(0)$  is a bounded linear functional and hence  $z_x(0) = \langle \psi_0, x \rangle$  for some  $\psi_0 \in X^*$ .

It follows from (3.9)<sub>2</sub> and Theorem 3.4 that the map  $\bar{\Psi} : [0, T] \rightarrow X^*$ , given by

$$\langle \bar{\Psi}(t), x \rangle := \left\langle \psi_0 + \star \int_0^t \Psi'_0(s) ds, x \right\rangle = z_x(t),$$

is absolutely continuous with G-weak derivative  $\Psi'_0$ , and so  $[\bar{\Psi}] \in W_{w*}^{1,q}(0, T; X^*)$ . Moreover, since  $\langle \bar{\Psi}(t), x \rangle = z_x(t)$ , (3.9)<sub>1</sub> implies that  $\bar{\Psi} \in [\Psi]$  and this provides (i).

The statement (ii) follows directly from (i) and Theorem 3.4.  $\square$



To simplify **the** notation in the sequel, we adopt the convention that, unless otherwise specified, we identify  $[\Phi] \in L_{w*}^q(0, T; X^*)$  with a norm-measurable representative, and if  $[\Psi] \in W_{w*}^{1,q}(0, T; X^*)$ , we identify  $[\Psi]$  with its absolutely continuous representative and  $[\Psi']$  with a norm-measurable representative.

**Corollary 3.7.** *Suppose that  $u \in W_{w*}^{1,q}(0, T; X^*)$  with continuous representative  $\bar{u}$ . If there exists a strongly measurable  $f : [0, T] \rightarrow X^*$  such that  $f \in [u']$ , then  $\bar{u}$  is strongly measurable and  $\bar{u} \in W^{1,q}(0, T; X^*)$ .*

*Proof.* Since  $f \in [u']$  and  $\bar{u}$  is continuous, we have

$$\bar{u}(t) = \bar{u}(0) + \star \int_0^t f(s) ds, \quad t \in [0, T]. \quad (3.10)$$

Moreover, since  $f$  is strongly measurable, it is norm-measurable, and

$$\int_0^T \|f(t)\|^q dt = \|[u']\|_q^q < \infty,$$

so that  $f \in L^q(0, T; X^*)$ , (3.10) is a Bochner integral, and  $\bar{u} \in W^{1,q}(0, T; X^*)$ .  $\square$

Our next calculus theorem is an integration by parts formula.

**Theorem 3.8.** *Let  $X$  be a Banach space,  $1 \leq p \leq \infty$ ,  $p^{-1} + q^{-1} = 1$ , and suppose  $f \in W^{1,p}(0, T; X)$  and  $\Psi \in W_{w*}^{1,q}(0, T; X^*)$ . The function  $\langle \Psi(t), f(t) \rangle : [0, T] \rightarrow \mathbb{R}$  is in  $W^{1,1}(0, T)$ , with weak derivative*

$$D\langle \Psi(t), f(t) \rangle = \langle \Psi'(t), f(t) \rangle + \langle \Psi(t), f'(t) \rangle \quad \lambda\text{-a.e.}, \quad (3.11)$$

where  $\Psi'$  and  $f'$  are the  $G$ -weak and  $B$ -weak derivatives of  $\Psi$  and  $f$ , respectively. Moreover,

$$\int_0^T \langle \Psi'(t), f(t) \rangle dt = \langle \bar{\Psi}(s), \bar{f}(s) \rangle \Big|_0^T - \int_0^T \langle \Psi(t), f'(t) \rangle dt, \quad (3.12)$$

where  $\bar{f}$  and  $\bar{\Psi}$  are the continuous representatives of  $f$  and  $\Psi$ , respectively.

*Proof.* Set  $z(t) = \langle \Psi(t), f(t) \rangle$ , which is in  $L^1(0, T)$  by Theorem 3.3, assume that  $f$  and  $\Psi$  are continuous representatives, and fix representatives  $f'$  and norm-measurable  $\Psi'$ . Let  $E = \bigcup_{n=1}^N (a_n, b_n)$  be a disjoint union of intervals. By Theorems 2.5 and 3.6 we have

$$f(b_n) - f(a_n) = \int_{a_n}^{b_n} f'(t) dt, \quad \langle \Psi(b_n) - \Psi(a_n), f(b_n) \rangle = \int_{a_n}^{b_n} \langle \Psi'(t), f(b_n) \rangle dt,$$

and so, since  $f$  and  $\Psi$  are continuous, we get

$$\begin{aligned} \sum_{n=1}^N |z(b_n) - z(a_n)| &\leq \sum_{n=1}^N \left| \int_{a_n}^{b_n} \langle \Psi'(t), f(b_n) \rangle dt \right| + \sum_{n=1}^N \left| \int_{a_n}^{b_n} \langle \Psi(a_n), f'(t) \rangle dt \right| \\ &\leq \|f\|_{L^\infty(0, T; X)} \int_E \|\Psi'(t)\| dt + \|\Psi\|_{L^\infty(0, T; X^*)} \int_E \|f'(t)\| dt. \end{aligned}$$

Absolute continuity the Lebesgue integral now implies that  $z(t)$  is absolutely continuous, so also  $z \in W^{1,1}(0, T)$ , with weak derivative  $Dz = \frac{dz}{dt}$  almost everywhere.

Next, Theorem 3.5, (2.5) and absolute continuity of  $z(t)$  imply that the set

$$E = \left\{ t : \lim_{h \rightarrow 0} \frac{1}{h} \int_0^T f'(s) ds = f'(t), \quad \frac{1}{h} \int_t^{t+h} \langle \Psi'(s) f(s) \rangle ds = \langle \Psi'(t) f(t) \rangle, \quad \frac{dz}{dt} \text{ exists} \right\}$$

has full measure,  $\lambda([0, T] \setminus E) = 0$ . For each  $t \in E$ , we have

$$\begin{aligned} \frac{dz}{dt}(t) &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \langle \Psi(t+h) - \Psi(t), f(t) \rangle + \langle \Psi(t), f(t+h) - f(t) \rangle \right) \\ &= \langle \Psi'(t), f(t) \rangle + \langle \Psi(t), f'(t) \rangle, \end{aligned}$$

which is (3.11). Next, by absolute continuity,

$$z(T) - z(0) = \int_0^T Dz(t) dt,$$

which is (3.12). □

Since representatives  $\bar{\Psi} \in W_{w*}^{1,q}(0, T; X^*)$  are continuous, we easily define the trace of  $\bar{\Psi}$  on the boundary  $\{0, T\}$ , and we define  $W_{0,w*}^{1,q}(0, T; X^*)$  to be the subset of those maps  $\Psi$  whose continuous representative  $\bar{\Psi}$  satisfies  $\bar{\Psi}(0) = \bar{\Psi}(T) = 0$ . Then the obvious simplification of (3.12) without boundary values follows if either  $f \in W_0^{1,p}(0, T; X)$  or  $\Psi \in W_{0,w*}^{1,q}(0, T; X^*)$ .

## 4 Application to hyperbolic conservation laws

We now apply the calculus we have developed to the Cauchy problem for systems of hyperbolic conservation laws in one space dimension,

$$\partial_t u(x, t) + \partial_x F(u(x, t)) = 0, \quad u(x, 0) = u^0(x), \quad (4.1)$$

where  $u \in \mathbb{R}^n$ ,  $x \in \mathbb{R}$ , and  $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ .

Recall that a *distributional solution* of (4.1) is a measurable function  $u(x, t) : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}^n$  which satisfies

$$\int_0^T \int_{\mathbb{R}} \left( u(x, t)^\top \partial_t \varphi(x, t) + F(u(x, t))^\top \partial_x \varphi(x, t) \right) dx + \int_{\mathbb{R}} u^0(x)^\top \varphi(0, x) dx = 0, \quad (4.2)$$

for every function  $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R}^n)$ ; here it is implicitly assumed that  $F(u(x, t))$  is locally integrable. A distributional solution which is locally bounded is a *weak solution*.

In defining distributional and weak solutions, there is no distinction between the roles of time and space, with both the solution  $u$  and test functions  $\varphi$  regarded as functions on  $\mathbb{R} \times [0, T]$ . Our approach is different, in that we wish to understand the PDE (4.1) as an evolution equation, so we will regard the solution as a function of time taking values in a Banach space. In particular, we regard the (spatial) test functions as elements of a Banach space  $X$ , which contains  $C_c^\infty$  as a dense subspace, and, since  $u(t)$  acts linearly on test functions, we regard  $u \in L_{w*}^q(0, T; X^*)$ . Taking this point of view, we use the convention that  $u(t) \in X^*$  stands for  $u(\cdot, t)$ , and  $F(u(t)) \in X^*$  for  $F(u(\cdot, t))$ . Also, throughout this section, we fix the constant  $1 \leq q \leq \infty$ .

The critical issue at this point is to make sense of the nonlinear flux  $F(u)$  and its derivative in the space  $X^*$ . To this end, we define a distributional derivative of an element in  $X^*$ .

**Definition 4.1.** Let  $\Omega \subseteq \mathbb{R}$  be open. Let  $C_c^\infty(\Omega)$  be dense in  $X$ . We say that  $f \in X^*$  has an  $X^*$ -valued distributional derivative, written  $\mathbb{D}_x f \in X^*$ , if, for all  $\phi \in C_c^\infty(\Omega) \subset X$ , we have

$$|\langle f, \phi' \rangle| \leq C \|\phi\|_X,$$

and in this case we define  $\mathbb{D}_x f$  by

$$\langle \mathbb{D}_x f, \phi \rangle := -\langle f, \phi' \rangle.$$

**Definition 4.2.** Let  $C_c^\infty(\Omega)$  be dense in  $X$ . Let  $F : X^* \rightarrow X^*$  be a flux mapping and  $u^0 \in X^*$ . The function

$$u \in W_{w*}^{1,q}(0, T; X^*)$$

is called a weak\* solution of the system (4.1) if

$$u'(t) + \mathbb{D}_x F(u(t)) = 0 \quad \text{in} \quad L_{w*}^q(0, T; X^*) \quad (4.3)$$

and such that  $\bar{u}(0) = u^0$  in  $X^*$ , where  $\bar{u}$  is the time continuous representative of  $u$ .

**Remark 4.3.** Equivalently, the equation (4.3) reads

$$\bar{u}(t) - \bar{u}(s) = \star \int_s^t \mathbb{D}_x F(u(\tau)) d\tau \quad \text{in } X^* \quad \text{for } t, s \in [0, T]. \quad (4.4)$$

**Remark 4.4.** In principle, by using  $X = C_0$  as the space of spatial test functions, we could admit time varying measures as solutions of (4.1) in  $X^* = M^n$ , the space of Radon measures. This point of view is taken in [13], where the notion of weak\* solution allows for vacuums in gas dynamics and fractures in elasticity. These solutions are not weak solutions as vacuums and fractures are represented by delta measures; the flux mapping  $F$  as well as the entropy and entropy flux in that case is extended to measures (see [13]). In our present work, we primarily focus on weak\* solutions  $u \in X^* \cap BV^n$ , with  $X = C_0$  and  $X^* = M^n$ .

## 4.1 BV weak\* solutions of hyperbolic conservation laws

Our intention is to regard  $u$  as a function of  $t$  with values in  $X^*$ , and to interpret  $\partial_t u$  in (4.1) as a G-weak derivative of  $u$ . Here, the state  $u(x, t)$  takes values in  $\mathbb{R}^n$ , so we use the convention (2.4), and referring to Corollary 2.3, we take

$$X = C_0(\mathbb{R})^n, \quad \text{so that} \quad X^* = M_{loc}(\mathbb{R})^n.$$

Also, recalling that if  $w \in BV_{loc}$  then  $\mathbb{D}w \in M_{loc}$ , we will further suppose that  $u \in BV_{loc}^n \subset M_{loc}^n$ , and that  $u$  has G-weak derivative in  $M_{loc}^n$ .

More generally, if  $Y \subset X^*$ , we define the set

$$W_{w*}^{1,q}(0, T^-; Y, X^*) := L_{w*}^q(0, T^-; Y) \bigcap W_{w*}^{1,q}(0, T^-; X^*),$$

where  $T^-$  means up to but not including  $T$ , namely

$$L_{w*}^q(0, T^-; Y) := \{\phi : [0, T] \rightarrow Y : \phi \in L_{w*}^q(0, \tau; Y) \text{ for each } 0 < \tau < T\},$$

and similarly for  $W_{w*}^{1,q}(0, T^-; X^*)$ . In our current application, we have  $X = (C_0)^n$ ,  $X^* = M_{loc}^n$ , and  $Y = BV_{loc}^n = BV_{loc}(\mathbb{R}; \mathbb{R}^n)$ .

**Definition 4.5.** Suppose that  $u^0 \in M_{loc}(\mathbb{R}; \mathbb{R}^n)$ . The function

$$u \in W_{w*}^{1,q}(0, T^-; BV_{loc}^n, M_{loc}^n)$$

is called a BV weak\* solution to the Cauchy problem (4.1) if for a.e.  $t \in (0, T)$

$$u'(t) + \mathbb{D}_x F(u(t)) = 0 \quad \text{in } M_{loc}^n \quad (4.5)$$

and such that  $\bar{u}(0) = u^0$  in  $M_{loc}^n$ , where  $\bar{u}$  is the time continuous representative of  $u$ .

In (4.5),  $u'$  is the G-weak derivative of  $u$  in the space  $W_{w*}^{1,q}(0, T^-; M_{loc}^n)$ , and  $\mathbb{D}_x F(u(t))$  is the distributional derivative of the function  $x \rightarrow F(u(x, t))$ . Since  $u(t) \in BV_{loc}^n$  for a.e.  $t$ , and  $F$  is  $C^1$  and so Lipschitz, for these  $t$  we also have  $F(u(\cdot, t)) \in BV_{loc}^n$ , and Lemma 2.1 implies that  $\mathbb{D}_x F(u(t)) \in M_{loc}^n$ , so we can understand the equation in  $M_{loc}^n$ . Moreover, in view of separability of  $C_0$ , each element of  $L_{w*}^q(0, T^-; M_{loc}^n)$  is norm-measurable, and (4.5) is equivalent to requiring that  $[u'] + [\mathbb{D}_x F(u)] = 0$  as equivalence classes of  $L_{w*}^q(0, T; M^n(\Omega))$ .

We note that our definition of weak\* solutions implicitly requires some regularity, namely that  $u$  is absolutely continuous as a function  $t \rightarrow u(t) \in M_{loc}^n$ . Also, recall that we require only that  $u'$  (and thus  $\mathbb{D}_x F(u)$ ) be weak\*-measurable, rather than strongly measurable, and the parameter  $q$  allows a range of growth rates of weak\* solutions. In this sense, our weak\* solutions are different from weak solutions, which are locally bounded locally integrable functions  $u(x, t)$  satisfying (4.2). Although the two notions of solution are different, we show they yield the same solutions in the most important case.

**Theorem 4.6.** Suppose  $u \in W_{w*}^{1,q}(0, T^-; BV_{loc}^n, M_{loc}^n)$  is a BV weak\* solution to the Cauchy problem (4.1), with continuous representative  $\bar{u}$ . Then  $\bar{u}$  is Hölder continuous as a function into  $L_{loc}^1(\mathbb{R}; \mathbb{R}^n)$ , that is,

$$\bar{u} \in C^{0, 1-1/q}(0, T^-; L_{loc}^1) \quad \text{or} \quad \bar{u} \in Lip(0, T^-; L_{loc}^1), \quad (4.6)$$

for  $1 \leq q < \infty$  or  $q = \infty$ , respectively. The function  $\bar{u}(x, t)$  is a distributional solution of the Cauchy problem (4.1). In particular, if  $u$  is locally bounded,  $u \in L_{w*}^\infty(0, T^-; L_{loc}^\infty(\mathbb{R}; \mathbb{R}^n))$ , then  $\bar{u}(x, t)$  is a weak solution to the Cauchy problem (4.1).

*Proof.* According to Theorem A.2 below,  $\bar{u} \in \widetilde{W}^{1,q}(0, T^-; M_{loc}^n)$ , which implies that for each open  $\Omega \subset \subset \mathbb{R}$ , there exists  $\phi \in L^q$  such that

$$\|\bar{u}(t) - \bar{u}(s)\|_{M^n(\Omega)} \leq \int_s^t \phi(\tau) d\tau, \quad t, s \in [0, \tau].$$

Since  $\phi \in L^q$ , we apply Hölder's inequality, together with (2.1), to get

$$\|\bar{u}(t) - \bar{u}(s)\|_{L^1(\Omega)} = \|\bar{u}(t) - \bar{u}(s)\|_{M^n(\Omega)} \leq \|\phi\|_q |t - s|^{1-1/q},$$

which implies (4.6).

To show that  $\bar{u}$  is a distributional solution, take any  $\psi \in C_c^1(\mathbb{R} \times (-\infty, T); \mathbb{R}^n)$ . Restricting  $\psi$  to the time interval  $[0, T)$ , we have  $\psi \in W^{1,\infty}(0, \tau; C_0((a, b); \mathbb{R}^n))$  for some  $0 < \tau < T$  and finite open interval  $(a, b) \in \mathbb{R}$ . From (4.5) it follows that  $u' = -\mathbb{D}_x F(u) \in L_{w*}^1(0, \tau; M^n(a, b))$ , and by Theorem 3.8, the functions  $t \rightarrow \langle \mathbb{D}_x F(u(t)), \psi(t) \rangle$  and  $t \rightarrow \langle u(t), \psi(t) \rangle$  are Lebesgue integrable and

$$\int_0^\tau \langle \mathbb{D}_x F(u(t)), \psi(t) \rangle dt = - \int_0^\tau \langle u'(t), \psi(t) \rangle dt = \langle \bar{u}(0), \psi(0) \rangle + \int_0^\tau \langle u(t), \psi'(t) \rangle dt.$$

Next, for each  $t \in [0, T]$  we have  $u(t) \in BV_{loc}^n$ , and since  $F$  is  $C^1$ , we have also  $F(u(t)) \in BV_{loc}^n$ , so that  $\mathbb{D}_x F(u(t)) \in M_{loc}^n$  and

$$\langle \mathbb{D}_x F(u(t)), \psi(t) \rangle = \int_{\mathbb{R}} \psi(x, t) \mathbb{D}_x (F(u))(dx, t) = - \int_{\mathbb{R}} F(u(x, t))^\top \partial_x \psi(x, t) dx.$$

Similarly, since  $u(t) \in M_{loc}^n \cap BV_{loc}^n \subset L_{loc}^1(\mathbb{R}; \mathbb{R}^n)$ ,

$$\begin{aligned} \langle u(t), \psi'(t) \rangle &= \int_{\mathbb{R}} \partial_t \psi(x, t) \cdot u(dx, t) = \int_{\mathbb{R}} u(x, t)^\top \partial_t \psi(x, t) dx, \quad \text{and} \\ \langle \bar{u}(0), \psi(0) \rangle &= \int_{\mathbb{R}} \psi(x, 0) \cdot \bar{u}(dx, 0) = \int_{\mathbb{R}} \bar{u}(x, 0)^\top \psi(x, 0) dx. \end{aligned}$$

Combining the above relations we obtain

$$\int_0^T \int_{\mathbb{R}} \left( u(x, t)^\top \partial_t \psi(x, t) + F(u(x, t))^\top \partial_x \psi(x, t) \right) dx dt + \int_{\mathbb{R}} u^0(x)^\top \psi(x, 0) dx = 0,$$

which is (4.2). Finally, since  $\bar{u}(t)$ , and hence also  $F(\bar{u}(t))$ , is continuous with values in  $L_{loc}^1$ , both  $\bar{u}(x, t)$  and  $F(\bar{u})(x, t)$  are locally integrable, and the proof is complete.  $\square$

Having shown that any weak\* solution is a distributional solution, we now show that a weak solution with sufficiently regular growth is also a weak\* solution. In particular, the global weak solutions generated by Glimm's method, front tracking, and vanishing viscosity, all of which have uniformly bounded total variation, are all weak\* solutions. It follows that the uniqueness and  $L^1$ -stability results of Bressan et.al. hold unchanged in the framework of weak\* solutions.

**Theorem 4.7.** *Let  $u(x, t)$  be a weak solution of the Cauchy problem (4.1), with  $u(\cdot, t) \in BV_{loc}^n$  for each  $t \in [0, T]$ . Suppose also that for each open interval  $\Omega \subset \subset \mathbb{R}$ , there is some  $g_\Omega \in L^q(0, T^-)$  such that*

$$\mathbb{V}(u(\cdot, t); \Omega) \leq g_\Omega(t), \quad \text{a.e. } t \in (0, T). \quad (4.7)$$

*Then  $u \in W_{w*}^{1,q}(0, T^-; BV_{loc}^n, M_{loc}^n)$ , and  $u$  is a BV weak\* solution to the Cauchy problem.*

*Proof.* Let  $I = (a, b) \subset \subset (0, T)$  and  $\Omega \subset \mathbb{R}$  be finite intervals. Since  $u(x, t)$  is locally bounded and integrable, for each  $\alpha \in C_0^1(\Omega)$ , the map  $t \rightarrow \langle u(t), \alpha \rangle := \int_{\Omega} u(x, t) \alpha(x) dx$  is measurable and integrable over  $I$ . Also,

$$\operatorname{ess\,sup}_{t \in I} \|u(\cdot, t)\|_{M^n(\Omega)} = \operatorname{ess\,sup}_{t \in I} \int_{\Omega} |u(x, t)| dx \leq \lambda(\Omega) \|u\|_{L^\infty(I \times \Omega)},$$

so that  $u \in L_{w*}^\infty(a, b; M(\Omega))$ .

Next, for each  $t$ , we have  $u(t) := u(\cdot, t) \in BV_{loc}^n$ , and so since  $F \in C^1$ , also  $F(u(t)) \in BV_{loc}^n$ , and in particular,  $x \rightarrow F(u(x, t))$  is integrable over  $\Omega$  for each  $t \in I$ . Lemma 2.1 now implies that  $\mathbb{D}_x F(u(t)) \in M_{loc}^n$  for each  $t \in I$ , and

$$t \rightarrow \langle \mathbb{D}_x F(u(t)), \alpha \rangle = - \int_{\mathbb{R}} F(u(x, t)) \alpha'(x) dx$$

is measurable and integrable on  $I$ . Moreover, since  $F$  is locally Lipschitz and  $u$  is locally bounded, our assumption (4.7) implies that there is some  $\tilde{g}_\Omega \in L^q(0, T^-)$  such that

$$\|\mathbb{D}_x F(u(t))\|_{M^n(\Omega)} = \mathbb{V}(F(u(\cdot, t)); \Omega) \leq \tilde{g}_\Omega(t), \quad \text{a.e. } t \in (0, T),$$

and so also

$$|\langle \mathbb{D}_x F(u(t)), \alpha \rangle| \leq \|\mathbb{D}_x F(u(t))\|_{M^n(\Omega)} \|\alpha\|_{L^\infty(\Omega)} \leq \tilde{g}_\Omega(t) \|\alpha\|_{L^\infty(\Omega)}.$$

Since  $\alpha$  is arbitrary and  $g_\Omega \in L^q(0, T^-)$ , Theorem A.1 below implies that

$$\mathbb{D}_x F(u(t)) \in L_{w*}^q(0, T^-; M^n(\Omega)).$$

Now let  $\psi \in C_c^1(I)$  and  $\alpha \in C_0^1(\Omega)$ , and set  $\varphi(x, t) = \psi(t)\alpha(x)$ . Using (4.2), we get

$$\int_I \left( \int_\Omega u(x, t)^\top \alpha(x) dx \right) \psi'(t) dt + \int_I \left( \int_\Omega F(u(x, t))^\top \alpha'(x) dx \right) \psi(t) dt = 0,$$

or equivalently

$$\int_I \langle u(t), \alpha \rangle \psi'(t) dt = \int_I \langle \mathbb{D}_x F(u(t)), \alpha \rangle \psi(t) dt.$$

Since  $\psi$  is arbitrary, it follows that the G-weak derivative  $u'$  exists and

$$\langle u'(t), \alpha \rangle = -\langle \mathbb{D}_x F(u(t)), \alpha \rangle \quad \text{a.e. } t \in I.$$

Since  $\alpha$  and  $I = (a, b) \subset \subset (0, T)$  are arbitrary we conclude that

$$u' + \mathbb{D}_x(F(u(t))) = 0 \quad \text{in } M^n(\Omega), \text{ a.e. } t \in (0, T)$$

and  $u \in W_{w*}^{1,q}(0, T^-; M^n(\Omega))$ . Moreover, (4.7) and Theorem A.1 yield  $u \in L_{w*}^q(0, T^-; BV^n(\Omega))$ , so also  $u \in W_{w*}^{1,q}(0, T^-; BV^n(\Omega), M^n(\Omega))$ , and  $u$  is a BV weak\* solution.

It remains to show that the initial data is taken on in an appropriate sense. Let  $\bar{u}(t)$  be the continuous representative of the weak solution  $u(\cdot, t) \in M^n(\Omega)$ . We have shown that this is a weak\* solution, with initial data  $\bar{u}(0) \in BV_{loc}^n$ . By Theorem 4.6, this is a continuous distributional solution. Moreover,  $u(t) = \bar{u}(t)$  a.e.  $t$  in  $M^n(\Omega)$ , and so also in  $L_{loc}^1(\Omega)$ , and this in turn implies  $u(x, t) = \bar{u}(x, t)$  almost surely as functions of both space and time. Since both  $u$  and  $\bar{u}$  are distributional solutions, for any  $\varphi \in C_c^1(\Omega \times (-\infty, \tau))$ , use of (4.2) then yields

$$\int_{\mathbb{R}} \bar{u}(x, 0)^\top \varphi(0, x) dx = \int_{\mathbb{R}} u^0(x)^\top \varphi(0, x) dx,$$

so that  $\bar{u}(0, x) = u^0(x)$  a.e.  $x$ , and thus  $\bar{u}(0) = u^0 \in M_{loc}^n$ .  $\square$

## 4.2 Shock waves and entropy

It is well known that discontinuities in weak solutions satisfy the Rankine-Hugoniot jump conditions, and that entropy conditions are required to select the unique, physically relevant solution when discontinuities are present. Here we restate these conditions from the point of view of weak\* solutions. In particular, we note the ease with which the Rankine-Hugoniot conditions are derived.

In deriving the shock conditions, we wish to understand the local pointwise structure of BV weak\* solutions. To do so, we make some reasonable simplifying assumptions. For a given  $\Omega \subset \mathbb{R}$

and  $t \in [0, T)$ , we assume  $u(t) \in SBV(\Omega)$ , so that  $u$  has no singular continuous part, and we can write

$$u(x, t) = \sum_j v_j(t) H(x - x_j(t)) + u_c(x, t), \quad \text{a.e. } x \in \Omega, \quad (4.8)$$

where  $u_c$  is the absolutely continuous part of  $u$ , and there are jumps of size  $v_j(t)$  located at points  $x_j(t) \in \mathcal{J} \subset \Omega$ . The jump  $v_j(t)$  is

$$v_j(t) = u(x_j(t)+, t) - u(x_j(t)-, t),$$

while  $u_c(\cdot, t) \in W^{1,1}(\mathbb{R})$  with  $\partial_x u = \partial_x u_c$  almost surely, where we have used  $\partial_x$  to denote the pointwise (partial) derivative. It follows that  $F(u)$  has the same form, namely

$$F(u(x, t)) = \sum_j g_j(t) H(x - x_j(t)) + f_c(x, t), \quad \text{a.e. } x \in \Omega, \quad (4.9)$$

where  $g_j(t)$  is the jump in  $F(u)$  at  $x_j(t)$ ,

$$g_j(t) = F(u(x_j(t)+, t)) - F(u(x_j(t)-, t)),$$

and  $f_c$  is given by

$$f_c(x, t) = F\left(u_c(x, t) + \sum_{x_j \leq x} v_j(t)\right) - \sum_{x_j \leq x} g_j(t),$$

so also  $f_c(\cdot, t) \in W^{1,1}(\mathbb{R})$ . From (4.9), we calculate the distributional derivative

$$\mathbb{D}_x F(u) = \sum_j g_j(t) \delta_{x_j(t)} + \partial_x f_c(\cdot, t) \in M^n(\Omega), \quad (4.10)$$

where  $\partial_x f_c(\cdot, t)$  is defined  $\lambda$ -a.e., and in fact, for almost all  $x \neq x_j(t)$ ,

$$\partial_x f_c(x, t) = DF(u) \partial_x u_c(x, t) = DF(u) \partial_x u(x, t) \in L^1(\mathbb{R}),$$

where  $DF(u)$  is the derivative of  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

We now make the further assumption that  $u_c(x, \cdot)$ ,  $v_j$  and  $x_j$  are absolutely continuous functions of  $t$ , a.e  $x \in \Omega$ . Thus for a full measure set of  $t$ , say  $t \in E$ , the weak derivative  $\mathbb{D}_t u$  can be calculated as a measure, and we have

$$\mathbb{D}_t u = \sum_j v_j(t) \frac{d(-x_j)}{dt} \delta_{x_j(t)} + \sum_j \frac{dv_j}{dt} H(x - x_j(t)) + \partial_t u_c(x, t) \in M^n(\Omega). \quad (4.11)$$

Since  $u$  is assumed to be a weak\* solution, it follows that its G-weak derivative  $u' \in M^n(\Omega)$ , and  $u' = \mathbb{D}_t u$  whenever  $\mathbb{D}_t u$  exists as a measure, and therefore that for  $t \in E$ ,

$$\mathbb{D}_t u + \mathbb{D}_x F(u) = 0 \in M^n(\Omega).$$

For two measures to be equal their atomic parts must coincide, and comparing (4.10) and (4.11) yields  $v_j(t) \frac{dx_j}{dt} = g_j(t)$ , which is the Rankine-Hugoniot condition,

$$F(u(x_j(t)+, t)) - F(u(x_j(t)-, t)) = x_j'(t) (u(x_j(t)+, t) - u(x_j(t)-, t)), \quad j \in \mathcal{J}. \quad (4.12)$$

Moreover, away from the jump set, the absolutely continuous parts of the measures must agree a.e., so that for a.e.  $x \neq x_j(t)$ ,

$$\sum_{x_j < x} \frac{dv_j}{dt} + \partial_t u_c(x, t) + \partial_x f_c(x, t) = 0,$$

and using (4.8), this yields

$$\partial_t u(x, t) + DF(u) \partial_x u(x, t) = 0, \quad \text{a.e. } x \neq x_j, \quad (4.13)$$

so that, as is well known, the absolutely continuous part of the solution satisfies the quasilinear form of the equation almost everywhere. Note that the jump conditions (4.12) are trivially satisfied at any point of continuity of  $u$ .

Having confirmed that discontinuities in weak\* solutions satisfy the Rankine-Hugoniot condition (4.12), we must now give an entropy selection criterion which will ensure uniqueness of solutions with shocks. For abstract systems that are genuinely nonlinear and/or linearly degenerate, there are two such conditions, namely the Lax entropy condition and that obtained from a convex entropy/flux pair. Here we give the appropriate statement of these entropy conditions for weak\* solutions.

Recall that the Lax entropy condition holds for a single isolated jump associated with a specific wave family. We can rewrite (4.12) for a single isolated jump located at  $x = \xi(t)$  as

$$F(u_+) - F(u_-) = \xi'(t) (u_+ - u_-), \quad \text{where } u_{\pm} = u(\xi(t) \pm, t), \quad (4.14)$$

which is an eigenvalue problem, with  $n$  different solutions, one for each wave family. The wave families are in turn distinguished by the associated nonlinear wave speed  $\lambda_k$ , the  $k$ -th eigenvalue of the matrix  $DF(u)$ . Lax's entropy criterion for a  $k$ -shock is then usually written as

$$\lambda_k(u_-) > \xi'(t) > \lambda_k(u_+), \quad (4.15)$$

so the associated  $k$ -th characteristics impinge on both sides of the shock. We continue to use this condition for isolated shocks in weak\* solutions, for which the left and right limits are well-defined functions.

On the other hand, if we are given a  $C^1$  entropy/flux pair  $(\eta, q)$ , we require that the map

$$t \mapsto \eta(u(\cdot, t)) \in W_{w*}^{1,1}(0, T; M_{loc}),$$

and the usual entropy condition holds, namely

$$\mu_{\eta} := \eta(u)' + \mathbb{D}_x q(u) \leq 0 \quad \text{in } M. \quad (4.16)$$

This can be interpreted as a measure, since every signed distribution is representable as a signed measure. Note that since  $\eta$  and  $q$  are  $C^1$  functions, use of (4.13) at points of absolute continuity of  $u(x, t)$  means that the measure  $\mu_{\eta}$  vanishes at those points, so that  $\mu_{\eta}$  is supported on the jump set of the solution. We note that the requirement that  $\eta \in W_{w*}^{1,1}(0, T; M_{loc})$  is stronger than the usual assumption that the sum  $\eta_t + q_x = \text{div}(\eta, q) \leq 0$  in  $M$ , because we require each term to be a measure separately.



### 4.3 The Riemann Problem

As an illustrative example, we describe the well-known solution of the Riemann problem as a weak\* solution. Recall that the solution of the Riemann problem consists of constant states separated by  $n$  centered elementary waves, these being (centered) shocks or rarefactions for genuinely nonlinear families, and (centered) contact discontinuities for linearly degenerate families.

Recall that a centered  $k$ -shock consists of two constant states  $u_- = u_{k-1}$  and  $u_+ = u_k$  satisfying (4.14) and (4.15), which has constant shock speed  $\xi'(t) = \sigma_k(u_{k-1}, u_k)$ , so that the position of the shock at time  $t$  is given by  $\xi(t) = \sigma_k(u_{k-1}, u_k) t$ . The centered  $k$ -shock can thus be written as

$$u(x, t) = u_{k-1} + (u_k - u_{k-1}) H(x - \sigma_k(u_{k-1}, u_k) t). \quad (4.17)$$

A centered rarefaction wave is a solution of the form  $u(x, t) = w(x/t)$ , which satisfies (4.13), which reduces to

$$-\varepsilon \frac{dw}{d\varepsilon} + DF(w(\varepsilon)) \frac{dw}{d\varepsilon} = 0, \quad \varepsilon = x/t.$$

This is an eigenvalue problem, and the centered  $k$ -rarefaction wave is  $u(x, t) = w_k(\varepsilon)$ , corresponding to the  $k$ -th eigenpair,

$$\frac{dw_k}{d\varepsilon} = r_k(\varepsilon) \quad \text{and} \quad \varepsilon = \frac{x}{t} = \lambda_k(w_k(\varepsilon)), \quad (4.18)$$

and such that  $\lambda_k(w_k(\varepsilon))$  is monotone increasing. The  $k$ -rarefaction between states  $u_{k-1}$  and  $u_k$  is then described by

$$u(x, t) = \begin{cases} u_{k-1}, & x/t \leq \lambda_k(u_{k-1}), \\ w_k(\varepsilon), & \lambda_k(u_{k-1}) \leq x/t = \varepsilon \leq \lambda_k(u_k), \\ u_k, & \lambda_k(u_k) \leq x/t. \end{cases} \quad (4.19)$$

If the  $k$ -th family is linearly degenerate, that is  $r_k \cdot \nabla \lambda_k \equiv 0$ , then the wave speed does not change across a  $k$ -wave, and both (4.17) and (4.19) degenerate and coincide, and the  $k$ -contact discontinuity is given by

$$u(x, t) = u_{k-1} + (u_k - u_{k-1}) H(x - \lambda_k(u_k) t), \quad \lambda_k(u_k) = \lambda_k(u_{k-1}). \quad (4.20)$$

The general Riemann problem with data  $u^0(x) = u_L + (u_R - u_L) H(x)$  is solved by finding (unique) states  $\{u_\ell\}_{\ell=0}^n$ , with  $u^0 = u_L$  and  $u_n = u_R$ , such that each  $u_{k-1}$  is connected to  $u_k$  by a centered  $k$ -wave. We describe this as a weak\* solution by explicitly calculating the appropriate derivatives. It suffices to consider the waves separately.

We first consider a  $k$ -shock satisfying (4.17). Clearly

$$F(u(x, t)) = F(u_{k-1}) + (F(u_k) - F(u_{k-1})) H(x - \sigma_k(u_{k-1}, u_k) t),$$

and differentiating yields

$$u' = \mathbb{D}_t u = -(u_k - u_{k-1}) \sigma_k \delta_{\sigma_k t} \quad \text{and} \quad \mathbb{D}_x F(u) = (F(u_k) - F(u_{k-1})) \delta_{\sigma_k t},$$

where  $u'$ , being strongly measurable in  $M_{loc}^n$ , is the B-weak derivative of  $u$ . The jump condition (4.14) implies (4.5), and  $u'$  is clearly bounded,

$$|\langle u', \alpha \rangle| = |u_k - u_{k-1}| |\sigma_k| \|\alpha\|, \quad \text{and} \quad \|u'\|_{M^n} = |u_k - u_{k-1}| |\sigma_k|,$$

so  $u \in W^{1,\infty}(0, \infty^-; BV_{loc}^n, M_{loc}^n)$ . The same estimates hold for  $k$ -contacts.

We now describe the  $k$ -rarefaction similarly. For  $\lambda_k(u_{k-1}) \leq x/t \leq \lambda_k(u_k)$ , we have

$$u' = \frac{dw_k}{d\varepsilon} \partial_t \varepsilon \quad \text{and} \quad \mathbb{D}_x F(u) = DF \frac{dw_k}{d\varepsilon} \partial_x \varepsilon,$$

where  $x = \lambda_k(w_k(\varepsilon))t$ , and  $u' = \mathbb{D}_x F(u) = 0$  otherwise. It follows that

$$1 = \dot{\lambda}_k t \partial_x \varepsilon \quad \text{and} \quad 0 = \lambda_k + \dot{\lambda}_k t \partial_t \varepsilon,$$

where  $\dot{\lambda}_k := \frac{d\lambda_k}{d\varepsilon} = \frac{dw_k}{d\varepsilon} \cdot \nabla \lambda_k$ , and we can calculate the action of  $u'$  on a test function  $\alpha$ , by

$$\langle u', \alpha \rangle = \int_{t\lambda_k(u_{k-1})}^{t\lambda_k(u_k)} \frac{dw_k}{d\varepsilon} \frac{-\lambda_k}{t\dot{\lambda}_k} \alpha(x) dx = - \int_{\lambda_k(u_{k-1})}^{\lambda_k(u_k)} \frac{dF}{d\lambda} \alpha(\lambda t) d\lambda,$$

by change of variables, where now  $\lambda$  parameterizes the integral curve. It follows that  $u'$  is bounded,

$$|\langle u', \alpha \rangle| \leq \int_{\lambda_k(u_{k-1})}^{\lambda_k(u_k)} \left| \frac{dF}{d\lambda} \right| d\lambda \|\alpha\|, \quad \text{so that} \quad \|u'\|_{M^n} \leq \mathbb{V}(F),$$

the variation being taken along the integral curve, and again  $u \in W^{1,\infty}(0, \infty^-; BV_{loc}^n, M_{loc}^n)$ .

Having described the simple waves separately, we now combine them into Lax's well-known solution of the general Riemann problem. That is, we identify states  $u_0 = u_L$ ,  $u_1, \dots, u_n = u_R$ , such that each pair  $(u_{k-1}, u_k)$  is joined by a  $k$ -wave, satisfying (4.17) or (4.19) for genuinely nonlinear fields, and (4.20) for linearly degenerate fields. Our assumption of strict hyperbolicity means that for each  $k = 1, \dots, n$ , we have  $\lambda_k(u_k) < \lambda_{k+1}(u_k)$ , so the waves can be consistently pieced together for positive times  $t > 0$ . This means that the calculations above all hold locally, and for  $t > 0$ , the (Bochner) derivative  $u'$  is simply the sum of each of the individual terms.

It remains to state the sense on which the initial data is taken on. Recall that the solution is absolutely continuous in  $M^n$ , and as  $t \rightarrow 0$ , each individual wave converges in  $M^n$  to the limit  $u_{k-1} + (u_k - u_{k-1})H(x)$ , so the full solution converges to  $u_L + (u_R - u_L)H(x)$ , as required. Here, since for each fixed  $t$ ,  $u(\cdot, t) \in BV \subset L_{loc}^1 \subset M_{loc}^n$ , we note that  $\|u(\cdot, t)\|_{M^n} = \|u(\cdot, t)\|_{L^1}$ , so we have continuity in  $L^1$  as a function of  $t$ , namely  $u(\cdot, t) \rightarrow u^0 \in L^1$ . We note that generally, the solution is not continuous at  $t = 0$  in  $BV$ , but only in the larger space  $L^1$ . However, the Riemann solution is continuous as a function of  $t$  in  $BV$  (with constant  $BV$  norm) for all positive times  $t > 0$ .

## A G-weak differentiability

We now show that if a function  $\Psi \in L_{w*}^q(0, T; X^*)$  is bounded by a sufficiently regular integral, then it is G-weakly differentiable.

**Theorem A.1.** *Fix  $\psi_0 \in X^*$  and suppose that  $\Psi \in L_{w*}^q(0, T; X^*)$  is such that for each  $x \in X$ , there are functions  $v_x \in L^1(0, T)$  such that*

$$\langle \Psi(t), x \rangle = \langle \psi_0, x \rangle + \int_0^t v_x(s) ds, \quad \text{a.e. } t \in [0, T], \quad (\text{A.1})$$

*and suppose there is a non-negative  $v \in L^q(0, T)$  such that*

$$|v_x(t)| \leq v(t) \|x\|, \quad \text{a.e. } t \in [0, T]. \quad (\text{A.2})$$

*Then, if  $1 < q \leq \infty$  or if  $X$  is separable and  $q = 1$ , we have  $[\Psi] \in W_{w*}^{1,q}(0, T; X^*)$ .*

*Proof.* For  $p$  given by  $p^{-1} + q^{-1} = 1$ , recalling that  $L^p(0, T; X)^* \simeq L_{w*}^q(0, T; X^*)$ , we construct a bounded linear functional on the space  $L^p(0, T; X)$  which is a G-weak derivative of  $[\Psi]$ .

We begin by observing that  $v_x$  is almost linear in  $x$ ; that is, for fixed  $x_1, x_2 \in X$  and scalars  $\alpha_1, \alpha_2$ , we have

$$\int_0^t v_{\alpha_1 x_1 + \alpha_2 x_2}(s) ds = \int_0^t \alpha_1 v_{x_1}(s) + \alpha_2 v_{x_2}(s) ds, \quad \text{a.e. } t,$$

and so absolute continuity of the integral implies  $v_{\alpha_1 x_1 + \alpha_2 x_2}(t) = \alpha_1 v_{x_1}(t) + \alpha_2 v_{x_2}(t)$ , a.e.  $t$ .

First, assume  $q > 1$  and define the functional  $\Gamma$  on simple functions  $h : [0, T] \rightarrow X$ , as follows. For  $E_n$  measurable and  $h(t) = \sum_{n=1}^N x_n \chi_{E_n}(t) \in X$ , we set

$$\langle \Gamma, h \rangle := \sum_{n=1}^N \int_{E_n} v_{x_n}(t) dt. \quad (\text{A.3})$$

It is easy to verify that  $\langle \Gamma, h \rangle$  is independent of the representation of  $h$  as a simple function, and

$$\langle \Gamma, \alpha_1 h_1 + \alpha_2 h_2 \rangle = \alpha_1 \langle \Gamma, h_1 \rangle + \alpha_2 \langle \Gamma, h_2 \rangle, \quad (\text{A.4})$$

so that  $\Gamma$  is a linear functional on the space of simple functions.

Recall that for simple  $h(t) = \sum_{n=1}^N x_n \chi_{E_n}(t)$  we can take the  $E_n$  disjoint, in which case

$$\|h\|_{L^p(0, T; X)} = \left( \int_0^T \left\| \sum_n x_n \chi_{E_n}(t) \right\|^p dt \right)^{1/p} = \left( \sum_n \|x_n\|^p \lambda(E_n) \right)^{1/p},$$

and for such  $h$ , we have

$$\begin{aligned} |\langle \Gamma, h \rangle| &\leq \sum_{n=1}^N \|x_n\| \int_{E_n} v(t) dt = \int_0^T v(t) \left( \sum_{n=1}^N \|x_n\| \chi_{E_n}(t) \right) dt \\ &\leq \|v\|_{L^q(0, T)} \left( \sum_{n=1}^N \|x_n\|^p \lambda(E_n) \right)^{1/p} = \|v\|_{L^q(0, T)} \|h\|_{L^p(0, T; X)}, \end{aligned} \quad (\text{A.5})$$

where we have used (A.2) and Young's inequality.

If  $f \in L^p(0, T; X)$ , take a sequence  $\{h_n\}_{n \geq 1}$  such that  $\|f - h_n\|_{L^p(0, T; X)} \rightarrow 0$ . By (A.5), the sequence  $\{\langle \Gamma, h_n \rangle\}_{n \geq 1}$  is Cauchy and hence converges, so we set  $\langle \Gamma, f \rangle := \lim_{n \rightarrow \infty} \langle \Gamma, h_n \rangle$ , this limit being independent of the sequence  $h_n$ . Using (A.4), it follows that  $\Gamma$  is linear, and from (A.5) it is bounded,

$$|\langle \Gamma, f \rangle| \leq \|v\|_{L^q(0, T)} \lim_{n \rightarrow \infty} \|h_n\|_{L^p(0, T; X)} = \|v\|_{L^q(0, T)} \|f\|_{L^p(0, T; X)},$$

so that  $\Gamma \in L^p(0, T; X)^*$ . By Theorem 3.3, there exists  $[\Phi] \in L_{w*}^q(0, T; X^*)$  such that

$$\langle \Gamma, f \rangle = \int_0^T \langle \Phi(s), f(s) \rangle ds \quad \text{for all } f \in L^p(0, T; X).$$

Define  $\bar{\Psi} : [0, T] \rightarrow X^*$  by  $\bar{\Psi}(t) = \psi_0 + \star_0^t \Phi(s) ds$ , so that  $[\bar{\Psi}] \in W_{w*}^{1, q}(0, T; X^*)$ . Then, recalling (A.1) and (A.3), we have for each  $x \in X$

$$\begin{aligned} \langle \bar{\Psi}(t), x \rangle &= \langle \psi_0, x \rangle + \int_0^t \langle \Phi(s), x \chi_{[0, t]}(s) \rangle ds = \langle \psi_0, x \rangle + \langle \Gamma, x \chi_{[0, t]} \rangle \\ &= \langle \psi_0, x \rangle + \int_0^t v_x(s) ds = \langle \Psi(t), x \rangle, \quad \text{a.e. } t \in [0, T], \end{aligned}$$

so that  $\Psi \in [\bar{\Psi}]$  and thus  $\Psi \in W_{w*}^{1,q}(0, T; X^*)$ , completing the proof for  $q > 1$ .

Now set  $q = 1$  and suppose that  $\{x_n\}_{n \geq 1}$  is dense in  $X$ . Then the set

$$E = \bigcup_{n \geq 1} \left\{ t \in (0, T) : \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} v_{x_n}(s) ds = v_{x_n}(t), \quad |v_{x_n}(t)| \leq v(t) \|x_n\| \right\}$$

has full measure. For  $t_0 \in E$ , define  $\Phi(t_0)$  by  $\langle \Phi(t_0), x_n \rangle = v_{x_n}(t_0)$  for each  $n$ . Then, as above, we check that  $\Phi(t_0)$  is a bounded linear functional on the linear span of the set  $\{x_n\}_{n \geq 1}$ , with norm bounded by  $v(t_0)$ . This can be extended by continuity to a linear functional on all of  $X$ , with norm  $\|\Phi(t_0)\|_{X^*} \leq v(t_0)$ . Setting  $\Phi(t_0) = 0$  for  $t_0 \notin E$ , we obtain a mapping  $\Phi : [0, T] \rightarrow X^*$  such that  $\langle \Phi(t), x_n \rangle = v_{x_n}(t)$  a.e.  $t \in [0, T]$  for each  $n$ , and  $\lim_{n \rightarrow \infty} \langle \Phi(t), x_n \rangle = \langle \Phi(t), x \rangle$  for each  $t$ . This implies that  $\Phi$  is weak\* measurable, and  $\|\Phi(t)\| \leq v(t)$  implies that  $[\Phi] \in L_{w*}^1(0, T; X^*)$ . Again define  $\bar{\Psi} : [0, T] \rightarrow X^*$  by  $\bar{\Psi} = \psi_0 + \star \int_0^t \Phi(s) ds$ , so that  $[\bar{\Psi}] \in W^{1,1}(0, T; X^*)$ .

For fixed  $x \in X$ , the set

$$A_x = \left\{ t : \langle \Psi(t), z \rangle = \langle \psi_0, z \rangle + \int_0^t v_z(s) ds, \quad \text{for } z = x \text{ and each } x_n \right\}$$

has full measure. For  $t \in A_x$  and any  $n$ , we have  $\langle \bar{\Psi}(t), x_n \rangle = \langle \Psi(t), x_n \rangle$ , so that

$$\begin{aligned} |\langle \bar{\Psi}(t) - \Psi(t), x \rangle| &= |\langle \bar{\Psi}(t), x - x_n \rangle - \langle \Psi(t), x - x_n \rangle| \\ &\leq \|\bar{\Psi}(t)\| \|x - x_n\| + \|\Psi(t)\| \|x - x_n\|, \end{aligned}$$

so that  $\langle \bar{\Psi}(t), x \rangle = \langle \Psi(t), x \rangle$  for all  $t \in A_x$ . Since  $x \in X$  is arbitrary,  $\Psi$  and  $\bar{\Psi}$  are weak\*-equivalent and so  $\Psi \in [\bar{\Psi}] \in W_{w*}^{1,1}(0, T; X^*)$ .  $\square$

## A.1 Relation to Brezis' space

Given a Banach space  $X$  and  $1 \leq p \leq \infty$ , Brezis defined the space  $\widetilde{W}^{1,p}(0, T; X)$  to be the set of all absolutely continuous functions  $\Psi : [0, T] \rightarrow X$  for which the total variation function  $t \rightarrow \mathbb{V}_\Psi(t)$  is absolutely continuous on  $[0, T]$ , and such that its pointwise derivative  $\frac{d\mathbb{V}_\Psi}{dt} \in L^p(0, T)$ . In particular, for  $\Psi \in \widetilde{W}^{1,p}(0, T; X)$ , for all  $0 \leq s < t \leq T$ , we have

$$\|\Psi(t) - \Psi(s)\| \leq \mathbb{V}_\Psi(t) - \mathbb{V}_\Psi(s) = \int_s^t \frac{d}{d\tau} \mathbb{V}_\Psi(\tau) d\tau. \quad (\text{A.6})$$

Brezis proved that  $W^{1,p}(0, T; X) \subset \widetilde{W}^{1,p}(0, T; X)$ , and if  $X$  is reflexive, the two spaces coincide,  $W^{1,p}(0, T; X) = \widetilde{W}^{1,p}(0, T; X)$  [3].

We see from Theorems 3.6 and 3.4 that each  $\Psi \in W_{w*}^{1,q}(0, T; X^*)$  has an absolutely continuous representative  $\bar{\Psi}$ , which in turn satisfies  $\bar{\Psi} \in \widetilde{W}^{1,q}(0, T; X^*)$ . We now show that the converse is also true, that is, if  $\Psi \in \widetilde{W}^{1,q}(0, T; X^*)$ , then  $[\Psi] \in W_{w*}^{1,q}(0, T; X^*)$ .

**Theorem A.2.** *For  $1 < q \leq \infty$ , the canonical mapping  $\widetilde{W}^{1,q}(0, T; X^*) \rightarrow W_{w*}^{1,q}(0, T; X)$ , given by  $\Psi \mapsto [\Psi]$ , is injective, onto and norm-preserving. The same conclusion holds for  $q = 1$  provided  $X$  is separable.*

*Proof.* Suppose that  $\Psi \in \widetilde{W}^{1,q}(0, T; X^*)$  and fix  $x \in X$ . Then the numerical function

$$z_x(t) := \langle \Psi(t), x \rangle : [0, T] \rightarrow \mathbb{R}$$

is absolutely continuous, so its pointwise derivative  $\frac{dz_x}{dt}$  is defined  $\lambda$ -a.e., belongs to  $L^1(0, T)$ , and

$$\langle \Psi(t), x \rangle = \langle \Psi(s), x \rangle + \int_s^t \frac{dz_x}{dt}(\tau) d\tau \quad \text{for all } 0 \leq s < t \leq T.$$

Now, using (A.6), we have for all  $0 \leq s < t \leq T$ ,

$$\left| \int_s^t \frac{dz_x}{dt}(\tau) d\tau \right| = |\langle \Psi(t) - \Psi(s), x \rangle| \leq \|x\| \int_s^t \frac{d}{dt} \mathbb{V}_\Psi(\tau) d\tau,$$

which in turn implies

$$\left| \frac{dz_x}{dt}(t) \right| \leq \|x\| \frac{d}{dt} \mathbb{V}_\Psi(t) \quad \text{a.e. } t \in [0, T].$$

Recalling that  $\frac{d}{dt} \mathbb{V}_\Psi(\cdot) \in L^q(0, T)$ , the result now follows from Theorem A.1.  $\square$

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