KENV

V. Fedorov (mailto:fuodorov1998@gmail.com), D. Nikiforov (mailto:nikdanila@bk.ru), A. Petrenko (http://www.inp.nsk.su/~petrenko/), (Novosibirsk, 2019)

Theory

Maxwell Equations

Bulk current density of the beam $\jmath = \rho \upsilon$, where ρ is the bulk charge density, υ is the beam velocity. We write the differential equations of Maxwell:

$$\nabla \vec{D} = 4\pi\rho,$$
$$\nabla \times \vec{H} = \frac{4\pi\vec{\jmath}}{c},$$

where $\vec{D}$ is the induction of the electric field, $\vec{H}$ is the magnetic field strength, c is the speed of light. We use the Stokes theorem on the integration of differential forms to obtain Maxwell's equations in integral form:

$$\oint_{\eth V} \vec{D}\overrightarrow{dS} = 4\pi \int_V \rho dV,$$
$$\oint_{\eth S} \vec{H}\overrightarrow{dl} = \frac{4\pi}{c} \int_S \vec{\jmath}\overrightarrow{dS}.$$

Find D_r for a cylindrical beam of radius a with constant density ρ_0:

$$D_r = \frac{4\pi}{r} \int_0^r \rho(\xi)\xi d\xi = \begin{cases} 2\pi\rho_0 r, r < a, \\ \dfrac{2\pi\rho_0 a^2}{r}, r > a. \end{cases}$$

Given that in a vacuum $D = E$, $H = B$, E - electric field strength, B - magnetic field induction, and in flat space in a Cartesian coordinate system $H_\alpha = \beta D_r$, where $\beta = \frac{\upsilon}{c}$, the radial component of the force F_r from the Lorentz force $\vec{F} = e\vec{E} + \frac{e}{c}\vec{\upsilon} \times \vec{B}$:

$$F_r = eE_r - \frac{e\upsilon_z B_\alpha}{c} = eE_r(1 - \frac{\upsilon^2}{c^2}) = \frac{eE_r}{\gamma^2}.$$

It is useful to express the field through the current $I = \rho_0 \upsilon \pi a^2$ then:

$$E_r = \begin{cases} \dfrac{2Ir}{a^2\upsilon}, r < a, \\ \dfrac{2I}{r\upsilon}, r > a. \end{cases}$$

Equations of motion

Newton's second law $\dot{p}_r = F_r$, we use the paraxial approximation and consider $\gamma = const$:

$$\gamma m\ddot{r} = \frac{eE_r}{\gamma^2} = \frac{2Ie}{\gamma^2 a^2 v} r,$$

we got a linear equation, but since all the particles are moving, we need to take into account that $a = a(t)$. The solution of the linear equation can be represented as a linear transformation of the phase plane. Since a segment on the phase plane under a non-degenerate linear transformation goes into a segment, it can be characterized by a single point. Therefore, we choose the extreme point $r = a$, which will characterize the extreme trajectory:

$$\gamma m\ddot{r} = \frac{2Ie}{\gamma^2 av}.$$

We proceed to differentiation with respect to z, taking into account that $dt = \dfrac{dz}{v}$, then:

$$a'' = \frac{e}{a}\frac{2I}{m\gamma^3 v^3}.$$

We introduce the characteristic Alfven current $I_a = \dfrac{mc^3}{e} \approx$ 17 kA, therefore:

$$a'' = \frac{2I}{I_a(\beta\gamma)^3}\frac{1}{a}.$$

We take into account external focusing, assuming a superposition of fields (it is not always true, for example, in nonlinear media this is not true), we obtain:

$$a'' + k(z)a - \frac{2I}{I_a(\beta\gamma)^3}\frac{1}{a} = 0,$$

which resembles the equation of the envelope:

$$w'' + kw - \frac{1}{w^3} = 0,$$

where $w = \sqrt{\beta}$.

Envelope equations for an elliptical beam with Kapchinsky-Vladimirsky distribution with external focusing by linear fields

Kapchinsky-Vladimirsky distribution:

$$f = A\delta(1 - \frac{\beta_x x'^2 + 2\alpha_x x x' + \gamma_x x^2}{\epsilon_x} - \frac{\beta_y y'^2 + 2\alpha_y y y' + \gamma_y y^2}{\epsilon_y}),$$

where A is the Courant-Snyder invariant. Axes of the ellipse:

$$a = \sqrt{\epsilon_x \beta_x}, b = \sqrt{\epsilon_y \beta_y}.$$

The field is obtained linearly inside the charged elliptical cylinder:

$$E_x = \frac{4I}{v}\frac{x}{a(a+b)},$$
$$E_y = \frac{4I}{v}\frac{y}{b(a+b)}.$$

Check that $\nabla \vec{E} = 4\pi\rho$:

$$I = \rho v \pi ab,$$
$$\nabla \vec{E} = \frac{4I(a+b)}{\pi(a+b)ab} = \frac{4I}{\pi ab} = 4\pi\rho.$$

Since the fields are linear, they are added to the fields of the focusing lens. Substitute $a = \sqrt{\epsilon}_x w_x, b = \sqrt{\epsilon}_y w_y$: in the envelope equation

$$a'' + k_{xt} a - \frac{\epsilon_x^2}{a^3} = 0,$$

where $k_{xt} = k_x + k_{xsc}$ is the total stiffness, k_x is the stiffness of the lens, and $k_{xsc} = \frac{4I}{I_a(\beta\gamma)^3}\frac{1}{a(a+b)}$. As a result, we obtain a system of equations connected through a space charge:

$$\begin{cases} a'' + k_x a - \frac{4I}{I_a(\beta\gamma)^3}\frac{1}{(a+b)} - \frac{\epsilon_x^2}{a^3} = 0, \\ b'' + k_y b - \frac{4I}{I_a(\beta\gamma)^3}\frac{1}{(a+b)} - \frac{\epsilon_y^2}{b^3} = 0. \end{cases}$$

Quantitative criterion for the applicability of the flow laminarity approximation

This system of equations makes it possible to take into account 2 effects that interfere with focusing the beam at a point — the finiteness of the emmittance and the space charge. Members work the same way, so you can compare these values.

When the current I is small - weak repulsion, if the current I is large - strong repulsion, therefore, the emmittance can be discarded and the flow can be considered laminar. Obviously, a quantitative criterion for the applicability of laminar flow is as follows:

$$\sqrt{\frac{2I}{I_a(\beta\gamma)^3}} \gg \sqrt{\frac{\epsilon}{\beta}}.$$

It can be seen that the space charge affects the envelope more where the β function is greater, and in the vicinity of the focus, the influence of the space charge can be neglected.

Bush's Theorem

As an addition to the derivation of the equation of the envelope of the beam, we prove an important auxiliary theorem called the Bush theorem. It connects the angular velocity of a charged particle moving in an axially symmetric magnetic field with a magnetic flux enveloped in a circle centered on the axis and passing through the point at which the particle is located.

Consider the charge q moving in a magnetic field $\vec{B} = (B_r, 0, B_z)$. We equate the θ component of the Lorentz force with the time derivative of the momentum divided by r:

$$F_\theta = -q(\ddot{r} B_z - \dot{z} B_r) = \frac{d}{rdt}(\gamma m r^2 \dot{\theta}).$$

The flow penetrating the area covered by a circle of radius r, the center of which is located on the axis, and it passes through the point at which the charge is located, is written in the form $\psi = \int\limits_0^r 2\pi r B_z dr$. When a particle moves by $\overrightarrow{dl} = (dr, dz)$, the rate of change of the flow covered by this circle can be found from the second Maxwell equation $\nabla \vec{B} = 0$. Thus,

$$\dot{\psi} = 2\pi r(-B_r \dot{z} + B_z \dot{r}).$$

After integration over time, from the above equations we obtain the following expression:

$$\dot{\theta} = (-\frac{q}{2\pi\gamma m r^2})(\psi - \psi_0).$$

Paraxial Ray Equation

Here we write the equation of the paraxial ray in the form corresponding to a system with axial symmetry under the assumptions made earlier. To derive the equation of the paraxial ray, we equate the force of radial acceleration to the forces of electric and magnetic from the side of external fields. It must be remembered that the value $\gamma = \gamma(t)$.

$$\frac{d}{dt}(\gamma m \dot{r}) - \gamma m r(\dot{\theta})^2 = q(E_r + r\dot{\theta} B_z).$$

We apply the Bush theorem and the independence of B_z from r :

$$-\dot{\theta} = \frac{q}{2\gamma m}(B_z - \frac{\psi_0}{\pi r}).$$

Exclude $\dot{\theta}$ and substitute $\dot{\gamma} \approx \frac{\beta q E_z}{mc}$:

$$\ddot{r} + \frac{\beta q E_z}{\gamma m c}\dot{r} + \frac{q^2 B_z^2}{4\gamma^2 m^2} r - \frac{q^2 \psi_0^2}{4\pi^2 \gamma^2 m^2}(\frac{1}{r^3}) - \frac{q E_r}{\gamma m} = 0.$$

Envelope equation for round and elliptical beams

Given that:

$$\dot{r} = \beta c r',$$
$$\ddot{r} = r''(\dot{z})^2 + r'\ddot{z} \approx r''\beta^2 c^2 + r'\beta'\beta c^2.$$

And also, if there are no charges in the beam region, then expanding into a Taylor series in the vicinity of the axis and leaving only the first term, taking

$$\nabla \vec{E} = 0$$
$$E_r = -0.5 r E_z' \approx -0.5 r \gamma'' mc^2/q.$$

Then we can finally write the envelope equation for a round beam of radius r with a Kapchinsky-Vladimirsky distribution with external focusing by linear fields:

$$r'' + \frac{1}{\beta^2\gamma}\gamma' r' + \frac{1}{2\beta^2\gamma}\gamma'' r + kr - \frac{2I}{I_a(\beta\gamma)^3}\frac{1}{r} - \frac{\epsilon^2}{r^3} = 0;$$

and for an elliptical beam:

$$\begin{cases} a'' + \dfrac{1}{\beta^2\gamma}\gamma' a' + \dfrac{1}{2\beta^2\gamma}\gamma'' a + k_x a - \dfrac{4I}{I_a(\beta\gamma)^3}\dfrac{1}{(a+b)} - \dfrac{\epsilon_x^2}{a^3} = 0, \\ b'' + \dfrac{1}{\beta^2\gamma}\gamma' b' + \dfrac{1}{2\beta^2\gamma}\gamma'' b + k_y b - \dfrac{4I}{I_a(\beta\gamma)^3}\dfrac{1}{(a+b)} - \dfrac{\epsilon_y^2}{b^3} = 0. \end{cases}$$

Magnetic Lenses

The focusing elements can be: solenoids and magnetic quadrupole lenses.

Solenoids

$k_x = k_y = k_s$ - stiffness of the solenoid:

$$k_s = \left(\frac{eB_z}{2m_e c\beta\gamma}\right)^2 = \left(\frac{eB_z}{2\beta\gamma \cdot 0.511 \cdot 10^6 e \cdot \mathrm{volt}/c}\right)^2 = \left(\frac{cB_z[\mathrm{T}]}{2\beta\gamma \cdot 0.511 \cdot 10^6 \cdot \mathrm{volt}}\right)^2.$$

Quadrupoles

$k_q = \dfrac{eG}{pc}$ is the quadrupole stiffness, where $G = \dfrac{\partial B_x}{\partial y} = \dfrac{\partial B_y}{\partial x}$ is the magnetic field gradient, and $k_x = k_q, k_y = -k_q$.

$$k_q = \left(\frac{eG}{m_e c\beta\gamma}\right) = \left(\frac{eG}{\beta\gamma \cdot 0.511 \cdot 10^6 e \cdot \mathrm{volt}/c}\right) = \left(\frac{cG}{\beta\gamma \cdot 0.511 \cdot 10^6 \cdot \mathrm{volt}}\right).$$

Longitudinal beam dynamics

The equation for the longitudinal dynamics of the beam can be solved independently of the equation for the envelope, so that the ready-made function of the beam energy of z is already used in the equation for the envelope. Assuming that the electron velocity is close enough to the speed of light and therefore its longitudinal coordinate is $z \approx ct$, and the momentum is $p_z \approx \gamma mc$

$$\frac{d\gamma}{dz} \approx \frac{eE_z}{mc^2},$$

Then it is enough to integrate the function $E_z(z)$ once.

Solution of the envelope equation for an elliptical beam with focusing elements

The envelope equation for an elliptical beam with half axes a, b with the Kapchinsky-Vladimirsky distribution with external focusing by linear fields:

$$\begin{cases} a'' + \dfrac{1}{\beta^2\gamma}\gamma' a' + \dfrac{1}{2\beta^2\gamma}\gamma'' a + ka - \dfrac{2P}{(a+b)} - \dfrac{\epsilon_x^2}{a^3} = 0, \\ b'' + \dfrac{1}{\beta^2\gamma}\gamma' b' + \dfrac{1}{2\beta^2\gamma}\gamma'' b - kb - \dfrac{2P}{(a+b)} - \dfrac{\epsilon_y^2}{b^3} = 0. \end{cases}$$

Let $x = \dfrac{da}{dz}, y = \dfrac{db}{dz}, \dfrac{d\gamma}{dz} \approx \dfrac{eE_z}{mc^2}$, then:

$$\begin{cases} \dfrac{dx}{dz} = -\dfrac{1}{\beta^2\gamma}\gamma' a' - \dfrac{1}{2\beta^2\gamma}\gamma' a - ka + \dfrac{2P}{(a+b)} + \dfrac{\epsilon_x^2}{a^3} \\ \dfrac{da}{dz} = x \\ \dfrac{dy}{dz} = -\dfrac{1}{\beta^2\gamma}\gamma' b' - \dfrac{1}{2\beta^2\gamma}\gamma'' b + kb + \dfrac{2P}{(a+b)} + \dfrac{\epsilon_x^2}{b^3} \\ \dfrac{db}{dz} = y \end{cases}$$

Let $\vec{X} = \begin{bmatrix} x \\ a \\ y \\ b \end{bmatrix}$, we now compose the differential equation $X' = F(X)$.

Trajectory equation for one particle

The vector potential in the solenoid has only one component and is easily expressed in terms of $B_z(z)$:

$$A_\phi(z,r) = \sum_0^{inf} \frac{(-1)^n}{n!(N+1)!} B_z^{2n} (\frac{r}{2})^{2n+1/2}.$$

Therefore, the Lagrangian in a cylindrical coordinate system has the form:

$$L = -mc^2(1-\beta^2)^{1/2} + \frac{e}{c} A_\phi r \dot{\phi}.$$

Axial symmetry implies the existence of an integral of motion:

$$P_\phi = \gamma m r^2 \dot{\phi} + \frac{e}{c} r A_\phi = \gamma m r^2 \dot{\phi}_0.$$

Differentiating by z we get:

$$\phi' - \phi'_0 = -\frac{e}{pc}\frac{A_\phi}{r}.$$

And in the paraxial approximation:

$$\phi' - \phi'_0 = -\frac{e}{pc} B_z(z).$$

From:

$$\delta\phi = -\int \frac{e}{2pc} B_z(z) dz.$$

Then the equations of the trajectory in the given coordinates look like this:

$$\begin{cases} \tilde{x}'' + \frac{1}{\beta^2\gamma}\gamma'\tilde{x}' + \frac{1}{2\beta^2\gamma}\gamma''\tilde{x} + k_s\tilde{x} = 0, \\ \tilde{y}'' + \frac{1}{\beta^2\gamma}\gamma'\tilde{y}' + \frac{1}{2\beta^2\gamma}\gamma''\tilde{y} + k_s\tilde{y} = 0, \end{cases}$$

Literature

1. J. Lawson "Physics of charged particle beams" (https://books.google.ru/books/about/The_Physics_of_Charged_particle_Beams.html?id=j87vAAAAMAAJ&redir_esc=y)
2. N.A. Vinokurov "Lectures on electronic optics for accelerating physicists" / Н.А. Винокуров "Лекции по электронной оптике для ускорительных физиков" (http://accel.inp.nsk.su/refs/Elopt24.pdf)
3. A.A. Zholents "On compensation of the influence of the solenoid field by quadrupole lenses" / А.А. Жоленц "О компенсации влияния поля соленоида квадрупольными линзами" (http://www.inp.nsk.su/images/preprint/1981_080.pdf)