HW3

a) $\int \sqrt{\cos 2x} \sin x \, dx = -\int \sqrt{2\cos^2 x - 1} \, d\cos x =$

$= -\int \sqrt{2t^2 - 1} \, dt = -\sqrt{2} \int \sqrt{t^2 - \frac{1}{2}} \, dt =$

$= -\sqrt{2} \left(\int \frac{t^2}{\sqrt{t^2 - \frac{1}{2}}} dt - \frac{1}{2} \int \frac{dt}{\sqrt{t^2 - \frac{1}{2}}} \right) =$

$= -\sqrt{2} \left(\int t \, d\sqrt{t^2 - \frac{1}{2}} - \frac{1}{2} \int \frac{dt}{\sqrt{t^2 - \frac{1}{2}}} \right) =$

$= -\sqrt{2} \left(t\sqrt{t^2 - \frac{1}{2}} - \int \sqrt{t^2 - \frac{1}{2}} \, dt - \frac{1}{2} \int \frac{dt}{\sqrt{t^2 - \frac{1}{2}}} \right) =$

$= -\sqrt{2} t \sqrt{t^2 - \frac{1}{2}} + \sqrt{2} \int \sqrt{t^2 - \frac{1}{2}} dt + \frac{\sqrt{2}}{2} \int \frac{dt}{\sqrt{t^2 - \frac{1}{2}}} =$

$= -t\sqrt{2t^2 - 1} + \int \sqrt{2t^2 - 1} \, dt + \frac{\sqrt{2}}{2} \int \frac{dt}{\sqrt{t^2 - \frac{1}{2}}}$

$\Rightarrow \int \sqrt{\cos 2x} \sin x \, dx = -\int \sqrt{2t^2 - 1} \, dt =$

$= -\frac{1}{2} t \sqrt{2t^2 - 1} + \frac{\sqrt{2}}{4} \int \frac{dt}{\sqrt{t^2 - \frac{1}{2}}} = -\frac{1}{2} t \sqrt{2t^2 - 1} + \frac{\sqrt{2}}{4} \cdot$

$\cdot \ln \left| t + \sqrt{t^2 - \frac{1}{2}} \right| + C = -\frac{1}{2} \cos x \sqrt{\cos 2x} +$

$+ \frac{\sqrt{2}}{4} \ln \left| \cos x + \frac{\sqrt{\cos 2x}}{\sqrt{2}} \right| + C$

b) (из 50 инт-лов)

c) $\triangle$ $I_n^k = \int \frac{dx}{x^n(x-2)^k} = -\frac{1}{k-1}\int \frac{d\frac{1}{(x-2)^{k-1}}}{x^n} =$ $\quad n \geq 2,\ k \geq 2$

$$= -\frac{1}{k-1}\left(\frac{1}{(x-2)^{k-1}x^n} - \int \frac{d\frac{1}{x^n}}{(x-2)^{k-1}}\right) =$$

$$= -\frac{1}{(k-1)(x-2)^{k-1}x^n} - \frac{n}{k-1} I_{n+1}^{k-1}$$

$\triangle$ $I_6^1 = \int \frac{dx}{x^6(x-2)} = \int \frac{1-\frac{1}{2}x+\frac{1}{2}x-\frac{1}{4}x^2+\ldots+\frac{1}{64}x^6}{x^6(x-2)}dx =$

$$= -\frac{1}{2}\int\frac{dx}{x^6} - \frac{1}{4}\int\frac{dx}{x^5} - \ldots - \frac{1}{64}\int\frac{dx}{x} + \frac{1}{64}\int\frac{dx}{x-2}$$

Т.о. из I_6^1 получаем I_5^2, потом из I_5^2 получаем I_4^3 - искомый интеграл

d) $\int \frac{dx}{(x+2)^2(x-3)^2} = \int \frac{A(x+2)^2+B(x-3)^2+C(x+2)(x-3)}{(x+2)^2(x-3)^2}dx =$

$A(x+2)^2 = Ax^2+4Ax+4A$

$B(x-3)^2 = Bx^2-6Bx+9B$

$C(x+2)(x-3) = C(x^2-x-6) = Cx^2-Cx-6C$

$$\begin{cases} A+B+C=0 \\ 4A-6B-C=0 \\ 4A+9B-6C=1 \end{cases} \quad \text{I} + \text{II} \Rightarrow 5A-5B=0 \Rightarrow A=B$$

$$\begin{cases} -2A-C=0 \\ 13A-6C=1 \end{cases} \Rightarrow \begin{cases} 12A+6C=0 \\ 13A-6C=1 \end{cases} \Rightarrow A=B=\frac{1}{25}$$

$$C = -\frac{2}{25}$$

$$= \frac{1}{25}\int\frac{dx}{(x-3)^2} + \frac{1}{25}\int\frac{dx}{(x+2)^2} + \frac{2}{25}\int\frac{dx}{x^2-x-6} =$$

$$= -\frac{1}{25(x-3)} - \frac{1}{25(x+2)} + \frac{2}{25}\int\frac{d(x-\frac{1}{2})}{(x-\frac{1}{2})^2-2{,}5^2} =$$

$$= -\frac{1}{25(x-3)} - \frac{1}{25(x+2)} + \frac{2}{25}\cdot\frac{1}{5}\ln\left|\frac{x-\frac{1}{2}+2{,}5}{x-\frac{1}{2}-2{,}5}\right| + C =$$

$$= -\frac{1}{25(x-3)} - \frac{1}{25(x+2)} + \frac{2}{125}\ln\left|\frac{x+2}{x-3}\right| + C$$

e) $$\int\frac{3x^7+5x^3}{(x^8+4)^2}dx = 3\int\frac{x^7}{(x^8+4)^2}dx + 5\int\frac{x^3dx}{(x^8+4)^2} =$$

$$= \frac{3}{8(x^8+4)} + \frac{5}{4}\int\frac{dx^4}{(x^8+4)^2} = \frac{3}{8(x^8+4)} + \frac{5}{4}\int\frac{dt}{(t^2+4)^2} =$$

$$= \frac{3}{8(x^8+4)} + \frac{5}{16}\left(\int\frac{t^2+4}{(t^2+4)^2}dt - \int\frac{t^2dt}{(t^2+4)^2}\right) =$$

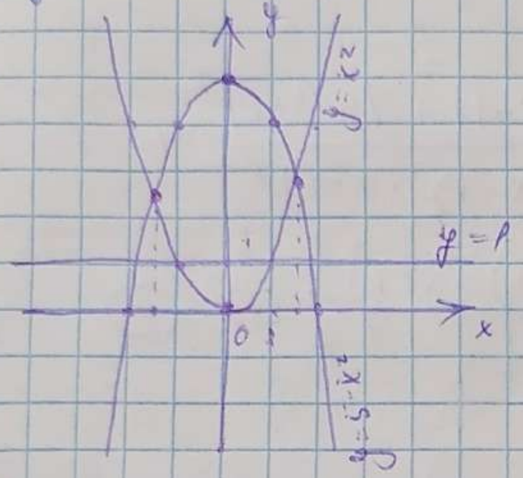
$$= \frac{3}{8(x^8+4)} + \frac{5}{32}\operatorname{arctg}\frac{t}{2} - \frac{5}{16}\int -\frac{1}{2}t\, d\frac{1}{t^2+4} =$$

$$= \frac{3}{8(x^8+4)} + \frac{5}{32}\operatorname{arctg}\frac{t}{2} + \frac{5}{32}\left(\frac{t}{t^2+4} - \int\frac{dt}{t^2+4}\right) =$$

$$= \frac{3}{8(x^8+4)} + \frac{5}{32}\operatorname{arctg}\frac{t}{2} + \frac{5t}{32(t^2+4)} - \frac{5}{64}\operatorname{arctg}\frac{t}{2} + C =$$

$$= \frac{3}{8(x^8+4)} + \frac{5}{64}\,\text{arctg}\,\frac{x^4}{2} + \frac{5x^4}{32(x^8+4)} + C$$

f) $\int \max(5-x^2, 1, x^2)\,dx = \int \max(5-x^2, x^2)\,dx =$



$x^2 = 5 - x^2$

$x = \pm\sqrt{2{,}5}$

Функция $\max(5-x^2, x^2)$ непрерывна, тогда хотим, чтобы $\int \max(5-x^2, x^2)\,dx$ также была непр-а.

$$\int (5-x^2)\,dx = 5x - \frac{x^3}{3} + C_1 \quad (\mathbb{R} \to \mathbb{R}),\ x \in (-\infty; -\sqrt{2{,}5}] \cup [\sqrt{2{,}5}; +\infty)$$

$$\int x^2\,dx = \frac{x^3}{3} + C_2 \quad (\mathbb{R} \to \mathbb{R}),\ x \in [-\sqrt{2{,}5}, \sqrt{2{,}5}]$$

$$\begin{cases} 5\sqrt{2{,}5} - \dfrac{\sqrt{2{,}5}^3}{3} + C_1 = \dfrac{\sqrt{2{,}5}^3}{3} + C_2 \\ 5(-\sqrt{2{,}5}) - \dfrac{(-\sqrt{2{,}5})^3}{3} + C_1' = \dfrac{(-\sqrt{2{,}5})^3}{3} + C_2 \end{cases}$$

$$\begin{cases} \dfrac{10}{3}\sqrt{2{,}5} + C_1 = C_2 \\ -\dfrac{10}{3}\sqrt{2{,}5} + C_1' = C_2 \end{cases}$$

$$\text{Пусть } C_1 = 0 \Rightarrow C_2 = \frac{10\sqrt{2{,}5}}{3} \Rightarrow C_1' = \frac{20\sqrt{2{,}5}}{3}$$

$$\stackrel{\circ}{=} \left(\begin{cases} 5x - \frac{x^3}{3} + \frac{20\sqrt{2,5}}{3}, & x \le -\sqrt{2,5} \\ \frac{x^3}{3} + \frac{10\sqrt{2,5}}{3}, & x \in [-\sqrt{2,5}, \sqrt{2,5}] \\ 5x - \frac{x^3}{3}, & x \ge \sqrt{2,5} \end{cases} \right) + C$$

2) $$\int \frac{dx}{(x-1)^2(x+1)^3} = \int \left(\frac{A(x-1)^2 + B(x+1)^3}{(x-1)^2(x+1)^3} + \right.$$

$$\left. + \frac{C(x-1)(x+1)^2 + D(x-1)^2(x+1)}{(x-1)^2(x+1)^3} \right) dx \stackrel{\circ}{=}$$

$$A(x-1)^2 = A(x^2 - 2x + 1)$$
$$B(x+1)^3 = B(x^3 + 3x^2 + 3x + 1)$$
$$C(x-1)(x+1)^2 = C(x^3 + x^2 - x - 1)$$
$$D(x-1)^2(x+1) = D(x^3 - x^2 - x + 1)$$

$$\begin{cases} B + C + D = 0 \\ A + 3B + C - D = 0 \\ -2A + 3B - C - D = 0 \\ A + B - C + D = 1 \end{cases}$$

$$\begin{cases} A + 4B + 2C = 0 \\ -2A + 4B = 0 \\ A - 2C = 1 \end{cases}$$

$$\begin{cases} 2C = A - 1 \\ 4B = 2A \\ A + 2A + A - 1 = 0 \end{cases}$$

$$\Rightarrow A = \frac{1}{4}, \quad B = \frac{1}{8}, \quad C = -\frac{3}{8}, \quad D = \frac{1}{4}$$

$$\stackrel{\circ}{=} \frac{1}{4}\int \frac{dx}{(x+1)^3} + \frac{1}{8}\int \frac{dx}{(x-1)^2} - \frac{3}{8}\int \frac{dx}{x^2-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} =$$

$$= -\frac{1}{8(x+1)^2} - \frac{1}{8(x-1)} + \frac{3}{16}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{4(x+1)} + C$$