Problems on Optimal and Efficient Mechanisms.

Problem 1. An Optimal Mechanism.

There is a single object for sale and there are two potential buyers. The value assigned by buyer 1 to the object is $v_1 \sim U[0, 1+k]$, while the value of the object for buyer 2 is $v_2 \sim U[0, 1-k]$ independently distributed, where $k \in (0, 1)$.

(a) Calculate the optimal mechanism. That is, describe the optimal allocation as a function of virtual valuations and describe the payment rule. Is the optimal mechanism efficient?

(b) Now assume that there is a third buyer with independent value $V_3 \sim exp(1)$. Describe the optimal allocation. What is the minimum sample value v_3 such that the third buyer gets the object for sure, that is, independently of the values of the first two buyers. Describe the optimal payment rule for the third buyer. Is it bounded?

Problem 2. VCG Example.

Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a; b\}$ to three buyers. A buyer can buy none, one, or both of the objects. For simplicity, assume the valuation function of each buyer depends only on the set of objects assigned to that buyer. The values are:

$$u_1(\varnothing) = 0, u_1(\{a\}) = 10, u_1(\{b\}) = 3, u_1(\{a, b\}) = 13$$
$$u_2(\varnothing) = 0, u_2(\{a\}) = 2, u_2(\{b\}) = 8, u_2(\{a, b\}) = 10$$
$$u_3(\varnothing) = 0, u_3(\{a\}) = 3, u_3(\{b\}) = 2, u_3(\{a, b\}) = 14$$

Since we do not know each buyer's lowest possible value for each product, consider an alternative VCG payment rule: each buyer i pays m_i, which is the maximum welfare of the other buyers minus the realized welfare of the other buyers, both computed using the reported valuation functions.

(a) Determine the assignment of objects to buyers and the payments of the buyers, under truthful bidding.

(b) Discuss why buyer 3 might have an objection to the outcome.

Problem 3. Issues with the VCG Mechanism.

(*Final Exam 2020.*) In this problem, you will explore various issues with the VCG mechanism.

1. Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a; b\}$ to three buyers. A buyer can buy none, one, or both of the objects. The buyers' valuations are:

$$u_1(\varnothing) = 0, u_1(\{a\}) = 9, u_1(\{b\}) = 0, u_1(\{a, b\}) = 9$$
$$u_2(\varnothing) = 0, u_2(\{a\}) = 0, u_2(\{b\}) = 9, u_2(\{a, b\}) = 9$$
$$u_3(\varnothing) = 0, u_3(\{a\}) = 2, u_3(\{b\}) = 3, u_3(\{a, b\}) = 10$$

Consider a version of the VCG payment rule: each buyer i pays m_i, which is the maximum welfare of the other buyers minus the realized welfare of the other buyers given the reported values.

(a) Determine the assignment of objects to buyers and the payments, under truthful bidding. (4 pts.)

(b) Discuss what is the issue with the VCG mechanism in this case. (1 pt.)

2. Consider the same setup but with different valuations:

$$u_1(\varnothing) = 0, u_1(\{a\}) = 3, u_1(\{b\}) = 0, u_1(\{a, b\}) = 3$$
$$u_2(\varnothing) = 0, u_2(\{a\}) = 0, u_2(\{b\}) = 4, u_2(\{a, b\}) = 4$$
$$u_3(\varnothing) = 0, u_3(\{a\}) = 2, u_3(\{b\}) = 3, u_3(\{a, b\}) = 10$$

   (a) Determine the assignment of objects to buyers and the payments, under truthful bidding. (1 pts.)

   (b) This example highlights another flaw of the VCG mechanism. State this flaw and discuss it. (4 pts.)

3. Consider a similar setup but with two buyers and the following valuations:

$$u_1(\varnothing) = 0, u_1(\{a\}) = 1, u_1(\{b\}) = 1, u_1(\{a, b\}) = 10$$
$$u_2(\varnothing) = 0, u_2(\{a\}) = 1, u_2(\{b\}) = 1, u_2(\{a, b\}) = 9$$

   (a) Determine the assignment of objects to buyers and the payments, under truthful bidding. (1 pts.)

   (b) Suppose buyer 2 thinks about entering the system as two different buyers, 2A and 2B. Can it be profitable fir her to do this? Discuss whether you think such splitting schemes are a flaw of the VCG mechanism. Can it be problematic in practice? (4 pts.)

Problem 4. Properties of the VCG Mechanism.

Consider the following buyer/seller framework. Let $x \in [0, 100] = X$ denote the amount of a good that is produced by a seller, which a buyer then consumes. Let $\theta_B, \theta_S \in [1, 2] = \Theta_B = \Theta_S$ denote the types of the buyer and seller, respectively. Types are independently distributed and privately known by each agent. Assume that utility functions are quasilinear in money, and are given by

$$U_B(x, t, \theta) = \theta_B x - t_B$$
$$U_S(x, t, \theta) = -\frac{x^2}{2\theta_S} - t_S$$

where $t = (t_B, t_S)$ denotes a vector of transfers that the agents pay. (If $t_i < 0$, then agent i is receiving a transfer.)

(a) What is the efficient level of production $x(\theta)$? What social surplus is attained by it?

(b) Define a *least charitable type* of agent i to be any type $\underline{\theta}_i \in \Theta_i$ that minimizes the interim expected social surplus, where the expectation is taken over others' types. What are the least charitable types of each agent?

(c) Using the least charitable types as the default types, what are the VCG transfers? What are the buyer's and seller's utilities in the mechanism? Does the mechanism run an ex post budget surplus for all θ? Is the mechanism interim individually rational?

(d) Show directly that, in the direct revelation game with the VCG transfer and allocation functions you computed, it is weakly dominant for each player to truthfully reveal his type. That is, let $U_S(\hat{\theta}_S, \hat{\theta}_B; \theta_S)$ denote the seller's utility when his type is θ_S, he announces $\hat{\theta}_S$, and the buyer announces $\hat{\theta}_B$. Show that $\theta_S \in \arg\max_{\hat{\theta}_S} U_S(\hat{\theta}_S, \hat{\theta}_B; \theta_S)$ for all $\hat{\theta}_B, \theta_S$. Show the similar statement for the buyer. (*Note: Every VCG mechanism is dominant strategy incentive compatible, so this verifies in this special case something that we know in general.*)

Problem 5. Third-Price Auction.

(*Final Exam 2020.*) Consider an auction for a single object among three bidders. The format is a *third price* auction. The high bidder wins and pays the third-highest bid (here the lowest bid because there are three bidders.) If there is a tie, each high bidder will have an equal chance of being selected as the winner. Losers pay nothing. Assume that the bidders' private valuations v_i are independently drawn from the uniform distribution on the unit interval.

(a) Prove that it is a weakly dominated strategy for bidder i to bid strictly less than his true value v_i. (5 pts.)

(b) Prove that it is not a weakly dominant strategy for a bidder i to bid his true value v_i. (5 pts.)

(c) Let's consider a symmetric (interim) Bayesian Nash equilibrium in which a bidder with value v bids according to the strategy $\beta(v)$.

   (i) Let $q(b)$ be the probability, in equilibrium, that a bidder wins when he bids b and $m(b)$ be the expected payment made by a bidder who bids b. Explain why the following is a necessary condition of equilibrium: (2 pts.)

$$vq(b(v)) - m(b(v)) \geq vq(b') - m(b') \; \forall b' < b(v).$$

   (ii) Based on the previous observation prove that the equilibrium bidding strategy β is weakly increasing, i.e, if $v' > v$ then $b(v') \geq b(v)$. (8 pts.)

   (iii) Let's consider a symmetric equilibrium which is in fact strictly increasing. Apply the revenue equivalence theorem to prove that $b(v) = 2v$. (*Whether or not you can complete the calculations, you should describe clearly the steps you would take to verify this and how you would use revenue equivalence.*) (15 pts.)

   - (BONUS QUESTION: What is the optimal bidding strategy in a general case with N bidders?)

Relevant Problems from Sergey Kovbasyuk's Version of the Class

Problem 1. Screening.

A buyer needs to procure a delivery of certain product/service from a supplier. The value of the product for the buyer is $v > 0$ per unit of product, if quantity q is supplied the total value is vq. The buyer pays for the product to the provider an amount T, so that the net payoff of the buyer is $vq - T$.

Providers are heterogeneous and differ in their costs, type θ provider bears a cost $c(q, \theta) = \theta q^2/2$ if he provides quantity q. The cost parameter is his private information. It is drawn from a uniform distribution on the unit interval $[\varepsilon, 1 + \varepsilon]$, $\varepsilon > 0$. Suppliers' outside options are zero.

The buyer attempts to screen suppliers and offers a contract described by pairs T, q, we refer to this contract as a menu T, q. The objective of the buyer is to maximize his expected payoff

$$E[vq - T].$$

(a) How many pairs T, q the buyer can consider offering in order to maximize his payoff? How can this set of pairs be described on T, q plane?

(b) Write down the buyer's optimization problem and the constraints she is facing.

(c) Is the single-crossing condition satisfied? What does it say about possible relationship between q and θ in the optimal contract?

(d) Denote by $W(\theta)$ the profit of a supplier of type θ under the optimal procurement contract, how does $W(\theta)$ change with θ?

(e) Reformulate the buyer's optimization problem using $W(\theta)$.

(f) Find the first order necessary condition for optimal q as a function of θ. Does it satisfy the relationship between q and θ established in question 3?

(g) Characterize the candidate optimal contract(menu) T, q, that is characterize how q and T depend on θ, and how T depends on q. Can this contract be interpreted as quantity discounts (per unit price drops with the quantity) or not?

(h) Suppose the buyer decides to exclude some types of suppliers, what types of suppliers he may want to exclude? Why he might want to do that? Does he do so in the optimal contract? Why?

(i) Compare the quantities bought under the optimal contract, and in the first-best (when types are known). What is the expected welfare loss due to asymmetric information? For simplicity, compute welfare as a sum of all payoffs.

Problem 2. Moral Hazard.

Suppose the principal faces an agent with unknown linear production costs $c(x, \theta) = x\theta$, here θ is the agents private information. The principal offers a payment schedule to the agent $t(x)$, and the agent responds by producing quantity $x(\theta)$. Suppose the principal wants to induce production $x(\theta) = 1/\theta^\gamma$, $\gamma > 1$, what kind of payment schedule $t(x)$ the principal should offer?

Problem 3. Auction and Mechanism Design.

Consider an auction with two participants $i = 1, 2$. The valuation of each participant i is

$$u_i(v_i, v_{-i}) = v_i + \gamma v_{-i}.$$

Here, $\gamma \in (0, 1)$ and v_i is private information of agent i, $v_i \sim U[0, 1]$.

(a) Find the efficient allocation rule q.

(b) Find a direct mechanism which implements the efficient allocation rule as a BNE, and has transfers that depend only on $q(\hat{v}_1, v_2)$ and the report of the other agent, that is $m_i(q, \hat{v}_{-i})$.

(c) Can the efficient allocation be implemented in dominant strategies with transfers that you found in (c)?

Problem 4. Bilateral Trade and Mechanism Design.

Consider a buyer and a seller that have private valuations of an object $v \sim U[0, 1]$ and $c \sim U[0, 1]$ correspondingly. The object can be sold $q = 1$ (goes from the seller to the buyer) or not $q = 0$. The seller's payoff is $m_S - cq$ and the buyer's payoff is $vq - m_B$. Realization of v is known to the buyer only, whereas c is known to the seller only.

(a) Characterize the efficient physical allocation of the object q^*. (1 point)

(b) Suppose the efficient allocation is being implemented via a direct mechanism, in which the seller reports $\hat{c} \in [0, 1]$, while the buyer reports $\hat{v} \in [0, 1]$. Characterize possible direct mechanisms that implement the efficient physical allocation in dominant strategies (VCG mechanisms).

(c) Show that there is a unique VCG mechanism which commands zero transfers for situations when the project is not sold optimally, i.e. $m_S = m_B = 0$ when $q^* = 0$. Show that this mechanism requires external finance for certain realizations of reports $m_S - m_B > 0$.

(d) Show that there is no VCG mechanism which exactly balances the budget ($m_S - m_B = 0$) for all possible reports.

(e) Consider now a BNE implementation of the efficient allocation in a direct mechanism. Write down IC constraints for the buyer and the seller, and show that they can be written in a form analogous to

$$\theta_i = \underset{\{\hat{\theta}_i [0,1]\}}{\arg\max}\, \theta_i Q_i(\hat{\theta}_i) + M_i(\hat{\theta}_i).$$

(f) Find the direct efficient mechanism that balances the budget for all possible reports, and characterize functions analogous to $Q_i(\hat{\theta}_i)$ and $M_i(\hat{\theta}_i)$ above.

Final Exam (2019) – By Sergey Kovbasyuk

Problem 1. Financial advice and communication (15 points)

Consider a client who meets financial advisor and asks about an investment in a risky financial product at $t = 0$, and the product delivers return at $t = 1$. The suitability (quality) of the financial product to the client is $\theta \sim U[0, 1]$. At $t = 0$ the client is uninformed about θ, while the advisor privately knows θ. At $t = 0$ the client needs to decide one the amount of investment in the risky product $I \geq 0$. If he invests I, his final payoff at $t = 1$ is

$$U = \theta I - \frac{I^2}{2}.$$

The adviser can communicate to the client at $t = 1$, and she uses two messages. With no loss of generality assume that she sends message L for $\theta \in [0, \theta^*]$ and another message H for $\theta \in (\theta^*, 1]$. The advisor partially cares about the payoff of the client with a weight $\alpha \in (0, 1)$, but also gets a fraction $\beta \in [0, 1)$ of the amount the client invests in the financial product from the creator of the financial product, her payoff is

$$V = \alpha U + \beta I = \alpha \left(\theta I - \frac{I^2}{2} \right) + \beta I.$$

Questions:

(a) **Cheap talk:** characterize communication equilibrium with two messages. How does θ^* depend on β, what happens when $\beta = 0$? What does it imply about the informativeness of equilibria for different β. (5 points) (*Alexey's note: Since we didn't go over cheap talk in our class, you won't see it on this year's final exam.*)

(b) **Bayesian persuasion:** suppose now that the advisor can commit to a disclosure rule with two messages at $t = 0$, i.e. she can credibly promise to send message L only if $\theta \in [0, \theta']$ and H if $\theta \in [\theta', 1]$. What optimal θ' the advisor would choose? Is it different from what you obtained in the previous question? Why? (5 points)

(c) **Mechanism design:** suppose now that the advisor can't commit to a disclosure rule, but, instead, the client can commit to a payment rule at $t = 0$, i.e. he can credibly promise to pay to the advisor $t_L \geq 0$ and $t_H \geq 0$ for messages L and H accordingly. For simplicity assume $\alpha = \beta \to 0$. Which t_L and t_H the client will offer? How does the resulting equilibrium partition compares to the ones you obtained in questions 1 and 2? (5 points)

Problem 2. Second-Price Sealed-Bid Auction (10 points)

(*Alexey's note: This problem may be too easy for you since I spent a lot more time on auctions that Sergey did last year. Still, this could be good for practice.*) Consider an auction with two participants $i = 1, 2$. The valuation of each participant i is his private information $v_i \sim U[0, 1]$. Participants simultaneously submit bids b_i, the highest bid wins, and the winner pays the second highest bid, i.e. if $b_i > b_{-i}$, then $p = b_{-i}$.

Questions:

(a) Find an efficient allocation rule q. (1 point)

(b) Consider a BNE implementation. Write down the expected payoff of a participant i as a function of v_i and b_i. Does the auction implement the efficient allocation rule? (1 point)

(c) Characterize the equilibrium bidding strategy of participant i, i.e., find $b_i(v_i)$. (3 points)

(d) Find the expected payoff of a participant i as a function of v_i only. (5 points)

Problem 3. Weighted Auction (10 points)

(*Alexey's note: This problem is a special case of a mixed auction set-up we considered in class.*) Consider now an auction with two participants as in Problem 2, with one key difference: the final price paid by the winner is a weighted average of the bids. In other words participants simultaneously submit bids b_i, the highest bid wins, and the winner pays the price which is equal to his bid with weight 3/4 and the second highest bid with weight 1/4, i.e. if b_i is the highest then $p = \frac{3b_i}{4} + \frac{b_{-i}}{4}$.

Questions:

(a) Find an efficient allocation rule q. (1 point)

(b) Consider a BNE implementation. Write down the expected payoff of a participant i as a function of v_i and b_i. Does the auction implement the efficient allocation rule? (1 point)

(c) Find the expected payoff of a participant i as a function of v_i only. (5 points)

(d) Characterize the equilibrium bidding strategy of a participant i, i.e. find $b_i(v_i)$. (3 points)

Problem 4. Monopolist choosing quality (15 points)

Consider a monopolists who can produce a good of different qualities. The cost of producing quality q is q^2. Each consumer buys at most one unit of the good. The utility of the consumer is $U(q, \theta) = \theta q$ if she consumes one unit of product of quality q, and zero otherwise.

The monopolist decides on the qualities he is going to offer and prices. Consumers observe qualities and prices and decide whether to buy one unit of the product, and of which quality.

Questions:

(a) Characterize the first-best solution. (3 points)

(b) Suppose that the monopolist does not know θ, that is θ is private information of a consumer. Assume $\theta \in \{\theta_L, \theta_H\}$ and $\Pr(\theta = \theta_L) = \beta$. Formulate the monopolist's maximization problem. (3 points)

(c) Solve the problem, and characterize optimal prices and qualities offered by the monopolist. (3 points)

(d) Suppose now that $\theta \sim U[0, 1]$. Formulate the monopolist's maximization problem. (3 points)

(e) Solve the problem, and characterize optimal prices and qualities offered by the monopolist. (3 points)