

# From Auctions to Mechanism Design

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# Introduction

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- This week, we abstract away from the details of any particular selling format and ask: *“What is the best way to allocate an object?”*

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- This week: Optimal allocation rule to sell an object?

# From Auctions to a General Mechanism Design Problem

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- Buyers have independently distributed private valuations.
- Buyer  $i$ 's valuation  $V_i$  is distributed over the interval  $\mathcal{V}_i = [0, \omega_i]$  according to c.d.f.  $F_i$  with density  $f_i$ .

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- Let  $\mathcal{V} = \times_{j=1}^N \mathcal{V}_j$  denote the product of the sets of buyers' values and, for all  $i$ , let  $\mathcal{V}_{-i} = \times_{j \neq i} \mathcal{V}_j$



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- Similarly, define  $f_{-i}(\mathbf{v}_{-i})$  to be the joint density of  $\mathbf{v}_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_N)$

# Mechanism

## Definition

A selling *mechanism*  $(\mathcal{B}, \pi, \mu)$  is a combination of:

- ① A set of possible *messages* (or “bids”)  $\mathcal{B}_i$  for each buyer  $i$ ;
- ② An *allocation rule*  $\pi : \mathcal{B} \rightarrow \Delta$  where  $\Delta$  is the set of probability distributions over the set of buyers  $\mathcal{N}$ ;
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Intuitively:

- Allocation rule determines, as a function of all  $N$  messages, the probability  $\pi_i(\mathbf{b})$  that  $i$  will get the object.
- Payment rule determines, as a function of all  $N$  messages, for each buyer  $i$ , the expected payment  $\mu_i(\mathbf{b})$  that  $i$  must make.

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$$\pi_i(\mathbf{b}) = \begin{cases} 1 & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

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- Second-price auction:

$$\mu_i^{II}(\mathbf{b}) = \begin{cases} \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

# Game Within a Mechanism

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  - Strategies:  $\beta_i : [0, \omega_i] \rightarrow \mathcal{B}_i$
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- A strategy profile  $\beta(\cdot)$  is a **Bayesian Nash Equilibrium** of a mechanism if for all  $i$  and for all  $v_i$ , given strategies  $\beta_{-i}$  of other buyers,  $\beta_i(v_i)$  maximizes  $i$ 's expected payoff.

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- *Note:* Today, we focus on **Bayesian Mechanism Design**, i.e., with BNE as an underlying equilibrium concept. Later, we will come back and consider **Dominant Strategies Mechanism Design** which relies on the Dominant Strategy Equilibrium.

# Direct Mechanisms and Revelation Principle

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- Such mechanisms are called *direct*, since every buyer is asked to directly report a value.

# Direct Mechanisms

- Formally, direct mechanism  $(\mathbf{Q}, \mathbf{M})$  consists of functions  $\mathbf{Q} : \mathcal{V} \rightarrow \Delta$  and  $\mathbf{M} : \mathcal{V} \rightarrow \mathbb{R}^N$ , where  $Q_i(\mathbf{v})$  is the probability that  $i$  will get the object and  $M_i(\mathbf{v})$  is the expected payment by  $i$ .

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- *Revelation principle*: outcomes of any mechanism's equilibrium can be replicated by a truthful equilibrium of a direct mechanism.
- Hence, no loss of generality in focusing on direct mechanisms.

## Proposition. Revelation Principle.

*Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.*

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- Now suppose that some buyer finds it profitable to be untruthful and report a value  $\hat{v}_i$  when his value is  $v_i$ .
- Then in the original mechanism same buyer would have found it profitable to submit  $\beta_i(\hat{v}_i)$  instead of  $\beta_i(v_i)$ . Contradiction. ■

# Buyer's Payoff Function

Given a direct mechanism  $(\mathbf{Q}, \mathbf{M})$ :

$$q_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} Q_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

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$$m_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} M_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

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is expected payment when  $i$  reports  $\hat{v}_i$  and others tell the truth. Then:

$$q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)$$

is the expected payoff of  $i$  when he reports  $\hat{v}_i$  and others tell the truth.

# Incentive Compatibility

The direct revelation mechanism is incentive compatible (IC) if:

$$U_i(v_i) \equiv q_i(v_i)v_i - m_i(v_i) \geq q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)$$

$$\forall i \in \mathcal{N}; \forall v_i, \hat{v}_i \in [0, \omega_i]$$

where  $U_i$  is the *equilibrium payoff function*.



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- That is,  $U_i$  is a maximum of a family of affine functions, therefore  $U_i$  is a *convex function*.

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- Second, we can rewrite:

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- Since the above inequality has to hold for all  $v_i$  and  $\hat{v}_i$ ,  $q_i(v_i)$  is the subgradient of the function  $U_i$  at  $v_i$ .



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$$\begin{aligned}U_i(\hat{v}_i) &= q_i(\hat{v}_i)\hat{v}_i - m_i(\hat{v}_i) \geq q_i(v_i)\hat{v}_i - m_i(v_i) \\&= q_i(v_i)v_i - m_i(v_i) + q_i(v_i)(\hat{v}_i - v_i) \\&= U_i(v_i) + q_i(v_i)(\hat{v}_i - v_i)\end{aligned}$$

- Since the above inequality has to hold for all  $v_i$  and  $\hat{v}_i$ ,  $q_i(v_i)$  is the subgradient of the function  $U_i$  at  $v_i$ .
- Since  $U_i$  is convex, it must be that  $q_i$  is *non-decreasing*.

# Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

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- In other words,  $(\mathbf{Q}, \mathbf{M})$  and  $(\mathbf{Q}, \bar{\mathbf{M}})$  are *payoff equivalent*.

# Revenue Equivalence Strikes Again!

## Generalized Revenue Equivalence

*If the direct mechanism  $(\mathbf{Q}, \mathbf{M})$  is incentive compatible, then for all  $i$  and  $v_i$ , the expected payment is*

$$m_i(v_i) = m_i(0) + q_i(v_i)v_i - \int_0^{v_i} q_i(x_i)dx_i$$

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