Some Relevant Last Years' HW Problems

Problem 1. Screening.

A buyer needs to procure a delivery of certain product/service from a supplier. The value of the product for the buyer is $v > 0$ per unit of product, if quantity q is supplied the total value is vq. The buyer pays for the product to the provider an amount T, so that the net payoff of the buyer is $vq - T$.

Providers are heterogeneous and differ in their costs, type θ provider bears a cost $c(q, \theta) = \theta q^2/2$ if he provides quantity q. The cost parameter is his private information. It is drawn from a uniform distribution on the unit interval $[\varepsilon, 1 + \varepsilon]$, $\varepsilon > 0$. Suppliers' outside options are zero.

The buyer attempts to screen suppliers and offers a contract described by pairs T, q, we refer to this contract as a menu T, q. The objective of the buyer is to maximize his expected payoff

$$E[vq - T].$$

(a) How many pairs T, q the buyer can consider offering in order to maximize his payoff? How can this set of pairs be described on T, q plane?

(b) Write down the buyer's optimization problem and the constraints she is facing.

(c) Is the single-crossing condition satisfied? What does it say about possible relationship between q and θ in the optimal contract?

(d) Denote by $W(\theta)$ the profit of a supplier of type θ under the optimal procurement contract, how does $W(\theta)$ change with θ?

(e) Reformulate the buyer's optimization problem using $W(\theta)$.

(f) Find the first order necessary condition for optimal q as a function of θ. Does it satisfy the relationship between q and θ established in question 3?

(g) Characterize the candidate optimal contract(menu) T, q, that is characterize how q and T depend on θ, and how T depends on q. Can this contract be interpreted as quantity discounts (per unit price drops with the quantity) or not?

(h) Suppose the buyer decides to exclude some types of suppliers, what types of suppliers he may want to exclude? Why he might want to do that? Does he do so in the optimal contract? Why?

(i) Compare the quantities bought under the optimal contract, and in the first-best (when types are known). What is the expected welfare loss due to asymmetric information? For simplicity, compute welfare as a sum of all payoffs.

Problem 2. Moral Hazard.

Suppose the principal faces an agent with unknown linear production costs $c(x, \theta) = x\theta$, here θ is the agents private information. The principal offers a payment schedule to the agent $t(x)$, and the agent responds by producing quantity $x(\theta)$. Suppose the principal wants to induce production $x(\theta) = 1/\theta^\gamma$, $\gamma > 1$, what kind of payment schedule $t(x)$ the principal should offer?

Problem 3. Auction and Mechanism Design.

Consider an auction with two participants $i = 1, 2$. The valuation of each participant i is

$$u_i(v_i, v_{-i}) = v_i + \gamma v_{-i}.$$

Here, $\gamma \in (0, 1)$ and v_i is private information of agent i, $v_i \sim U[0, 1]$.

(a) Find the efficient allocation rule q.

(b) Find a direct mechanism which implements the efficient allocation rule as a BNE, and has transfers that depend only on $q(\hat{v}_1, v_2)$ and the report of the other agent, that is $m_i(q, \hat{v}_{-i})$.

(c) Can the efficient allocation be implemented in dominant strategies with transfers that you found in (c)?

Problem 4. Bilateral Trade and Mechanism Design.

Consider a buyer and a seller that have private valuations of an object $v \sim U[0, 1]$ and $c \sim U[0, 1]$ correspondingly. The object can be sold $q = 1$ (goes from the seller to the buyer) or not $q = 0$. The seller's payoff is $m_S - cq$ and the buyer's payoff is $vq - m_B$. Realization of v is known to the buyer only, whereas c is known to the seller only.

(a) Characterize the efficient physical allocation of the object q^*. (1 point)

(b) Suppose the efficient allocation is being implemented via a direct mechanism, in which the seller reports $\hat{c} \in [0, 1]$, while the buyer reports $\hat{v} \in [0, 1]$. Characterize possible direct mechanisms that implement the efficient physical allocation in dominant strategies (VCG mechanisms).

(c) Show that there is a unique VCG mechanism which commands zero transfers for situations when the project is not sold optimally, i.e. $m_S = m_B = 0$ when $q^* = 0$. Show that this mechanism requires external finance for certain realizations of reports $m_S - m_B > 0$.

(d) Show that there is no VCG mechanism which exactly balances the budget ($m_S - m_B = 0$) for all possible reports.

(e) Consider now a BNE implementation of the efficient allocation in a direct mechanism. Write down IC constraints for the buyer and the seller, and show that they can be written in a form analogous to

$$\theta_i = \underset{\{\hat{\theta}_i[0,1]\}}{\arg\max}\, \theta_i Q_i(\hat{\theta}_i) + M_i(\hat{\theta}_i).$$

(f) Find the direct efficient mechanism that balances the budget for all possible reports, and characterize functions analogous to $Q_i(\hat{\theta}_i)$ and $M_i(\hat{\theta}_i)$ above.

Final Exam (2019).

Problem 1. Financial advice and communication (15 points)

Consider a client who meets financial advisor and asks about an investment in a risky financial product at $t = 0$, and the product delivers return at $t = 1$. The suitability (quality) of the financial product to the client is $\theta \sim U[0, 1]$. At $t = 0$ the client is uninformed about θ, while the advisor privately knows θ. At $t = 0$ the client needs to decide one the amount of investment in the risky product $I \geq 0$. If he invests I, his final payoff at $t = 1$ is

$$U = \theta I - \frac{I^2}{2}.$$

The adviser can communicate to the client at $t = 1$, and she uses two messages. With no loss of generality assume that she sends message L for $\theta \in [0, \theta^*]$ and another message H for $\theta \in (\theta^*, 1]$. The advisor partially cares about the payoff of the client with a weight $\alpha \in (0, 1)$, but also gets a fraction $\beta \in [0, 1)$ of the amount the client invests in the financial product from the creator of the financial product, her payoff is

$$V = \alpha U + \beta I = \alpha \left(\theta I - \frac{I^2}{2} \right) + \beta I.$$

Questions:

(a) **Cheap talk:** characterize communication equilibrium with two messages. How does θ^* depend on β, what happens when $\beta = 0$? What does it imply about the informativeness of equilibria for different β. (5 points) (*Alexey's note: Since we didn't go over it in our class, it is unlikely that you will see it on this year's final exam. However, since you went over cheap talk in Francesco's class, you should be able to solve this.*)

(b) **Bayesian persuasion:** suppose now that the advisor can commit to a disclosure rule with two messages at $t = 0$, i.e. she can credibly promise to send message L only if $\theta \in [0, \theta']$ and H if $\theta \in [\theta', 1]$. What optimal θ' the advisor would choose? Is it different from what you obtained in the previous question? Why? (5 points)

(c) **Mechanism design:** suppose now that the advisor can't commit to a disclosure rule, but, instead, the client can commit to a payment rule at $t = 0$, i.e. he can credibly promise to pay to the advisor $t_L \geq 0$ and $t_H \geq 0$ for messages L and H accordingly. For simplicity assume $\alpha = \beta \to 0$. Which t_L and t_H the client will offer? How does the resulting equilibrium partition compares to the ones you obtained in questions 1 and 2? (5 points)

Problem 2. Second-Price Sealed-Bid Auction (10 points)

(*Alexey's note: This problem may be too easy for you since I spent a lot more time on auctions that Sergey did last year. Still, this could be good for practice.*) Consider an auction with two participants $i = 1, 2$. The valuation of each participant i is his private information $v_i \sim U[0, 1]$. Participants simultaneously submit bids b_i, the highest bid wins, and the winner pays the second highest bid, i.e. if $b_i > b_{-i}$, then $p = b_{-i}$.

Questions:

(a) Find an efficient allocation rule q. (1 point)

(b) Consider a BNE implementation. Write down the expected payoff of a participant i as a function of v_i and b_i. Does the auction implement the efficient allocation rule? (1 point)

(c) Characterize the equilibrium bidding strategy of participant i, i.e., find $b_i(v_i)$. (3 points)

(d) Find the expected payoff of a participant i as a function of v_i only. (5 points)

Problem 3. Weighted Auction (10 points)

(*Alexey's note: This problem is a special case of a mixed auction set-up we considered in Section 3.*) Consider now an auction with two participants as in Problem 2, with one key difference: the final price paid by the winner is a weighted average of the bids. In other words participants simultaneously submit bids b_i, the highest bid wins, and the winner pays the price which is equal to his bid with weight $3/4$ and the second highest bid with weight $1/4$, i.e. if b_i is the highest then $p = \frac{3b_i}{4} + \frac{b_{-i}}{4}$.

Questions:

(a) Find an efficient allocation rule q. (1 point)

(b) Consider a BNE implementation. Write down the expected payoff of a participant i as a function of v_i and b_i. Does the auction implement the efficient allocation rule? (1 point)

(c) Find the expected payoff of a participant i as a function of v_i only. (5 points)

(d) Characterize the equilibrium bidding strategy of a participant i, i.e. find $b_i(v_i)$. (3 points)

Problem 4. Monopolist choosing quality (15 points)

Consider a monopolists who can produce a good of different qualities. The cost of producing quality q is q^2. Each consumer buys at most one unit of the good. The utility of the consumer is $U(q, \theta) = \theta q$ if she consumes one unit of product of quality q, and zero otherwise.

The monopolist decides on the qualities he is going to offer and prices. Consumers observe qualities and prices and decide whether to buy one unit of the product, and of which quality.

Questions:

(a) Characterize the first-best solution. (3 points)

(b) Suppose that the monopolist does not know θ, that is θ is private information of a consumer. Assume $\theta \in \{\theta_L, \theta_H\}$ and $\Pr(\theta = \theta_L) = \beta$. Formulate the monopolist's maximization problem in this case (3 points)

(c) Solve the problem, and characterize optimal prices and qualities offered by the monopolist in this case. (3 point)

(d) Suppose now that $\theta \sim U[0, 1]$. Formulate the monopolist's maximization problem in this case (3 points)

(e) Solve the problem, and characterize optimal prices and qualities offered by the monopolist in this case. (3 point)