From Auctions to Mechanism Design

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(In)Efficiency of the Optimal Mechanism

The optimal mechanism has two separate sources of inefficiency:

1. In *efficient* mechanisms, seller cannot retain the object if at least some buyer i has valuation above seller's valuation, i.e., $v_i > v_0$.
    - Here, seller keeps object if highest virtual valuation is negative.
    - But buyer values are always nonnegative and seller's value is zero.
    - So it's always socially optimal to give the object to some buyer.
2. *Efficient* mechanisms give object to buyer with highest value.
    - Here, it is allocated to the buyer with highest *virtual* valuation.
    - In the asymmetric case, this need not be the highest-value buyer.

Interpreting Virtual Valuations

Why is it optimal to allocate the object on the basis of virtual valuations. And what the hell are virtual valuations anyway?

- Consider buyer i whose values are distributed according to F.
- Suppose seller makes a take-it-or-leave it offer to i at price p.
- The probability that i accepts this offer is $1 - F(p)$.
- Think of this probability as the "quantity" demanded by i and write the implied "demand curve" as $q(p) \equiv 1 - F(p)$.
- The inverse demand curve is then $p(q) \equiv F^{-1}(1 - q)$.

Interpreting Virtual Valuations

- Then the "revenue function" that the seller is facing is:

$$p(q) \times q = qF^{-1}(1-q)$$

- Differentiating with respect to q:

$$\frac{\partial TR}{\partial q} = F^{-1}(1-q) - \frac{q}{F'(F^{-1}(1-q))}$$

$$\implies MR(p) \equiv p - \frac{1-F(p)}{f(p)} = \psi(p)$$

- Thus, virtual valuation of a buyer can be interpreted as a *marginal revenue*. (Recall that ψ is strictly increasing.)

Interpreting Virtual Valuations

- Facing one buyer, seller would set a "monopoly price" of r^* by equating: $MR(r^*) = MC = 0 \implies r^* = \psi^{-1}(0)$.
- Facing many buyers, the optimal mechanism calls for the seller to set *discriminatory reserve prices* of $r_i^* = \psi_i^{-1}(0)$ to the buyers.
- If no buyer's value v_i exceeds his/her reserve price r_i^*, the seller keeps the object.
- Otherwise, object is allocated to buyer with highest MR and he is asked to pay $p_i = y_i(\mathbf{v}_{-i})$, smallest value such that he still wins.

Interpreting Optimal Mechanism

Optimal mechanism favors *disadvantaged* buyers. Why?

- Suppose there are two buyers, $v_1 \sim F_1[0, \omega]$ and $v_2 \sim F_2[0, \omega]$.
- Suppose further that for all v, $\lambda_1(v) \leq \lambda_2(v)$.
- Buyer 2 is relatively disadvantaged since his values are likely to be lower: in particular, F_1 (first-order) stochastically dominates F_2.
- But if $v_1 = v_2 = v$, virtual valuation of buyer 2 is higher:

$$\psi_1(v) = v - \frac{1}{\lambda_1(v)} \leq v - \frac{1}{\lambda_2(v)} = \psi_2(v)$$

- Thus, buyer 2 will "win" more often than is dictated by a comparison of actual values alone.

Interpreting Optimal Mechanism

Note: In the optimal mechanism, buyers have a positive surplus.

- "Winning" buyer pays $y_i(\mathbf{v}_{-i}) \leq v_i$.
- Expected surplus $\mathbb{E}[V_i - y_i(\mathbf{V}_{-i})]$ is called *informational rent*, which accrues to buyer i due to his private knowledge of v_i.
- Because of informational asymmetry, seller is unable to perfectly price discriminate and extract all the surplus.
- Buyers must be given informational rents to get them to reveal their private information.

Auctions vs. Mechanisms

- By definition (Krishna Ch.1), an auction has to be:
  - *Universal*: can be used to sell any good;
  - *Anonymous*: bidder's identity plays no role.
- Optimal mechanism is neither!
  - *Not universal*: rules depend on buyers' value distributions;
  - *Not anonymous*: buyers with different virtual valuations are treated differently (e.g., face different reserve prices).
- Thus, the optimal mechanism does not satisfy two important properties of auctions.
- Since these properties are important from a practical standpoint, one might want to restrict attention to mechanisms that satisfy universality and anonymity (*Wilson's doctrine*).

Efficient Mechanisms

Defining Efficient Mechanism

- Let us generalize the setup by allowing agents' values to lie in $\mathcal{V}_i = [\alpha_i, \omega_i]$, i.e., allow for negative values when $\alpha_i < 0$.
- An allocation rule $\boldsymbol{Q}^* : \mathcal{V} \to \boldsymbol{\Delta}$ is *efficient* if it maximizes "social welfare"—that is, for all $\mathbf{v} \in \mathcal{V}$:

$$\boldsymbol{Q}^*(\mathbf{v}) \in \arg\max_{\boldsymbol{Q} \in \boldsymbol{\Delta}} \sum_{j \in \mathcal{N}} Q_j v_j$$

- When there are no ties, an efficient rule allocates the object to the person who values it the most.
- Any mechanism with an efficient allocation rule is called *efficient*.

Defining Maximal Social Welfare

- Given an efficient allocation rule $\boldsymbol{Q}^*$, define the maximized value of social welfare:

$$W(\mathbf{v}) \equiv \sum_{j \in \mathcal{N}} Q_j^*(\mathbf{v}) v_j$$

- Similarly, define welfare of agents other than i as:

$$W_{-i}(\mathbf{v}) \equiv \sum_{j \neq i} Q_j^*(\mathbf{v}) v_j$$

The VCG Mechanism

- The *VCG (Vickrey-Clarke-Groves)* mechanism $(\boldsymbol{Q}^*, \boldsymbol{M}^V)$ is an efficient mechanism with payment rule $\boldsymbol{M}^V : \mathcal{V} \to \mathbb{R}^N$ given by:

$$\boldsymbol{M}_i^V(\mathbf{v}) = W(\alpha_i, \mathbf{v}_{-i}) - W_{-i}(\mathbf{v})$$

- Thus, payment is the difference between *total* social welfare at i's lowest possible value α_i and welfare of *other* agents at v_i.
- (Of course, both calculated assuming efficient allocation rule $\boldsymbol{Q}^*$.)

The VCG Mechanism

Lemma 1.

Assume $\alpha_i = 0 \; \forall i \in \mathcal{N}$. Then the VCG mechanism is equivalent to a second-price auction.

Proof: First, the allocation rule is the same—object goes to the person with the highest value.
Second, can show that the payment rule is the same too:

$$M_i^V(\mathbf{v}) = W(0, \mathbf{v}_{-i}) - W_{-i}(\mathbf{v}) = W_{-i}(0, \mathbf{v}_{-i}) - W_{-i}(\mathbf{v})$$
$$= \begin{cases} \max_{j \neq i} v_j & \text{if } v_i > \max_{j \neq i} v_j \\ 0 & \text{if } v_i < \max_{j \neq i} v_j \end{cases}$$

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The VCG Mechanism

Lemma 2.

The VCG mechanism is (dominant strategy) incentive compatible.

<u>*Proof:*</u> Suppose other buyers report values $\mathbf{v}_{-i}$. Then by reporting value z_i, agent i earns:

$$Q_i^*(z_i, \mathbf{v}_{-i})v_i - M_i^V(z_i, \mathbf{v}_{-i}) = \sum_{j \in \mathcal{N}} Q_j^*(z_i, \mathbf{v}_{-i})v_j - W(\alpha_i, \mathbf{v}_{-i})$$

Note that the second term does not depend on z_i. The first term is just the total welfare of all agents. It is maximized by setting $z_i = v_i$ because if $z_i > v_i$ or $z_i < v_i$, then the object could potentially be allocated inefficiently. ■

Intuitively, this payment rule makes i internalize the externality of him lying about his value. Then i's equilibrium payoff is:

$$Q_i^*(z_i, \mathbf{v}_{-i})v_i - M_i^V(z_i, \mathbf{v}_{-i}) = W(\mathbf{v}) - W(\alpha_i, \mathbf{v}_{-i})$$

The VCG Mechanism

- Since the VCG mechanism is incentive compatible, it has the properties of IC mechanisms derived earlier.
- In particular, equilibrium expected payoff function U_i^V associated with the VCG mechanism:

$$U_i^V(v_i) = \mathbb{E}[W(v_i, \boldsymbol{V}_{-i}) - W(\alpha_i, \boldsymbol{V}_{-i})]$$

  is convex and increasing in v_i.
- Since $U_i^V(\alpha_i) = 0$, VCG mechanism is also individually rational.

The VCG Mechanism

Proposition.

Among all mechanisms for allocating a single object that are efficient, incentive compatible, and individually rational, the VCG mechanism maximizes the expected payment of each agent.

Proof:

- Let $(\boldsymbol{Q}^*, \boldsymbol{M})$ be some other efficient, IC, and IR mechanism.
- By revenue equivalence, for all i, expected payoff function for this mechanism U_i differs from U_i^V by at most a constant c_i.
- Since $(\boldsymbol{Q}^*, \boldsymbol{M})$ is IR, it must be that $c_i = U_i - U_i^V \geq 0$. Otherwise would have $U_i(\alpha_i) < U_i^V(\alpha_i) = 0$, contradicting IR.
- Since expected payoffs in $(\boldsymbol{Q}^*, \boldsymbol{M})$ are greater than in VCG, and allocation is the same, expected payments must be lower. ■

Balancing the Budget

- Frequently, it is desirable to consider mechanisms that do not require injection of funds from the mechanism designer—that is, the designer's budget is exactly balanced *ex post*.
- A mechanism is said to *balance the budget* if for every realization of values, net payments from agents sum to zero:

$$\sum_{i \in \mathcal{N}} M_i(\mathbf{v}) = 0 \quad \forall \mathbf{v}$$

- While VCG does not always have a balanced budget, it is still helpful in determining when this is feasible (more on this later).

The AGV Mechanism

- The *AGV (Arrow-d'Aspremont-Gérard-Varet)* mechanism, or the 'expected externality' mechanism, $(\boldsymbol{Q}^*, \boldsymbol{M}^A)$ is defined by:

$$M_i^A(\mathbf{v}) = \frac{1}{N-1} \sum_{j \neq i} \mathbb{E}_{\boldsymbol{V}_{-j}} \left[W_{-j}(v_j, \boldsymbol{V}_{-j}) \right] - \mathbb{E}_{\boldsymbol{V}_{-i}} \left[W_{-i}(v_i, \boldsymbol{V}_{-i}) \right]$$

so that budget is balanced *ex post* for all $\mathbf{v}$:

$$\sum_{i \in \mathcal{N}} M_i^A(\mathbf{v}) = 0$$