Home Assigment 3. Exercise 3.

Egor

Part A

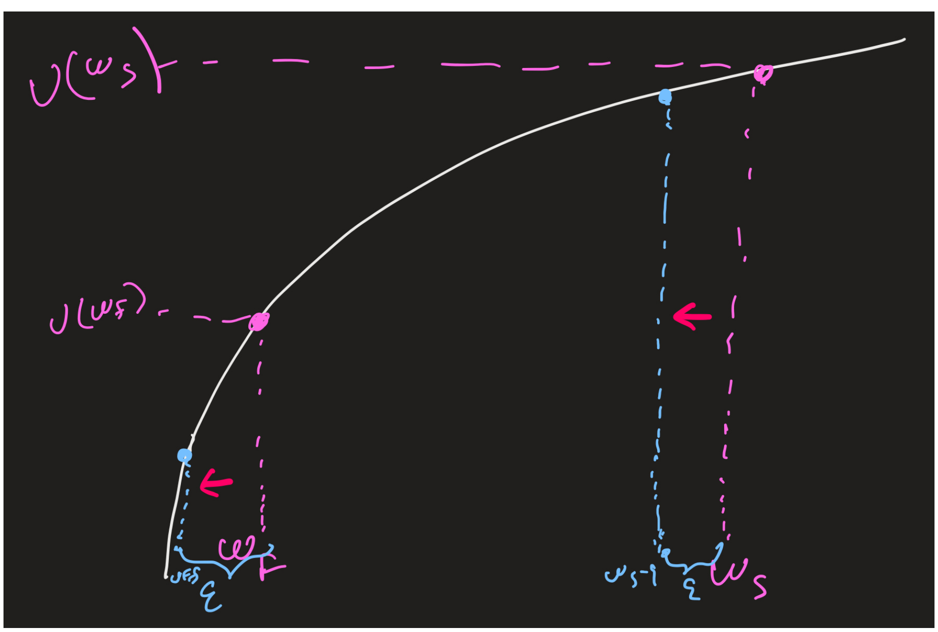
If ε is small enough or $S - F$ is enough huge, it is better to hire Alexey. I write all the constraints for Alexey, but the same procedure can be realised with Georgii by substituting $(p-\varepsilon) \Rightarrow p$ and $r \Rightarrow q$.
So, we have:

$$\text{IC} : (p-\varepsilon) \cdot v(w_S) + (1-p+\varepsilon) \cdot v(w_F) - c \geq r \cdot v(w_S) + (1-r) \cdot v(w_F)$$

$$\text{IR} : (p-\varepsilon) \cdot v(w_S) + (1-p+\varepsilon) \cdot v(w_F) - c \geq 0$$

Let's look at the IC:
$v(w_S)(p-\varepsilon-r) - c \geq v(w_G)(1-r-1+p-\varepsilon)$
$v(w_S)(p-\varepsilon-r) + v(w_F)(-1+r+1-p+\varepsilon) \geq c$
$v(w_S)(p-\varepsilon-r) + v(w_F)(r-p+\varepsilon) \geq c$
$(p-\varepsilon-r)(v(w_S)-v(w_F)) \geq c$
since $p-\varepsilon > r$,
$v(w_S) - v(w_F) \geq \frac{c}{p-\varepsilon-r}$

An obvious property of concave functions is presented on the following picture



If we decrease both w_S and w_F by γ, the distance between $v(w_S)$ and $v(w_F) \uparrow$. More formally, $v(w_S) - v(w_F) < v(w_S - \varepsilon) - v(w_F - \varepsilon)$.

Now we can show that IR binds.

If IR does not, we can $\downarrow$ both w_S and w_F by γ. IC will be still satisfied and Liyan is going to pay less. So, IR binds.

Therefore, we have

IC : $(p-\varepsilon) \cdot v(w_S) + (1-p+\varepsilon) \cdot v(w_F) - c \geq r \cdot v(w_S) + (1-r) \cdot v(w_F)$

IR : $(p-\varepsilon) \cdot v(w_S) + (1-p+\varepsilon) \cdot v(w_F) - c = 0$

We can derive $u(w_S)$ from IR:

$u(w_S) = u(w_F) \cdot \frac{p-\varepsilon-1}{p-\varepsilon} + \frac{c}{p-\varepsilon}$

Now let's back to IC

$(p-\varepsilon) \cdot v(w_S) + (1-p+\varepsilon) \cdot v(w_F) - c \geq r \cdot v(w_S) + (1-r) \cdot v(w_F)$

$(p-\varepsilon) \cdot [u(w_F) \cdot \frac{p-\varepsilon-1}{p-\varepsilon} + \frac{c}{p-\varepsilon}] + (1-p+\varepsilon) \cdot v(w_F) - c \geq r \cdot [u(w_F) \cdot \frac{p-\varepsilon-1}{p-\varepsilon} + \frac{c}{p-\varepsilon}] + (1-r) \cdot v(w_F)$

I open the brackets and rearrange slightly

$0 \geq r \cdot v(w_F) \cdot \frac{p-\varepsilon-1}{p-\varepsilon} + r \cdot \frac{c}{p-\varepsilon} + (1-r) \cdot v(w_F)$

Note that $r \cdot v(w_S) + (1-r)v(w_F) \leq 0$. If $v(w_F) \geq 0$, $v(w_S) > 0 \Rightarrow r \cdot v(w_S) + (1-r)v(w_F) > 0$ contradiction. So, $v(w_F) < 0$.

Back to the equation above,

$0 \geq r \cdot v(w_F) \cdot \frac{p-\varepsilon-1}{p-\varepsilon} + r \cdot \frac{c}{p-\varepsilon} + (1-r) \cdot v(w_F)$ $0 \leq r \cdot \frac{p-\varepsilon-1}{p-\varepsilon} + \frac{rc}{p-\varepsilon} \cdot \frac{1}{v(w_F)} + (1-r)$

Since $v(w_F) < 0$, if we $\downarrow v(w_F)$, RHS also $\downarrow$.

So, at the optimum (from Liyan's point of view),

$0 = r \cdot \frac{p-\varepsilon-1}{p-\varepsilon} + \frac{rc}{p-\varepsilon} \cdot \frac{1}{v(w_F)} + (1-r)$

$0 = r \cdot (1 - \frac{1}{p-\varepsilon}) + \frac{rc}{p-\varepsilon} \cdot \frac{1}{v(w_F)} + (1-r)$

$0 = r - \frac{r}{p-\varepsilon} + \frac{rc}{p-\varepsilon} \cdot \frac{1}{v(w_F)} + (1-r)$

$0 = -\frac{r}{p-\varepsilon} + \frac{rc}{p-\varepsilon} \cdot \frac{1}{v(w_F)} + 1$

$\frac{r}{p-\varepsilon} - 1 = \frac{rc}{p-\varepsilon} \cdot \frac{1}{v(w_F)}$

$\frac{r-p+\varepsilon}{p-\varepsilon} = \frac{rc}{p-\varepsilon} \cdot \frac{1}{v(w_F)}$

$r - p + \varepsilon = \frac{rc}{v(w_F)}$

$v(w_F) = \frac{rc}{r-p+\varepsilon}$

$w_F = v^{-1}(\frac{rc}{r-p+\varepsilon})$

So, if we had considered Georgii, we would have obtained

$w_F = v^{-1}(\frac{qc}{q-p})$

$v(w_S) = \frac{qc}{q-p} \cdot \frac{p-1}{p} + \frac{c}{p}$

$v(w_S) = \frac{c}{p} \cdot \frac{q(p-1)}{q-p} + \frac{c}{p}$

$v(w_S) = \frac{c}{p} \cdot (\frac{q(p-1)}{q-p} + 1)$

$v(w_S) = \frac{c}{p} \cdot (\frac{q(p-1)}{q-p} + \frac{q-p}{q-p})$

$v(w_S) = \frac{c}{p} \cdot (\frac{q(p-1)+q-p}{q-p})$

$v(w_S) = \frac{c}{p} \cdot (\frac{qp-q+q-p}{q-p})$

$v(w_S) = \frac{c}{p} \cdot (\frac{qp-p}{q-p})$
$v(w_S) = c \cdot (\frac{q-1}{q-p})$
$w_S = v^{-1}(c \cdot \frac{q-1}{q-p})$

Part B

Ok. Now we have the following:
$w_F^A = v^{-1}(\frac{rc}{r-p+\varepsilon})$
$w_S^A = v^{-1}(c \cdot \frac{r-1}{r-p+\varepsilon})$
$w_F^G = v^{-1}(\frac{qc}{q-p})$
$w_S^G = v^{-1}(c \cdot \frac{q-1}{q-p})$

I want to show that $w_S^A < w_S^G$.
To do this, let's consider
$\frac{r-1}{r-p+\varepsilon} - \frac{q-1}{q-p}$
as $\varepsilon \to 0$,
we have $\frac{r-1}{r-p} - \frac{q-1}{q-p}$?<? 0
$\frac{(r-1)(q-p)-(q-1)(r-p)}{(r-p)(q-p)}$?<? 0
note that $(r-p) < 0$ and $(q-p) < 0$
$(r-1)(q-p) - (q-1)(r-p)$?<? 0
$-rp - q + r + qp$?<? 0
$(1-p)(r-1)$?<? 0
Yes.
So, if ε is sufficiently small, $c \cdot \frac{r-1}{r-p+\varepsilon} < c \cdot \frac{q-1}{q-p}$
Therefore, $w_S^A < w_S^G$

What do we have?
We have that if ε is sufficiently small, Alexey's $w_S^A <$ than Georgii's w_S^G. Since we created Alexey's w_S^A in such a way that he would like to play "Effort", we can conclude the following. If Liyan hires Alexey insted of Georgii, she can create a contract, s.t. Alexey will accept it and play "Effort". Moreover, she can significantly decrease w_S^A, comparing with w_S^G. If ε is sufficiently small, $\varepsilon \cdot (S-F)$ is also sufficiently small. So, if Liyan hires Alexey instead of Georgii, she decreases her costs by a significant amount $(w_S^G - w_S^A)$ but her expected decrease of revenue is extremely close to 0.
Done.
Some intuition. Alexey has less motivation to play "No Effort", compared to Georgii. In other words, Alexey has more incentives to play "Effort". Therefore, Liyan can exploit this feature and to significantly decrease Alexey's reward for S.