Auctions

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Introduction

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- That said, first ever recorded auctions took place in Babylon around 500 B.C.

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- A variety of auction mechanisms were developed, including *English auction*, *Dutch auction*, and so-called *auction by the candle*.

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- Public sector:
  - Privatization and public resource allocation (ex: famous FCC Spectrum Auction in 1993 designed by Paul Milgrom and others);
  - Reverse auctions: Trillions of dollars of goods *bought* by governments on e-procurement auctions around the globe.

Questions in Auction Theory

- Why are auctions so prevalent, historically and today?
- In which situations auctions are preferred to other selling mechanisms, e.g., to a fixed posted price?
- Bidders: for a given auction, what are good bidding strategies?
- Sellers: are there particular types of auctions that would bring greater revenues than others?

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- Sellers want to maximize revenues and organize sales in such a way that would elicit this private information (e.g., such that bidders with the highest valuations offer the highest price).
- Auctions are one set of *mechanisms* in a general *mechanism design* problem: how to organize a game such that a certain objective is achieved?
  - Typically two competing objectives: revenue vs. efficiency.

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2. Interdependent values
   - Ex: auctioned object is an asset that can be resold later
3. Pure common value
   - Ex: value of the auctioned object is derived from a market price that is unknown at the time of the auction

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Common auction forms:

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   - Bids are private; highest bidder gets object at highest price.
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4. Sealed-bid second-price auction (SPSB, *Vickrey auction*).
   - Bids are private; highest bidder gets object at 2nd-highest price.
   - Example: Google ad auctions (before 2019).

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- English auction $\approx$ Sealed-bid second-price (under *private values*).
  - In both formats, best to bid your valuation.

Private Value Auctions

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- N potential risk-neutral buyers
  - v_i — valuation of buyer $i \in N$
  - $v_i \sim$ i.i.d. $F[0, \omega]$ — independent *symmetric* values
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- Bidder i knows v_i, $F[0, \omega]$, and N, but not v_j where $j \neq i$
- Since bidders are symmetric, focus on *symmetric* equilibria — equilibria in which all bidders follow the same strategy

FPSB vs. SPSB

In this setup, we will consider two auction formats:

1. *First-price sealed bid auction (I)*: highest bidder gets the object and pays the amount he bids
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Questions:

- What are equilibrium bidding strategies $\beta_i : [0, \omega] \to \mathbb{R}_+$ in these auction formats?
- Which of the two formats is better for the seller?

Second-Price Sealed-Bid Auction

Payoffs in SPSB

Bidders' payoffs are:

$$\mathbb{E}U_i = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{K}[v_i - \max_{j \neq i} b_j] & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

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First-Price Sealed-Bid Auction

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- Basic trade-off: increasing one's bid increases the probability of winning but reduces the gains from winning
- Suppose that bidders $j \neq i$ follow a symmetric, increasing, and differentiable equilibrium strategy β.
- Bidder i has valuation v and bids b. What is the optimal b?

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- Taking FOC:

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(v - b) - G[\beta^{-1}(b)] = 0$$

(where $g = G'$ is the density of Y_1).

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- Clearly, the player should never choose $b > \beta(\omega)$ since bidding $b = \beta(\omega)$ already ensures the victory.
- Thus, need to show that a player of type v is at least as well off choosing $\beta(v)$ as $\beta(\hat{v})$ for any $\hat{v} \in [0, \omega]$.

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Symmetric equilibrium strategies in a first-price auction are given by:

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where Y_1 is the highest of $N-1$ independently drawn valuations.

Proof (cont'd): Substituting $b = \beta(\hat{v})$ into buyer's utility function:

$$\begin{aligned} G(\beta^{-1}(b)) \times (v - b) &= G(\hat{v})[v - \beta(\hat{v})] \\ &= G(\hat{v})v - \int_0^{\hat{v}} x dG(x) = \int_0^{\hat{v}} (v - x) dG(x) \end{aligned}$$

This is clearly maximized at $\hat{v} = v$. $\blacksquare$

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- So, the degree of "shading" goes down as N increases!