Problem set #1

Problem 1

1. $N=2$, $F \sim \mathcal{U}(0,1)$, $b(\theta) = \alpha + B\theta$

$\alpha \in (0,1)$, $B \in (1-\alpha, 2-2\alpha)$

claim: $p_1 = p_2 = \frac{2\alpha}{2-B}$ is the only possible SPNE

Prood: $E(b(\theta) \mid \theta \leq p) = \alpha + BE(\theta \mid \theta \leq p) = \alpha + B \frac{\int_0^p \theta \, d\theta}{p} =$

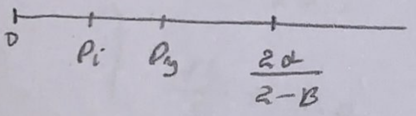
$= \alpha + \frac{Bp}{2}$

$E(b(\theta) \mid \theta \leq p) = p \Rightarrow p^* = \frac{2\alpha}{2-B}$ (max WTP of buyers)

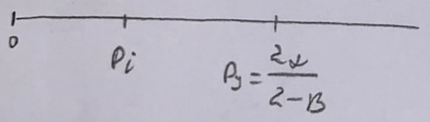
so in any SPNE $p_1 = p_2 = p^*$

why? Consider the following cases

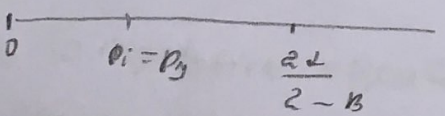
case:

I) 

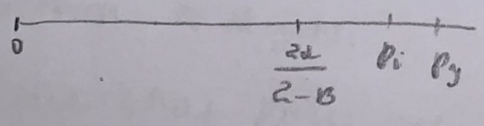
Profitable deviation.
i wants to offer $p_j + \varepsilon$

II) 

j wants to offer $p_i + \varepsilon$

III) 

i or j want to deviate to $p_i + \varepsilon$
because $E(b(\theta) \mid \theta \leq p) \geq \frac{E(b(\theta) \mid \theta \leq p)}{2}$

IV) 

j wants to deviate to $p=0$ or $p_i - \varepsilon$

So $p_1 = p_2 = \frac{2\alpha}{2-B}$ is the only possible SPNE strategy of the buyer.

In any SPNE the seller's strategy is as follows:

$$\begin{cases} A \text{ if } \max\{P_i, P_y\} \geq \theta & \text{and give it to } i \text{ if } P_i > P_y \\ & \text{choose randomly} (\tfrac{1}{2}) \text{ if } P_i = P_y \\ R \text{ if } \max\{P_i, P_y\} < \theta \end{cases}$$

In any SPNE we have that Seller A if $\max\{P_i, P_y\} > \theta$ and R if $\max\{P_i, P_y\} < \theta$. But if the type $\theta = \max\{P_i, P_y\}$ Rejects, then there would be no equilibrium where trade occurs.

Also, if seller's strategy is $\begin{cases} A \text{ if } P > \frac{2d}{2-B} & \text{and give to } i \text{ if } P_i > P_y \\ R \text{ if } P < \frac{2d}{2-B} \end{cases}$

No trade occurs for these prices.

2) $F \sim U(0,1)$, $\theta \in [0,1]$

$$b(\theta) = \begin{cases} \frac{1}{6} + \frac{1}{2}\theta, & \theta \in [0, \frac{1}{3}) \\ -1 + 4\theta, & \theta \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{14}{9} + \frac{1}{6}\theta, & \theta \in [\frac{2}{3}, 1] \end{cases}$$

a) Competitive equilibrium: (N players)

In any CE $p^* = E(b(\theta) \mid \theta \leq p^*)$, which requires rational expectations of buyers.

Let's check what prices are possible in equilibrium:

$0 < p < \frac{1}{3}$: $E(b(\theta) \mid \theta \leq p) = \dfrac{\int_0^p (\frac{1}{6} + \frac{1}{2}\theta)\, d\theta}{F(p)} = \dfrac{\frac{p}{6} + \frac{p^2}{4}}{p} = \frac{1}{6} + \frac{p}{4}$

$E(b(\theta) \mid \theta \leq p) = p \Rightarrow \frac{1}{6} + \frac{p}{4} = p \Rightarrow p^* = \frac{2}{9}$.

$\frac{1}{3} < p \leq \frac{2}{3}$: (buyer expects that all $\theta \leq \frac{2}{3}$ types will trade)

$$E(b(\theta) \mid \theta \leq p) = \frac{\int_0^{1/3} \left(\frac{1}{6} + \frac{1}{2}\theta\right) d\theta + \int_{1/3}^{p} (-1 + 4\theta) d\theta}{F(p)} =$$

$$= \frac{\left(\frac{1}{6}\theta + \frac{1}{4}\theta^2\right)\Big|_0^{1/3} + (2\theta^2 - \theta)\Big|_{1/3}^{p}}{p} = \frac{\frac{1}{18} + \frac{1}{36} + 2p^2 - p - \frac{2}{9} + \frac{1}{3}}{p} =$$

$$= \frac{\frac{7}{36} - p + 2p^2}{p}$$

$$E(b(\theta) \mid \theta \leq p) = \frac{\frac{7}{36} - p + 2p^2}{p} = p \Rightarrow$$

$$p^2 - p + \frac{7}{36} = 0 \Rightarrow p_1 = \frac{3 + \sqrt{2}}{6} \approx 0.736$$

$$p_2 = \frac{3 - \sqrt{2}}{6} \approx 0.264 \quad \notin \left[\frac{1}{3}, \frac{2}{3}\right)$$

So no such CE price.

$\frac{2}{3} < p \leq 1$:

$$E(b(\theta) \mid \theta \leq p) = \frac{\int_0^{1/3} \left(\frac{1}{6} + \frac{1}{2}\theta\right) d\theta + \int_{1/3}^{2/3} (4\theta - 1) d\theta + \int_{2/3}^{p} \left(\frac{14}{9} + \frac{1}{6}\theta\right) d\theta}{p} =$$

$$= \frac{\frac{1}{12} + (2\theta^2 - \theta)\Big|_{1/3}^{2/3} + \left(\frac{14}{9}\theta + \frac{1}{12}\theta^2\right)\Big|_{2/3}^{p}}{p} =$$

$$= \frac{\frac{1}{12} + \frac{8}{9} - \frac{2}{3} - \frac{2}{9} + \frac{1}{3} + \frac{14}{9}p + \frac{1}{12}p^2 - \frac{28}{27} - \frac{4}{108}}{p} =$$

$$\frac{\frac{1}{12}p^2 + \frac{14}{9}p - \frac{71}{108}}{p}$$

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$$\frac{1}{12}p^2 + \frac{14}{9}p - \frac{71}{108} = p^2$$

| Note: $\max \dfrac{\frac{14}{9}p - \frac{71}{108} + p^2}{p} - p$
is at $p^m = \dfrac{\sqrt{\frac{71}{11}}}{3} \approx 0.846$

$$P_1 = \frac{28+\sqrt{3}}{33} \approx 0.901$$

$$\in \left(\frac{2}{3}, 1\right]$$

$$P_2 = \frac{28-\sqrt{3}}{33} \approx 0.796$$

So we have $p_1^* = \frac{2}{9}$, $p_2^* = \frac{28+\sqrt{3}}{33}$, $p_3^* = \frac{28-\sqrt{3}}{33}$

$\underline{p \geq 1}$ is not possible since $E(b(\theta)) = \frac{53}{54} < 1$

6) $N = 2$, SPNE

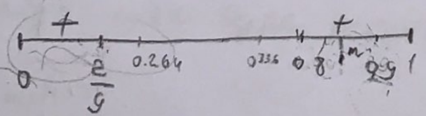
claim: in SPNE sellers strategy is

$$\begin{cases} A \text{ if } \max\{p_i, p_j\} \geq 0 \text{ and give to } i \text{ if } p_i > p_j \\ \qquad\qquad (\tfrac{1}{2}, \tfrac{1}{2}) \text{ rand if } p_i = p_j \\ R \text{ if } \max\{p_i, p_j\} < 0 \end{cases}$$

While $p_1^* = p_2^* = \dfrac{28+\sqrt{3}}{33}$ which is the max CE price.

Proof: First, note that no buyer will pay $p \geq 1$, because $E(b(\theta)) = \frac{53}{54}$.

So



$E(b(\theta) \mid \theta \leq p) > 0$ for $p < \frac{2}{9}$ and $\frac{28-\sqrt{3}}{33} < p < \frac{28+\sqrt{3}}{33}$

Cases:

1) if $p_i \leq p_j < \frac{2}{9} \Rightarrow i$ wants to offer $p_j + \varepsilon$.

$p_i = p_j = \frac{2}{9}$: i or j wants to deviate to $p^m \approx 0.846$

$\frac{28-\sqrt{3}}{33} < p_i < p_j < p^m \Rightarrow i$ wants to offer p^m and get the car

$p^m \leq p_i < p_j < \frac{28+\sqrt{3}}{33} \Rightarrow i$ wants to offer $p_j + \varepsilon$.

so the only SPE is $\boxed{p_i = p_j = \dfrac{28+\sqrt{3}}{33}}$

2)

W - wealth

$\theta \sim U[0,1]$

$u(x) = \sqrt{x}$

$Eu(x) = \theta\sqrt{w-e} + (1-\theta)\sqrt{w}$

Insurance costs p, pays $\begin{cases} e & \text{if loss occurs} \\ 0 & \text{if no loss} \end{cases}$

$w = 9,\ e = 5$

a) Individual buys insurance iff $Eu(w)^I \geq Eu(w)^0$

$\sqrt{W-p} \geq \theta\sqrt{w-e} + (1-\theta)\sqrt{w}$

$\sqrt{9-p} \geq 2\theta + (1-\theta)3 = 3-\theta$

$9 - p \geq 9 - 6\theta + \theta^2 \implies p \leq 6\theta - \theta^2$ (max WTP)

So $p(\theta) = 6\theta - \theta^2 = 5 - 4(1-\theta) - (1-\theta)^2$

$\frac{\partial p(\theta)}{\partial \theta} = 6 - 2\theta > 0$ for $\theta \in [0,1]$ so decreasing with $1-\hat{\theta}$

$p\uparrow$ 0 $\hat{\theta}$ 1

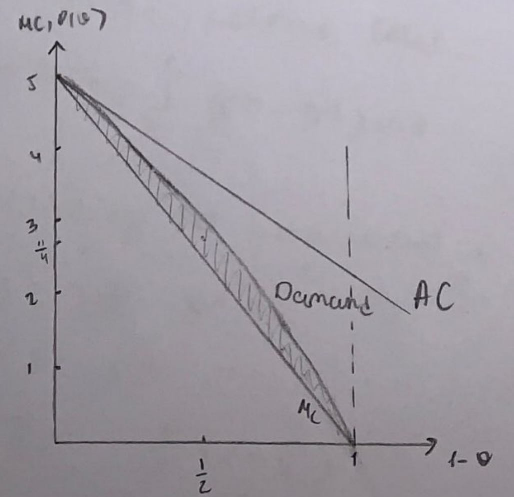
All individuals with $\theta \geq \hat{\theta}$ get insured. $\hat{\theta}$ s.t. $p = 6\hat{\theta} - \hat{\theta}^2$

$Q = P(\theta \geq \hat{\theta}) = 1 - \hat{\theta}$

b) $MC(\theta) = 5\theta$

It is decreasing withe the number of insured $(1-\hat{\theta})$ since the high risk type people are willing to pay the highest amount,

c)



d) $AC(\hat{\theta}) = E(5\theta \mid \theta > \hat{\theta}) = 5E(\theta \mid \theta > \hat{\theta}) =$

$$= \frac{5\int_{\hat{\theta}}^{1} \theta d\theta}{1-F(\hat{\theta})} = \frac{\frac{5}{2}(1-\hat{\theta}^2)}{1-\hat{\theta}} = \frac{5}{2}(1+\hat{\theta})$$

e) CE: $\underline{AC = P} \Rightarrow \frac{5}{2}(1+\hat{\theta}) = 6\theta - \hat{\theta}^2$

$$5 + 5\hat{\theta} = 12\hat{\theta} - 2\hat{\theta}^2$$

$$2\hat{\theta}^2 - 7\hat{\theta} + 5 = 0$$

$$\hat{\theta} = 1 \Rightarrow P^* = 5$$

$$\theta = 2.5 \ \cancel{\theta} > 1.$$

$P = AC$ because of perfect competition assumption in insurance market. This means $P \cdot Q = AC \cdot Q = TC \Rightarrow \Pi = 0$, zero profit conditio. $P > AC$. New entrances to market would drive price down to $P = AC$. If $P < AC$, exit from market drives price up to $P = AC$.

f) Yes market unravels, since with $P = 5$ only high risk $(\theta = 1)$ would want to insure, but their mass is negligible, so complete welfare loss:

$$DWL = \int_0^1 (6\theta - \theta^2)d\theta - \int_0^1 (5\theta)d\theta = \frac{1}{6}.$$

g) Subsidy to insurance companies

$P(\hat{\theta}) = 6\hat{\theta} - \hat{\theta}^2$

$MC(\theta) = 3\theta$

$AC(\hat{\theta}) = \frac{3}{2}(1+\hat{\theta})$

$AC = P \Rightarrow 3(1+\theta) = 2(6\theta - \theta^2)$

$2\theta^2 - 9\theta + 3 = 0$

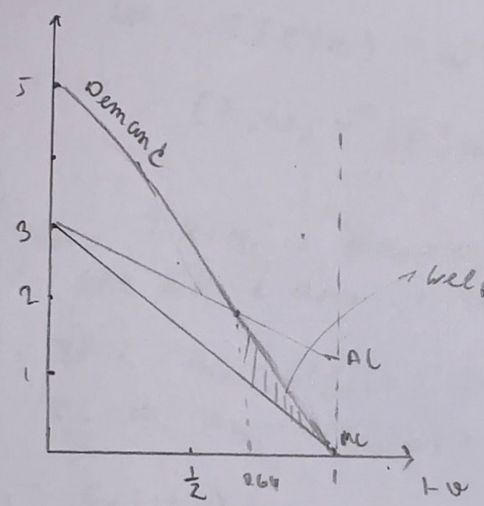
$\hat{\theta} = \dfrac{9 \pm \sqrt{57}}{4} \approx 0.36 \Rightarrow P^* = \dfrac{39 - 3\sqrt{57}}{8} \approx 2.04$

$\hat{\theta} = 4.13 > 1 \; \emptyset$

$MC = P \Rightarrow 3\theta = 6\theta - \theta^2 \Rightarrow \underline{\theta = 0}$ or

$\theta = 3 > 1 \; \emptyset$

No, under insurance problem is still present.



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(Q 3)

<u>Lemma</u>: Single Crossing Point property

If $(e,w) \succsim^L (e',w')$ and $e \geq e'$ $\Rightarrow$ $(e,w) \succ^H (e',w')$

Proof: $(e,w) \succsim^L (e',w')$ means that

$$w - c(e|L) \geq w' - c(e'|L)$$

$$w - w' \geq c(e|L) - c(e'|L)$$

Since $c_e(\cdot|L) > c_e(\cdot|H)$

$$w - w' \geq c(e|L) - c(e'|L) > c(e|H) - c(e'|H)$$

Then, $w - c(e|H) > w' - c(e'|H)$ which means that $(e,w) \succ^H (e',w')$. ∎

<u>Corollary</u>: IR-H is redundant.

If the IR-L and IC-H are satisfied, IR-H is also satisfied.

Proof: IR-L $w_L - c(e_L|L) \geq 0$

IC-H: $w_H - c(e_H|H) \geq w_L - c(e_L|H)$

Since $c_e(\cdot|H) < c_e(\cdot|L)$ and $c(0|\theta) = 0$ $\forall e$, it follows that $c(e_L|L) \geq c(e_L|H)$ and $c(e_H|L) \geq c(e_H|H)$

Then

$$0 \leq w_L - c(e_L|L) \overset{\text{by cost function property}}{\leq} w_L - c(e_L|H) \overset{\text{by IC-H}}{\leq} w_H - c(e_H|H)$$

i.e

$$w_H - c(e_H|H) \geq 0$$ ∎

<u>Corollary</u>: Optimal contract is monotone

If $(e_L, w_L), (e_H, w_H)$ satisfy IC constraints, then $e_H \geq e_L$.

Proof: Suppose $e_L > e_H$. Then by single crossing property it follows

$$c(e_L|H) - c(e_H|H) < c(e_L|L) - c(e_L|L) \quad (*)$$

By IC constraints we have: $w_L - w_H \geq c(e_L|L) - c(e_H|L)$

$w_L - w_H \leq c(e_L|H) - c(e_H|H)$

Contradiction!

$$\Rightarrow c(e_L|L) - c(e_H|L) \leq c(e_L|H) - c(e_H|H) \quad (**)$$ ∎

Lemma: No slack: A constrained profit maximizing contract satisfies the IR-L and IC-H constraints with equality.

Proof by contradiction:
Suppose at least one of them is not satisfied with equality.

1) $w_L - c(e_L|L) > 0$

$$w_H - c(e_H|H) = w_L - c(e_L|H) \overset{\text{by } c_e(\cdot|H) \leq c_e(\cdot|L)}{\geq} w - c(e_L|L) > 0$$

Then firm can $\downarrow$ w_L by ε until $w = c(e_L|L)$ because

$$\lambda(F(e_L) - (w_L - \varepsilon)) + (1-\lambda)[f(e_H) - w_H] > \lambda(f(e_L) - w_L) + (1-\lambda)(F(e_H) - w_H)$$

So firm is not profit maximizing

ii) Suppose IC-H is not satisfied

$$w_H - c(e_H|H) > w_L - c(e_L|H)$$

Firm can $\downarrow$ w_H by ε so that IC-H is still satisfied

This process continues up to $w_H - c(e_H|H) = w_L - c(e_L|H)$.

Otherwise firm is not maximizing ■

Lemma: IC-L is redundant.

If the contract is monotone and IC-H is satisfied with equality IC-L is also satisfied.

Proof: Since $e_H \geq e_L$ and $w_H - c(e_H|H) = w_L - w(e_L|H)$

$$\Rightarrow w_H - w_L = c(e_H|H) - c(e_L|H)$$

By single cross property, $\underbrace{c(e_H|H) - c(e_L|H)}_{= w_H - w_L} \leq c(e_H|L) - c(e_L|L)$

$$\Rightarrow c(e_H|L) - c(e_L|L) \geq w_H - w_L$$

so we get IC-L: $w_L - c(e_L|L) \geq w_H - c(e_H|L)$ ■

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2) $e_H \neq e_L$ for profit max firm:

Proof:

$$\max \ \lambda(F(e_L)-w_L) + (1-\lambda)[F(e_H)-w_H]$$

$$\text{s.t.} \begin{cases} w_H - c(e_H|H) = w_L - c(e_L|H) \\ w_L - c(e_L|L) = 0 \\ e_H \geq e_L \end{cases}$$

$$\max \ \lambda[F(e_L) - c(e_L|L)] + (1-\lambda)[F(e_H) - c(e_H|H) - c(e_L|L) + c(e_L|H)]$$

FOC:

e_L: $\lambda F_e(e_L) - C_e(e|L) + (1-\lambda) C_e(e_L|H) = 0$

e_H: $(1-\lambda) F_e(e_H) + (1-\lambda)(-C_e(e_H|H)) = 0$

$$\Rightarrow \ F_e(e_H) = C_e(e_H|H)$$

$$\lambda[\underbrace{F_e(e_L) - C_e(e_L|H)}_{0}] = C_e(e_L|L) - C_e(e_L|H)$$

If $e_H = e_L = e \Rightarrow C_e(e_L|L) = C_e(e_L|H)$ which contradicts single crossing property. So firm will never set $e_H = e_L$. ∎