Problem 1. Exchange with Many Buyers and Sellers.

Consider the following model of exchange. There are k sellers and k buyers. Each seller has a single indivisible object and these objects are perfect substitutes. Each buyer demands at most one unit. The cost to seller s of selling his object is $c_s \in [0,1]$ privately known to seller s. The value to buyer b from buying an object is $v_b \in [0,1]$, privately known to buyer b. All agents have quasi-linear utility.

(a) What is the efficient allocation rule as a function of all agents' types?

(b) Prove that there does not exist a mechanism that implements an efficient allocation rule in dominant strategies.

(c) Find a (direct-revelation) mechanism whose dominant strategy solution implements an allocation rule which gets very close to the efficient one as k grows large:

(i) Order the buyers in decreasing order of their *reported* valuations $v^1 \geq v^2 \geq \ldots \geq v^k$, and the sellers in increasing order of their *reported* costs $c^1 \leq c^2, \leq \ldots \leq c^k$. Suppose that we want the l highest-value buyers to buy units from the l lowest-value sellers. We can do this with separate auctions for the buyers and sellers. The l buyers will pay v^{l+1}, the l sellers will receive c^{l+1}. Prove that this mechanism is dominant-strategy incentive compatible for any l.

(ii) For a given profile of reported values and costs, what is the largest l such that the above scheme generates a budget surplus?

(iii) Using your answers to these, construct the desired mechanism.

Problem 2. Designing Mechanisms Graphically.

Consider the auction environment with two buyers. Suppose that each buyer i has willingness to pay v_i drawn from the set $[0,1]$.

(a) Draw a square with v_1 on the horizontal axis and v_2 on the vertical axis. A point in the square represents a profile (v_1, v_2).

(b) Draw a downward sloping curve through the box.

(c) Draw an upward sloping curve through the box that intersects the downward sloping curve (exactly once.)

(d) The region above your downward sloping curve is divided into two subregions by your upward sloping curve. Label the subregion that is above your upward sloping curve with a 2 and label the other subregion with a 1. Label the entire region that is below (and to the left of) your downward sloping curve with a 0.

(e) Consider the allocation rule that is defined by your drawing. In the 1 region agent 1 gets the good, in the 2 region agent 2 gets the good and in the 0 region neither agent gets the good. (On the boundary between regions pick the allocation from one of the neighboring regions.) Find a transfer rule which, when coupled with your allocation rule, forms a DSIC mechanism.

(f) Is there any DSIC allocation rule that picks alternatives from the set $\{0, 1, 2\}$ that could not be represented by a drawing that follows the instructions given above?