Question 1

The results are in the order presented in the notes.

Lemma 1. *If (e, w) is (weakly) preferred by type L to (e', w') and $e > e'$, then (e, w) is strictly preferred by type H.*

Proof. Note that (e, w) preferred to (e', w') by type θ means $w - c(e|\theta) \geq w' - c(e'|\theta)$, which holds iff

$$w - w' \geq c(e|\theta) - c(e'|\theta) = \int_{e'}^{e} c_e(x|\theta) \, \mathrm{d}x.$$

By assumption, $c_e(e|L) > c_e(e|H)$. Since $e > e'$, we have

$$w - w' \geq \int_{e'}^{e} c_e(x|L) \, \mathrm{d}x > \int_{e'}^{e} c_e(x|H) \, \mathrm{d}x,$$

which implies (e, w) is strictly preferred to (e', w') by H. $\square$

Corollary 1. *If the (IR-L) constraints and the (IC-H) constraints are satisfied then the (IR-H) constraint is also satisfied.*

Proof. By assumption we have $w_H - c(e_H|H) \geq w_L - c(e_L|H)$ and $w_L - c(e_L|L) \geq 0$. Noting that $c(e|H) = \int_0^e c_e(x|H) \, \mathrm{d}x \leq \int_0^e c_e(x|L) \, \mathrm{d}x = c(e|L)$ for all e, we have

$$w_H - c(e_H|H) \geq w_L - c(e_L|H) \geq w_L - c(e_L|L) \geq 0.$$

$\square$

Lemma 2. *If $(e_L, w_L), (e_H, w_H)$ satisfy the (IC) constraints, then $e_H \geq e_L$.*

Proof. The constraints imply

$$\begin{aligned} w_H - c(e_H|H) &\geq w_L - c(e_L|H) \\ w_L - c(e_L|L) &\geq w_H - c(e_H|L) \end{aligned}$$

Adding them and rearranging shows

$$c(e_L|H) - c(e_H|H) \geq c(e_L|L) - c(e_H|L),$$

or

$$\int_{e_H}^{e_L} c_e(x|H) \, \mathrm{d}x \geq \int_{e_H}^{e_L} c_e(x|L) \, \mathrm{d}x$$

Since $c_e(e|H) < c_e(e|L)$, the inequality holds iff $e_H \geq e_L$.

$\square$

Lemma 3. *A constrained profit-maximizing contract satisfies the (IR-L) and (IC-H) constraints with equality.*

Proof. Let (w_L, e_L, w_H, e_H) be any feasible contract. We will show that if either constraint does not bind with equality, then there exists a contract that is strictly better for the principal.

(IR-L) must bind with equality: Suppose that (IR-L) is slack: $w_L - c(e_L|L) > \epsilon > 0$ for some ϵ. Consider the alternative contract (w'_L, e_L, w'_H, e_H) where $w'_L = w_L - \epsilon$ and $w_H = w'_H - \epsilon$. The (IC) constraints are unchanged and hence still satisfied. (IR-L) is still satisfied. By the previous Corollary, (IR-H) is satisfied since (IC-H) and (IR-L) are satisfied. Therefore this contract is feasible. Furthermore, the principal's payoff from the contract is ϵ higher than the original contract. Hence this alternative contract is strictly better.

(IC-H) must bind with equality: Suppose that (IC-H) is slack: $w_H - c(e_H|H) > \epsilon + w_L - c(e_L|H)$ for some ϵ. Consider the alternative contract (w_L, e_L, w'_H, e_H) where $w'_H = w_H - \epsilon$. (IC-H), (IC-L), and (IR-L) are all still satisfied. By the previous Corollary, this implies that (IR-H) is also satisfied. Therefore this contract is feasible. Furthermore, the principal's payoff is $\epsilon \Pr(H)$ higher than under the original contract. Hence this alternative contract is strictly better. □

Lemma 4. *If the contract is monotone and the (IC-H) constraint is satisfied with equality, then the (IC-L) constraint is satisfied.*

Proof. We must show (IC-L): $w_L - c(e_L|L) \geq w_H - c(e_H|L)$, or equivalently $w_H - w_L \leq c(e_H|L) - c(e_L|L)$.

By assumption (IC-H) holds with equality: $w_H - c(e_H|H) = w_L - c(e_L|H)$, or equivalently $w_H - w_L = c(e_H|H) - c(e_L|H)$. By assumption $e_H \geq e_L$. Since $c_e(e|H) < c_e(e|L)$ for all e, we have

$$w_H - w_L = c(e_H|H) - c(e_L|H) = \int_{e_L}^{e_H} c_e(x|H)\ \mathrm{d}x \leq \int_{e_L}^{e_H} c_e(x|L)\ \mathrm{d}x = c(e_H|L) - c(e_L|L),$$

as desired. □

Question 2

Let $(\overline{w}, \overline{e})$ be a feasible contract that pools workers, and suppose that it is optimal. We show that there exists a feasible contract (w_L, e_L, w_H, e_H) that yields strictly higher profits for the seller.

If $(\overline{w}, \overline{e})$ is optimal, then by the previous question it must be that (IR-L) holds with equality: $\overline{w} - c(\overline{e}|L) = 0$. Since the (IC) constraints are vacuous and (IR-H) must be satisfied since $\overline{w} - c(\overline{e}|H) \geq \overline{w} - c(\overline{e}|L) = 0$, the contract must therefore satisfy

$$\max_e F(e) - c(e|L).$$

This implies that $F'(\overline{e}) = c_e(\overline{e}|L)$.

Consider the contract (w_L, e_L, w_H, e_H) where $w_L = \overline{w}$, $e_L = \overline{e}$, $w_H = \overline{w} + c(e_H|H) - c(\overline{e}|H)$, and $e_H > \overline{e}$. We show the following:

1. For any $e_H > \overline{e}$, the contract is feasible.

   Proof. (IR-L) holds with equality by construction: $w_L - c(e_L|L) = \overline{w} - c(\overline{e}|L) = 0$. (IC-H) also holds with equality by construction:

   $$w_H - c(e_H|H) = (\overline{w} + c(e_H|H) - c(\overline{e}|H)) - c(e_H|H) = \overline{w} - c(\overline{e}|H).$$

   By the Corollary in Question 1, this implies that (IR-H) is satisfied. Furthermore, since the contract is monotone by construction, by the final Lemma in Question 1 (IC-L) is satisfied. □

2. There exists $e_H > \overline{e}$ such that this contract is strictly better for the principal.

   Proof. The payoff to the principal from the new contract is

$$P(L)[F(\overline{e}) - \overline{w}] + P(H)[F(e_H) - w_H]$$

   The payoff to the principal from the original contract is $F(\overline{e}) - \overline{w}$. Hence the payoff to the new contract is strictly larger iff $F(e_H) - w_H > F(\overline{e}) - \overline{w}$. Since $w_H = \overline{w} + c(e_H|H) - c(\overline{e}|H)$, this is equivalent to $F(e_H) - c(e_H|H) > F(\overline{e}) - c(\overline{e}|H)$. The derivative of $F(e) - c(e|H)$ evaluated at $\overline{e}$ is $F'(\overline{e}) - c_e(\overline{e}|H) = c_e(\overline{e}|L) - c_e(\overline{e}|H) > 0$. By continuity of the derivative, $F'(e) - c_e(e|H) > 0$ for all e in some neighborhood $(\overline{e} - \delta, \overline{e} + \delta)$ of $\overline{e}$. Therefore letting $e_H = \overline{e} + \delta$, we have

$$F(e_H) - c(e_H|H) = F(\overline{e}) - c(\overline{e}|H) + \int_{\overline{e}}^{e_H} F'(x) - c_e(x|H)\, \mathrm{d}x > F(\overline{e}) - c(\overline{e}|H).$$

   This is what we wanted. □

Question 3

1. (Note: This question is written somewhat vaguely. I assume the monopolist must charge a linear price: $T(x) = px$.)

   A consumer of type θ facing price p solves

$$\max_x \theta \left(\frac{1-(1-x)^2}{2}\right) - px$$

   This yields demand function $x(p, \theta) = \max\{1 - p/\theta, 0\}$. Assuming that both consumers purchase, the aggregate demand function is $x(p) = \lambda x(p, \theta_L) + (1-\lambda)x(p, \theta_H) = 1 - p\left(\frac{\lambda}{\theta_L} + \frac{1-\lambda}{\theta_H}\right) := 1 - p/\theta^*$ where $1/\theta^* := \frac{\lambda}{\theta_L} + \frac{1-\lambda}{\theta_H}$.

   If the monopolist serves both consumers, he solves

$$\max_p (p-c)\left(1 - p/\theta^*\right)$$

   The FOC gives $p = \frac{c+\theta^*}{2}$. The profit is $\frac{(\theta^*-c)^2}{4\theta^*}$.

   Note that $x(p, \theta_H) \geq x(p, \theta_L)$, so the only other possibility is that the monopolist serves only θ_H types. If he does, he solves

$$\max_p (1-\lambda)(p-c)\left(1 - p/\theta_H\right)$$

   The FOC gives $p = \frac{c+\theta_H}{2}$, with profit $(1-\lambda)\frac{(\theta_H-c)^2}{4\theta_H}$.

   Both types are served iff

$$\frac{(\theta^*-c)^2}{4\theta^*} \geq (1-\lambda)\frac{(\theta_H-c)^2}{4\theta_H}$$

   For λ sufficiently large or θ^* sufficiently large (equivalently, θ_L sufficiently large), this inequality will hold.

2. The monopolist solves $p(\theta) \in \arg\max_p (p-c)x(p,\theta)$. This yields $p(\theta) = \frac{\theta+c}{2}$. Profits are

$$\lambda \frac{(\theta_L - c)^2}{4\theta_L} + (1-\lambda)\frac{(\theta_H - c)^2}{4\theta_H}$$

3. Let $S(p,\theta) := \theta v(x(p,\theta)) - px(p,\theta)$ denote the surplus of a type θ when the price is p. Suppose the monopolist charges a fee F and price p, and suppose that both types are served. Then, for any p, the largest fee that the monopolist can charge is $F = S(p,\theta_L)$, since $S(p,\theta_H) \geq S(p,\theta_L)$. Therefore the monopolist chooses p to solve

$$\max_p S(p,\theta_L) + (p-c)x(p).$$

   Plugging in, the problem is

$$\max_p \theta_L \left(\frac{1-(1-(1-p/\theta_L))^2}{2} \right) - p\,(1-p/\theta_L) + (p-c)(1-p/\theta^*).$$

   The FOC gives $p = \frac{c}{2-\theta^*/\theta_L}$.

   If the monopolist chooses to not serve the low types, he will set $p = c$ to sell the efficient quantity to the high types and extract the full surplus using the fee $F = S(p,\theta_H)$. The profit is $S(c,\theta_H) = \frac{(\theta_H-c)^2}{2\theta_H}$.

   The monopolist serves both types iff the profit from serving both types is at least $S(c,\theta_H)$.

4. The monopolist solves

$$\max_{(T_L,x_L,T_H,x_H)} \lambda\,(T_L - cx_L) + (1-\lambda)\,(T_H - cx_H)$$

   subject to (IC) and (IR):

$$\theta_H v(x_H) - T_H \geq \theta_H v(x_L) - T_L \tag{IC-H}$$
$$\theta_L v(x_L) - T_L \geq \theta_L v(x_H) - T_H \tag{IC-L}$$
$$\theta_H v(x_H) - T_H \geq 0 \tag{IR-H}$$
$$\theta_L v(x_L) - T_L \geq 0 \tag{IR-L}$$

   For arguments similar to those in the screening model, (IC-H) and (IR-L) will hold with equality. Therefore $T_L = \theta_L v(X_L)$ and $T_H = \theta_H v(x_H) - (\theta_H - \theta_L)v(x_L)$. This also implies (IR-H) will hold. (IC-L) holds since $0 \geq \theta_L v(X_H) - T_H$ iff $0 \geq (\theta_L - \theta_H)(v(x_H) - v(x_L))$, which holds iff $x_H \geq x_L$ which we will see is the case. Plugging T_L and T_H into the objective function, we solve

$$\max_{(x_L,x_H)} \lambda\,(\theta_L v(x_L) - cx_L) + (1-\lambda)\,(\theta_H v(x_H) - cx_H - (\theta_H - \theta_L)v(x_L)).$$

   The FOCs are

$$\theta_L(1-x_L) = c \bigg/ \left(1 - \frac{1-\lambda}{\lambda}\frac{\theta_H - \theta_L}{\theta_L}\right)$$

   and

$$\theta_H(1-x_H) = c.$$

   These imply that $x_H \geq x_L$.

Question 4

If a type θ chooses effort level e, his utility is $E_{\mu(e)}\theta - c(e|\theta)$. By choosing $e = 0$, he can guarantee himself a payoff of at least L (since $E_{\mu(e)}\theta \geq L$ for all μ). Since $E_{\mu(e)}\theta \leq H$ for all μ, we know the max wage is H.
Denote by e^* the solution to $L = H - c(e^*|_{\theta_l})$, then any education level above e^* is weakly dominated by $e = 0$for the low type as cost of eduction is monotone increasing which implies $L \geq H - c(e|_{\theta_l}) \quad \forall e \geq e^*$. This implies that $\mu(e) = H \quad \forall e \geq e^*$ after deleting the dominating strategies for the low type. This in turn implies that $e > e^*$ is also weakly dominated by e^* for the high type, as the wage is constant in e in this range.

Combing these claims implies that the unique surviving separating equilibrium is $e_h = e^*$, note this is the best separating equilibrium. Also note that in this equilibrium low workers are indifferent between the two on path actions.
As was the case in the example in class off-path beliefs must be low enough in the range $(0, e^*)$, but unlike the example in class in the range (e^*, e') beliefs are high with probability 1, where $L = H - c(e'|_{\theta_h})$. Above this range beliefs are free.

If we look at pooling equilibria, we can also eliminate some equilibria as the deletion of dominated strategies limits the off path beliefs of the firm. This restriction, is not a problem with regard to the low type. This is the case as in the construction of equilibria in class we demanded $E\theta - c(e|_{\theta_l}) \geq L$. Moreover the restriction on beliefs from iterative deletion only effects levels of education for which $H - c(e|_{\theta_l}) < L$ thus the condition required in class implies there is no profitable deviation no matter what the beliefs above e^* are.
However, these restrictions are important when checking optimality for the high type. As $\mu(e) = H \quad \forall e \geq e^*$ we must have the equilibrium level of education satisfying:

$$E\theta - c(e|_{\theta_h}) \geq H - c(e^*|_{\theta_h})$$

This implies that the maximal level of education in a pooling equilibrium is lower than if we did not delete dominated strategies.
For example $E\theta - c(\hat{e}|_{\theta_l}) = L$ is an pooling equilibrium in dominated strategies if the wage for all other levels of education is L. Deleting dominated strategies eliminates this equilibrium as this level of education makes L indifferent between $\hat{e}$ and choosing e^* and getting a wage H (which must be the wage after deletion). As the high type has a lower cost of education he will strictly prefer deviating to e^* rather than selecting $\hat{e}$.
Depending on the parameters of the question there may be no pooling equilibria. This will be the case if

$$E\theta - c(0|_{\theta_h}) < H - c(e^*|_{\theta_h})$$

As if this condition holds it is always profitable for the high type to separate himself by selecting e^* and get a wage of H.

Question 5

Let λ denote the probability of type H. In the pooling equilibrium, both types H and L choose $e = 0$ and get payoff $\lambda H + (1 - \lambda)L$. In a separating equilibrium, H types choose $e = e_H > 0$ and get utility $H - c(e_H|H)$, and L types choose $e = 0$ and get utility L.

For any λ, L types strictly prefer the pooling equilibrium. H types prefer the pooling equilibrium if $\lambda H + (1 - \lambda)L > H - c(e_H|H)$, or equivalently iff $c(e_H|H) > (1 - \lambda)(H - L)$. Since $e_H > 0$ so that $c(e_H|H) > 0$, for

λ sufficiently close to 1 this inequality will hold.

(Note: the firm makes 0 profit in all equilibria.)