From Auctions to Mechanism Design

Alexey Makarin

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Introduction

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  - Seller could hold an auction and then negotiate with winner; etc.
- This week, we abstract away from the details of any particular selling format and ask: *"What is the best way to allocate an object?"*

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- This week: Optimal allocation rule to sell an object?

From Auctions to a General Mechanism Design Problem

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- Buyer i's valuation V_i is distributed over the interval $\mathcal{V}_i = [0, \omega_i]$ according to c.d.f. F_i with density f_i.

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- Let $\mathcal{V} = \times_{j=1}^{N} \mathcal{V}_j$ denote the product of the sets of buyers' values and, for all i, let $\mathcal{V}_{-i} = \times_{j \neq i} \mathcal{V}_j$

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- Similarly, define $f_{-i}(\mathbf{v}_{-i})$ to be the joint density of $\mathbf{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N)$

Mechanism

Definition

A selling *mechanism* $(\mathcal{B}, \boldsymbol{\pi}, \boldsymbol{\mu})$ is a combination of:

1. A set of possible *messages* (or "bids") $\mathcal{B}_i$ for each buyer i;
2. An *allocation rule* $\boldsymbol{\pi} : \mathcal{B} \to \Delta$ where Δ is the set of probability distributions over the set of buyers $\mathcal{N}$;
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- Allocation rule determines, as a function of all N messages, the probability $\pi_i(\mathbf{b})$ that i will get the object.
- Payment rule determines, as a function of all N messages, for each buyer i, the expected payment $\mu_i(\mathbf{b})$ that i must make.

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$$\pi_i(\mathbf{b}) = \begin{cases} 1 & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

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- Second-price auction:

$$\mu_i^{II}(\mathbf{b}) = \begin{cases} \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

Game Within a Mechanism

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  - Strategies: $\beta_i : [0, \omega_i] \to \mathcal{B}_i$
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- *Note*: Today, we focus on **Bayesian Mechanism Design**, i.e., with BNE as an underlying equilibrium concept. Later, we will come back and consider **Dominant Strategies Mechanism Design** which relies on the Dominant Strategy Equilibrium.

Direct Mechanisms and Revelation Principle

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- A smaller and simpler class are mechanisms for which the set of messages is the same as the set of values—that is, $\mathcal{B}_i = \mathcal{V}_i \ \forall i$.
- Such mechanisms are called *direct*, since every buyer is asked to directly report a value.

Direct Mechanisms

- Formally, direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ consists of functions $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$, where $Q_i(\mathbf{v})$ is the probability that i will get the object and $M_i(\mathbf{v})$ is the expected payment by i.

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- *Revelation principle*: outcomes of any mechanism's equilibrium can be replicated by a truthful equilibrium of a direct mechanism.
- Hence, no loss of generality in focusing on direct mechanisms.

Proposition. Revelation Principle.

Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.

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- Now suppose that some buyer finds it profitable to be untruthful and report a value $\hat{v}_i$ when his value is v_i.
- Then in the original mechanism same buyer would have found it profitable to submit $\beta_i(\hat{v}_i)$ instead of $\beta_i(v_i)$. Contradiction. ■

Buyer's Payoff Function

Given a direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$:

$$q_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} Q_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

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$$m_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} M_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

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$$q_i(\hat{v}_i) v_i - m_i(\hat{v}_i)$$

is the expected payoff of i when he reports $\hat{v}_i$ and others tell the truth.

Incentive Compatibility

The direct revelation mechanism is incentive compatible (IC) if:

$$U_i(v_i) \equiv q_i(v_i)v_i - m_i(v_i) \geq q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)$$

$$\forall i \in \mathcal{N}; \forall v_i, \hat{v}_i \in [0, \omega_i]$$

where U_i is the *equilibrium payoff function*.

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- That is, U_i is a maximum of a family of affine functions, therefore U_i is a *convex function*.

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- Since the above inequality has to hold for all v_i and $\hat{v}_i$, $q_i(v_i)$ is the subgradient of the function U_i at v_i.

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- Since the above inequality has to hold for all v_i and $\hat{v}_i$, $q_i(v_i)$ is the subgradient of the function U_i at v_i.
- Since U_i is convex, it must be that q_i is *non-decreasing*.

Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

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- That is, up to an additive constant, buyer's expected payoff in an IC direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ depends only on allocation rule $\boldsymbol{Q}$.
- Thus, if $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are two IC mechanisms with same allocation rule but different payment rules, then their expected payoff functions, U_i and $\bar{U}_i$, differ by at most a constant.

Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

$$U_i'(v_i) = q_i(v_i) \implies U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(x_i)dx_i$$

- That is, up to an additive constant, buyer's expected payoff in an IC direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ depends only on allocation rule $\boldsymbol{Q}$.
- Thus, if $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are two IC mechanisms with same allocation rule but different payment rules, then their expected payoff functions, U_i and $\bar{U}_i$, differ by at most a constant.
- In other words, $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are *payoff equivalent*.

Revenue Equivalence Strikes Again!

Generalized Revenue Equivalence

If the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ is incentive compatible, then for all i and v_i, the expected payment is

$$m_i(v_i) = m_i(0) + q_i(v_i)v_i - \int_0^{v_i} q_i(x_i)dx_i$$

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Generalized Revenue Equivalence

Remarks:

- Given two BNE of two different auctions such that for each i:
  - For all $(v_1, \ldots, v_N)$, probability of i getting the object is the same,
  - Two equilibria have the same expected payment at 0 value.

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- This generalizes the result from last time:
  - Revenue equivalence at the symmetric equilibrium of standard auctions (object allocated to buyer with the highest bid).

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$$U_i(\hat{v}_i) \geq U(v_i) + q_i(v_i)(\hat{v}_i - v_i) \iff$$
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- The latter inequality certainly holds if q_i is nondecreasing.

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- (We implicitly assume that by not participating, a buyer can guarantee himself a payoff of zero.)
- If the mechanism is IC, then IR is equivalent to $U_i(0) \geq 0$, and since $U_i(0) = -m_i(0)$ this is equivalent to $m_i(0) \leq 0$.

Optimal Mechanisms

Finding an Optimal Mechanism

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where, in the last step, we used the definition of q_i as the expected allocation probability intergrated over valuations of all other players.

Optimal Mechanism Design Problem

$$\sum_{i \in \mathcal{N}} m_i(0) + \sum_{i \in \mathcal{N}} \int_{\mathcal{V}} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] Q_i(\mathbf{v}) f(\mathbf{v}) d\mathbf{v} \to \max_{\boldsymbol{Q}, \boldsymbol{M}}$$

s.t. *IC* constraint ($\Leftrightarrow q_i$ is nondecreasing)

IR constraint ($\Leftrightarrow m_i(0) \leq 0$)

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- Let's define **virtual valuation** of a buyer with value v_i as:

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- We say that design problem is regular if $\psi_i(v_i)$ is increasing in v_i (it is sufficient that hazard rate $\lambda_i(v_i)$ is increasing in v_i).
- Let us temporarily neglect the IC and the IR constraints and circle back to them later.

Optimal Mechanism Design Problem

- The seller should choose $(\boldsymbol{Q}, \boldsymbol{M})$ to maximize:

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{V}} \left(\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v}) \right) f(\mathbf{v}) d\mathbf{v}$$

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- This approach would maximize this expression at every point $\mathbf{v}$ and so would also maximize its integral.

Optimal Mechanism

Claim

The following is an optimal mechanism:

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1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

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First, let's check IC and IR:

- IC: Check if q_i is nondecreasing. Suppose $\hat{v}_i < v_i$. Then by the regularity condition, $\psi_i(\hat{v}_i) < \psi_i(v_i)$ and, thus, for all v_{-i}, it is also the case that $Q_i(\hat{v}_i, v_{-i}) \leq Q(\mathbf{v})$. Thus, q_i is nondecreasing.

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First, let's check IC and IR:

- IR: From the payment rule, it is clear that $M_i(0, \mathbf{v}_{-i}) = 0$ for all $\mathbf{v}_{-i}$, and thus $m_i(0) = 0$.

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Second, note that this mechanism separately maximizes two terms in seller's expected revenue: (i) It gives positive weight only to nonnegative and maximal terms in the second expression; (ii) It sets maximal $m_i(0)$ given the IR constraint.

Optimal Mechanism

This implies that the maximized value of the expected revenue is:

$$\mathbb{E}[\max\{\psi_1(V_1), \ldots, \psi_N(V_N), 0\}]$$

In other words, it is the expectation of the highest virtual valuation, provided it is nonnegative.

Optimal Mechanism

To obtain more intuitive formulas for $(\boldsymbol{Q}, \boldsymbol{M})$, we define:

$$y_i(\mathbf{v}_{-i}) = \inf\{x_i : \psi_i(x_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(x_i) \geq \psi_j(v_j)\}$$

as the smallest value for i that "wins" against $\mathbf{v}_{-i}$.

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as the smallest value for i that "wins" against $\mathbf{v}_{-i}$. Thus, can rewrite the optimal allocation rules as:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } x_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } x_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

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$$\implies M_i(\mathbf{v}) = \begin{cases} y_i(\mathbf{v}_{-i}) & \text{if } Q_i(\mathbf{v}) = 1 \\ 0 & \text{if } Q_i(\mathbf{v}) = 0 \end{cases}$$

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Thus, only the 'winning' buyer pays anything. He pays the smallest value that would result in his winning.

Optimal Mechanism

Proposition.

Suppose the design problem is regular. Then the following is an optimal mechanism:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } \psi_i(v_i) > \max_{j \neq i} \psi_j \text{ and } \psi_i(v_i) \geq 0 \\ 0 & \text{if } \psi_i(v_i) < \max_{j \neq i} \psi_j \end{cases}$$

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Illustration

Optimal Mechanism in a Symmetric Case

Suppose the problem is symmetric, i.e., $f_i = f$, and hence $\psi_i = \psi \ \forall i$.

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Proposition.

Suppose that the seller's design problem is regular and symmetric. Then a second-price auction with a reserve price $r^* = \psi^{-1}(0)$ is an optimal mechanism.

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Thus, the optimal mechanism is SPSB with reserve price $r^* = \psi^{-1}(0)$.

Proposition.

Suppose that the seller's design problem is regular and symmetric. Then a second-price auction with a reserve price $r^* = \psi^{-1}(0)$ is an optimal mechanism.

Note that $\psi^{-1}(0)$ is the optimal reserve price we derived earlier!

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2. Efficient mechanisms give objects to buyer with highest value.
   - Here, it is allocated to the buyer with highest *virtual* valuation.
   - In the asymmetric case, this need not be the highest-value buyer.

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- The inverse demand curve is then $p(q) \equiv F^{-1}(1 - q)$.

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- Thus, virtual valuation of a buyer can be interpreted as a *marginal revenue*. (Recall that ψ is strictly increasing.)

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- If no buyer's value v_i exceeds his reserve price r_i^*, the seller keeps the object.
- Otherwise, object is allocated to buyer with highest MR and he is asked to pay $p_i = y_i(\mathbf{v}_{-i})$, smallest value such that he still wins.

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- Thus, buyer 2 will "win" more often than is dictated by a comparison of actual values alone.

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- Because of informational asymmetry, seller is unable to perfectly price discriminate and extract all the surplus.
- Buyers must be given informational rents to get them to reveal their private information.

Auctions vs. Mechanisms

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- Thus, the optimal mechanism does not satisfy two important properties of auctions.
- Since these properties are important from a practical standpoint, one might want to restrict attention to mechanisms that satisfy universality and anonymity (*Wilson's doctrine*).

Efficient Mechanisms

Defining Efficient Mechanism

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- An allocation rule $\boldsymbol{Q}^* : \mathcal{V} \to \boldsymbol{\Delta}$ is *efficient* if it maximizes "social welfare"—that is, for all $\mathbf{v} \in \mathcal{V}$:

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- Any mechanism with an efficient allocation rule is called *efficient*.

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- Similarly, define welfare of agents other than i as:

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The VCG Mechanism

- The *VCG (Vickrey-Clarke-Groves)* mechanism $(\boldsymbol{Q}^*, \boldsymbol{M}^V)$ is an efficient mechanism with payment rule $\boldsymbol{M}^V : \mathcal{V} \to \mathbb{R}^N$ given by:

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- (Of course, both calculated assuming efficient allocation rule $\boldsymbol{Q}^*$.)

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Intuitively, this payment rule makes i internalize the externality of him lying about his value. Then i's equilibrium payoff is:

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- Since expected payoffs in $(\boldsymbol{Q}^*, \boldsymbol{M})$ are greater than in VCG, and allocation is the same, expected payments must be lower. $\blacksquare$

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- While VCG does not always have a balanced budget, it is still helpful in determining when this is feasible (more on this later).

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so that budget is balanced *ex post* for all $\mathbf{v}$:

$$\sum_{i \in \mathcal{N}} M_i^A(\mathbf{v}) = 0$$

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Note that the second term does not depend on z_i. The first term is simply expected total welfare. It is maximized by setting $z_i = v_i$ because if $z_i > v_i$ or $z_i < v_i$, then the object could potentially be allocated inefficiently. ■

(Note that AGV may not satisfy individual rationality).

Existence of Efficient, IC, IR, and Balanced Mechanisms

Proposition

There exists an efficient, incentive compatible, and individually rational mechanism that balances the budget if and only if the VCG mechanism results in an expected surplus.

Proof: *Necessity* follows immediately: if VCG runs a deficit, then all other efficient, IC, and IR mechanisms run a deficit too. (Why?).

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Need to show *sufficiency*. For that, let's construct an efficient, IC, and IR mechanism that balances the budget.

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Proof (cont'd): Define $d_i = c_i^A - c_i^V \ \forall i > 1$. Let $d_1 = -\sum_{i=2}^{N} d_i$.

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Proof (cont'd): For all $i \neq 1$:

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Left to check $i = 1$. By construction, $\sum_{i=1}^N d_i = 0$, so observe that:

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This proposition is quite useful in considering various efficient allocation problems. We now turn to an example of bilateral trade.

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- Is there some way to guarantee that trade will take place whenever it should?

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- At the moment, we do not restrict P or R to be positive or negative, nor do we assume that the budget is balanced—that is, P may not equal R.
- A mechanism is efficient if whenever $V > C$, the object is produced and allocated to the buyer.

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Unsurprisingly, VCG is efficient—object is transferred whenever $v > c$.

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  - For any realization of V and C such that $\overline{c} > V > C > \underline{v}$, the deficit $R - P = V - C$, i.e., exactly the ex-post gains from trade.

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- But since the VCG mechanism runs a deficit, every other mechanism also runs a deficit.
- Thus, in a bilateral trade problem, there does not exist an efficient mechanism that is IC, IR, and balances the budget. ∎

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   - FPSB and SPSB auctions: strategies, revenues, etc.
   - Revenue equivalence: basic (FPSB vs. SPSB) and general (symmetric and increasing equilibria in any standard auction).
   - FPSB and SPSB with reserve prices, optimal reserve price.
2. Mechanisms:
   - Object allocation as a mechanism design problem.
   - Revelation principle.
   - Optimal mechanism: virtual valuations, individual reserve prices.
     - Optimal mechanism with symmetric buyers: SPSB with an optimal reserve price.
   - Efficient mechanisms: VCG, AGV, balanced budget.
     - Myerson-Satterthwaite theorem: in a bilateral trade problem, no mechanism is efficient, IC, IR, and budget-balanced.

Extra: Online Auctions

Use of Auctions on the Internet

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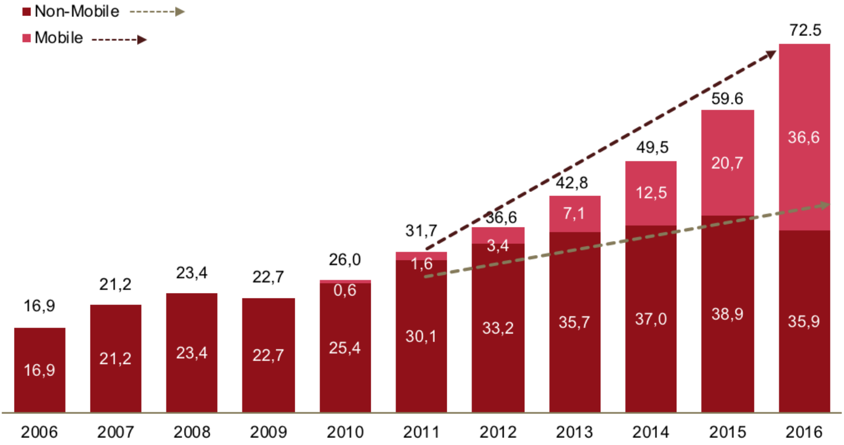
- This is a big market. How ads/paid search positions are sold?
- Typically bid per click for certain key words, like "Rome," "master in economics," etc.

IAB Internet Advertising Revenue Report 2016, US$ bln



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Why first-price auctions?

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Now auctions are taught in best computer science programs as they are they key source of revenue for many internet companies (ads generate more than 90% of Google's revenue).

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Since then the format was dominant on the Internet and is used to sell most online ads. (However, in 2019, Google went back to a version of a first-price auction, allegedly to increase transparency and boost competition.)