FINAL EXAM

Ground rules:

1. Your webcam must be on at ALL times.
2. Using your devices for purposes other than reading the exam or communicating with me is prohibited.
3. The exam is closed book. My goal is to test your understanding of the material and not your memorization skills. If you forgot some definition or formula from the lecture, you can ask me for it directly.
4. Ask for permission if you need to leave the desk.
5. When you finish, send me a scan copy of your exam via CamScan. Please mark your pages by number.
6. After the end of the exam at 18:00 CET, you need to put down the pen and stop writing. I will be deducting one point per one minute I see you going over time.
7. Finally, I would like to remind you of the RoME honour code which specifies that students may be expelled from the program in case of cheating or improper behavior.

Problem 1. Issues with the VCG Mechanism (15 pts.)

In this problem, you will explore various issues with the VCG mechanism.

1. Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a; b\}$ to three buyers. A buyer can buy none, one, or both of the objects. The buyers' valuations are:

$$u_1(\varnothing) = 0, u_1(\{a\}) = 9, u_1(\{b\}) = 0, u_1(\{a,b\}) = 9$$
$$u_2(\varnothing) = 0, u_2(\{a\}) = 0, u_2(\{b\}) = 9, u_2(\{a,b\}) = 9$$
$$u_3(\varnothing) = 0, u_3(\{a\}) = 2, u_3(\{b\}) = 3, u_3(\{a,b\}) = 10$$

   As in class, consider a version of the VCG payment rule: each buyer i pays m_i, which is the maximum welfare of the other buyers minus the realized welfare of the other buyers, both computed using the reported valuation functions.

   (a) Determine the assignment of objects to buyers and the payments, under truthful bidding. (4 pts.)

   (b) Discuss what is the issue with the VCG mechanism in this case. (1 pt.)

2. Consider the same setup but with different valuations:

$$u_1(\varnothing) = 0, u_1(\{a\}) = 3, u_1(\{b\}) = 0, u_1(\{a,b\}) = 3$$
$$u_2(\varnothing) = 0, u_2(\{a\}) = 0, u_2(\{b\}) = 4, u_2(\{a,b\}) = 4$$
$$u_3(\varnothing) = 0, u_3(\{a\}) = 2, u_3(\{b\}) = 3, u_3(\{a,b\}) = 10$$

   (a) Determine the assignment of objects to buyers and the payments, under truthful bidding. (1 pts.)

   (b) This example highlights another flaw of the VCG mechanism. State this flaw and discuss it. (4 pts.)

3. Consider a similar setup but with two buyers and the following valuations:

$$u_1(\varnothing) = 0, u_1(\{a\}) = 1, u_1(\{b\}) = 1, u_1(\{a,b\}) = 10$$
$$u_2(\varnothing) = 0, u_2(\{a\}) = 1, u_2(\{b\}) = 1, u_2(\{a,b\}) = 9$$

   (a) Determine the assignment of objects to buyers and the payments, under truthful bidding. (1 pts.)

   (b) Suppose buyer 2 thinks about entering the system as two different buyers, 2A and 2B. Can it be profitable fir her to do this? Discuss whether you think such splitting schemes are a flaw of the VCG mechanism. Can it be problematic in practice? (4 pts.)

Problem 2. Moral Hazard and Head-Hunting (30 pts.)

A risk-neutral small firm hires an agent whose effort is unobservable to the firm. The agent's effort $e \in \{0, 1\}$ determines the probability distribution of revenues $x \in \{x_L, x_H\}$, $x_H > x_L$, as follows: $Pr(x = x_H|e = 1) = 1/2$ and $Pr(x = x_H|e = 0) = 0$.

However, if the agent exerts high effort, then with probability $q \in (0, 1)$ he is scouted and hired away by a big firm before any revenue for the small firm is realized. If the agent is hired away, he gets utility of 1 (note that this includes his effort cost). If the agent stays at the firm and exerts effort level e for wage w, he gets utility $u(w) - e$, where $u(w)$ is continuous and strictly concave with $u(0) = 0$. The firm's profits are $x - w$ if the agent is not scouted, and 0 if he is. Assume that the agent's reservation utility is zero and that wages are non-negative. Suppose the firm offers a contract to the agent, which the agent may accept or reject. A contract is a profile of wages (w_L, w_H) to be paid to the agent after revenue is realized. If the agent accepts, he then chooses his effort.

(a) Suppose the firm wants to induce low effort. What do they set w_H to? Find the minimum w_L such that the agent is willing to comply with the firm's wish. Provide the intuition for your findings. (10 pts.)

(b) Now suppose $q = 5/7$ and $u(w) = \sqrt{w}$. Find necessary and sufficient conditions on x_L and x_H such that determines whether the firm would wish to induce low effort, and interpret. (20 pts.)

Problem 3. Third-Price Auction (35 pts.).

Consider an auction for a single object among three bidders. The format is a *third price* auction. The high bidder wins and pays the third-highest bid (here the lowest bid because there are three bidders.) If there is a tie, each high bidder will have an equal chance of being selected as the winner. Losers pay nothing. Assume that the bidders' private valuations v_i are independently drawn from the uniform distribution on the unit interval.

(a) Prove that it is a weakly dominated strategy for bidder i to bid strictly less than his true value v_i. (5 pts.)

(b) Prove that it is not a weakly dominant strategy for a bidder i to bid his true value v_i. (5 pts.)

(c) Let's consider a symmetric (interim) Bayesian Nash equilibrium in which a bidder with value v bids according to the strategy $\beta(v)$.

  (i) Let $q(b)$ be the probability, in equilibrium, that a bidder wins when he bids b and $m(b)$ be the expected payment made by a bidder who bids b. Explain why the following is a necessary condition of equilibrium: (2 pts.)

$$vq(b(v)) - m(b(v)) \geq vq(b') - m(b') \ \forall b' < b(v).$$

  (ii) Based on the previous observation prove that the equilibrium bidding strategy β is weakly increasing, i.e, if $v' > v$ then $b(v') \geq b(v)$. (8 pts.)

  (iii) Let's consider a symmetric equilibrium which is in fact strictly increasing. Apply the revenue equivalence theorem to prove that $b(v) = 2v$. (*Whether or not you can complete the calculations, you should describe clearly the steps you would take to verify this and how you would use revenue equivalence.*) (15 pts.)

   - (BONUS QUESTION: What is the optimal bidding strategy in a general case with N bidders?)

Problem 4. Bayesian Persuasion with Lie Detection (20 pts.)

A bank depositor is deciding whether to run from the bank or stay put. Her decision depends on the state of nature $\theta = \{\theta_G, \theta_B\}$. Depositor's payoffs are as follows:

Payoff	θ_G	θ_B
Stay	1	-2
Run	0	0

Depositor knows the probability of bad state, which is 1/2, so without any additional information she would choose to run. Bank of Italy (BdI, for *Banca d'Italia*) knows the true state and decides to intervene. It wants to induce depositor to stay as often as possible. BdI can give depositor a recommendation $s = \{Stay, Run\}$, thus potentially revealing some information about the true state. BdI pre-commits to a certain signal structure which then becomes known to the depositor.

(a) What is the best information structure that BdI selects? Make sure to detail all logical steps in your answer. (*Hint: You can safely assume that BdI wouldn't recommend 'Run' when the state is good.*) (5 pts.)

(b) Suppose now that if BdI lies about the true state, depositor learns about it with probability q. What is the best information structure for BdI as a function of q? Is the probability of BdI lying increasing or decreasing with the probability of lie detection q? Provide the intuition for this comparative statics. (*Hint: Again, you can safely assume that BdI wouldn't lie about the good state.*) (15 pts.)