1. Discussion of Problem Set #1

2. Screening on Subway

The Chicago CTA has decided that it needs to do more to maximize its revenue. As such it has hired you to design its new price and service scheme. There are two types of customers, High-class and Low-class. They have preferences over the fare P and the degree of bad smell in the train car they ride in, denoted by B. They have told you that they are able to charge different fares depending on the car a customer rides in (i.e., to have different classes of service).

The type of a customer is not observable; the fraction of high-class customers is λ. Customers' utility functions are $u_i(P, B) = v - \theta_i P - B$, for $i = H, L$, where $\theta_L > \theta_H > 0$. All customers get utility (normalized) of 0 from walking (their next best alternative) instead of taking the CTA train.

Making train cars smell bad is not costless (workers need to be hired to make the cars smell worse): the CTA incurs a cost of $\gamma B > 0$ per customer who rides in a car that has smell level B.

1. Write down the problem you would solve for determining the CTA's profit-maximizing scheme. Assume throughout that the CTA cannot charge negative prices; i.e., that $P \geq 0$. Assume also that the CTA wants to serve both high and low class customers.
2. Solve the maximization problem.
   (a) Are there any constraints that are redundant?
   (b) Are there any constraints that must bind?
   (c) Can we say something about B_L and B_H right away? Why or why not?
   (d) What is the CTA's profit-maximizing scheme? How does it depend on the parameters of the problem?

Solution

1. Appealing to the Revelation Principle, the CTA's problem can be written as:

$$\begin{aligned}
\max_{(P_L,B_L)\geq 0,(P_H,B_H)\geq 0} \quad & \lambda(P_H - \gamma B_H) + (1-\lambda)(P_L - \gamma B_L) & \\
\text{s.t.} \quad & v - \theta_L P_L - B_L \geq 0 & (IR_L)\\
& v - \theta_H P_H - B_H \geq 0 & (IR_H)\\
& v - \theta_L P_L - B_L \geq v - \theta_L P_H - B_H & (IC_L)\\
& v - \theta_H P_H - B_H \geq v - \theta_H P_L - B_L & (IC_H)
\end{aligned}$$

2.
   - Observe first that (IR_H) is redundant since it is implied by (IC_H) plus (IR_L).
   - Observe next that (IR_L) will bind: otherwise we could raise both P_L and P_H by some $\varepsilon > 0$ and not violate any constraints, thereby raising profit.
   - Observe next that $B_H = 0$. If not, then we can lower B_H and raise P_H so that $\theta_H P_H + B_H$ is unchanged. This violates no constraints (since it raises $\theta_L P_H + B_H$), but raises profit.
   - Next observation is that (IC_H) should bind as well. If not, increase P_H or decrease B_L.
   - Given everything said above we infer that (IC_L) should be slack. Indeed, given $B_H = 0$ we can rewrite

the binding (IC_H) as $B_L = \theta_H (P_H - P_L)$ and rewrite (IC_L) as $B_L \leq \theta_L (P_H - P_L)$. Since $\theta_H < \theta_L$, (IC_L) is implied by (IC_H).

In the end we get from (IR_L) and (IC_H) respectively that $P_L = \frac{v-B_L}{\theta_L}$ and $P_H = \frac{B_L}{\theta_H} - P_L = \frac{B_L}{\theta_H} - \frac{v-B_L}{\theta_L}$. So the CTA should choose $B_L \in [0, v]$ to solve

$$\max_{B_L} \lambda(\frac{B_L}{\theta_H} - \frac{v - B_L}{\theta_L}) + (1-\lambda)[\frac{v - B_L}{\theta_L} - \gamma B_L].$$

This problem is linear, so the solution is to set $B_L = v$ if $\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{1}{\theta_H} + \frac{1}{\theta_L}\right) - \left(\frac{1}{\theta_L} + \gamma\right) > 0$ and set $B_L = 0$ otherwise. Note that in the latter case, we have $P_L = P_H = \frac{v}{\theta_L}$, while in the former case we have $P_H - P_L > 0$. Also, observe that we are more likely to set $B_L = v$ when the fraction of high class consumers (λ) is high and the cost of making train cars smelly (γ) is low.