1. Discussion of Problem Set #5

2. An Optimal Mechanism

There is a single object for sale and there are two potential buyers. The value assigned by buyer 1 to the object is $v_1 \sim U[0, 1+k]$, while the value of the object for buyer 2 is $v_2 \sim U[0, 1-k]$ independently distributed, where $k \in (0, 1)$.

(a) Calculate the optimal mechanism. That is, describe the optimal allocation as a function of virtual valuations and describe the payment rule. Is the optimal mechanism efficient?

(b) Now assume that there is a third buyer with independent value $V_3 \sim exp(1)$. Describe the optimal allocation. What is the minimum sample value v_3 such that the third buyer gets the object for sure, that is, independently of the values of the first two buyers. Describe the optimal payment rule for the third buyer. Is it bounded?

3. VCG Example #1

Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a; b\}$ to three buyers. A buyer can buy none, one, or both of the objects. For simplicity, assume the valuation function of each buyer depends only on the set of objects assigned to that buyer. The values are:

$$u_1(\varnothing) = 0, u_1(\{a\}) = 10, u_1(\{b\}) = 3, u_1(\{a, b\}) = 13$$
$$u_2(\varnothing) = 0, u_2(\{a\}) = 2, u_2(\{b\}) = 8, u_2(\{a, b\}) = 10$$
$$u_3(\varnothing) = 0, u_3(\{a\}) = 3, u_3(\{b\}) = 2, u_3(\{a, b\}) = 14$$

Since we do not know each buyer's lowest possible value for each product, consider an alternative VCG payment rule: each buyer i pays m_i, which is the maximum welfare of the other buyers minus the realized welfare of the other buyers, both computed using the reported valuation functions.

(a) Determine the assignment of objects to buyers and the payments of the buyers, under truthful bidding.

(b) Discuss why buyer 3 might have an objection to the outcome.

4. VCG Example #2

Consider the following buyer/seller framework. Let $x \in [0, 100] = X$ denote the amount of a good that is produced by a seller, which a buyer then consumes. Let $\theta_B, \theta_S \in [1, 2] = \Theta_B = \Theta_S$ denote the types of the buyer and seller, respectively. Types are independently distributed and privately known by each agent. Assume that utility functions are quasilinear in money, and are given by

$$U_B(x, t, \theta) = \theta_B x - t_B$$
$$U_S(x, t, \theta) = -\frac{x^2}{2\theta_S} - t_S$$

where $t = (t_B, t_S)$ denotes a vector of transfers that the agents pay. (If $t_i < 0$, then agent i is receiving a transfer.)

(a) What is the efficient level of production $x(\theta)$? What social surplus is attained by it?

4. VCG Example #2 continued on next page. . .

(b) Define a *least charitable type* of agent i to be any type $\underline{\theta}_i \in \Theta_i$ that minimizes the interim expected social surplus, where the expectation is taken over others' types. What are the least charitable types of each agent?

(c) Using the least charitable types as the default types, what are the VCG transfers? What are the buyer's and seller's utilities in the mechanism? Does the mechanism run an ex post budget surplus for all θ? Is the mechanism interim individually rational?

(d) Show directly that, in the direct revelation game with the VCG transfer and allocation functions you computed, it is weakly dominant for each player to truthfully reveal his type. That is, let $U_S(\hat{\theta}_S, \hat{\theta}_B; \theta_S)$ denote the seller's utility when his type is θ_S, he announces $\hat{\theta}_S$, and the buyer announces $\hat{\theta}_B$. Show that $\theta_S \in \arg\max_{\hat{\theta}_S} U_S(\hat{\theta}_S, \hat{\theta}_B; \theta_S)$ for all $\hat{\theta}_B, \theta_S$. Show the similar statement for the buyer. (Note: Every VCG mechanism is dominant strategy incentive compatible, so this verifies in this special case something that we know in general.)