Auctions

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Revenue Comparison between FPSB and SPSB

Revenue Equivalence

Proposition

With i.i.d. private values, the expected payments of a type-v bidder and the seller's expected revenue are the same in a first-price sealed-bid auction as in a second-price sealed-bid auction.

Proof: (i) In FPSB, the winner pays his own bid, so the expected payment of a bidder with valuation v is:

$$m^I(v) = \text{Prob[Win]} \times \text{Amount bid} = G(v) \times \mathbb{E}\left[Y_1^{(N-1)} | Y_1^{(N-1)} < v\right]$$

Note that this amount is exactly the same as in SPSB!

(ii) Expected revenue is N times the *ex ante* payment of an individual bidder, so it too must be the same between FPSB and SPSB. From SPSB, it must be the expected second-highest of N valuations:

$$\mathbb{E}[R^I] = \mathbb{E}[R^{II}] = \mathbb{E}[Y_2^{(N)}]$$

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!
- Especially striking because, in specific realizations, revenue can be either higher or lower in FPSB vs. SPSB
    - Example: $v \sim U[0,1]$, $N = 2 \implies \beta^{I}(v) = \frac{1}{2}v$, $\beta^{II}(v) = v$
    - Case 1: $v_1 = 0.8$, $v_2 = 0.2$. Then FPSB brings more: $0.4 > 0.2$.
    - Case 2: $v_1 = 0.3$, $v_2 = 0.2$. Then SPSB brings more: $0.2 > 0.15$.
- But, *on average*, the revenues are the same!

Risk Equivalence?

- We established *expected* revenue equivalence, but are FPSB and SPSB revenue distributions the same in terms of *risk*?
- Note that FPSB prices are less variable than those in SPSB:
  - In SPSB, they can vary from 0 to ω
  - In FPSB, they are capped lower than ω:

$$\beta^I(v) = \mathbb{E}[Y_1^{(N-1)} | Y_1^{(N-1)} < v] \leq \mathbb{E}[Y_1^{(N-1)}]$$

  - Example: If $N = 2$, then $\mathbb{E}[Y_1^{(N-1)}] = \mathbb{E}[v]$.
- In fact, one can prove a more general result:

Proposition

With i.i.d. private values, the distribution of equilibrium prices in a SPSB auction is a mean-preserving spread of the distribution of equilibrium prices in a FPSB auction.

The Revenue Equivalence Principle

Revenue Equivalence

- So far, we've shown that, regardless of $F(v)$, expected selling prices in symmetric FPSB and SPSB auctions are the same.
- Thus, a risk-neutral seller is indifferent between the two formats.
- Again, this was not at all ex ante obvious — e.g., FPSB and SPSB auctions are not strategically equivalent.
- But we can prove a more general revenue-equivalence result.

General Revenue Equivalence with IPV

Def-n: Auction is '*standard*' if highest bidder gets the object.

Proposition

Suppose that values are i.i.d. and all bidders are risk neutral. Then any symmetric and increasing equilibrium of any standard auction, such that the expected payment of a bidder with value zero is zero, yields the same expected revenue to the seller.

Proof:

- Consider standard auction A and fix a symmetric equilibrium β.
- Let $m^A(v)$ be the equilibrium expected payment in auction A by a bidder with value v. Suppose that β is such that $m^A(0) = 0$.
- Consider bidder i with value v. Other bidders follow β.
- Bidder i can deviate by bidding $\beta(z)$ instead of $\beta(v)$.
- Bidder i wins if $\beta(z)$ exceeds $\beta(Y_1)$, i.e., when $z > Y_1$.

General Revenue Equivalence with IPV

Proof (cont-d): Bidder i's payoff is:

$$\mathbb{E}U^A(z,v) = G(z)v - m^A(z) \to \max_z$$

$$\frac{\partial \mathbb{E}U^A(z,v)}{\partial z} = g(z)v - \frac{d}{dz}m^A(z) = 0$$

In equilibrium, it must be that $z = v$, so for all x:

$$\frac{d}{dx}m^A(x) = g(x)x \implies m^A(v) = m^A(0) + \int_0^v xg(x)dx$$
$$= \int_0^v xg(x)dx = G(v) \times \mathbb{E}[Y_1 | Y_1 < v]$$

Note that the last expression does not depend on the particular auction form A. ■

Examples

Uniform Distribution

Valuations are uniformly distributed on [0,1]. What is the expected payment that seller expects to receive from any standard auction?

Since $v \sim U[0,1]$, $F(v) = v$, $G(v) = v^{N-1}$.

$$m^A(v) = \frac{N-1}{N} v^N$$
$$\mathbb{E}[m^A(V)] = \frac{N-1}{N(N+1)}$$
$$\mathbb{E}[R^A] = N \times \mathbb{E}[m^A(V)] = \frac{N-1}{N+1}$$

where $m^A(v)$ is the expected payment of bidder with value v, $\mathbb{E}[m^A(V)]$ is the *ex ante* expected payment of a bidder, and $\mathbb{E}[R^A]$ is the seller's expected revenue.

Revenue Equivalence with Uncertain Number of Bidders

Proposition

Suppose that values are i.i.d., all bidders are risk neutral, and that the number of bidders is uncertain. Then any symmetric and increasing equilibrium of any standard auction, such that the expected payment of a bidder with value zero is zero, yields the same expected revenue.

Proof:

- Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of *potential* bidders and $\mathcal{A} \subseteq \mathcal{N}$ be the set of *actual* bidders. Consider bidder $i \in \mathcal{A}$.
- Let p_n be the probability that any participating bidder assigns to the event that he is facing n other bidders. Thus, with p_n the number of actual bidders is $n + 1$.
- Suppose that all bidders hold the same beliefs p_n. Then the logic of the previous proof goes through.

Revenue Equivalence with Uncertain Number of Bidders

Proof (cont-d): Everything is same as before but the probability of i winning is now:

$$G(z) = \sum_{n=0}^{N-1} p_n [F(z)]^n$$

Then the bidder's payoff is:

$$\mathbb{E}U^A(z, v) = G(z)v - m^A(z) \to \max_z$$

and the previous analysis goes through. Thus, the revenue equivalence principle holds even if there is uncertainty about the number of bidders. ■

Leveraging Revenue Equivalence

Besides being a very useful result in practice, revenue equivalence is also helpful when trying to derive optimal strategies in weird or complicated auction setups.

Example 1. FPSB with Uncertain Number of Bidders.

Suppose that values are i.i.d., all bidders are risk neutral, and that the number of bidders is uncertain. What is the optimal bidding strategy in a first-price sealed-bid auction?

First, it is useful to derive expected payment in SPSB. Even with uncertain N, bidding $\beta^{II}(v) = v$ is optimal. Hence:

$$m^{II}(v) = \sum_{n=0}^{N-1} p_n [F(v)]^n \mathbb{E}\left[Y_1^{(n)} | Y_1^{(n)} < v\right]$$

Now suppose the object is sold via FPSB. The expected payment is:

$$m^{I}(v) = G(v)\beta^{I}(v)$$

Since $m^{I}(v) = m^{II}(v)$ due to revenue equivalence, we have:

$$\beta^{I}(v) = \sum_{n=0}^{N-1} \frac{p_n [F(v)]^n}{G(v)} \mathbb{E}\left[Y_1^{(n)} | Y_1^{(n)} < v\right] = \sum_{n=0}^{N-1} \frac{p_n [F(v)]^n}{G(v)} \beta^{(n)}(v)$$

Example 2. All-Pay Auction.

Consider an auction in which each bidder submits a bid, the highest bidder wins the object, but all bidders pay what they submitted.

- Although this is an unusual auction, it is still *standard*.
- Assume a symmetric increasing equilibrium such that the expected payment of a bidder with value 0 is 0.
- Hence, expected payment is that of all other standard auctions.
- Also, expected payment of type-v bidder is simply his bid! So:

$$\beta^{AP}(v) = m^A(v) = \int_0^v xg(x)dx$$

- Can show that β^{AP} is also sufficient. Consider alternative $\beta(\hat{v})$:

$$\mathbb{E}U(v, \hat{v}) = G(\hat{v})v - \beta(\hat{v}) = G(\hat{v})v - \int_0^{\hat{v}} xg(x)dx = \int_0^{\hat{v}} (v - x)dG(x)$$

- As earlier, this is maximized at $\hat{v} = v$.