From Auctions to Mechanism Design

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Introduction

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  - Seller could hold an auction and then negotiate with winner; etc.
- This week, we abstract away from the details of any particular selling format and ask: *"What is the best way to allocate an object?"*

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- This week: Optimal allocation rule to sell an object?

From Auctions to a General Mechanism Design Problem

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- Buyer i's valuation V_i is distributed over the interval $\mathcal{V}_i = [0, \omega_i]$ according to c.d.f. F_i with density f_i.

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- Let $\mathcal{V} = \times_{j=1}^{N} \mathcal{V}_j$ denote the product of the sets of buyers' values and, for all i, let $\mathcal{V}_{-i} = \times_{j \neq i} \mathcal{V}_j$

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- Similarly, define $f_{-i}(\mathbf{v}_{-i})$ to be the joint density of $\mathbf{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N)$

Mechanism

Definition

A selling *mechanism* $(\mathcal{B}, \boldsymbol{\pi}, \boldsymbol{\mu})$ is a combination of:

1. A set of possible *messages* (or "bids") $\mathcal{B}_i$ for each buyer i;
2. An *allocation rule* $\boldsymbol{\pi} : \mathcal{B} \to \Delta$ where Δ is the set of probability distributions over the set of buyers $\mathcal{N}$;
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- Allocation rule determines, as a function of all N messages, the probability $\pi_i(\mathbf{b})$ that i will get the object.
- Payment rule determines, as a function of all N messages, for each buyer i, the expected payment $\mu_i(\mathbf{b})$ that i must make.

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$$\pi_i(\mathbf{b}) = \begin{cases} 1 & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

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- Second-price auction:

$$\mu_i^{II}(\mathbf{b}) = \begin{cases} \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

Game Within a Mechanism

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- *Note*: Today, we focus on **Bayesian Mechanism Design**, i.e., with BNE as an underlying equilibrium concept. Later, we will come back and consider **Dominant Strategies Mechanism Design** which relies on the Dominant Strategy Equilibrium.

Direct Mechanisms and Revelation Principle

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- A smaller and simpler class are mechanisms for which the set of messages is the same as the set of values—that is, $\mathcal{B}_i = \mathcal{V}_i \ \forall i$.
- Such mechanisms are called *direct*, since every buyer is asked to directly report a value.

Direct Mechanisms

- Formally, direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ consists of functions $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$, where $Q_i(\mathbf{v})$ is the probability that i will get the object and $M_i(\mathbf{v})$ is the expected payment by i.

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- *Revelation principle*: outcomes of any mechanism's equilibrium can be replicated by a truthful equilibrium of a direct mechanism.
- Hence, no loss of generality in focusing on direct mechanisms.

Proposition. Revelation Principle.

Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.

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- Now suppose that some buyer finds it profitable to be untruthful and report a value $\hat{v}_i$ when his value is v_i.
- Then in the original mechanism same buyer would have found it profitable to submit $\beta_i(\hat{v}_i)$ instead of $\beta_i(v_i)$. Contradiction. ■

Buyer's Payoff Function

Given a direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$:

$$q_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} Q_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

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$$m_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} M_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

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$$q_i(\hat{v}_i) v_i - m_i(\hat{v}_i)$$

is the expected payoff of i when he reports $\hat{v}_i$ and others tell the truth.

Incentive Compatibility

The direct revelation mechanism is incentive compatible (IC) if:

$$U_i(v_i) \equiv q_i(v_i)v_i - m_i(v_i) \geq q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)$$

$$\forall i \in \mathcal{N}; \forall v_i, \hat{v}_i \in [0, \omega_i]$$

where U_i is the *equilibrium payoff function*.

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- That is, U_i is a maximum of a family of affine functions, therefore U_i is a *convex function*.

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- Second, we can rewrite:

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- Since the above inequality has to hold for all v_i and $\hat{v}_i$, $q_i(v_i)$ is the subgradient of the function U_i at v_i.

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- Since the above inequality has to hold for all v_i and $\hat{v}_i$, $q_i(v_i)$ is the subgradient of the function U_i at v_i.
- Since U_i is convex, it must be that q_i is *non-decreasing*.

Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

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- That is, up to an additive constant, buyer's expected payoff in an IC direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ depends only on allocation rule $\boldsymbol{Q}$.
- Thus, if $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are two IC mechanisms with same allocation rule but different payment rules, then their expected payoff functions, U_i and $\bar{U}_i$, differ by at most a constant.

Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

$$U_i'(v_i) = q_i(v_i) \implies U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(x_i)dx_i$$

- That is, up to an additive constant, buyer's expected payoff in an IC direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ depends only on allocation rule $\boldsymbol{Q}$.
- Thus, if $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are two IC mechanisms with same allocation rule but different payment rules, then their expected payoff functions, U_i and $\bar{U}_i$, differ by at most a constant.
- In other words, $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are *payoff equivalent*.

Revenue Equivalence Strikes Again!

Generalized Revenue Equivalence

If the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ is incentive compatible, then for all i and v_i, the expected payment is

$$m_i(v_i) = m_i(0) + q_i(v_i)v_i - \int_0^{v_i} q_i(x_i)dx_i$$

Thus, expected payments in any two incentive compatible mechanisms with the same allocation rule are equivalent up to a constant.

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Proof: Since $U_i(v_i) = q_i(v_i)v_i - m_i(v_i)$ and $U_i(0) = -m_i(0)$, then:

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Generalized Revenue Equivalence

Remarks:

- Given two BNE of two different auctions such that for each i:
  - For all $(v_1, \ldots, v_N)$, probability of i getting the object is the same,
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- This generalizes the result from last time:
  - Revenue equivalence at the symmetric equilibrium of standard auctions (object allocated to buyer with the highest bid).

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- To see that nondecreasing q_i implies IC, note that:

$$U_i(\hat{v}_i) \geq U(v_i) + q_i(v_i)(\hat{v}_i - v_i) \iff$$
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- The latter inequality certainly holds if q_i is nondecreasing.

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- (We implicitly assume that by not participating, a buyer can guarantee himself a payoff of zero.)
- If the mechanism is IC, then IR is equivalent to $U_i(0) \geq 0$, and since $U_i(0) = -m_i(0)$ this is equivalent to $m_i(0) \leq 0$.

Optimal Mechanisms

Finding an Optimal Mechanism

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$$\begin{aligned}
\mathbb{E}[m_i(V_i)] &= \int_0^{\omega_i} m_i(v_i) f_i(v_i) dv_i \\
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\end{aligned}$$

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$$= m_i(0) + \int_{\mathcal{V}}\left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right] Q_i(\mathbf{v})f(\mathbf{v})d\mathbf{v}$$

where, in the last step, we used the definition of q_i as the expected allocation probability intergrated over valuations of all other players.

Optimal Mechanism Design Problem

$$\sum_{i \in \mathcal{N}} m_i(0) + \sum_{i \in \mathcal{N}} \int_{\mathcal{V}} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] Q_i(\mathbf{v}) f(\mathbf{v}) d\mathbf{v} \to \max_{\boldsymbol{Q}, \boldsymbol{M}}$$

s.t. IC constraint ($\Leftrightarrow q_i$ is nondecreasing)

IR constraint ($\Leftrightarrow m_i(0) \leq 0$)

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- Let's define **virtual valuation** of a buyer with value v_i as:

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- We say that design problem is regular if $\psi_i(v_i)$ is increasing in v_i (it is sufficient that hazard rate $\lambda_i(v_i)$ is increasing in v_i).
- Let us temporarily neglect the IC and the IR constraints and circle back to them later.

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- The seller should choose $(\boldsymbol{Q}, \boldsymbol{M})$ to maximize:

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{V}} \left(\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v}) \right) f(\mathbf{v}) d\mathbf{v}$$

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- This approach would maximize this expression at every point $\mathbf{v}$ and so would also maximize its integral.

Optimal Mechanism

Claim

The following is an optimal mechanism:

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1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

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First, let's check IC and IR:

- IC: Check if q_i is nondecreasing. Suppose $\hat{v}_i < v_i$. Then by the regularity condition, $\psi_i(\hat{v}_i) < \psi_i(v_i)$ and, thus, for all v_{-i}, it is also the case that $Q_i(\hat{v}_i, v_{-i}) \leq Q(\mathbf{v})$. Thus, q_i is nondecreasing.

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First, let's check IC and IR:

- IR: From the payment rule, it is clear that $M_i(0, \mathbf{v}_{-i}) = 0$ for all $\mathbf{v}_{-i}$, and thus $m_i(0) = 0$.

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Second, note that this mechanism separately maximizes two terms in seller's expected revenue: (i) It gives positive weight only to nonnegative and maximal terms in the second expression; (ii) It sets maximal $m_i(0)$ given the IR constraint.

Optimal Mechanism

This implies that the maximized value of the expected revenue is:

$$\mathbb{E}[\max\{\psi_1(V_1), \ldots, \psi_N(V_N), 0\}]$$

In other words, it is the expectation of the highest virtual valuation, provided it is nonnegative.

Optimal Mechanism

To obtain more intuitive formulas for $(\boldsymbol{Q}, \boldsymbol{M})$, we define:

$$y_i(\mathbf{v}_{-i}) = \inf\{x_i : \psi_i(x_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(x_i) \geq \psi_j(v_j)\}$$

as the smallest value for i that "wins" against $\mathbf{v}_{-i}$.

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as the smallest value for i that "wins" against $\mathbf{v}_{-i}$. Thus, can rewrite the optimal allocation rules as:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } x_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } x_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

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$$\implies M_i(\mathbf{v}) = \begin{cases} y_i(\mathbf{v}_{-i}) & \text{if } Q_i(\mathbf{v}) = 1 \\ 0 & \text{if } Q_i(\mathbf{v}) = 0 \end{cases}$$

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Thus, only the 'winning' buyer pays anything. He pays the smallest value that would result in his winning.

Optimal Mechanism

Proposition.

Suppose the design problem is regular. Then the following is an optimal mechanism:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } \psi_i(v_i) > \max_{j \neq i} \psi_j \text{ and } \psi_i(v_i) \geq 0 \\ 0 & \text{if } \psi_i(v_i) < \max_{j \neq i} \psi_j \end{cases}$$

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