Auctions

Alexey Makarin

April 30 & May 3, 2021

Introduction

Ancient (Italian) Selling Mechanism

- Fact #1: "Auction" comes from Latin (*auctum*, "I increase")

Ancient (Italian) Selling Mechanism

- Fact #1: "Auction" comes from Latin (*auctum*, "I increase")
- Fact #2: Single most important auction ever took place in Italy!

Ancient (Italian) Selling Mechanism

- Fact #1: "Auction" comes from Latin (*auctum*, "I increase")
- Fact #2: Single most important auction ever took place in Italy!
  - A.D. 193: Having killed the Emperor Pertinax, Praetorian Guard sold the entire (!) Roman Empire by means of an auction.

Ancient (Italian) Selling Mechanism

- Fact #1: "Auction" comes from Latin (*auctum*, "I increase")
- Fact #2: Single most important auction ever took place in Italy!
  - A.D. 193: Having killed the Emperor Pertinax, Praetorian Guard sold the entire (!) Roman Empire by means of an auction.
  - The winner, Didius Julianus, was declared emperor and reigned for two months.

Ancient (Italian) Selling Mechanism

- Fact #1: "Auction" comes from Latin (*auctum*, "I increase")
- Fact #2: Single most important auction ever took place in Italy!
  - A.D. 193: Having killed the Emperor Pertinax, Praetorian Guard sold the entire (!) Roman Empire by means of an auction.
  - The winner, Didius Julianus, was declared emperor and reigned for two months.
- That said, first ever recorded auctions took place in Babylon around 500 B.C.

Revival of Auctions

- England in 18th century: auctions revived for selling artwork and other rare items.

Revival of Auctions

- England in 18th century: auctions revived for selling artwork and other rare items.
- Largest auction houses were founded:
  - Sotheby's (2nd largest auction house now): founded in 1744 to sell off a library of scarce books;
  - Christie's (largest auction house now): founded in 1766 to sell paintings and other artwork.

Revival of Auctions

- England in 18th century: auctions revived for selling artwork and other rare items.
- Largest auction houses were founded:
  - Sotheby's (2nd largest auction house now): founded in 1744 to sell off a library of scarce books;
  - Christie's (largest auction house now): founded in 1766 to sell paintings and other artwork.
- A variety of auction mechanisms were developed, including *English auction*, *Dutch auction*, and so-called *auction by the candle*.

Modern Auction Extravaganza

- Auctions became *extremely* popular in the 21st century.

Modern Auction Extravaganza

- Auctions became *extremely* popular in the 21st century.
- Private sector:
  - eBay — online auction platform with 182 mln users worldwide;
  - Google — selling ad space through an auction system; etc.

Modern Auction Extravaganza

- Auctions became *extremely* popular in the 21st century.
- Private sector:
  - eBay — online auction platform with 182 mln users worldwide;
  - Google — selling ad space through an auction system; etc.
- Public sector:
  - Privatization and public resource allocation (ex: famous FCC Spectrum Auction in 1993 designed by Paul Milgrom and others);
  - Reverse auctions: Trillions of dollars of goods *bought* by governments on e-procurement auctions around the globe.

Questions in Auction Theory

- Why are auctions so prevalent, historically and today?
- In which situations auctions are preferred to other selling mechanisms, e.g., to a fixed posted price?
- Bidders: for a given auction, what are good bidding strategies?
- Sellers: are there particular types of auctions that would bring greater revenues than others?

Auctions and Information Economics

- Auctions help to organize markets with information asymmetries.

Auctions and Information Economics

- Auctions help to organize markets with information asymmetries.
- Buyers possess private info—their valuations of the good.

Auctions and Information Economics

- Auctions help to organize markets with information asymmetries.
- Buyers possess private info—their valuations of the good.
- Sellers want to maximize revenues and organize sales in such a way that would elicit this private information (e.g., such that bidders with the highest valuations offer the highest price).

Auctions and Information Economics

- Auctions help to organize markets with information asymmetries.
- Buyers possess private info—their valuations of the good.
- Sellers want to maximize revenues and organize sales in such a way that would elicit this private information (e.g., such that bidders with the highest valuations offer the highest price).
- Auctions are one set of *mechanisms* in a general *mechanism design* problem: how to organize a game such that a certain objective is achieved?
  - Typically two competing objectives: revenue vs. efficiency.

Bidders' Valuations

Bidders' valuations can be:

1. Private (independent) values
    - Ex: value is derived from consumption alone

Bidders' Valuations

Bidders' valuations can be:

1. Private (independent) values
   - Ex: value is derived from consumption alone
2. Interdependent values
   - Ex: auctioned object is an asset that can be resold later

Bidders' Valuations

Bidders' valuations can be:

1. Private (independent) values
   - Ex: value is derived from consumption alone
2. Interdependent values
   - Ex: auctioned object is an asset that can be resold later
3. Pure common value
   - Ex: value of the auctioned object is derived from a market price that is unknown at the time of the auction

Auction Variations

Common auction forms:

1. Open ascending auction (*English auction*);
   - Bids are public; price is ascending until no one bids more.
   - Example: Standard auctions for rare items you see on TV.

Auction Variations

Common auction forms:

1. Open ascending auction (*English auction*);
   - Bids are public; price is ascending until no one bids more.
   - Example: Standard auctions for rare items you see on TV.
2. Open descending auction (*Dutch auction*);
   - Bids are public; price is descending until someone buys the object.
   - Example: Tulip auctions in the Netherlands.

Auction Variations

Common auction forms:

1. Open ascending auction (*English auction*);
   - Bids are public; price is ascending until no one bids more.
   - Example: Standard auctions for rare items you see on TV.
2. Open descending auction (*Dutch auction*);
   - Bids are public; price is descending until someone buys the object.
   - Example: Tulip auctions in the Netherlands.
3. Sealed-bid first-price auction (FPSB);
   - Bids are private; highest bidder gets object at highest price.
   - Example: E-procurement auctions, eBay auctions.

Auction Variations

Common auction forms:

1. Open ascending auction (*English auction*);
   - Bids are public; price is ascending until no one bids more.
   - Example: Standard auctions for rare items you see on TV.
2. Open descending auction (*Dutch auction*);
   - Bids are public; price is descending until someone buys the object.
   - Example: Tulip auctions in the Netherlands.
3. Sealed-bid first-price auction (FPSB);
   - Bids are private; highest bidder gets object at highest price.
   - Example: E-procurement auctions, eBay auctions.
4. Sealed-bid second-price auction (SPSB, *Vickrey auction*).
   - Bids are private; highest bidder gets object at 2nd-highest price.
   - Example: Google ad auctions (before 2019).

Equivalent Auctions

In game-theoretic terms, some auctions are similar to each other:

- Dutch auction $\equiv$ Sealed-bid first-price auction,

Equivalent Auctions

In game-theoretic terms, some auctions are similar to each other:

- Dutch auction $\equiv$ Sealed-bid first-price auction,
  - In FPSB, buyers choose a price at which they want to buy the object conditional on it still being available.

Equivalent Auctions

In game-theoretic terms, some auctions are similar to each other:

- Dutch auction $\equiv$ Sealed-bid first-price auction,
  - In FPSB, buyers choose a price at which they want to buy the object conditional on it still being available.
  - Dutch auction is open, but the only information it reveals to buyers is that the object is still available at a given price.

Equivalent Auctions

In game-theoretic terms, some auctions are similar to each other:

- Dutch auction $\equiv$ Sealed-bid first-price auction,
  - In FPSB, buyers choose a price at which they want to buy the object conditional on it still being available.
  - Dutch auction is open, but the only information it reveals to buyers is that the object is still available at a given price.
  - Thus, two formats are strategically equivalent.

Equivalent Auctions

In game-theoretic terms, some auctions are similar to each other:

- Dutch auction $\equiv$ Sealed-bid first-price auction,
  - In FPSB, buyers choose a price at which they want to buy the object conditional on it still being available.
  - Dutch auction is open, but the only information it reveals to buyers is that the object is still available at a given price.
  - Thus, two formats are strategically equivalent.
- English auction $\approx$ Sealed-bid second-price (under *private values*).

Equivalent Auctions

In game-theoretic terms, some auctions are similar to each other:

- Dutch auction $\equiv$ Sealed-bid first-price auction,
    - In FPSB, buyers choose a price at which they want to buy the object conditional on it still being available.
    - Dutch auction is open, but the only information it reveals to buyers is that the object is still available at a given price.
    - Thus, two formats are strategically equivalent.
- English auction $\approx$ Sealed-bid second-price (under *private values*).
    - In both formats, best to bid your valuation.

Private Value Auctions

Setup

- A single object is for sale

Setup

- A single object is for sale
- N potential risk-neutral buyers
  - v_i — valuation of buyer $i \in N$
  - $v_i \sim$ i.i.d. $F[0, \omega]$ — independent *symmetric* values
  - b_i — bid of buyer $i \in N$
  - Bidders don't face any budget constraints (have 'deep pockets')

Setup

- A single object is for sale
- N potential risk-neutral buyers
  - v_i — valuation of buyer $i \in N$
  - $v_i \sim$ i.i.d. $F[0, \omega]$ — independent *symmetric* values
  - b_i — bid of buyer $i \in N$
  - Bidders don't face any budget constraints (have 'deep pockets')
- Bidder i knows v_i, $F[0, \omega]$, and N, but not v_j where $j \neq i$

Setup

- A single object is for sale
- N potential risk-neutral buyers
  - v_i — valuation of buyer $i \in N$
  - $v_i \sim$ i.i.d. $F[0, \omega]$ — independent *symmetric* values
  - b_i — bid of buyer $i \in N$
  - Bidders don't face any budget constraints (have 'deep pockets')
- Bidder i knows v_i, $F[0, \omega]$, and N, but not v_j where $j \neq i$
- Since bidders are symmetric, focus on *symmetric* equilibria — equilibria in which all bidders follow the same strategy

FPSB vs. SPSB

In this setup, we will consider two auction formats:

1. *First-price sealed bid auction (I)*: highest bidder gets the object and pays the amount he bids
2. *Second-price sealed bid auction (II)*: highest bidder gets the object and pays the second highest bid

FPSB vs. SPSB

In this setup, we will consider two auction formats:

1. *First-price sealed bid auction (I)*: highest bidder gets the object and pays the amount he bids
2. *Second-price sealed bid auction (II)*: highest bidder gets the object and pays the second highest bid

Questions:

- What are equilibrium bidding strategies $\beta_i : [0, \omega] \to \mathbb{R}_+$ in these auction formats?
- Which of the two formats is better for the seller?

Second-Price Sealed-Bid Auction

Payoffs in SPSB

Bidders' payoffs are:

$$\mathbb{E}U_i = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{K}[v_i - \max_{j \neq i} b_j] & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

, where K is the number of winning bidders in case of a tie.

Bidding Strategies in SPSB

Proposition

In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(v) = v$.

Bidding Strategies in SPSB

Proposition

In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(v) = v$.

Proof: Consider bidder i and suppose that $p_i = \max_{j \neq i} b_j$ is the highest competing bid.

- If i bids v_i, he wins if $v_i > p_i$ and not if $v_i < p_i$ (if $v_i = p_i$, indifferent between winning and losing).

Bidding Strategies in SPSB

Proposition

In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(v) = v$.

Proof: Consider bidder i and suppose that $p_i = \max_{j \neq i} b_j$ is the highest competing bid.

- If i bids v_i, he wins if $v_i > p_i$ and not if $v_i < p_i$ (if $v_i = p_i$, indifferent between winning and losing).
- Suppose i bids z_i where $z_i < v_i$.

Bidding Strategies in SPSB

Proposition

In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(v) = v$.

Proof: Consider bidder i and suppose that $p_i = \max_{j \neq i} b_j$ is the highest competing bid.

- If i bids v_i, he wins if $v_i > p_i$ and not if $v_i < p_i$ (if $v_i = p_i$, indifferent between winning and losing).
- Suppose i bids z_i where $z_i < v_i$.
    - If $v_i > z_i > p_i$, i still wins, and his profit is still $v_i - p_i$.

Bidding Strategies in SPSB

Proposition

In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(v) = v$.

Proof: Consider bidder i and suppose that $p_i = \max_{j \neq i} b_j$ is the highest competing bid.

- If i bids v_i, he wins if $v_i > p_i$ and not if $v_i < p_i$ (if $v_i = p_i$, indifferent between winning and losing).
- Suppose i bids z_i where $z_i < v_i$.
  - If $v_i > z_i > p_i$, i still wins, and his profit is still $v_i - p_i$.
  - If $p_i > v_i > z_i$, i still loses and gets zero.

Bidding Strategies in SPSB

Proposition

In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(v) = v$.

<u>*Proof:*</u> Consider bidder i and suppose that $p_i = \max_{j \neq i} b_j$ is the highest competing bid.

- If i bids v_i, he wins if $v_i > p_i$ and not if $v_i < p_i$ (if $v_i = p_i$, indifferent between winning and losing).
- Suppose i bids z_i where $z_i < v_i$.
  - If $v_i > z_i > p_i$, i still wins, and his profit is still $v_i - p_i$.
  - If $p_i > v_i > z_i$, i still loses and gets zero.
  - But if $v_i > p_i > z_i$, then i loses where he could have made a positive profit if he bid v_i.

Bidding Strategies in SPSB

Proposition

In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(v) = v$.

Proof: Consider bidder i and suppose that $p_i = \max_{j \neq i} b_j$ is the highest competing bid.

- If i bids v_i, he wins if $v_i > p_i$ and not if $v_i < p_i$ (if $v_i = p_i$, indifferent between winning and losing).
- Suppose i bids z_i where $z_i < v_i$.
  - If $v_i > z_i > p_i$, i still wins, and his profit is still $v_i - p_i$.
  - If $p_i > v_i > z_i$, i still loses and gets zero.
  - But if $v_i > p_i > z_i$, then i loses where he could have made a positive profit if he bid v_i.
- Thus, bidding less than v_i is weakly dominated.

Bidding Strategies in SPSB

Proposition

In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(v) = v$.

Proof: Consider bidder i and suppose that $p_i = \max_{j \neq i} b_j$ is the highest competing bid.

- If i bids v_i, he wins if $v_i > p_i$ and not if $v_i < p_i$ (if $v_i = p_i$, indifferent between winning and losing).
- Suppose i bids z_i where $z_i < v_i$.
  - If $v_i > z_i > p_i$, i still wins, and his profit is still $v_i - p_i$.
  - If $p_i > v_i > z_i$, i still loses and gets zero.
  - But if $v_i > p_i > z_i$, then i loses where he could have made a positive profit if he bid v_i.
- Thus, bidding less than v_i is weakly dominated.
- Applying same logic, bidding $z_i > v_i$ is also weakly dominated. ■

Expected Payment in SPSB

- Fix a bidder i.

Expected Payment in SPSB

- Fix a bidder i.
- Let random variable $Y_1 \equiv Y_1^{N-1}$ denote the highest value among other $N-1$ bidders. (Y_1 is the *highest-order statistic* of $V_1, \ldots V_{i-1}, V_{i+1}, \ldots, V_N$.)

Expected Payment in SPSB

- Fix a bidder i.
- Let random variable $Y_1 \equiv Y_1^{N-1}$ denote the highest value among other $N-1$ bidders. (Y_1 is the *highest-order statistic* of $V_1, \ldots V_{i-1}, V_{i+1}, \ldots, V_N$.)
- Let G be the c.d.f. of Y_1.

Expected Payment in SPSB

- Fix a bidder i.
- Let random variable $Y_1 \equiv Y_1^{N-1}$ denote the highest value among other $N-1$ bidders. (Y_1 is the *highest-order statistic* of $V_1, \ldots V_{i-1}, V_{i+1}, \ldots, V_N$.)
- Let G be the c.d.f. of Y_1.
- $G(v) = F(v)^{N-1}$ since v's are independently drawn. (Why?)

Expected Payment in SPSB

- Fix a bidder i.
- Let random variable $Y_1 \equiv Y_1^{N-1}$ denote the highest value among other $N-1$ bidders. (Y_1 is the *highest-order statistic* of $V_1, \dots V_{i-1}, V_{i+1}, \dots, V_N$.)
- Let G be the c.d.f. of Y_1.
- $G(v) = F(v)^{N-1}$ since v's are independently drawn. (Why?)
- Then the expected payment by a bidder with value v is:

$$m^{II}(v) = \text{Prob[Win]} \times \mathbb{E}[\text{2nd highest bid}|v \text{ is the highest bid}]$$

Expected Payment in SPSB

- Fix a bidder i.
- Let random variable $Y_1 \equiv Y_1^{N-1}$ denote the highest value among other $N-1$ bidders. (Y_1 is the *highest-order statistic* of $V_1, \dots V_{i-1}, V_{i+1}, \dots, V_N$.)
- Let G be the c.d.f. of Y_1.
- $G(v) = F(v)^{N-1}$ since v's are independently drawn. (Why?)
- Then the expected payment by a bidder with value v is:

$$\begin{aligned} m^{II}(v) &= \text{Prob[Win]} \times \mathbb{E}[\text{2nd highest bid} | v \text{ is the highest bid}] \\ &= \text{Prob[Win]} \times \mathbb{E}[\text{2nd highest value} | v \text{ is the highest value}] \end{aligned}$$

Expected Payment in SPSB

- Fix a bidder i.
- Let random variable $Y_1 \equiv Y_1^{N-1}$ denote the highest value among other $N-1$ bidders. (Y_1 is the *highest-order statistic* of $V_1, \dots V_{i-1}, V_{i+1}, \dots, V_N$.)
- Let G be the c.d.f. of Y_1.
- $G(v) = F(v)^{N-1}$ since v's are independently drawn. (Why?)
- Then the expected payment by a bidder with value v is:

$$
\begin{aligned}
m^{II}(v) &= \text{Prob[Win]} \times \mathbb{E}[\text{2nd highest bid}|v \text{ is the highest bid}] \\
&= \text{Prob[Win]} \times \mathbb{E}[\text{2nd highest value}|v \text{ is the highest value}] \\
&= G(v) \times \mathbb{E}[Y_1|Y_1 < v] = F(v)^{N-1} \times \mathbb{E}[Y_1|Y_1 < v]
\end{aligned}
$$

First-Price Sealed-Bid Auction

Payoffs in FPSB

Bidders' payoffs are:

$$\mathbb{E}U_i = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{K}[v_i - b_i] & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

, where K is the number of winning bidders in case of a tie.

Bidding Strategies in FPSB

- Not as trivial as in SPSB auction.
- Strategy $\beta(v_i) = v_i$ is definitely sub-optimal. (Why?)

Bidding Strategies in FPSB

- Not as trivial as in SPSB auction.
- Strategy $\beta(v_i) = v_i$ is definitely sub-optimal. (Why?)
- Basic trade-off: increasing one's bid increases the probability of winning but reduces the gains from winning

Bidding Strategies in FPSB

- Not as trivial as in SPSB auction.
- Strategy $\beta(v_i) = v_i$ is definitely sub-optimal. (Why?)
- Basic trade-off: increasing one's bid increases the probability of winning but reduces the gains from winning
- Suppose that bidders $j \neq i$ follow a symmetric, increasing, and differentiable equilibrium strategy β.
- Bidder i has valuation v and bids b. What is the optimal b?

Finding Bidding Strategies in FPSB

- Bidder i wins whenever $\max_{j \neq i} \beta(V_j) < b$

Finding Bidding Strategies in FPSB

- Bidder i wins whenever $\max_{j \neq i} \beta(V_j) < b$
- Since β is increasing, $\max_{j \neq i} \beta(V_j) = \beta(\max_{j \neq i} V_j) = \beta(Y_1)$

Finding Bidding Strategies in FPSB

- Bidder i wins whenever $\max_{j \neq i} \beta(V_j) < b$
- Since β is increasing, $\max_{j \neq i} \beta(V_j) = \beta(\max_{j \neq i} V_j) = \beta(Y_1)$
- Probability of $Y_1 < \beta^{-1}(b)$ is $G[\beta^{-1}(b)]$. Hence:

$$\mathbb{E}U(b, v) = (v - b)G[\beta^{-1}(b)] \to \max_b$$

Finding Bidding Strategies in FPSB

- Bidder i wins whenever $\max_{j \neq i} \beta(V_j) < b$
- Since β is increasing, $\max_{j \neq i} \beta(V_j) = \beta(\max_{j \neq i} V_j) = \beta(Y_1)$
- Probability of $Y_1 < \beta^{-1}(b)$ is $G[\beta^{-1}(b)]$. Hence:

$$\mathbb{E}U(b, v) = (v - b)G[\beta^{-1}(b)] \to \max_b$$

- Taking FOC:

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(v - b) - G[\beta^{-1}(b)] = 0$$

(where $g = G'$ is the density of Y_1).

Finding Bidding Strategies in FPSB

- In symmetric equilibrium $b = \beta(v)$, so:

$$G(v)\beta'(v) + g(v)\beta(v) = vg(v) \implies \frac{d}{dv}[G(v)\beta(v)] = vg(v)$$

Finding Bidding Strategies in FPSB

- In symmetric equilibrium $b = \beta(v)$, so:

$$G(v)\beta'(v) + g(v)\beta(v) = vg(v) \implies \frac{d}{dv}[G(v)\beta(v)] = vg(v)$$

- Integrating both sides, get:

$$\beta(v) = \frac{1}{G(v)} \int_0^v yg(y)dy = \mathbb{E}[Y_1 | Y_1 < v]$$

Finding Bidding Strategies in FPSB

- In symmetric equilibrium $b = \beta(v)$, so:

$$G(v)\beta'(v) + g(v)\beta(v) = vg(v) \implies \frac{d}{dv}[G(v)\beta(v)] = vg(v)$$

- Integrating both sides, get:

$$\beta(v) = \frac{1}{G(v)} \int_0^v yg(y)dy = \mathbb{E}[Y_1 | Y_1 < v]$$

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N - 1$ independently drawn valuations.

Finding Bidding Strategies in FPSB

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N-1$ independently drawn valuations.

Finding Bidding Strategies in FPSB

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N-1$ independently drawn valuations.

Proof: We already proved *necessity*, i.e., that all symmetric equilibrium strategies have to be of this form. Now let's prove *sufficiency*.

Finding Bidding Strategies in FPSB

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N-1$ independently drawn valuations.

Proof: We already proved *necessity*, i.e., that all symmetric equilibrium strategies have to be of this form. Now let's prove *sufficiency*.

- Need to show that $\beta \equiv \beta^I(v)$ produce a symmetric equilibrium.

Finding Bidding Strategies in FPSB

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N-1$ independently drawn valuations.

<u>*Proof:*</u> We already proved *necessity*, i.e., that all symmetric equilibrium strategies have to be of this form. Now let's prove *sufficiency*.

- Need to show that $\beta \equiv \beta^I(v)$ produce a symmetric equilibrium.
- Suppose that player's opponents chose this strategy.

Finding Bidding Strategies in FPSB

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N-1$ independently drawn valuations.

<u>*Proof:*</u> We already proved *necessity*, i.e., that all symmetric equilibrium strategies have to be of this form. Now let's prove *sufficiency*.

- Need to show that $\beta \equiv \beta^I(v)$ produce a symmetric equilibrium.
- Suppose that player's opponents chose this strategy.
- Clearly, the player should never choose $b > \beta(\omega)$ since bidding $b = \beta(\omega)$ already ensures the victory.

Finding Bidding Strategies in FPSB

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N-1$ independently drawn valuations.

Proof: We already proved *necessity*, i.e., that all symmetric equilibrium strategies have to be of this form. Now let's prove *sufficiency*.

- Need to show that $\beta \equiv \beta^I(v)$ produce a symmetric equilibrium.
- Suppose that player's opponents chose this strategy.
- Clearly, the player should never choose $b > \beta(\omega)$ since bidding $b = \beta(\omega)$ already ensures the victory.
- Thus, need to show that a player of type v is at least as well off choosing $\beta(v)$ as $\beta(\hat{v})$ for any $\hat{v} \in [0, \omega]$.

Finding Bidding Strategies in FPSB

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N-1$ independently drawn valuations.

Proof (cont'd): Substituting $b = \beta(\hat{v})$ into buyer's utility function:

$$\begin{aligned} G(\beta^{-1}(b)) \times (v - b) &= G(\hat{v})[v - \beta(\hat{v})] \\ &= G(\hat{v})v - \int_0^{\hat{v}} x dG(x) = \int_0^{\hat{v}} (v - x) dG(x) \end{aligned}$$

This is clearly maximized at $\hat{v} = v$. ■

Properties of FPSB Bidding Strategies

- Integrating by parts, the equilibrium bid can be rewritten as:

$$\beta^I(v) = v - \int_0^v \frac{G(x)}{G(v)} dx$$

Properties of FPSB Bidding Strategies

- Integrating by parts, the equilibrium bid can be rewritten as:

$$\beta^I(v) = v - \int_0^v \frac{G(x)}{G(v)} dx$$

- So, optimal bid in FPSB auction is lower than one's valuation!

Properties of FPSB Bidding Strategies

- Integrating by parts, the equilibrium bid can be rewritten as:

$$\beta^I(v) = v - \int_0^v \frac{G(x)}{G(v)} dx$$

- So, optimal bid in FPSB auction is lower than one's valuation!
- Also recall that, in independent private values (IPV) setting:

$$\frac{G(x)}{G(v)} = \left[\frac{F(x)}{F(v)}\right]^{N-1}$$

Properties of FPSB Bidding Strategies

- Integrating by parts, the equilibrium bid can be rewritten as:

$$\beta^I(v) = v - \int_0^v \frac{G(x)}{G(v)} dx$$

- So, optimal bid in FPSB auction is lower than one's valuation!
- Also recall that, in independent private values (IPV) setting:

$$\frac{G(x)}{G(v)} = \left[\frac{F(x)}{F(v)}\right]^{N-1}$$

- So, the degree of "shading" goes down as N increases!

Examples

Example 1

Valuations are uniformly distributed on [0,1].

In this case, $F(v) = v$, $G(v) = v^{N-1}$, and $\beta^I(v) =$

Examples

Example 1

Valuations are uniformly distributed on [0,1].

In this case, $F(v) = v$, $G(v) = v^{N-1}$, and $\beta^I(v) = \frac{N-1}{N}v$.

Examples

Example 1

Valuations are uniformly distributed on [0,1].

In this case, $F(v) = v$, $G(v) = v^{N-1}$, and $\beta^I(v) = \frac{N-1}{N} v$.

Example 2

Valuations are exponentially distributed on $[0,\infty)$, and there are only two bidders.

If $F(x) = 1 - \exp(-\lambda x)$, for some $\lambda > 0$, and $N = 2$, then:

$$\beta^I(v) = v - \int_0^v \frac{F(x)}{F(v)} dx =$$

Examples

Example 1

Valuations are uniformly distributed on [0,1].

In this case, $F(v) = v$, $G(v) = v^{N-1}$, and $\beta^I(v) = \frac{N-1}{N}v$.

Example 2

Valuations are exponentially distributed on [0,∞), and there are only two bidders.

If $F(x) = 1 - \exp(-\lambda x)$, for some $\lambda > 0$, and $N = 2$, then:

$$\beta^I(v) = v - \int_0^v \frac{F(x)}{F(v)}dx = \frac{1}{\lambda} - \frac{v\exp(-\lambda v)}{1 - \exp(-\lambda v)}$$

Revenue Comparison between FPSB and SPSB

Revenue Equivalence

Proposition

With i.i.d. private values, the expected payments of a type-v bidder and the seller's expected revenue are the same in a first-price sealed-bid auction as in a second-price sealed-bid auction.

Revenue Equivalence

Proposition

With i.i.d. private values, the expected payments of a type-v bidder and the seller's expected revenue are the same in a first-price sealed-bid auction as in a second-price sealed-bid auction.

Proof: (i) In FPSB, the winner pays his own bid, so the expected payment of a bidder with valuation v is:

$$m^I(v) = \text{Prob[Win]} \times \text{Amount bid} =$$

Revenue Equivalence

Proposition

With i.i.d. private values, the expected payments of a type-v bidder and the seller's expected revenue are the same in a first-price sealed-bid auction as in a second-price sealed-bid auction.

Proof: (i) In FPSB, the winner pays his own bid, so the expected payment of a bidder with valuation v is:

$$m^I(v) = \text{Prob[Win]} \times \text{Amount bid} = G(v) \times \mathbb{E}\left[Y_1^{(N-1)} | Y_1^{(N-1)} < v\right]$$

Revenue Equivalence

Proposition

With i.i.d. private values, the expected payments of a type-v bidder and the seller's expected revenue are the same in a first-price sealed-bid auction as in a second-price sealed-bid auction.

Proof: (i) In FPSB, the winner pays his own bid, so the expected payment of a bidder with valuation v is:

$$m^I(v) = \text{Prob[Win]} \times \text{Amount bid} = G(v) \times \mathbb{E}\left[Y_1^{(N-1)} | Y_1^{(N-1)} < v\right]$$

Note that this amount is exactly the same as in SPSB!

Revenue Equivalence

Proposition

With i.i.d. private values, the expected payments of a type-v bidder and the seller's expected revenue are the same in a first-price sealed-bid auction as in a second-price sealed-bid auction.

<u>*Proof:*</u> (i) In FPSB, the winner pays his own bid, so the expected payment of a bidder with valuation v is:

$$m^I(v) = \text{Prob[Win]} \times \text{Amount bid} = G(v) \times \mathbb{E}\left[Y_1^{(N-1)} | Y_1^{(N-1)} < v\right]$$

Note that this amount is exactly the same as in SPSB!

(ii) Expected revenue is N times the *ex ante* payment of an individual bidder, so it too must be the same between FPSB and SPSB. From SPSB, it must be the expected second-highest of N valuations:

$$\mathbb{E}[R^I] = \mathbb{E}[R^{II}] = \mathbb{E}[Y_2^{(N)}]$$

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!
- Especially striking because, in specific realizations, revenue can be either higher or lower in FPSB vs. SPSB

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!
- Especially striking because, in specific realizations, revenue can be either higher or lower in FPSB vs. SPSB
  - Example: $v \sim U[0,1]$, $N = 2 \implies \beta^{I}(v) = \frac{1}{2}v$, $\beta^{II}(v) = v$

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!
- Especially striking because, in specific realizations, revenue can be either higher or lower in FPSB vs. SPSB
    - Example: $v \sim U[0,1]$, $N = 2 \implies \beta^{I}(v) = \frac{1}{2}v$, $\beta^{II}(v) = v$
    - Case 1: $v_1 = 0.8$, $v_2 = 0.2$.

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!
- Especially striking because, in specific realizations, revenue can be either higher or lower in FPSB vs. SPSB
  - Example: $v \sim U[0,1]$, $N = 2 \implies \beta^{I}(v) = \frac{1}{2}v$, $\beta^{II}(v) = v$
  - Case 1: $v_1 = 0.8$, $v_2 = 0.2$. Then FPSB brings more: $0.4 > 0.2$.

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!
- Especially striking because, in specific realizations, revenue can be either higher or lower in FPSB vs. SPSB
  - Example: $v \sim U[0,1]$, $N = 2 \implies \beta^{I}(v) = \frac{1}{2}v$, $\beta^{II}(v) = v$
  - Case 1: $v_1 = 0.8$, $v_2 = 0.2$. Then FPSB brings more: $0.4 > 0.2$.
  - Case 2: $v_1 = 0.3$, $v_2 = 0.2$.

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!
- Especially striking because, in specific realizations, revenue can be either higher or lower in FPSB vs. SPSB
  - Example: $v \sim U[0,1]$, $N = 2 \implies \beta^{I}(v) = \frac{1}{2}v$, $\beta^{II}(v) = v$
  - Case 1: $v_1 = 0.8$, $v_2 = 0.2$. Then FPSB brings more: $0.4 > 0.2$.
  - Case 2: $v_1 = 0.3$, $v_2 = 0.2$. Then SPSB brings more: $0.2 > 0.15$.

Revenue Equivalence

- The fact that expected revenues in FPSB and SPSB is the same is striking — these are quite different formats!
- Especially striking because, in specific realizations, revenue can be either higher or lower in FPSB vs. SPSB
  - Example: $v \sim U[0,1]$, $N = 2 \implies \beta^{I}(v) = \frac{1}{2}v$, $\beta^{II}(v) = v$
  - Case 1: $v_1 = 0.8$, $v_2 = 0.2$. Then FPSB brings more: $0.4 > 0.2$.
  - Case 2: $v_1 = 0.3$, $v_2 = 0.2$. Then SPSB brings more: $0.2 > 0.15$.
- But, *on average*, the revenues are the same!

Risk Equivalence?

- We established *expected* revenue equivalence, but are FPSB and SPSB revenue distributions the same in terms of *risk*?

Risk Equivalence?

- We established *expected* revenue equivalence, but are FPSB and SPSB revenue distributions the same in terms of *risk*?
- Note that FPSB prices are less variable than those in SPSB:

Risk Equivalence?

- We established *expected* revenue equivalence, but are FPSB and SPSB revenue distributions the same in terms of *risk*?
- Note that FPSB prices are less variable than those in SPSB:
  - In SPSB, they can vary from 0 to ω

Risk Equivalence?

- We established *expected* revenue equivalence, but are FPSB and SPSB revenue distributions the same in terms of *risk*?
- Note that FPSB prices are less variable than those in SPSB:
  - In SPSB, they can vary from 0 to ω
  - In FPSB, they are capped lower than ω:

$$\beta^I(v) = \mathbb{E}[Y_1^{(N-1)} | Y_1^{(N-1)} < v] \leq \mathbb{E}[Y_1^{(N-1)}]$$

Risk Equivalence?

- We established *expected* revenue equivalence, but are FPSB and SPSB revenue distributions the same in terms of *risk*?
- Note that FPSB prices are less variable than those in SPSB:
  - In SPSB, they can vary from 0 to ω
  - In FPSB, they are capped lower than ω:

$$\beta^I(v) = \mathbb{E}[Y_1^{(N-1)} | Y_1^{(N-1)} < v] \leq \mathbb{E}[Y_1^{(N-1)}]$$

  - Example: If $N = 2$, then $\mathbb{E}[Y_1^{(N-1)}] = \mathbb{E}[v]$.

Risk Equivalence?

- We established *expected* revenue equivalence, but are FPSB and SPSB revenue distributions the same in terms of *risk*?
- Note that FPSB prices are less variable than those in SPSB:
  - In SPSB, they can vary from 0 to ω
  - In FPSB, they are capped lower than ω:

$$\beta^I(v) = \mathbb{E}[Y_1^{(N-1)} | Y_1^{(N-1)} < v] \leq \mathbb{E}[Y_1^{(N-1)}]$$

  - Example: If $N = 2$, then $\mathbb{E}[Y_1^{(N-1)}] = \mathbb{E}[v]$.
- In fact, one can prove a more general result:

Proposition

With i.i.d. private values, the distribution of equilibrium prices in a SPSB auction is a mean-preserving spread of the distribution of equilibrium prices in a FPSB auction.

The Revenue Equivalence Principle

Revenue Equivalence

- So far, we've shown that, regardless of $F(v)$, expected selling prices in symmetric FPSB and SPSB auctions are the same.
- Thus, a risk-neutral seller is indifferent between the two formats.

Revenue Equivalence

- So far, we've shown that, regardless of $F(v)$, expected selling prices in symmetric FPSB and SPSB auctions are the same.
- Thus, a risk-neutral seller is indifferent between the two formats.
- Again, this was not at all ex ante obvious — e.g., FPSB and SPSB auctions are not strategically equivalent.

Revenue Equivalence

- So far, we've shown that, regardless of $F(v)$, expected selling prices in symmetric FPSB and SPSB auctions are the same.
- Thus, a risk-neutral seller is indifferent between the two formats.
- Again, this was not at all ex ante obvious — e.g., FPSB and SPSB auctions are not strategically equivalent.
- But we can prove a more general revenue-equivalence result.

General Revenue Equivalence with IPV

Def-n: Auction is '*standard*' if highest bidder gets the object.

General Revenue Equivalence with IPV

Def-n: Auction is '*standard*' if highest bidder gets the object.

Proposition

Suppose that values are i.i.d. and all bidders are risk neutral. Then any symmetric and increasing equilibrium of any standard auction, such that the expected payment of a bidder with value zero is zero, yields the same expected revenue to the seller.