From Auctions to Mechanism Design

Alexey Makarin

May 11, 2020

Introduction

Is Auction The Best Allocation Mechanism?

- Auction is one of many ways to sell an object.

Is Auction The Best Allocation Mechanism?

- Auction is one of many ways to sell an object.
- In an auction, object is sold at a price determined by competition among buyers according to rules set out by the seller.

Is Auction The Best Allocation Mechanism?

- Auction is one of many ways to sell an object.
- In an auction, object is sold at a price determined by competition among buyers according to rules set out by the seller.
- Other possible methods include:

Is Auction The Best Allocation Mechanism?

- Auction is one of many ways to sell an object.
- In an auction, object is sold at a price determined by competition among buyers according to rules set out by the seller.
- Other possible methods include:
  - Seller could post a fixed price and sell the object to the first arrival;

Is Auction The Best Allocation Mechanism?

- Auction is one of many ways to sell an object.
- In an auction, object is sold at a price determined by competition among buyers according to rules set out by the seller.
- Other possible methods include:
  - Seller could post a fixed price and sell the object to the first arrival;
  - Seller could negotiate with one buyer (e.g., chosen at random);

Is Auction The Best Allocation Mechanism?

- Auction is one of many ways to sell an object.
- In an auction, object is sold at a price determined by competition among buyers according to rules set out by the seller.
- Other possible methods include:
  - Seller could post a fixed price and sell the object to the first arrival;
  - Seller could negotiate with one buyer (e.g., chosen at random);
  - Seller could hold an auction and then negotiate with winner; etc.

Is Auction The Best Allocation Mechanism?

- Auction is one of many ways to sell an object.
- In an auction, object is sold at a price determined by competition among buyers according to rules set out by the seller.
- Other possible methods include:
  - Seller could post a fixed price and sell the object to the first arrival;
  - Seller could negotiate with one buyer (e.g., chosen at random);
  - Seller could hold an auction and then negotiate with winner; etc.
- This week, we abstract away from the details of any particular selling format and ask: *"What is the best way to allocate an object?"*

General Mechanism Design Problem

- Design of a game (structure, rules, sequence of actions) among agents that implements certain desirable outcomes.

General Mechanism Design Problem

- Design of a game (structure, rules, sequence of actions) among agents that implements certain desirable outcomes.
- Agents have private information (adverse selection).

General Mechanism Design Problem

- Design of a game (structure, rules, sequence of actions) among agents that implements certain desirable outcomes.
- Agents have private information (adverse selection).
- Principal wants to achieve certain outcomes and has commitment power: commits to a decision rule.

General Mechanism Design Problem

- Design of a game (structure, rules, sequence of actions) among agents that implements certain desirable outcomes.
- Agents have private information (adverse selection).
- Principal wants to achieve certain outcomes and has commitment power: commits to a decision rule.
- Typical mechanism design questions:

General Mechanism Design Problem

- Design of a game (structure, rules, sequence of actions) among agents that implements certain desirable outcomes.
- Agents have private information (adverse selection).
- Principal wants to achieve certain outcomes and has commitment power: commits to a decision rule.
- Typical mechanism design questions:
  1. Which decision rules (social choice functions) are implementable?

General Mechanism Design Problem

- Design of a game (structure, rules, sequence of actions) among agents that implements certain desirable outcomes.
- Agents have private information (adverse selection).
- Principal wants to achieve certain outcomes and has commitment power: commits to a decision rule.
- Typical mechanism design questions:
  1. Which decision rules (social choice functions) are implementable?
  2. Which decision rule is optimal, i.e., preferred by the principal?

General Mechanism Design Problem

- Design of a game (structure, rules, sequence of actions) among agents that implements certain desirable outcomes.
- Agents have private information (adverse selection).
- Principal wants to achieve certain outcomes and has commitment power: commits to a decision rule.
- Typical mechanism design questions:
  1. Which decision rules (social choice functions) are implementable?
  2. Which decision rule is optimal, i.e., preferred by the principal?
  3. Which decision rule is efficient, i.e., maximizes overall surplus?

General Mechanism Design Problem

- Design of a game (structure, rules, sequence of actions) among agents that implements certain desirable outcomes.
- Agents have private information (adverse selection).
- Principal wants to achieve certain outcomes and has commitment power: commits to a decision rule.
- Typical mechanism design questions:
  1. Which decision rules (social choice functions) are implementable?
  2. Which decision rule is optimal, i.e., preferred by the principal?
  3. Which decision rule is efficient, i.e., maximizes overall surplus?
- This week: Optimal allocation rule to sell an object?

From Auctions to a General Mechanism Design Problem

Setup

- As before, seller has one indivisible object to sell.

Setup

- As before, seller has one indivisible object to sell.
- For simplicity, suppose the value of the object to the seller is 0.

Setup

- As before, seller has one indivisible object to sell.
- For simplicity, suppose the value of the object to the seller is 0.
- N risk-neutral buyers come from the set $\mathcal{N} = \{1, \ldots, N\}$.

Setup

- As before, seller has one indivisible object to sell.
- For simplicity, suppose the value of the object to the seller is 0.
- N risk-neutral buyers come from the set $\mathcal{N} = \{1, \ldots, N\}$.
- Buyers have independently distributed private valuations.

Setup

- As before, seller has one indivisible object to sell.
- For simplicity, suppose the value of the object to the seller is 0.
- N risk-neutral buyers come from the set $\mathcal{N} = \{1, \ldots, N\}$.
- Buyers have independently distributed private valuations.
- Buyer i's valuation V_i is distributed over the interval $\mathcal{V}_i = [0, \omega_i]$ according to c.d.f. F_i with density f_i.

Setup

- Let $\mathcal{V} = \times_{j=1}^{N} \mathcal{V}_j$ denote the product of the sets of buyers' values and, for all i, let $\mathcal{V}_{-i} = \times_{j \neq i} \mathcal{V}_j$

Setup

- Let $\mathcal{V} = \times_{j=1}^{N} \mathcal{V}_j$ denote the product of the sets of buyers' values and, for all i, let $\mathcal{V}_{-i} = \times_{j \neq i} \mathcal{V}_j$
- Define $f(\mathbf{v})$ be the joint density of $\mathbf{v} = (v_1, \ldots, v_N)$.

Setup

- Let $\mathcal{V} = \times_{j=1}^{N} \mathcal{V}_j$ denote the product of the sets of buyers' values and, for all i, let $\mathcal{V}_{-i} = \times_{j \neq i} \mathcal{V}_j$
- Define $f(\mathbf{v})$ be the joint density of $\mathbf{v} = (v_1, \ldots, v_N)$.
- Since valuations are independent, $f(\mathbf{v}) = f_1(v_1) \times \ldots \times f_N(v_N)$.

Setup

- Let $\mathcal{V} = \times_{j=1}^{N} \mathcal{V}_j$ denote the product of the sets of buyers' values and, for all i, let $\mathcal{V}_{-i} = \times_{j \neq i} \mathcal{V}_j$
- Define $f(\mathbf{v})$ be the joint density of $\mathbf{v} = (v_1, \ldots, v_N)$.
- Since valuations are independent, $f(\mathbf{v}) = f_1(v_1) \times \ldots \times f_N(v_N)$.
- Similarly, define $f_{-i}(\mathbf{v}_{-i})$ to be the joint density of $\mathbf{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N)$

Mechanism

Definition

A selling *mechanism* $(\mathcal{B}, \boldsymbol{\pi}, \boldsymbol{\mu})$ is a combination of:

1. A set of possible *messages* (or "bids") $\mathcal{B}_i$ for each buyer i;
2. An *allocation rule* $\boldsymbol{\pi} : \mathcal{B} \to \Delta$ where Δ is the set of probability distributions over the set of buyers $\mathcal{N}$;
3. A *payment rule* $\boldsymbol{\mu} : \mathcal{B} \to \mathbb{R}^N$.

Mechanism

Definition

A selling *mechanism* $(\mathcal{B}, \boldsymbol{\pi}, \boldsymbol{\mu})$ is a combination of:

1. A set of possible *messages* (or "bids") $\mathcal{B}_i$ for each buyer i;
2. An *allocation rule* $\boldsymbol{\pi} : \mathcal{B} \to \Delta$ where Δ is the set of probability distributions over the set of buyers $\mathcal{N}$;
3. A *payment rule* $\boldsymbol{\mu} : \mathcal{B} \to \mathbb{R}^N$.

Intuitively:

- Allocation rule determines, as a function of all N messages, the probability $\pi_i(\mathbf{b})$ that i will get the object.

Mechanism

Definition

A selling *mechanism* $(\mathcal{B}, \boldsymbol{\pi}, \boldsymbol{\mu})$ is a combination of:

1. A set of possible *messages* (or "bids") $\mathcal{B}_i$ for each buyer i;
2. An *allocation rule* $\boldsymbol{\pi} : \mathcal{B} \to \Delta$ where Δ is the set of probability distributions over the set of buyers $\mathcal{N}$;
3. A *payment rule* $\boldsymbol{\mu} : \mathcal{B} \to \mathbb{R}^N$.

Intuitively:

- Allocation rule determines, as a function of all N messages, the probability $\pi_i(\mathbf{b})$ that i will get the object.
- Payment rule determines, as a function of all N messages, for each buyer i, the expected payment $\mu_i(\mathbf{b})$ that i must make.

FPSB and SPSB as Mechanisms

- First- and second-price auctions are mechanisms.

FPSB and SPSB as Mechanisms

- First- and second-price auctions are mechanisms.
- In both, the set of possible bids can be assumed to be $\mathcal{B}_i = \mathcal{V}_i$

FPSB and SPSB as Mechanisms

- First- and second-price auctions are mechanisms.
- In both, the set of possible bids can be assumed to be $\mathcal{B}_i = \mathcal{V}_i$
- In both, the allocation rule is (ignoring ties):

$$\pi_i(\mathbf{b}) = \begin{cases} 1 & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

FPSB and SPSB as Mechanisms

- First- and second-price auctions are mechanisms.
- In both, the set of possible bids can be assumed to be $\mathcal{B}_i = \mathcal{V}_i$
- In both, the allocation rule is (ignoring ties):

$$\pi_i(\mathbf{b}) = \begin{cases} 1 & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

- However, the payment rules are different. First-price auction:

$$\mu_i^I(\mathbf{b}) = \begin{cases} b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

FPSB and SPSB as Mechanisms

- First- and second-price auctions are mechanisms.
- In both, the set of possible bids can be assumed to be $\mathcal{B}_i = \mathcal{V}_i$
- In both, the allocation rule is (ignoring ties):

$$\pi_i(\mathbf{b}) = \begin{cases} 1 & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

- However, the payment rules are different. First-price auction:

$$\mu_i^I(\mathbf{b}) = \begin{cases} b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

- Second-price auction:

$$\mu_i^{II}(\mathbf{b}) = \begin{cases} \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

Game Within a Mechanism

- Every mechanism defines a game of incomplete information among the buyers.
  - Strategies: $\beta_i : [0, \omega_i] \to \mathcal{B}_i$
  - Payoffs: Expected payoff for a given strategy profile and selling mechanism

Game Within a Mechanism

- Every mechanism defines a game of incomplete information among the buyers.
  - Strategies: $\beta_i : [0, \omega_i] \to \mathcal{B}_i$
  - Payoffs: Expected payoff for a given strategy profile and selling mechanism
- A strategy profile $\beta(\cdot)$ is a Bayesian Nash Equilibrium of a mechanism if for all i and for all v_i, given strategies $\boldsymbol{\beta}_{-i}$ of other buyers, $\beta_i(v_i)$ maximizes i's expected payoff.

Game Within a Mechanism

- Every mechanism defines a game of incomplete information among the buyers.
  - Strategies: $\beta_i : [0, \omega_i] \to \mathcal{B}_i$
  - Payoffs: Expected payoff for a given strategy profile and selling mechanism

- A strategy profile $\beta(\cdot)$ is a Bayesian Nash Equilibrium of a mechanism if for all i and for all v_i, given strategies $\boldsymbol{\beta}_{-i}$ of other buyers, $\beta_i(v_i)$ maximizes i's expected payoff.

- *Note*: Today, we focus on **Bayesian Mechanism Design**, i.e., with BNE as an underlying equilibrium concept. Later, we will come back and consider **Dominant Strategies Mechanism Design** which relies on the Dominant Strategy Equilibrium.

Direct Mechanisms and Revelation Principle

Direct Mechanisms

- Mechanisms could be quite complicated since we made no assumptions on the sets of "bids" or "messages" $\mathcal{B}_i$.

Direct Mechanisms

- Mechanisms could be quite complicated since we made no assumptions on the sets of "bids" or "messages" $\mathcal{B}_i$.
- A smaller and simpler class are mechanisms for which the set of messages is the same as the set of values—that is, $\mathcal{B}_i = \mathcal{V}_i \; \forall i$.

Direct Mechanisms

- Mechanisms could be quite complicated since we made no assumptions on the sets of “bids” or “messages” $\mathcal{B}_i$.
- A smaller and simpler class are mechanisms for which the set of messages is the same as the set of values—that is, $\mathcal{B}_i = \mathcal{V}_i \; \forall i$.
- Such mechanisms are called *direct*, since every buyer is asked to directly report a value.

Direct Mechanisms

- Formally, direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ consists of functions $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$, where $Q_i(\mathbf{v})$ is the probability that i will get the object and $M_i(\mathbf{v})$ is the expected payment by i.

Direct Mechanisms

- Formally, direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ consists of functions $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$, where $Q_i(\mathbf{v})$ is the probability that i will get the object and $M_i(\mathbf{v})$ is the expected payment by i.
- If it is an equilibrium for each buyer to reveal his true value, then the direct mechanism is said to have a *truthful* equilibrium.

Direct Mechanisms

- Formally, direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ consists of functions $\boldsymbol{Q} : \mathcal{V} \rightarrow \Delta$ and $\boldsymbol{M} : \mathcal{V} \rightarrow \mathbb{R}^N$, where $Q_i(\mathbf{v})$ is the probability that i will get the object and $M_i(\mathbf{v})$ is the expected payment by i.

- If it is an equilibrium for each buyer to reveal his true value, then the direct mechanism is said to have a *truthful* equilibrium.

- *Revelation principle*: outcomes of any mechanism's equilibrium can be replicated by a truthful equilibrium of a direct mechanism.

Direct Mechanisms

- Formally, direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ consists of functions $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$, where $Q_i(\mathbf{v})$ is the probability that i will get the object and $M_i(\mathbf{v})$ is the expected payment by i.
- If it is an equilibrium for each buyer to reveal his true value, then the direct mechanism is said to have a *truthful* equilibrium.
- *Revelation principle*: outcomes of any mechanism's equilibrium can be replicated by a truthful equilibrium of a direct mechanism.
- Hence, no loss of generality in focusing on direct mechanisms.

Proposition. Revelation Principle.

Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.

Proposition. Revelation Principle.

Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.

Proof: Let $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$ be defined as follows: $\boldsymbol{Q}(\boldsymbol{v}) = \boldsymbol{\pi}(\beta(\boldsymbol{v}))$ and $\boldsymbol{M}(\boldsymbol{v}) = \boldsymbol{\mu}(\beta(\boldsymbol{v}))$. Then both statements must be true.

Proposition. Revelation Principle.

Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.

Proof: Let $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$ be defined as follows: $\boldsymbol{Q}(\boldsymbol{v}) = \boldsymbol{\pi}(\beta(\boldsymbol{v}))$ and $\boldsymbol{M}(\boldsymbol{v}) = \boldsymbol{\mu}(\beta(\boldsymbol{v}))$. Then both statements must be true. Intuitively:

- Fix a mechanism and equilibrium β of that mechanism.

Proposition. Revelation Principle.

Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.

Proof: Let $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$ be defined as follows: $\boldsymbol{Q}(\boldsymbol{v}) = \boldsymbol{\pi}(\beta(\boldsymbol{v}))$ and $\boldsymbol{M}(\boldsymbol{v}) = \boldsymbol{\mu}(\beta(\boldsymbol{v}))$. Then both statements must be true. Intuitively:

- Fix a mechanism and equilibrium β of that mechanism.
- Instead of buyers submitting messages $b_i = \beta_i(v_i)$, we directly ask buyers to report their values v_i and then make sure that the outcomes are the same as if they had submitted bids $\beta_i(v_i)$.

Proposition. Revelation Principle.

Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.

Proof: Let $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$ be defined as follows: $\boldsymbol{Q}(\boldsymbol{v}) = \boldsymbol{\pi}(\beta(\boldsymbol{v}))$ and $\boldsymbol{M}(\boldsymbol{v}) = \boldsymbol{\mu}(\beta(\boldsymbol{v}))$. Then both statements must be true. Intuitively:

- Fix a mechanism and equilibrium β of that mechanism.
- Instead of buyers submitting messages $b_i = \beta_i(v_i)$, we directly ask buyers to report their values v_i and then make sure that the outcomes are the same as if they had submitted bids $\beta_i(v_i)$.
- Now suppose that some buyer finds it profitable to be untruthful and report a value $\hat{v}_i$ when his value is v_i.

Proposition. Revelation Principle.

Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his valuation truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.

Proof: Let $\boldsymbol{Q} : \mathcal{V} \to \Delta$ and $\boldsymbol{M} : \mathcal{V} \to \mathbb{R}^N$ be defined as follows: $\boldsymbol{Q(v)} = \boldsymbol{\pi}(\beta(\boldsymbol{v}))$ and $\boldsymbol{M(v)} = \boldsymbol{\mu}(\beta(\boldsymbol{v}))$. Then both statements must be true. Intuitively:

- Fix a mechanism and equilibrium β of that mechanism.
- Instead of buyers submitting messages $b_i = \beta_i(v_i)$, we directly ask buyers to report their values v_i and then make sure that the outcomes are the same as if they had submitted bids $\beta_i(v_i)$.
- Now suppose that some buyer finds it profitable to be untruthful and report a value $\hat{v}_i$ when his value is v_i.
- Then in the original mechanism same buyer would have found it profitable to submit $\beta_i(\hat{v}_i)$ instead of $\beta_i(v_i)$. Contradiction. ■

Buyer's Payoff Function

Given a direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$:

$$q_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} Q_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

is the probability that i gets the object when he reports $\hat{v}_i$ and all other buyers report their values truthfully.

Buyer's Payoff Function

Given a direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$:

$$q_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} Q_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

is the probability that i gets the object when he reports $\hat{v}_i$ and all other buyers report their values truthfully. Similarly,

$$m_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} M_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

is expected payment when i reports $\hat{v}_i$ and others tell the truth.

Buyer's Payoff Function

Given a direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$:

$$q_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} Q_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

is the probability that i gets the object when he reports $\hat{v}_i$ and all other buyers report their values truthfully. Similarly,

$$m_i(\hat{v}_i) = \int_{\mathcal{V}_{-i}} M_i(\hat{v}_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

is expected payment when i reports $\hat{v}_i$ and others tell the truth. Then:

$$q_i(\hat{v}_i) v_i - m_i(\hat{v}_i)$$

is the expected payoff of i when he reports $\hat{v}_i$ and others tell the truth.

Incentive Compatibility

The direct revelation mechanism is incentive compatible (IC) if:

$$U_i(v_i) \equiv q_i(v_i)v_i - m_i(v_i) \geq q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)$$

$$\forall i \in \mathcal{N}; \forall v_i, \hat{v}_i \in [0, \omega_i]$$

where U_i is the *equilibrium payoff function*.

Incentive Compatibility: Implications

- Incentive compatibility has several important implications.

Incentive Compatibility: Implications

- Incentive compatibility has several important implications.
- First, for each reported value $\hat{v}_i$, expected payoff $q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)$ is an affine function of true value v_i.

Incentive Compatibility: Implications

- Incentive compatibility has several important implications.
- First, for each reported value $\hat{v}_i$, expected payoff $q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)$ is an affine function of true value v_i. Thus, IC implies that:

$$U_i(v_i) = \max_{\hat{v}_i \in \mathcal{V}_i} \{q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)\}$$

Incentive Compatibility: Implications

- Incentive compatibility has several important implications.
- First, for each reported value $\hat{v}_i$, expected payoff $q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)$ is an affine function of true value v_i. Thus, IC implies that:

$$U_i(v_i) = \max_{\hat{v}_i \in \mathcal{V}_i} \{q_i(\hat{v}_i)v_i - m_i(\hat{v}_i)\}$$

- That is, U_i is a maximum of a family of affine functions, therefore U_i is a *convex function*.

Incentive Compatibility: Implications

- Second, we can rewrite:

$$U_i(\hat{v}_i) = q_i(\hat{v}_i)\hat{v}_i - m_i(\hat{v}_i) \geq q_i(v_i)\hat{v}_i - m_i(v_i)$$

Incentive Compatibility: Implications

- Second, we can rewrite:

$$\begin{aligned} U_i(\hat{v}_i) = q_i(\hat{v}_i)\hat{v}_i - m_i(\hat{v}_i) &\geq q_i(v_i)\hat{v}_i - m_i(v_i) \\ &= q_i(v_i)v_i - m_i(v_i) + q_i(v_i)(\hat{v}_i - v_i) \end{aligned}$$

Incentive Compatibility: Implications

- Second, we can rewrite:

$$\begin{aligned} U_i(\hat{v}_i) = q_i(\hat{v}_i)\hat{v}_i - m_i(\hat{v}_i) &\geq q_i(v_i)\hat{v}_i - m_i(v_i) \\ &= q_i(v_i)v_i - m_i(v_i) + q_i(v_i)(\hat{v}_i - v_i) \\ &= U_i(v_i) + q_i(v_i)(\hat{v}_i - v_i) \end{aligned}$$

Incentive Compatibility: Implications

- Second, we can rewrite:

$$\begin{aligned} U_i(\hat{v}_i) = q_i(\hat{v}_i)\hat{v}_i - m_i(\hat{v}_i) &\geq q_i(v_i)\hat{v}_i - m_i(v_i) \\ &= q_i(v_i)v_i - m_i(v_i) + q_i(v_i)(\hat{v}_i - v_i) \\ &= U_i(v_i) + q_i(v_i)(\hat{v}_i - v_i) \end{aligned}$$

- Since the above inequality has to hold for all v_i and $\hat{v}_i$, $q_i(v_i)$ is the subgradient of the function U_i at v_i.

Incentive Compatibility: Implications

- Second, we can rewrite:

$$\begin{aligned} U_i(\hat{v}_i) = q_i(\hat{v}_i)\hat{v}_i - m_i(\hat{v}_i) &\geq q_i(v_i)\hat{v}_i - m_i(v_i) \\ &= q_i(v_i)v_i - m_i(v_i) + q_i(v_i)(\hat{v}_i - v_i) \\ &= U_i(v_i) + q_i(v_i)(\hat{v}_i - v_i) \end{aligned}$$

- Since the above inequality has to hold for all v_i and $\hat{v}_i$, $q_i(v_i)$ is the subgradient of the function U_i at v_i.
- Since U_i is convex, it must be that q_i is *non-decreasing*.

Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

$$U_i'(v_i) = q_i(v_i) \implies U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(x_i)dx_i$$

Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

$$U_i'(v_i) = q_i(v_i) \implies U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(x_i)dx_i$$

- That is, up to an additive constant, buyer's expected payoff in an IC direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ depends only on allocation rule $\boldsymbol{Q}$.

Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

$$U_i'(v_i) = q_i(v_i) \implies U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(x_i)dx_i$$

- That is, up to an additive constant, buyer's expected payoff in an IC direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ depends only on allocation rule $\boldsymbol{Q}$.
- Thus, if $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are two IC mechanisms with same allocation rule but different payment rules, then their expected payoff functions, U_i and $\bar{U}_i$, differ by at most a constant.

Incentive Compatibility: Implications

- Third, since convexity implies differentiability almost everywhere:

$$U_i'(v_i) = q_i(v_i) \implies U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(x_i)dx_i$$

- That is, up to an additive constant, buyer's expected payoff in an IC direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ depends only on allocation rule $\boldsymbol{Q}$.
- Thus, if $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are two IC mechanisms with same allocation rule but different payment rules, then their expected payoff functions, U_i and $\bar{U}_i$, differ by at most a constant.
- In other words, $(\boldsymbol{Q}, \boldsymbol{M})$ and $(\boldsymbol{Q}, \bar{\boldsymbol{M}})$ are *payoff equivalent*.

Revenue Equivalence Strikes Again!

Generalized Revenue Equivalence

If the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ is incentive compatible, then for all i and v_i, the expected payment is

$$m_i(v_i) = m_i(0) + q_i(v_i)v_i - \int_0^{v_i} q_i(x_i)dx_i$$

Thus, expected payments in any two incentive compatible mechanisms with the same allocation rule are equivalent up to a constant.

Revenue Equivalence Strikes Again!

Generalized Revenue Equivalence

If the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ is incentive compatible, then for all i and v_i, the expected payment is

$$m_i(v_i) = m_i(0) + q_i(v_i)v_i - \int_0^{v_i} q_i(x_i)dx_i$$

Thus, expected payments in any two incentive compatible mechanisms with the same allocation rule are equivalent up to a constant.

<u>*Proof*</u>: Since $U_i(v_i) = q_i(v_i)v_i - m_i(v_i)$ and $U_i(0) = -m_i(0)$, then:

$$U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(x_i)dx_i \implies$$

Revenue Equivalence Strikes Again!

Generalized Revenue Equivalence

If the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ is incentive compatible, then for all i and v_i, the expected payment is

$$m_i(v_i) = m_i(0) + q_i(v_i)v_i - \int_0^{v_i} q_i(x_i)dx_i$$

Thus, expected payments in any two incentive compatible mechanisms with the same allocation rule are equivalent up to a constant.

Proof: Since $U_i(v_i) = q_i(v_i)v_i - m_i(v_i)$ and $U_i(0) = -m_i(0)$, then:

$$U_i(v_i) = U_i(0) + \int_0^{v_i} q_i(x_i)dx_i \implies$$

$$\implies m_i(v_i) = m_i(0) + q_i(v_i)v_i - \int_0^{v_i} q_i(x_i)dx_i \quad \blacksquare$$

Generalized Revenue Equivalence

Remarks:

- Given two BNE of two different auctions such that for each i:
  - For all $(v_1, \ldots, v_N)$, probability of i getting the object is the same,
  - Two equilibria have the same expected payment at 0 value.

These auctions generate same expected revenue for the seller.

Generalized Revenue Equivalence

Remarks:

- Given two BNE of two different auctions such that for each i:
  - For all $(v_1, \ldots, v_N)$, probability of i getting the object is the same,
  - Two equilibria have the same expected payment at 0 value.

  These auctions generate same expected revenue for the seller.

- This generalizes the result from last time:
  - Revenue equivalence at the symmetric equilibrium of standard auctions (object allocated to buyer with the highest bid).

Incentive Compatibility: Implications

- Finally, can show that a mechanism is incentive compatible *if and only if* the associated q_i is nondecreasing.

Incentive Compatibility: Implications

- Finally, can show that a mechanism is incentive compatible *if and only if* the associated q_i is nondecreasing.
- We have shown that IC implies that q_i is nondecreasing.

Incentive Compatibility: Implications

- Finally, can show that a mechanism is incentive compatible *if and only if* the associated q_i is nondecreasing.
- We have shown that IC implies that q_i is nondecreasing.
- To see that nondecreasing q_i implies IC, note that:

$$U_i(\hat{v}_i) \geq U(v_i) + q_i(v_i)(\hat{v}_i - v_i) \iff$$
$$\iff \int_{v_i}^{\hat{v}_i} q_i(x_i)dx_i \geq q_i(v_i)(\hat{v}_i - v_i)$$

Incentive Compatibility: Implications

- Finally, can show that a mechanism is incentive compatible *if and only if* the associated q_i is nondecreasing.
- We have shown that IC implies that q_i is nondecreasing.
- To see that nondecreasing q_i implies IC, note that:

$$U_i(\hat{v}_i) \geq U(v_i) + q_i(v_i)(\hat{v}_i - v_i) \iff$$
$$\iff \int_{v_i}^{\hat{v}_i} q_i(x_i) dx_i \geq q_i(v_i)(\hat{v}_i - v_i)$$

- The latter inequality certainly holds if q_i is nondecreasing.

Individual Rationality

- Mechanism's payments may be too high and can scare off buyers.

Individual Rationality

- Mechanism's payments may be too high and can scare off buyers.
- Direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ is *individually rational* if equilibrium expected payoffs are: $U_i(v_i) \geq 0 \; \forall i \in N, v_i \in [0, \omega_i]$.

Individual Rationality

- Mechanism's payments may be too high and can scare off buyers.
- Direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ is *individually rational* if equilibrium expected payoffs are: $U_i(v_i) \geq 0 \ \forall i \in N, v_i \in [0, \omega_i]$.
- (We implicitly assume that by not participating, a buyer can guarantee himself a payoff of zero.)

Individual Rationality

- Mechanism's payments may be too high and can scare off buyers.
- Direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$ is *individually rational* if equilibrium expected payoffs are: $U_i(v_i) \geq 0 \ \forall i \in N, v_i \in [0, \omega_i]$.
- (We implicitly assume that by not participating, a buyer can guarantee himself a payoff of zero.)
- If the mechanism is IC, then IR is equivalent to $U_i(0) \geq 0$, and since $U_i(0) = -m_i(0)$ this is equivalent to $m_i(0) \leq 0$.

Optimal Mechanisms

Finding an Optimal Mechanism

- Seller's goal is to design an optimal mechanism that maximizes expected revenue among all mechanisms that are IC and IR.

Finding an Optimal Mechanism

- Seller's goal is to design an optimal mechanism that maximizes expected revenue among all mechanisms that are IC and IR.
- Without loss of generality we can focus on direct revelation mechanisms.

Finding an Optimal Mechanism

- Seller's goal is to design an optimal mechanism that maximizes expected revenue among all mechanisms that are IC and IR.
- Without loss of generality we can focus on direct revelation mechanisms.
- Consider the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$.

$$\mathbb{E}[R] = \sum_{i \in N} \mathbb{E}[m_i(V_i)]$$

Finding an Optimal Mechanism

- Seller's goal is to design an optimal mechanism that maximizes expected revenue among all mechanisms that are IC and IR.
- Without loss of generality we can focus on direct revelation mechanisms.
- Consider the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$.

$$\mathbb{E}[R] = \sum_{i \in N} \mathbb{E}[m_i(V_i)]$$

where the ex ante expected payment of buyer i is

$$\mathbb{E}[m_i(V_i)] = \int_0^{\omega_i} m_i(v_i) f_i(v_i) dv_i$$

Finding an Optimal Mechanism

- Seller's goal is to design an optimal mechanism that maximizes expected revenue among all mechanisms that are IC and IR.
- Without loss of generality we can focus on direct revelation mechanisms.
- Consider the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$.

$$\mathbb{E}[R] = \sum_{i \in N} \mathbb{E}[m_i(V_i)]$$

where the ex ante expected payment of buyer i is

$$\begin{aligned} \mathbb{E}[m_i(V_i)] &= \int_0^{\omega_i} m_i(v_i) f_i(v_i) dv_i \\ &= m_i(0) + \int_0^{\omega_i} q_i(v_i) v_i f(v_i) dv_i - \int_0^{\omega_i} \int_0^{v_i} q_i(x_i) f(x_i) dx_i dv_i \end{aligned}$$

Finding an Optimal Mechanism

Changing the order of integration in the last term:

$$\int_0^{\omega_i} \int_0^{v_i} q_i(x_i) f(x_i) dx_i dv_i =$$

Finding an Optimal Mechanism

Changing the order of integration in the last term:

$$\int_0^{\omega_i} \int_0^{v_i} q_i(x_i) f(x_i) dx_i dv_i = \int_0^{\omega_i} \left[\int_{x_i}^{\omega_i} f(v_i) dv_i \right] q_i(x_i) dx_i$$

Finding an Optimal Mechanism

Changing the order of integration in the last term:

$$\int_0^{\omega_i} \int_0^{v_i} q_i(x_i) f(x_i) dx_i dv_i = \int_0^{\omega_i} \left[\int_{x_i}^{\omega_i} f(v_i) dv_i \right] q_i(x_i) dx_i$$
$$= \int_0^{\omega_i} \left[1 - F_i(v) \right] q_i(x_i) dx_i$$

Finding an Optimal Mechanism

Changing the order of integration in the last term:

$$\int_0^{\omega_i} \int_0^{v_i} q_i(x_i) f(x_i) dx_i dv_i = \int_0^{\omega_i} \left[\int_{x_i}^{\omega_i} f(v_i) dv_i \right] q_i(x_i) dx_i$$
$$= \int_0^{\omega_i} [1 - F_i(v)]\, q_i(x_i) dx_i$$

Thus, can write:

$$\mathbb{E}[m_i(V_i)] = m_i(0) + \int_0^{\omega_i} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] q_i(v_i) f(v_i) dv_i$$

Finding an Optimal Mechanism

Changing the order of integration in the last term:

$$\int_0^{\omega_i} \int_0^{v_i} q_i(x_i) f(x_i) dx_i dv_i = \int_0^{\omega_i} \left[\int_{x_i}^{\omega_i} f(v_i) dv_i \right] q_i(x_i) dx_i$$
$$= \int_0^{\omega_i} [1 - F_i(v)] \, q_i(x_i) dx_i$$

Thus, can write:

$$\mathbb{E}[m_i(V_i)] = m_i(0) + \int_0^{\omega_i} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] q_i(v_i) f(v_i) dv_i$$
$$= m_i(0) + \int_{\mathcal{V}} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] Q_i(\mathbf{v}) f(\mathbf{v}) d\mathbf{v}$$

where, in the last step, we used the definition of q_i as the expected allocation probability intergrated over valuations of all other players.

Optimal Mechanism Design Problem

$$\sum_{i \in \mathcal{N}} m_i(0) + \sum_{i \in \mathcal{N}} \int_{\mathcal{V}} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] Q_i(\mathbf{v}) f(\mathbf{v}) d\mathbf{v} \to \max_{\boldsymbol{Q}, \boldsymbol{M}}$$

s.t. IC constraint ($\Leftrightarrow q_i$ is nondecreasing)

IR constraint ($\Leftrightarrow m_i(0) \leq 0$)

Optimal Mechanism Design Problem

- Let's define **virtual valuation** of a buyer with value v_i as:

$$\psi_i(v_i) \equiv v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Optimal Mechanism Design Problem

- Let's define **virtual valuation** of a buyer with value v_i as:

$$\psi_i(v_i) \equiv v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

- Note that $\mathbb{E}[\psi_i(v_i)] = 0$. (Why?)

Optimal Mechanism Design Problem

- Let's define **virtual valuation** of a buyer with value v_i as:

$$\psi_i(v_i) \equiv v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

- Note that $\mathbb{E}[\psi_i(v_i)] = 0$. (Why?)
- We say that design problem is regular if $\psi_i(v_i)$ is increasing in v_i (it is sufficient that hazard rate $\lambda_i(v_i)$ is increasing in v_i).

Optimal Mechanism Design Problem

- Let's define **virtual valuation** of a buyer with value v_i as:

$$\psi_i(v_i) \equiv v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

- Note that $\mathbb{E}[\psi_i(v_i)] = 0$. (Why?)
- We say that design problem is regular if $\psi_i(v_i)$ is increasing in v_i (it is sufficient that hazard rate $\lambda_i(v_i)$ is increasing in v_i).
- Let us temporarily neglect the IC and the IR constraints and circle back to them later.

Optimal Mechanism Design Problem

- The seller should choose $(\boldsymbol{Q}, \boldsymbol{M})$ to maximize:

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{V}} \left(\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v}) \right) f(\mathbf{v}) d\mathbf{v}$$

Optimal Mechanism Design Problem

- The seller should choose $(\boldsymbol{Q}, \boldsymbol{M})$ to maximize:

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{V}} \left(\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v}) \right) f(\mathbf{v}) d\mathbf{v}$$

- Consider the expression from the second term:

$$\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v})$$

Optimal Mechanism Design Problem

- The seller should choose $(\boldsymbol{Q}, \boldsymbol{M})$ to maximize:

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{V}} \left(\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v}) \right) f(\mathbf{v}) d\mathbf{v}$$

- Consider the expression from the second term:

$$\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v})$$

- Here, $\mathbf{Q}$ is akin to a weighting function, and clearly it is best to give weight only to those $\psi_i(v_i)$ that are maximal (and positive).

Optimal Mechanism Design Problem

- The seller should choose $(\boldsymbol{Q}, \boldsymbol{M})$ to maximize:

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{V}} \left(\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v}) \right) f(\mathbf{v}) d\mathbf{v}$$

- Consider the expression from the second term:

$$\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v})$$

- Here, $\mathbf{Q}$ is akin to a weighting function, and clearly it is best to give weight only to those $\psi_i(v_i)$ that are maximal (and positive).
- This approach would maximize this expression at every point $\mathbf{v}$ and so would also maximize its integral.

Optimal Mechanism

Claim

The following is an optimal mechanism:

Optimal Mechanism

Claim

The following is an optimal mechanism:

1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

$$Q_i(\mathbf{v}) > 0 \Leftrightarrow \psi_i(v_i) = \max_{j \in \mathcal{N}} \psi_j(v_j) \geq 0$$

Optimal Mechanism

Claim

The following is an optimal mechanism:

1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

$$Q_i(\mathbf{v}) > 0 \Leftrightarrow \psi_i(v_i) = \max_{j \in \mathcal{N}} \psi_j(v_j) \geq 0$$

2. The payment rule $\boldsymbol{M}$ is as follows:

$$M_i(\mathbf{v}) = Q_i(\mathbf{v}) v_i - \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i$$

Optimal Mechanism

Claim

The following is an optimal mechanism:

1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

$$Q_i(\mathbf{v}) > 0 \Leftrightarrow \psi_i(v_i) = \max_{j \in \mathcal{N}} \psi_j(v_j) \geq 0$$

2. The payment rule $\boldsymbol{M}$ is as follows:

$$M_i(\mathbf{v}) = Q_i(\mathbf{v}) v_i - \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i$$

First, let's check IC and IR:

Optimal Mechanism

Claim

The following is an optimal mechanism:

1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

$$Q_i(\mathbf{v}) > 0 \Leftrightarrow \psi_i(v_i) = \max_{j \in \mathcal{N}} \psi_j(v_j) \geq 0$$

2. The payment rule $\boldsymbol{M}$ is as follows:

$$M_i(\mathbf{v}) = Q_i(\mathbf{v})v_i - \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i})dx_i$$

First, let's check IC and IR:

- IC: Check if q_i is nondecreasing. Suppose $\hat{v}_i < v_i$. Then by the regularity condition, $\psi_i(\hat{v}_i) < \psi_i(v_i)$ and, thus, for all v_{-i}, it is also the case that $Q_i(\hat{v}_i, v_{-i}) \leq Q(\mathbf{v})$. Thus, q_i is nondecreasing.

Optimal Mechanism

Claim

The following is an optimal mechanism:

1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

$$Q_i(\mathbf{v}) > 0 \Leftrightarrow \psi_i(v_i) = \max_{j \in \mathcal{N}} \psi_j(v_j) \geq 0$$

2. The payment rule $\boldsymbol{M}$ is as follows:

$$M_i(\mathbf{v}) = Q_i(\mathbf{v}) v_i - \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i$$

First, let's check IC and IR:

- IR: From the payment rule, it is clear that $M_i(0, \mathbf{v}_{-i}) = 0$ for all $\mathbf{v}_{-i}$, and thus $m_i(0) = 0$.

Optimal Mechanism

Claim

The following is an optimal mechanism:

1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

$$Q_i(\mathbf{v}) > 0 \Leftrightarrow \psi_i(v_i) = \max_{j \in \mathcal{N}} \psi_j(v_j) \geq 0$$

2. The payment rule $\boldsymbol{M}$ is as follows:

$$M_i(\mathbf{v}) = Q_i(\mathbf{v}) v_i - \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i$$

Second, note that this mechanism separately maximizes two terms in seller's expected revenue: (i) It gives positive weight only to nonnegative and maximal terms in the second expression; (ii) It sets maximal $m_i(0)$ given the IR constraint.

Optimal Mechanism

This implies that the maximized value of the expected revenue is:

$$\mathbb{E}[\max\{\psi_1(V_1), \ldots, \psi_N(V_N), 0\}]$$

In other words, it is the expectation of the highest virtual valuation, provided it is nonnegative.

Optimal Mechanism

To obtain more intuitive formulas for $(\boldsymbol{Q}, \boldsymbol{M})$, we define:

$$y_i(\mathbf{v}_{-i}) = \inf\{x_i : \psi_i(x_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(x_i) \geq \psi_j(v_j)\}$$

as the smallest value for i that "wins" against $\mathbf{v}_{-i}$.

Optimal Mechanism

To obtain more intuitive formulas for $(\boldsymbol{Q}, \boldsymbol{M})$, we define:

$$y_i(\mathbf{v}_{-i}) = \inf\{x_i : \psi_i(x_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(x_i) \geq \psi_j(v_j)\}$$

as the smallest value for i that "wins" against $\mathbf{v}_{-i}$. Thus, can rewrite the optimal allocation rules as:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } x_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } x_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

Optimal Mechanism

To obtain more intuitive formulas for $(\boldsymbol{Q}, \boldsymbol{M})$, we define:

$$y_i(\mathbf{v}_{-i}) = \inf\{x_i : \psi_i(x_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(x_i) \geq \psi_j(v_j)\}$$

as the smallest value for i that "wins" against $\mathbf{v}_{-i}$. Thus, can rewrite the optimal allocation rules as:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } x_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } x_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

$$\implies \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i = \begin{cases} v_i - y_i(\mathbf{v}_{-i}) & \text{if } v_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } v_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

Optimal Mechanism

To obtain more intuitive formulas for $(\boldsymbol{Q}, \boldsymbol{M})$, we define:

$$y_i(\mathbf{v}_{-i}) = \inf\{x_i : \psi_i(x_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(x_i) \geq \psi_j(v_j)\}$$

as the smallest value for i that "wins" against $\mathbf{v}_{-i}$. Thus, can rewrite the optimal allocation rules as:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } x_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } x_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

$$\implies \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i = \begin{cases} v_i - y_i(\mathbf{v}_{-i}) & \text{if } v_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } v_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

$$\implies M_i(\mathbf{v}) = \begin{cases} y_i(\mathbf{v}_{-i}) & \text{if } Q_i(\mathbf{v}) = 1 \\ 0 & \text{if } Q_i(\mathbf{v}) = 0 \end{cases}$$

Optimal Mechanism

To obtain more intuitive formulas for $(\boldsymbol{Q}, \boldsymbol{M})$, we define:

$$y_i(\mathbf{v}_{-i}) = \inf\{x_i : \psi_i(x_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(x_i) \geq \psi_j(v_j)\}$$

as the smallest value for i that "wins" against $\mathbf{v}_{-i}$. Thus, can rewrite the optimal allocation rules as:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } x_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } x_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

$$\implies \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i = \begin{cases} v_i - y_i(\mathbf{v}_{-i}) & \text{if } v_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } v_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

$$\implies M_i(\mathbf{v}) = \begin{cases} y_i(\mathbf{v}_{-i}) & \text{if } Q_i(\mathbf{v}) = 1 \\ 0 & \text{if } Q_i(\mathbf{v}) = 0 \end{cases}$$

Thus, only the 'winning' buyer pays anything. He pays the smallest value that would result in his winning.

Optimal Mechanism

Proposition.

Suppose the design problem is regular. Then the following is an optimal mechanism:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } \psi_i(v_i) > \max_{j \neq i} \psi_j \text{ and } \psi_i(v_i) \geq 0 \\ 0 & \text{if } \psi_i(v_i) < \max_{j \neq i} \psi_j \end{cases}$$

$$M_i(\mathbf{v}) = \begin{cases} y_i(\mathbf{v}_{-i}) & \text{if } Q_i(\mathbf{v}) = 1 \\ 0 & \text{if } Q_i(\mathbf{v}) = 0 \end{cases}$$

Illustration

Optimal Mechanism in a Symmetric Case

Suppose the problem is symmetric, i.e., $f_i = f$, and hence $\psi_i = \psi \ \forall i$.

Optimal Mechanism in a Symmetric Case

Suppose the problem is symmetric, i.e., $f_i = f$, and hence $\psi_i = \psi \ \forall i$. Now we have that:

$$y_i(\mathbf{v}_{-i}) = \max\left\{\psi^{-1}(0), \max_{j \neq i} v_j\right\}$$

Optimal Mechanism in a Symmetric Case

Suppose the problem is symmetric, i.e., $f_i = f$, and hence $\psi_i = \psi \ \forall i$. Now we have that:

$$y_i(\mathbf{v}_{-i}) = \max \left\{ \psi^{-1}(0), \max_{j \neq i} v_j \right\}$$

Thus, the optimal mechanism is SPSB with reserve price $r^* = \psi^{-1}(0)$.

Optimal Mechanism in a Symmetric Case

Suppose the problem is symmetric, i.e., $f_i = f$, and hence $\psi_i = \psi \ \forall i$. Now we have that:

$$y_i(\mathbf{v}_{-i}) = \max\left\{\psi^{-1}(0), \max_{j \neq i} v_j\right\}$$

Thus, the optimal mechanism is SPSB with reserve price $r^* = \psi^{-1}(0)$.

Proposition.

Suppose that the seller's design problem is regular and symmetric. Then a second-price auction with a reserve price $r^* = \psi^{-1}(0)$ is an optimal mechanism.

Optimal Mechanism in a Symmetric Case

Suppose the problem is symmetric, i.e., $f_i = f$, and hence $\psi_i = \psi \ \forall i$. Now we have that:

$$y_i(\mathbf{v}_{-i}) = \max\left\{\psi^{-1}(0), \max_{j \neq i} v_j\right\}$$

Thus, the optimal mechanism is SPSB with reserve price $r^* = \psi^{-1}(0)$.

Proposition.

Suppose that the seller's design problem is regular and symmetric. Then a second-price auction with a reserve price $r^* = \psi^{-1}(0)$ is an optimal mechanism.

Note that $\psi^{-1}(0)$ is the optimal reserve price we derived earlier!

(In)Efficiency of the Optimal Mechanism

The optimal mechanism has two separate sources of inefficiency:

1. Efficient mechanisms give the object away whenever $v > v_0$.

(In)Efficiency of the Optimal Mechanism

The optimal mechanism has two separate sources of inefficiency:

1. Efficient mechanisms give the object away whenever $v > v_0$.
   - Here, seller retains it if highest virtual valuation is negative.

(In)Efficiency of the Optimal Mechanism

The optimal mechanism has two separate sources of inefficiency:

1. Efficient mechanisms give the object away whenever $v > v_0$.
    - Here, seller retains it if highest virtual valuation is negative.
    - But buyers' values are nonnegative and seller's value is zero.

(In)Efficiency of the Optimal Mechanism

The optimal mechanism has two separate sources of inefficiency:

1. Efficient mechanisms give the object away whenever $v > v_0$.
   - Here, seller retains it if highest virtual valuation is negative.
   - But buyers' values are nonnegative and seller's value is zero.
   - So it's always socially optimal to give the object to some buyer.

(In)Efficiency of the Optimal Mechanism

The optimal mechanism has two separate sources of inefficiency:

1. Efficient mechanisms give the object away whenever $v > v_0$.
    - Here, seller retains it if highest virtual valuation is negative.
    - But buyers' values are nonnegative and seller's value is zero.
    - So it's always socially optimal to give the object to some buyer.
2. Efficient mechanisms give objects to buyer with highest value.

(In)Efficiency of the Optimal Mechanism

The optimal mechanism has two separate sources of inefficiency:

1. Efficient mechanisms give the object away whenever $v > v_0$.
   - Here, seller retains it if highest virtual valuation is negative.
   - But buyers' values are nonnegative and seller's value is zero.
   - So it's always socially optimal to give the object to some buyer.
2. Efficient mechanisms give objects to buyer with highest value.
   - Here, it is allocated to the buyer with highest *virtual* valuation.

(In)Efficiency of the Optimal Mechanism

The optimal mechanism has two separate sources of inefficiency:

1. Efficient mechanisms give the object away whenever $v > v_0$.
   - Here, seller retains it if highest virtual valuation is negative.
   - But buyers' values are nonnegative and seller's value is zero.
   - So it's always socially optimal to give the object to some buyer.
2. Efficient mechanisms give objects to buyer with highest value.
   - Here, it is allocated to the buyer with highest *virtual* valuation.
   - In the asymmetric case, this need not be the highest-value buyer.

Interpreting Virtual Valuations

Why is it optimal to allocate the object on the basis of virtual valuations. And what the hell are virtual valuations anyway?

Interpreting Virtual Valuations

Why is it optimal to allocate the object on the basis of virtual valuations. And what the hell are virtual valuations anyway?

- Consider buyer i whose values are distributed according to F.

Interpreting Virtual Valuations

Why is it optimal to allocate the object on the basis of virtual valuations. And what the hell are virtual valuations anyway?

- Consider buyer i whose values are distributed according to F.
- Suppose seller makes a take-it-or-leave it offer to i at price p.

Interpreting Virtual Valuations

Why is it optimal to allocate the object on the basis of virtual valuations. And what the hell are virtual valuations anyway?

- Consider buyer i whose values are distributed according to F.
- Suppose seller makes a take-it-or-leave it offer to i at price p.
- The probability that i accepts this offer is $1 - F(p)$.

Interpreting Virtual Valuations

Why is it optimal to allocate the object on the basis of virtual valuations. And what the hell are virtual valuations anyway?

- Consider buyer i whose values are distributed according to F.
- Suppose seller makes a take-it-or-leave it offer to i at price p.
- The probability that i accepts this offer is $1 - F(p)$.
- Think of this probability as the "quantity" demanded by i and write the implied "demand curve" as $q(p) \equiv 1 - F(p)$.

Interpreting Virtual Valuations

Why is it optimal to allocate the object on the basis of virtual valuations. And what the hell are virtual valuations anyway?

- Consider buyer i whose values are distributed according to F.
- Suppose seller makes a take-it-or-leave it offer to i at price p.
- The probability that i accepts this offer is $1 - F(p)$.
- Think of this probability as the "quantity" demanded by i and write the implied "demand curve" as $q(p) \equiv 1 - F(p)$.
- The inverse demand curve is then $p(q) \equiv F^{-1}(1 - q)$.

Interpreting Virtual Valuations

- Then the "revenue function" that the seller is facing is:

$$p(q) \times q = qF^{-1}(1 - q)$$

Interpreting Virtual Valuations

- Then the "revenue function" that the seller is facing is:

$$p(q) \times q = qF^{-1}(1-q)$$

- Differentiating with respect to q:

$$\frac{\partial TR}{\partial q} = F^{-1}(1-q) - \frac{q}{F'(F^{-1}(1-q))}$$

Interpreting Virtual Valuations

- Then the “revenue function” that the seller is facing is:

$$p(q) \times q = qF^{-1}(1-q)$$

- Differentiating with respect to q:

$$\frac{\partial TR}{\partial q} = F^{-1}(1-q) - \frac{q}{F'(F^{-1}(1-q))}$$

$$\implies MR(p) \equiv p - \frac{1-F(p)}{f(p)} = \psi(p)$$

Interpreting Virtual Valuations

- Then the "revenue function" that the seller is facing is:

$$p(q) \times q = qF^{-1}(1-q)$$

- Differentiating with respect to q:

$$\frac{\partial TR}{\partial q} = F^{-1}(1-q) - \frac{q}{F'(F^{-1}(1-q))}$$

$$\implies MR(p) \equiv p - \frac{1-F(p)}{f(p)} = \psi(p)$$

- Thus, virtual valuation of a buyer can be interpreted as a *marginal revenue*. (Recall that ψ is strictly increasing.)

Interpreting Virtual Valuations

- Facing one buyer, seller would set a "monopoly price" of r^* by setting: $MR(r^*) = MC = 0 \implies r^* = \psi^{-1}(0)$.

Interpreting Virtual Valuations

- Facing one buyer, seller would set a "monopoly price" of r^* by setting: $MR(r^*) = MC = 0 \implies r^* = \psi^{-1}(0)$.
- Facing many buyers, the optimal mechanism calls for the seller to set *discriminatory reserve prices* of $r_i^* = \psi_i^{-1}(0)$ for the buyers.

Interpreting Virtual Valuations

- Facing one buyer, seller would set a "monopoly price" of r^* by setting: $MR(r^*) = MC = 0 \implies r^* = \psi^{-1}(0)$.
- Facing many buyers, the optimal mechanism calls for the seller to set *discriminatory reserve prices* of $r_i^* = \psi_i^{-1}(0)$ for the buyers.
- If no buyer's value v_i exceeds his reserve price r_i^*, the seller keeps the object.

Interpreting Virtual Valuations

- Facing one buyer, seller would set a "monopoly price" of r^* by setting: $MR(r^*) = MC = 0 \implies r^* = \psi^{-1}(0)$.
- Facing many buyers, the optimal mechanism calls for the seller to set *discriminatory reserve prices* of $r_i^* = \psi_i^{-1}(0)$ for the buyers.
- If no buyer's value v_i exceeds his reserve price r_i^*, the seller keeps the object.
- Otherwise, object is allocated to buyer with highest MR and he is asked to pay $p_i = y_i(\mathbf{v}_{-i})$, smallest value such that he still wins.