Auctions

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Introduction

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- That said, first ever recorded auctions took place in Babylon around 500 B.C.

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- A variety of auction mechanisms were developed, including *English auction*, *Dutch auction*, and so-called *auction by the candle*.

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- Public sector:
  - Privatization and public resource allocation (ex: famous FCC Spectrum Auction in 1993 designed by Paul Milgrom and others);
  - Reverse auctions: Trillions of dollars of goods *bought* by governments on e-procurement auctions around the globe.

Questions in Auction Theory

- Why are auctions so prevalent, historically and today?
- In which situations auctions are preferred to other selling mechanisms, e.g., to a fixed posted price?
- Bidders: for a given auction, what are good bidding strategies?
- Sellers: are there particular types of auctions that would bring greater revenues than others?

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- Sellers want to maximize revenues and organize sales in such a way that would elicit this private information (e.g., such that bidders with the highest valuations offer the highest price).
- Auctions are one set of *mechanisms* in a general *mechanism design* problem: how to organize a game such that a certain objective is achieved?
    - Typically two competing objectives: revenue vs. efficiency.

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1. Private (independent) values
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2. Interdependent values
   - Ex: auctioned object is an asset that can be resold later
3. Pure common value
   - Ex: value of the auctioned object is derived from a market price that is unknown at the time of the auction

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   - Bids are private; highest bidder gets object at highest price.
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4. Sealed-bid second-price auction (SPSB, *Vickrey auction*).
   - Bids are private; highest bidder gets object at 2nd-highest price.
   - Example: Google ad auctions (before 2019).

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- English auction $\approx$ Sealed-bid second-price (under *private values*).
  - In both formats, best to bid your valuation.

Private Value Auctions

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- N potential risk-neutral buyers
    - v_i — valuation of buyer $i \in N$
    - $v_i \sim$ i.i.d. $F[0, \omega]$ — independent *symmetric* values
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- Bidder i knows v_i, $F[0, \omega]$, and N, but not v_j where $j \neq i$
- Since bidders are symmetric, focus on *symmetric* equilibria — equilibria in which all bidders follow the same strategy

FPSB vs. SPSB

In this setup, we will consider two auction formats:

1. *First-price sealed bid auction (I)*: highest bidder gets the object and pays the amount he bids
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Questions:

- What are equilibrium bidding strategies $\beta_i : [0, \omega] \to \mathbb{R}_+$ in these auction formats?
- Which of the two formats is better for the seller?

Second-Price Sealed-Bid Auction

Payoffs in SPSB

Bidders' payoffs are:

$$\mathbb{E}U_i = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{K}[v_i - \max_{j \neq i} b_j] & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

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Proof: Consider bidder i and suppose that $p_i = \max_{j \neq i} b_j$ is the highest competing bid.

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&= G(v) \times \mathbb{E}[Y_1 | Y_1 < v] = F(v)^{N-1} \times \mathbb{E}[Y_1 | Y_1 < v]
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First-Price Sealed-Bid Auction

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- Suppose that bidders $j \neq i$ follow a symmetric, increasing, and differentiable equilibrium strategy β.
- Bidder i has valuation v and bids b. What is the optimal b?

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- Taking FOC:

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(v-b) - G[\beta^{-1}(b)] = 0$$

(where $g = G'$ is the density of Y_1).

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- Suppose that player's opponents chose this strategy.
- Clearly, the player should never choose $b > \beta(\omega)$ since bidding $b = \beta(\omega)$ already ensures the victory.
- Thus, need to show that a player of type v is at least as well off choosing $\beta(v)$ as $\beta(\hat{v})$ for any $\hat{v} \in [0, \omega]$.

Finding Bidding Strategies in FPSB

Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where Y_1 is the highest of $N-1$ independently drawn valuations.

Proof (cont'd): Substituting $b = \beta(\hat{v})$ into buyer's utility function:

$$\begin{aligned} G(\beta^{-1}(b)) \times (v - b) &= G(\hat{v})[v - \beta(\hat{v})] \\ &= G(\hat{v})v - \int_0^{\hat{v}} x dG(x) = \int_0^{\hat{v}} (v - x) dG(x) \end{aligned}$$

This is clearly maximized at $\hat{v} = v$. $\blacksquare$

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- So, the degree of "shading" goes down as N increases!

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Valuations are uniformly distributed on [0,1].

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Valuations are exponentially distributed on $[0,\infty)$, and there are only two bidders.

If $F(x) = 1 - \exp(-\lambda x)$, for some $\lambda > 0$, and $N = 2$, then:

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Revenue Comparison between FPSB and SPSB

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With i.i.d. private values, the expected payments of a type-v bidder and the seller's expected revenue are the same in a first-price sealed-bid auction as in a second-price sealed-bid auction.

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(ii) Expected revenue is N times the *ex ante* payment of an individual bidder, so it too must be the same between FPSB and SPSB. From SPSB, it must be the expected second-highest of N valuations:

$$\mathbb{E}[R^I] = \mathbb{E}[R^{II}] = \mathbb{E}[Y_2^{(N)}]$$

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- In fact, one can prove a more general result:

Proposition

With i.i.d. private values, the distribution of equilibrium prices in a SPSB auction is a mean-preserving spread of the distribution of equilibrium prices in a FPSB auction.

The Revenue Equivalence Principle

Revenue Equivalence

- So far, we've shown that, regardless of $F(v)$, expected selling prices in symmetric FPSB and SPSB auctions are the same.
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- But we can prove a more general revenue-equivalence result.

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- Bidder i wins if $\beta(z)$ exceeds $\beta(Y_1)$, i.e., when $z > Y_1$.

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Note that the last expression does not depend on the particular auction form A. ■

Examples

Uniform Distribution

Valuations are uniformly distributed on [0,1]. What is the expected payment that seller expects to receive from any standard auction?

Since $v \sim U[0,1]$, $F(v) = v$, $G(v) = v^{N-1}$.

$$m^A(v) =$$

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where $m^A(v)$ is the expected payment of bidder with value v, $\mathbb{E}[m^A(V)]$ is the *ex ante* expected payment of a bidder, and $\mathbb{E}[R^A]$ is the seller's expected revenue.

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Revenue Equivalence with Uncertain Number of Bidders

Proposition

Suppose that values are i.i.d., all bidders are risk neutral, and that the number of bidders is uncertain. Then any symmetric and increasing equilibrium of any standard auction, such that the expected payment of a bidder with value zero is zero, yields the same expected revenue.

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- Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of *potential* bidders and $\mathcal{A} \subseteq \mathcal{N}$ be the set of *actual* bidders. Consider bidder $i \in \mathcal{A}$.

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- Suppose that all bidders hold the same beliefs p_n. Then the logic of the previous proof goes through.

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Proof (cont-d): Everything is same as before but the probability of i winning is now:

$$G(z) = \sum_{n=0}^{N-1} p_n [F(z)]^n$$

Then the bidder's payoff is:

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and the previous analysis goes through. Thus, the revenue equivalence principle holds even if there is uncertainty about the number of bidders. ■

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Since $m^I(v) = m^{II}(v)$ due to revenue equivalence, we have:

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Example 1. FPSB with Uncertain Number of Bidders.

Suppose that values are i.i.d., all bidders are risk neutral, and that the number of bidders is uncertain. What is the optimal bidding strategy in a first-price sealed-bid auction?

First, it is useful to derive expected payment in SPSB. Even with uncertain N, bidding $\beta^{II}(v) = v$ is optimal. Hence:

$$m^{II}(v) = \sum_{n=0}^{N-1} p_n [F(v)]^n \mathbb{E}\left[Y_1^{(n)} | Y_1^{(n)} < v \right]$$

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- As earlier, this is maximized at $\hat{v} = v$.

Reserve Prices

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- Let us now examine how r affects the two auction formats.

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- The expected payment of a bidder with value $v \geq r$ is:

$$m^{II}(v, r) = rG(r) + \int_r^v xg(x)dx$$

(since the winner pays the reserve price r whenever the second-highest bid is below r)

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, where $\lambda(r) = f(r)/(1 - F(r))$ is the *hazard rate* function.

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  - Intuitively, reserve price comes into play only when there is a single bidder with a value that exceeds the reserve price.

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- Rest of analysis stays the same, so the two tools are equivalent.

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- Reserve price requires credible commitment from the seller:
  - Seller leaves some money on the table by setting $r > v_0$.
  - Hence, needs to credibly commit not to sell the product with a lower price and not to lower the reserve price in the future.