

# Auctions

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# Introduction

# Ancient (Italian) Selling Mechanism

- Fact #1: “Auction” comes from Latin (*auctum*, “I increase”)
- Fact #2: Single most important auction ever took place in Italy!
  - A.D. 193: Having killed the Emperor Pertinax, Praetorian Guard sold the entire (!) Roman Empire by means of an auction.
  - The winner, Didius Julianus, was declared emperor and reigned for two months.
- That said, first ever recorded auctions took place in Babylon around 500 B.C. (e.g., auctions of women for marriage).

# Revival of Auctions

- England in 18th century: auctions revived for selling artwork and other rare items.
- Largest auction houses were founded:
  - Sotheby's (2nd largest auction house now): founded in 1744 to sell off a library of scarce books;
  - Christie's (largest auction house now): founded in 1766 to sell paintings and other artwork.
- A variety of auction mechanisms were developed, including *English auction*, *Dutch auction*, and so-called *auction by the candle*.

# Modern Auction Extravaganza

- Auctions became *extremely* popular in the 21st century.
- Private sector:
  - eBay — online auction platform with 182 mln users worldwide;
  - Google — selling ad space through an auction system; etc.
- Public sector:
  - Privatization and public resource allocation (ex: famous FCC Spectrum Auction in 1993 designed by Paul Milgrom and others);
  - Reverse auctions: Trillions of dollars of goods *bought* by governments on e-procurement auctions around the globe.

# Questions in Auction Theory

- Why are auctions so prevalent, historically and today?
- In which situations auctions are preferred to other selling mechanisms, e.g., to a fixed posted price?
- Bidders: for a given auction, what are good bidding strategies?
- Sellers: are there particular types of auctions that would bring greater revenues than others?

# Auctions and Information Economics

- Auctions help to organize markets with information asymmetries.
- Buyers possess private info—their valuations of the good.
- Sellers want to maximize revenues and organize sales in such a way that would elicit this private information (e.g., such that bidders with the highest valuations offer the highest price).
- Auctions are one set of *mechanisms* in a general *mechanism design* problem: how to organize a game such that a certain objective is achieved?
  - Typically two competing objectives: revenue vs. efficiency.

# Bidders' Valuations

Bidders' valuations can be:

- ① Private (independent) values
  - Ex: value is derived from consumption alone
- ② Interdependent values
  - Ex: auctioned object is an asset that can be resold later
- ③ Pure common value
  - Ex: value of the auctioned object is derived from a market price that is unknown at the time of the auction



# Auction Variations

Common auction forms:

- ① Open ascending auction (*English auction*);
  - Bids are public; price is ascending until no one bids more.
  - Example: Standard auctions for rare items you see on TV.
- ② Open descending auction (*Dutch auction*);
  - Bids are public; price is descending until someone buys the object.
  - Example: Tulip auctions in the Netherlands.
- ③ Sealed-bid first-price auction (FPSB);
  - Bids are private; highest bidder gets object at highest price.
  - Example: E-procurement auctions, eBay auctions.
- ④ Sealed-bid second-price auction (SPSB, *Vickrey auction*).
  - Bids are private; highest bidder gets object at 2nd-highest price.
  - Example: Google ad auctions (before 2019).

# Equivalent Auctions

In game-theoretic terms, some auctions are similar to each other:

- Dutch auction  $\equiv$  Sealed-bid first-price auction,
  - In FPSB, buyers choose a price at which they want to buy the object conditional on it still being available.
  - Dutch auction is open, but the only information it reveals to buyers is that the object is still available at a given price.
  - Thus, two formats are strategically equivalent.
- English auction  $\approx$  Sealed-bid second-price (under *private values*).
  - In both formats, best to bid your valuation.

# Private Value Auctions

# Setup

- A single object is for sale
- $N$  potential risk-neutral buyers
  - $v_i$  — valuation of buyer  $i \in N$
  - $v_i \sim \text{i.i.d. } F[0, \omega]$  — independent *symmetric* values
  - $b_i$  — bid of buyer  $i \in N$
  - Bidders don't face any budget constraints (have 'deep pockets')
- Bidder  $i$  knows  $v_i$ ,  $F[0, \omega]$ , and  $N$ , but not  $v_j$  where  $j \neq i$
- Since bidders are symmetric, focus on *symmetric* equilibria — equilibria in which all bidders follow the same strategy

# FPSB vs. SPSB

In this setup, we will consider two auction formats:

- 1 *First-price sealed bid auction (I)*: highest bidder gets the object and pays the amount he bids
- 2 *Second-price sealed bid auction (II)*: highest bidder gets the object and pays the second highest bid

Questions:

- What are equilibrium bidding strategies  $\beta_i : [0, \omega] \rightarrow \mathbb{R}_+$  in these auction formats?
- Which of the two formats is better for the seller?

# Second-Price Sealed-Bid Auction

# Payoffs in SPSB

Bidders' payoffs are:

$$\mathbb{E}U_i = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{K}[v_i - \max_{j \neq i} b_j] & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

, where  $K$  is the number of winning bidders in case of a tie.

# Bidding Strategies in SPSB

## Proposition

*In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to  $\beta^H(v) = v$ .*

Proof: Consider bidder  $i$  and suppose that  $p_i = \max_{j \neq i} b_j$  is the highest competing bid.

- If  $i$  bids  $v_i$ , he wins if  $v_i > p_i$  and not if  $v_i < p_i$  (if  $v_i = p_i$ , indifferent between winning and losing).
- Suppose  $i$  bids  $z_i$  where  $z_i < v_i$ .
  - If  $v_i > z_i > p_i$ ,  $i$  still wins, and his profit is still  $v_i - p_i$ .
  - If  $p_i > v_i > z_i$ ,  $i$  still loses and gets zero.
  - But if  $v_i > p_i > z_i$ , then  $i$  loses where he could have made a positive profit if he bid  $v_i$ .
- Thus, bidding less than  $v_i$  is weakly dominated.
- Applying same logic, bidding  $z_i > v_i$  is also weakly dominated. ■



## Expected Payment in SPSB

- Fix a bidder  $i$ .
- Let random variable  $Y_1 \equiv Y_1^{N-1}$  denote the highest value among other  $N - 1$  bidders. ( $Y_1$  is the *highest-order statistic* of  $V_1, \dots, V_{i-1}, V_{i+1}, \dots, V_N$ .)
- Let  $G$  be the c.d.f. of  $Y_1$ .
- $G(v) = F(v)^{N-1}$  since  $v$ 's are independently drawn. (Why?)
- Then the expected payment by a bidder with value  $v$  is:

$$\begin{aligned} m''(v) &= \text{Prob}[\text{Win}] \times \mathbb{E}[\text{2nd highest bid} | v \text{ is the highest bid}] \\ &= \text{Prob}[\text{Win}] \times \mathbb{E}[\text{2nd highest value} | v \text{ is the highest value}] \\ &= G(v) \times \mathbb{E}[Y_1 | Y_1 < v] = F(v)^{N-1} \times \mathbb{E}[Y_1 | Y_1 < v] \end{aligned}$$

# First-Price Sealed-Bid Auction

# Payoffs in FPSB

Bidders' payoffs are:

$$\mathbb{E}U_i = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{K}[v_i - b_i] & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

, where  $K$  is the number of winning bidders in case of a tie.

# Bidding Strategies in FPSB

- Not as trivial as in SPSB auction.
- Strategy  $\beta(v_i) = v_i$  is definitely sub-optimal. (Why?)
- Basic trade-off: increasing one's bid increases the probability of winning but reduces the gains from winning
- Suppose that bidders  $j \neq i$  follow a symmetric, increasing, and differentiable equilibrium strategy  $\beta$ .
- Bidder  $i$  has valuation  $v$  and bids  $b$ . What is the optimal  $b$ ?

## Finding Bidding Strategies in FPSB

- Bidder  $i$  wins whenever  $\max_{j \neq i} \beta(V_j) < b$
- Since  $\beta$  is increasing,  $\max_{j \neq i} \beta(V_j) = \beta(\max_{j \neq i} V_j) = \beta(Y_1)$
- Probability of  $Y_1 < \beta^{-1}(b)$  is  $G[\beta^{-1}(b)]$ . Hence:

$$\mathbb{E}U(b, v) = (v - b)G[\beta^{-1}(b)] \rightarrow \max_b$$

- Taking FOC:

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(v - b) - G[\beta^{-1}(b)] = 0$$

(where  $g = G'$  is the density of  $Y_1$ ).

## Finding Bidding Strategies in FPSB

- In symmetric equilibrium  $b = \beta(v)$ , so:

$$G(v)\beta'(v) + g(v)\beta(v) = vg(v) \implies \frac{d}{dv}[G(v)\beta(v)] = vg(v)$$

- Integrating both sides, get:

$$\beta(v) = \frac{1}{G(v)} \int_0^v yg(y)dy = \mathbb{E}[Y_1 | Y_1 < v]$$

### Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta'(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where  $Y_1$  is the highest of  $N - 1$  independently drawn valuations.

# Finding Bidding Strategies in FPSB

## Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta^I(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where  $Y_1$  is the highest of  $N - 1$  independently drawn valuations.

Proof: We already proved *necessity*, i.e., that all symmetric equilibrium strategies have to be of this form. Now let's prove *sufficiency*.

- Need to show that  $\beta \equiv \beta^I(v)$  produce a symmetric equilibrium.
- Suppose that player's opponents chose this strategy.
- Clearly, the player should never choose  $b > \beta(\omega)$  since bidding  $b = \beta(\omega)$  already ensures the victory.
- Thus, need to show that a player of type  $v$  is at least as well off choosing  $\beta(v)$  as  $\beta(\hat{v})$  for any  $\hat{v} \in [0, \omega]$ .

# Finding Bidding Strategies in FPSB

## Proposition

Symmetric equilibrium strategies in a first-price auction are given by:

$$\beta'(v) = \mathbb{E}[Y_1 | Y_1 < v]$$

where  $Y_1$  is the highest of  $N - 1$  independently drawn valuations.

Proof (cont'd): Substituting  $b = \beta(\hat{v})$  into buyer's utility function:

$$\begin{aligned} G(\beta^{-1}(b)) \times (v - b) &= G(\hat{v})[v - \beta(\hat{v})] \\ &= G(\hat{v})v - \int_0^{\hat{v}} x dG(x) = \int_0^{\hat{v}} (v - x) dG(x) \end{aligned}$$

This is clearly maximized at  $\hat{v} = v$ . ■



# Properties of FPSB Bidding Strategies

- Integrating by parts, the equilibrium bid can be rewritten as:

$$\beta^I(v) = v - \int_0^v \frac{G(x)}{G(v)} dx$$

- So, optimal bid in FPSB auction is lower than one's valuation!
- Also recall that, in independent private values (IPV) setting:

$$\frac{G(x)}{G(v)} = \left[ \frac{F(x)}{F(v)} \right]^{N-1}$$

- So, the degree of “shading” goes down as  $N$  increases!