PS 3

1

AGENT $\rightarrow$ utility $\ln w - g(e)$

reservation utility 1

$e \in \{e_L, e_H\}$

$g(e_L) = 0$, $g(e_H) = 1$

x^B, x^G

$P(x^G|e_H) = \frac{2}{3}$ $\quad P(x^G|e_L) = \frac{1}{3}$

a) There are 2 constraints

- IC (he must be better off by working hard)

$$\underbrace{\frac{2}{3}\left(\log w_G - 1\right) + \frac{1}{3}\left(\log w_F - 1\right)}_{\text{expected utility with high effort}} \geq \underbrace{\frac{1}{3}\log w_G + \frac{2}{3}\log w_F}_{\text{expected utility with low effort}}$$

- IR (he doesn't have to work to quit)

$$\frac{2}{3}\log w_G + \frac{1}{3}\log w_F \geq 2$$

b) Claim: Both are binding.

- $\boxed{\text{IR binds}}$

$\Downarrow$

PROOF - by contraddiction

1°- $\frac{2}{3}\log(w_G) + \frac{1}{3}\log(w_F) > 2$

So we can $\downarrow w_G$ and $\downarrow w_F$ by ε st IR still respected

2°- I verify that IC still holds.

IC: $\left(\frac{2}{3} - \frac{1}{3}\right)\log(w_G) \geq 1 + \log(w_B)\left(\frac{2}{3} - \frac{1}{3}\right)$

$\frac{1}{3}\log w_G \geq 1 + \frac{1}{3}\log w_B$.

Since log is a monotone, concave function and $w_G > w_B$ (by assumption) the $\downarrow$ in w has a higher effect on w_B than on w_G guaranteeing that IC still holds.

3°- So Claudio can $\uparrow \bar{u}$ respecting the IC constraint.

• $\boxed{\text{IC binds}}$ - By contradiction

1°- Suppose not

$$\frac{1}{3} \log w_G - 1 > \frac{1}{3} \log w_B$$

Then we could $\downarrow w_G$ by δ and $\uparrow w_B$ by $\delta \frac{2}{\frac{3}{3}} = 2\delta$

because we want the expected π unchanged

$$\frac{2}{3}(x_G - w_G) + \frac{1}{3}(x_B - w_B)$$

2°- But then IR_1 gets relaxed because of the concavity of the log

$$\frac{2}{3} \log(w_G - \delta) + \frac{1}{3} \log(w_B + 2\delta) - 1 > 1$$

so we can lower the wage by ε as before

3°- Thus, the original contract $\{w_G, w_B\}$ isn't π maximizing

c) Optimal contract

$$\begin{cases} \left(\frac{2}{3} - \frac{1}{3}\right)[\log w_G - \log w_B] = 1 & IC \\ \frac{2}{3} \log w_G + \frac{1}{3} \log w_B = 2 & IR \end{cases}$$

$$\begin{cases} \frac{1}{3}(\log w_G - \log w_B) = 1 \Rightarrow \log w_G = 3 + \log w_B \\ \frac{2}{3} \log w_G + \frac{1}{3} \log w_B = 2 \end{cases}$$

$$\frac{2}{3}(3 + \log w_B) + \frac{1}{3} \log w_B = 2$$

$$2 + \frac{2}{3} \log w_B + \frac{1}{3} \log w_B = 2$$

$$\log w_B = 0$$

$\boxed{w_B = 1}$ $\quad \log w_G = 3 \quad$ $\boxed{w_G = e^3}$

d) Since we had found that $w_B = 1$ and it cannot be accepted Claudio will ↑ $\boxed{w_B = e}$ (the minimum acceptable)

Claudio still wants to pay the lowest wage possible to induce Alexander to make the high effort. This IC remain binding.

⇓

$$\log w_G = 3 + \frac{\log e}{1}$$

$\boxed{w_G = e^4}$ (The ↑ is higher than the one for the w_B because of the concavity of the logarithm)

Instead the IR is not binding, the individual will always want to work if $w_B = e$ since, by not working he gets 1 and by working he gets e

2 Risk neutral firm

$e \in \{0, 1\}$, $x \in \{x_L, x_H\}$, $x_H > x_L$

$P(x = x_H \mid e = 1) = \frac{1}{2}$, $P(x = x_H \mid e = 0) = 0$

- if agent e_H → with probability $q \in (0,1)$ he's scouted and hired away before any revenue for the firm is realized and he gets $u = 1$

- if scouted $u = 1$
- if stays $u = u(w) - e$ → $u(w)$ continuous, strictly concave, with $u(0) = 0$
- $\bar{U} = 0$

Firm → $\pi = \begin{cases} 0 & \text{if scouted} \\ x - w & \text{if not scouted} \end{cases}$

chooses (w_L, w_H)

a) To induce low effort what do they set w_H to? Find minimum w_L st the agent is willing to comply with the firm's wish

$$E[u_H] = q + (1-q)\left[\frac{1}{2}u(w_H) + \frac{1}{2}u(w_L) - 1\right]$$

$$E[u_L] = u(w_L)$$

$$E[R] \begin{cases} q \cdot 0 + (1-q)\left[\frac{1}{2}(x_H - w_H) + \frac{1}{2}(x_L - w_L)\right] & e = 1 \\ x_L - w_L & e = 0 \end{cases}$$

$u(w_L) - 0 \geq q + (1-q)\left[\frac{1}{2}(u(w_L) - 1) + \frac{1}{2}\left[u(w_H) - 1\right]\right]$ IC

$u(w_L) \geq 0$ IR

$\Downarrow$

$\boxed{w_H = 0}$

$q + (1-q)\left(\frac{1}{2}u(w_L) - 1\right) = u(w_L)$

$u(w_L)\left(\frac{1}{2} + \frac{q}{2}\right) = 2q - 1$

$u(w_L) = \dfrac{2q-1}{\frac{1}{2}+\frac{q}{2}}$

$\boxed{u(w_L) = \dfrac{4q-2}{1+q}}$

Since the utility is equal to 0 when $w = 0$ and it's continuous and strictly concave so I want to impose $u(w_L) \geq 0$. Thus when $u(w_L) \leq 0 \Rightarrow w_L = 0$.

$\Downarrow$

$\dfrac{4q-2}{1+q} \leq 0 \Rightarrow 4q - 2 \leq 0 \Rightarrow 4q \leq \frac{1}{2}$

$\Downarrow$

$$w_L = \begin{cases} 0 & \text{if } q \leq \frac{1}{2} \\ u^{-1}\left(\dfrac{4q-2}{1+q}\right) & q > \frac{1}{2}. \end{cases}$$

Intuition: In order to minimize any possible incentive for the worker to make a high effort the firm will set $w_H = 0$. The firm wants to minimize the wage by setting them to 0. However when the probability of being employed by the other firm is high the firm will $\uparrow$ w_L to avoid any incentive of the worker to deviate to high type (indeed $\uparrow$ in q when $q > \frac{1}{2}$)

b) On the base of point a $w_H = 0$

$$\sqrt{w_L} = \frac{4q-2}{1+q}$$

$$\sqrt{w_L} = \frac{4\cdot\frac{5}{7}-2}{1+\frac{5}{7}}$$

$$\sqrt{w_L} = \frac{\frac{20}{7}-2}{\frac{12}{7}} = \frac{\cancel{6}}{\cancel{7}}\cdot\frac{\cancel{7}}{\cancel{12}_{2}}$$

$$\sqrt{w_L} = \frac{1}{2} \Rightarrow w_L = \frac{1}{4}$$

$\Downarrow$

- So the $\pi = x_L - \frac{1}{4}$

- Now let's compare it to the profit where the firm wants to induce high effort.

$$\pi = \frac{2}{7}\left(\frac{1}{2}(x_H - w_H) + \frac{1}{2}(x_L - w_L)\right)$$

$$\text{st } \frac{2}{7}\left(\frac{1}{2}(\sqrt{w_L}-1) + \frac{1}{2}(\sqrt{w_H}-1)\right) + \frac{5}{7} \geq \sqrt{w_L} \qquad \text{IC-H}$$

$$\frac{2}{7}\left(\frac{1}{2}(\sqrt{w_L}-1) + \frac{1}{2}(\sqrt{w_H}-1)\right) + \frac{5}{7} \geq 0 \qquad \text{IR-H}$$

The firm would like to set the lowest possible price that she can induce high effort.
If $w_H = w_L = 0$, the worker will always choose high effort since with probability q he'll get positive utility.

$\Downarrow$

Then, to see when the firm would want to induce low effort we need to impose

$$x_L - \frac{1}{4} > \frac{2}{7}\cdot\frac{1}{2}x_H + \frac{2}{7}\cdot\frac{1}{2}x_L$$

$$x_L - \frac{1}{4} > \frac{1}{7}x_H + \frac{1}{7}x_L$$

$$28x_L - 7 > 4x_H + 4x_L$$

$$x_L > \frac{4}{24}x_H + \frac{7}{24}$$

$$\boxed{x_L > \frac{1}{6}x_H + \frac{7}{24}}$$