Problem 1. An Optimal Mechanism.

There is a single object for sale and there are two potential buyers. The value assigned by buyer 1 to the object is $v_1 \sim U[0, 1+k]$, while the value of the object for buyer 2 is $v_2 \sim U[0, 1-k]$ independently distributed, where $k \in (0, 1)$.

(a) Calculate the optimal mechanism. That is, describe the optimal allocation as a function of virtual valuations and describe the payment rule. Is the optimal mechanism efficient?

(b) Now assume that there is a third buyer with independent value $V_3 \sim exp(1)$. Describe the optimal allocation. What is the minimum sample value v_3 such that the third buyer gets the object for sure, that is, independently of the values of the first two buyers. Describe the optimal payment rule for the third buyer. Is it bounded?

Problem 2. VCG.

Consider the following buyer/seller framework. Let $x \in [0, 100] = X$ denote the amount of a good that is produced by a seller, which a buyer then consumes. Let $\theta_B, \theta_S \in [1, 2] = \Theta_B = \Theta_S$ denote the types of the buyer and seller, respectively. Types are independently distributed and privately known by each agent. Assume that utility functions are quasilinear in money, and are given by

$$U_B(x, t, \theta) = \theta_B x - t_B$$
$$U_S(x, t, \theta) = -\frac{x^2}{2\theta_S} - t_S$$

where $t = (t_B, t_S)$ denotes a vector of transfers that the agents pay. (If $t_i < 0$, then agent i is receiving a transfer.)

(a) What is the efficient level of production $x(\theta)$? What social surplus is attained by it?

(b) Define a *least charitable type* of agent i to be any type $\underline{\theta}_i \in \Theta_i$ that minimizes the interim expected social surplus, where the expectation is taken over others' types. What are the least charitable types of each agent?

(c) Using the least charitable types as the default types, what are the VCG transfers? What are the buyer's and seller's utilities in the mechanism? Does the mechanism run an ex post budget surplus for all θ? Is the mechanism interim individually rational?

(d) Show directly that, in the direct revelation game with the VCG transfer and allocation functions you computed, it is weakly dominant for each player to truthfully reveal his type. That is, let $U_S(\hat{\theta}_S, \hat{\theta}_B; \theta_S)$ denote the seller's utility when his type is θ_S, he announces $\hat{\theta}_S$, and the buyer announces $\hat{\theta}_B$. Show that $\theta_S \in \arg\max_{\hat{\theta}_S} U_S(\hat{\theta}_S, \hat{\theta}_B; \theta_S)$ for all $\hat{\theta}_B, \theta_S$. Show the similar statement for the buyer. (*Note: Every VCG mechanism is dominant strategy incentive compatible, so this verifies in this special case something that we know in general.*)