From Auctions to Mechanism Design

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Optimal Mechanisms

Finding an Optimal Mechanism

- Seller's goal is to design an optimal mechanism that maximizes expected revenue among all mechanisms that are IC and IR.
- Without loss of generality we can focus on direct revelation mechanisms.
- Consider the direct mechanism $(\boldsymbol{Q}, \boldsymbol{M})$.

$$\mathbb{E}[R] = \sum_{i \in N} \mathbb{E}[m_i(V_i)]$$

where the ex ante expected payment of buyer i is

$$\begin{aligned}\mathbb{E}[m_i(V_i)] &= \int_0^{\omega_i} m_i(v_i) f_i(v_i) dv_i \\ &= m_i(0) + \int_0^{\omega_i} q_i(v_i) v_i f(v_i) dv_i - \int_0^{\omega_i} \int_0^{v_i} q_i(x_i) f(x_i) dx_i dv_i\end{aligned}$$

Finding an Optimal Mechanism

Changing the order of integration in the last term:

$$\int_0^{\omega_i} \int_0^{v_i} q_i(x_i) f(x_i) dx_i dv_i = \int_0^{\omega_i} \left[\int_{x_i}^{\omega_i} f(v_i) dv_i \right] q_i(x_i) dx_i$$
$$= \int_0^{\omega_i} \left[1 - F_i(v) \right] q_i(x_i) dx_i$$

Thus, can write:

$$\mathbb{E}[m_i(V_i)] = m_i(0) + \int_0^{\omega_i} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] q_i(v_i) f(v_i) dv_i$$
$$= m_i(0) + \int_{\mathcal{V}} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] Q_i(\mathbf{v}) f(\mathbf{v}) d\mathbf{v}$$

where, in the last step, we used the definition of q_i as the expected allocation probability intergrated over valuations of all other players.

Optimal Mechanism Design Problem

$$\sum_{i\in\mathcal{N}} m_i(0) + \sum_{i\in\mathcal{N}} \int_{\mathcal{V}} \left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \right] Q_i(\mathbf{v}) f(\mathbf{v}) d\mathbf{v} \to \max_{\boldsymbol{Q},\boldsymbol{M}}$$

s.t. *IC* constraint ($\Leftrightarrow q_i$ is nondecreasing)

IR constraint ($\Leftrightarrow m_i(0) \leq 0$)

Optimal Mechanism Design Problem

- Let's define **virtual valuation** of a buyer with value v_i as:

$$\psi_i(v_i) \equiv v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

- Note that $\mathbb{E}[\psi_i(v_i)] = 0$. (Why?)
- We say that design problem is regular if $\psi_i(v_i)$ is increasing in v_i (it is sufficient that hazard rate $\lambda_i(v_i)$ is increasing in v_i).
- Let us temporarily neglect the IC and the IR constraints and circle back to them later.

Optimal Mechanism Design Problem

- The seller should choose $(\boldsymbol{Q}, \boldsymbol{M})$ to maximize:

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{V}} \left(\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v}) \right) f(\mathbf{v}) d\mathbf{v}$$

- Consider the expression from the second term:

$$\sum_{i \in \mathcal{N}} \psi_i(v_i) Q_i(\mathbf{v})$$

- Here, **Q** is akin to a weighting function, and clearly it is best to give weight only to those $\psi_i(v_i)$ that are maximal (and positive).
- This approach would maximize this expression at every point $\mathbf{v}$ and so would also maximize its integral.

Optimal Mechanism

Claim

The following is an optimal mechanism:

1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

$$Q_i(\mathbf{v}) > 0 \Leftrightarrow \psi_i(v_i) = \max_{j \in \mathcal{N}} \psi_j(v_j) \geq 0$$

2. The payment rule $\boldsymbol{M}$ is as follows:

$$M_i(\mathbf{v}) = Q_i(\mathbf{v}) v_i - \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i$$

First, let's check IC and IR:

- IC: Check if q_i is nondecreasing. Suppose $\hat{v}_i < v_i$. Then by the regularity condition, $\psi_i(\hat{v}_i) < \psi_i(v_i)$ and, thus, for all v_{-i}, it is also the case that $Q_i(\hat{v}_i, v_{-i}) \leq Q(\mathbf{v})$. Thus, q_i is nondecreasing.

Optimal Mechanism

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$$M_i(\mathbf{v}) = Q_i(\mathbf{v}) v_i - \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i$$

First, let's check IC and IR:

- IR: From the payment rule, it is clear that $M_i(0, \mathbf{v}_{-i}) = 0$ for all $\mathbf{v}_{-i}$, and thus $m_i(0) = 0$.

Optimal Mechanism

Claim

The following is an optimal mechanism:

1. The allocation rule $\boldsymbol{Q}$ is such that the object goes to buyer i with positive probability if and only if $\psi_i = \max_{j \in N} \psi_j(v_j) \geq 0$:

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Second, note that this mechanism separately maximizes two terms in seller's expected revenue: (i) It gives positive weight only to nonnegative and maximal terms in the second expression; (ii) It sets maximal $m_i(0)$ given the IR constraint.

Optimal Mechanism

This implies that the maximized value of the expected revenue is:

$$\mathbb{E}[\max\{\psi_1(V_1), \ldots, \psi_N(V_N), 0\}]$$

In other words, it is the expectation of the highest virtual valuation, provided it is nonnegative.

Optimal Mechanism

To obtain more intuitive formulas for $(\boldsymbol{Q}, \boldsymbol{M})$, we define:

$$y_i(\mathbf{v}_{-i}) = \inf\{x_i : \psi_i(x_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(x_i) \geq \psi_j(v_j)\}$$

as the smallest value for i that "wins" against $\mathbf{v}_{-i}$. Thus, can rewrite the optimal allocation rules as:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } x_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } x_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

$$\implies \int_0^{v_i} Q_i(x_i, \mathbf{v}_{-i}) dx_i = \begin{cases} v_i - y_i(\mathbf{v}_{-i}) & \text{if } v_i > y_i(\mathbf{v}_{-i}) \\ 0 & \text{if } v_i < y_i(\mathbf{v}_{-i}) \end{cases}$$

$$\implies M_i(\mathbf{v}) = \begin{cases} y_i(\mathbf{v}_{-i}) & \text{if } Q_i(\mathbf{v}) = 1 \\ 0 & \text{if } Q_i(\mathbf{v}) = 0 \end{cases}$$

Thus, only the 'winning' buyer pays anything. He pays the smallest value that would result in his winning.

Optimal Mechanism

Proposition.

Suppose the design problem is regular. Then the following is an optimal mechanism:

$$Q_i(x_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } \psi_i(v_i) > \max_{j \neq i} \psi_j \text{ and } \psi_i(v_i) \geq 0 \\ 0 & \text{if } \psi_i(v_i) < \max_{j \neq i} \psi_j \end{cases}$$

$$M_i(\mathbf{v}) = \begin{cases} y_i(\mathbf{v}_{-i}) & \text{if } Q_i(\mathbf{v}) = 1 \\ 0 & \text{if } Q_i(\mathbf{v}) = 0 \end{cases}$$

Illustration

Optimal Mechanism in a Symmetric Case

Suppose the problem is symmetric, i.e., $f_i = f$, and hence $\psi_i = \psi\ \forall i$. Now we have that:

$$y_i(\mathbf{v}_{-i}) = \max\left\{\psi^{-1}(0), \max_{j\neq i} v_j\right\}$$

Thus, the optimal mechanism is SPSB with reserve price $r^* = \psi^{-1}(0)$.

Proposition.

Suppose that the seller's design problem is regular and symmetric. Then a second-price auction with a reserve price $r^* = \psi^{-1}(0)$ is an optimal mechanism.

Note that $\psi^{-1}(0)$ is the optimal reserve price we derived earlier!