Derivation of Time Derivative of Magnetization Component M_z

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August 2024

1 Introduction

In this document, we derive the time derivative of the z-component of the magnetization, M_z, for a system subjected to an alternating magnetic field. The derivation involves differentiating the magnetization magnitude and accounting for the time dependence of the magnetic field components.

2 Expression for M_z

We start with the expression for the magnetization magnitude, M_{mag}, given by:

$$M_{\text{mag}} = c_m \left(\coth(\beta H_{\text{mag}}) - \frac{1}{\beta H_{\text{mag}}} \right),$$

where H_{mag} is the magnitude of the applied magnetic field, given by:

$$H_{\text{mag}} = \sqrt{H_{\text{DC}}^2 + (H_d \cos(w_{\text{drive}} t))^2}.$$

3 Time Derivative of M_z

To find $\frac{dM_z}{dt}$, we first need to compute $\frac{dM_{\text{mag}}}{dt}$ and then account for the time dependence of the magnetic field components.

3.1 Derivative of M_{mag}

Differentiate M_{mag} with respect to time:

$$\frac{dM_{\text{mag}}}{dt} = c_m \left(\frac{d}{dt} \left[\coth(\beta H_{\text{mag}}) - \frac{1}{\beta H_{\text{mag}}} \right] \right).$$

Using the chain rule:

$$\frac{d}{dt} \coth(\beta H_{\text{mag}}) = \text{csch}^2(\beta H_{\text{mag}}) \cdot \frac{d}{dt}(\beta H_{\text{mag}}),$$

$$\frac{d}{dt}\left(-\frac{1}{\beta H_{\text{mag}}}\right) = \frac{1}{\beta^2 H_{\text{mag}}^2} \cdot \frac{d}{dt}(\beta H_{\text{mag}}).$$

Thus:

$$\frac{dM_{\text{mag}}}{dt} = c_m \left(-\text{csch}^2(\beta H_{\text{mag}}) + \frac{1}{\beta^2 H_{\text{mag}}^2}\right) \cdot \frac{d}{dt}(\beta H_{\text{mag}}).$$

3.2 Derivative of H_{mag}

To compute $\frac{d}{dt} H_{\text{mag}}$:

$$\frac{d}{dt} H_{\text{mag}} = \frac{d}{dt}\sqrt{H_{\text{DC}}^2 + (H_d \cos(w_{\text{drive}} t))^2}.$$

Using the chain rule:

$$\frac{d}{dt} H_{\text{mag}} = \frac{1}{2H_{\text{mag}}} \cdot \frac{d}{dt}\left[H_{\text{DC}}^2 + (H_d \cos(w_{\text{drive}} t))^2\right],$$

$$\frac{d}{dt}\left[H_{\text{DC}}^2 + (H_d \cos(w_{\text{drive}} t))^2\right] = -2H_d^2 \cos(w_{\text{drive}} t) \sin(w_{\text{drive}} t) \cdot w_{\text{drive}}.$$

Thus:

$$\frac{d}{dt} H_{\text{mag}} = \frac{-H_d^2 w_{\text{drive}} \sin(w_{\text{drive}} t) \cos(w_{\text{drive}} t)}{H_{\text{mag}}}.$$

3.3 Final Expression for $\frac{dM_z}{dt}$

To get $\frac{dM_z}{dt}$, we must also account for the time dependence of the z-component of the field:

$$\frac{dM_z}{dt} = \frac{dM_{\text{mag}}}{dt} \cdot \frac{H_d \cos(w_{\text{drive}} t)}{H_{\text{mag}}} + M_{\text{mag}} \cdot \frac{d}{dt}\left(\frac{H_d \cos(w_{\text{drive}} t)}{H_{\text{mag}}}\right).$$

$$\frac{d}{dt}\left(\frac{H_d \cos(w_{\text{drive}} t)}{H_{\text{mag}}}\right) = \frac{-H_d w_{\text{drive}} \sin(w_{\text{drive}} t) H_{\text{mag}} - H_d \cos(w_{\text{drive}} t) \frac{d}{dt} H_{\text{mag}}}{H_{\text{mag}}^2}.$$

Combining these:

$$
\begin{aligned}
\frac{dM_z}{dt} = c_m & \left(-\operatorname{csch}^2 \left(\beta \sqrt{H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2} \right) + \frac{1}{\beta^2 \left(H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2 \right)} \right) \\
& \cdot \left(\frac{-\beta H_d^2 w_{\mathrm{drive}} \sin(w_{\mathrm{drive}} t) \cos(w_{\mathrm{drive}} t)}{\sqrt{H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2}} \right) \\
& \cdot \left(\frac{H_d \cos(w_{\mathrm{drive}} t)}{\sqrt{H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2}} \right) \\
+ & \left(\frac{-H_d w_{\mathrm{drive}} \sin(w_{\mathrm{drive}} t) \sqrt{H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2} + H_d^3 w_{\mathrm{drive}} \cos^2(w_{\mathrm{drive}} t) \sin(w_{\mathrm{drive}} t) / \sqrt{H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2}}{\left(H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2 \right)} \right) \\
\cdot & \left(c_m \left(\coth \left(\beta \sqrt{H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2} \right) - \frac{1}{\beta \sqrt{H_{\mathrm{DC}}^2 + \left(H_d \cos(w_{\mathrm{drive}} t)\right)^2}} \right) \right)
\end{aligned}
\tag{1}
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