MAC 2313 Lecture Note

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Note 7

15 Multiple Integrals

15.1 Double Integrals over Rectangles

15.1.1 Review of the Definite Integral

If $f(x)$ is defined on $[a, b]$, by dividing the interval $[a, b]$ into n sub-intervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b-a)/n$, we define the Riemann sum as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where x_i^* are the sample points in the sub-interval $[x_{i-1}, x_i]$. If the limit of the Riemann sum as $n \to \infty$ exits, we call the limit is the definite integral of f from a to b, denoted by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

15.1.2 Volumes and Double Integrals

Consider the function $f(x, y)$ of two variables x and y is defined on a closed rectangle $R = [a, b] \times [c, d]$. First suppose that $f(x, y) \geq 0$. The graph of f is a surface with $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, i.e.

$$S = \{(x, y, z) \in \mathbb{R}^3 |\, 0 \leq z \leq f(x, y), (x, y) \in R\}$$

In order to find the volume of S, we do the following.

(a). Divide the rectangle R into mn sub-rectangles by diving the interval $[a, b]$ into m sub-intervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and diving the interval $[c, d]$ into n sub-intervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. Each of the sub-rectangle R_{ij} is the set

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) |\, x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

(b). If we choose a sample point $f(x_{ij}^*, y_{ij}^*)$ in each R_{ij}, then we approximate the part of S that lies above each R_{ij} by a thin rectangular box with base R_{ij} and the height $f(x_{ij}^*, y_{ij}^*)$. If we denote the area of R_{ij} by ΔA, the volume of this thin rectangle box is given by

$$f(x_{ij}^*, y_{ij}^*)\Delta A$$

(c). If we approximate all parts of S over all R_{ij} and add the corresponding volumes, we have an approximation of S as follows

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

The approximation is called a **double Riemann sum**.

(d). Sending m, n to infinity, the limit of double Riemann sum is the volume of the solid S, i.e.,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

The limits of the type shown above appear frequently, so we have the following definition.

Definition 15.1. The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

if this limits exists. We also could write the double integral as $\iint_R f(x,y)dxdy$.

Remark 15.2. If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)dA$$

Example 15.3. If $R = \{(x,y)| -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1-x^2}dA$$

Solution: $\frac{1}{2}\pi(1)^2 \times 4 = 2\pi$.

Remark 15.4. The sample points (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the sub-rectangle R_{ij}. If we chose the sample points as the midpoint of R_{ij}, the approximation is called the Midpoint Rule for double integrals, i.e.

$$\iint_R f(x,y)dA \approx \sum_{i=1}^{n}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_i)\Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_i$ is the midpoint of $[y_{i-1}, y_i]$.

15.1.3 Iterated Integrals

Suppose $f(x,y)$ is integrable on the rectangle $R = [a,b]\times[c,d]$. Consider the integral $\int_c^d f(x,y)dy$, which means that $f(x,y)$ is integrated w.r.t y from c to d when x is fixed. It is called *partial integration w.r.t y*. Now, the partial integration defines a function of x,

$$A(x) = \int_c^d f(x,y)dy$$

If we integrate $A(x)$ from a to b, we obtain

$$\int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right]dx,$$

which is called an **iterated integral**. Usually, the square brackets are omitted as

$$\int_a^b \left[\int_c^d f(x,y)dy\right]dx = \int_a^b\int_c^d f(x,y)dydx$$

Similarly, we have another iterated integral

$$\int_c^d \left[\int_a^b f(x,y)dx\right]dy = \int_c^d\int_a^b f(x,y)dxdy$$

Example 15.5. Evaluate the iterated integral

$$\int_0^1\int_1^2 (x+e^{-y})dxdy$$

The following theorem provides a method for evaluating a double integral on a rectangle.

Theorem 15.6 (Fubini's Theorem)**.** If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_A f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

More generally, it still holds if f is bounded on R and f is only discontinuous on a finite number of smooth curves, and the iterated integrals exist.

Example 15.7. If $R = \{(x, y) |\, 0 \le x \le 2, 1 \le y \le 2\}$, evaluate the integral

$$\iint_R (x - 3y^2)dA$$

Solution: -12.

Theorem 15.8. If $f(x, y)$ is multiplicatively separable, i.e. $f(x, y) = g(x)h(y)$, then

$$\iint_R f(x,y)dA = \iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy$$

where $R = [a, b] \times [c, d]$.

Example 15.9. Evaluate

$$\iint_{[0,\pi/2]\times[0,\pi/2]} \sin x \cos y dA$$

15.2 Double Integrals over General Regions

In general, we want to integrate a function of $f(x, y)$ not just over rectangles bu also over regions D of more general shape. We introduce two types of regions D.

A plane region D is said to be of **type I** if it lies between the graph of two continuous functions of x, i.e.

$$D = \{(x, y) |\, a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1, g_2 are continuous on $[a, b]$. To evaluate the double integral over type I region D, we choose a rectangle $R = [a, b] \times [c, d]$ such that $c \le g_1(x) \le g_2(x) \le d$ for all $x \in [a, b]$ and define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

By Fubini's Theorem, we have

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA = \int_a^b \int_c^d F(x,y)dydx$$

Observe that $F(x, y) = 0$ if $y > g_2(x)$ or $y < g_1(x)$ since $(x, y) \notin D$. Then

$$\int_c^d F(x,y)dy = \int_c^{g_1(x)} F(x,y)dy + \int_{g_1(x)}^{g_2(x)} F(x,y)dy + \int_{g_2(x)}^d F(x,y)dy = \int_{g_1(x)}^{g_2(x)} f(x,y)dy$$

Therefore, we have that

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

The **type II** region is defined as

$$D = \{(x, y) |\, c \le y \le d, h_1(y) \le x \le h_2(x)\}$$

Similarly, we have

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Remark 15.10. If D is of **type I**, i.e. $D = \{(x,y)|\, a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

If D is of **type II**, i.e. $D = \{(x,y)|\, c \leq y \leq d, h_1(y) \leq x \leq h_2(x)\}$, then

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Example 15.11. Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1+x^2$.

Example 15.12. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and $y = x^2$.

Example 15.13. Evaluate $\iint_D xydA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Example 15.14. Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)dydx$$

15.2.1 Properties of Double Integrals

In general, the following properties of double integrals are true

$$\iint_D [f(x,y) + g(x,y)]dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$$

$$\iint_D cf(x,y)dA = c\iint_D f(x,y)dA, \quad \text{where } c \text{ is constant}$$

If $f(x,y) \geq g(x,y)$, then

$$\iint_D f(x,y)dA \geq \iint_D g(x,y)dA$$

If $D = D_1 \cup D_2$ where D_1, D_2 do not overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

If we integrate $f(x,y) = 1$ over D, we get the area of D:

$$\iint_D 1dA = A(D)$$

If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$, then

$$mA(D) \leq \iint_D f(x,y)dA \leq MA(D)$$