MAC 2313 Lecture Note

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Note 1

12 Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

12.1.1 3D space

The three coordinate axes determine the three coordinate planes. The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants**.

$$\text{First Octant: } \{(x,y,z)|x>0, y>0, z>0\} \tag{1}$$

We represent the point P by the ordered triple of real numbers (a,b,c) and we call a, b and c the **coordinates** of P. a is the (directed) distance from the yz-plane to P, b is the (directed) distance from the xz-plane to P, c is the (directed) distance from the xy-plane to P.

The Cartesian product is $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x,y,z)|x,y,z\in\mathbb{R}\}$, denoted by $\mathbb{R}^3$. We give a one-to-one correspondence between all points in space and ordered triples in $\mathbb{R}^3$. It is called a **3D rectangular coordinate system**. In three-dimensional analytic geometry, an equation in x, y, and z represents a surface in $\mathbb{R}^3$.

12.1.2 Surfaces

In three-dimensional analytic geometry, an equation in x, y, and z represents a surface in $\mathbb{R}^3$.

Example 12.1

a. $z=0$: xy-plane; $y=0$: xz-plane; $x=0$: yz-plane.

b. $z=1$: a plane parallel to $z=0$ and one unite above it.

c. $y=-2$: a plane parallel to $y=0$ (xz-plane) and two units to the left.

Example 12.2

a. The points (x,y,z) that satisfy $x^2+y^2=1, z=0$. The points of the set $\{(x,y,z)|x^2+y^2=1, z=0\}$ form a circle on the xy-plane centered at $(0,0,0)$ with radius 1.

b. The points (x,y,z) that satisfy $x^2+y^2=1, z=3$. The points of the set $\{(x,y,z)|x^2+y^2=1, z=3\}$ form a circle on the xy-plane centered at $(0,0,3)$ with radius 1.

c. The points of the set $\{(x,y,z)|x^2+y^2=1\}$ form a cylinder with radius 1, axis is the z-axis. See Figure 1a.

d. The equation $z=\cos y$ where $-\pi\le y\le\pi$ and $0\le x\le 2$ defines part of a cylindrical surface in three dimensions. See Figure 1b

12.1.3 Distance and Spheres

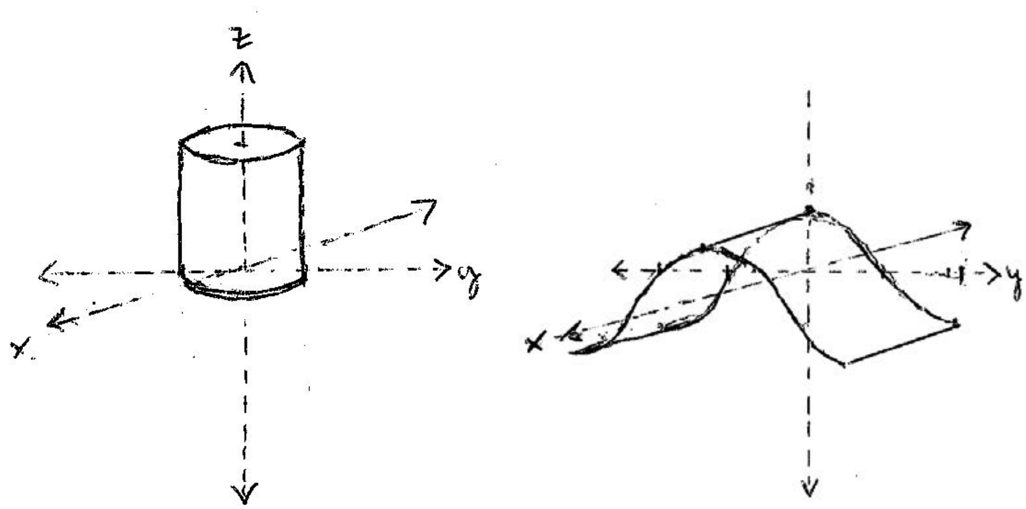
Theorem 12.3 The distance $|P_1P_2|$ between the points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \tag{2}$$

Theorem 12.4 The midpoint of the line segment joining the two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$.

Definition 12.5 A **sphere** is the set of all points (x,y,z) that are a fixed distance r from a given point (a,b,c). The distance r is the radius of the sphere. The point (a,b,c) is the **center** of the sphere. The equation of the sphere with radius $r(r>0)$ and center (a,b,c) is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \tag{3}$$



(a) $x^2 + y^2 = 1$ (b) $z = \cos y$ where $-\pi \le y \le \pi$ and $0 \le x \le 2$.

Figure 1: Visualizations of Example 12.3

Example 12.6 Find the center and radius of the sphere defined by the equation $3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$.

Proof. Rearrange the equation

$$\begin{aligned}
3x^2 + 3y^2 - 6y + 3z^2 - 12z &= 10 \\
3x^2 + 3(y^2 - 2y) + 3(z^2 - 4z) &= 10 \\
3x^2 + 3(y^2 - 2y + 1) + 3(z^2 - 4z + 4) &= 10 + 3 + 12 \\
3x^2 + 3(y-1)^2 + 3(z-2)^2 &= 25 \\
x^2 + (y-1)^2 + (z-2)^2 &= \frac{25}{3}
\end{aligned}$$

Then the center is $(0, 1, 2)$ and the radius is $\sqrt{\frac{25}{3}} = \frac{3\sqrt{5}}{3}$ □

Example 12.7 Find an equation for the sphere where one of its diameters has endpoints at $(2, 1, 4)$ and $(4, 3, 10)$.

Proof. The center of the sphere is the midpoint $(3, 2, 7)$. The radius is half of the diameter,

$$r = \frac{\sqrt{(4-2)^2 + (3-1)^2 + (10^2 - 4^2)}}{2} = \frac{\sqrt{44}}{2} = \sqrt{11}$$

So, the standard form the equation of the sphere is

$$(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$$

□

12.2 Vectors

Definition 12.8 A **nonzero vector** $\mathbf{v} = \overrightarrow{PQ}$ is a directed line segment from an initial point P to a terminal point Q. The zero vector $\mathbf{0}$ is the vector with length zero and unspecified direction.

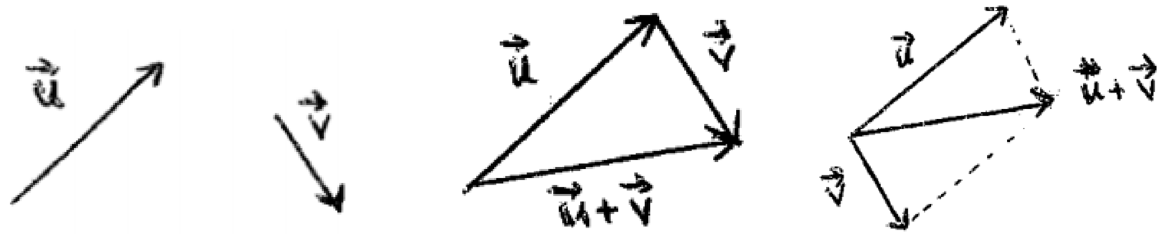
Remark 12.9 A vector is either written in boldface type or with an arrow above it. This is done to avoid confusing a vector with a scalar (which is a real number).

Definition 12.10 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are equal if and only if they have the same length and direction. Note that two vectors do not have to be in the same position to be equal.

Definition 12.11 Let c be a scalar and $\mathbf{v}$ be a vector. The vector $c\mathbf{v}$ is called a **scalar multiple** of $\mathbf{v}$ whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

12.2.1 Vector Addition & Subtraction

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors as drawn below (see Figure 2a). There are two methods for geometrically adding the vectors to find $\mathbf{u} + \mathbf{v}$. The methods are called the Triangle Method (or Tip-to-Tail Method) and the Parallelogram Method.



(a) $\mathbf{u}$ and $\mathbf{v}$ (b) Triangle Method (c) Parallelogram Method

Figure 2: Vector Addition

You can geometrically find $\mathbf{u} - \mathbf{v}$ by rewriting the difference as a sum and using the Triangle or Parallelogram Method.

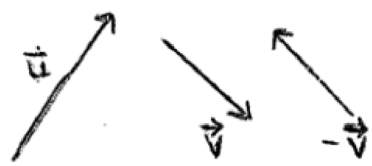


Figure 3: Vector Subtraction

12.2.2 Components

Definition 12.12 A vector $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ with initial point at the origin is called a **position vector** and is said to be in **standard position**.

Definition 12.13 The length of $\mathbf{v}$ is called the magnitude of $\mathbf{v}$ and is denoted by $|\mathbf{v}|$. In $\mathbb{R}^2$, the magnitude of the vector $\mathbf{v} = \langle x_1, y_1 \rangle$ is $|\mathbf{v}| = \sqrt{x_1^2 + y_1^2}$. In $\mathbb{R}^3$, the magnitude of the vector $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ is $|\mathbf{v}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

Remark 12.14 We use angled brackets for a vector to avoid confusing it with the ordered pair (x_1, y_1) which is a point in space. This notation for the vector is called the component form of the vector.

GENERAL REPRESENTATION OF A VECTOR IN $\mathbb{R}^3$: Let $\mathbf{v}$ be the vector with initial point at $P(x_1, y_1, z_1)$ and terminal point at $Q(x_2, y_2, z_2)$. Then

$$\mathbf{v} = \overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$
$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 12.15 Let $\mathbf{v}$ be the vector with initial point at $A(4, 0, -2)$ and terminal point at $B(4, 2, 1)$. Then

$$\mathbf{v} = \langle 4 - 4, 2 - 0, 1 - (-2) \rangle = \langle 0, 2, 3 \rangle$$
$$|\mathbf{v}| = \sqrt{0^2 + 2^2 + 3^2} = \sqrt{13}$$

Theorem 12.16 (Properties of Vectors) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors and c, d are scalars, then

(a). $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(b). $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

(c). $\mathbf{a} + \mathbf{0} = \mathbf{a}$

(d). $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

(e). $c(\mathbf{a}+\mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

(f). $(c+d)\mathbf{a} = c\mathbf{a} + c\mathbf{b}$

(g). $(cd)\mathbf{a} = c(d\mathbf{a})$

(h). $1\mathbf{a} = \mathbf{a}$

Definition 12.17 The standard basis vecotrs for $\mathbb{R}^3$ are

$$\mathbf{i} = \langle 1,0,0\rangle, \quad \mathbf{j} = \langle 0,1,0\rangle, \quad \mathbf{k} = \langle 0,0,1\rangle$$

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, then we can also write

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Definition 12.18 A **unit vector** is a vector with length one. Note that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors.

Theorem 12.19 (Normalization) Let $\mathbf{v}$ be a nonzero vector. The unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v}$ is given by $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$. The vector $\mathbf{u}$ is called the normalized vector of $\mathbf{v}$.

Example 12.20 Find the normalized vector of $\mathbf{v} = \langle -2, 4, 2\rangle$. Followed by the definition, one has

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle -2,4,2\rangle}{\sqrt{(-2)^2+4^2+2^2}} = \frac{\langle -2,4,2\rangle}{2\sqrt{6}} = \langle -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\rangle$$

Example 12.21 (Orthogonal Decomposition of a vector in $\mathbb{R}^2$) Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ be a nonzero vector in $\mathbb{R}^2$. Then

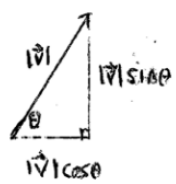


Figure 4: Orthogonal Decomposition

$\cos\theta = \frac{x}{|\mathbf{v}|}, \sin\theta = \frac{y}{|\mathbf{v}|}, \tan\theta = \frac{y}{x}, x \neq 0$. Thus,

$$\begin{aligned}\mathbf{v} &= x\mathbf{i} + y\mathbf{j} = |\mathbf{v}|\cos\theta\mathbf{i} + |\mathbf{v}|\sin\theta\mathbf{j}\\ &= |\mathbf{v}|\cos\theta\langle 1,0\rangle + |\mathbf{v}|\sin\theta\langle 0,1\rangle = |\mathbf{v}|\langle\cos\theta,\sin\theta\rangle\end{aligned}$$

12.3 The Dot Product

Definition 12.22 If $\mathbf{a} = \langle a_1,a_2,a_3\rangle$ and $\mathbf{b} = \langle b_1,b_2,b_3\rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a}\cdot\mathbf{b}$ given by

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \tag{4}$$

Example 12.23 Let $\mathbf{a} = \langle 6,-2,3\rangle$ and $\mathbf{b} = \langle 2,5,-1\rangle$. Then $\mathbf{a}\cdot\mathbf{b} = -1$.

Theorem 12.24 (Properties of the Dot Product) If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^3$ and c is a scalar, then

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$
5. $\mathbf{0}\cdot\mathbf{a} = 0$

Proof of Theorem 12.24.(1) in $\mathbb{R}^3$. Let $\mathbf{a} = \langle a_1,a_2,a_3\rangle$. Then

$$\mathbf{a}\cdot\mathbf{a} = \langle a_1,a_2,a_3\rangle\cdot\langle a_1,a_2,a_3\rangle = a_1^2+a_2^2+a_3^2 = |\mathbf{a}|^2$$

Other properties could be proven similarly. □

Consider the following example to see that some properties that work for scalar products do not work for the vector dot product.

Example 12.25 (a). *Prove or disprove:* For all real numbers (scalars) a, b, c, if $ac = bc$, then $a = b$.

Disproof: Consider the case $c = 0$.

(b). *Prove or disprove:* For all real numbers (scalars) a, b, c, if $ac = bc$ and $c \neq 0$, then $a = b$.

Proof: ac-bc=(a-b)c=0. Since $c \neq 0$, then $a - b = 0$.

(c). *Prove or disprove:* For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, if $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$ and $\mathbf{c} \neq \mathbf{0}$, then $\mathbf{a} = \mathbf{b}$.

Disproof: $\mathbf{a} = \mathbf{i}, \mathbf{b} = \mathbf{j}, \mathbf{c} = \mathbf{k}$ is a counterexample.

(d). *Prove or disprove:* For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, if $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \neq 0$ then $\mathbf{a} = \mathbf{b}$.

Disproof: $\mathbf{a} = \langle 1, 0, 1 \rangle, \mathbf{b} = \langle 1, 0, -1 \rangle, \mathbf{c} = \langle 1, 0, 0 \rangle$ is a counterexample.

12.3.1 Direction Angles and Direction Cosines

Theorem 12.26 If θ is the smallest angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$, where $\theta \in [0, \pi]$.

Proof. Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be nonzero vectors. By the Law of Cosines, one has

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta,$$

which gives

$$\begin{aligned} 2|\mathbf{u}||\mathbf{v}|\cos\theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2 \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - (|\mathbf{u}|^2 - 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2) \\ &= 2\mathbf{u} \cdot \mathbf{v} \end{aligned}$$

Simplifying the equation above completes the proof. $\square$

The **direction angles** of a nonzero vector $\mathbf{v} = (v_1, v_2, v_3)$ are the angles α, β, and γ that $\mathbf{v}$ makes with the positive $x-, y-$ and $z-$ axes, respectively. Thus, to find the direction cosines $(\cos\alpha, \cos\beta, \cos\gamma)$, we only replace $\mathbf{u}$ by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively, i.e.

$$\begin{aligned} \cos\alpha &= \frac{\mathbf{v} \cdot \mathbf{i}}{|\mathbf{v}||\mathbf{i}|} = \frac{v_1}{|\mathbf{v}|} \\ \cos\beta &= \frac{\mathbf{v} \cdot \mathbf{j}}{|\mathbf{v}||\mathbf{j}|} = \frac{v_2}{|\mathbf{v}|} \\ \cos\gamma &= \frac{\mathbf{v} \cdot \mathbf{k}}{|\mathbf{v}||\mathbf{k}|} = \frac{v_3}{|\mathbf{v}|} \end{aligned}$$

Definition 12.27 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be perpendicular (or orthogonal) iff the angle between them is $\theta = \frac{\pi}{2}$.

Definition 12.28 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be parallel iff the angle between them is $\theta = 0$ or $\theta = \pi$.

Corollary 12.29 (Test for orthogonal vectors) Since $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}, 0 \leq \theta \leq \pi$ for two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v} = 0$ iff $\theta = \frac{\pi}{2}$. Therefore, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal iff $\mathbf{u} \cdot \mathbf{v} = 0$.

Corollary 12.30 (Test for parallel vectors) Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other iff the angle between Therefore, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel iff $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.

Example 12.31 Find the angle between the given vectors $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a} = \langle 1, 2, -2 \rangle$ and $\mathbf{b} = \langle 4, 0, -3 \rangle$.

Using the Theorem 12.26 yields

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{1 \cdot 4 + 2 \cdot 0 + (-2) \cdot (-3)}{\sqrt{1^2 + 2^2 + (-2)^2} \cdot \sqrt{4^2 + 0^2 + (-3)^2}} = \frac{10}{15} = \frac{2}{3}$$

Since $\theta \in [0, \pi]$, we have $\theta = \arccos(2/3)$.

Example 12.32 (a). Let $\mathbf{u} = \langle -3, 9, 6\rangle$ and $\mathbf{v} = \langle 4, -12, -8\rangle$. Note that $\mathbf{u} = -3\langle 1, -3, -2$ and $\mathbf{v} = 4\langle 1, -3, -2\rangle$. Thus, $\mathbf{u} = -\frac{3}{4}\mathbf{v}$ and therefore they are parallel.

(b). Let $\mathbf{u} = \langle 1, 2, -2\rangle$ and $\mathbf{v} = \langle 4, 0, -3\rangle$. Since $\mathbf{u} \cdot \mathbf{v} = 10$, they are not orthogonal. Also, they are not scalar multiples of each other and are therefore not parallel.

(c). Let $\mathbf{u} = \langle a, b, c\rangle$ and $\mathbf{v} = \langle -b, a, 0\rangle$. Since $\mathbf{u} \cdot \mathbf{v} = -ab + ab + 0 = 0$, they are orthogonal.

12.3.2 Projections

Definition 12.33 The **vector projection** of $\mathbf{u}$ into $\mathbf{v}$ is defined by $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}^2|}\right)\mathbf{v}$.

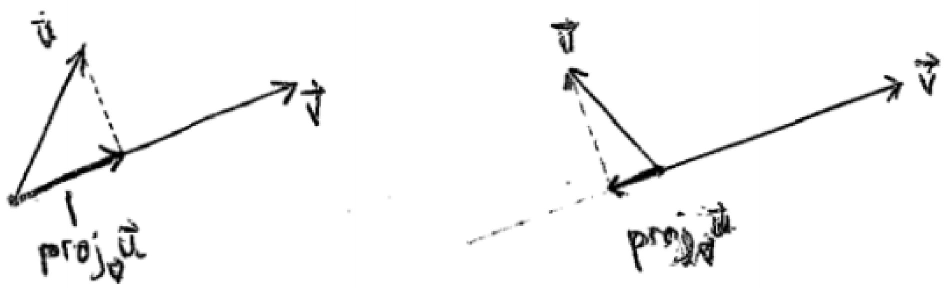


Figure 5: Visual of projections when $\mathbf{u}, \mathbf{v}$ vary.

Definition 12.34 The **scalar projection** of $\mathbf{u}$ into $\mathbf{v}$ (also called the **component of u along v**) is defined to be the signed magnitude of the vector projection, $|\mathbf{u}|\cos\theta$. Then

$$\operatorname{comp}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}|\cos\theta = |\mathbf{u}| \cdot \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

Note that $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = (\operatorname{comp}_{\mathbf{v}} \mathbf{u})\frac{\mathbf{v}}{|\mathbf{v}|}$.

Example 12.35 Let $\mathbf{a} = \langle 1, 4\rangle$ and $\mathbf{a} = \langle 2, 3\rangle$. Then, $\mathbf{a} \cdot \mathbf{b} = 14$ and $|\mathbf{a}| = \sqrt{17}$. So,

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\right)\mathbf{a} = \frac{14}{17}\mathbf{a}$$
$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \frac{14}{13}\mathbf{b}$$

Acknowledgement

I would like to thank Mr. Patrick Higgins for providing his lecture notes. The drawings are taken from his notes.