MAC 2313 Lecture Note

Feng Yu

Note 4

13.3 Arc Length and Curvature

13.3.1 Arc Length and Arc Length Function

In $\mathbb{R}^2$, consider a plane curve with parametric equations $x = f(t), y = g(t), t \in [a, b]$ where the derivatives f', g' are continuous. Then the length of the curve is given by

$$L = \int_a^b \sqrt{|f'(t)|^2 + |g'(t)|^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The length of a space curve is defined in exactly the same way. Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), z(t)\rangle, t \in [a, b]$, where f', g', h' are continuous. If the curve is travserd exactly once from $t = a$ to $t = b$, its length is

$$L = \int_a^b \sqrt{|f'(t)|^2 + |g'(t)|^2 + |h'(t)|^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Note that the integrand is exactly the gratitude of the tangent vector of $\mathbf{r}(t)$, the length of the curve is equivalent to

$$\boxed{L = \int_a^b |\mathbf{r}'(t)| dt}$$

Example 13.1. Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Solution. The arc from $(1, 0, 0)$ to $(1, 0, 2\pi)$ is described by the parameter interval $t \in [0, 2\pi]$. Since $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$, then

$$L = \int_0^{2\pi} \sqrt{|-\cos t|^2 + |\sin t|^2 + 1^2} dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$$

□

13.3.2 Curvature

Suppose that C is a curve given by a vector function $\mathbf{r}(t) = \langle f(t), g(t), z(t)\rangle, t \in [a, b]$ where $\mathbf{r}'$ is continuous and C is traversed exactly once as t increases from $t = a$ to $t = b$. The **arc length function** s is defined by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I is $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ on I. A curve is called **smooth** if it has a smooth parametrization. If the curve C is smooth and defined by $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and $\mathbf{T}$ indicates the direction of the curve.

We use the curvature of C at a given point to measure how quickly the curve changes direction at that point. Precisely, the **curvature** is defined to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. Since $\mathbf{T}$ has length 1, only the changes in direction of $\mathbf{T}$ contribute to the rate change of $\mathbf{T}$.

Definition 13.2. The **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$.

Corollary 13.3. If the curve has a parametrization $\mathbf{r}(t)$. The arc length function is given by $s(t) = \int_a^b |\mathbf{r}'(t)|dt$. So the derivative of s is $ds/dt = |\mathbf{r}'(t)|$. Applying the Chain Rule of vector functions yields

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

Therefore, the curvature can be computed as follows

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \left|\frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$$

Note that $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$. We could also write κ as

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \tag{1}$$

Proof. We show eq. (1) is true. It is equivalent to show that the following formula

$$|\mathbf{T}'(t)| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}$$

By the definition of the unit tangent vector, one has $\mathbf{r}'(t) = |\mathbf{r}'(t)|\mathbf{T}(t)$. Taking derivative on both sides and from the Chain Rule of vector functions gives

$$\mathbf{r}''(t) = (|\mathbf{r}'(t)|)'\mathbf{T}(t) + |\mathbf{r}'(t)|\mathbf{T}'(t)$$

Therefore,

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \Big(|\mathbf{r}'(t)|\mathbf{T}(t)\Big) \times \mathbf{r}''(t) = \Big(|\mathbf{r}'(t)|\mathbf{T}(t)\Big) \times \Big((|\mathbf{r}'(t)|)'\mathbf{T}(t) + |\mathbf{r}'(t)|\mathbf{T}'(t)\Big) \\ &= \Big(|\mathbf{r}'(t)|\mathbf{T}(t)\Big) \times \Big((|\mathbf{r}'(t)|)'\mathbf{T}(t)\Big) + \Big(|\mathbf{r}'(t)|\mathbf{T}(t)\Big) \times \Big((|\mathbf{r}'(t)|\mathbf{T}'(t)\Big) \end{aligned} \tag{2}$$

Note that the first term of the formula above is zero because the cross product of two parallel vectors is zero. Then, eq. (2) becomes

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \Big(|\mathbf{r}'(t)|\mathbf{T}(t)\Big) \times \Big((|\mathbf{r}'(t)|\mathbf{T}'(t)\Big) = |\mathbf{r}'(t)|^2\Big(\mathbf{T}(t) \times \mathbf{T}'(t)\Big)$$

Since $\mathbf{T}$ is defined as a unit vector, i.e. $|\mathbf{T}(t)| = 1$. By taking derivatives on $|\mathbf{T}(t)|^2 = 1$ yields $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, which implies that the angle between $\mathbf{T}$ and $\mathbf{T}'(t)$ is $\pi/2$. Then, we have that

$$\begin{aligned}|\mathbf{r}'(t) \times \mathbf{r}''(t)| &= |\mathbf{r}'(t)|^2\Big|\mathbf{T}(t) \times \mathbf{T}'(t)\Big| \\ &= |\mathbf{r}'(t)|^2|\mathbf{T}(t)||\mathbf{T}'(t)|\sin\left(\frac{\pi}{2}\right) \\ &= |\mathbf{r}'(t)|^2||\mathbf{T}'(t)|,\end{aligned}$$

which gives us

$$|\mathbf{T}'(t)| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}.$$

Then,

$$\kappa = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

□

Example 13.4. Find the curvature of the curve $\mathbf{r}(t) = \langle a\cos tt, a\sin t\rangle$.

Solution. The calculation is straightforward,

$$\kappa = \frac{|\mathbf{T}'|}{|s'|} = \frac{1}{|a|}$$

$\square$

Definition 13.5. By definition, $\mathbf{T}(t)$ is a unit tangent vector. From Example 13.11, for any fixed length vector function, it is orthogonal to its tangent. So $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal to each other, but it does not imply that $\mathbf{T}'(t)$ is a unit vector. At any point where $\kappa \neq 0$, we define the principal unit normal vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The vector $\mathbf{N}$ indicates the direction in which the curve is turning at that point. We also define the binormal vector as

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Example 13.6. Let $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, t \geq 0$. Find the unit normal and binormal vector.

Solution. First, we find the tangent vector of $\mathbf{r}$ and its unit tangent vector,

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad \mathbf{T}(t) = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

The tangent of $\mathbf{T}$ is given by

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$$

So the normalized vector $\mathbf{T}'$, i.e. the unit normal vector $\mathbf{N}$, is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{|\mathbf{T}'|} = \langle -\cos t, -\sin t, 0 \rangle$$

The binormal vector $\mathbf{N}$ is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \left\langle \frac{\sqrt{2}}{2} \sin t, -\frac{\sqrt{2}}{2} \cos t, \frac{\sqrt{2}}{2} \right\rangle$$

$\square$