MAC 2313 Lecture Note

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Note 2

12.4 Cross Product

Lemma 12.1 (Determinants in $\mathbb{R}^2$ and $\mathbb{R}^3$)**.** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. The determinants of $\mathbf{A}$ and $\mathbf{B}$ are given by

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and

$$\begin{aligned}
|\mathbf{B}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} \\
&= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - hf) - b(di - gf) + c(dh - eg) \\
&= aei + bgf + cdh - ahf - bdi - ceg
\end{aligned}$$

Definition 12.2. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} \langle v_1, v_2, v_3 \rangle$. The **cross product** $\mathbf{u} \times \mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Then

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\
&= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}
\end{aligned}$$

Example 12.3. Let $\mathbf{u} = \langle 1, 3, -2 \rangle$ and $\mathbf{v} = \langle 2, 1, 1 \rangle$. Compute $\mathbf{u} \times \mathbf{v}, \mathbf{v} \times \mathbf{u}, \mathbf{u} \times \mathbf{u}$.

Theorem 12.4 (Properties of Cross Product)**.** Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$ and let c be a scalar. Then,

(a). $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(b). $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$

(c). $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

(d). $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

(e). $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

(f). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

Theorem 12.5. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Proof. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v}\langle v_1, v_2, v_3\rangle$. By Corollary 12.29 (Test for orthogonal vectors), we only need to check $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})$ and $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})$. Thus,

$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) &= \langle u_1, u_2, u_3\rangle \cdot \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle \\
&= u_1(u_2v_3 - u_3v_2) - u_2(u_1v_3 - u_3v_1) + u_3(u_1v_2 - u_2v_1) = 0 \\
\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) &= \langle v_1, v_2, v_3\rangle \cdot \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle \\
&= v_1(u_2v_3 - u_3v_2) - v_2(u_1v_3 - u_3v_1) + v_3(u_1v_2 - u_2v_1) = 0
\end{aligned}$$

Since $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0$ and $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$, $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. ☐

Example 12.6. Let $\mathbf{u} = \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j}$. Find two unit vectors that are orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Solution. We can use $\frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$ and $\frac{\mathbf{v}\times\mathbf{u}}{|\mathbf{v}\times\mathbf{u}|}$. Then

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k} = \langle 1, -1, -1\rangle \\
\mathbf{v}\times\mathbf{u} &= -\mathbf{u}\times\mathbf{v} = \langle -1, 1, 1\rangle \\
|\mathbf{u}\times\mathbf{v}| &= |\mathbf{v}\times\mathbf{u}| = \sqrt{3}
\end{aligned}$$

The unite vectors $\frac{\sqrt{3}}{3}\langle 1, -1, -1\rangle$ and $\frac{\sqrt{3}}{3}\langle -1, 1, 1\rangle$ are orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. ☐

Theorem 12.7. If θ is the smallest angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then $\sin\theta = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$ where $0 \le \theta \le \pi$.

Proof. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle, \mathbf{v} = \langle v_1, v_2, v_3\rangle$ be nonzero vectors. Then,

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \\
|\mathbf{u}\times\mathbf{v}|^2 &= (u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2v_3u_3v_2 + u_3^2v_2^2 + u_1^2v_3^2 - 2u_1v_3u_3v_1 + u_3^2v_1^2 + u_1^2v_2^2 - 2u_1v_2u_2v_1 + u_2^2v_1^2 \qquad (1)
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3 \\
(\mathbf{u}\cdot\mathbf{v})^2 &= (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2(u_2v_3u_3v_2 + u_1v_3u_3v_1 + u_1v_2u_2v_1) \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
|\mathbf{u}|^2|\mathbf{v}|^2 &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) \\
&= u_2^2v_3^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 \qquad (3)
\end{aligned}$$

Combing eqs. (1) to (3) yields

$$|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2 = |\mathbf{u}\times\mathbf{v}|^2$$

Therefore,

$$\begin{aligned}
|\mathbf{u}\times\mathbf{v}|^2 &= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - |\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta \\
&= |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta
\end{aligned}$$

Since $\sin\theta \in [0, 1]$, then $\sin\theta = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$. ☐

Theorem 12.8. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel iff $\mathbf{u}\times\mathbf{v} = 0$.

Proof. $\mathbf{u}$ and $\mathbf{v}$ are parallel $\Longleftrightarrow \theta = 0$ or $\theta = \pi \Longleftrightarrow \sin\theta = 0 \Longleftrightarrow |\mathbf{u}\times\mathbf{v}| = 0$. ☐

Theorem 12.9. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors that are not scalar multiples of each other (not parallel). The area of the parallelogram formed by using $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides is $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$, where θ is the smallest angle between $\mathbf{u}$ and $\mathbf{v}$.

Hint: Area=base×height.

12.4.1 Triple Products

The product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is called the scalar triple product of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \mathbf{w} = \langle w_1, w_2, w_3 \rangle$. Then

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \cdot ((v_2w_3 - w_3v_2)\mathbf{i} - (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}) \\
&= (v_2w_3 - w_3v_2)u_1 + (v_3w_1 - v_1w_3)u_2 + (v_1w_2 - v_2w_1)u_3 \\
&= (v_2w_3 - w_3v_2)u_1 - (v_1w_w - v_3w_1)u_2 + (v_1w_2 - v_2w_1)u_3 \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
\end{aligned}$$

Theorem 12.10. The volume of the parallelepiped determined by the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the magnitude of their scalar triple product:

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

12.5 Equations of Lines and Planes

12.5.1 Lines

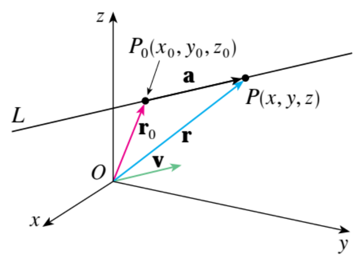


Figure 1: Forming a line by vectors (Fig. 1 [1, Page 863])

A line L is determined when a point $P_0(x_0, y_0, z_0)$ is fixed and the direction of L is known. Let $P(x, y, z)$ be an arbitrary point on L and $\mathbf{r}_0, \mathbf{r}$ be position vectors of P_0, P respectively. By Triangle Law of vectors addition, one has $\mathbf{a} = \mathbf{r}_0 + \mathbf{r}$. Let $\mathbf{v}$ be a fixed parallel vector to $\mathbf{a}$, i.e. $\mathbf{a} = t\mathbf{v}$ for some $t \in \mathbb{R}$. Then the **vector equation** of L is given by

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

If $\mathbf{v} = \langle a, b, c \rangle$, then we obtain the **parametric equations**

$$\boxed{x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct} \tag{4}$$

Equivalently, we can write the line of L as

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

In general, if $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of L, a, b, c are called **direction numbers** of L. If we represent t for the parametric equations (4) as follows

$$t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c},$$

which implies that

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} \tag{5}$$

The equation (5) is called **symmetric equations** of L.

Example 12.11. Find an equation for the line passing through the points $(1, 2, 1)$ and $(3, 0, -1)$.
Hint: Let $(x_0, y_0, z_0) = (1, 2, 1)$ and determine the direction vector $\mathbf{v} = \langle 2, -2, -2 \rangle$.

12.5.2 Planes

Let $P_0(x_0, y_0, z_0)$ be a fixed point in space and $\mathbf{n} = \langle a, b, c\rangle$ be a nonzero vector. The set of all points $P(x, y, z)$ such that $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called a **plane**. $\mathbf{n}$ is called a **normal vector** of the plane.

Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P respectively. $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$ is orthogonal to $\mathbf{n}$, which implies

$$\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0} \tag{6}$$

Or, equivalently, it can be written as

$$\boxed{\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0} \tag{7}$$

The eqs. (6) and (7) are called **vector equation of the plane**. Writing eq. (6) in the component form gives

$$\boxed{\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0} \tag{8}$$

or

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \tag{9}$$

The standard form for an equation of the plane containing the point (x_0, y_0, z_0) with a nonzero vector $\mathbf{n} = \langle a, b, c\rangle$ is

$$\boxed{ax + by + cz = d}$$

where $d = ax_0 + by_0 + cz_0$.

Example 12.12. Sketch the part of the plane $2x + 3y + z = 6$ that is in the first octant.

Proof. Find the $x-, y-, z-$intercepts. Let $y = z = 0$, then $x = 3$ implies $x-$intercept is $(3, 0, 0)$. Similarly, $y-$intercept is $(0, 2, 0)$ and $z-$intercept is $(0, 0, 6)$. Connecting these three points gives the wanted plane in the first octant. See Fig. 2. □

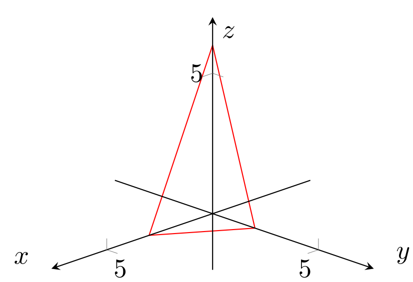


Figure 2: The plane $2x + 3y + z = 6$ in the first octant.

Example 12.13. Find an equation for the plane containing the points $P(1, 3, 2), Q(3, -1, 6)$ and $R(5, 2, 0)$.

Proof. Let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$. Then $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Use eq. (7) to obtain the plane. Then $\mathbf{u} = \langle 2, -4, 4\rangle$ and $\mathbf{v} = \langle 4, -1, -2\rangle$.

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

Thus, the plane containing P, Q, R is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

□

Definition 12.14. Two planes are **parallel** if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their vectors.

Example 12.15. (a). Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b). Find symmetric equations for the line of intersection L of these two planes.

Proof. (a). The normal vectors of these two planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle, \quad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

So the angle θ between two planes is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$,

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{2}{\sqrt{42}} = \frac{\sqrt{42}}{21}$$
$$\theta = \cos^{-1}\left(\frac{\sqrt{42}}{21}\right)$$

(b). To find the intersection line L, first locate one pointe of the piont. Solve the system,

$$\begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases} \tag{10}$$

Subtracting two equations of eq. (10), we have that $3y - 2z = 0$. One solution is $y = z = 0$. Substitute $y = z = 0$ into any equation of eq. (10) and obtain $x = 1$. We find one point $(1, 0, 0)$ on the intersection line L. Observe that L on both planes, then L is orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Thus, $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to L. An direction vector of L is given by

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

Using eq. (5) yields the symmetric equations for the intersection line L

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

□

12.6 Cylinders and Quadric Surfaces

12.6.1 Cylinders

A **cylinder** is a surface that consists of all lines that are paralell to a given line and pass through a given plane curve.

Example 12.16. (a). Sketch the surface $z = y^2$ in $\mathbb{R}^3$. *Hint: $z = y^2$ does not involve x. This means that any vertical plane with $x = k$ intersects the graph in a cureve with $z = y^2$.* See Fig. 3. The graph is called a **parabolic cylinder**.

(b). Sketch the surface $z = e^y$ in $\mathbb{R}^3$. See 4.

12.6.2 Quadric Surfaces

Definition 12.17. A **quadric surface** is the graph of a second-degree equation in x, y, z. A generic formula is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + j = 0$$

Example 12.18 (Ellipsoid)**.** Use traces to sketch the quadric surface with

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$$

See Fig. 5a. Find $xy-, yz-, xz-$traces as follows:

The surface $z = y^2$ in $\mathbb{R}^3$ | Vertical trace of $z = y^2$, when $x = 0$

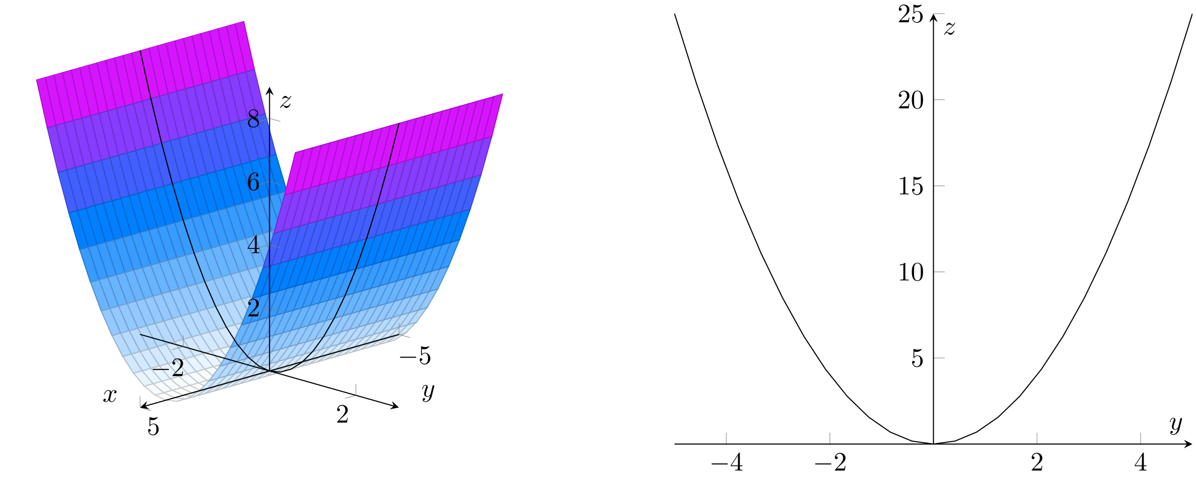


Figure 3: The surface $z = y^2$ in $\mathbb{R}^3$

The surface $z = e^y$ in $\mathbb{R}^3$ | Vertical trace of $z = e^y$, when $x = 0$

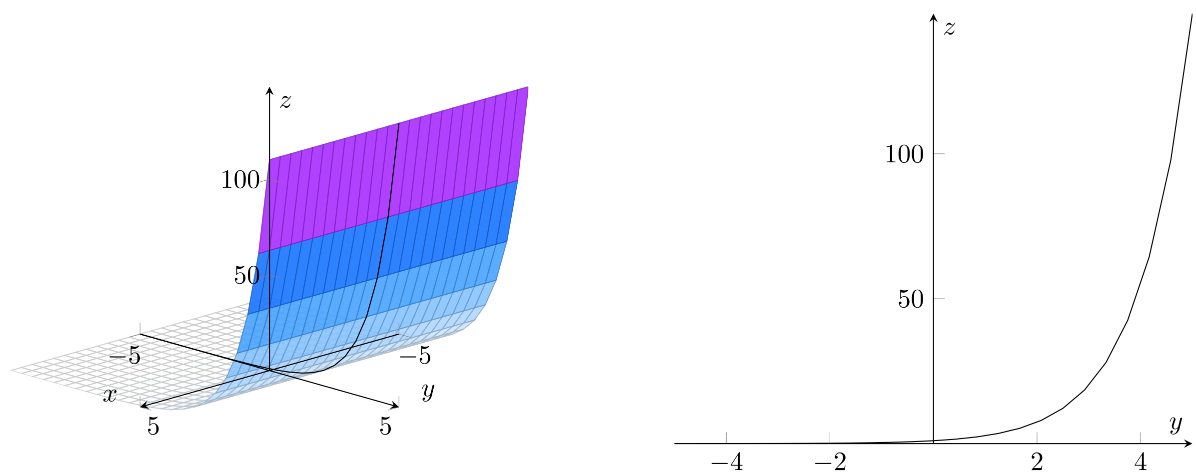


Figure 4: The surface $z = e^y$ in $\mathbb{R}^3$

1. $xy-$trace (set $z = 0$): $x^2 + \frac{y^2}{9} = 1$.
2. $yz-$trace (set $x = 0$): $\frac{y^2}{9} + \frac{z^2}{4} = 1$.
3. $xz-$trace (set $y = 0$): $x^2 + \frac{z^2}{4} = 1$.

Example 12.19 (Elliptic Paraboloid)**.** Use traces to sketch the surface $z = 4x^2 + y^2$.

Note that $z \geq 0$, thus the graph only stays over the $xy-$plane. Observe that for any $z = k$ where $k > 0$, $4x^2 + y^2 = k$ is an ellipse. Fix any value of x, the graph intersect $yz-$plane with a parabola. Fix any value of y, the graph intersect $xz-$plane with a parabola.

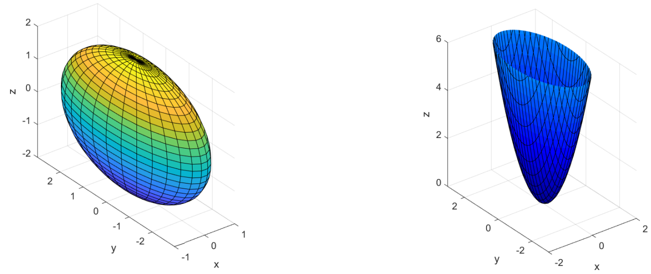
Example 12.20 (Hyperbolic Paraboloid)**.** Sketch the surface $z = y^2 - x^2$. See Fig. 6a.

The traces in vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which are open upward. The traces in vertical planes $y = k$ are the parabolas, which are open downward. $z = k^2 - x^2$. The traces in horizontal planes $z = k$ are the hyperbolas $y^2 - x^2 = k$.

Example 12.21 (Cone)**.** Sketch the surface $z^2 = x^2 + y^2$. See Fig. 6b.

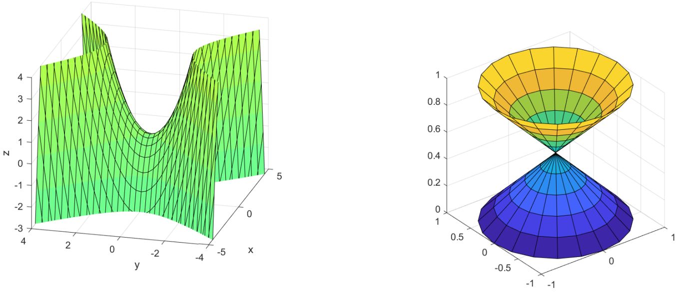
Example 12.22 (Hyperboloid of One Sheet)**.** Sketch the surface $x^2 + y^2 - z^2 = 1$. See Fig. 7a.

The traces in vertical planes $x = k$ are the parabolas $y^2 - z^2 = 1 - k^2$. The traces in vertical planes $y = k$ are the parabolas $x^2 - z^2 = 1 - k^2$. The traces in horizontal planes $z = k$ are circles $x^2 + y^2 = 1 + k^2$.



(a) Ellipsoid of $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$. (b) Elliptic Paraboloid $z = 4x^2 + y^2$.

Figure 5: Ellipsoid and Elliptic Paraboloid.

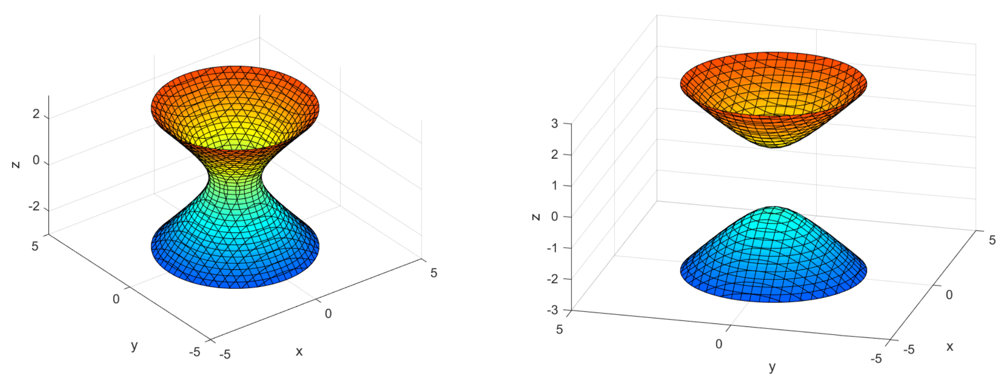


(a) Hyperbolic Paraboloid $z = y^2 - x^2$. (b) Elliptic Cone $z^2 = x^2 + y^2$.

Figure 6: Ellipsoid and Elliptic Paraboloid and Cone.

Example 12.23 (Hyperboloid of Two Sheet)**.** Sketch the surface $z^2 - x^2 - y^2 = 1$. See Fig. 7b.

The traces in vertical planes $x = k$ are the parabolas $z^2 - y^2 = 1 + k^2$. The traces in vertical planes $y = k$ are the parabolas $z^2 - x^2 = 1 + k^2$. The traces in horizontal planes $z = k$ are circles $x^2 + y^2 = k^2 - 1$. Note that $z \geq 1$ on last case.



(a) Hyperboloid of One Sheet $x^2 + y^2 - z^2 = 1$. (b) Hyperboloid of Two Sheet $z^2 - x^2 - y^2 = 1$.

Figure 7: Hyperboloid of One Sheet and Hyperboloid of Two Sheet

Example 12.24. Sketch the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$

Proof. These are two elliptic paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$. The solid See Fig. 8. First we find the interst of these two surfaces by solving the following system

$$\begin{cases} z = x^2 + y^2 \\ z = 2 - x^2 - y^2 \end{cases}$$

whose solution is $\{(x, y, z) | x^2 + y^2 = 1, z = 1\}$. The intersection is a circle on the plane $z = 1$. $\square$

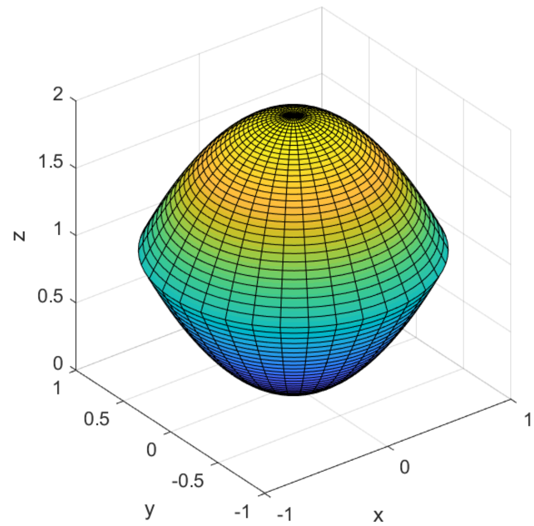


Figure 8: Two elliptic paraboloids of $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$

References

[1] James Stewart. *Multivariable calculus.* Nelson Education, 2015.