MAC 2313 Lecture Note

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Note 5

14 Partial Derivatives

14.1 Functions of Several Variables

14.1.1 Functions of Two Variables

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). A function f of two variables is a mapping that assigns to each ordered pair of real numbers (x, y) in a set D a **unique** real value z. The set D is the domain of f, denoted by $D = \{(x, y) : \ f(x, y) \text{ is well defined.}\}$. The range of f is the set $\{f(x, y) : (x, y) \in D\}$. The variables x, y are called **independent variables** and z is the dependent variable.

Example 14.1. Consider the function $f(x, y) = x + y$. The domain of f is $D = \mathbb{R}^2$ and the range is $\mathbb{R}$.

Example 14.2. Consider $f(x, y) = \ln(xy)$. Since f is a logarithmic function, so it is defined only if $xy > 0$. Thus the domain of f is given by

$$D = \{(x, y)|xy > 0\} = \{(x, y)|x > 0, y > 0\} \cup \{(x, y)|x < 0, y < 0\},$$

which is the union of the first and the third quadrant of xy-plane.

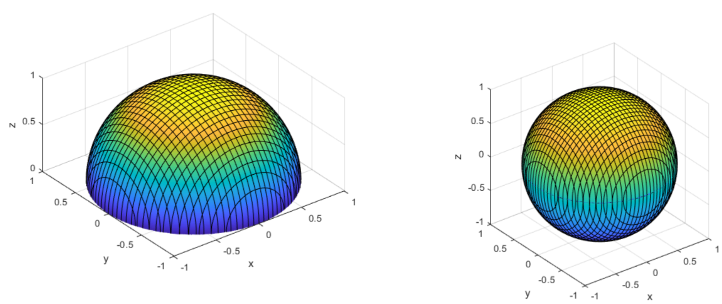
14.1.2 Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definition 14.3. The **graph** of $f(x, y)$ is defined as the set of points

$$\{(x, y, f(x, y)) : (x, y) \in D_f\}$$

Example 14.4. Sketch the graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$. The graph of f is just the top half of the sphere



(a) The graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$

(b) The surface $x^2 + y^2 + z^2 = 9$.

Figure 1: The graph of f and a surface in space.

$x^2 + y^2 + z^2 = 9$.

Remark 14.5. In general, a surface may not be a graph of some function f as Example 14.4 shown. A surface in $\mathbb{R}^3$ is the graph of a function of (x, y) if and only no line parallel to the z-axis intersects the surface more than once. This law can be verified that if there is a line parallel to the z-axis intersects with the surface more than once, it means that there exits one point (x, y) such that $f(x, y)$ is assigned with two different values, which is against the fact that f is a function.

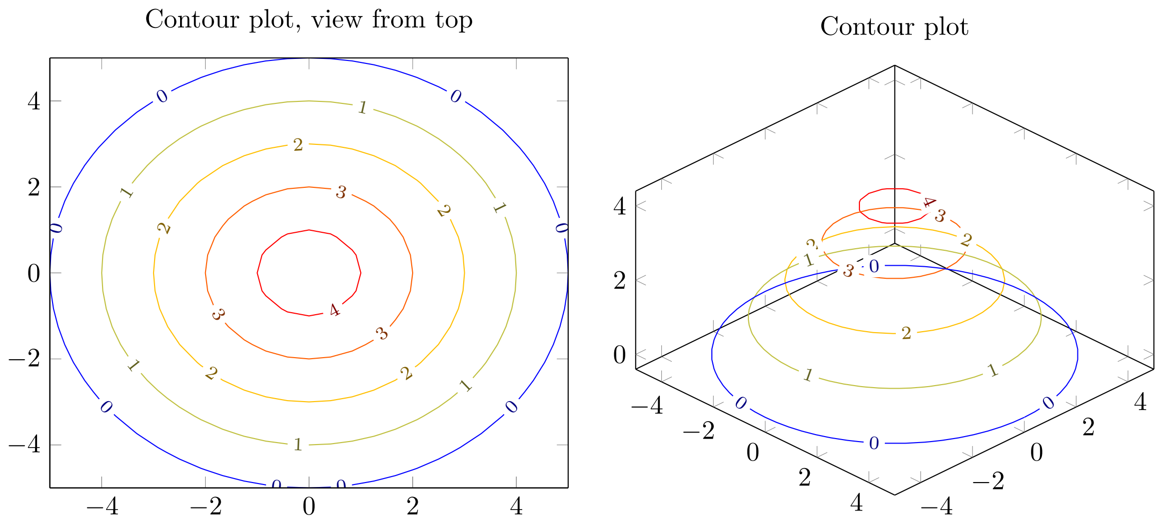


Figure 2: Level Curves of $z = 5 - \sqrt{x^2 + y^2}$

14.1.3 Level Curves

A third may to observe the behavior of a function of two variables is evaluating the *contour curves*, or *level level curves* of f.

Definition 14.6. The **level curves** of $f(x, y)$ are the curves with equations $f(x, y) = k$, where the value of the constant k is taken from the range of f.

Example 14.7. Consider $f(x, y) = 5 - \sqrt{x^2 + y^2}$. The domain of f is $D_f = \{(x, y) : x^2 + y^2 \geq 0\} = \mathbb{R}^2$ and the range of f is $(-\infty, 5]$. So the level curves of f is given by

$$x^2 + y^2 = (5 - k)^2, \quad k \in (-\infty, 5]$$

14.2 Limits and Continuity*

In general, we use the notation

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches to (a, b). A precise definition of the limit of a function of two variables is given by the follows:

Definition 14.8. Let $f(x, y)$ be a function whose domain D includes the neighborhood of (a, b). Then we say that the limit of f as (xy) approaches to (a, b) is L and write is as

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon$$

if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$.

As we define the limit of a function of one variable, the limit of $f(x)$ exists iff the left-hand limit and the right-hand limit of f exist and they are equal. The limit of $f(x, y)$ exists as (x, y) approaches (a, b) iff the limit of $f(x, y)$ exists as (x, y) approaches (a, b) along any path C and all these limits are equal.

Example 14.9. Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

Proof. Consider two paths C_1 and C_2, where C_1 the path that approach $(0, 0)$ along x-axis and C_2 is the path that approach $(0, 0)$ along y-axis. The two limits are 1 and -1, which implies that the limit does not exist. □

14.3 Partial Derivatives

In general, if $f(x,y)$ is a function of two variables x and y, we only consider the behavior of f as x varies while the value of y is fixed. Namely, if we fix $y=b$, we focus the function $g(x)=f(x,b)$. If g has a derivative at a, then the **partial derivative of f w.r.t x at** (a,b), denoted by $f_x(a,b)=g'(a)$ is $g'(a)$.

$$f_x(a,b)=g'(a)$$

where $g(x)=f(x,b)$. So the partial derivative is also defined as

$$f_x(a,b)=\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

Similarly, we can define the partial derivative of f w.r.t y at (a,b). If $z=f(x,y)$, there are many alternative notations for partial derivatives, such as

$$f_x(x,y)=f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial z}{\partial x}=f_1=D_1f=D_xf$$

Example 14.10. If $f(x,y)=x^3+x^2y^3-2y^2$. Find $f_x(2,1)$ and $f_y(2,1)$.

Solution. We differentiate f w.r.t x and y, respectively, and obtain that

$$f_x(x,y)=3x^2+2xy^3,\quad f_y(x,y)=3x^2y^2-4y$$

So

$$f_x(2,1)=16, f_y(2,1)=8$$

□

14.3.1 Interpretations of Partial Derivatives

If we fix $y=b$, the curve C_1 is the intersection of $z=f(x,y)$ and $y=b$, which is the trace of $z=f(x,y)$ in the plane $y=b$. Note that the curve C_1 is also the graph of the function $g(x)=f(x,b)$, so the slop of its tangent at $P(a,b,f(a,b))$ is $g'(a)=f_x(a,b)$. Thus the partial derivative $f_x(a,b)$ can be interpreted geometrically as the slop of the tangent line at $P(a,b,f(a,b))$ to the trace C_1 in the plane $y=b$.

If $z=f(x,y)$, the partial derivative $\partial z/\partial x$ can be also represent the rate of change of z w.r.t x when y is fixed. Similarly, $\partial z/\partial x$ can be represent the rate of change of z w.r.t y when x is fixed.

Example 14.11. If $f(x,y)=\sin\left(\frac{x}{1+y}\right)$, calculate $\partial f/\partial x$ and $\partial f/\partial y$.

Partial derivatives can also be defined for functions of three or more variables. In general, if u is a function of n variables, $u=f(x_1,\cdots,x_n)$, its partial derivatives w.s.t the i-th variable x_i is given by

$$\frac{\partial u}{\partial x}=\lim_{h\to 0}\frac{f(x_1,\cdots,x_{i-1},x_i+h,x_{i+1},\cdots,x_n)-f(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)}{h}$$

if the limit exists.

Example 14.12. Find f_x,f_y,f_z if $f(x,y,z)=e^{xy}\ln z$.

14.3.2 Higher Derivatives

If f is a function of two variables x and y, its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives. We call the derivatives of f_x,f_y, namely, $(f_x)_x,(f_x)_y,(f_y)_x,(f_y)_y$, the **second partial derivatives** of f. If $z=f(x,y)$, we have the following notations

$$(f_x)_x=f_{xx}=f_{11}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 z}{\partial x^2}$$

Example 14.13. If $f(x,y)=x^3+x^2y^3-2y^2$. Find f_xx and f_yy.

Theorem 14.14 (Clairaut's Theorem)**.** Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both exist and continuous on D, then

$$f_{xy}(a,b)=f_{yx}(a,b)$$

14.3.3 Partial Differential Equations

A partial differential equation is a differential equation that contains unknown multivariable functions and their partial derivatives.

Example 14.15. Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

Proof. We need find the second partial derivatives of $u(x, y) = e^x \sin y$,

$$u_x = e^x \sin y, \quad u_{xx} = e^x \sin y$$
$$u_y = e^x \cos y, \quad u_{yy} = -e^x \sin y$$

So

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0,$$

which implies that u satisfies Laplace's equation. □

Example 14.16. Verity that the functon $u(x, t) = \sin(x - at)$ satisfies the wave equation $u_{tt} = a^2 u_{xx}$.