MAC 2313 Lecture Note

Feng Yu

Note 3

13.1 Vector Functions & Space Curves

A **vector-valued function**, or **vector function**, is a function whose domain is set of real numbers and whose range is a set of vectors, i.e. $\mathbf{r} : \mathbb{R} \to \mathbf{V}$, where $\mathbf{V}$ is the set of all vectors. In $\mathbb{R}^3$, if f, g, h are components of the vector $\mathbf{r}(t)$, they are called the component functions of $\mathbf{r}$, and

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Example 13.1 (Circle)**.** Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle, t \in [0, 2\pi]$. Then after the elimination of parameter, one has $x^2 + y^2 = 1$, which is a circle.

Example 13.2 (Circular Helix)**.** Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, t \geq 0$. The parametric equations for this curve are $x = \cos t, y = \sin t, z = t$. Since $x^2 + y^2 = 1$, the curve lies on the cylinder $x^2 + y^2 = 1$. The curve is called a helix.

Example 13.3. Let $\mathbf{r}(t) = \langle t \cos t, t \sin t, t \rangle, t \geq 0$. After the elimination of parameter, one obtain $x^2 + y^2 = z^2$. So the curve lies on the cone $x^2 + y^2 = z^2$.

By usual convention, the domain of $\mathbf{r}$ consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined. See the following example.

Example 13.4. Let $\mathbf{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t} \rangle$. The component functions are $f(t) = t^3, g(t) = \ln(3 - t), h(t) = \sqrt{t}$. Then, these three components functions have three domains,

$$D_f = (-\infty, \infty), \quad D_g = (-\infty, 3), \quad D_h = [0, \infty)$$

Then the domain of $\mathbf{r}(t)$ is the intersection of three domains of f, g, h, i.e.

$$D_{\mathbf{r}} = D_f \cap D_g \cap D_h = [0, 3)$$

13.1.1 Limits and Continuity

The limit of $\mathbf{r}$ is defined by taking the limits of its component functions. If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided the limits of f, g, h at $t = a$ exists.

Example 13.5. Let $\mathbf{r}(t) = \langle \frac{t^2 - t}{t - 1}, \sqrt{t + 8}, \frac{\sin \pi t}{\ln t} \rangle$. Find the limit $\lim_{t \to 1} \mathbf{r}(t)$

Solution. We need to find the component limits as follows

$$\lim_{t \to 1} \frac{t^2 - t}{t - 1} = \lim_{t \to 1} t = 1, \quad \lim_{t \to 1} \sqrt{t + 8} = 3, \quad \lim_{t \to 1} \frac{\sin \pi t}{\ln t} \overset{H}{=} \lim_{t \to 1} \frac{\pi \cos \pi t}{\frac{1}{t}} = -\pi \text{ (L'Hopital's Rule)}$$

Then $\lim_{t \to 1} \mathbf{r}(t) = \langle 1, 3, -\pi \rangle$. □

Example 13.6. Find the vale $L = \lim_{t \to \infty} \left\langle \frac{1+t^2}{1-t^2}, \tan^{-1} t, \frac{1-e^{-2t}}{t} \right\rangle$.

Solution. We have that

$$\begin{aligned} L_1 &= \lim_{t \to \infty} \frac{1 + t^2}{1 - t^2} = \lim_{t \to \infty} \frac{t^2}{-t^2} = -1 \quad \text{(or use L'Hopital's Rule)}, \\ L_2 &= \lim_{t \to \infty} \tan^{-1} t = \frac{\pi}{2} \\ L_3 &= \lim_{t \to \infty} \frac{1 - e^{-2t}}{t} = 0 \end{aligned}$$

Therefore, $L = \langle -1, \frac{\pi}{2}, 0 \rangle$. □

A vector function $\mathbf{r}$ is continuous at a if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

13.2 Derivatives and Integrals of Vector Functions

13.2.1 Derivatives

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined as similar for the real-valued function:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists. The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point $\mathbf{r}(t)$, provided $\mathbf{r}'(t)$ exists and is nonzero vector. The **tangent line** to the curve at this point is defined as the line through this point and parallel to the tangent vector $\mathbf{r}'(t)$. The **unit tangent vector** is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The following theorem gives a way of computing the derivative of $\mathbf{r}$.

Theorem 13.7. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, h are differentiable, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Example 13.8. (a). Find the derivative of $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$. (b). Find the unit tangent vector at the point where $t = 0$.

Solution. (a). The derivative is $\mathbf{r}'(t) = 3t^2\mathbf{i} + (1-t)e^{-t}\mathbf{j} + 2\cos 2t\mathbf{k}$. (b). The unit tangent vector $\mathbf{T}(t)$ at $t = 0$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\langle 0, 1, 2\rangle}{|\langle 0, 1, 2\rangle|} = \left\langle 0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle$$

□

Example 13.9. Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t, \quad y = \sin t, \quad z = t$$

at the point $(0, 1, \pi/2)$.

Solution. The tangent vector of $\mathbf{r}$ is given by

$$\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1\rangle$$

The tangent line is the line through $(0, 1, \pi/2)$ parallel to the vector $\mathbf{r}'(\pi/2) = \langle -2, 0, 1\rangle$. Thus, its parametric equations are

$$x = -2t, \quad y = 1, \quad z = \frac{\pi}{2} + t$$

□

13.2.2 Differentiation Rules

Theorem 13.10. Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Example 13.11. Show that if $|\mathbf{r}(t)| = c$ for some constant c, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Proof. Since $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2 = c^2$, then

$$[\mathbf{r}(t) \cdot \mathbf{r}(t)]' = 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0.$$

Therefore, $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t. Indeed, since $|\mathbf{r}(t)|^2 = x^2(t) + y^2(t) + z^t(t) = c^2$, the curve of $\mathbf{r}(t)$ lies on the sphere with radius c and the center $(0, 0, 0)$. □

13.2.3 Integrals

The **definite integral** of a continuous vector function $\mathbf{r}$ can be defined as the same way of the integral of a real-valued function. So, if the component function f, g, h of $\mathbf{r}$ are integrable on $[a, b]$, we have the following

$$\begin{aligned}\int_a^b \mathbf{r}(t)dt &= \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle \\ &= \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}\end{aligned}$$

We can also extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)\Big]_b^a = \mathbf{R}(b) - \mathbf{R}(a),$$

where $\mathbf{R}$ is the anti-derivative of $\mathbf{r}$.

Example 13.12. Let $\mathbf{r}(t) = \frac{4}{1+t^2}\mathbf{j} + \frac{2t}{1+t^2}\mathbf{k}$. Find $\mathbf{I}(t) = \int \mathbf{r}(t)dt$ and evaluate $\int_0^1 \mathbf{r}(t)dt$.

Solution. We could write

$$\mathbf{r}(t) = \left\langle 0, \frac{4}{1+t^2}, \frac{2t}{1+t^2} \right\rangle$$

Then

$$\begin{aligned}\int \mathbf{r}(t)dt &= \left\langle \int 0dt, \int \frac{4}{1+t^2}dt, \int \frac{2t}{1+t^2}dt \right\rangle \\ &= \langle C_1, 4\tan^{-1} t + C_2, \ln(1+t^2) + C_3 \rangle \\ &= \langle 0, 4\tan^{-1} t, \ln(1+t^2) \rangle + \langle C_1, C_2, C_3 \rangle\end{aligned}$$

and

$$\int_0^1 \mathbf{r}(t)dt = \mathbf{I}(1) - \mathbf{I}(0) = \langle 0, \pi, \ln 2 \rangle$$

$\square$

Example 13.13. Evaluate

$$\int_1^2 (t^2\mathbf{i} + t\sqrt{t-1}\mathbf{j} + t\sin(\pi t)\mathbf{k})dt$$

Appendix

The followings are Matlab codes for Example 13.9.

```
x=0:0.01:4*pi;
y=-1:0.01:1;

figure;
plot3(2*cos(x),sin(x),x)
hold on
plot3(-2*y,ones(size(y)),pi/2+y)
hold on
plot3(0,1,pi/2,'.r')

xlabel('x')
ylabel('y')
zlabel('z')
legend('Helix','Tangent line')
```