MAC 2313 Lecture Note

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Note 6

14.5 The Chain Rule

Let $z = f(x, y)$ and each of the variables x and y is a function of t. Then z is indirectly a function of t, i.e., $z = f(x(t), y(t))$. Assume f is differentiable. Recall that in this case, f_x and f_y exist and are continuous.

Theorem 14.1 (The Chain Rule (Case 1))**.** Suppose that $z = f(x, y)$ is differentiable of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 14.2. If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

Theorem 14.3 (The Chain Rule (Case 2))**.** Suppose that $z = f(x, y)$ is differentiable of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 14.4. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Theorem 14.5 (The Chain Rule (General Version))**.** Suppose that u is differentiable of n variables $x_1, \cdots, x_n$, each x_i is a differentiable function of the m variables $t_1, \cdots, t_m$. Then u is function of $t_1, \cdots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \cdots, m$.

Example 14.6. If $u = x^4y + y^2z^3$, where $x = rse^t, y = rs^2e^{-t}, z = r^2s\sin t$, find the value $\partial u/\partial s$ when $r = 2, s = 1, t = 0$.

14.5.1 Implicit Differentiation

Suppose an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x. If F is differentiable and we have $F(x, f(x)) = 0$ for some f, then we obtain that

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Hence,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example 14.7. Find y' if $x^3 + y^3 = 6xy$.

14.6 Directional Derivatives and the Gradient Vector

Recall that the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ of a function $z = f(x, y)$ are defined as

$$f_x(a, b) = \lim_{h\to 0}\frac{f(a + h, b) - f(a, b)}{h}$$
$$f_y(a, b) = \lim_{h\to 0}\frac{f(a, b + h) - f(a, b)}{h}$$

These partial derivatives also can be represented the rates of changes of z in the $x-$ and $y-$ directions.

Definition 14.8. The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Remark 14.9. If we take $\mathbf{u} = \mathbf{i} = \langle 1, 0\rangle$ or $\mathbf{u} = \mathbf{j} = \langle 0, 1\rangle$, we can see that $D_{\mathbf{i}} = f_x$ and $D_{\mathbf{j}} = f_y$.

Theorem 14.10. If f is differentiable of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Proof. Let $g(h) = f(x + ah, y + bh)$. By the definition of the directional derivative, we have that

$$D_{\mathbf{u}} f(x, y) = \lim_{h\to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0)$$

By the Chain Rule, one has

$$g'(h) = f_x(x + ah, y + bh)a + f_y(x + ah, y + bh)b$$

Thus, we have that

$$D_{\mathbf{u}} f(x, y) = g'(0) = f_x(x, y)a + f_y(x, y)b$$

which completes the proof.

□

14.6.1 Gradient Vector

Definition 14.11. If f is function of x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = f_x\mathbf{i} + f_y\mathbf{j}$$

Example 14.12. If $f(x, y) = \sin x + e^{xy}$, find the gradient vector of f at $(0, 1)$.

Remark 14.13. From Theorem 14.10, we could also write

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example 14.14. Find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at $(2, -1)$ in the direction of $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

The directional derivatives and the gradient vector can also be defined similarly for functions of three variables. The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if the limit exists. The gradient vector of $f(x, y, z)$ is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

Theorem 14.15. Suppose f is differentiable of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and the maximum is reached when $\mathbf{u} = \nabla f(\mathbf{x})/|\nabla f(\mathbf{x})|$.

Proof. Form Theorem 14.10, we have that

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta$$

Notice that $|\nabla f|$ is fixed if f is given and $|\mathbf{u}| = 1$. Since $\theta \in [0, \pi]$, its maximum is reached at $\theta = 0$. In this case, $\mathbf{u}$ has the same direction as ∇f. Then $\mathbf{u} = \nabla f/|\nabla f|$. □

14.6.2 Tangent Planes to Level Surfaces

Suppose S is a surface with $F(x, y, z) = k$. Let $P(x_0, y_0, z_0)$ be a point on S and C be the any curve that lies on S and passes through P. The curve C is given by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Since C passes through the point P, then there exists t_0 such that $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C is on the level surface S, then $\mathbf{r}(t)$ satisfies that

$$F(x(t), y(t), z(t)) = k$$

If x, y, z are differentiable of t and F is also differentiable, then apply the Chain Rule to obtain

$$\frac{d}{dt}F(x(t), y(t), z(t)) = 0,$$

which implies that

$$F_x x'(t) + F_y y'(t) + F_z z'(t) = 0$$

Since $\nabla F = \langle F_x, F_y, F_z\rangle$ and $\mathbf{r}'(t) = \langle x', y', z'\rangle$, the equation above is equivalent as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

In particular, when $t = t_0$, we have that

$$\nabla F(x(t_0), z(t_0), y(t_0)) \cdot \mathbf{r}'(t_0) = \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

It is clear that $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t)$ to any curve C on S that passes through P. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, we define the **tangent plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. So the equation of this tangent plane is given by

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

The line that passing through P and being perpendicular to the tangent plane is the **normal line** to S. So we can find the symmetric equations for the normal line as

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Consider the level surface $F(x, y, z) = z - f(x, y) = 0$, which is, indeed, the graph of $z = f(x, y)$. Then

$$F_x = -f_x, F_y = -f_y, F_z = 1.$$

So the tangent plane to $z - f(x, y) = 0$ is

$$(z - z_0) - f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) = 0,$$

which is exactly the tangent plane of the surface $z = f(x, y)$ at (x_0, y_0). (Followed from Section 14.4)

14.7 Maximum and Minimum Values

Definition 14.16. The function $f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) and $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b).

If $f(x, y) \leq f(a, b)$ for any (x, y) in the domain, we call f has an absolute maximum. Similarly, we could define that f has an absolute minimum if $f(x, y) \geq f(a, b)$ for any (x, y) in the domain.

Definition 14.17. A point (x_0, y_0) is called a **critical point** of $f(x, y)$ if and only if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ or if any of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ does not exists.

Theorem 14.18. If f has a local maximum or minimum at (a, b) and the first-order partial derivatives exist, then $f_x(a, b) = f_y(a, b) = 0$, or, $\nabla f(a, b) = \mathbf{0}$.

Theorem 14.19 (Second Derivative Test)**.** Suppose the second partial derivatives of f exist and are continuous on a disk D with the center (a, b), and suppose that $f_x(a, b) = f_y(a, b) = 0$. Let D be the determinant of the Hessian matrix,

$$D = D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Then we have the following criterion:

(a). If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b). If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c). If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

In case (c), the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

Example 14.20. Find the local minimum and maximum values of $f(x, y) = x^2 + xy + y^2 + y$.

Example 14.21. Find the local minimum and maximum values of $f(x, y) = y \cos x$.