

MAC 2313 Lecture Note

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Note 4

13.3 ARC LENGTH AND CURVATURE

13.3.1 ARC LENGTH AND ARC LENGTH FUNCTION

In \mathbb{R}^2 , consider a plane curve with parametric equations $x = f(t), y = g(t), t \in [a, b]$ where the derivatives f', g' are continuous. Then the length of the curve is given by

$$L = \int_a^b \sqrt{|f'(t)|^2 + |g'(t)|^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The length of a space curve is defined in exactly the same way. Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), z(t) \rangle, t \in [a, b]$, where f', g', h' are continuous. If the curve is traversed exactly once from $t = a$ to $t = b$, its length is

$$L = \int_a^b \sqrt{|f'(t)|^2 + |g'(t)|^2 + |h'(t)|^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Note that the integrand is exactly the magnitude of the tangent vector of $\mathbf{r}(t)$, the length of the curve is equivalent to

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

Example 13.1. Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Solution. The arc from $(1, 0, 0)$ to $(1, 0, 2\pi)$ is described by the parameter interval $t \in [0, 2\pi]$. Since $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$, then

$$L = \int_0^{2\pi} \sqrt{|-\cos t|^2 + |\sin t|^2 + 1^2} dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$$

□

13.3.2 CURVATURE

Suppose that C is a curve given by a vector function $\mathbf{r}(t) = \langle f(t), g(t), z(t) \rangle, t \in [a, b]$ where \mathbf{r}' is continuous and C is traversed exactly once as t increases from $t = a$ to $t = b$. The **arc length function** s is defined by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq 0$ on I . A curve is called **smooth** if it has a smooth parametrization. If the curve C is smooth and defined by \mathbf{r} , recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and \mathbf{T} indicates the direction of the curve.

We use the curvature of C at a given point to measure how quickly the curve changes direction at that point. Precisely, the **curvature** is defined to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. Since \mathbf{T} has length 1, only the changes in direction of \mathbf{T} contribute to the rate change of \mathbf{T} .

Definition 13.2. The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where \mathbf{T} is the unit tangent vector $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$.

Corollary 13.3. If the curve has a parametrization $\mathbf{r}(t)$. The arc length function is given by $s(t) = \int_a^t |\mathbf{r}'(t)| dt$. So the derivate of s is $ds/dt = |\mathbf{r}'(t)|$. Applying the Chain Rule of vector functions yields

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt}$$

Therefore, the curvature can be computed as follows

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right| = \left| \frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|} \right| = \left| \frac{\mathbf{T}'(t)}{\mathbf{r}'(t)} \right|$$

Note that $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$. We could also write κ as

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad (1)$$

Proof. We show eq. (1) is true. It is equivalent to show that the following formula

$$|\mathbf{T}'(t)| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}$$

By the definition of the unit tangent vector, one has $\mathbf{r}'(t) = |\mathbf{r}'(t)|\mathbf{T}(t)$. Taking derivative on both sides and from the Chain Rule of vector functions gives

$$\mathbf{r}''(t) = (|\mathbf{r}'(t)|)'\mathbf{T}(t) + |\mathbf{r}'(t)|\mathbf{T}'(t)$$

Therefore,

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= (|\mathbf{r}'(t)|\mathbf{T}(t)) \times \mathbf{r}''(t) = (|\mathbf{r}'(t)|\mathbf{T}(t)) \times ((|\mathbf{r}'(t)|)'\mathbf{T}(t) + |\mathbf{r}'(t)|\mathbf{T}'(t)) \\ &= (|\mathbf{r}'(t)|\mathbf{T}(t)) \times ((|\mathbf{r}'(t)|)'\mathbf{T}(t)) + (|\mathbf{r}'(t)|\mathbf{T}(t)) \times (|\mathbf{r}'(t)|\mathbf{T}'(t)) \end{aligned} \quad (2)$$

Note that the first term of the formula above is zero because the cross product of two parallel vectors is zero. Then, eq. (2) becomes

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = (|\mathbf{r}'(t)|\mathbf{T}(t)) \times (|\mathbf{r}'(t)|\mathbf{T}'(t)) = |\mathbf{r}'(t)|^2 (\mathbf{T}(t) \times \mathbf{T}'(t))$$

Since \mathbf{T} is defined as a unit vector, i.e. $|\mathbf{T}(t)| = 1$. By taking derivatives on $|\mathbf{r}(t)|^2 = 1$ yields $\mathbf{T}(t) \cdot \mathbf{T}(t) = 0$, which implies that the angle between \mathbf{T} and $\mathbf{T}'(t)$ is $\pi/2$. Then, we have that

$$\begin{aligned} |\mathbf{r}'(t) \times \mathbf{r}''(t)| &= |\mathbf{r}'(t)|^2 |\mathbf{T}(t) \times \mathbf{T}'(t)| \\ &= |\mathbf{r}'(t)|^2 |\mathbf{T}(t)| |\mathbf{T}'(t)| \sin\left(\frac{\pi}{2}\right) \\ &= |\mathbf{r}'(t)|^2 |\mathbf{T}'(t)|, \end{aligned}$$

which gives us

$$|\mathbf{T}'(t)| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}.$$

Then,

$$\kappa = \left| \frac{\mathbf{T}'(t)}{\mathbf{r}'(t)} \right| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

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