Decoupled Potential Integral Equation Model

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Purpose

The purpose for this Decoupled Potential Integral Equation (DPIE) model is to allow for solving the Maxwell equations, for problems involving perfect conductors, in a manner that should have better conditioning and scope than tackling the same problems with the Magnetic Field Integral Equation (MFIE).

Additionally, this model should be able to handle different surface topologies that would otherwise lead to inaccuracies in vanilla MFIE formulations.

Mathematical Formulation

The mathematical formulation can be found in the paper found here. A brief summary is that we aim to solve the following electromagnetic scattering problem defined below.

For a fixed frequency ω, the electric and magnetic fields, $\mathcal{E}$ and $\mathcal{H}$, take the form:

$$
\begin{aligned}
\mathcal{E}(\boldsymbol{x}, t) &= \mathcal{R}\{\boldsymbol{E}(\boldsymbol{x}) e^{-i\omega t}\} \\
\mathcal{H}(\boldsymbol{x}, t) &= \mathcal{R}\{\boldsymbol{H}(\boldsymbol{x}) e^{-i\omega t}\}
\end{aligned}
$$

where $\mathcal{R}\{z\}$ returns the real part of z. We can then represent $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$ as a sum of incident (known) and scattered (unknown) fields:

$$
\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}^{inc} + \boldsymbol{E}^{scat} \\
\boldsymbol{H} &= \boldsymbol{H}^{inc} + \boldsymbol{H}^{scat}
\end{aligned}
$$

The resulting system of equations we wish to solve then becomes the following:

- Maxwell Equations

$$\nabla \times \boldsymbol{E} = i\omega\mu\boldsymbol{H}, \;\; \nabla \times \boldsymbol{H} = -i\omega\epsilon\boldsymbol{E}$$

- Sommerfield-Silver-Müller Radiation Condition

$$\boldsymbol{H}^{scat}(\boldsymbol{x}) \times \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \sqrt{\frac{\mu}{\epsilon}} \boldsymbol{E}^{scat}(\boldsymbol{x}) = o(|\boldsymbol{x}|^{-1}), \quad |\boldsymbol{x}| \to \infty$$

- Perfect Conductor Boundary Conditions

$$\left(\boldsymbol{n} \times \boldsymbol{E}^{scat}\right)|_{\partial D} = -\left(\boldsymbol{n} \times \boldsymbol{E}^{inc}\right)|_{\partial D}$$
$$\left(\boldsymbol{n} \cdot \boldsymbol{H}^{scat}\right)|_{\partial D} = -\left(\boldsymbol{n} \cdot \boldsymbol{H}^{inc}\right)|_{\partial D}$$

We can then arrive at a new system of integral equations for the scalar and vector potentials, ϕ and $\boldsymbol{A}$, defined via the Lorentz Gauge. We can go about this using the Lorentz Gauge choice for relating E-M fields to the potentials

$$\boldsymbol{E}^{scat} = i\omega \boldsymbol{A}^{scat} - \nabla \phi^{scat}$$
$$\boldsymbol{H}^{scat} = \frac{1}{\mu} \nabla \times \boldsymbol{A}^{scat}$$

with the Lorenz gauge relationship defined as

$$\nabla \cdot \boldsymbol{A}^{scat} = i\omega\mu\epsilon\phi^{scat}$$

We can then set the boundary conditions for the vector and scalar potentials to be:

$$\left(\boldsymbol{n} \times \boldsymbol{A}^{scat}\right)\Big|_{\partial D} = -\left(\boldsymbol{n} \times \boldsymbol{A}^{inc}\right)\Big|_{\partial D}$$
$$\left(\boldsymbol{n} \times \nabla\phi^{scat}\right)\Big|_{\partial D} = -\left(\boldsymbol{n} \times \nabla\phi^{inc}\right)\Big|_{\partial D}$$

which are shown in the paper mentioned in the beginning to ensure the original E-M boundary conditions are satisfied. Using these boundary conditions and some variables $\{V_j\}$ used to impose uniqueness on the integral equation solution, we can arrive at the following system of equations for the scalar potential:

$$\frac{\sigma}{2} + D_k\sigma - ikS_k\sigma - \sum_{j=1}^{N} V_j\chi_j = f$$
$$\int_{\partial D_j} \left(\frac{1}{k} D_k'\sigma + i\,\frac{\sigma}{2} - iS_k'\sigma\right) ds = \frac{1}{k}\, Q_j$$
$$f = -\phi^{inc}|_{\partial D_j}$$
$$Q_j = -\int_{\partial D_j} \left(\nabla\phi^{inc} \cdot \boldsymbol{n}\right) ds$$

with unknowns $\{V_j\}$, σ for a representation of $\phi^{scat}(\boldsymbol{x})$ as

$$\phi^{scat}(\boldsymbol{x}) = D_k[\sigma](\boldsymbol{x}) - ikS_k[\sigma](\boldsymbol{x})$$

Note that $\{\chi_j\}$ are characteristic functions that return 1 when you are evaluating the integral equation on the j^{th} connected component of the boundary, ∂D_j, and returns 0 otherwise. Using these boundary conditions and some variables $\{v_j\}$ used to impose uniqueness on the integral equation solution, we can also arrive at the following system of equations for the vector potential:

$$\frac{1}{2}\begin{pmatrix}\boldsymbol{a}\\ \rho\end{pmatrix}+\bar{L}\begin{pmatrix}\boldsymbol{a}\\ \rho\end{pmatrix}+i\bar{R}\begin{pmatrix}\boldsymbol{a}\\ \rho\end{pmatrix}+\begin{pmatrix}\boldsymbol{0}\\ \sum_{j=1}^{N} v_j\chi_j\end{pmatrix}=\begin{pmatrix}\boldsymbol{f}\\ h\\ \overline{k}\end{pmatrix}$$

$$\int_{\partial D_j}\left(\boldsymbol{n}\cdot\nabla\times S_k\boldsymbol{a}-k\boldsymbol{n}\cdot S_k\left(\boldsymbol{n}\rho\right)\right)ds+$$
$$i\int_{\partial D_j}\left(k\boldsymbol{n}\cdot S_k\left(\boldsymbol{n}\times\boldsymbol{a}\right)-\frac{\rho}{2}+S_k'\rho\right)ds=q_j$$

$$\boldsymbol{f}=-\boldsymbol{n}\times\boldsymbol{A}^{inc}|_{\partial D_j}$$
$$h=-\nabla\cdot\boldsymbol{A}^{inc}|_{\partial D}$$
$$q_j=-\int_{\partial D_j}\left(\boldsymbol{n}\cdot\boldsymbol{A}^{inc}\right)ds$$

with $\bar{L}$, $\bar{R}$ are defined as

$$\bar{L}\begin{pmatrix}\boldsymbol{a}\\ \rho\end{pmatrix}=\begin{pmatrix}\bar{L}_{11}\boldsymbol{a}+\bar{L}_{12}\rho\\ \bar{L}_{21}\boldsymbol{a}+\bar{L}_{22}\rho\end{pmatrix}$$
$$\bar{R}\begin{pmatrix}\boldsymbol{a}\\ \rho\end{pmatrix}=\begin{pmatrix}\bar{R}_{11}\boldsymbol{a}+\bar{R}_{12}\rho\\ \bar{R}_{21}\boldsymbol{a}+\bar{R}_{22}\rho\end{pmatrix}$$

where

$$\begin{aligned}\bar{L}_{11}\boldsymbol{a}&=\boldsymbol{n}\times S_k\boldsymbol{a} & \bar{R}_{11}\boldsymbol{a}&=k\boldsymbol{n}\times S_k\boldsymbol{n}\times\boldsymbol{a}\\ \bar{L}_{12}\rho&=-k\boldsymbol{n}\times S_k\left(\boldsymbol{n}\rho\right) & \bar{R}_{12}\rho&=\boldsymbol{n}\times\nabla S_k\left(\rho\right)\\ \bar{L}_{21}\boldsymbol{a}&=0 & \boldsymbol{R}_{21}\boldsymbol{a}&=\nabla\cdot S_k\left(\boldsymbol{n}\times\boldsymbol{a}\right)\\ \bar{L}_{22}\rho&=D_k\rho & \bar{R}_{22}\rho&=-kS_k\rho\end{aligned}$$

and with unknowns $\{v_j\}, \boldsymbol{a}, \rho$ for a representation of $\boldsymbol{A}^{scat}(\boldsymbol{x})$ as

$$\begin{aligned}\boldsymbol{A}^{scat}(\boldsymbol{x})=&\ \nabla\times S_k[\boldsymbol{a}](\boldsymbol{x})-kS_k[\boldsymbol{n}\rho](\boldsymbol{x})\\ &+i\left(kS_k[\boldsymbol{n}\times\boldsymbol{a}](\boldsymbol{x})+\nabla S_k[\rho](\boldsymbol{x})\right)\end{aligned}$$

With the system of equations all defined above to find the densities and scalars needed to represents the scattered scalar and vector potentials, one can then use the representation for the scalar and vector potentials and the associated Lorentz Gauge choice to obtain the scattered electric and magnetic fields.

Handling of Multiple Disjoint Objects

The approach used to handled disjoint objects the equations are being solved over can be illustrated using the below Integral Equation (IE):

$$\frac{1}{2}\sigma(x)+D\sigma(x)=f(x)\qquad\forall x\in\partial\Omega$$

where $D\sigma(x)$ represents the double layer potential integral operator on some density $\sigma(x)$. Note that we can define the following quantities based on a k^{th} boundary component $\partial\Omega_k$:

$$
\begin{aligned}
\partial\Omega &= \bigcup_{i=1}^{n} \partial\Omega_i \\
\chi_k(x) &= \begin{cases} 1 & x \in \partial\Omega_k \\ 0 & x \notin \partial\Omega_k \end{cases}
\end{aligned}
$$

With the above definitions, we can define the density $\sigma(x)$, such that it generalizes across all disjoint surfaces, as the following:

$$
\sigma(x) = \sum_{k=1}^{n} \chi_k(x)\sigma_k(x)
$$

Note that using this expression, we can also rewrite $D\sigma(x)$ in the following manner:

$$
\begin{aligned}
D\sigma(x) &= \int_{\partial\Omega} \hat{n}(y) \cdot \nabla g(x-y)\sigma(y)dA_y \\
&= \sum_{k=1}^{n} \int_{\partial\Omega_k} \hat{n}(y) \cdot \nabla g(x-y)\sigma(y)dA_y \\
&= \sum_{k=1}^{n} \int_{\partial\Omega_k} \hat{n}(y) \cdot \nabla g(x-y)\sigma_k(y)dA_y \\
&= \sum_{k=1}^{n} D^{(k)}\sigma_k(x)
\end{aligned}
$$

where $D^{(j)}\rho(x) = \int_{\partial\Omega_j} \hat{n}(y) \cdot \nabla g(x-y)\rho(y)dA_y$. Using this, we can rewrite the original integral equation into a system of n coupled integral equations of the following form:

$$
\frac{1}{2}\sigma_k(x) + \sum_{j=1}^{n} D^{(j)}\sigma_j(x) = f(x) \qquad \forall x \in \partial\Omega_k
$$

This resulting system can now be used to solve for the densities defined on each disjoint surface and stitched together, if needed, using the characteristic functions $\{\chi_k\}$.

Software Implementation

Model

The DPIE model can be found in the file:
`pytential/symbolic/pde/maxwell/dpie.py`

This model is represented as a class called `DPIEOperator` which essentially represents the underlying integral equations to be solved and then the representations for the scalar and vector potentials, ϕ and $\boldsymbol{A}$. For convenience, one can also generate the scattered Electric and Magnetic fields using the representations for the scalar and vector potentials.

Tests

The current tests for the DPIE model reside in the file:
`test/test_maxwell_dpie.py`

As of this version of this documentation, this test script is not complete. This will be completed in the near future.