Financial Engineering

MSc in Risk Management and Financial Engineering, BS1031

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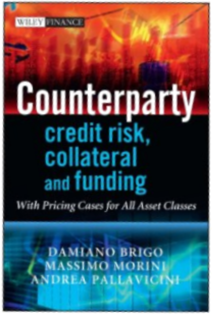
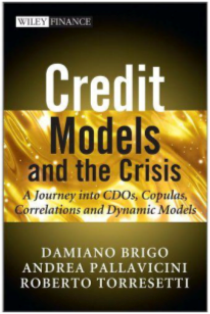
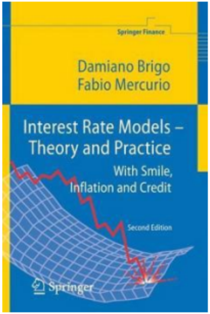
This course is mostly based on the books:







especially the first one (most recent) and the last one (least recent).
Also: “Principles of Financial Engineers” by R. Kosowski and S. Neftci (2014)

PART 1. OPTION PRICING AND ITS SIGNIFICANCE

In this introductory part we introduce the Black Scholes and Merton result, their precursors (Bachelier, DeFinetti...) and the refinements of their theory (Harrison, Kreps, Pliska....), pointing out its significance, successes and failures.

We also look at the derivatives markets and their significance

The Black Scholes and Merton Analysis

- Portfolio replication theory plus Ito's formula to derive the Black and Scholes PDE under certain assumptions on the dynamics of the stock price.
- The Feynman-Kac theorem to interpret the solution of the Black and Scholes PDE as an expected value of a function of the stock price with different dynamics.
- The Girsanov theorem to interpret the different dynamics of the stock price as a dynamics under a different (martingale) probability measure.

Description of the economy

We consider:

- A probability space with a r.c. filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t : 0 \leq t \leq T), P)$.
- In the given economy, two securities are traded continuously from time 0 until time T. The first one (a bond) is riskless and its (deterministic) price B_t evolves according to

$$dB_t = B_t r dt, \quad B_0 = 1, \tag{1}$$

which is equivalent to

$$B_t = e^{rt}, \tag{2}$$

where r is a nonnegative number. To state it differently, the short term interest rate is assumed to be constant and equal to r through time.

Description of the economy

- As for the second one, given the $(\mathcal{F}_t, P)$-Wiener process W_t, consider the following stochastic differential equation

$$dS_t = S_t[\mu dt + \sigma dW_t], \quad 0 \leq t \leq T, \tag{3}$$

with initial condition $S_0 > 0$, and where μ and σ are positive constants. Equation (3) has a unique (strong) solution which is given by

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\}, \quad 0 \leq t \leq T. \tag{4}$$

The risky asset, The B e S Assumptions, and Contingent Claims

$$dB_t = B_t r dt, \quad B_0 = 1,$$

$$dS_t = S_t[\mu dt + \sigma dW_t], \quad 0 \leq t \leq T,$$

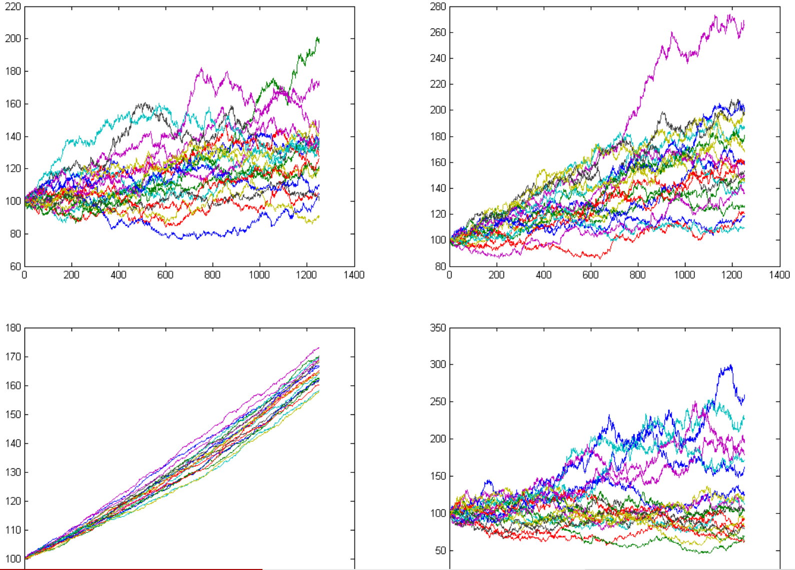
The second asset (a stock) is risky and its price is described by the process S_t. Furthermore, it is assumed that

- (i) there are no transaction costs in trading the stock;
- (ii) the stock pays no dividends or other distributions;
- (iii) shares are infinitely divisible;
- (iv) short selling is allowed without any restriction or penalty.

We refer to these assumptions as to Black and Scholes' *ideal conditions*.

Example of risky asset dynamics over 5 years:

$S_0 = 100, \quad (\mu, \sigma) = (5\%, 10\%), (10, 10), (10, 1), (1, 20)$



Contingent claim, Pricing problem, Complete Market

A **contingent claim** Y for the maturity T is any random variable which is $\mathcal{F}_T$–measurable.
We limit ourselves to *simple* contingent claims, i.e. claims of the form $Y = f(S_T)$.
The idea behind a claim is that it represents an amount that will be paid at maturity to the holder of the contract.
The **Pricing Problem** is giving a fair price to such a contract.
Loosely speaking, the market is said to be **complete** if every contingent claim has a price.

Trading strategies, Value process, gain process, self–financing

A **trading strategy** $\phi = (\phi^B, \phi^S)$ is a pair of functions $\mathcal{F}$–adapted. The pair (ϕ^B_t, ϕ^S_t) represents respectively amounts of bond and stock to be held at time t.

The **value process** is the process V describing the value of the portfolio constructed by following the strategy ϕ,

$$V_t(\phi) = \phi^B_t \; B_t + \phi^S_t \; S_t \; .$$

The **gain process** is defined as

$$G_t(\phi) = \int_0^t \phi^B_u \; dB_u + \int_0^t \phi^S_u \; dS_u \; .$$

and represents the income one obtains thanks to price movements in bond and stock when following the trading strategy ϕ.

Trading strategies, Value process, gain process, self–financing

The strategy is said to be *self–financing* if

$$\phi_t^B\, B_t + \phi_t^S\, S_t \; - (\phi_0^B\, B_0 + \phi_0^S\, S_0) \; = G_t(\phi) \; ,$$

or, in differential terms, $d\; V_t(\phi) = d\; G_t(\phi)$, i.e.

$$d(\phi_t^B\, B_t + \phi_t^S\, S_t) \; = \; \phi_t^B\, dB_t + \phi_t^S\, dS_t \; . \tag{5}$$

Intuitively, this means that the changes in value of the portfolio described by the strategy ϕ are only due to gains/losses coming from price movements, i.e. to changes in the prices B and S, without any cash inflow and outflow.

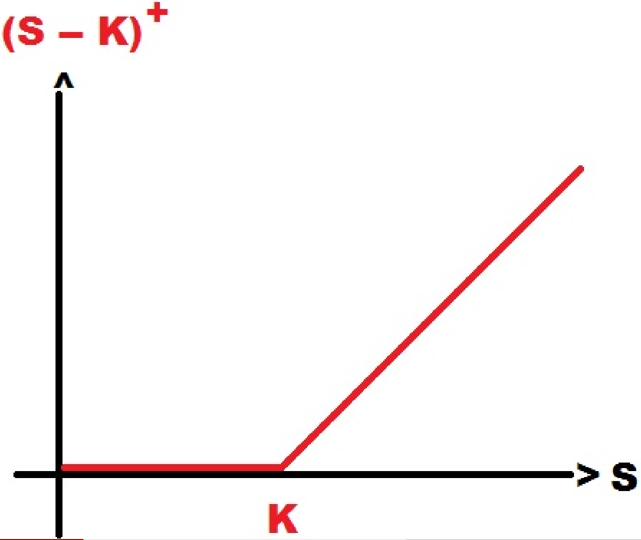
Arbitrage opportunity, arbitrage–free market

An **arbitrage opportunity** is a self–financing strategy ϕ such that

$$\phi_0^B\, B_0 + \phi_0^S\, S_0 \; = 0\,, \;\; \phi_T^B\, B_T + \phi_T^S\, S_T > 0 \;\text{ a.s.}$$

Basically, an arbitrage opportunity is a strategy which creates an almost surely positive cash inflow from nothing. It is sometimes called a **free lunch**.
The market is said to be **arbitrage–free** if there are no arbitrage opportunities.

Example of Claim: European Call Option



Example of Claim: European Call Option I

Suppose we have to price a simple claim $Y = f(S_T)$ at time t.

We focus on the case of a European call option: Let K be its strike price and T its maturity. The option payoff (to a long position) is represented by $Y = (S_T - K)^+ = \max(S_T - K, 0)$.

This is a contract which at maturity-time T pays nothing if the risky–asset price S_T is smaller than the strike price K, whereas it pays the difference between the two prices in the other case.

An investor who expects the risky–asset value to increase considerably can speculate by buying a call option.

Example of Claim: European Call Option II

An example of use of a call option is the following. Suppose now we are at time 0 and we plan to buy one share (unit) of a certain stock at time T. We wish to pay this stock the same price $K = S_0$ it has now, rather than the price it will have at time T, which could be much higher. What one can do in this situation is to buy a call option on the stock with maturity time T and strike price S_0.

He then buys the stock at time T paying S_T and receives $(S_T - S_0)^+$ from the option payoff. Clearly, the total amount he pays in T is then $S_T - (S_T - S_0)^+$ which equals S_T if $S_T \leq S_0$ and equals S_0 if $S_T \geq S_0$. Therefore, an European call option can be seen as a contract which locks the stock price at a desired value to be paid at maturity time T. This *locking* has of course a price, which we wish to determine.

The Black and Scholes PDE

Let $V_t = V(t, S_t)$ be the candidate claim (option) value at time t.
Assume the function $V(t, S_t)$ of time t and of the stock price S_t to have regularity $V \in C^{1,2}([0, T] \times R)$.
Apply Ito's Lemma to V so as to obtain

$$\begin{aligned} dV(t, S_t) &= \left(\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 \right) dt \\ &+ \frac{\partial V}{\partial S}(t, S_t)\sigma S_t dW_t. \end{aligned} \tag{6}$$

Set, for each $0 \le t \le T$,

$$\phi_t^S = \frac{\partial V}{\partial S}(t, S_t), \quad \phi_t^B = (V_t - \phi_t^S S_t)/B_t. \tag{7}$$

By construction, the value of this strategy at time t is V itself, since clearly $V(t, S_t) = \phi_t^B\, B_t + \phi_t^S\, S_t$.

The Black and Scholes PDE

Now assume ϕ to be self–financing. Since ϕ is self-financing

$$\begin{aligned} dV_t &= \phi_t^B dB_t + \phi_t^S dS_t & (8) \\ &= \left[V(t, S_t) - \frac{\partial V}{\partial S}(t, S_t) S_t \right] r dt + \frac{\partial V}{\partial S}(t, S_t) S_t (\mu dt + \sigma dW_t). \end{aligned}$$

Then by equating (6) and (8) (ITO + SELF FINANCING), we obtain that V_t satisfies

$$\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t) r S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t) \sigma^2 S_t^2 = rV(t, S_t), \qquad (9)$$

which is the celebrated Black and Scholes partial differential equation with terminal condition $V_T = (S_T - K)^+$.

Black and Scholes' famous formula

The strategy (ϕ^B, ϕ^S) has final value equal to the claim Y we wish to price, and during its life the strategy does not involve cash inflows or outflows (self–financing condition). As a consequence, its initial value V_t at time t must be equal to the unique claim price to avoid arbitrage opportunities.
The solution of the above equation is given by

$$V_{BS}(t) = V_{BS}(t, S_t, K, T, \sigma, r) := S_t \Phi(d_1(t)) - K e^{-r(T-t)} \Phi(d_2(t)), \quad (10)$$

where

$$d_1(t) := \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2(t) := d_1(t) - \sigma\sqrt{T-t},$$

and $\Phi(\cdot)$ denotes the cumulative standard normal distribution function.

Black and Scholes' famous formula

Expression (10) is the celebrated Black and Scholes option pricing formula which provides the unique no-arbitrage price for the given European call option.

Notice that the coefficient μ does not appear in (10), indicating that investors, though having different risk preferences or predictions about the future stock price behaviour, must yet agree on this unique option price.

MORE ON THE SIGNIFICANCE OF THIS LATER.

Numerical example

Suppose the current stock value is $S_0 = 100$.
Suppose the risk free interest rate is $r = 2\% = 0.02$.
Suppose that the strike $K = 100$ (at the money option).
Assume the volatility $\sigma = 0.2 = 20\%$.
Take a maturity of $T = 5y$. CALL PRICE IS $V_{BS}(0) = 22.02$.

For example, in Matlab this is obtained through commands

```
S0=100; sig=0.2; r=0.02; K=100; T=5;
d1 = (r + 0.5*sig*sig)*T/(sig*sqrt(T));
d2 = (r - 0.5*sig*sig)*T/(sig*sqrt(T));
V0 = S0*normcdf(d1)-K*exp(-r*T)*normcdf(d2);
```

The same calculation with lower volatility $\sigma = 0.05 = 5\%$ would give

$$V_{BS}(0)|_{\sigma=0.05} = 10.5943, \quad V_{BS}(0)|_{\sigma=0.0001} = 9.52.$$

The last value is very close to the intrinsic value $S_0 - Ke^{-rT}$.

Numerical example

- Acme today is worth $S_0 = 100$.
- The more the value of acme goes up in 5 years, the more we gain as $S_{5y} - S_0$ grows. In a scenario where $S_{5y} = 200$, we gain 100.
- If however Acme goes down instead, $S_{5y} - S_0$ goes negative but the option $(S_{5y} - S_0)^+$ caps it at zero and we lose nothing. For example, in a scenario where Acme goes down to 60, we get $(60 - 100)^+ = (-40)^+ = 0$ ie we lose nothing
- With the original data, entering the gamble costs initially 22 USD out of 100 of stock notional. It is expensive. On the other hand, it is a gamble where we can only win and in principle have scenarios with unlimited profit.
- You will notice that:

$$\uparrow \sigma \Rightarrow V_{CallBS} \uparrow, \quad \uparrow S_0 \Rightarrow V_{CallBS} \uparrow, \quad \downarrow K \Rightarrow V_{CallBS} \uparrow \ldots.$$

Another numerical example

Take one more example where now the strike K is at the money forward and volatility very low, namely

```
S0=100; sig=0.0001; r=0.02; T=5; K=S0*exp(r*T);
```

Then

$$V_{BS}(0) = 0 \approx S_0 - Ke^{-rT} = S_0 - S_0 = 0.$$

Verifying the Self financing condition

Going back to the general Black Scholes result, we then prove that the strategy

$$\phi_t^S = \frac{\partial V_{BS}}{\partial S}(t, S_t), \;\; \phi_t^B = (V_{BS}(t) - \phi_t^S S_t)/B_t$$

$$\Big(V_{BS}(t) = V_{BS}(t, S_t, K, T, \sigma, r) := S_t \Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t)$$

is indeed self-financing. By Ito's Lemma, in fact, we have

$$dV_{BS}(t) = \frac{\partial}{\partial t} V_{BS}(t)dt + \frac{\partial}{\partial S} V_{BS}(t)dS_t + \frac{1}{2}\frac{\partial^2}{\partial S^2} V_{BS}(t)\sigma^2 S_t^2 dt. \qquad (11)$$

Verifying the Self financing condition

Since straightforward differentiation of V_{BS} expression leads to

$$\frac{\partial}{\partial t}V_{BS}(t) = -\frac{S_t\Phi'(d_1(t))\sigma}{2\sqrt{T-t}} - rXe^{-r(T-t)}\Phi(d_2(t)),$$

$$\frac{\partial^2}{\partial S^2}V_{BS}(t) = \frac{\Phi'(d_1(t))}{S_t\sigma\sqrt{T-t}},$$

where $\Phi'(x) := \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, then it is enough to substitute ϕ^S and ϕ^B expressions given above to obtain from (11) that $dV_{BS}(t) = \phi_t^S dS_t + \phi_t^B dB_t$, which is the self–financing condition in differential form.

The Feynman Kac theorem for Risk Neutral Valuation

Different interpretation: the Feynman-Kac Theorem allows to interpret the solution of a parabolic PDE such as the Black and Scholes PDE in terms of expected values of a diffusion process. In general, given suitable regularity and integrability conditions, the solution of the PDE

$$\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)b(x)+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,x)\sigma^2(x)=rV(t,x), \quad V(T,x)=f(x), \tag{12}$$

can be expressed as

$$V(t,x)=e^{-r(T-t)}\mathbb{E}^{Q}_{t,x}\{f(X_T)|\mathcal{F}_t\} \tag{13}$$

where the diffusion process X has dynamics starting from x at time t

$$dX_s=b(X_s)ds+\sigma(X_s)dW^Q_s\ , s\geq t,\ X_t=x \tag{14}$$

under the probability measure $\mathbb{Q}$ under which the expectation $\mathbb{E}^{Q}_{t,x}\{\cdot\}$ is taken. The process W^Q is a standard Brownian motion under $\mathbb{Q}$.

Risk Neutral interpretation of the B e S's formula

By applying this theorem to the Black and Scholes setup, with $b(x) = rx$, $\sigma(x) = \sigma\, x$ (so that the general PDE of the theorem coincides with the BeS PDE) we obtain:
The unique no-arbitrage price of the integrable contingent claim $Y = (S_T - K)^+$ (European call option) at time t, $0 \leq t \leq T$, is given by

$$V_{BS}(t) = \mathbb{E}^Q \left(e^{-r(T-t)} Y | \mathcal{F}_t \right). \tag{15}$$

The expectation is taken with respect to the so-called martingale measure $\mathbb{Q}$, i.e. a probability measure under which the risky–asset price $S_t/B_t = e^{-rt} S_t$ measured with respect to the risk-free asset price B_t is a martingale, i.e.

$$dS_t = S_t[rdt + \sigma dW_t^Q], \quad 0 \leq t \leq T, \tag{16}$$

An expression for $\mathbb{Q}$: Girsanov's theorem

Consider on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a stochastic differential equation

$$d\ X_t = b(X_t)\ dt + \ v(X_t)\ dW_t, \ \ X_0.$$

Define the measure $\mathbb{Q}$ by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{b^Q(X_s) - b(X_s)}{v(X_s)}\right)^2 ds + \int_0^t \frac{b^Q(X_s) - b(X_s)}{v(X_s)}\, dW_s\right\}$$

Then under $\mathbb{Q}$

$$dW_t^Q = -(b^Q(X_t) - b(X_t))/v(X_t)dt + dW_t$$

is a Brownian motion and

$$d\ X_t = b^Q(X_t)\ dt + \ v(X_t)\ dW_t^Q, \ \ X_0.$$

The Risk Neutral measure via Girsanov's theorem

We apply Girsanov's theorem to move from

$$d\ S_t = \mu S_t\ dt + \sigma S_t\ dW_t$$

to

$$d\ S_t = rS_t\ dt + \sigma S_t\ dW_t^Q$$

We obtain

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{ -\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \frac{\mu - r}{\sigma} W_T \right\}. \tag{17}$$

No arbitrage: Main steps followed so far I

1. Self Financing Condition (Portfolio replication theory) plus Ito's formula to derive the Black and Scholes PDE for any payout claim in S_T:

$$d\ S_t = \mu S_t\ dt + \sigma S_t\ dW_t$$

$$\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t) r S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 = rV(t, S_t),$$

$$V_T = \phi(S_T)$$

2. If each such claim can be replicated with a unique self financing strategy then there is clearly no arbitrage and we have a unique arbitrage free claim price equal to the initial cost of the strategy (and given by the PDE).

No arbitrage: Main steps followed so far II

3. The Feynman-Kac theorem to interpret the solution of the Black and Scholes PDE as an expected value of a function of the stock price with different dynamics

$$V(t, S_t) = \mathbb{E}^Q\{e^{-r(T-t)}\phi(S_T)|\mathcal{F}_t\}$$

$$d\ S_t = rS_t\ dt + \sigma S_t\ dW_t^Q$$

4. The Girsanov theorem to interpret the different dynamics of the stock price as a dynamics under a new (Risk neutral or martingale) probability measure $\mathbb{Q}$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \frac{\mu - r}{\sigma} W_T\right\}.$$

No arbitrage: Main steps followed so far III

5. Hence the PDE coming from the no-arbitrage self-financing condition / replication approach implies that prices are expectations under a risk neutral measure where the risky asset local mean grows at the risk free rate. This is a second way to express no-arbitrage via the condition that S/B is a martingale (more on martingales in a minute), ie a fair game. Hence no arbitrage will be related to the market for the underlying risky asset S to be a fair game.
6. Hence we have two ways to express no arbitrage: (i) Every claim can be replicated by a self financing trading strategy and (ii) the underlying risky asset measured with respect to the bank account is a martingale under the measure used to price claims with expectations.

The idea behind the martingale approach

Why martingales?

A martingale is a stochastic process representing a fair game. Loosely speaking, the above proposition states that in order to price under uncertainty one must price in a world where the probability measure is such that the risky asset evolves as a fair game when expressed in units of the risk–free asset.

Hence in our case S_t/B_t must be a fair game, ie a martingale.

martingales: local mean =0

For regular diffusion processes X_t martingale means ”zero-drift”, no up or down local direction: $dX_t = 0dt + \sigma(t, X_t)dW_t$.

Indeed, show that the drift of the SDE for $d(S_t/B_t)$ is zero under $\mathbb{Q}$.

The idea behind the martingale approach

Numeraire

When we consider S_t/B_t we may say that we are looking at S measured with respect to the numeraire B_t.
In general, as we shall see later on, it is possible to adopt any non-dividend paying asset price as numeraire, and price under the particular probability measure associated with that numeraire.
However, the canonical numeraire is the bank account B we have used now and the probability measure associated with the numeraire B is the risk neutral measure $\mathbb{Q}$.

The above analysis is easily generalized from a call option to any integrable claim $Y = f(S_T)$ different from a Call Option. We give a few examples now.

Examples of simple claims: Digital options I

Digital options include Cash or Nothing options where the final payout at T is

$$Y^{CN} = K 1_{\{S_T > K\}},$$

where K can be easily replaced by any other fixed amount at time 0, and Asset or Nothing options

$$Y^{AN} = S_T 1_{\{S_T > K\}}.$$

Here the indicator 1_A is one in scenarios where condition A is true, and zero in scenarios where it is false. Hence, a CN option pays a fixed amount K at maturity T only if the underlying asset S_T is above the treshold K at maturity. Similarly, an AN options pays the asset price at maturity S_T but only if said asset price is above the threshold K. These

Examples of simple claims: Digital options II

options are called also binary options. We compute both option prices via Risk Neutral Valuation,

$$V_{BS}(0) = \mathbb{E}^Q \left(e^{-rT} Y \right).$$

We notice immediately that

$$Y^{AN} - Y^{CN} = (S_T - K)1_{\{S_T > K\}} = (S_T - K)1_{\{S_T - K > 0\}} = (S_T - K)^+,$$

namely a call option payoff. Hence it is enough to derive the price of one of the two options and the remaining one can be derived using the fact that $\mathbb{E}^Q \left(e^{-rT} Y^{CALL} \right) =$

$$= \mathbb{E}^Q \left(e^{-rT} (Y^{AN} - Y^{CN}) \right) = \mathbb{E}^Q \left(e^{-rT} Y^{AN} \right) - \mathbb{E}^Q \left(e^{-rT} Y^{CN} \right)$$

Examples of simple claims: Digital options III

or

$$V_{BS}^{CALL}(0) = V_{BS}^{AN}(0) - V_{BS}^{CN}(0).$$

It is easier to compute the cash or nothing option price. Compute

$$V_{BS}^{CN}(0) = \mathbb{E}^{Q}\left[e^{-rT}K1_{\{S_T>K\}}\right] = e^{-rT}K\mathbb{E}^{Q}\left[1_{\{S_T>K\}}\right] = e^{-rT}K\mathbb{Q}(S_T > K)$$

since from basic probability we know that $E^{Q}[1_A] = \mathbb{Q}(A)$. We are now left with computing $\mathbb{Q}(S_T > K)$. We recall the SDE for S under the risk neutral measure $\mathbb{Q}$:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0.$$

Ito's formula for the natural logarithm $\ln S_t$ gives easily (exercise, write this in detail)

$$d\ln(S_t) = (r - \sigma^2/2)dt + \sigma dW_t$$

Examples of simple claims: Digital options IV

from which, writing in integral form and recalling that $W_0 = 0$

$$\ln S_T - \ln S_0 = (r - \sigma^2/2)T + \sigma W_T \sim (r - \sigma^2/2)T + \sigma\sqrt{T}\mathcal{N}(0,1)$$

Now we write

$$\mathbb{Q}(S_T > K) = \mathbb{Q}(\ln S_T > \ln K) = ...$$

because logarithm is an increasing function; by substituing our expression for $\ln S_T$

$$... = \mathbb{Q}(\ln S_0 + (r - \sigma^2/2)T + \sigma\sqrt{T}\mathcal{N}(0,1) > \ln K) =$$

$$= \mathbb{Q}(\sigma\sqrt{T}\mathcal{N}(0,1) > -\ln(S_0/K) - (r - \sigma^2/2)T) =$$

$$= \mathbb{Q}\left(-\mathcal{N}(0,1) < \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) =$$

Examples of simple claims: Digital options V

$$= \Phi\left(\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) = \Phi(d_2(0))$$

where Φ is the cdf of the standard normal and where we used the fact that the opposite of a standard normal is still a standard normal.
Going back to our formula

$$V_{BS}^{CN}(0) = e^{-rT} K \mathbb{Q}(S_T > K) = e^{-rT} K \Phi(d_2(0)).$$

Recall the BS formula for a Call option we wrote earlier, written at time 0

$$V_{BS}^{CALL}(0) = S_0 \Phi(d_1(0)) - e^{-rT} K \Phi(d_2(0)).$$

Hence

$$V_{BS}^{CALL}(0) = S_0 \Phi(d_1(0)) - V_{BS}^{CN}(0).$$

Examples of simple claims: Digital options VI

Since we have seen a few lines above than

$$V_{BS}^{CALL}(0) = V_{BS}^{AN}(0) - V_{BS}^{CN}(0)$$

it follows immediately by difference that

$$V_{BS}^{AN}(0) = S_0 \Phi(d_1(0)).$$

Finally, if we need to price a cash or nothing option for a fixed amount different from the strike K, namely

$$Y^{CN1} = X 1_{\{S_T > K\}},$$

with X fixed at time 0 but possibly different from K, we just write

$$Y^{CN1} = (X/K) K 1_{\{S_T > K\}} = (X/K) Y^{CN},$$

so that the price for CN1 is the price for the standard CN multiplied by X/K.

The idea behind the martingale approach

No need to know the real expected return

We noticed earlier that the coefficient μ does not appear in (10), indicating that investors, though having different risk preferences or predictions about the future stock price behaviour, must yet agree on this unique option price.

This property can also be inferred from (16), since, under $\mathbb{Q}$, the drift rate of the stock price process equals the risk-free interest rate while the variance rate is unchanged. For this reason the pricing rule (15) is often referred to as **risk-neutral valuation**, and the measure $\mathbb{Q}$ defines what is called **the risk-neutral world**.

Intuitively, in a risk-neutral world the expected rate of return on all securities is the risk-free interest rate, implying that investors do not require any risk premium for trading stocks.

Weak point of the derivation: Uniqueness of ϕ

The above derivation, however, is still not fully satisfactory, since we have implicitly assumed that (ϕ^B, ϕ^S) is the *unique* self-financing strategy replicating the claim with payoff $f(S_T)$. This uniqueness, anyway, can be obtained by applying the more general theory on complete markets, which is beyond the scope of this introduction.

Dynamic Hedging I

In the process of deriving the BS formula, we have also found a way to perfectly hedge the risk embedded in this contract.

Indeed look at the option pricing problem from the following point of view:

- You are the bank and you just sold a call option to the client.
- At the future time T you will have to pay $(S_T - K)^+$ to your client
- You client pays you V_0 for the option now, at time 0
- Clearly, if the equity goes up a lot in the future, $(S_T - K)^+$ could be very large
- You wish to avoid any risks and decide to hedge away the risk in this contract you sold.
- How should you do that?

Dynamic Hedging II

The answer to this question is in our derivation above.

- You cash in V_0 from the client and use it to buy, at time 0,

$$\frac{\partial V_0}{\partial S_0} = \Phi(d_1(0)) =: \phi_0^S =: \Delta_0 \quad \text{stock and}$$

$$\phi_0^B = (V_0 - \Delta_0 S_0)/B_0 \quad \text{bank account / bond (cash).}$$

- You then implement the self-financing trading strategy, **rebalancing continuously** (hence *dynamic hedging*) your ϕ_t^S, ϕ_t^B amounts of S and B according to

$$\phi_t^S = \frac{\partial V_t}{\partial S_t} = \Phi(d_1(t)) =: \Delta_t \quad \text{stock and}$$

$$\phi_t^B = (V_t - \Delta_t S_t)/B_t \quad \text{bank account / bond (cash).}$$

Dynamic Hedging III

- Because the strategy is self-financing, this rebalancing can be financed thanks to price movements of B and S and you need not add any cash or assets from outside.
- At final maturity we know that the final value will be $V_T = (S_T - K)^+$ as we posed this as boundary condition in our pricing problem.
- Hence by following the above strategy, set up with the initial V_0 and with no subsequent cost, we end up with the payout $(S_T - K)^+$ at maturity.
- We can then deliver this payout to our client and face no risk.
- Basically, our self financing trading strategy in the underlying S, set up with the initial payment V_0, completely replicated the claim we sold to our client.

Dynamic Hedging IV

- An obvious but often overlooked point it this: If we are perfectly hedged, all the money we received from the client (V_0) is spent to set up the hedge, and we as a bank make no gain.
- That's why in reality only partial hedges are often implemented, in an attempt not to erode all potential profit.

The above framework is called "**delta-hedging**".

Basically one holds an amount of risky asset equal to the sensitivity of the contract price to the risky asset itself (delta).

This strategy is possible only in markets where all risks are directly linked to tradable assets and viceversa (roughly: "complete markets").

Dynamic Hedging V

Metatheorem/folklore: A market is complete if there are as many assets as independent sources of randomness.

In reality markets are incomplete, as there are some risks that are covered by no direct assets, and there are more risks than assets.

This can be partly addressed by including a few derivatives themselves among the basic assets, but it is hard to keep the market complete

For example, in credit risk with intensity models, where the default time is $\tau = \Lambda^{-1}(\xi)$, and Λ is the cumulated instantaneous credit spread and ξ is the jump to default exponential variable, we have that ξ cannot be hedged unless we introduce a credit derivative depending on ξ itself in the pool of our basic assets. And even then the hedge remains partial. We cannot hedge recovery rates, correlations...

Dynamic Hedging VI

A further problem is that continuous rebalancing does not happen. Real hedging happens in discrete time and this will imply an hedging error with respect to the idealized case

In the end hedging is more an art than a science, and it involves many pragmatic choices and rules of thumbs. However, a sound understanding of the idealized case is crucial to appreciate the subtleties in real market applications.

The sensitivities (or greeks) I

When hedging derivatives, often sensitivities (or greeks) are used in practice.

A sensitivity is the partial derivative of the price or of another sensitivity with respect to one of the parameters. It tells us how much a small change of the parameter impacts a change in the price or sensitivity we are examining.

We have already met one of the most important sensitivities, delta.

$$\Delta(t) = \frac{\partial V(t)}{\partial S}$$

which, for a call option price under Black Scholes, is equal to $\Phi(d_1(t))$, as we have seen above. Delta measures how much the option price V changes when there is a small change in the underlying asset price S.

The sensitivities (or greeks) II

In general a large sensitivity with respect to a parameter means that the trade is quite sensitive to that parameter, and the trader may consider trades that reduce the sensitivity if she wishes to be more prudent with respect to that parameter. If the trader is more aggressive she may decide to trade to increase the sensitivity further.

Other sensitivities or greeks are: Time decay or Θ, sensitivity to time,

$$\Theta_t = \frac{\partial V(t)}{\partial t} = -\frac{\partial V(t)}{\partial (T-t)}$$

Gamma, the sensitivity of delta to the underlying:

$$\Gamma_t = \frac{\partial \Delta(t)}{\partial S} = \frac{\partial^2 V(t)}{\partial S^2}$$

The sensitivities (or greeks) III

At this point we may write an equation linking the three sensitivities just introduced. Recall Ito's formula we have seen earlier

$$dV(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)dt + \frac{\partial V}{\partial S}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 dt.$$

We can rewrite this as

$$dV(t, S_t) = \Theta_t dt + \frac{1}{2}\Gamma_t \sigma^2 S_t^2 dt + \Delta_t dS_t$$

On the other hand we have, from the self financing condition,

$$dV(t, S_t) = \Delta_t dS_t + [(V(t, S_t) - \Delta_t S_t)/B_t]dB_t$$

The sensitivities (or greeks) IV

(the quantity inside square brackets being previously called η_t). By matching the two expressions we have

$$[(V(t,S_t)-\Delta_t S_t)/B_t]dB_t = \Theta_t dt + \frac{1}{2}\Gamma_t \sigma^2 S_t^2 dt$$

or

$$[(V(t,S_t)-\Delta_t S_t)]r_t = \Theta_t dt + \frac{1}{2}\Gamma_t \sigma^2 S_t^2 dt$$

or

$$r_t V(t,S_t) = r_t \Delta_t S_t + \Theta_t dt + \frac{1}{2}\Gamma_t \sigma^2 S_t^2$$

Back to definitions, Vega is the sensitivity to volatility, namely

$$\nu_t = \frac{\partial V(t)}{\partial \sigma}$$

The sensitivities (or greeks) V

ρ is the sensitivity to interest rates r, namely

$$\rho_t = \frac{\partial V(t)}{\partial r}$$

These greeks can be computed in closed form in Black Scholes for call and put options, for example. There are further higher order greeks Vanna, Charm, Vomma/volga, Veta, Yoghurt, Speed, Zomma, Color Ultima, Totto... (sounds crazy I know... and one on this list is fake)

The higher the order of the greeks we use, the smoother we are assuming prices to be. For example, Speed = $\partial^3 V/\partial S^3$ requires the price V to be three times differentiable with respect to the underlying S. While this may hold in simple models like Black Scholes for specific payoffs, in general assuming excessive smoothness is not realistic,

The sensitivities (or greeks) VI

and therefore using high order greeks has to be done very carefully, especially when the greeks are computed with numerical methods.

What does it all mean

So far we have tried to follow a technical path, but it is time to appreciate the significance of what we have done so far.

We now ask ourselves: What are the implications of what we have calculated on the big picture?

Financial Engineering deals in large part with Derivatives. So, following our derivation above, why are derivatives so important, so popular and, often, unpopular?

Why is there a course in Financial Engineering in this Master programme?

What does it all mean? Call option and Gambling

Assume we wish to enter into a gamble (call option) against a bank, where:

- If the future price of the ACME stock in 1y is larger than the value of ACME today, we receive from the bank the difference between the two prices (on a given notional).
- If the future price of the ACME stock in 1y is smaller or equal than the value of ACME today, nothing happens.

The bank will charge us for entering this wage, since we can only win or get into a draw, whereas the bank can only lose or get to a draw.

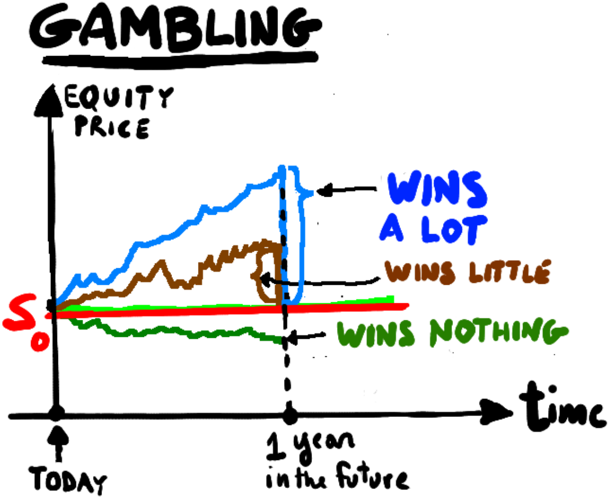


Figure: A one-year maturity Gamble on an equity stock. Call Option.

Call option and Gambling

We have an investor buying a call option on ACME with a 1y maturity.

The Bank's problem is finding the correct price of this option today. This price will be charged to the investor, who may also go to other banks.

This is an option pricing problem.

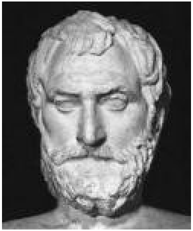
The market introduced options and more generally financial derivatives that may be much more complex than the above example. Such derivatives often work on different sectors: Foreign Exchange Rates, Interest Rates, Default Events, Metheorological events, Energy, etc.

Derivatives can be bought to protect or hedge some risk, but also for speculation or ”gambling”.

Options and Derivatives

Derivatives outstanding notional as of June 2011 (BIS) is estimated at **708 trillions USD** (US GDP 2011: 15 Trillions; World GDP: 79 Trillions)

$$708000 \text{ billions}, \quad 708,000,000,000,000, \quad 7.08 \times 10^{14} \text{ USD}$$



How did it start? It has always been there. Around 580 B.C., Thales purchased options on the future use of olive presses and made a fortune when the olives crop was as abundant as he had predicted, and presses were in high demand. (Thales is also considered to be the father of the sciences and of western philosophy, as you know).

Options and Derivatives valuation: precursors

- **Louis Bachelier** (1870 – 1946) (First to introduce Bronwnian motion W_t in Finance, First in the modern study of Options);
- **Bruno de Finetti** (1906 – 1985) (Father of the subjective interpret of probability; defines the risk neutral measure in a way that is very similar to current theories: first to derive no arbitrage (ante-litteram!) through inequalities constraints, discrete setting, consistent betting quotients, see also Frank Ramsey (1903-1930).

Modern theory follows Nobel awarded **Black, Scholes and Merton** (and then Harrison and Kreps etc) on the correct pricing of options

Black and Scholes: What does it mean?

We have derived the Black Scholes formula for a call option earlier. Let us recall the key points.

Let S_t be the equity price for ACME at time t.
For the value of the ACME stock S_t let us assume, as before, a SDE
$dS_t = \mu S_t dt + \sigma S_t dW_t$ or also

$$\underbrace{\frac{dS_t}{S_t}}_{\substack{\text{relative change}\\ \text{in stock ACME}\\ \text{between}\\ t \text{ and } t+dt}} = \underbrace{\mu}_{\substack{\text{instantaneous}\\ \text{"mean" return}\\ \text{of ACME}\\ \text{between } t \text{ and } t+dt}} dt + \underbrace{\sigma}_{\substack{\text{volatility}\\ \text{for ACME}}} \underbrace{dW_t}_{\substack{\text{New}\\ \text{random}\\ \text{shock}}}$$

Black and Scholes: What does it mean?

Then we have seen there exists a formula (Black and Scholes') providing a unique fair price for the above gamble (option) on ACME in one year.

This Black Scholes formula **depends on the volatility** σ of ACME, and from the initial value S_0 of ACME today, but **does NOT depend on the expected return** μ of ACME.

This means that two investors with very different expectations on the future performance of ACME (for example one investor believes ACME will grow, the other one that ACME will go down) will be charged the same price from the bank to enter into the option.

The Gamble price does not depend on the investor perception of future markets. One would think that Red Investor should be willing to pay a higher price for the option with respect to Blue Investor. Instead, both will have to pay the gamble according to the green scenarios, where ACME grows with the same returns as a riskless asset

Derivatives prices independent of expected returns???

This seemingly counterintuitive result renders derivatives pricing independent of the expected returns of their underlying assets.

This makes derivatives valuations quite objective, and has contributed to derivatives growth worldwide.

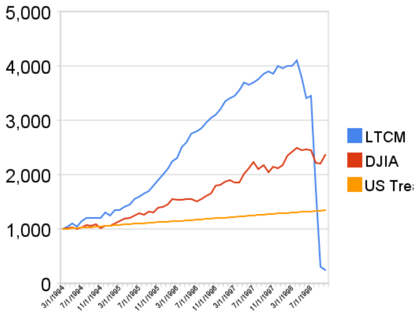
Today, derivatives are used for several purposes by banks and corporates all over the world

A mathematical result has contributed to create new markets that reached 708 trillions (US GDP: 15 Trillions)

But keep in mind that the derivation of the Black Scholes result holds only under the 4 ideal conditions and actually many more assumptions:

The Black Scholes Merton analysis assumptions

- Short selling is allowed without restrictions
- Infinitely divisible shares
- No transaction costs
- No dividends in the stock
- No default risk of the parties in the deal
- No funding costs: Cash can be borrowed or lent at the risk free rate r
- Continuous time and continuous trading/hedging
- Perfect market information
- ....



Sometimes the timing of the Nobel committee is funny, and we are not talking about the peace Nobel prize. Warning: anedoctal

1997: Nobel award.

1998: the US Long-Term Capital Management hedge fund has to be bailed out after a huge loss. The fund had Merton and Scholes in their board and made high use of leverage (derivatives). This leads us to...

The Credit Crisis: Is this Mathematics fault?

Quantitative Analysts (”quants”) and Academics guilty?

Over the past few years a number of articles has disputed the role of Mathematics in Finance, especially in relationship with Counterparty Credit Risk and Credit Derivatives (especially CDOs).
Quants have been accused to be unaware of models limitations and to have provided the market with a false sense of security.

- “The formula that killed Wall Street”[1]
- “The formula that fell Wall Street”[2]
- “Wall Street Math Wizards forgot a few variables”[3]
- “Misplaced reliance on sophisticated (mathematical) models”[4]
- BUT WHAT IS THIS FORMULA PRECISELY?

[1] Recipe for disaster. Wired Magazine, 17.03.

[2] The Financial Times, Jones, S. (2009). April 24 2009.

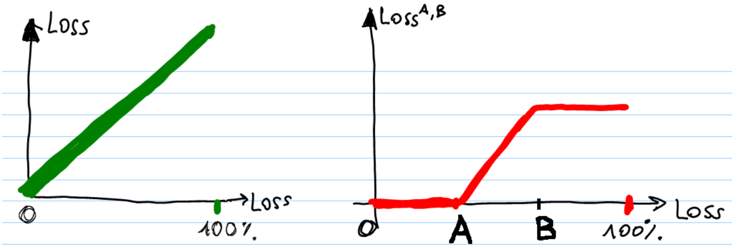
[3] Lohr (2009), New York Times of September 12.

[4] Turner, J.A. (2009). The Turner Review. 03/2009. FSA, UK. www.fsa.gov.uk/pubs/other/turner_review.pdf

CDOs: The standard synthetic case I

- Portfolio of names, say 125. Names may default, generating losses.
- A tranche is a portion of the loss between two percentages. The $3\% - 6\%$ tranche focuses on the losses between 3% (attachment point) and 6% (detachment point).
- The CDO protection seller agrees to pay to the buyer all notional default losses (minus the recoveries) in the portfolio whenever they occur due to one or more defaults, within 3% and 6% of the total pool loss.
- In exchange for this, the buyer pays the seller a periodic fee on the notional given by the portion of the tranche that is still “alive” in each relevant period.
- Valuation problem: What is the fair price of this “insurance”?

CDOs: The standard synthetic case II



- Pricing (marking to market) a tranche: taking expectation of the future tranche losses under the pricing measure.
- From nonlinearity, the tranche expectation will depend on the loss distribution: marginal distributions of the single names defaults **and** dependency among different names' defaults. Dependency is commonly called "correlation".
- Abuse of language: correlation is a complete description of dependence for jointly Gaussians, but more generally it is not.

Copulas

The complete description is either the whole multivariate distribution or the so-called “copula function” (marginal distributions have been standardized to uniform distributions).

CDO Valuation: The culprit.

One-factor Gaussian copula

$$\boxed{\int_{-\infty}^{+\infty} \prod_{i=1}^{125} \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Lambda_i(T))) - \sqrt{\rho_i}m}{\sqrt{1-\rho_i}}\right)\varphi(m)dm.}$$

“MEA COPULA!” From Nobel award to universal scapegoat

Introduced in Credit Risk modeling by David X. Li. Commentators went from suggesting a Nobel award to blaming Li for the whole Crisis.

The scapegoat

David Li, 2005, Wall Street Journal

[...] ”The most dangerous part,” Mr. Li himself says of the model, ”is when people believe everything coming out of it.” Investors who put too much trust in it or don’t understand all its subtleties may think they’ve eliminated their risks when they haven’t.

Indeed, these models are static. they ignore Credit Spread Volatilities, that in Credit can be 100%; this has further paradoxical consequences in copula models for wrong way risk, as we will see later on.

Tranches and Correlations

The dependence of the tranche on “correlation” is crucial. The market assumes a Gaussian Copula connecting the defaults of the 125 names, parametrized by a correlation matrix with 125*124/2 = 7750 entries. However, when looking at a tranche:

$$7750 \;\; \text{parameters} \longrightarrow \;\; 1 \;\; \text{parameter.}$$

The unique parameter is reverse-engineered to reproduce the price of the liquid tranche under examination. ”Implied correlation”. Once obtained it is used to value related products.

Problem with this implied ”compound correlation”

If at a given time the $3\% - 6\%$ tranche for a five year maturity has a given implied correlation, the $6\% - 9\%$ tranche for the same maturity will have a different one. The two tranches on the *same pool* are priced (and hedged!!!) with two inconsistent loss distributions

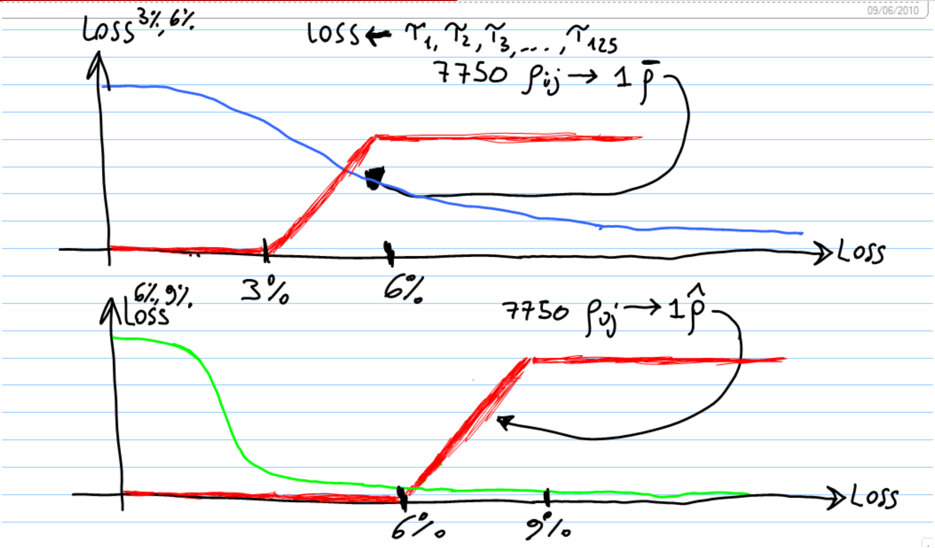


Figure: Compound correlation inconsistency

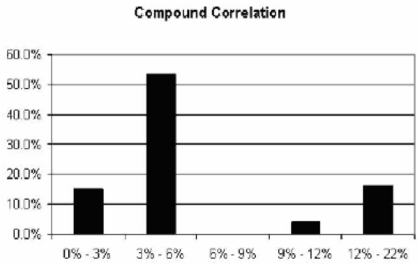


Figure: (After Edvard Munch's The Scream; Compound correlation DJ-iTraxx S5, 10y on 3 Aug 2005)

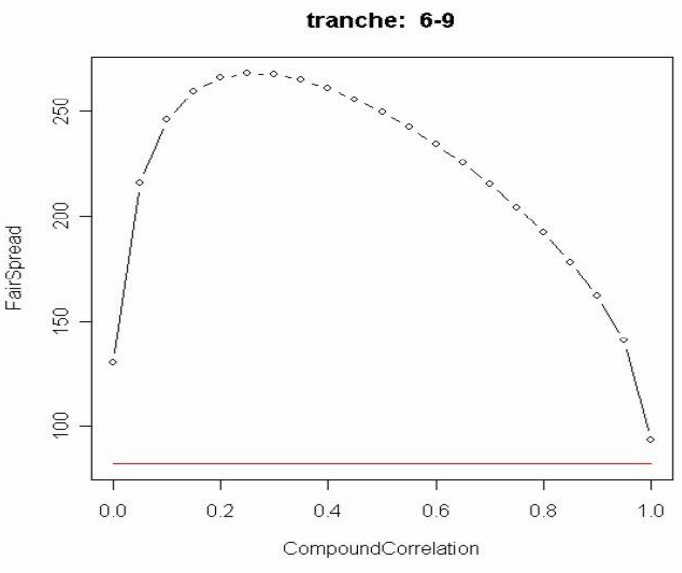


Figure: Non-invertibility compound correl DJ-iTraxx S5, 10y on 3 Aug 2005

Base correlation I

As a possible remedy for non-invertibility of compound correlation and other matters, the market introduced Base Correlation, which is still prevailing in the market.

Problems with base correlation

Base correlation is easier to interpolate but is inconsistent even at single tranche level, in that it prices the $3\% - 6\%$ tranche by decomposing it into the $0\% - 3\%$ tranche and $0\% - 6\%$ tranche and using two different correlations (and hence distributions) for those. This inconsistency shows up occasionally in negative losses (i.e. in defaulted names resurrecting).

[in the graph we use put-call parity to simplify]

Base correlation II

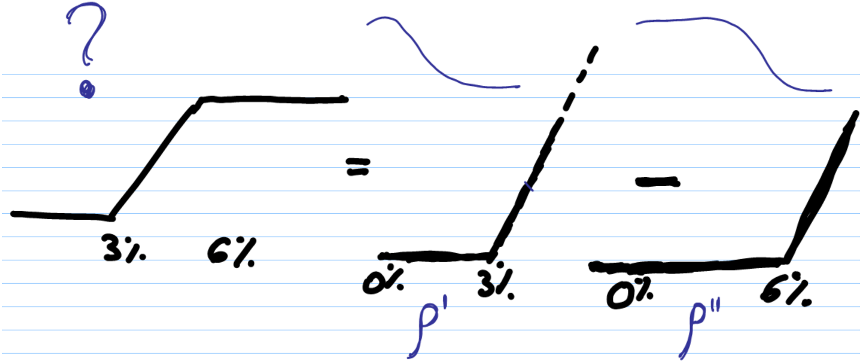
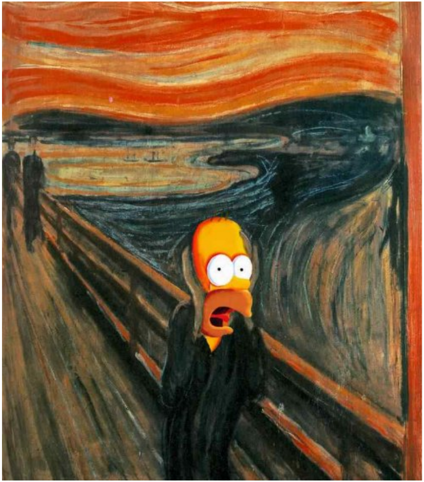


Figure: Base correlation inconsistency

Base correlation III



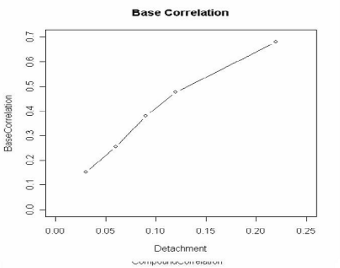


Figure: (Base correl DJ-iTraxx S5, 10y on 3 Aug 2005)

Base correlation

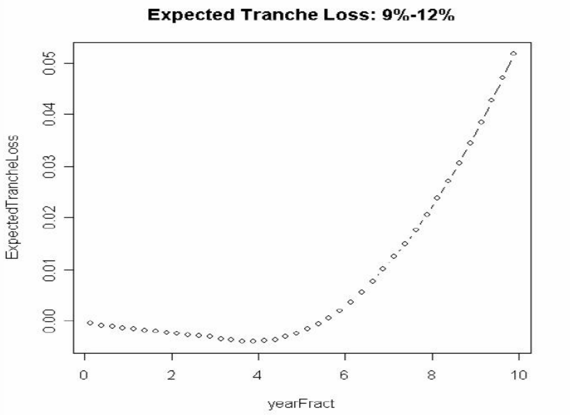
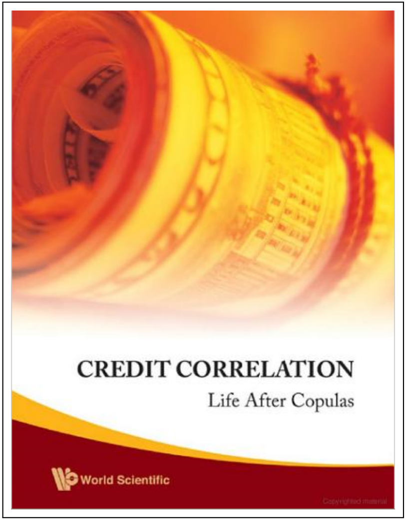


Figure: Expected tranche loss coming from Base correlation calibration, 3d August 2005, First published in 2006. The locally negative loss distribution means there are defaulted names RESURRECTING with positive probability

Some facts

Proceedings of a Conference held in London in 2006 by Merrill Lynch.
A number of proposals to improve the static copula models used (and abused) for credit derivatives have been presented. I was there.
Quants and Academics were well aware (and had been for years) of the models limitations and were trying to overcome them.



A few journalist have very short memory...

12 Sept 2005. Wall Street Journal

How a Formula [Base correlation + Gaussian Copula] Ignited Market That Burned Some Big Investors.

There are many other publications preceeding the crisis started in 2007. Such publications questioned the use of the Gaussian copula and the notion of implied and base correlation. For example, see our 2006 article

Implied Correlation: A paradigm to be handled with care, 2006, SSRN, `http://ssrn.com/abstract=946755`

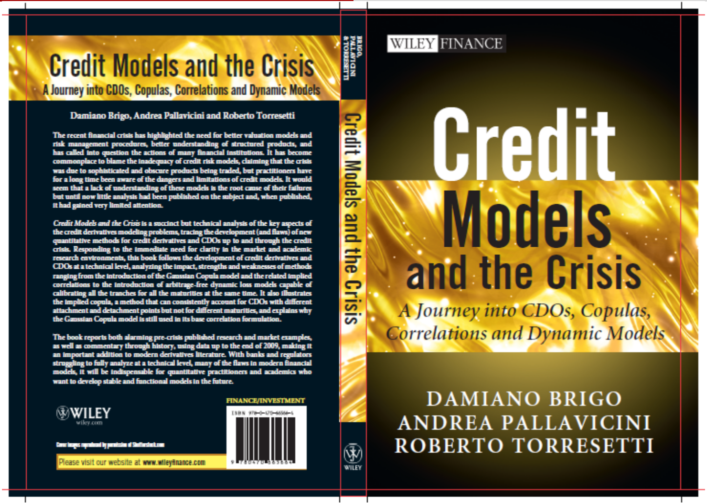


Figure: This book collects research published originally in 2006, warning against the flaws of the industry credit derivatives models. Related papers in the journals *Mathematical Finance, Risk Magazine, IJTAF*

Beyond copulas: GPL and GPCL Models (2006-on)

We model the total number of defaults in the pool by t as

$$Z_t := \sum_{j=1}^{n} \delta_j Z_j(t)$$

(for integers δ_j) where Z_j are independent Poissons. This is consistent with the Common Poisson Shock framework, where defaults are linked by a Marshall Olkin copula (Lindskog and McNeil).

$$\text{Example}: \quad n = 125, \;\; Z_t = 1\ Z_1(t) + 2\ Z_2(t) + \ldots + 125\ Z_{125}(t).$$

If Z_1 jumps there is just one default (idiosyncratic), if Z_{125} jumps there are 125 ones and the whole pool defaults one shot (total systemic risk), otherwise for other Z_i's we have intermediate situations (sectors).

The GPL and GPCL Models: Default clusters?

- Thrifts in the early 90s at the height of the loan and deposit crisis.
- Airliners after 2001.
- Autos and financials more recently. From the September, 7 2008 to the October, 8 2008, we witnessed seven credit events: Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir, Kaupthing.

S&P ratings and default clusters

Moreover, S&P issued a request for comments related to changes in the rating criteria of corporate CDO. Tranches rated 'AAA' should be able to withstand the default of the largest single industry in the pool with zero recoveries. Stressed but plausible scenario that a cluster of defaults in the objective measure exists.

The GPL and GPCL Models

Problem: infinite defaults. Solution 1: **GPL:** Modify the aggregated pool default counting process so that this does not exceed the number of names, by simply capping Z_t to n, regardless of cluster structures:

$$C_t := \min(Z_t, n)$$

Solution 2: **GPCL.** Force clusters to jump only once and deduce single names defaults consistently.
The first choice is ok at top level but it does not really go down towards single names. The second choice is a real top down model, but combinatorially more complex.

Calibration

The GPL model is calibrated to the market quotes observed on March 1 and 6, 2006. Deterministic discount rates are listed in Brigo, Pallavicini and Torresetti (2006). Tranche data and DJi-TRAXX fixings, along with bid-ask spreads, are (I=index,T=Tranche,Tl=Tranchelet)

	Att-Det	**March, 1 2006**		**March, 6 2006**		
		5y	7y	3y	5y	7y
I		35(1)	48(1)	20(1)	35(1)	48(1)
T	0-3	2600(50)	4788(50)	500(20)	2655(25)	4825(25)
	3-6	71.00(2.00)	210.00(5.00)	7.50(2.50)	67.50(1.00)	225.50(2.50)
	6-9	22.00(2.00)	49.00(2.00)	1.25(0.75)	22.00(1.00)	51.00(1.00)
	9-12	10.00(2.00)	29.00(2.00)	0.50(0.25)	10.50(1.00)	28.50(1.00)
	12-22	4.25(1.00)	11.00(1.00)	0.15(0.05)	4.50(0.50)	10.25(0.50)
Tl	0-1	6100(200)	7400(300)			
	1-2	1085(70)	5025(300)			
	2-3	393(45)	850(60)			

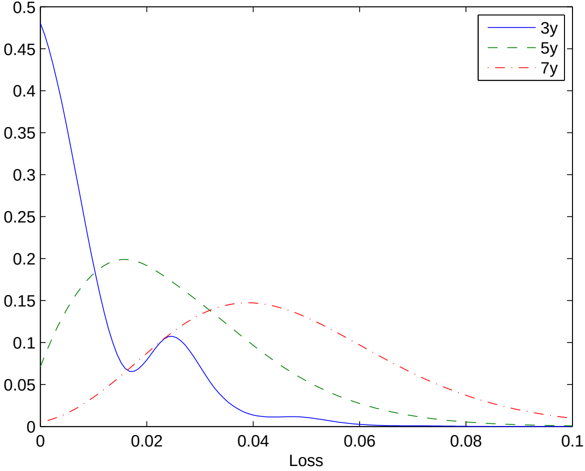
Calibration: All standard tranches up to seven years

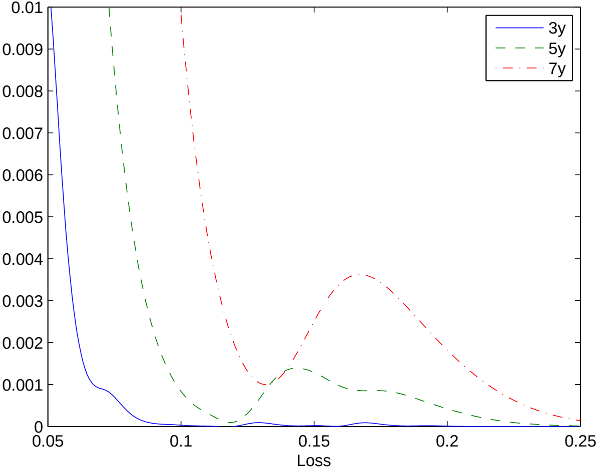
As a first calibration example we consider standard DJi-TRAXX tranches up to a maturity of 7y with constant recovery rate of 40%. The calibration procedure selects five Poisson processes. The 18 market quotes used by the calibration procedure are almost perfectly recovered. In particular all instruments are calibrated within the bid-ask spread (we show the ratio calibration error / bid ask spread).

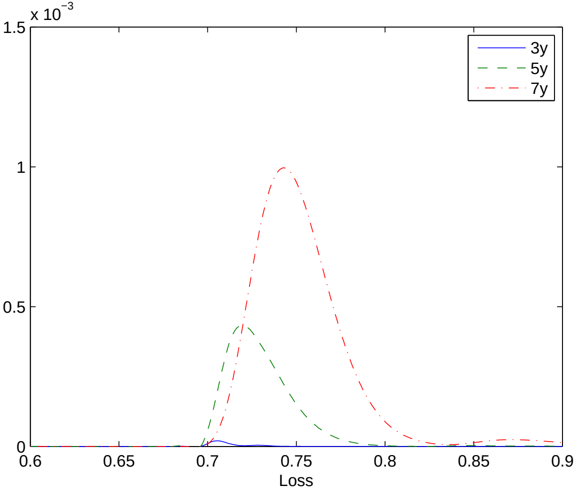
	Att-Det	**Maturities**		
		3y	5y	7y
Index		-0.4	-0.2	-0.9
Tranche	0-3	0.1	0.0	-0.7
	3-6	0.0	0.0	0.7
	6-9	0.0	0.0	-0.2
	9-12	0.0	0.0	0.0
	12-22	0.0	0.0	0.2

δ	$\boldsymbol{\Lambda(T)}$		
	3y	5y	7y
1	0.535	2.366	4.930
3	0.197	0.266	0.267
16	0.000	0.007	0.024
21	0.000	0.003	0.003
88	0.000	0.002	0.007

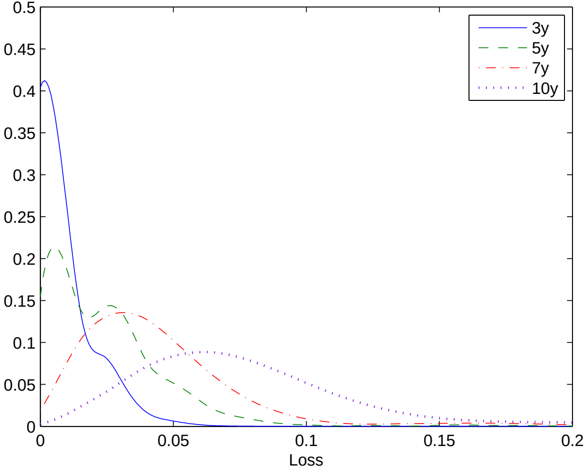
Loss distribution of the calibrated GPL model at different times



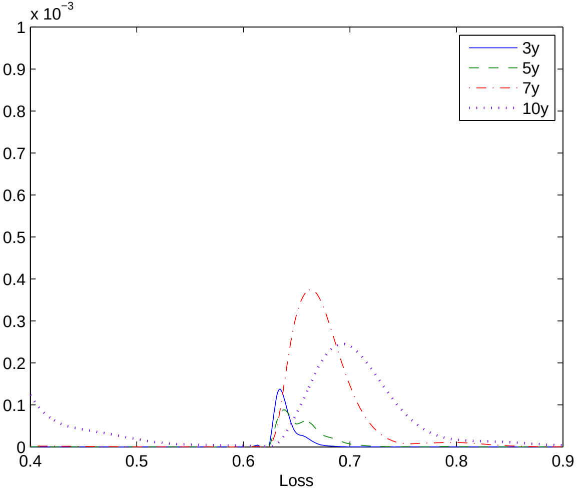

0.01
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
0.05
0.1
0.15
0.2
0.25
Loss
3y
5y
7y


x 10^−3
1.5
1
0.5
0
0.6
0.65
0.7
0.75
0.8
0.85
0.9
Loss
3y
5y
7y

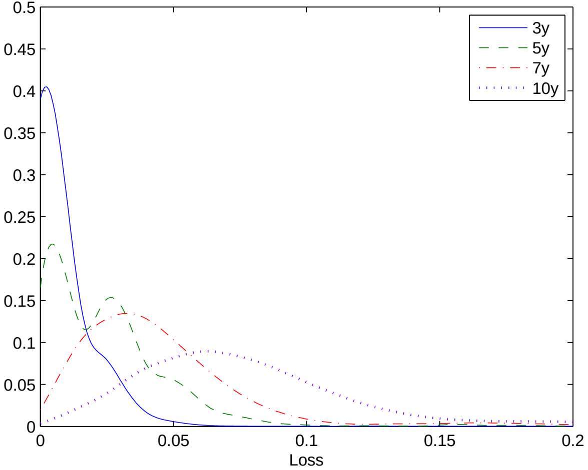
October 2 2006, GPL, Calibration up to 10y



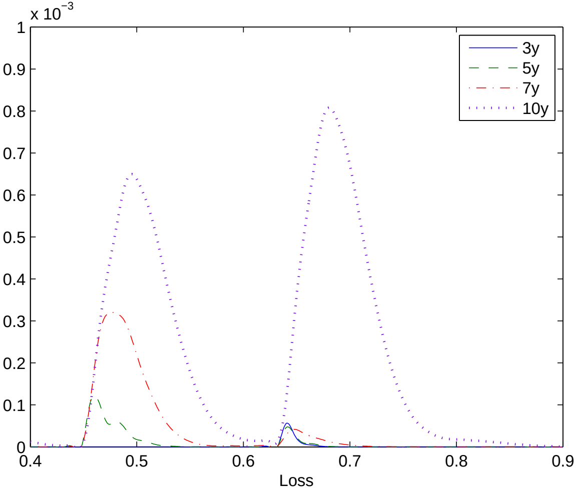
October 2 2006, GPL tail



October 2 2006, GPCL, Calibration up to 10y



October 2 2006, GPCL tail



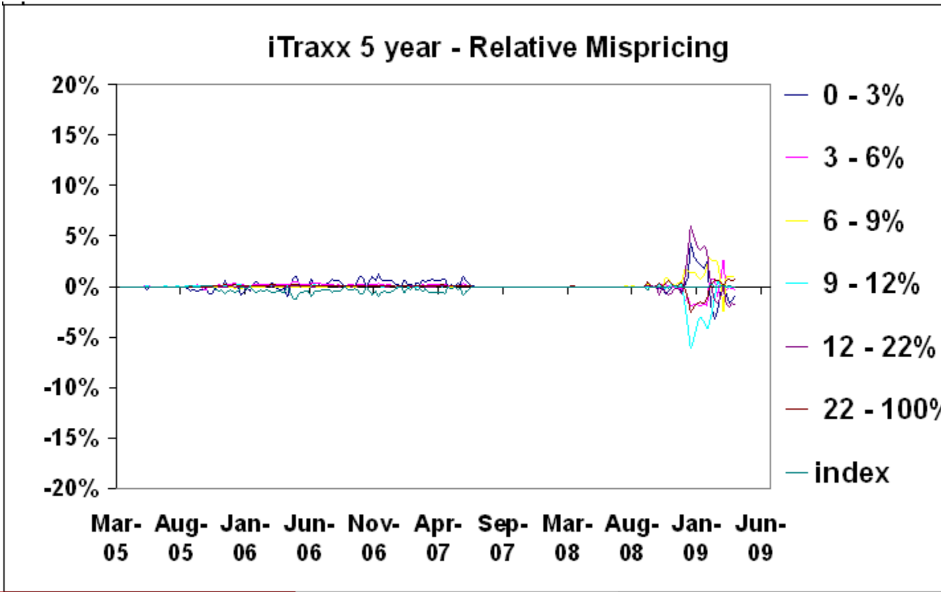
Calibration comments I

Sector / systemic calibration:

Notice the large portion of mass concentrated near the origin, the subsequent modes (default clusters) when moving along the loss distribution for increasing values, and the bumps in the far tail. Modes in the tail represent risk of default for large sectors. This is systemic risk as perceived by the dynamical model from the CDO quotes. With the crisis these probabilities have become larger, but they were already observable pre-crisis. Difficult to get this with parametric copula models.

History of calibration in-crisis with a different parametrization (α's fixed a priori):

Calibration comments II

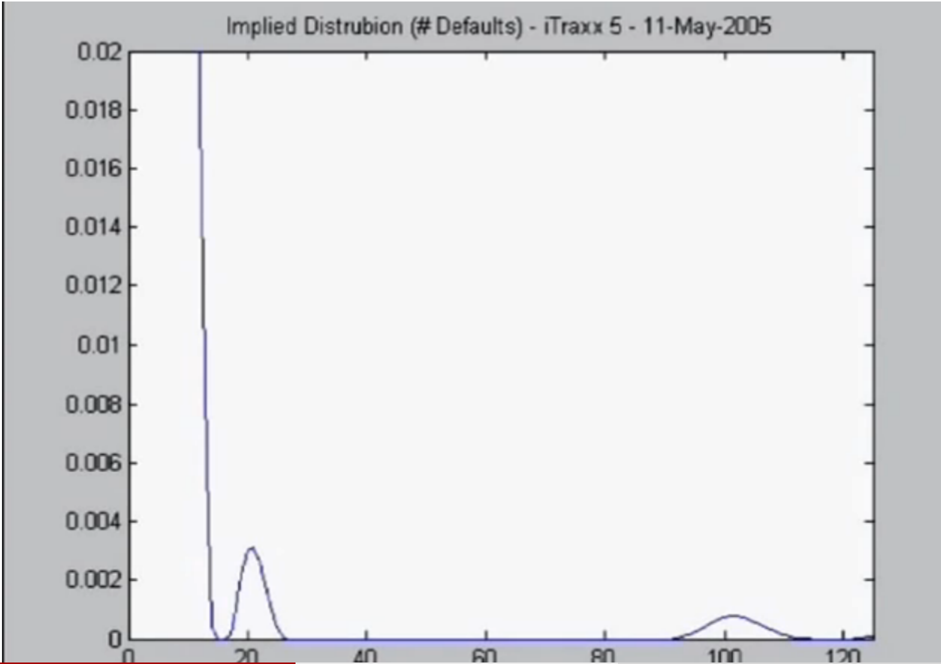


Loss distribution in GPL through the crisis

The loss distribution in the GPL model is arbitrage free, rich in structure and consistent with all market quotes, a feat impossible for implied correlation models.

The following movie shows how structured the loss dynamics can be, as highlighted by the GPL model.

Animation showing how the loss distribution evolved in 2005+ is here
`http://www.youtube.com/watch?v=YZO-HeaGHkk&t=62m40s`


Implied Distrubion (# Defaults) - iTraxx 5 - 11-May-2005
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0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
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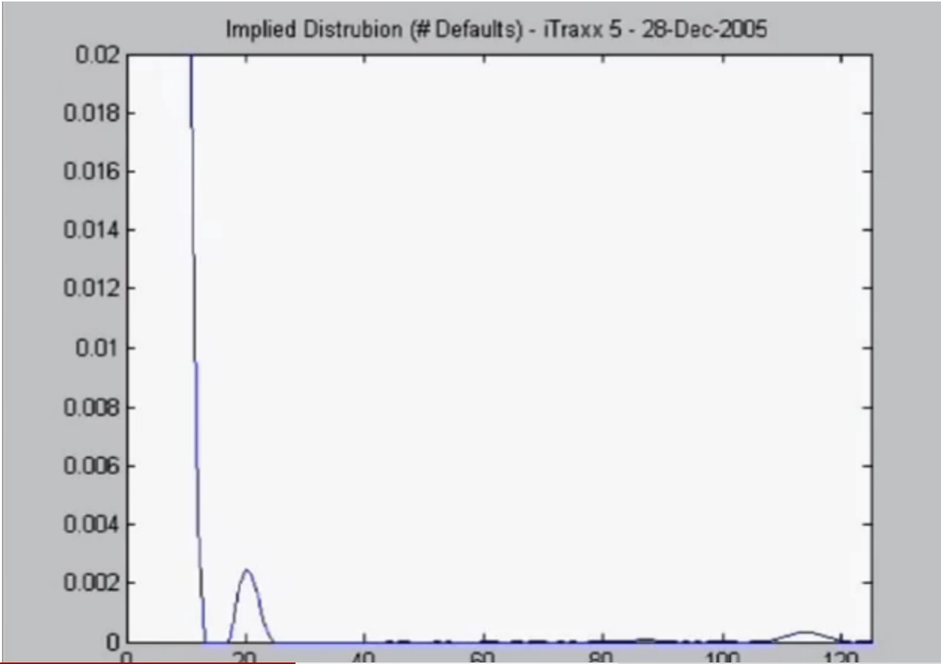
40
60
80
100
120



Implied Distrubion (# Defaults) - iTraxx 5 - 28-Dec-2005
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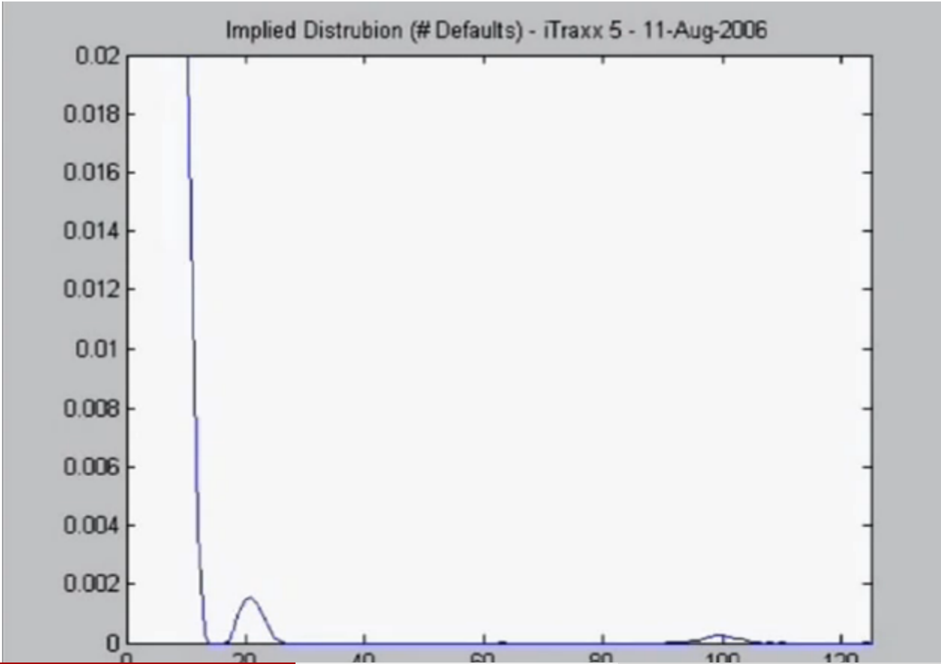
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 11-Aug-2006
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0.016
0.014
0.012
0.01
0.008
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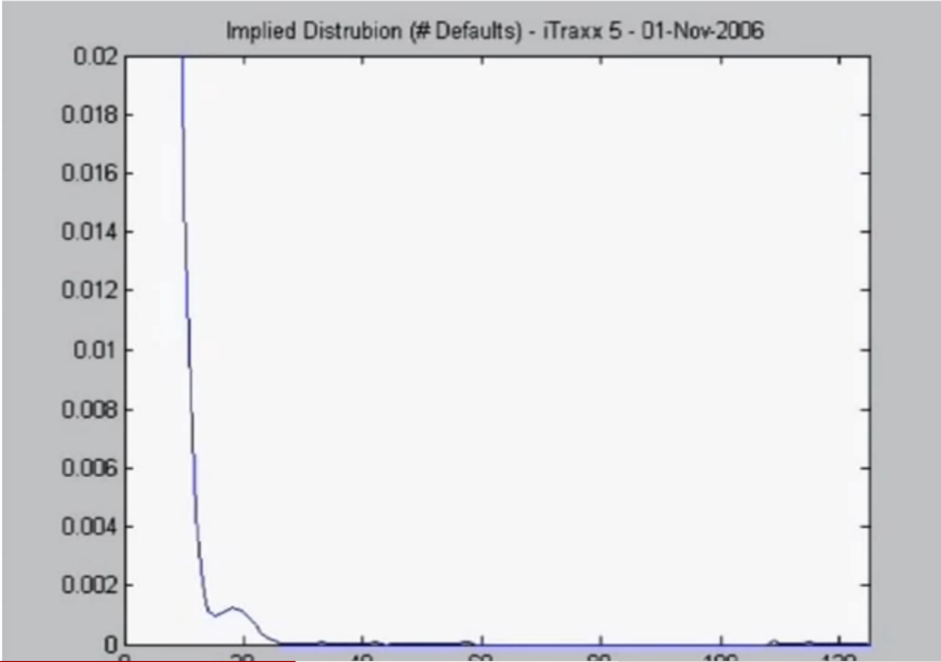
0.004
0.002
0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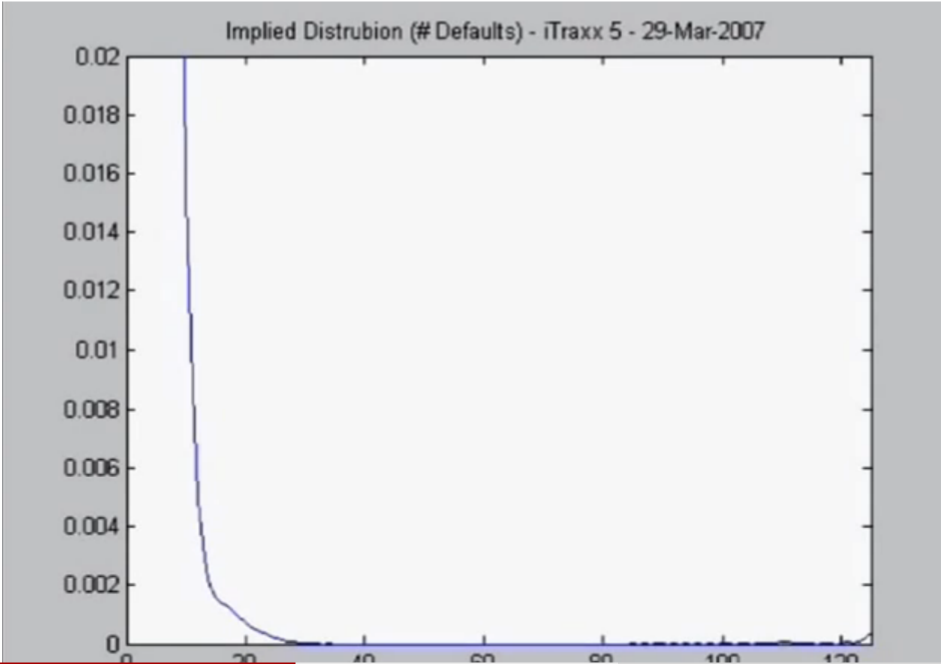
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 29-Mar-2007
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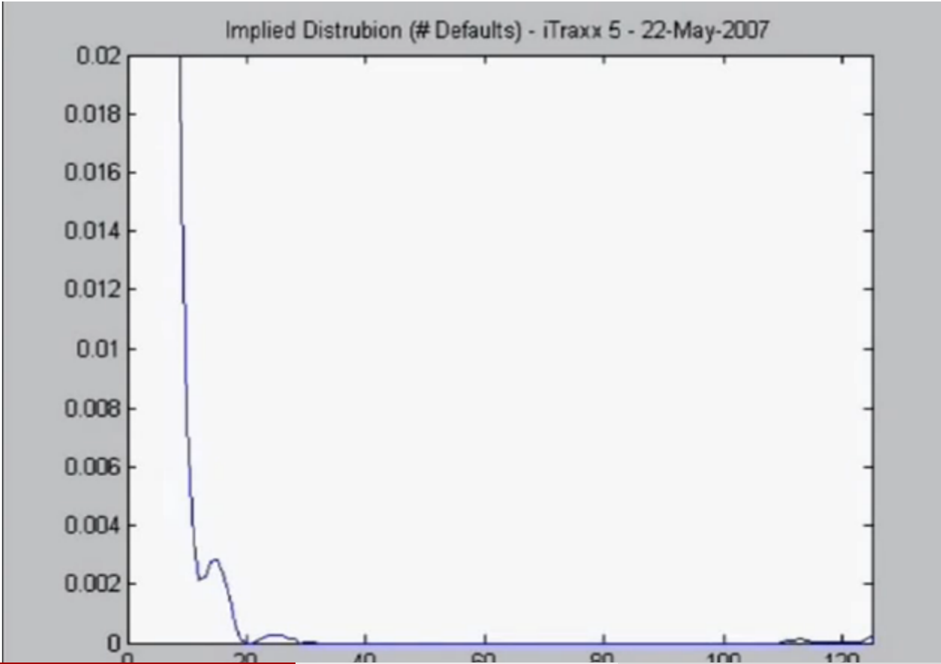
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0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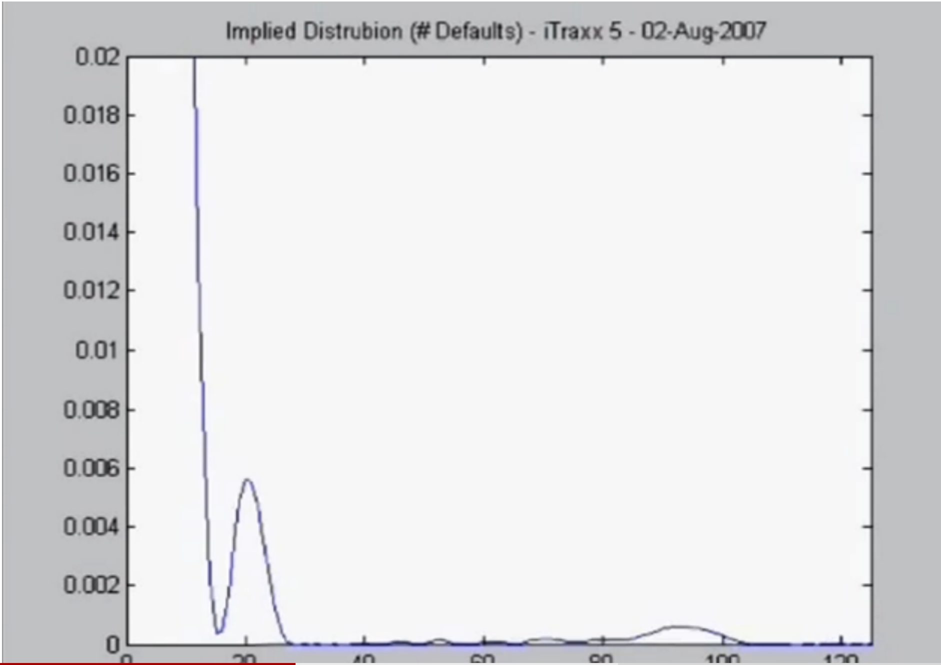
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 02-Aug-2007
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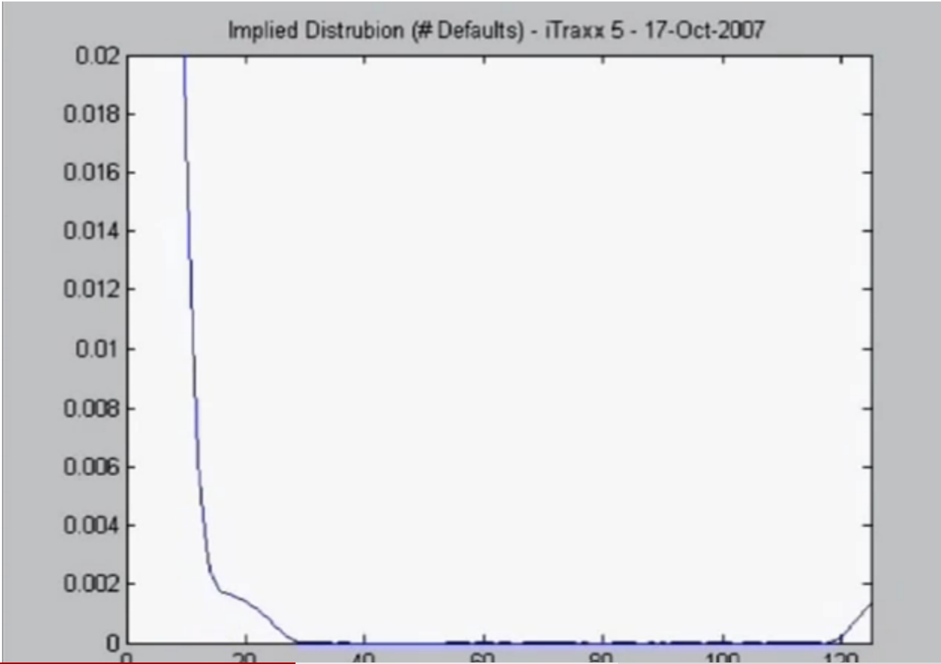
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 17-Oct-2007
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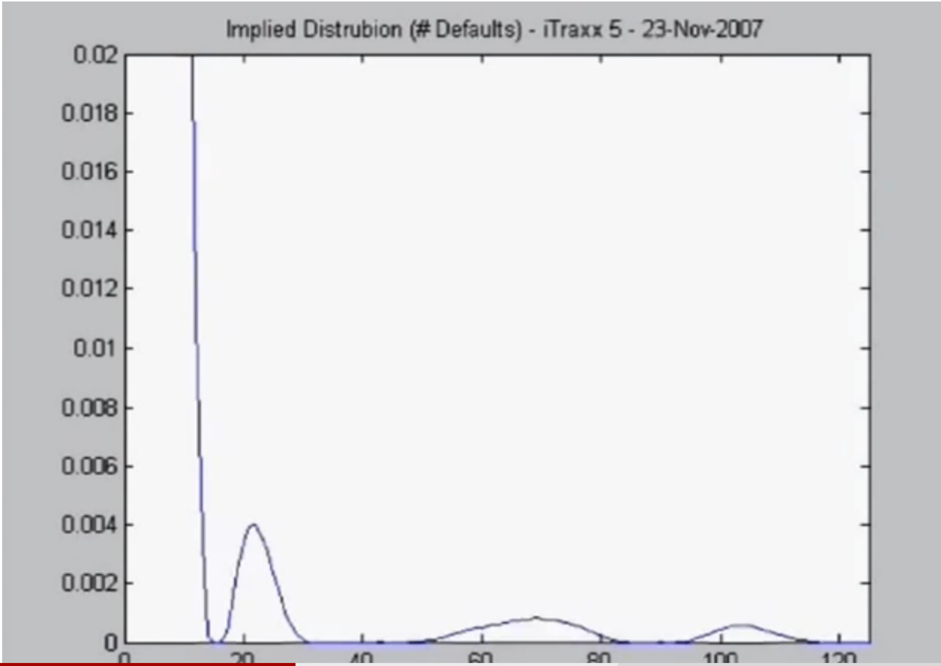
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 23-Nov-2007
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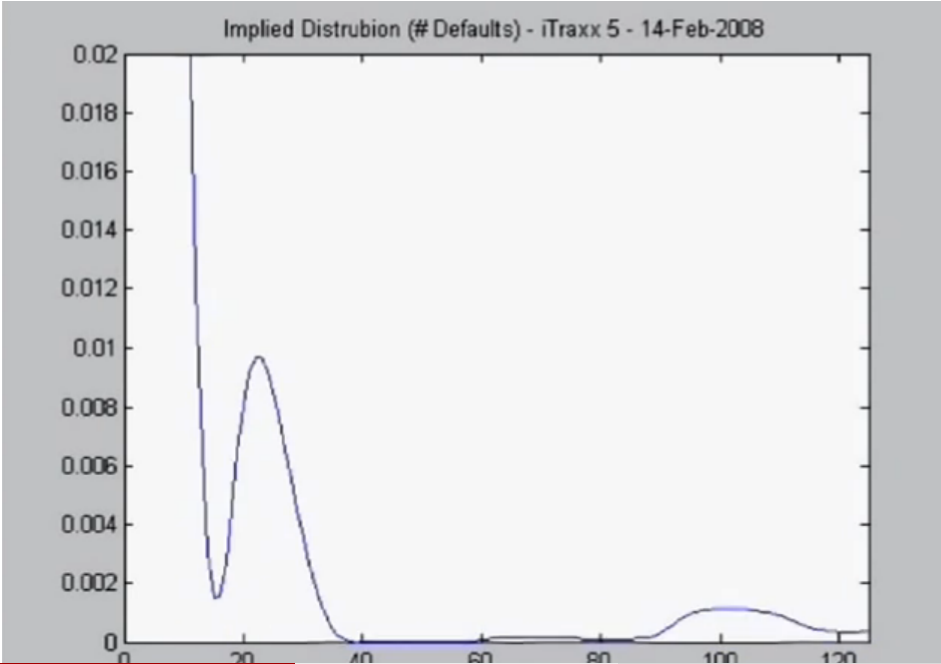
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
20
40
60
80
100
120


Implied Distrubion (# Defaults) - iTraxx 5 - 14-Feb-2008
0.02
0.018
0.016
0.014
0.012
0.01
0.008
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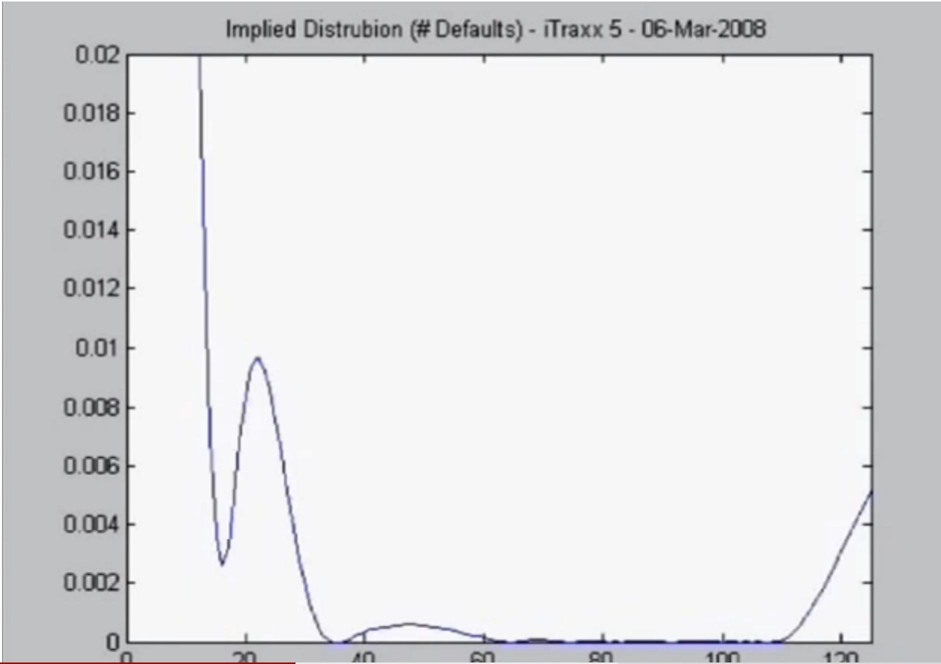
40
60
80
100
120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0.014
0.012
0.01
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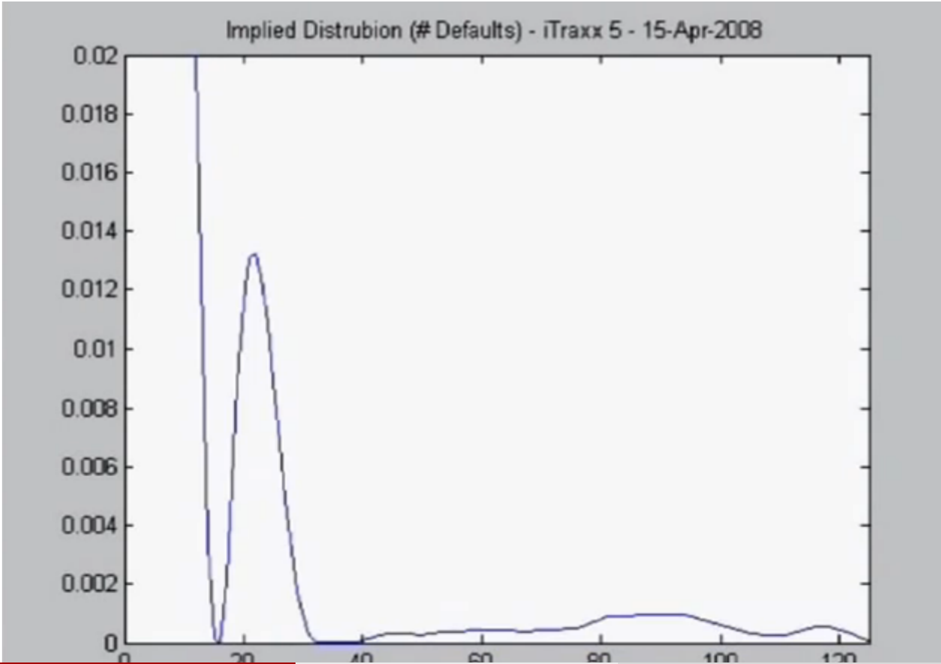
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 15-Apr-2008
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0.016
0.014
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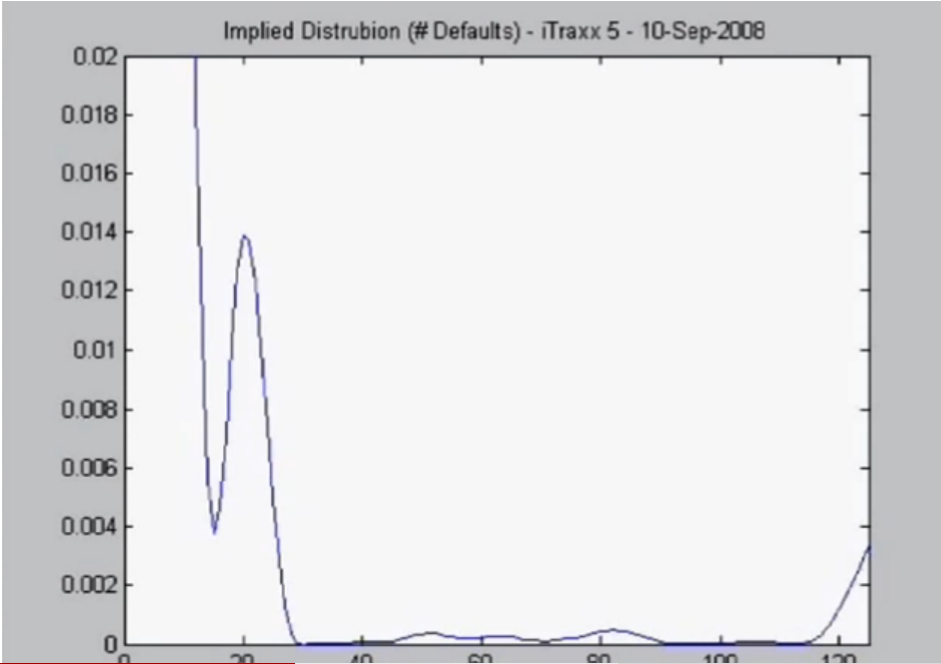
0.008
0.006
0.004
0.002
0
0
20
40
60
80
100
120


Implied Distrubion (# Defaults) - iTraxx 5 - 10-Sep-2008
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0.016
0.014
0.012
0.01
0.008
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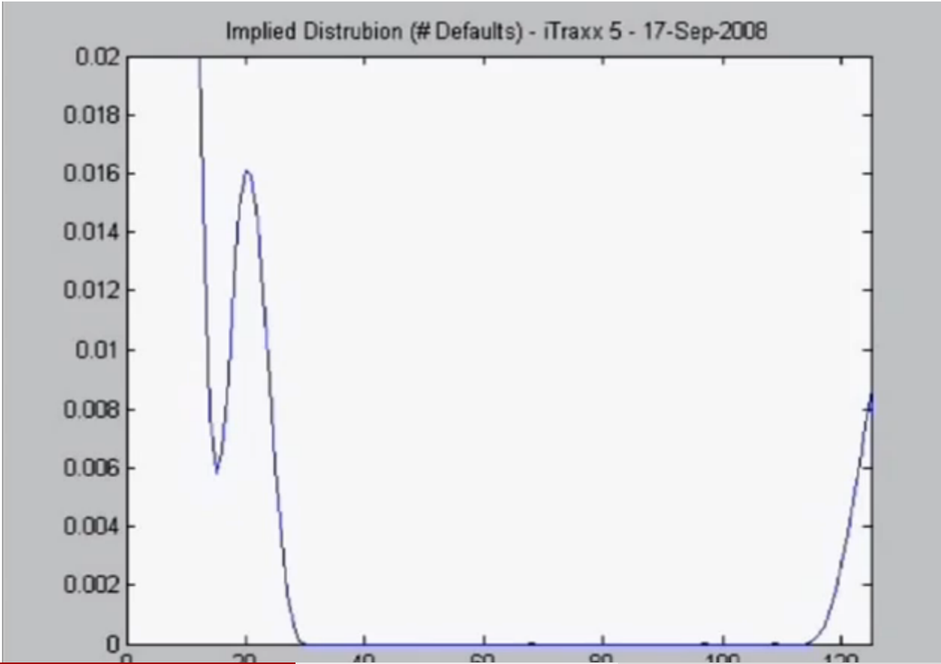
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 17-Sep-2008
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0.016
0.014
0.012
0.01
0.008
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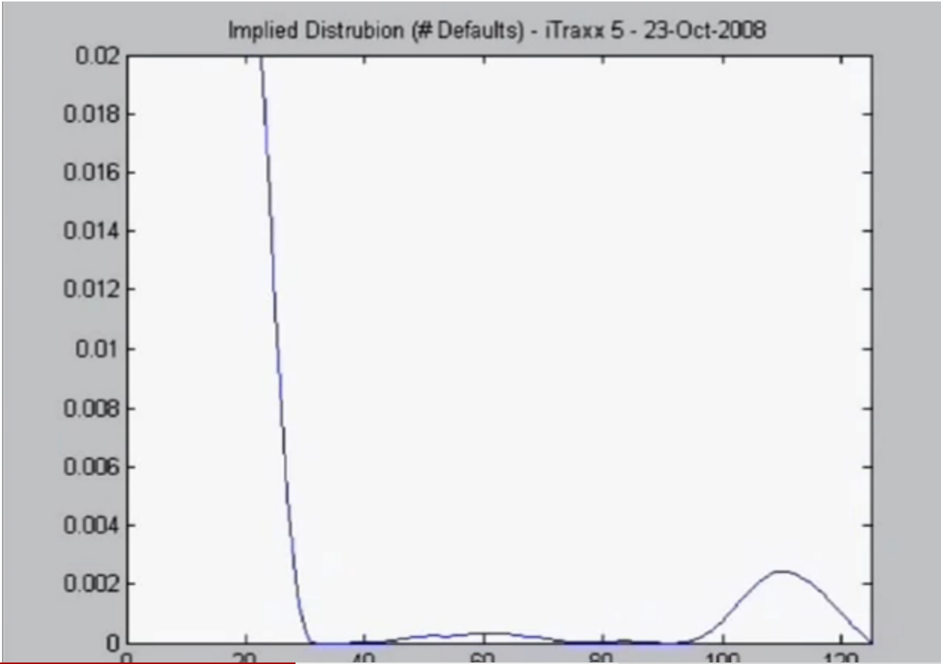
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 23-Oct-2008
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0.016
0.014
0.012
0.01
0.008
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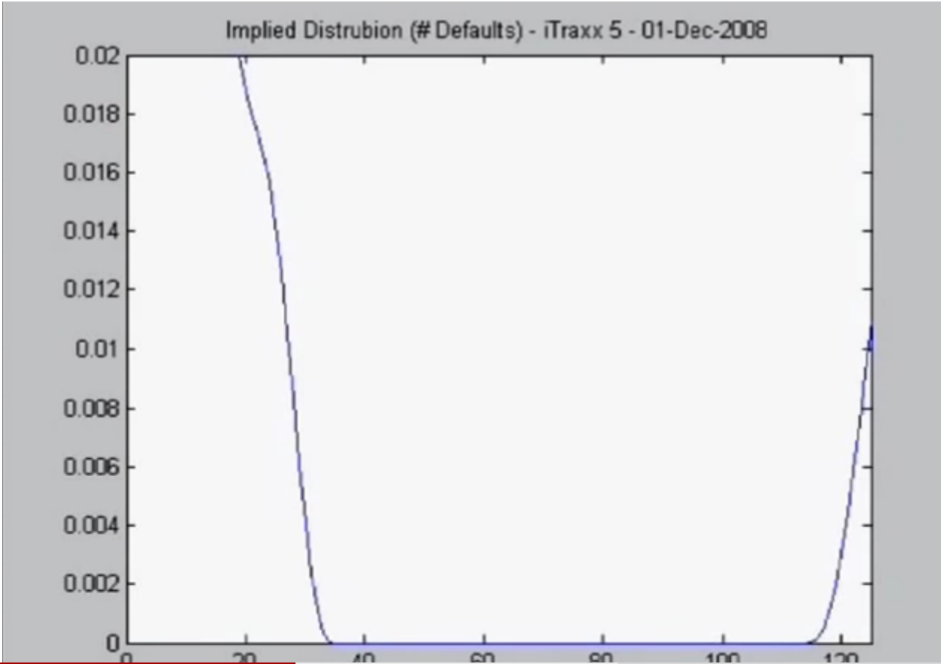
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 01-Dec-2008
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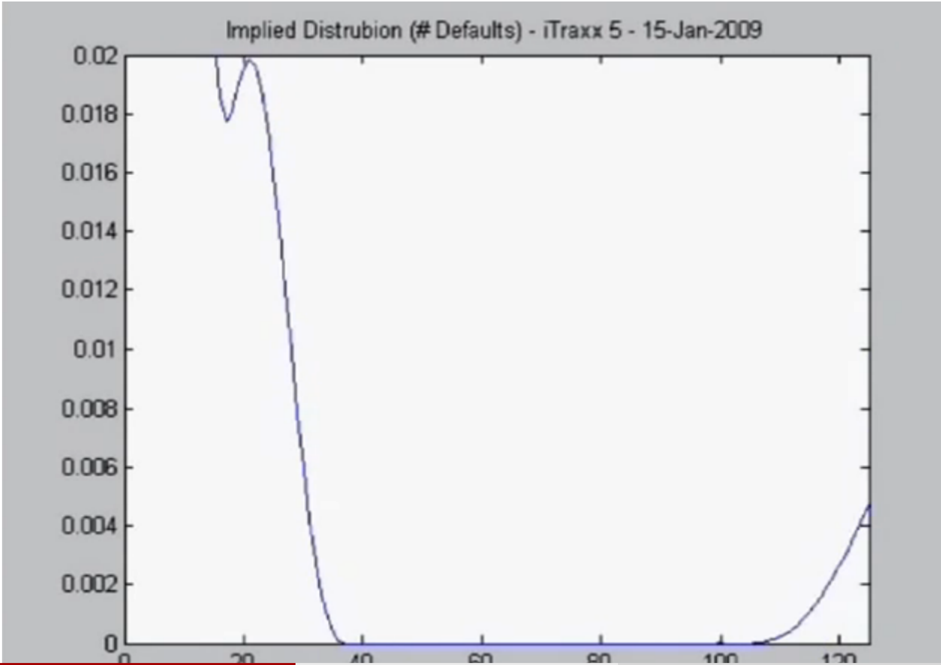
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0


Implied Distrubion (# Defaults) - iTraxx 5 - 15-Jan-2009
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
20
40
60
80
100
120

Calibration in-crisis

A full treatment of the calibration in crisis and a model extension is given in the book ”Credit Models and the Crisis” by Brigo, Pallavicini and Torresetti (2010), Wiley.

The synthetic CDO case?

- We have illustrated how a complex situation in CDO markets has been trivialized by media and even regulators
- Models (such as base correlation) were indeed inadequate, but the industry and researchers had been looking for much more powerful and consistent alternatives
- We have seen the example of the GPL model, a fully consistent arbitrage free dynamic model for CDOs
- So why didn't the media pick this up? Why didn't the media realize the glitches they were signalling were the same the Wall Street Journal had reported years earlier in 2005?
- We hope the CDO case study illustrates the lack of rigour in a broad part of investigative journalism, especially in connection with complex and technical subjects.
- We cannot blame (even poor) modeling for policy, regulation, incentives, banking model, governance, lack of culture...
- We have a duty to make our research visible and heard to society

Is Maths Guilty and Wrong?

- Mathematics is not wrong. We have to be careful in understanding what is meant when saying that one uses *mathematical models*.
- Mathematical models are a simplification of reality, and as such, are always ”wrong”, even if they try to capture the salient features of the problem at hand.
- **”All models are wrong, but some models are useful”** (Prof. George E.P. Box)
- The core mathematical theory behind derivatives valuation is correct, but the assumptions on which the theory is based may not reflect the real world when the market evolves over the years.

Is Mathematics guilty?

- Although the models used in Credit Derivatives and counterparty risk have limits that have been highlighted before the crisis by several researchers, the ongoing crisis is due to factors that go well beyond any methodological inadequacy: the killer formula

$$\boxed{\int_{-\infty}^{+\infty} \prod_{i=1}^{125} \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Lambda_i(T))) - \sqrt{\rho_i}\, m}{\sqrt{1-\rho_i}}\right) \varphi(m) dm.}$$

Versus

The Crisis:
US real estate policy, Originate to Distribute (to Hold?) system fragility, volatile monetary policies,
myopic compensation and incentives system, lack of homogeneity in regulation, underestimation of liquidity risk, lack of data, fraud corrupted data...(Szegö 2009, The crash sonata in D major, JRMFI).

And what about the data?

Data and Inputs quality

For many financial products, and especially RMBS (Residential Mortgage Backed Securities), quite related to the asset class that triggered the crisis, the problem is in the data rather than in the models.

Risk of fraud

At times data for valuation in mortgages CDOs (RMBS and CDO of RMBS) can be distorted by fraud (see for example the FBI Mortgage fraud report, 2007, www.fbi.gov/publications/fraud/mortgage_fraud07.htm.

Pricing a CDO on this underlying:

Figure: The above photos are from condos that were involved in a mortgage fraud. The appraisal described ”recently renovated condominiums” to include Brazilian hardwood, granite countertops, and a value of 275,000 USD

And what about the data?

At times it is not even clear what is in the portfolio: From the offering circular of a huge RMBS (more than 300.000 mortgages)

Type of property	% of Total
Detached Bungalow	2.65%
Detached House	16.16%
Flat	13.25%
Maisonette	1.53%
Not Known	$\boxed{2.49}$%
New Property	0.02%
Other	0.21%
Semi Detached Bungalow	1.45%
Semi Detached House	27.46%
Terraced House	34.78%
Total	100.00%

Mathematics or Magic?

All this is before modeling. Models obey a simple rule that is popularly summarized by the acronym GIGO (Garbage In $\rightarrow$ Garbage Out). As Charles Babbage (1791–1871) famously put it:

On two occasions I have been asked,
"Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?"
I am not able rightly to apprehend
the kind of confusion of ideas
that could provoke such a question.

So, in the end, how can the crisis be mostly due to models inadequacy, and to quantitative analysts and academics pride and unawareness of models limitations?

Interesting times...

We are indeed going through very interesting times. New derivatives are appearing, eg Longevity swaps, but there's much more beyond derivatives: *We need better models, not no models.*

We need to model risks that were absent/neglected in classical theory: Counterparty credit risk, liquidity risk, funding risk... Nonlinearities!

We need to understand systemic risk, contagion, the dynamics of dependence, and how to deal with scarcity of data and data proxying...

We need to enhance consistency of models in different areas

Optimal execution, algo trading, high freq trading, risk optimization...

All these areas, and many more, require quantitative input and good quantitative finance.

Interesting times...

This is not a good idea:

Rather then accusing financial engineering for failures that are more managerial, political and behavioural in nature, we should derive better models that may account for the types of risks that had been neglected earlier.
But before doing that, we need to learn the classical theory pretty well.

... we need to learn the classical theory pretty well...

So let's get started!

Once upon a time....

PART 2. INTEREST RATES AND TERM STRUCTURE

In this first part we deal with the theory of interest rates, especially as concerns derivatives and pricing/hedging.

Risk Neutral Valuation

Bank account $dB(t) = r_t B(t) dt$, $B(t) = B_0 \exp\left(\int_0^t r_s ds\right)$.

Risk neutral measure Q associated with numeraire B, $Q = Q^B$.

Recall shortly the risk-neutral valuation paradigm of Harrison et al (1983), generalizing the result of Black and Scholes we have seen above, characterizing no-arbitrage theory:

A future stochastic payoff H_T, built on an underlying fundamental asset, paid at a future time T and satisfying some technical conditions, has as unique price at current time t the *risk neutral world* expectation

$$E_t^B\left[\frac{B(t)}{B(T)} H(\text{Asset})_T\right] = E_t^Q\left[\exp\left(-\int_t^T r_s\, ds\right) H(\text{Asset})_T\right]$$

Risk neutral valuation I

$$E_t^B\left[\frac{B(t)}{B(T)}H(\text{Asset})_T\right] = E_t^Q\left[\exp\left(-\int_t^T r_s\, ds\right) H(\text{Asset})_T\right]$$

As we have seen above. “Risk neutral world” means that all fundamental underlying assets must have as locally deterministic drift rate the risk-free interest rate r:

$$d\ \text{Asset}_t = \boxed{r_t}\ \text{Asset}_t\ dt +$$

$$+\text{Asset-Volatility}_t\ (d\ \text{Brownian-motion-under-Q})_t$$

Nothing strange at first sight. To value **future unknown** quantities now, we discount at the relevant interest rate and then take **expectation.** The mean is a reasonable estimate of unknown quantities with known distributions.

Risk neutral valuation I

But what is surprising is that we do not take the mean in the **real world**, where statistics and econometrics based on the observed data are used. Indeed, in the real world probability measure P, we have

$$d\ \text{Asset}_t = \boxed{\mu_t}\ \text{Asset}_t\ dt +$$

$$+\text{Asset-Volatility}_t\ (d\ \text{Brownian-motion-under-P})_t.$$

But when we consider risk-neutral valuation, or no-arbitrage pricing, we do not use the real-world P-dynamics with μ but rather the risk-neutral world Q-dynamics with r.

We have a feeling for why this happens, since we derived the Black Scholes formula, a special case of the above framework, earlier. Basically we can avoid μ thanks to a replicating self-financing strategy in the underlying asset whose value does not depend on μ.

Risk neutral valuation II

From the risk neutral valuation formula we see that one fundamental quantity is r_t, the instantaneous interest rate.

As a very important special case of the general valuation formula, if we take $H_T = 1$, we obtain the Zero-Coupon Bond

Zero-coupon Bond, LIBOR rate I

A T**–maturity zero–coupon bond** is a contract which guarantees the payment of one unit of currency at time T. The contract value at time $t < T$ is denoted by $P(t, T)$:

$$P(T, T) = 1,$$

$$P(t, T) = E_t^Q \left[\frac{B(t)}{B(T)} 1 \right] = E_t^Q \exp\left(- \int_t^T r_s \, ds \right) = E_t^Q D(t, T)$$

All kind of rates can be expressed in terms of zero–coupon bonds and vice-versa. ZCB's can be used as fundamental quantities.
The **spot–Libor rate** at time t for the maturity T is the constant rate at which an investment has to be made to produce an amount of one unit

Zero-coupon Bond, LIBOR rate II

of currency at maturity, starting from $P(t, T)$ units of currency at time t, when accruing occurs **proportionally** to the investment time.

$$P(t, T)(1 + \ (T - t)\ L(t, T)) = 1, \quad L(t, T) = \frac{1 - P(t, T)}{(T - t)\ P(t, T)} \quad .$$

Notice:

$$r(t) = \lim_{T \to t^+} L(t, T) \approx L(t, t + \epsilon),$$

ϵ small.

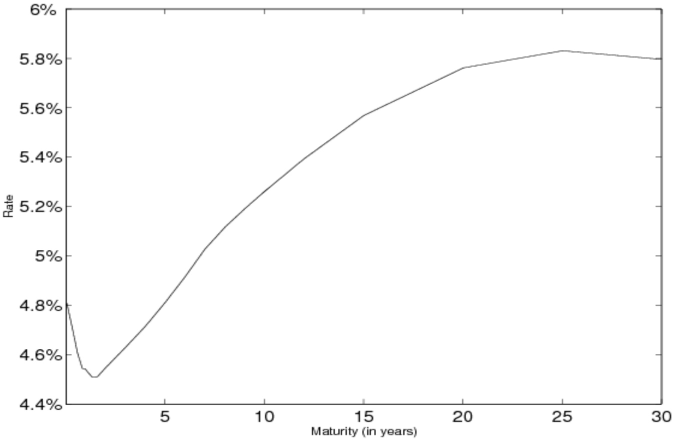
LIBOR, zero coupon curve (term structure) I

The **zero–coupon curve** (often referred to as "yield curve" or "term structure") at time t is the graph of the function

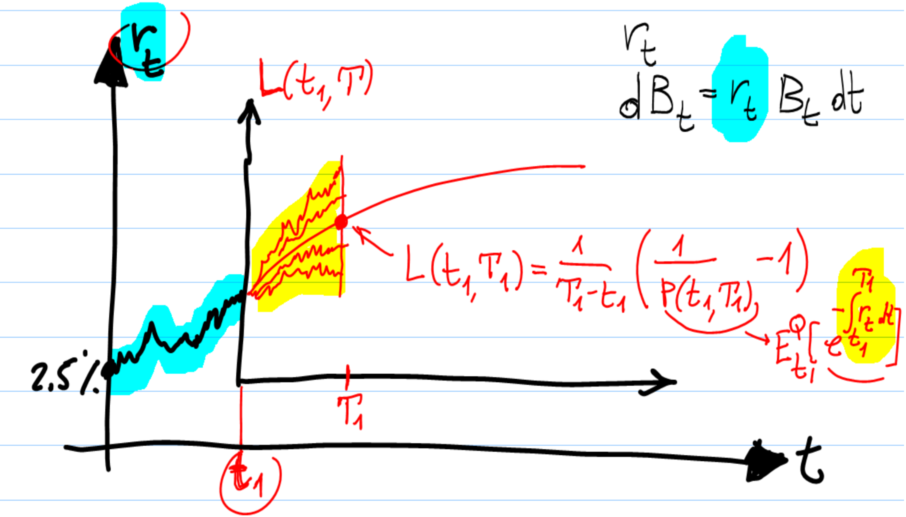
$$T \mapsto L(t,T), \quad \text{initial point} \;\; r_t \approx L(t, t+\epsilon).$$

This function is called *term structure of interest rates* at time t.

Zero-coupon curve $T \mapsto L(t, t + T)$ stripped from market EURO rates on 13 Feb 2001 I



Zero-coupon curve $T \mapsto L(t, t+T)$ stripped from market EURO rates on 13 Feb 2001 II



LIBOR, zero coupon curve (term structure) I

This figure illustrates the different variables at play:

- the fundamental process is the short rate $t \mapsto r_t$. We show one path (in black, with a cyan contour) of the short rate r from time 0 (starting from $r_0 = 2.5\% = 0.025$) to t_1.
- Then at t_1 we show the term structure of interest rates $T \mapsto L(t_1, T)$ (in red), highlighting a point $L(t_1, T_1)$.
- As we have seen before, $L(t_1, T_1)$ is a function of $P(t_1, T_1)$ which, in turn, is $E_{t_1}[\exp(-\int_{t_1}^{T_1} r_t dt)]$.
- This means that the point $L(t_1, T_1)$ of the term structure is obtained through an expectation of an integral of every path of r from t_1 to T_1.
- Some of these paths are shown as zig-zagging lines in red from t_1 to T_1 in the picture.

Products not depending on the curve dynamics: FRA

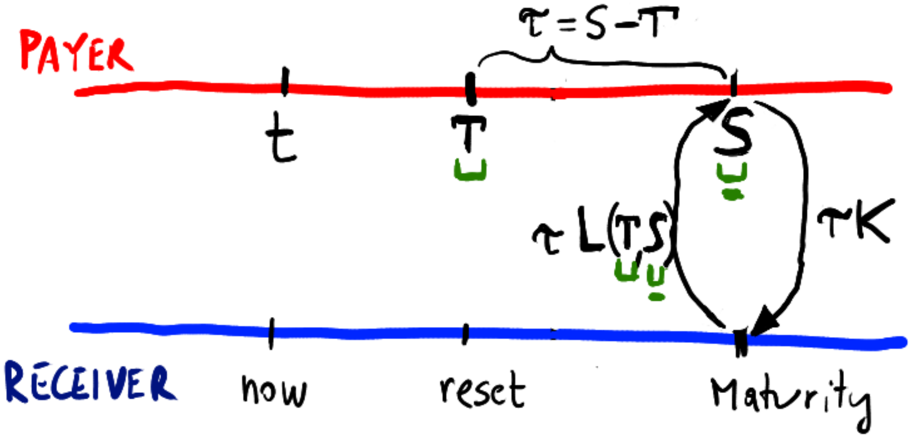
$$\boxed{\text{At time } S, \ \text{ with reset time } T \ \ (S > T)}$$

Fixed payment $\quad \longrightarrow \ (S-T)K \ \longrightarrow$

$\longleftarrow \ (S-T) \ L(T,S) \ \longleftarrow \quad$ Float. payment

A **forward rate agreement** FRA is a contract involving three time instants: The current time t, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest rate payment for the period $T \mapsto S$ with fixed rate K at maturity S against an interest rate payment over the same period with rate $L(T,S)$.
Basically, this contract allows one to lock–in the interest rate between T and S at a desired value K.

Products not depending on the curve dynamics: FRA



Products not depending on the curve dynamics: FRA I

The FRA is said to be a Receiver FRA if we pay floating $L(T, S)$ and receive Fixed K. It is a Payer FRA if we pay K and receive floating $L(T, S)$.
By easy static no-arbitrage arguments, the price of a receiver FRA is:

$$\mathrm{FRA}(t, T, S, K) = P(t, S)(S - T)K - P(t, T) + P(t, S)\ .$$

$(S - T)$ may be replaced by a year fraction τ. The price of a payer FRA is exactly the opposite, since cash flows go into the opposite direction.
The Proof is as follows.
The Receiver Fra Price is obtained by taking the risk neutral expectation of the FRA Discounted Cash Flows. As payments happen in S, we need to discount them back to t through $D(t, S)$.

$$\mathrm{FRA}(t, T, S, K) = E_t[D(t, S)\tau K - D(t, S)\tau L(T, S)] =$$

FRA Pricing: I

FRA Pricing: II

$$E_t[D(t,S)\tau K - D(t,S)\tau L(T,S)] =$$
$$= \tau K E_t[D(t,S)] - E_t[D(t,S)\tau L(T,S)] =$$
$$= \tau K P(t,S) - E_t[D(t,S)\tau L(T,S)] =$$

now use $D(t,S) = D(t,T)D(T,S)$ (ok for D, not for P)

$$= \tau K P(t,S) - E_t[\tau D(t,T)D(T,S)L(T,S)] =$$
$$= \tau K P(t,S) - E_t[E_T\{\tau D(t,T)D(T,S)L(T,S)\}] =$$
$$= \tau K P(t,S) - E_t[\tau D(t,T)L(T,S)E_T\{D(T,S)\}] =$$
$$= \tau K P(t,S) - E_t[\tau D(t,T)L(T,S)P(T,S)] =$$
$$= \tau K P(t,S) - E_t[D(t,T)P(T,S)(1/P(T,S) - 1)] =$$
$$= \tau K P(t,S) - E_t[D(t,T)] + E_t[D(t,T)P(T,S)] =$$
$$= \tau K P(t,S) - E_t[D(t,T)] + E_t[D(t,T)E_T[D(T,S)]] =$$
$$= \tau K P(t,S) - E_t[D(t,T)] + E_t[E_T[D(t,T)D(T,S)]] =$$

FRA Pricing I

Note that this derivation did not require any modeling assumptions. We have made no assumption on the dynamics of interest rates. We have only used very general no-arbitrage principles to derive this formula.

The value of K which makes the contract fair (=0) is the **forward LIBOR interest rate** prevailing at time t for the expiry T and maturity S: $K = F(t; T, S)$. This is derived by solving in K

$$\tau K P(t, S) - P(t, T) + P(t, S) = 0.$$

$$K = F(t; T, S) := \frac{1}{S - T}\left(\frac{P(t, T)}{P(t, S)} - 1\right) .$$

Notice that incidentally we have found, with the above derivation, that

$$E_t[D(t, S)L(T, S)] = P(t, S)F(t, T, S).$$

Are Forward rates expectations of future interest rates? I

It is important to notice that while

$$\mathbb{E}_t^Q[D(t,S)L(T,S)] = P(t,S)F(t,T,S),$$

we also have

$$\mathbb{E}_t^Q[L(T,S)] \neq F(t,T,S).$$

The second one would follow from the first one only if D and L were independent. Clearly this is not the case. We will be able to write

$$\mathbb{E}_t^{Q^S}[L(T,S)] = F(t,T,S)$$

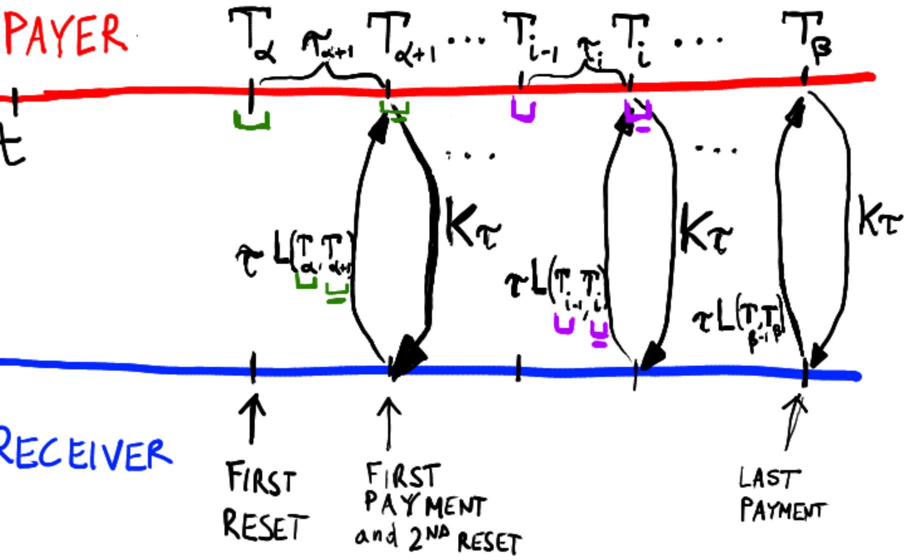
only under a different probability measure Q^S, called S forward measure. We'll deal with this later in the course.

Products not depending on the curve dynamics: IRS I

An Interest Rate Swap (PFS) is a contract that exchanges payments between two differently indexed legs, starting from a future time–instant. At future dates $T_{\alpha+1}, ..., T_\beta$,

$$\begin{array}{cccc} & \longrightarrow \;\; \tau_j K \;\; \longrightarrow & \\ \text{at } T_j: \;\; \text{Fixed Leg} & & \text{Float. Leg} \\ & \longleftarrow \;\; \tau_j \;\; L(T_{j-1}, T_j) \;\; \longleftarrow & \\ & \text{or taking } \; E_{T_\alpha}[\cdot]: & \\ & \tau_j \;\; F(T_\alpha; T_{j-1}, T_j) & \end{array}$$

where $\tau_i = T_i - T_{i-1}$. The IRS is called “payer IRS” from the company paying K and “receiver IRS” from the company receiving K.


PAYER
T_α $\tau_{\alpha+1}$ $T_{\alpha+1}$ ... T_{i-1} τ_i T_i ... T_β
t
$K\tau$
$\tau L(T_\alpha, T_{\alpha+1})$
$K\tau$
$\tau L(T_{i-1}, T_i)$
$K\tau$
$\tau L(T_{\beta-1}, T_\beta)$
RECEIVER
FIRST RESET
FIRST PAYMENT and 2ND RESET
LAST PAYMENT

Products not depending on the curve dynamics: IRS I

The *discounted* payoff at a time $t < T_\alpha$ of a receiver IRS is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)\, \tau_i (K - L(T_{i-1}, T_i)), \quad \text{or alternatively}$$

we may proceed as follows. (i) value the swap at the future first reset T_α. (ii) Take the T_α IRS price, which is a random payoff when seen from t, and dicount it back at t. This will help later with swaptions and this is why we do this. We obtain

$$D(t, T_\alpha) E_{T_\alpha}[\sum_{i=\alpha+1}^{\beta} D(T_\alpha, T_i)\, \tau_i (K - L(T_{i-1}, T_i))] =$$

$$= D(t, T_\alpha) \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\, \tau_i (K - F(T_\alpha; T_{i-1}, T_i)).$$

Products not depending on the curve dynamics: FRA's and IRS's I

Now rather than taking risk neutral expectations and going through the calculations, we simply note that IRS can be valued as a collection of FRAs. In particular, a receiver IRS can be valued as a collection of (receiver) FRAs.

$$\text{ReceiverIRS}(t, [T_\alpha, \dots, T_\beta], K) = \sum_{i=\alpha+1}^{\beta} \text{FRA}(t, T_{i-1}, T_i, K) =$$

$$= \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) - P(t, T_\alpha) + P(t, T_\beta), \quad \text{or alternatively}$$

$$= \sum_{i=\alpha+1}^{\beta} P(t, T_i)\, \tau_i (K - F(t; T_{i-1}, T_i)).$$

Products not depending on the curve dynamics: FRA's and IRS's II

Analogously,

$$\text{PayerIRS}(t, [T_\alpha, \dots, T_\beta], K) =$$

$$= \sum_{i=\alpha+1}^{\beta} P(t, T_i)\, \tau_i (F(t; T_{i-1}, T_i) - K), \quad \text{or alternatively}$$

$$= -\sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) + P(t, T_\alpha) - P(t, T_\beta).$$

The value $K = S_{\alpha,\beta}(t)$ which makes
$\text{IRS}(t, [T_\alpha, \dots, T_\beta], K) = 0$
is the **forward swap rate.**
Denote $F_i(t) := F(t; T_{i-1}, T_i)$.

Products not depending on the curve dynamics: FRA's and IRS's I

Three possible formulas for the forward swap rate:

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t), \quad w_i(t) = \frac{\tau_i P(t, T_i)}{\sum_{j=\alpha+1}^{\beta} \tau_j P(t, T_j)}$$

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j\, F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j\, F_j(t)}}.$$

The second expression is a "weighted" average: $0 \leq w_i \leq 1$, $\sum_{j=\alpha+1}^{\beta} w_j = 1$. The weights are functions of the F's and thus random at future times.

Products not depending on the curve dynamics: FRA's and IRS's I

Recall the Receiver IRS Formula

$$\text{ReceiverIRS}(t, [T_\alpha, \ldots, T_\beta], K) =$$
$$= \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) - P(t, T_\alpha) + P(t, T_\beta)$$

and combine it with

$$S_{\alpha,\beta}(t) \quad = \quad \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

Products not depending on the curve dynamics: FRA's and IRS's II

to obtain

$$\text{ReceiverIRS}(t,[T_\alpha,\ldots,T_\beta],K) =$$
$$= (K - S_{\alpha,\beta}(t)) \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$$

Analogously,

$$\text{PayerIRS}(t,[T_\alpha,\ldots,T_\beta],K) =$$
$$= (S_{\alpha,\beta}(t) - K) \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$$

Products depending on the curve dynamics: Caplets and CAPS I

A **cap** can be seen as a payer IRS where each exchange payment is executed only if it has positive value.

$$\text{Cap discounted payoff:} \quad \sum_{i=\alpha+1}^{\beta} D(t, T_i)\, \tau_i (L(T_{i-1}, T_i) - K)^+ \,.$$

$$= \sum_{i=\alpha+1}^{\beta} D(t, T_i)\, \tau_i (F_i(T_{i-1}) - K)^+ \,.$$

Suppose a company is Libor–indebted and has to pay at $T_{\alpha+1}, \ldots, T_\beta$ the Libor rates resetting at $T_\alpha, \ldots, T_{\beta-1}$.

Products depending on the curve dynamics: Caplets and CAPS II

The company has a view that libor rates will increase in the future, and wishes to protect itself

$$\text{buy a cap: } (L-K)^+ \longrightarrow^{CAP} \text{Company} \longrightarrow^{DEBT} L$$

$$\text{or Company} \longrightarrow^{NET} L-(L-K)^+ = \min(L,K)$$

The company pays at most K at each payment date.
A cap contract can be decomposed additively: Indeed, the discounted payoff is a sum of terms (**caplets**)

$$D(t,T_i)\,\tau_i(L(T_{i-1},T_i)-K)^+$$

$$= D(t,T_i)\,\tau_i(F_i(T_{i-1})-K)^+ \; .$$

Products depending on the curve dynamics: Caplets and CAPS III

Each caplet can be evaluated separately, and the corresponding values can be added to obtain the cap price (notice the “call option” structure!).
However, even if separable, the payoff is not linear in the rates. This implies that, roughly speaking, we need the whole distribution of future rates, and not just their means, to value caplets. This means that the dynamics of interest rates is needed to value caplets: We cannot value caplets at time t based only on the current zero curve $T \mapsto L(t, T)$, but we need to specify how this infinite-dimensional object moves, in order to have its distribution at future times. This can be done for example by specifying how r moves.

Products depending on the curve dynamics: Floors

A **floor** can be seen as a receiver IRS where each exchange payment is executed only if it has positive value.

$$\text{Floor discounted payoff:} \quad \sum_{i=\alpha+1}^{\beta} D(t, T_i)\, \tau_i (K - L(T_{i-1}, T_i))^+ \, .$$

$$= \sum_{i=\alpha+1}^{\beta} D(t, T_i)\, \tau_i (K - F_i(T_{i-1}))^+ \, .$$

The floor price is the risk neutral expectation E of the above discounted payoff.

Products depending on the curve dynamics: SWAPTIONS I

Finally, we introduce options on IRS's **(swaptions)**.
A (payer) swaption is a contract giving the right to enter at a future time a (payer) IRS.
The time of possible entrance is the maturity.
Usually maturity is first reset of underlying IRS.
IRS value at its first reset date T_α, i.e. at maturity, is, by our above formulas,

$$\text{PayerIRS}(T_\alpha, [T_\alpha, \ldots, T_\beta], K) =$$

$$= \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\, \tau_i (F(T_\alpha; T_{i-1}, T_i) - K) =$$

$$= (S_{\alpha,\beta}(T_\alpha) - K) \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)$$

Products depending on the curve dynamics: SWAPTIONS II

Call $C_{\alpha,\beta}(T_\alpha)$ the summation on the right hand side.
The option will be excercised only if this IRS value is positive. There results the payer–swaption discounted–payoff at time t:

$$D(t, T_\alpha) C_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha) - K)^+ =$$

$$D(t, T_\alpha) \left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\, \tau_i (F(T_\alpha; T_{i-1}, T_i) - K) \right)^+ .$$

Unlike Caps, this payoff **cannot be decomposed** additively.
Caps can be decomposed in caplets, each with a single fwd rate.
Caps: Deal with each caplet **separately**, and put results together.
Only **marginal** distributions of different fwd rates are involved.
Not so with swaptions: The summation is *inside* the positive part operator $()^+$, and not outside.

Products depending on the curve dynamics: SWAPTIONS III

With swaptions we will need to consider the *joint* action of the rates involved in the contract.
The **correlation** between rates is fundamental in handling swaptions, contrary to the cap case.

Which variables do we model? I

For some products (Forward Rate Agreements, Interest Rate Swaps) the **dynamics** of interest rates is not necessary for valuation, the current curve being enough.
For caps, swaptions and more complex derivatives a dynamics is necessary.
Specifying a stochastic dynamics for interest rates amounts to choosing an **interest-rate model**.

- Which quantities do we model? Short rate r_t? LIBOR rates $L(t, T)$? Forward LIBOR rates $F_i(t) = F(t; T_{i-1}, T_i)$? Forward Swap rates $S_{\alpha,\beta}(t)$? Bond Prices $P(t, T)$?
- How is randomness modeled? i.e: What kind of stochastic process or stochastic differential equation do we select for our model? (Markov diffusions)

Which variables do we model? II

- What are the consequences of our choice in terms of valuation of market products, ease of implementation, goodness of calibration to real data, pricing complicated products with the calibrated model, possibilities for diagnostics on the model outputs and implications, stability, robustness, etc?

First Choice: short rate r I

This approach is based on the fact that the zero coupon curve at any instant, or the (informationally equivalent) zero bond curve

$$T \mapsto P(t,T) = E_t^Q \exp\left(-\int_t^T \boxed{r_s}\, ds\right)$$

is completely characterized by the probabilistic/dynamical properties of r.
So we write a model for r, the initial point of the curve $T \mapsto L(t,T)$ for $T = t$ at every instant t.
Typically a stochastic differential equation for r is chosen.

$$d\ r_t = \text{local\_mean}(t, r_t)dt +$$

$$+\text{local\_standard\_deviation}(t, r_t) \times \boxed{\text{stochastic\_change}_t}$$

First Choice: short rate r II

which we write

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)\boxed{dW_t}$$

The local mean b is called the "drift" and the local standard deviation σ is the "diffusion coefficient"

First Choice: short rate r I

Dynamics of $r_t = x_t$ under the risk–neutral-world probability measure

1. **Vasicek (1977):**

$$dx_t = k(\theta - x_t)dt + \sigma dW_t, \ \ \alpha = (k, \theta, \sigma).$$

2. **Cox-Ingersoll-Ross (CIR, 1985):**

$$dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \ \ \alpha = (k, \theta, \sigma), \ \ 2k\theta > \sigma^2 \ .$$

3. **Dothan / Rendleman and Bartter:**

$$dx_t = ax_t dt + \sigma x_t dW_t, \ (x_t = x_0 \ e^{(a - \frac{1}{2}\sigma^2)t + \sigma W_t}, \ \alpha = (a, \sigma)).$$

4. **Exponential Vasicek:**

$$x_t = \exp(z_t), \ \ dz_t = k(\theta - z_t)dt + \sigma dW_t, \ \ \alpha = (k, \theta, \sigma).$$

Every different choice has important consequences.

First Choice: short rate r. Example: Vasicek I

$$dx_t = k(\theta - x_t)dt + \sigma dW_t, \quad r_t = x_t.$$

The Vasicek model has some peculiarities that make it attractive.
The equation is linear and can be solved explicitly.
Joint distributions of many important quantities are Gaussian. Many formula for prices (i.e. expectations)
The model is mean reverting: The expected value of the short rate tends to a constant value θ with velocity depending on k as time grows towards infinity, while its variance does not explode.
However, this model features also some drawbacks.
Rates can assume negative values with positive probability.
Gaussian distributions for the rates are not compatible with the market implied distributions.

First Choice: short rate r. Example: Vasicek II

The choice of a particular dynamics has several important consequences, which must be kept in mind when designing or choosing a particular short-rate model. A typical comparison is for example with the Cox Ingersoll Ross (CIR) model.

First Choice: short rate r. Example: CIR I

$$dy(t) = \kappa[\mu - y(t)]dt + \nu \boxed{\sqrt{y(t)}} dW(t), \quad r_t = y_t$$

For the parameters κ, μ and ν ranging in a reasonable region, this model implies **positive** interest rates, but the instantaneous rate is characterized by a **noncentral chi-squared distribution**.

The model is mean reverting: The expected value of the short rate tends to a constant value μ with velocity depending on κ as time grows towards infinity, while its variance does not explode.

This model maintains a certain degree of analytical tractability, but is **less tractable** than Vasicek, especially as far as the extension to the multifactor case with correlation is concerned

CIR is usually closer to market implied distributions of rates than Vasicek.

First Choice: short rate r. Example: CIR II

Therefore, the CIR dynamics has both some advantages and disadvantages with respect to the Vasicek model.

CIR and Vasicek models: some intuition I

The parameters of the CIR model are similar to those of the Vasicek model in terms of interpretation.

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW_t$$

κ: speed of mean reversion
μ: long term mean reversion level
ν: volatility.

CIR model I

$$E[y_t] = y_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$$

$$\mathrm{VAR}(y_t) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa t})^2$$

After a long time the process reaches (asymptotically) a stationary distribution around the mean μ and with a corridor of variance $\mu\nu^2/2\kappa$. The largest κ, the fastest the process converges to the stationary state. So, ceteris paribus, increasing κ kills the volatility of the interest rate. The largest μ, the highest the long term mean, so the model will tend to higher rates in the future in average.

The largest ν, the largest the volatility. Notice however that κ and ν fight each other as far as the influence on volatility is concerned. We see some plots of scenarios now

CIR model II

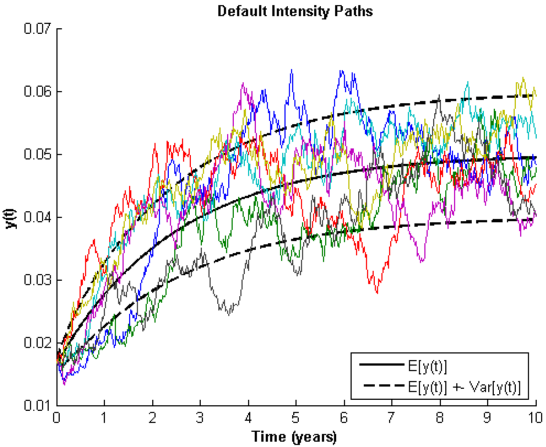


Figure: $y0 = 0.0165, \kappa = 0.4, \mu = 0.05, \nu = 0.04$

Case Study: Vasicek I

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \;\; \alpha = (k, \theta, \sigma).$$

Compute

$$d[e^{kt} r_t] = k e^{kt} r_t dt + e^{kt} dr_t = \ldots = e^{kt}[k\theta \; dt + \sigma dW_t]$$

Integrating both sides between s and t we obtain, for each $s \leq t$,

$$e^{kt} r_t - e^{ks} r_s = \int_s^t e^{ku} k\theta \; du + \int_s^t e^{ku} \sigma dW_u$$

Now, multiplying both sides by e^{-kt} we get

$$r(t) = r(s) e^{-k(t-s)} + \theta \left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)} dW(u), \tag{18}$$

Case Study: Vasicek II

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)} dW(u), \quad (19)$$

so that $r(t)$ conditional on r_s is normally distributed with mean and variance given respectively by

$$E\{r(t)|r_s\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\text{VAR}\{r(t)|r_s\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right].$$

(Ito isometry: for deterministic $v(t)$ we have
$\text{VAR}(\int v(u)dW_u) = E[(\int v(u)dW_u)^2] = \int v(u)^2 du)$
This implies that, for each time t, the rate $r(t)$ can be negative with positive probability.

Case Study: Vasicek III

$$E\{r(t)|r_s\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\mathrm{VAR}\{r(t)|r_s\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right],$$

and r is normally distributed. The possibility of negative rates is indeed a major drawback of the Vasicek model. However, the analytical tractability that is implied by a Gaussian density is hardly achieved when assuming other distributions for r.

The short rate r is mean reverting, since the expected rate tends, for t going to infinity, to the value θ.

The price of a pure-discount bond can be derived by computing the expectation $P(t,T) = E_t \exp(-\int_t^T r_u du)$.

Case Study: Vasicek IV

We will base our calculation on the following fact:

$$\int_t^T r_u du \;\Big|\; r_t \text{ is a Gaussian random variable.}$$

To show this, recall that we had derived earlier the formula

$$r(u) = r(t)e^{-k(u-t)} + \theta\left(1 - e^{-k(u-t)}\right) + \sigma \int_t^u e^{-k(u-s)} dW(s),$$

Then, integrating both sides

$$\int_t^T r(u)du = \int_t^T r(t)e^{-k(u-t)} du + \int_t^T \theta\left(1 - e^{-k(u-t)}\right) du +$$

$$+\sigma \int_t^T \left(\int_t^u e^{-k(u-s)} dW(s)\right) du,$$

Case Study: Vasicek V

or

$$\int_t^T r(u)du = \int_t^T r(t)e^{-k(u-t)}du + \int_t^T \theta\left(1 - e^{-k(u-t)}\right)du +$$

$$+\sigma \int_t^T \left(\int_t^T 1_{\{s \leq u\}} e^{-k(u-s)} dW(s)\right) du;$$

now we switch integrals:

$$\int_t^T r(u)du = \int_t^T r(t)e^{-k(u-t)}du + \int_t^T \theta\left(1 - e^{-k(u-t)}\right)du +$$

$$+\sigma \int_t^T (\int_t^T 1_{\{s \leq u\}} e^{-k(u-s)} du) dW(s)$$

Case Study: Vasicek VI

or

$$\int_t^T r_u du = \int_t^T r_t e^{-k(u-t)} du + \int_t^T \theta \left(1 - e^{-k(u-t)}\right) du + \sigma \int_t^T g(s) dW_s$$

where

$$g(s) = \int_t^T 1_{\{s \le u\}} e^{-k(u-s)} du$$

is a deterministic function. Now, given $r(t)$, everything in the right hand side of the last expression for $\int_t^T r(u)du$ is deterministic, except for the last integral. The last integral is an Ito integral with deterministic integrand $g(s)$, and is therefore a Gaussian random variable.

We have thus shown that $\int_t^T r(u)du$ is Gaussian. This means that to completely characterize its distribution and compute the bond price we just need its mean and variance.

Case Study: Vasicek VII

Mean and variance can be computed from

$$E_t \int_t^T r_u du = \int_t^T E_t[r_u] du =$$

$$\int_t^T [r(t)e^{-k(u-t)} + \theta\left(1 - e^{-k(u-t)}\right)] du = \ldots$$

$$E_t\left[\left(\int_t^T r_u du\right)^2\right] = E_t\left[\int_t^T \int_t^T r_u r_v du dv\right] = \int_t^T \int_t^T E_t[r_u r_v] du dv =$$

$$= \int_t^T \int_t^T E_t\left\{\left[r_t e^{-k(u-t)} + \theta(1 - e^{-k(u-t)}) + \sigma \int_t^u e^{-k(u-z)} dW_z\right]\right.$$

$$\left.\left[r_t e^{-k(v-t)} + \theta(1 - e^{-k(v-t)}) + \sigma \int_t^v e^{-k(v-z)} dW_z\right]\right\} du dv$$

Case Study: Vasicek VIII

...

This can be computed by using the isometry

$$E[\int_t^u f(z)dW_z \int_t^v g(z)dW_z] = \int_t^{\min(u,v)} f(z)g(z)dz.$$

One obtains (moment generating function of a Gaussian)

$$X := -\int_t^T r_u du \sim \mathcal{N}(M, V^2),$$

$$P(t, T) = E[e^X] = \exp(M + V^2/2)$$

By completing the (now trivial) computations we have

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$

Case Study: Vasicek IX

$$A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\right\}$$

$$B(t,T) = \frac{1}{k}\left[1 - e^{-k(T-t)}\right].$$

Put Option on a S-maturity Zero coupon bond. Payoff at T (discounted back at t)

$$\exp\left(-\int_t^T r_u du\right)(X - P(T,S))^+$$

The price at time t of a European option with strike X, maturity T and written on a pure discount bond maturing at time S is the risk neutral expectation of the above quantity, and is denoted by ZBP(t,T,S,X). Here is how one can compute it.

Case Study: Vasicek. Bond Option I

$$E_t \left[\exp\left(-\int_t^T r_u du \right) (X - P(T,S))^+ \right]$$

Recall:

$$r(T) = r(t)e^{-k(T-t)} + \theta \left(1 - e^{-k(T-t)} \right) + \sigma \int_t^T e^{-k(T-u)} dW(u),$$

and $P(T,S) = A(T,S)e^{-B(T,S)r(T)}$. Moreover, integrating both sides of $dr = k(\theta - r)dt + \sigma dW$ we get

$$-\int_t^T r_u du = (r_T - r_t)/k - \theta(T-t) - (\sigma/k) \int_t^T dW_u.$$

Case Study: Vasicek. Bond Option II

The above expectation depends only on the random vector

$$\left[\int_t^T dW(u), \int_t^T e^{-k(T-u)} dW(u)\right]$$

which is normally distributed (isometry)

$$\mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} T-t & (1-e^{-k(T-t)})/k \\ . & (1-e^{-2k(T-t)})/(2k) \end{bmatrix}\right),$$

$$E_t\left[\exp\left(-\int_t^T r_u du\right)(X - P(T,S))^+\right]$$
$$= E_t\left[e^{aY_2+bY_1+c}\left(X - \alpha e^{\gamma Y_2}\right)^+\right]$$

Case Study: Vasicek. Bond Option III

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} T-t & (1-e^{-k(T-t)})/k \\ . & (1-e^{-2k(T-t)})/(2k) \end{bmatrix} \right),$$

so that we know how to compute the expectation explicitly. One obtains, after a lot of computations (but there are easier ways)

$$\mathrm{ZBP}(t,T,S,X) = \left[XP(t,T)\Phi(\sigma_p - h) - P(t,S)\Phi(-h) \right],$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function, and

$$\sigma_p = \sigma \sqrt{\frac{1-e^{-2k(T-t)}}{2k}} B(T,S), \quad h = \frac{1}{\sigma_p} \ln \frac{P(t,S)}{P(t,T)X} + \frac{\sigma_p}{2}.$$

Case Study: Vasicek. Caplet I

A caplet can be seen as a put option on a zero bond.
If N is the notional amount, and $\tau = S - T$, we have

$$
\begin{aligned}
\mathrm{Cpl}(t, T, S, X, N) &= E\left(e^{-\int_t^S r_s ds} N\tau(L(T,S) - X)^+ | \mathcal{F}_t\right) \\
&= E\left(E\left[e^{-\int_t^S r_s ds} N\tau(L(T,S) - X)^+ | \mathcal{F}_T\right] | \mathcal{F}_t\right) \\
&= E\left(E\left[e^{-\int_t^T r_s ds} e^{-\int_T^S r_s ds} N\tau(L(T,S) - X)^+ | \mathcal{F}_T\right] | \mathcal{F}_t\right) \\
&= E\left(e^{-\int_t^T r_s ds} E\left[e^{-\int_T^S r_s ds} | \mathcal{F}_T\right] N\tau(L(T,S) - X)^+ | \mathcal{F}_t\right) \\
&= NE\left(e^{-\int_t^T r_s ds} P(T,S)\tau(L(T,S) - X)^+ | \mathcal{F}_t\right),
\end{aligned}
$$

Case Study: Vasicek. Caplet II

where we used iterated conditioning. Using the definition of the LIBOR rate $L(T, S)$, we obtain

$$= NE\left(e^{-\int_t^T r_s ds}P(T,S)\left[\frac{1}{P(T,S)} - 1 - X\tau\right]^+ |\mathcal{F}_t\right)$$

$$= N(1+X\tau)E\left(e^{-\int_t^T r_s ds}[1/(1+X\tau) - P(T,S)]^+|\mathcal{F}_t\right),$$

We have thus seen that a caplet can be expressed as a put option on a bond, for which we derived a formula earlier.

$$\mathrm{Cpl}(t, T, S, X, N) = N(1+X\tau)\mathrm{ZBP}(t, T, S, 1/(1+X\tau))$$

Case Study: Vasicek. Summary I

In Vasicek's model we can:

- Solve explicitly the SDE for r

$$dr_t = k(\theta - r_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma)$$

  because it is **linear**, and find the **normal** distribution of r;
- Find the price of a bond $P(t, T) = P(t, T; \alpha; r_t)$ thanks to the fact that adding up jointly normal variables one obtains a **normal** random variable, so that $\int r_s ds$ is normal;
- Find the price of a put option on a zero coupon bond

$$\mathrm{ZBP}(t, T, S, X) = \mathrm{ZBP}(t, T, S, X; \alpha, r_t)$$

  by means of the expectation of a certain random variable based on a **bivariate normal** distribution coming from properties of Brownian motions;

Case Study: Vasicek. Summary II

- Find the price of a caplet

  $$\mathrm{Cpl}(t, T, S, X, N) = N(1 + X\tau)\mathrm{ZBP}((t, T, S, 1/(1 + \tau X); \alpha, r_t)$$

  as a price of a zero-bond put option thanks to **iterated conditioning** (property of conditional expectations).

Even this simple example shows that in order to price financial products one needs to master probability and statistics. Also, **analytical tractability** is often related to **linearity** and **normality.**

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation I

We can consider the objective measure $\mathbb{P} = Q_0$-dynamics of the Vasicek model as a process of the form

$$dr(t) = [k\,\theta - (k + \lambda\,\sigma)r(t)]dt + \sigma dW^0(t), \;\; r(0) = r_0 \; ,$$

where λ is a new parameter, contributing to the market price of risk. Compare this Q_0 dynamics to the risk-neutral Q-dynamics

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t), \;\; r(0) = r_0 \; .$$

Notice that for $\lambda = 0$ the two dynamics coincide. More generally, the above Q_0-dynamics is expressed again as a linear Gaussian stochastic differential equation, although it depends on the new parameter λ.

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation II

Requiring that the dynamics be of the same nature under the two measures (linear-Gaussian), imposes a Girsanov change of measure:

$$\frac{dQ}{dQ_0}\Big|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \lambda^2\, r(s)^2 ds + \int_0^t \lambda\, r(s) dW^0(s)\right)$$

although λ has to be assumed to be constant and not depending on r, which is not true in general. However, under this choice we obtain a short rate process that is tractable under both measures.
Important: In traditional finance, one first postulates a dynamics under the objective measure Q^0, and then writes the risk neutral dynamics by adding one or more parameters. For example, one would write

$$dr(t) = k(\theta - r(t))dt + \sigma dW^0(t), \quad r(0) = r_0 \; .$$

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation III

under the objective measure Q^0 and then

$$dr(t) = [k\,\theta - (k - \lambda\,\sigma)r(t)]dt + \sigma dW(t), \;\; r(0) = r_0$$

under the risk neutral measure.
We did the contrary because in pricing practice one starts from the risk neutral dynamics first.

$$dr(t) = [k\,\theta - (k + \lambda\,\sigma)r(t)]dt + \sigma dW^0(t)$$

(Statistics, historical estimation, econometrics).

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t)$$

(Pricing, risk neutral valuation).

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation IV

It is clear why tractability under the risk-neutral measure is a desirable property: claims are priced under that measure, so that the possibility to compute expectations in a tractable way with the Q-dynamics is important. Yet, why do we find it desirable to have a tractable dynamics under Q_0 too?

If we are provided with a series $r_0, r_1, r_2, \ldots, r_n$ of daily observations of a proxy of $r(t)$ (say a monthly rate, $r(t) \approx L(t, t + 1m)$), and we wish to incorporate information from this series in our model, we can estimate the model parameters on the basis of this daily series of data.

However, data are collected in the real world, and their statistical properties characterize the distribution of our interest-rate process $r(t)$ under the objective measure Q_0. Therefore, what is to be estimated from historical observations is the Q_0 dynamics. The estimation

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation V

technique can provide us with estimates for the objective parameters k, λ, θ and σ, or more precisely for combinations thereof.

If we are provided with a series $r_0, r_1, r_2, \ldots, r_n$ of daily observations of a proxy of $r(t)$, their statistical properties characterize the distribution of our interest-rate process $r(t)$ under the objective measure Q_0.

Therefore, what is to be estimated from historical observations is the Q_0 dynamics, with the objective parameters k, λ, θ and σ.

On the other hand, prices are computed through expectations under the risk-neutral measure. When we observe prices, we observe expectations under the measure Q. Therefore, when we calibrate the model to derivative prices we need to use the Q dynamics, thus finding the parameters k, θ and σ involved in the Q-dynamics and reflecting current market prices of derivatives.

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation VI

We could then combine the two approaches. For example, since the diffusion coefficient is the same under the two measures, we might estimate σ from historical data through a maximum-likelihood estimator, while finding k and θ through calibration to market prices. However, this procedure may be necessary when very few prices are available. Otherwise, it might be used to deduce historically a σ which can be used as initial guess when trying to find the three parameters that match the market prices of a given set of instruments.
Maximum-likelihood estimator for the Vasicek model. Write

$$dr(t) = [b - ar(t)]dt + \sigma dW^0(t),$$

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation VII

with b and a suitable constants.

$$r(t) = r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-u)} dW^0(u).$$

Given $\mathcal{F}_s$ the variable $r(t)$ is normally distributed with mean $r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e^{-a(t-s)})$ and variance $\frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})$.
It is natural to estimate the following functions of the parameters: $\beta := b/a$, $\alpha := e^{-a\delta}$ and $V^2 = \frac{\sigma^2}{2a}(1 - e^{-2a\delta})$, where δ denotes the time-step of the observed proxies. The maximum likelihood estimators for α, β and V^2 are easily derived as

$$\widehat{\alpha} = \frac{n\sum_{i=1}^n r_i r_{i-1} - \sum_{i=1}^n r_i \sum_{i=1}^n r_{i-1}}{n\sum_{i=1}^n r_i^2 - \left(\sum_{i=1}^n r_{i-1}\right)^2}, \quad \widehat{\beta} = \frac{\sum_{i=1}^n [r_i - \widehat{\alpha} r_{i-1}]}{n(1 - \widehat{\alpha})},$$

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation VIII

$$\widehat{V^2} = \frac{1}{n}\sum_{i=1}^{n}\left[r_i - \widehat{\alpha} r_{i-1} - \widehat{\beta}(1-\widehat{\alpha})\right]^2.$$

The estimated quantities give complete information on the δ-transition probability for the process r under Q_0, thus allowing for example simulations at one-day spaced future discrete time instants.

First Choice: short rate r. Questions to ask. I

Back to short rate models in general. When choosing a model, one should ask:

- Does the dynamics imply positive rates, i.e., $r(t) > 0$ a.s. for each t?
- What distribution does the dynamics imply for the short rate r? Is it, for instance, a fat-tailed distribution?
- Are bond prices $P(t,T) = E_t\left\{e^{-\int_t^T r(s)ds}\right\}$ (and therefore spot rates, forward rates and swap rates) explicitly computable from the dynamics?
- Are bond-option (and cap, floor, swaption) prices explicitly computable from the dynamics?
- Is the model mean reverting, in the sense that the expected value of the short rate tends to a constant value as time grows towards infinity, while its variance does not explode?

First Choice: short rate *r*. Questions to ask. II

- How do the volatility structures implied by the model look like?
- Does the model allow for explicit short-rate dynamics under the forward measures?
- How suited is the model for Monte Carlo simulation?
- How suited is the model for building recombining lattices (trees)?
- Does the chosen dynamics allow for historical estimation techniques to be used for parameter estimation purposes?

First Choice: Modeling r. Endogenous models. I

Model	**Dist**	**Analytic** $P(t,T)$	**Analytic Options**	**Multif**	**M-R**	$r>0$?
Vasicek	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
CIR	n.c. χ^2	Yes	Yes	Yes	Yes	Yes
Dothan	$e^{\mathcal{N}}$	"Yes"	No	No	"Yes"	Yes
Exp. Vasicek	$e^{\mathcal{N}}$	No	No	No	Yes	Yes

These models are **endogenous.** $P(t,T) = E_t(e^{-\int_t^T r(s)ds})$ can be computed as an expression (or numerically in the last two) depending on the model parameters.
For example, in Vasicek and CIR, given k, θ, σ and $r(t)$, once the function $T \mapsto P(t,T; k, \theta, \sigma, r(t))$ is known, we know the whole interest-rate curve at time t. At $t = 0$ (initial time), the interest rate curve is an **output** of the model, rather than an input, depending on k, θ, σ, r_0 in the dynamics.

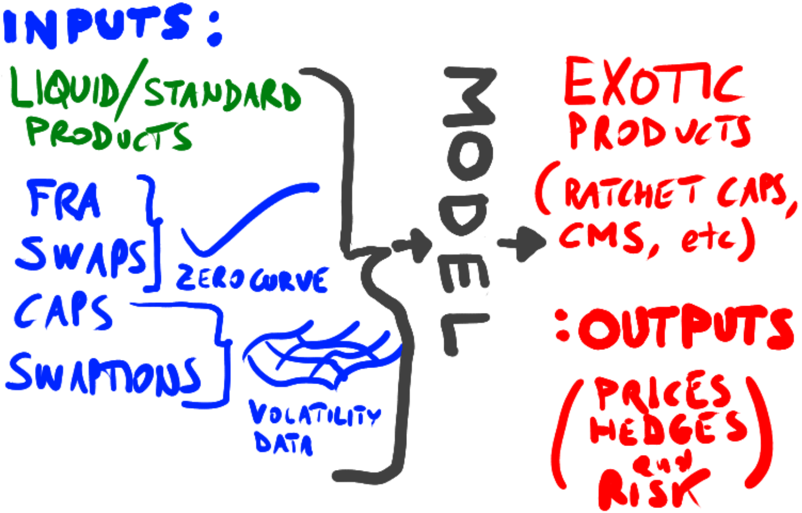
First Choice: Modeling r. Endogenous models. II

If we have the initial curve $T \mapsto P^M(0,T)$ from the market, and we wish our model to incorporate this curve, we need forcing the model parameters to produce a curve as close as possible to the market curve. This is the **calibration of the model to market data**. In the Vasicek case, run an optimization to have

$$\text{Fit } \; T \mapsto P(0,T;k,\theta,\sigma,r_0) \; \text{ to } \; T \mapsto P^M(0,T) \; \text{ through } k,\theta,\sigma,r_0.$$

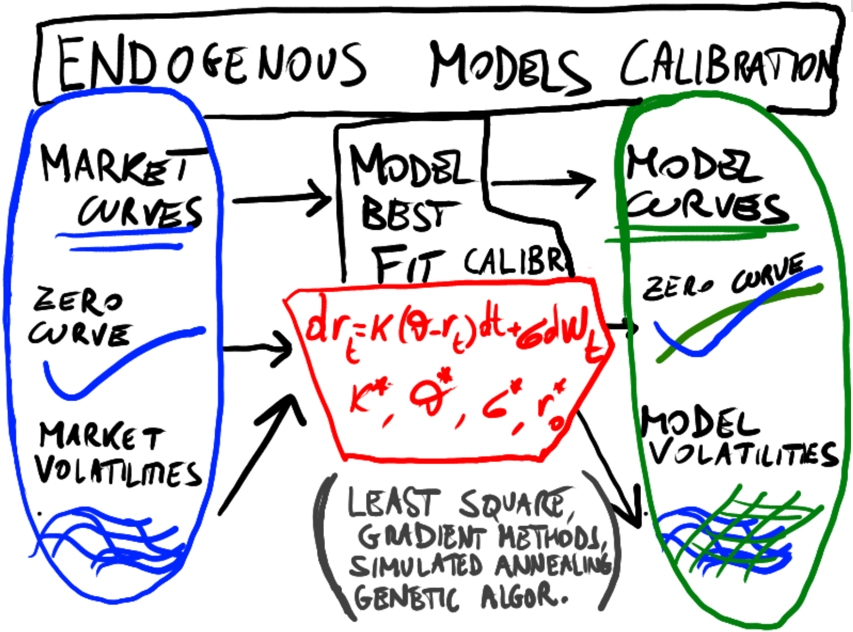
Too few parameters. Some shapes of $T \mapsto L^M(0,T)$ (like an inverted shape) can never be obtained, no matter the values of the parameters in the dynamics. To improve this situation and calibrate also **caplet** data, **exogenous** term structure models are usually considered.

THE MODEL CALIBRATION



Calibration

- A particularly important part of a model's operations is the calibration.
- Our aim is pricing, hedging and possibly risk managing a complex EXOTIC financial product whose quotations are not liquid or easily found
- To do so we plan to use a model
- The model needs to reflect as many available liquid market data as possible when these data are pertinent to the financial product to be analyzed
- In the interest rate market ususally one starts from the zero curve (FRA, Swaps) and a few vanilla options (Caps, Swaptions), imposing the model to fit them
- Once the model has been fit as well as possible to such data, the model is used to price the complex product


ENDOGENOUS MODELS CALIBRATION
MARKET CURVES
ZERO CURVE
MARKET VOLATILITIES
MODEL BEST FIT CALIBR
$dr_t = k(\theta - r_t)dt + \sigma dW_t$
$k^*, \theta^*, \sigma^*, r_0^*$
LEAST SQUARE, GRADIENT METHODS, SIMULATED ANNEALING GENETIC ALGOR.
MODEL CURVES
ZERO CURVE
MODEL VOLATILITIES

Endogenous Models Calibration Figure

- We are given a market zero coupon curve of interest rates at time 0, the blue curve ”zero curve” for $T \mapsto L^M(0, T)$.
- We are given also a number of options volatilities possibly, the blue surface ”market volatilities”
- We best fit the Red Vasicek model formula for the curve $L(0, T; k, \theta, \sigma, r_0)$ and perhaps a few options formulas to get the best parameters we can in matching the market data. These will be the red parameters $k^*, \theta^*, \sigma^*, r_0^*$.
- The best fit can occur through the grey optimization methods, either local (gradient method) or global (simulated annealing, genetic algorithms...)
- The resulting fit is usually poor. For example, Vasicek cannot reproduce an inverted curve, compare the green (model) and blue (market) zero curves on the right hand side of the figure...
- The volatility structure is also poorly fit, as you see comparing the blue and the green surfaces on the right hand side.

First Choice: Modeling r. Exogenous models. I

Exogenous short-rate models are built by suitably modifying the above endogenous models. The basic strategy that is used to transform an endogenous model into an exogenous model is the inclusion of "time-varying" parameters. Typically, in the Vasicek case, one does the following:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t) \longrightarrow dr(t) = k[\boxed{\vartheta(t)} - r(t)]dt + \sigma dW(t) \; .$$

Now the function of time $\vartheta(t)$ can be defined in terms of the market curve $T \mapsto L^M(0, T)$ in such a way that the model reproduces exactly the curve itself at time 0.

The remaining parameters may be used to calibrate CAPS/Swaptions data. We no longer price caps, since they are very liquid, but wish the model to "absorb" them to price more difficult things.

First Choice: Modeling r. Exogenous models. I

Dynamics of $r_t = x_t$ under the risk–neutral measure:

1. **Ho-Lee:**
$$dx_t = \theta(t)\, dt + \sigma\, dW_t.$$
2. **Hull-White (Extended Vasicek):**
$$dx_t = k(\theta(t) - x_t)dt + \sigma dW_t.$$
3. **Hull-White (Extended CIR):**
$$dx_t = k(\theta(t) - x_t)dt + \sigma \sqrt{x_t}\, dW_t\ .$$
4. **Black-Derman-Toy (Extended Dothan):**
$$x_t = x_0\ e^{u(t)+\sigma(t)W_t}$$

First Choice: Modeling *r*. Exogenous models. II

5. **Black-Karasinski (Extended exponential Vasicek):**

$$x_t = \exp(z_t), \quad dz_t = k\left[\theta(t) - z_t\right] dt + \sigma dW_t.$$

6. **CIR++ (Shifted CIR model, Brigo & Mercurio (2000)):**

$$r_t = x_t + \phi(t;\alpha), \quad dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t$$

Now parameters are used to fit volatility structures.
In general other parameters can be chosen to be time–varying so as to improve fitting of the volatility term–structure (but...)

Reference Model	**Dist**	**ABP**	**AOP**	**Multif**	**M-R**	$r > 0$?
Vasicek	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
CIR	n.c. χ^2	Yes	Yes	Yes	Yes	Yes
Dothan	$e^{\mathcal{N}}$	"Yes"	No	No	"Yes"	Yes
Exp. Vasicek	$e^{\mathcal{N}}$	No	No	No	Yes	Yes

Classical extended models:
Distribution (Distr)
Analytical bond prices (ABP)
Analytical bond–option prices (AOP)
Mean Reversion (MR)
Tractable Multi Factor Extension (Multif)

Extended Model	**Distr**	**ABP**	**AOP**	**Multif**	**M-R**	$r > 0$?
Ho-Lee	$\mathcal{N}$	Yes	Yes	Yes	No	No
Hull-White (Vas.)	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
Hull-White (CIR)	n.c. χ^2	No	No	No	Yes	Yes-but
BDT	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
Black Karasinski	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
CIR++ Brigo Mercurio	s.n.c. χ^2	Yes	Yes	Yes	Yes	Yes

Short rate models: Which model? I

Extended Model	Distr	ABP	AOP	Multif	M-R	$r > 0$?
Ho-Lee	$\mathcal{N}$	Yes	Yes	Yes	No	No
Hull-White (Vas.)	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
Hull-White (CIR)	n.c. χ^2	No	No	No	Yes	Yes-but
BDT	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
Black Karasinski	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
CIR++ Brigo Mercurio	s.n.c. χ^2	Yes	Yes	Yes	Yes	Yes

- Ho Lee: very tractable; stylized, simplistic, negative rates;
- Hull-White (Vasicek): Very tractable, formulas, easy to implement and calibrate, trees easy, Monte Carlo possible; possibly negative rates; can give pathological calibrations under certain market situations.
- Hull-White (CIR): Not tractable, numerical problems...

Short rate models: Which model? II

- BDT: No tractability, some mean reversion but linked to the volatility, excellent distribution and good calibration to the market rates implied distributions, explosion problem of bank account in continuos version: $EB(\epsilon) = E(\exp(\int_0^{\epsilon} r_u du)) = \infty$.
  Need trinomial trees (discretization in time and space) to have it work. No reasonable Monte Carlo simulation possible.

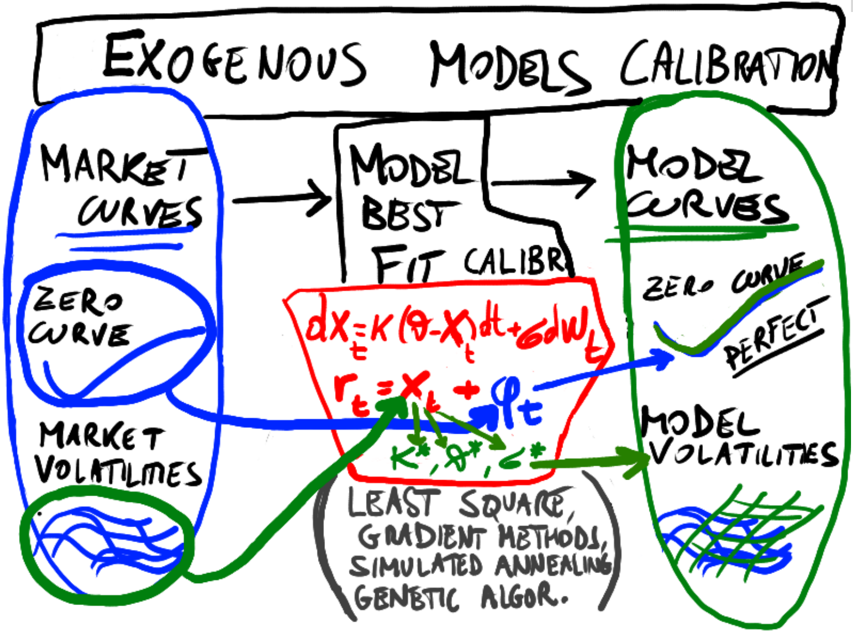
Short rate models: Which model? III

Extended Model	Distr	ABP	AOP	Multif	M-R	$r > 0$?
Ho-Lee	$\mathcal{N}$	Yes	Yes	Yes	No	No
Hull-White (Vas.)	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
Hull-White (CIR)	n.c. χ^2	No	No	No	Yes	Yes-but
BDT	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
Black Karasinski	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
CIR++ Brigo Mercurio	s.n.c. χ^2	Yes	Yes	Yes	Yes	Yes

- Black Karasinski: No tractability, mean reversion, excellent distribution and good calibration to the market rates distributions, explosion problem of bank account in continuos version (as in all lognormal short-rate models). Need trinomial trees (discretization in time and space) to have it work. No reasonable Monte Carlo simulation possible.

Short rate models: Which model? IV

- CIR++: Tractable, many formulas, easy to implement and calibrate, trees are not so easy but feasible, Monte Carlo possible, positive rates, can give pathological calibrations under certain market situations (as most one-dimensional short-rate models)


EXOGENOUS MODELS CALIBRATION
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MODEL CURVES
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$dx_t = k(\theta - x_t)dt + \sigma dW_t$
$r_t = x_t + \varphi_t$
k^*, θ^*, σ^*
ZERO CURVE
PERFECT
MARKET VOLATILITIES
MODEL VOLATILITIES
LEAST SQUARE,
GRADIENT METHODS,
SIMULATED ANNEALING,
GENETIC ALGOR.

Exogenous Models Calibration Figure

- We are given a market zero coupon curve of interest rates at time 0, the blue curve ”zero curve” for $T \mapsto L^M(0, T)$.
- We are given also a number of vanilla options volatilities (typically caps and a few swaptions), possibly. This is the blue surface ”market volatilities”
- We now use a time dependent ”parameter” $\vartheta(t)$ or shift $\varphi(t)$ to fit the zero curve exactly, and this is represented by the blue arrow.
- Then we use the parameters k, θ, σ, r_0 in the x part of r to best fit the vanilla option data, and this is the green arrow.
- The best fit of the options data can occur through the grey optimization methods, either local (gradient method) or global (simulated annealing, genetic algorithms...)
- The resulting fit is usually not too good, as you see comparing the blue and the green surfaces on the right hand side. If we fit just a few options, the fit improves

Extension: Shifted Vasicek I

We have seen extensions of

$$dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t \ ,$$

obtained through time varying coefficients,

$$r_t = x_t, \ \ dx_t = \mu(x_t;\alpha(t))dt + \sigma(x_t;\alpha(t))dW_t \ .$$

Instead, consider the following alternative (Shifted Vasicek):

$$r_t = x_t + \phi(t;\alpha), \ \ dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t \ ,$$

with x_0 **a further parameter** we include augmenting α**.**

Extension: Shifted Vasicek II

We have the following bond and option prices

$$P^r(t,T,r_t;\alpha) = \exp\left[-\int_t^T \phi(s;\alpha)ds\right] P^x(t,T,x_t;\alpha)$$

$$\mathrm{ZBP}^r(0,T,s,K,r_0;\alpha) = E_0\left\{\exp\left[-\int_0^T r_u du\right](K - P^r(T,s,r_T;\alpha))^+\right\}$$
$$= \exp\left[-\int_0^s \phi(u;\alpha)du\right]\mathrm{ZBP}^x\left(0,T,s,K\exp\left[\int_T^s \phi(u;\alpha)\,du\right],x_0^\alpha;\alpha\right)$$

Calibration of the market zero curve ($T \mapsto P^M(0,T)$) and of Caplet data: select α and $\phi(\cdot,\alpha)$ (for details see the course in Fixed Income by Prof Lara Cathcart in this same Master Programme, or the book ”Interest Rate Models” by Brigo and Mercurio (2006).)

Monte Carlo and Trinomial Trees I

In the market there are products featuring path dependent payoffs and early exercise payoffs.
When we aim at pricing derivatives whose payout at final maturity T is a function not only of interest rates at a final time related to the final maturity T but also of interest rates related to earlier times $t_i < T$, then we say that we have a path dependent payout. More precisely, this happens if the payout cannot be decomposed into a sum of payouts each referencing a single maturity interest rate at the time.
For these path dependent payouts, except for a few exceptions, it may be necessary to price unsing Monte Carlo simulation.

There are also products that can be exercised at times t_i preceding the final maturity of the payout. The typical example is bermudan swaptions, which are swaptions that can be exercised every year rather than at a single maturity T_α. For such products Monte Carlo simulation

Monte Carlo and Trinomial Trees II

is not suitable. Indeed, simulating forward in time does not allow us to know or propagate the optimal exercise strategy for the option. On the contrary, this can be know at terminal time and be propagated backwards in time along a tree, similarly for how American options on equity are priced using binomial trees and backward induction.

Monte Carlo Simulation I

Since for Vasicek we know that
$r(t)$ conditional on r_s is normally distributed with mean and variance given respectively by

$$E\{r(t)|r_s\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\mathrm{VAR}\{r(t)|r_s\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right],$$

this means that the short rate can be simulated exactly across large intervals t_{i-1}, t_i without further discretization. Monte Carlo simulation is easy because we know the exact normal distribution for the transition probability of the short rate between times t_{i-1} and t_i. A further advantage of the Vasicek model is that if we know the short rate at t_i we have a formula for the bond price $P(t_i, T)$ for every maturity T.

Monte Carlo Simulation II

Hence from the short rate simulation we can immediately get as a bonus Libor rates, forward and swap rates for any maturity. This makes the model handy in pricing path dependent payoffs via simulation. This reasoning of course applies as well to the shifted Vasicek model.

Trinomial Tree I

We now illustrate a procedure for the construction of a trinomial tree that approximates the evolution of the process x. It can be then extended to the shifted Vasicek model by suitably adjusting the tree (see for example Brigo and Mercurio 2006).
This is a two-stage procedure that is basically based on those suggested by Hull and White (1993d, 1994a).
Let us fix a time horizon T and the times $0 = t_0 < t_1 < \cdots < t_N = T$, and set $\Delta t_i = t_{i+1} - t_i$, for each i. The time instants t_i need not be equally spaced. This is an essential feature when employing the tree for practical purposes.
The first stage consists in constructing a trinomial tree for the process x

$$dx_t = -kx_t dt + \sigma dW_t$$

We explain how to build a tree for a generic diffusion process X first

Trinomial Tree II

Let us consider the diffusion process X

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

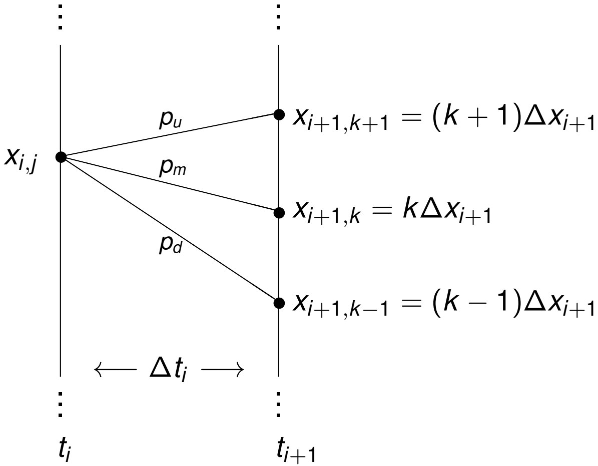
where μ and σ are smooth scalar real functions and W is a scalar standard Brownian motion.

We want to discretize this dynamics both in time and in space.

Precisely, we want to construct a trinomial tree that suitably approximates the evolution of the process X.

To this end, we fix a finite set of times $0 = t_0 < t_1 < \cdots < t_n = T$ and we set $\Delta t_i = t_{i+1} - t_i$. At each time t_i, we have a finite number of equispaced states, with constant vertical step Δx_i to be suitably determined. We set $x_{i,j} = j\Delta x_i$.

Trinomial Tree III



Trinomial Tree IV

Assuming that at time t_i we are on the j-th node with associated value $x_{i,j}$, the process can move to $x_{i+1,k+1}$, $x_{i+1,k}$ or $x_{i+1,k-1}$ at time t_{i+1} with probabilities p_u, p_m and p_d, respectively. The central node is therefore the k-th node at time t_{i+1}, where also the level k is to be suitably determined.

Denoting by $M_{i,j}$ and $V_{i,j}^2$ the mean and the variance of X at time t_{i+1} conditional on $X(t_i) = x_{i,j}$, i.e.,

$$
\begin{aligned}
E\left\{X(t_{i+1})|X(t_i) = x_{i,j}\right\} &= M_{i,j} \\
\mathrm{Var}\left\{X(t_{i+1})|X(t_i) = x_{i,j}\right\} &= V_{i,j}^2,
\end{aligned}
$$

we want to find p_u, p_m and p_d such that these conditional mean and variance match those in the tree.

Trinomial Tree V

Precisely, noting that $x_{i+1,k+1} = x_{i+1,k} + \Delta x_{i+1}$ and $x_{i+1,k-1} = x_{i+1,k} - \Delta x_{i+1}$, we look for positive constants p_u, p_m and p_d summing up to one and satisfying

$$\begin{cases} p_u(x_{i+1,k} + \Delta x_{i+1}) + p_m x_{i+1,k} + p_d(x_{i+1,k} - \Delta x_{i+1}) = M_{i,j} \\ p_u(x_{i+1,k} + \Delta x_{i+1})^2 + p_m x_{i+1,k}^2 + p_d(x_{i+1,k} - \Delta x_{i+1})^2 = \\ = V_{i,j}^2 + M_{i,j}^2. \end{cases}$$

Simple algebra leads to

$$\begin{cases} x_{i+1,k} + (p_u - p_d)\Delta x_{i+1} = M_{i,j} \\ x_{i+1,k}^2 + 2x_{i+1,k}\Delta x_{i+1}(p_u - p_d) + \Delta x_{i+1}^2(p_u + p_d) \\ = V_{i,j}^2 + M_{i,j}^2. \end{cases}$$

Trinomial Tree VI

Setting $\eta_{j,k} = M_{i,j} - x_{i+1,k}$ (we omit to express the dependence on the index i to lighten the notation) we finally obtain

$$\begin{cases} (p_u - p_d)\Delta x_{i+1} = \eta_{j,k} \\ (p_u + p_d)\Delta x_{i+1}^2 = V_{i,j}^2 + \eta_{j,k}^2, \end{cases}$$

so that, remembering that $p_m = 1 - p_u - p_d$, the candidate probabilities are

$$\begin{cases} p_u = \frac{V_{i,j}^2}{2\Delta x_{i+1}^2} + \frac{\eta_{j,k}^2}{2\Delta x_{i+1}^2} + \frac{\eta_{j,k}}{2\Delta x_{i+1}}, \\ p_m = 1 - \frac{V_{i,j}^2}{\Delta x_{i+1}^2} - \frac{\eta_{j,k}^2}{\Delta x_{i+1}^2}, \\ p_d = \frac{V_{i,j}^2}{2\Delta x_{i+1}^2} + \frac{\eta_{j,k}^2}{2\Delta x_{i+1}^2} - \frac{\eta_{j,k}}{2\Delta x_{i+1}}. \end{cases}$$

Trinomial Tree VII

In general, there is no guarantee that p_u, p_m and p_d are actual probabilities, because the expressions defining them could be negative. We then have to exploit the available degrees of freedom in order to obtain quantities that are always positive. To this end, we make the assumption that $V_{i,j}$ is independent of j, so that from now on we simply write V_i instead of $V_{i,j}$. We then set $\Delta x_{i+1} = V_i\sqrt{3}$ (this choice, motivated by convergence purposes, is a standard one. See for instance Hull and White (1993, 1994)) and we choose the level k, and hence $\eta_{j,k}$, in such a way that $x_{i+1,k}$ is as close as possible to $M_{i,j}$. As a consequence,

$$k = \text{round}\left(\frac{M_{i,j}}{\Delta x_{i+1}}\right), \tag{20}$$

Trinomial Tree VIII

where round(x) is the closest integer to the real number x. Moreover,

$$\begin{cases} p_u = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i}, \\ p_m = \frac{2}{3} - \frac{\eta_{j,k}^2}{3V_i^2}, \\ p_d = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i}. \end{cases} \tag{21}$$

It is easily seen that both p_u and p_d are positive for every value of $\eta_{j,k}$, whereas p_m is positive if and only if $|\eta_{j,k}| \leq V_i\sqrt{2}$. However, defining k as above implies that $|\eta_{j,k}| \leq V_i\sqrt{3}/2$, hence the condition for the positivity of p_m is satisfied, too.
As a conclusion, the above are actual probabilities such that the corresponding trinomial tree has conditional (local) mean and variance that match those of the continuous-time process X.

Trinomial Tree IX

Going back to our x_t as in Vasicek, we have

$$
\begin{aligned}
E\{x(t_{i+1})|x(t_i) = x_{i,j}\} &= x_{i,j}e^{-a\Delta t_i} =: M_{i,j} \\
\mathrm{Var}\{x(t_{i+1})|x(t_i) = x_{i,j}\} &= \frac{\sigma^2}{2a}\left[1 - e^{-2a\Delta t_i}\right] =: V_i^2.
\end{aligned}
\tag{22}
$$

We then set $x_{i,j} = j\Delta x_i$, where

$$
\Delta x_i = V_{i-1}\sqrt{3} = \sigma\sqrt{\frac{3}{2a}\left[1 - e^{-2a\Delta t_{i-1}}\right]}. \tag{23}
$$

and we apply the above procedure.

Trinomial Tree X

Once we have the tree, pricing of (early exercise) Bermudan swaptions occurs by backward induction.

One first computes the final payout at each final node in the tree at T, and then starts rolling back the payout along the tree in time.

At each time where exercise is available one then compares the rolled back price down to that point/node (continuation value) to the price of exercise in that specific node, and takes the maximum.

This way we take care of comparing the immediate exercise with the possibility to hold the option for longer, and make the optimal choice

This is easily implemented once the tree is built.

Trinomial Tree XI

A (forward looking) monte carlo simulation would not work here, since we would not know, in a specific path at a point in time, the continuaton value, which can be computed going backwards but not forward

Special versions of the Monte Carlo method that approximate the continuation value as a function of the present state variables can be used. This is called Least Squared Monte Carlo.

The student has certainly seen continuation value calculations in trees for simple option pricing theory. This is completely analogous to the binomial-tree model of Cox Ross Rubinstein for american options on a stock.

Trinomial Tree XII

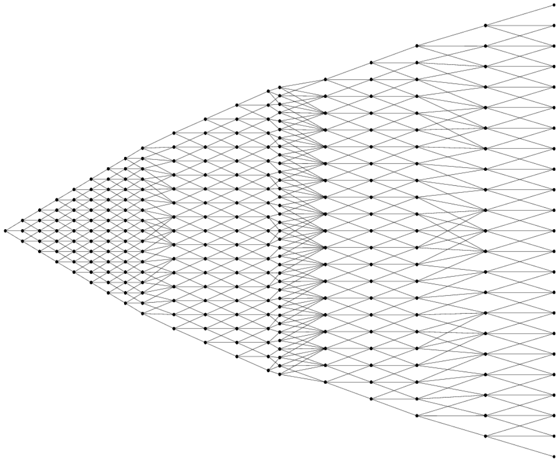


Figure: A possible geometry for the tree approximating x.

First choice: Modeling r. Multidimensional models I

In these models, typically (e.g. shifted two-factor Vasicek)

$$\begin{aligned} dx_t &= k_x(\theta_x - x_t)dt + \sigma_x dW_1(t), \\ dy_t &= k_y(\theta_y - y_t)dt + \sigma_y dW_2(t), \quad dW_1\, dW_2 = \rho\, dt, \\ r_t &= x_t + y_t + \phi(t,\alpha), \quad \alpha = (k_x, \theta_x, \sigma_x, x_0, k_y, \theta_y, \sigma_y, y_0) \end{aligned}$$

More parameters, can capture more flexible caps or swaptions structures in the market and especially gives **less correlated** rates at future times.
Indeed, suppose we define Continuously Compounded Spot Rates at time t for the maturity T as

$$R(t,T) := -\frac{1}{T-t} \ln P(t,T) \;\Rightarrow\; P(t,T) = e^{-R(t,T)(T-t)}.$$

First choice: Modeling r. Multidimensional models II

This is an alternative definition to the Simply Compounded (Libor) Spot rates we have seen earlier:

$$L(t,T) := \frac{1}{T-t}\left[\frac{1}{P(t,T)} - 1\right]$$

One dimensional models have
$\text{corr}_0(R(1y,2y), R(1y,30y)) = 1$,
due to the unique source of randomness dW.
Multidimensional models can lower this perfect correlation by playing with the instantaneous correlation ρ in the **two** sources of randomness W_1 and W_2.
We may retain analytical tractability.

Multidimensional models and correlations I

We now consider multidimensional models more in detail.
Recall that the Vasicek model assumes the evolution of the short-rate process r to be given by the linear-Gaussian SDE

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \ .$$

Recall also the bond price formula $P(t,T) = A(t,T)\exp(-B(t,T)r_t)$, from which all rates can be computed in terms of r. In particular, continuously-compounded spot rates are given by the following affine transformation of the fundamental quantity r

$$R(t,T) = -\ln(P(t,T))/(T-t) = -\frac{\ln A(t,T)}{T-t} + \frac{B(t,T)}{T-t} r_t =$$

$$=: a(t,T) + b(t,T) r_t \ .$$

Multidimensional models and correlations II

Consider now a payoff depending on the joint distribution of two such rates at time t. For example, we may set $T_1 = t + 1$ years and $T_2 = t + 10$ years. The payoff would then depend on the joint distribution of the one-year and ten-year continuously-compounded spot interest rates at “terminal” time t. In particular, since the joint distribution is involved, the correlation between the two rates plays a crucial role. With the Vasicek model such terminal correlation is easily computed as

$$\text{Corr}(R(t,T_1), R(t,T_2)) =$$

$$= \text{Corr}(a(t,T_1) + b(t,T_1)r_t, a(t,T_2) + b(t,T_2)r_t) = 1$$

so that at every time instant rates for all maturities in the curve are perfectly correlated. For example, the thirty-year interest rate at a given instant is perfectly correlated with the three-month rate at the same instant. This means that a shock to the interest rate curve at

Multidimensional models and correlations III

time t is transmitted equally through all maturities, and the curve, when its initial point (the short rate r_t) is shocked, moves almost rigidly in the same direction. Clearly, it is hard to accept this perfect-correlation feature of the model. Truly, interest rates are known to exhibit some decorrelation (i.e. non-perfect correlation), so that a more satisfactory model of curve evolution has to be found.

One-factor models such as HW, BK, CIR++, EEV may still prove useful when the product to be priced does not depend on the correlations of different rates but depends at every instant on a single rate of the whole interest-rate curve (say for example the six-month rate).

Otherwise, the approximation can still be acceptable, especially for “risk-management-like” purposes, when the rates that jointly influence the payoff at every instant are close (say for example the six-month and one-year rates). Indeed, the real correlation between such near

Multidimensional models and correlations IV

rates is likely to be rather high anyway, so that the perfect correlation induced by the one-factor model will not be unacceptable in principle. But in general, whenever the correlation plays a more relevant role, or when a higher precision is needed anyway, we need to move to a model allowing for more realistic correlation patterns. This can be achieved with multifactor models, and in particular with two-factor models. Indeed, suppose for a moment that we replace the Gaussian Vasicek model with its hypothetical two-factor version (G2):

$$r_t = x_t + y_t,$$

$$dx_t = k_x(\theta_x - x_t)dt + \sigma_x dW_1(t),$$

$$dy_t = k_y(\theta_y - y_t)dt + \sigma_y dW_2(t),$$

with instantaneously-correlated sources of randomness, $dW_1 dW_2 = \rho\, dt$. Again, we will see later on that also for this kind of

Multidimensional models and correlations V

models the bond price is an affine function, this time of the two factors x and y,

$$P(t,T) = A(t,T)\exp(-B^x(t,T)x_t - B^y(t,T)y_t),$$

where quantities with the superscripts "x" or "y" denote the analogous quantities for the one-factor model where the short rate is given by x or y, respectively. Taking this for granted at the moment, we can see easily that now

Multidimensional models and correlations VI

$$\mathrm{Corr}(R(t,T_1), R(t,T_2)) =$$
$$= \mathrm{Corr}(b^x(t,T_1)x_t + b^y(t,T_1)y_t, b^x(t,T_2)x_t + b^y(t,T_2)y_t),$$

and this quantity is not identically equal to one, but depends crucially on the correlation between the two factors x and y, which in turn depends, among other quantities, on the instantaneous correlation ρ in their joint dynamics.

How much flexibility is gained in the correlation structure and whether this is sufficient for practical purposes will be debated. It is however clear that the choice of a multi-factor model is a step forth in that correlation between different rates of the curve at a given instant is not necessarily equal to one.

Another question that arises naturally is: How many factors should one use for practical purposes? Indeed, what we have suggested with two

Multidimensional models and correlations VII

factors can be extended to three or more factors. The choice of the number of factors then involves a compromise between numerically-efficient implementation and capability of the model to represent realistic correlation patterns (and covariance structures in general) and to fit satisfactorily enough market data in most concrete situations.

Multidimensional models: how many factors? I

Usually, historical analysis of the whole yield curve, based on principal component analysis or factor analysis, suggests that under the objective measure two components can explain 85% to 90% of variations in the yield curve, as illustrated for example by Jamshidian and Zhu (1997, Finance and Stochastics 1, in their Table 1), who consider JPY, USD and DEM data. They show that one principal component explains from 68% to 76% of the total variation, whereas three principal components can explain from 93% to 94%. A related analysis is carried out in Chapter 3 of Rebonato (book on interest rate models, 1998, in his Table 3.2) for the UK market, where results seem to be more optimistic: One component explains 92% of the total variance, whereas two components already explain 99.1% of the total variance. In some works an interpretation is given to the components in terms of average level, slope and curvature of the zero-coupon curve, see for example again Jamshidian and Zhu (1997).

Multidimensional models: how many factors? II

What we learn from these analyses is that, in the objective world, a while back a two- or three-dimensional process was needed to provide a realistic evolution of the whole zero-coupon curve. Since the instantaneous-covariance structure of the same process when moving from the objective probability measure to the risk-neutral probability measure does not change, we may guess that also in the risk-neutral world a two- or three-dimensional process may be needed in order to obtain satisfactory results. This is a further motivation for introducing a two- or three-factor model for the short rate. Here, we have decided to focus on two-factor models for their better tractability and implementability. In particular, we will consider additive models of the form

$$r_t = x_t + y_t + \varphi(t), \tag{24}$$

Multidimensional models: how many factors? III

where φ is a deterministic shift which is added in order to fit exactly the initial zero-coupon curve, as in the one-factor case. This formulation encompasses the classical Hull and White two-factor model as a deterministically-shifted two-factor Vasicek (G2++), and an extension of the Longstaff and Schwartz (LS) model that is capable of fitting the initial term structure of rates (CIR2++), where the basic LS model is obtained as a two-factor additive CIR model.

Multidimensional models: volatility shape I

These are the two-factor models we will consider, and we will focus especially on the two-factor additive Gaussian model G2++. The main advantage of the G2++ model over the shifted Longstaff and Schwartz CIR2++ with x and y as in

$$dx_t = k_x(\theta_x - x_t)dt + \sigma_x\sqrt{x_t}dW_1(t),$$

$$dy_t = k_y(\theta_y - y_t)dt + \sigma_y\sqrt{y_t}dW_2(t),$$

is that in the latter we are forced to take $dW_1 dW_2 = 0\,dt$ in order to maintain analytical tractability, whereas in the former we do not need to do so. The reason why we are forced to take $\rho = 0$ in the CIR2++ case lies in the fact that square-root non-central chi-square processes do not work as well as linear-Gaussian processes when adding nonzero instantaneous correlations. Requiring $dW_1 dW_2 = \rho\,dt$ with $\rho \neq 0$ in the above CIR2++ model would indeed destroy analytical tractability: It

Multidimensional models: volatility shape II

would no longer be possible to compute analytically bond prices and rates starting from the short-rate factors. Moreover, the distribution of r would become more involved than that implied by a simple sum of independent non-central chi-square random variables. Why is the possibility that the parameter ρ be different than zero so important as to render G2++ preferable to CIR2++? As we said before, the presence of the parameter ρ renders the correlation structure of the two-factor model more flexible. Moreover, $\rho < 0$ allows for a humped volatility curve of the instantaneous forward rates. Indeed, if we consider at a given time instant t the graph of the T function

$$T \mapsto \sqrt{\mathrm{Var}[d\, f(t, T)]/dt}$$

where the instantaneous forward rate $f(t, T)$ comes from the G2++ model, it can be seen that for $\rho = 0$ this function is decreasing and

Multidimensional models: volatility shape III

upwardly concave. This function can assume a humped shape for suitable values of k_x and k_y only when $\rho < 0$. Since such a humped shape is a desirable feature of the model which is in agreement with market behaviour, it is important to allow for nonzero instantaneous correlation in the G2++ model. The situation is somewhat analogous in the CIR2++ case: Choosing $\rho = 0$ does not allow for humped shapes in the curve

$$T \mapsto \sqrt{\mathrm{Var}[d\, f(t,T)]/dt},$$

which consequently results monotonically decreasing and upwardly concave, exactly as in the G2++ case with $\rho = 0$.

Multidimensional models: G2++ vs CIR2++ I

In turn, the advantage of CIR2++ over G2++ is that, as in the one-factor case where HW is compared to CIR++, it can maintain positive rates through reasonable restrictions on the parameters. Moreover, the distribution of the short rate is the distribution of the sum of two independent noncentral chi-square variables, and as such it has fatter tails than the Gaussian distribution in G2++. This is considered a desirable property, especially because in such a way (continuously-compounded) spot rates for any maturity are affine transformations of such non-central chi-squared variables and are closer to the lognormal distribution than the Gaussian distribution for the same rates implied by the G2++ model. Therefore, both from a point of view of positivity and distribution of rates, the CIR2++ model would be preferable to the G2++ model. However, the humped shape for the instantaneous forward rates volatility curve is very important for the model to be able to fit market data in a satisfactory way.

Multidimensional models: G2++ vs CIR2++ II

Furthermore, the G2++ model is more analytically tractable and easier to implement. These overall considerations then imply that the G2++ model is more suitable for practical applications, even though we should not neglect the advantages that a model like CIR2++ may have. In general, when analyzing an interest rate model from a practical point of view, one should try to answer questions like the following. Is a two-factor model like G2++ flexible enough to be calibrated to a large set of swaptions, or even to caps and swaptions at the same time? How many swaptions can be calibrated in a sufficiently satisfactory way? What is the evolution of the term structure of volatilities as implied by the calibrated model? Is this realistic? How can one implement trees for models such as G2++? Is Monte Carlo simulation feasible? Can the model be profitably used for quanto-like products and for products depending on more than an interest rate curve when

Multidimensional models: G2++ vs CIR2++ III

taking into account correlations between different interest-rate curves and also with exchange rates?

Here we will focus mainly on the G2++ model and we will try to deal with some of the above questions.

The G2++ model I

We assume that the dynamics of the instantaneous-short-rate process under the risk-adjusted measure Q is given by

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0, \tag{25}$$

where the processes $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ satisfy

$$dx(t) = -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0,$$

$$dy(t) = -by(t)dt + \eta dW_2(t), \quad y(0) = 0,$$

where (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ as from

$$dW_1(t)dW_2(t) = \rho dt,$$

The G2++ model II

where r_0, a, b, σ, η are positive constants, and where $-1 \leq \rho \leq 1$. The function φ is deterministic and well defined in the time interval $[0, T^*]$, with T^* a given time horizon, typically 10 or 30 (years). In particular, $\varphi(0) = r_0$.
Parameters a, b, σ, η and ρ are used to calibrate a few liquid swaptions.

For a deeper discussion of G2++ including derivation of formulas for bonds and options, volatility structures, comparison with 2 factor CIR etc, see

- the MSc course "Interest Rate Models" by Prof Brigo at the Dept of Mathematics,
- or the course in Fixed Income by Prof Lara Cathcart in this same Master Programme, or
- the book "Interest Rate Models" by Brigo and Mercurio (2006).

First Choice: short rate r I

This approach is based on the fact that the zero coupon curve at any instant, or the (informationally equivalent) zero bond curve

$$T \mapsto P(t,T) = E_t^Q \exp\left(-\int_t^T \boxed{r_s}\, ds\right)$$

is completely characterized by the probabilistic/dynamical properties of r. So we write a model for r, the initial point of the curve $T \mapsto L(t,T)$ for $T = t$ at every instant t.

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t$$

- Unrealistic correlation patterns between points of the curve with different maturities. for example, in one-factor short-rate models

$$\mathrm{Corr}(dF_i(t), dF_j(t)) = 1;$$

First Choice: short rate r II

- Poor calibration capabilities: can only fit a low number of caps and swaptions unless dangerous and uncontrollable extensions are taken into account;
- Difficulties in expressing market views and quotes in terms of model parameters;
- Related lack of agreement with market valuation formulas for basic derivatives.
- Models that are good as distribution (lognormal models) are not analytically tractable and have problems of explosion for the bank account.

What do we model? Second Choice: instantaneous forward rates $f(t, T)$ I

Recall the forward LIBOR rate at time t between T and S, $F(t; T, S) = (P(t, T)/P(t, S) - 1)/(S - T)$, which makes the FRA contract to lock in at time t interest rates between T and S fair. When S collapses to T we obtain *instantaneous* forward rates:

$$f(t, T) = \lim_{S \to T^+} F(t; T, S) \approx -\frac{\partial \ln P(t, T)}{\partial T}, \quad \lim_{T \to t} f(t, T) = r_t.$$

Why should one be willing to model the f's at all? The f's are not observed in the market, so that there is no improvement with respect to modeling r in this respect.

What do we model? Second Choice: instantaneous forward rates $f(t, T)$ II

Moreover notice that f's are more structured quantities:

$$f(t, T) = -\frac{\partial \ln E_t \left[\exp \left(-\int_t^T r(s) ds \right) \right]}{\partial T},$$

$$P(t, T) = e^{-\int_t^T f(t,u) du}$$

Given the structure in r, we may expect some restrictions on the risk-neutral dynamics that are allowed for f.

What do we model? Second Choice: instantaneous forward rates $f(t, T)$ III

Indeed, there is a fundamental theoretical result: Set $f(0, T) = f^M(0, T)$. We have

$$df(t, T) = \boxed{\sigma(t, T)\left(\int_t^T \sigma(t, s)ds\right)} dt + \sigma(t, T)dW(t),$$

under the risk neutral world measure, if no arbitrage has to hold. Thus we find that the no-arbitrage property of interest rates dynamics is here clearly expressed as a **link between the local standard deviation (volatility or diffusion coefficient) and the local mean (drift)** in the dynamics.

What do we model? Second Choice: instantaneous forward rates $f(t, T)$ IV

Given the volatility, there is no freedom in selecting the drift, contrary to the more fundamental models based on dr_t, where the whole risk neutral dynamics was free:

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t$$

b and σ had no link due to no-arbitrage.

Second Choice, modeling f (HJM): is it worth it? I

$$df(t,T) = \boxed{\sigma(t,T)\left(\int_t^T \sigma(t,s)ds\right)} dt + \sigma(t,T)dW(t),$$

This can be useful to study arbitrage free properties of models, but when in need of writing a concrete model to price and hedge financial products, most useful models coming out of HJM are the already known short rate models seen earlier (these are particular HJM models, especially Gaussian models) or the market models we are going to see later.

Even though market models do not necessarily need the HJM framework to be derived, HJM may serve as a unifying framework in which all categories of no-arbitrage interest-rate models can be expressed.

What? 3d choice: MARKET MODELS. Intro I

Before market models were introduced, short-rate models used to be the main choice for pricing and hedging interest-rate derivatives. Short-rate models are still chosen for many applications and are based on modeling the instantaneous spot interest rate ("short rate" r_t) via a (possibly multi-dimensional) diffusion process. This diffusion process characterizes the evolution of the complete yield curve in time.
To introduce market models, recall the forward LIBOR rate at time t between T and S,

$$F(t; T, S) = \frac{1}{(S - T)}(P(t, T)/P(t, S) - 1),$$

which makes the FRA contract to lock in at time t interest rates between T and S fair (=0). **A family of such rates for $(T, S) = (T_{i-1}, T_i)$ spanning $T_0, T_1, T_2, \ldots, T_M$ is modeled in the LIBOR market model.**

What? 3d choice: MARKET MODELS. Intro II

These are rates associated to market payoffs (FRA's) and not abstract rates such as r_t or $f(t, T)$ (rates on infinitesimal maturities/tenors).

To further motivate market models, let us consider the time-0 price of a T_2-maturity caplet resetting at time T_1 ($0 < T_1 < T_2$) with strike X and a notional amount of 1. Let τ denote the year fraction between T_1 and T_2. Such a contract pays out at time T_2 the amount

$$\tau(L(T_1, T_2) - X)^+ = \tau(F_2(T_1) - X)^+.$$

On the other hand, the market has been pricing caplets (actually caps) with Black's formula for years. This formula can be derived under the LIBOR model dynamics (the only dynamical model that is consistent with it) using the following facts.

What? 3d choice: MARKET MODELS. Intro III

FACT ONE. *The price of any asset divided by a reference asset (called numeraire) is a martingale (no drift) under the measure associated with that numeraire.*
In particular,

$$F_2(t) = \frac{(P(t,T_1) - P(t,T_2))/(T_2 - T_1)}{P(t,T_2)},$$

is a portfolio of two zero coupon bonds divided by the zero coupon bond $P(\cdot, T_2)$. If we take the measure Q^2 associated with the numeraire $P(\cdot, T_2)$, by FACT ONE F_2 will be a martingale (no drift) under that measure.
F_2 is a martingale (no drift) under that Q^2 measure associated with numeraire $P(\cdot, T_2)$.

What? 3d choice: MARKET MODELS. Intro IV

FACT TWO: THE TIME-t RISK NEUTRAL PRICE

$$\text{Price}_t = E_t^{\boxed{B}}\left[\boxed{B(t)}\frac{\text{Payoff}(T)}{\boxed{B(T)}}\right]$$

IS INVARIANT BY CHANGE OF NUMERAIRE: IF S IS ANY OTHER NUMERAIRE, WE HAVE

$$\text{Price}_t = E_t^{\boxed{S}}\left[\boxed{S_t}\frac{\text{Payoff}(T)}{\boxed{S_T}}\right].$$

IN OTHER TERMS, IF WE SUBSTITUTE THE THREE OCCURRENCES OF THE NUMERAIRE WITH A NEW NUMERAIRE THE PRICE DOES NOT CHANGE.

What? 3d choice: MARKET MODELS. Intro V

The forward LIBOR rates F's are the quantities that are modeled instead of r and f in the LIBOR market model.

$$dF_2(t) = \boxed{\sigma_2(t)}\, F_2(t)dW_2(t), \quad \text{mkt } F_2(0)$$

where σ_2 is the instantaneous volatility, assumed here to be constant for simplicity, and W_2 is a standard Brownian motion under the measure Q^2.

What? 3d choice: MARKET MODELS. Intro VI

The classic market Black's formula for the $T_1 - T_2$ caplet is:

$$\mathrm{Cpl}(0, T_1, T_2, X) = P(0, T_2)\tau[F_2(0)\Phi(d_1) - X\Phi(d_2)],$$
$$d_{1,2} = \frac{\ln\frac{F_2(0)}{X} \pm \frac{1}{2}\int_0^{T_1}\sigma_2^2(t)dt}{\sqrt{\int_0^{T_1}\sigma_2^2(t)dt}}$$

The term in squared brackets can be also written as

$$= F_2(0)\Phi(d_1) - X\Phi(d_2), \quad d_{1,2} = \frac{\ln\frac{F_2(0)}{X} \pm \frac{1}{2}T_1 v_1(T_1)^2}{\sqrt{T_1}v_1(T_1)}$$

where $v_1(T_1)$ is the time-averaged quadratic volatility

$$v_1(T_1)^2 = \frac{1}{T_1}\int_0^{T_1}\sigma_2(t)^2 dt.$$

What? 3d choice: MARKET MODELS. Intro VII

Notice that in case $\sigma_2(t) = \sigma_2$ is constant we have $v_1(T_1) = \sigma_2$. Summing up: take

$$dF(t; T_1, T_2) = \sigma_2\, F(t; T_1, T_2) dW_2(t), \quad \text{mkt } F(0; T_1, T_2)$$

The current zero-curve $T \mapsto L(0, T)$ is calibrated through **the initial market** $F(0; T, S)$**'s.** This dynamics in **under the numeraire** $P(\cdot, T_2)$ (measure Q^2), where W_2 is a Brownian motion. Computing

$$E\left[\frac{B(0)}{B(T_2)}\tau(F(T_1; T_1, T_2) - X)^+\right]$$

gives the Black formula used in the market to convert Cpl prices in volatilities σ and vice-versa. This dynamical model is thus compatible with Black's market formula. The key property is **lognormality** of F when taking the expectation.

What? 3d choice: MARKET MODELS. Intro VIII

The example just introduced is a simple case of what is known as "lognormal forward-LIBOR model". It is known also as Brace-Gatarek-Musiela (1997) model, from the name of the authors of one of the first papers where it was introduced rigorously. This model was also introduced earlier by Miltersen, Sandmann and Sondermann (1997). Jamshidian (1997) also contributed significantly to its development. However, a common terminology is now emerging and the model is generally known as "LIBOR Market Model" (LMM).

What? 3d choice: MARKET MODELS. Intro IX

Question: Can this model be obtained as a special **short rate model?** Is there a choice for the equation of r that is consistent with the above market formula, or with the lognormal distribution of F's?
Again to fix ideas, let us choose a specific short-rate model and assume we are using the Vasicek model. The parameters k, θ, σ, r_0 are denoted by α.

$$r_t = x_t, \quad dx_t = k(\theta - x_t)dt + \sigma dW_t.$$

Such model allows for an analytical formula for forward LIBOR rates F,

$$F(t; T_1, T_2) \;\; = \;\; F^{VAS}(t; T_1, T_2; x_t, \alpha).$$

Question: Is there a short-rate model compatible with the Market model?

What? 3d choice: MARKET MODELS. Intro X

F^{VAS} is not lognormal, nor are F's associated to other known short rate models. So no known short rate model is consistent with the market formula. Short rate models are calibrated through their particular formulas for caplets, but these formulas are not Black's market formula (although some are close).
When Hull and White (extended VASICEK) is calibrated to caplets one has the values of k, θ, σ, x_0 consistent with caplet prices, but these parameters don't have an **immediate intuitive meaning** for traders, who don't know **how to relate them to Black's market formula**. On the contrary, the parameter σ_2 in the mkt model **has an immediate meaning as the Black caplet volatility of the market. There is an immediate link between model parameters and market quotes. Language is important**.

What? 3d choice: MARKET MODELS. Intro XI

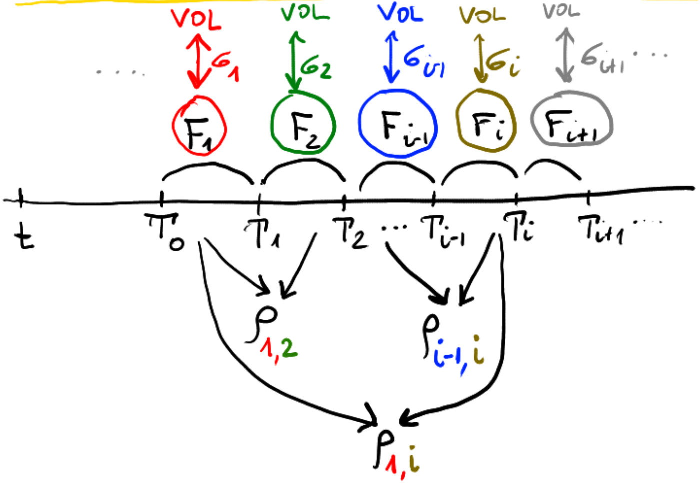
Modeling correlation is necessary for pricing payoffs depending on more than a single rate at a given time, such as swaptions.
One can select a different σ for each forward rate by assuming each forward rate to have a constant instantaneous volatility. Alternatively, one can select piecewise-constant instantaneous volatilities for each forward rate. Moreover, different forward rates can be modeled as each having different random sources Z that are **instantaneously correlated**. This implies that we have great freedom in modeling

$$corr(dF_k(t), dF_j(t)) = \rho_{k,j}$$

whereas in one-factor short rate models dr these correlations were fixed practically to 1.

What? 3d choice: MARKET MODELS. Intro XII

WITH THE LMM WE MAY SPECIFY PRECISE VOLATILITIES AND CORRELATIONS ACROSS THE TERM STRUCTURE



What? 3d choice: MARKET MODELS. Intro XIII

The LIBOR market model for *F*'s allows for:

- immediate and intuitive calibration of caplets (better than any short rate model)
- easy calibration to swaptions through algebraic approximation (again better than most short rate models)
- can virtually calibrate a high number of market products exactly or with a precision impossible to short rate models;
- clear correlation parameters, since these are intantaneous correlations of market forward rates;
- Powerful diagnostics: can check **future** volatility and terminal correlation structures (Diagnostics impossibile with most short rate models);
- Can be used for monte carlo simulation;

What? 3d choice: MARKET MODELS. Intro XIV

However the LIBOR market model is not the only market model. The simple market options on interest rates are divided in two markets CAPS/FLOORS and SWAPTION.

The LIBOR market model is the model of choice for caplets, as we have seen, since it produces the Black-Scholes type (Black's) caplet formula the market uses to quote implied volatilities.

But what about SWAPTIONS?

SWAPTIONS can be managed well in the LIBOR model only through approximations like drift freezing. To properly deal with swaptions, one would have to use a different market model, the SWAP market model (SMM).

We now present it briefly.

SWAP market model (SMM) I

Consider the payer swaption giving the right (and no obligation) to enter into the swap first resetting in T_α and paying at $T_{\alpha+1}$, $T_{\alpha+2}$... up to T_β, for a fixed rate K.

Recall that one way to write the payout of such option at maturity T_α is

$$(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i).$$

Let's define the annuity numeraire, also known as Present Value per Basis Point (PVPBP), PV01 or DV01, and the related measure:

$$U = C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i), \quad Q^U = Q^{\alpha,\beta}$$

SWAP market model (SMM) II

By FACT ONE the forward swap rate $S_{\alpha,\beta}$ is then a martingale under $Q^{\alpha,\beta}$:

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)} = \frac{P(t,T_\alpha) - P(t,T_\beta)}{C_{\alpha,\beta}(t)}$$

Take the usual martingale (zero drift) lognormal geometric brownian motion

$$d\ S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t)\ dW_t^{\alpha,\beta}, \quad Q^{\alpha,\beta} \quad \textbf{(SMM)},$$

SWAP market model (SMM) III

BY FACT TWO on the change of numeraire

$$E^{\boxed{B}}\left((S_{\alpha,\beta}(T_\alpha)-K)^+\ C_{\alpha,\beta}(T_\alpha)\frac{\boxed{B(0)}}{\boxed{B(T_\alpha)}}\right)=$$

$$= \ E^{\boxed{\alpha,\beta}}\left[(S_{\alpha,\beta}(T_\alpha)-K)^+ C_{\alpha,\beta}(T_\alpha)\frac{\boxed{C_{\alpha,\beta}(0)}}{\boxed{C_{\alpha,\beta}(T_\alpha)}}\right]$$

$$= C_{\alpha,\beta}(0)\ E^{\alpha,\beta}\left[(S_{\alpha,\beta}(T_\alpha)-K)^+\right]$$

$$= C_{\alpha,\beta}(0)\ [S_{\alpha,\beta}(0)\Phi(d_1)-K\Phi(d_2)],\ \ d_{1,2}=\frac{\ln\frac{S_{\alpha,\beta}(0)}{K}\pm\frac{1}{2}T_\alpha v^2_{\alpha,\beta}(T_\alpha)}{\sqrt{T_\alpha}v_{\alpha,\beta}(T_\alpha)}$$

$$v^2_{\alpha,\beta}(T)=\frac{1}{T}\int_0^T(\sigma^{(\alpha,\beta)}(t))^2\,dt\ .$$

SWAP market model (SMM) IV

This is the well known Black's formula for swaptions.

It is a Black Scholes type formula for swaptions.

It is the formula the market uses to convert swaptions prices into swaptions implied volatilities v.

SMM is the only model that is consistent with this market formula.

LMM is not compatible with the Black formula for Swaptions.

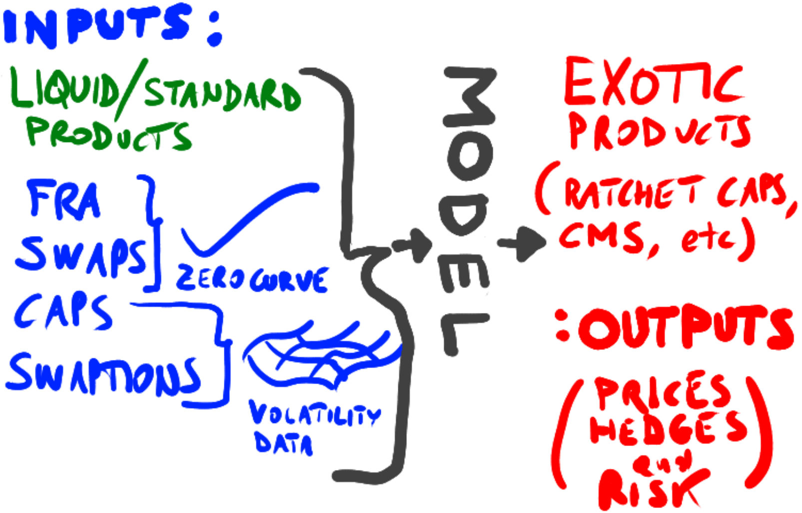
The SMM is not used as much as the LMM. The reason is that swap rates do not recombine as well as forward rates in describing other rates. Also, swaptions can be priced easily in the LMM through drift freezing with formulas that are very similar to the market swaptions formula. It follows that, even if in principle the two models are not compatible and consistent, in practice the LMM is quite close to the SMM even in terms of swap rate dynamics.

Giving rigor to Black's formulas: The LMM market model in general I

In order to derive the above dynamics of F_k under Q^i in general, one needs to use the change of numeraire technique. This is illustrated in detail, for example:

- In the Interest Rate Models course for MSc in Mathematics and Finance at the Dept of Mathematics (Prof. Brigo)
- In the Fixed Income elective course of this master (Prof. Lara Cathcart)
- In the Brigo Mercurio book "Interest Rate Models: Theory and Practice" available in the library

THE MODEL CALIBRATION



THE MODEL CALIBRATION

INPUTS:

LIQUID/STANDARD PRODUCTS

FRA
SWAPS
CAPS
SWAPTIONS

INITIAL FORWARD RATES $F_1(0), F_2(0), \ldots F_n(0)$

VECTOR OF CAPLET VOLS,

MATRIX OF SWAPTIONS

(+ HISTORICAL F CORREL ρ)

$\rightarrow$ MODEL $\rightarrow$

MODEL $\downarrow$ σ and ρ

EXOTIC PRODUCTS (RATCHET CAPS, CMS, etc)

:OUTPUTS

(PRICES HEDGES and RISK)

LMM instantaneous covariance structures I

LMM is natural for caps and SMM is natural for swaptions. **Choose.**
We could choose LMM and adapt it to price swaptions.
Recall: Under numeraire $P(\cdot, T_i) \neq P(\cdot, T_k)$:

$$dF_k(t) = \mu_k^i(t)\ F_k(t)\ dt + \boxed{\sigma_k(t)} F_k(t)\, dZ_k, \quad dZ\ dZ' = \boxed{\rho}\ dt$$

Model specification: Choice of $\sigma_k(t)$ and of ρ.
Only the σ's have impact on caplet (and cap) prices, the ρ's having no influence.

Powerful Vol term structure diagnostics I

Very important: The LMM allows us to inspect the future term structure of volatilities almost without effort.

- The caplet volatilities curve in the future is deterministic and we can compute its future values today.
- Once we calibrate the LMM to caplets (and swaptions, possibly, later), we not only make sure that the model is consistent with the *current* caplet volatility term structure, but we can immediately check *how this term structure will evolve in the future.*
- This is very important, as this helps deciding whether a calibration is good or not, and whether we should change the LMM instantaneous volatility parameterization.

Powerful Vol term structure diagnostics II

- In most models, and the short rate models we have seen earlier in particular, it is impossible to plot the future caplet volatilities term structure. This is random and a plot can be done only conditionally on a future realization of the short rate, and usually with great numerical effort.
- Market models instead allow to do this via summations and square root. Diagnostics could be easily implemented in a spreadsheet.

Instantaneous correlation: Parametric forms I

Swaptions depend on **terminal** correlation among forward rates (ie on both ρ's **and** σ's). How do we model ρ?

A long detailed treatment of parametric correlation matrices and of rank reduction is given in the Interest Rate Models Master Course at Mathematics or in the Brigo Mercurio "Interest Rate Models" book. Here we report:

Full Rank, Classical, two-parameters, exponentially decreasing parameterization

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty) \exp[-\beta|i - j|], \quad \beta \geq 0.$$

where now ρ_∞ is only asymptotically representing the correlation between the farthest rates in the family.

Instantaneous correlation: Parametric forms II

Correlations should preferably decrease when moving away from the diagonal (farther rates are less correlated).

Full Rank (Rebonato) three parameters exponential form

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty) \exp[-|i - j|(\beta - \alpha (\max(i, j) - 1))], \tag{26}$$

Here, the domain of positivity for the resulting matrix is not specified “off-line” in terms of $\alpha, \beta, \rho_\infty$.

Monte Carlo and variance reduction

See the Interest Rate Models course at Mathematics or the Brigo Mercurio book.

Analytical swaption prices with LMM I

Approximated method to compute swaption prices with the LMM LIBOR MODEL without resorting to Monte Carlo simulation.
This method is rather simple and its quality has been tested in Brace, Dun, and Barton (1999) and by Brigo et al.
Use Black's formula for swaptions with volatility $v^{\mathrm{LMM}}_{\alpha,\beta}$ to price swaptions **analytically** with the LMM.
It turns out that the approximation is not at all bad, as pointed out by Brace, Dun and Barton (1999) and by Brigo et al.
This pricing formula is ALGEBRAIC and very quick (compare with short-rate models)

Analytical terminal correlation I

By similar arguments (freezing the drift and collapsing all measures) one may find a formula for terminal correlation. ($\mathrm{Corr}(F_i(T_\alpha), F_j(T_\alpha))$ should be computed with MC simulation and depends on the chosen numeraire). Brigo and Mercurio (2001) obtain easily

$$\frac{\exp\left(\int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\rho_{i,j} dt\right) - 1}{\sqrt{\exp\left(\int_0^{T_\alpha} \sigma_i^2(t)dt\right) - 1}\sqrt{\exp\left(\int_0^{T_\alpha} \sigma_j^2(t)dt\right) - 1}}$$

$$\approx \rho_{i,j}\frac{\int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\, dt}{\sqrt{\int_0^{T_\alpha} \sigma_i^2(t)dt}\sqrt{\int_0^{T_\alpha} \sigma_j^2(t)dt}},$$

the second approximation as from Rebonato (1999). Schwartz's inequality: *terminal correlations are always smaller, in absolute value, than instantaneous correlations*.

Calibration: Summing up

Calibration targets (via low # of factors):

- A small rank for ρ (for Monte Carlo) and small calibration error;
- Positive and decreasing inst. and term. correlations;
- Smooth and stable evolution of the term structure of vols;

The one-to-one formulation is perhaps the most promising: Fitting to swaptions is exact; can fit caps by introducing infra-correlations; instantaneous correlation OK by construction; Terminal correlation not spoiled by the fitted σ's; Terms structure evolution smooth but not fully satisfactory qualitatively.

Requirements hardly checkable with general HJM or short-rate models

For the details and a refinement of the cascade calibration algorithm see Brigo Mercurio book ”Interest Rate Models”.

Zero Coupon Swaption I

A payer (receiver) zero-coupon swaption is a contract giving the right to enter a payer (receiver) zero-coupon IRS at a future time. A zero-coupon IRS is an IRS where a single fixed payment is due at the unique (final) payment date T_β for the fixed leg in exchange for a stream of usual floating payments $\tau_i L(T_{i-1}, T_i)$ at times T_i in $T_{\alpha+1}, T_{\alpha+2}, \ldots, T_\beta$ (usual floating leg). In formulas, the discounted payoff of a payer zero-coupon IRS is, at time $t \leq T_\alpha$:

$$\frac{B(t)}{B(T_\alpha)} \left[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta} K \right],$$

where $\tau_{\alpha,\beta}$ is the year fraction between T_α and T_β. The analogous payoff for a receiver zero-coupon IRS is obviously given by the opposite quantity.

Zero Coupon Swaption II

Taking risk-neutral expectation, we obtain easily the contract value as

$$P(t, T_\alpha) - P(t, T_\beta) - \tau_{\alpha,\beta} K P(t, T_\beta),$$

which is the typical value of a floating leg minus the value of a fixed leg with a single final payment.

Zero Coupon Swaption I

The value of K that renders the contract fair is obtained by equating to zero the above value. $K = F(t; T_\alpha, T_\beta)$. Indeed, the value of the swap is independent of the number of payments on the floating leg, since the floating leg always values at par, no matter the number of payments. Therefore, we might as well have taken a floating leg paying only in T_β the amount $\tau_{\alpha,\beta} L(T_\alpha, T_\beta)$. This would have given us again a standard swaption, standard in the sense that the two legs of the underlying IRS have the same payment dates (collapsing to T_β) and the unique reset date T_α. In such a one-payment case, the swap rate collapses to a forward rate, so that we should not be surprised to find out that the forward swap rate in this particular case is simply a forward rate.

Zero Coupon Swaption I

An option to enter a payer zero-coupon IRS is a payer zero-coupon swaption, and the related payoff is

$$\frac{B(t)}{B(T_\alpha)}[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta} K]^+,$$

or, equivalently, by expressing the F's in terms of discount factors,

$$\frac{B(t)}{B(T_\alpha)}\,[1 - P(T_\alpha, T_\beta) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta} K]^+ \,,$$

which in turn can be written as

$$\frac{B(t)}{B(T_\alpha)}\tau_{\alpha,\beta} P(T_\alpha, T_\beta)\,[F(T_\alpha; T_\alpha, T_\beta) - K]^+ \,.$$

Zero Coupon Swaption I

$$\frac{B(t)}{B(T_\alpha)}\tau_{\alpha,\beta}P(T_\alpha, T_\beta)\left[F(T_\alpha; T_\alpha, T_\beta) - K\right]^+ .$$

Notice that, from the point of view of the payoff structure, this is merely a caplet. As such, it can be priced easily through Black's formula for caplets. The problem, however, is that such a formula requires the integrated percentage variance (volatility) of the forward rate $F(\cdot; T_\alpha, T_\beta)$, which is a forward rate over a non-standard period. Indeed, $F(\cdot; T_\alpha, T_\beta)$ is not in our usual family of spanning forward rates, unless we are in the trivial case $\beta = \alpha + 1$.

Zero Coupon Swaption I

Therefore, since the market provides us (through standard caps and swaptions) with volatility data for standard forward rates, we need a formula for deriving the integrated percentage volatility of the forward rate $F(\cdot; T_\alpha, T_\beta)$ from volatility data of the standard forward rates $F_{\alpha+1}, \ldots, F_\beta$. The reasoning is once again based on the “freezing the drift” procedure, leading to an approximately lognormal dynamics for our standard forward rates.

Zero Coupon Swaption I

Denote for simplicity $F(t) := F(t; T_\alpha, T_\beta)$ and $\tau := \tau_{\alpha,\beta}$.
We begin by noticing that, through straightforward algebra, we have (write everything in terms of discount factors to check)

$$1 + \tau F(t) = \prod_{j=\alpha+1}^{\beta} (1 + \tau_j F_j(t)).$$

It follows that

$$\ln(1 + \tau F(t)) = \sum_{j=\alpha+1}^{\beta} \ln(1 + \tau_j F_j(t)),$$

so that $d \ln(1 + \tau F(t)) =$

$$= \sum_{j=\alpha+1}^{\beta} d \ln(1 + \tau_j F_j(t)) = \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1 + \tau_j F_j(t)} + (\ldots)dt.$$

Zero Coupon Swaption II

$$\text{Since } dF(t) = \frac{1 + \tau F(t)}{\tau} d \ln(1 + \tau F(t)) + (\ldots)dt,$$

we obtain from the above expression

$$dF(t) = \frac{1 + \tau F(t)}{\tau} \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1 + \tau_j F_j(t)} + (\ldots)dt.$$

Zero Coupon Swaption I

$$dF(t) = \frac{1+\tau F(t)}{\tau} \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1+\tau_j F_j(t)} + (\dots)dt.$$

Take variance (conditional on t) on both sides:

$$\mathrm{Var}\left(\frac{dF(t)}{F(t)}\right) = \left[\frac{1+\tau F(t)}{\tau F(t)}\right]^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{i,j} \sigma_i(t)\sigma_j(t) F_i(t) F_j(t)}{(1+\tau_i F_i(t))(1+\tau_j F_j(t))} dt.$$

Freeze all $t's$ to 0 except for the σ's, and integrate over $[0, T_\alpha]$:
$(v^{zc}_{\alpha,\beta})^2 := (1/T_\alpha)\times$

$$\left[\frac{1+\tau F(0)}{\tau F(0)}\right]^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{i,j} F_i(0) F_j(0)}{(1+\tau_i F_i(0))(1+\tau_j F_j(0))} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt.$$

Zero Coupon Swaption II

To price the zero-coupon swaption it is then enough to put this quantity into the related Black's Caplet formula:

$$\mathbf{ZCPS} = \tau P(0, T_\beta)[F(0)\Phi(d_1(F(0), K, v^{zc}_{\alpha,\beta}))$$

$$-K\Phi(d_2(F(0), K, v^{zc}_{\alpha,\beta}))].$$

All cases show the formula to be sufficiently accurate for practical purposes.
When using the formula notice that the at-the-money standard swaption has always a lower volatility (and hence price) than the corresponding at-the-money zero-coupon swaption.

Constant Maturity Swaps (CMS's) I

A constant-maturity swap is a financial product structured as follows. We assume a unit nominal amount. Let us denote by $\{T_0, \ldots, T_n\}$ a set of payment dates at which coupons are to be paid. At time T_{i-1} (in some variants at time T_i), $i \geq 1$, institution A pays to B the c-year swap rate resetting at time T_{i-1} in exchange for a fixed rate K. Formally, at time T_{i-1} institution A pays to B

$$S_{i-1,i-1+c}(T_{i-1})\,\tau_i\,,$$

instead of

$$L(T_{i-1}, T_i)\tau_i = F_i(T_{i-1})\,\tau_i\,,$$

as would be natural (standard Interest Rate Swap with model independent valuation, see earlier Lecture).

Constant Maturity Swaps (CMS's) I

The net value of the contract to B at time 0 is

$$E^B\left(\sum_{i=1}^{n}\frac{B(0)}{B(T_{i-1})}(S_{i-1,i-1+c}(T_{i-1})-K)\tau_i\right)$$

$$=\sum_{i=1}^{n}\tau_i E^{\boxed{B}}\left[\frac{\boxed{B(0)}}{\boxed{B(T_{i-1})}}S_{i-1,i-1+c}(T_{i-1})\right]-K\sum_{i=1}^{n}\tau_i P(0,T_{i-1})$$

We can change numeraire in two ways: choose a rolling numeraire in each different term, $P(\cdot, T_{i-1})$, or choose the single "final" numeraire

Constant Maturity Swaps (CMS's) II

$P(\cdot, T_n)$

$$1: \quad \rightarrow = \sum_{i=1}^{n} \tau_i P(0, T_{i-1}) \left[E^{i-1} \left(S_{i-1,i-1+c}(T_{i-1}) \right) - K \right]$$

$$2: \quad \rightarrow = \sum_{i=1}^{n} \tau_i \left(P(0, T_n) \, E^n \left(\frac{S_{i-1,i-1+c}(T_{i-1})}{P(T_{i-1}, T_n)} \right) - K P(0, T_{i-1}) \right).$$

CMS's I

We need only compute either

$$E^{i-1}\left[S_{i-1,i-1+c}(T_{i-1})\right] \quad \text{or} \quad E^{n}[S_{i-1,i-1+c}(T_{i-1})/P(T_{i-1},T_n)]$$

At first sight, one might think to discretize the dynamics of the forward swap rate in the swap model under the relevant forward measure, and compute the required expectation through a Monte Carlo simulation. However, notice that forward rates appear in the drift of such equation, so that we are forced to evolve forward rates anyway. As a consequence, we can build forward swap rates as functions of the forward LIBOR rates obtained by the Monte Carlo simulated dynamics of the LIBOR model. Find the swap rate $S_{i-1,i-1+c}(T_{i-1})$ from the T_{i-1} values of the (Monte Carlo generated) spanning forward rates

$$F_i(T_{i-1}), F_{i+1}(T_{i-1}), \ldots, F_{i-1+c}(T_{i-1}).$$

CMS's II

Analogously to earlier cases, such forward rates can be generated according to the usual discretized (Milstein) dynamics based on Gaussian shocks and under the unique measure Q^n for example. Alternatively, resort to $S_{\alpha,\beta}(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) F_i(T_\alpha)$ and compute

$$E^\alpha S_{\alpha,\beta}(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) E^\alpha F_i(T_\alpha)$$

$$\approx \sum_{i=\alpha+1}^{\beta} w_i(0) e^{\int_0^{T_\alpha} \bar{\mu}_{\alpha,i}(t)dt} F_i(0)$$

We have frozen again the drift in the F_i's dynamics of the F's under Q^α. This can be compared with classical market convexity adjustments. The two methods give similar results when volatilities are not too high.

CMS's III

Notation for $\bar{\mu}$ was given at the beginning of this unit.

The method is general and can be used whenever swap rates or forward rates are paid at times that are not "natural" in swaps and similar contracts. A dynamics can be obtained by the freezing procedures outlined above.

ADDING SMILE TO LIBOR MODELS

For extension of the LMM dynamics that incorporate the so called volatility smile of caps and swaptions, in a variety of models, see the MSc course "Interest Rate Models" at Mathematics or the Brigo Mercurio book "Interest Rate Models".

The crisis (2008-current). Multiple curves

Following the 7[8] credit events happening to Financials in one month of 2008,

Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir and Kaupthing [and Merrill Lynch]

the market broke up and interest rates that used to be very close to each other and were used to model risk free rates for different maturities started to diverge.

Multiple curves: LIBOR?

Credit/Default-free interest rates r_t, $L(t, T)$, $F(t, T_{i-1}, T_i)$ etc?

So it is not clear what is the risk free rate r_t anymore, but especially credit/default-free interest rates with finite (rather than infinitesimal) tenor $T - t$ are hard to define: What is the credit/default-free $L(t, T)$? In the above course we identified it with LIBOR interbank rates, ie interest rates banks charge each other for lending and borrowing. However, after the credit events above, banks can no longer be considered as default free, so that Interbank rates, and LIBOR rates in particular, are contaminated by counterparty credit risk and liquidity risk.

LIBOR has been also subject to illegal manipulation (see the LIBOR rigging scandal involving a number of major banks), but this is fraud risk and is another story.

Multiple curves: OIS?

Besides LIBOR, other rates have been considered as default/credit risk free rates in the past. One of the most popular is the overnight rate. This is an interest rate $O(t_{i-1}, t_i)$ applied at time t_{i-1} to a loan that is closed one or two days later at t_i. Hence the credit risk embedded in the overnight rate is only on one day and is limited. Furthermore, overnight rates are harder to manipulate illegally (some are quoted by central banks).

There are *swaps* built on overnight rates, and they are called Overnight Indexed Swaps (OIS).

Multiple curves: OIS?

OIS have been introduced back in the mid nineties. The maturities T of OISs range from 1 week to 2 years or longer.

Overnight swaps

At maturity T, the swap parties calculate the final payment as a difference between the accrued interest of the fixed rate K and the geometric average $L^O(0,T)$ of the floating index rates $O(t_{i-1},t_i)$ on the swap notional for t_i ranging from the initial time $t_{\text{first}} = 0$ to the swap maturity $t_{\text{last}} = T$. Since the net difference is exchanged, rather than swapping the actual rates, OISs have little counterparty credit risk.

Overnight swaps vs LIBOR indexed swaps: Counterparty risk

In a LIBOR based swap where we pay L and receive K, if our counterparty defaults (say with zero recovery) we still pay L and we lose the whole K. If the net rate were exhanged as in OIS, at default we would only lose $K - L$ if positive.

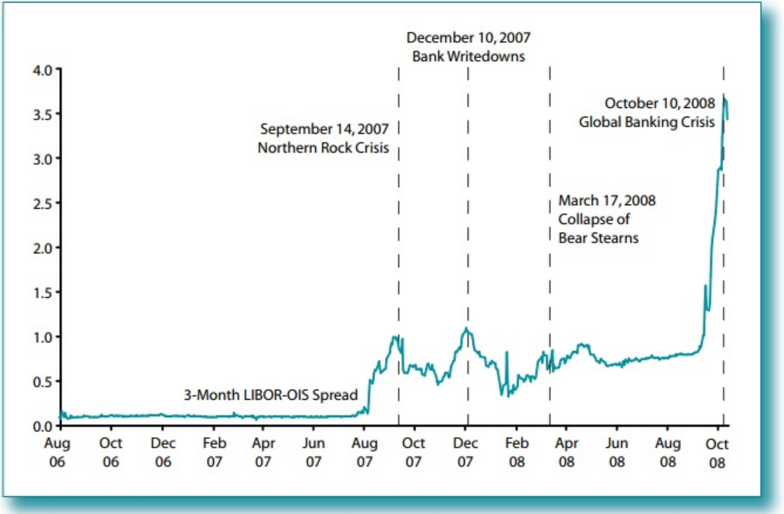


Figure: Spread between 3 months Libor and 3 months ONIA (OIS) swaps. Plotting $t \mapsto L(t, t+3m) - L^O(t, t+3m)$ (proxy of credit and liquidty risk). From Economic Synopses 2008, Number 25, FRB of St Louis

The crisis (2008-current). Multiple curves

At the moment it is no longer realistic to neglect credit risk and liquidity effects in interest rate modeling, pretending there is a risk free rate that is governing the LIBOR and interbank markets.

The OIS rate partly solves the problem as it is a best proxy for a default- and liquidity-free interest rate. Residual credit risk is still present and liquidity effects may still be visible, especially under strong stress scenarios.

These days one tends to use overnight swap rates as proxies for the risk free rates, whereas LIBOR and LIBOR-based swap rates have to be managed more carefully. **There are multiple curves that can be built for discounting, some LIBOR based, other OIS based, and yet other different ones.**

The following table is taken by a presentation of Marco Bianchetti (2011)

The crisis (2008-current). Multiple curves

	Libor	Euribor	Eonia	Eurepo
Definition	London InterBank Offered Rate	Euro InterBank Offered Rate	Euro OverNight Index Average	Euro Repurchase Ageement rate
Market	London Interbank	Euro Interbank	Euro Interbank	Euro Interbank
Side	Offer	Offer	Offer	Offer
Rate quotation specs	EURLibor = Euribor, Other currencies: minor differences (e.g. act/365, T+0, London calendar for GBPLibor).	TARGET calendar, settlement *T+2*, act/360, three decimal places, modified following, end of month, tenor variable.	TARGET calendar, settlement *T+1*, act/360, three decimal places, tenor 1d.	As Euribor
Maturities	1d-12m	1w, 2w, 3w,1m,...,12m	1d	T/N-12m
Publication time	12.30 CET	11:00 am CET	6:45-7:00 pm CET	As Euribor
Panel banks	8-16 banks (London based) per currency	42 banks from 15 EU countries + 4 international banks	Same as Euribor	34 EU banks plus some large international bank from non-EU countries
Calculation agent	Reuters	Reuters	European Central Bank	Reuters
Transactions based	No	No	Yes	No
Collateral	No (unsecured)	No (unsecured)	No (unsecured)	Yes (secured)
Counterparty risk	Yes	Yes	Low	Negligible
Liquidity risk	Yes	Yes	Low	Negligible
Tenor basis	Yes	Yes	No	No

The crisis (2008-current). Multiple curves

The uncertainty on which rate could be considered as a natural discounting rate is pushing banks to use multiple curves, trying to patch them together, at times in inconsistent ways.

Much work needs to be done to include consistently credit and liquidity effects in interest rate theory from the start, thus avoiding the confusion of unexplained multiple curves. The industry is looking at this now. I have a recent paper with Pallavicini where we try this.

Multiple curves explained as synthesis of more fundamental Credit, Liquidity and Funding effects

Multiple curves explained as synthesis of more fundamental Credit, Liquidity and Funding effects.

Rather than taking the curves as fundamental objects, we need to interpret them as incorporating fundamental effects that need to be modeled first.

These effects are Credit Risk and Liquidity Funding Risk.

We model these risks now.

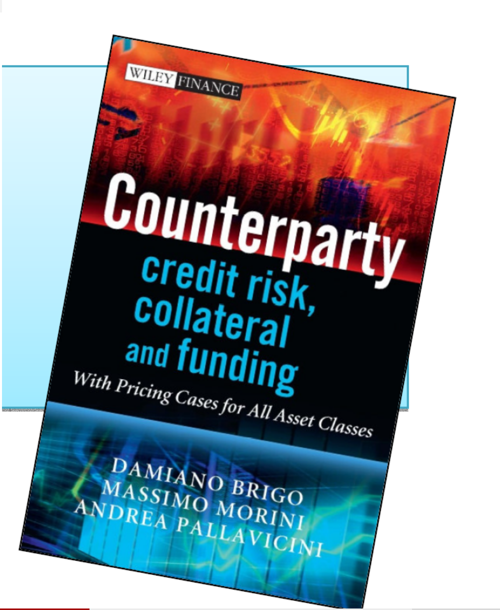
PART 3: PRICING CREDIT RISK, COLLATERAL AND FUNDING

In this Part we look at how we may include Counterparty Credit Risk into the Valuation from the start rather than through unexplained ad-hoc discount (multiple) curves.

This leads to the notions of Credit and Debit Valuation Adjustments (CVA DVA).

We also hint at Funding Valuation Adjustments (FVA).

Presentation based on the Book



Intro to Basic Credit Risk Products and Models

Before dealing with the current topical issues of Counterparty Credit Risk, CVA, DVA and Funding, we need to introduce some basic elements of Credit Risk Products and Credit Risk Modelling.

We now briefly look at:

- Products: Credit Default Swaps (CDS) and Defaultable Bonds
- Payoffs and prices of such products
- Market implied $\mathbb{Q}$ probabilities of default defined by such models
- Intensity models and probabilities of defaults as credit spreads
- Credit spreads as possibly constant, curved or even stochastic
- Credit spread volatility (stochastic credit spreads)

Defaultable (corporate) zero coupon bonds

We started this course by defining the zero coupon bond price $P(t, T)$. Similarly to $P(t, T)$ being one of the possible fundamental quantities for describing the interest-rate curve, we now consider a defaultable bond $\bar{P}(t, T)$ as a possible fundamental variable for describing the defaultable market.

DEFAULT FREE		
time t		time T
:	$\longleftarrow$	:
$P(t, T)$		1

with DEFAULT		
time t		time T :
:	$\longleftarrow$	NO DEFAULT: 1
$\bar{P}(t, T)$		DEFAULT: 0

When considering default, we have a random time τ representing the time at which the bond issuer defaults. $\boxed{\tau : \text{Default time of the issuer}}$

Defaultable (corporate) zero coupon bonds I

The value of a bond issued by the company and promising the payment of 1 at time T, as seen from time t, is the risk neutral expectation of the discounted payoff
BondPrice = Expectation[Discount x Payoff]

$$P(t,T) = \mathbb{E}\{D(t,T)\ 1\ |\mathcal{F}_t\}, \qquad \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T) := \mathbb{E}\{D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\}$$

where $\mathcal{G}_t$ represents the flow of information on whether default occurred before t and if so at what time exactly, and on the default free market variables (like for example the risk free rate r_t) up to t. The filtration of default-free market variables is denoted by $\mathcal{F}_t$. Formally, we assume

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \le u\},\ 0 \le u \le t).$$

D is the stochastic discount factor between two dates, depending on interest rates, and represents discounting.

Defaultable (corporate) zero coupon bonds II

The "indicator" function $\mathbf{1}_{\text{condition}}$ is 1 if "condition" is satisfied and 0 otherwise. In particular, $\mathbf{1}_{\{\tau > T\}}$ reads 1 if default τ did not occur before T, and 0 in the other case.

We understand then that (ignoring recovery) $\mathbf{1}_{\{\tau > T\}}$ is the correct payoff for a corporate bond at time T: the contract pays 1 if the company has not defaulted, and 0 if it defaulted before T, according to our earlier stylized description.

Defaultable (corporate) zero coupon bonds

If we include a recovery amount REC to be paid at default τ in case of early default, we have as discounted payoff at time t

$$D(t,T)\mathbf{1}_{\{\tau>T\}} + \mathrm{R\scriptstyle EC}\, D(t,\tau)\mathbf{1}_{\{\tau\le T\}}$$

If we include a recovery amount REC paid at maturity T, we have as discounted payoff

$$D(t,T)\mathbf{1}_{\{\tau>T\}} + \mathrm{R\scriptstyle EC}\, D(t,T)\mathbf{1}_{\{\tau\le T\}}$$

Taking $\mathbb{E}[\cdot|\mathcal{G}_t]$ on the above expressions gives the price of the bond.

A Lehman bond price example, maturity June 2046, Default on Sep 14, 2008 with indicative recovery 7.625 at the time (auction for CDS will give 8.62%, see below)

Fundamental Credit Derivatives: Credit Default Swaps

Credit Default Swaps are basic protection contracts that became quite liquid on a large number of entities after their introduction in 1991-94.

CDS reached a notional of \$3.7 trillion in 2003, \$62.2 trillion in 2007, \$38.6 trillion in 2008, \$25 tr 2012 (ISDA).

CDS's are now actively traded and are a sort of basic product of the credit derivatives area, analogously to interest-rate swaps and FRA's being basic products in the interest-rate derivatives world.

Then we don't need a model to value CDS's, but rather we need a model that can be *calibrated* to CDS's, i.e. to take CDS's as model inputs (rather than outputs), in order to price more complex derivatives.

As for options, single name CDS options have never been liquid, as there is more liquidity in the CDS index options. We may expect models will have to incorporate CDS index options quotes rather than price them, similarly to what happened to CDS themselves.

Fundamental Credit Derivatives: CDS's

A CDS contract ensures protection against default. Two companies "A" (Protection buyer) and "B" (Protection seller) agree on the following.
If a third company "C" (Reference Credit) defaults at time τ, with $T_a < \tau < T_b$, "B" pays to "A" a certain (deterministic) cash amount $\mathrm{L_{GD}}$.
In turn, "A" pays to "B" a rate R at times $T_{a+1}, \ldots, T_b$ or until default.
Set $\alpha_i = T_i - T_{i-1}$ and $T_0 = 0$.

Protection Seller B	$\rightarrow$ $\leftarrow$	protection $\mathrm{L_{GD}}$ at default τ_C if $T_a < \tau_C \leq T_b$ rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\rightarrow$ $\leftarrow$	Protection Buyer A

(protection leg and premium leg respectively). The cash amount $\mathrm{L_{GD}}$ is a *protection* for "A" in case "C" defaults. Typically $\mathrm{L_{GD}} =$ notional, or "notional - recovery" $= 1 - \mathrm{R_{EC}}$.

Fundamental Credit Derivatives: CDS's

A typical stylized case occurs when “A” has bought a corporate bond issued by “C” and is waiting for the coupons and final notional payment from “C”: If “C” defaults before the corporate bond maturity, “A” does not receive such payments. “A” then goes to “B” and buys some protection against this risk, asking “B” a payment that roughly amounts to the loss on the bond (e.g. notional minus deterministic recovery) that A would face in case “C” defaults.

Or again ”A” has a portfolio of several instruments with a large exposure to counterparty ”C”. To partly hedge such exposure, ”A” enters into a CDS where it buys protection from a bank ”B” against the default of ”C”.

Fundamental Credit Derivatives: CDS's I

What counts as a credit event triggering τ_C?

- Bankruptcy of "C"
- Failure to pay of "C";
- Obligation acceleration, when "C" is requested to pay debt ahead of schedule because "C" didn't meet the terms of the loan
- Restructuring, when "C" undergoes reorganization to consolidate its debt (there are several types of restructuring and definitions may be different in Europe and US)

Fundamental Credit Derivatives: CDS's II

- What happens in a CDS contract at default of "C"?
- Cash Settlement. Protection seller pays to the buyer the loss in value of the referenced instruments (e.g. "C" issued bonds) following the credit event. Bonds or loans are not transferred. When more instruments can be referenced the cheapest to deliver price variation is used (see below).
- Physical Settlement. The protection buyer receives a cash payment, typically the "insured" face value, from the protection seller, and the seller takes possession of the defaulted loan instrument or bonds for an equivalent notional amount.
- Physical S.: most CDS allow the protection buyer to choose deliverables from a pool of defaulted bonds with equal seniority. Cheapest to deliver bond is typically chosen (different value in a reorganization, higher accrued interest...)

Fundamental Credit Derivatives: CDS's III

- Physical S: Auction. If there are not enough bonds to match the insured face value, a credit event auction occurs, and the payment received is usually substantially less than the face value of the loan.
- Recovery rate REC is implicitly defined by these procedures and by market value decline after credit event and is very hard to estimate a priori.

Fundamental Credit Derivatives: CDS's IV

Recovery Rate?

- Prior to 2007 one would assume REC = 40% in most cases and REC = 50% for financials. Lehman recovery was 8.625%![a]
- ISDA big bang recommends REC = 20% or REC = 40%
- Analysis is mostly possible in aggregate on large pools of bonds or loans with similar ratings
- Only few studies available. In aggregate, an inverse relationship between Recovery rates and credit risk.
- The Higher the credit risk the lower the recovery
- This led some market operators to postulate inverse relationships between spreads and recoveries, but no consensus is available on how this relationship should be shaped precisely.

[a] A final value of 8.625% was set on the bonds of Lehman Brothers [...], in an auction intended to cash-settle credit default swap (CDS) trades linked to the toppled dealer. Over 350 firms participated in the auction protocol, according to ISDA. The final price is about four points lower than that for Lehmans actual defaulted debt, according to Morgan Stanley. It means protection sellers will pay 91.375% of par to settle defaulted CDSs (Risk Magazine)

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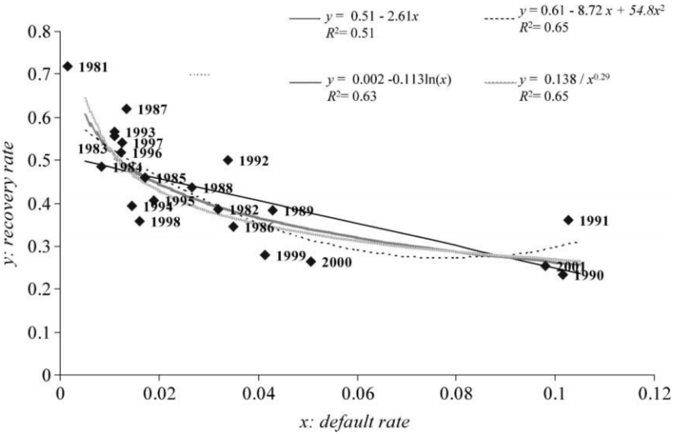


FIG. 1.—Univariate models. Results of a set of univariate regressions carried out between the recovery rate (BRR) or its natural log (BLRR) and the default rate (BDR) or its natural log (BLDR). See table 2 for more details.

Based on Corporate Bond data 1982-2001, Altman, Brady, Resti, Sironi, Journal of Business, 2005, vol. 78, no. 6, 2005

Fundamental Credit Derivatives: CDS's

Protection Seller B	$\rightarrow$ $\leftarrow$	protection $\mathrm{L_{GD}}$ at default τ_C if $T_a < \tau_C \leq T_b$ rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\rightarrow$ $\leftarrow$	Protection Buyer A

Formally we may write the (Running) CDS discounted payoff to "B" at time $t < T_a$ as

$$\Pi_{\mathrm{RCDS}_{a,b}}(t) := D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{T_a<\tau\leq T_b\}} D(t,\tau)\,\mathrm{L_{GD}}$$

where $T_{\beta(\tau)}$ is the first of the T_i's following τ.

CDS payout to Protection seller (receiver CDS)

The 3 terms in the payout are as follows (they are seen from the protection seller, receiver CDS):

- Discounted Accrued rate at default : This is supposed to compensate the protection seller for the protection he provided from the last T_i before default until default τ:

$$D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}$$

- CDS Rate premium payments if no default: This is the premium received by the protection seller for the protection being provided

$$\sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}}$$

- Payment of protection at default if this happens before final T_b

$$-\mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\ \mathrm{L}_{\mathrm{GD}}$$

These are random discounted cash flows, not yet the CDS price.

CDS's: Risk Neutral Valuation Formula

Denote by $\mathrm{CDS}_{a,b}(t, R, \mathrm{L_{GD}})$ the time t *price* of the above Running standard CDS's *payoffs*.

As usual, the price associated to a discounted payoff is its *risk neutral expectation*.

The resulting pricing formula depends on the assumptions on interest-rate dynamics and on the default time τ (reduced form models, structural models...).

CDS's: Risk Neutral Valuation

In general by risk-neutral valuation we can compute the CDS price at time 0 (or at any other time similarly):

$$\mathrm{CDS}_{a,b}(0, R, \mathrm{L_{GD}}) = \mathbb{E}\{\Pi_{\mathrm{RCDS}_{a,b}}(0)\},$$

with the CDS discounted payoffs defined earlier. As usual, $\mathbb{E}$ denotes the risk-neutral expectation, the related measure being denoted by $\mathbb{Q}$.

However, we will not use the formulas resulting from this approach to price CDS that are already quoted in the market. *Rather, we will invert these formulas in correspondence of market CDS quotes to calibrate our models to the CDS quotes themselves. We will give examples of this later.*

Now let us have a look at some particular formulas resulting from the general risk neutral approach through some simplifying assumptions.

CDS Model-independent formulas

Assume the stochastic discount factors $D(s,t)$ to be independent of the default time τ for all possible $0 < s < t$. The price of the premium leg of the CDS at time 0 is:

$$\begin{aligned}
\text{PremiumLeg}_{a,b}(R) &= \mathbb{E}[D(0,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}] + \\
&\qquad + \sum_{i=a+1}^{b} \mathbb{E}[D(0,T_i)\alpha_i R\mathbf{1}_{\{\tau \geq T_i\}}] \\
&= \mathbb{E}\left[\int_{t=0}^{\infty} D(0,t)(t - T_{\beta(t)-1})R\mathbf{1}_{\{T_a<t<T_b\}}\delta_\tau(t)dt\right] \\
&\qquad + \sum_{i=a+1}^{b} \mathbb{E}[D(0,T_i)]\alpha_i R\,\mathbb{E}[\mathbf{1}_{\{\tau \geq T_i\}}] =
\end{aligned}$$

For those who don't know the theory of distributions (Dirac's delta etc), read $\delta_\tau(t)dt = \mathbf{1}_{\{\tau \in [t,t+dt]\}}$.

CDS Model-independent formulas

$$\text{PremiumLeg}_{a,b}(R) = \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)(t - T_{\beta(t)-1})R\,\delta_\tau(t)dt] +$$

$$+ \sum_{i=a+1}^{b} P(0,T_i)\alpha_i R\,\mathbb{Q}(\tau \geq T_i) =$$

$$= \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)](t - T_{\beta(t)-1})R\,\mathbb{E}[\delta_\tau(t)dt] + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i R\,\mathbb{Q}(\tau \geq T_i)$$

$$= R \int_{t=T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})\mathbb{Q}(\tau \in [t, t+dt)) +$$

$$+R \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \mathbb{Q}(\tau \geq T_i),$$

where we have used independence in factoring terms. Again, read $\delta_\tau(t)dt = \mathbf{1}_{\{\tau \in [t,t+dt]\}}$

CDS Model-independent formulas

We have thus, by rearranging terms and introducing a "unit-premium" premium leg (sometimes called "DV01", "Risky duration" or "annuity"):

$$\text{PremiumLeg}_{a,b}(R; P(0,\cdot), \mathbb{Q}(\tau > \cdot)) = R \ \text{PremiumLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot))$$

$$\text{PremiumLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)) := -\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) d_t \boxed{\mathbb{Q}(\tau \geq t)}$$

$$+ \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)}$$

This model independent formula uses the initial market zero coupon curve (bonds) at time 0 (i.e. $P(0,\cdot)$) and the survival probabilities $\mathbb{Q}(\tau \geq \cdot)$ at time 0 (terms in the boxes).

A similar formula holds for the protection leg, again under independence between default τ and interest rates.

CDS Model-independent formulas

$$
\begin{aligned}
\text{ProtecLeg}_{a,b}(\text{L}_{\text{GD}}) &= \mathbb{E}[\mathbf{1}_{\{T_a<\tau\le T_b\}} D(0,\tau)\ \text{L}_{\text{GD}}] \\
&= \text{L}_{\text{GD}}\ \mathbb{E}\left[\int_{t=0}^{\infty} \mathbf{1}_{\{T_a<t\le T_b\}} D(0,t)\delta_\tau(t)dt\right] \\
&= \text{L}_{\text{GD}} \left[\int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)\delta_\tau(t)dt]\right] \\
&= \text{L}_{\text{GD}} \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)]\mathbb{E}[\delta_\tau(t)dt] \\
&= \text{L}_{\text{GD}} \int_{t=T_a}^{T_b} P(0,t)\mathbb{Q}(\tau \in [t,t+dt))
\end{aligned}
$$

(again interpret $\delta_\tau(t)dt = \mathbf{1}_{\{\tau\in[t,t+dt]\}}$)

CDS Model-independent formulas

so that we have, by introducing a "unit-notional" protection leg:

$$\text{ProtecLeg}_{a,b}(\text{L}_{\text{GD}}; P(0,\cdot), \mathbb{Q}(\tau > \cdot)) = \text{L}_{\text{GD}}\ \text{ProtecLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)),$$

$$\text{ProtecLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)) := -\int_{T_a}^{T_b} P(0,t)\, d_t \boxed{\mathbb{Q}(\tau \geq t)}$$

This formula too is model independent given the initial zero coupon curve (bonds) at time 0 observed in the market and given the survival probabilities at time 0 (term in the box).

CDS Model-independent formulas

The total (Receiver) CDS price can be written as

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L_{GD}}; \mathbb{Q}(\tau > \cdot)) = R\,\mathrm{PremiumLeg1}_{a,b}(\mathbb{Q}(\tau > \cdot))$$

$$-\mathrm{L_{GD}}\ \mathrm{ProtecLeg1}_{a,b}(\mathbb{Q}(\tau > \cdot))$$

$$= R\left[-\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})d_t \boxed{\mathbb{Q}(\tau \geq t)} + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)}\right] +$$

$$+\mathrm{L_{GD}}\left[\int_{T_a}^{T_b} P(0,t)\ d_t \boxed{\mathbb{Q}(\tau \geq t)}\right]$$

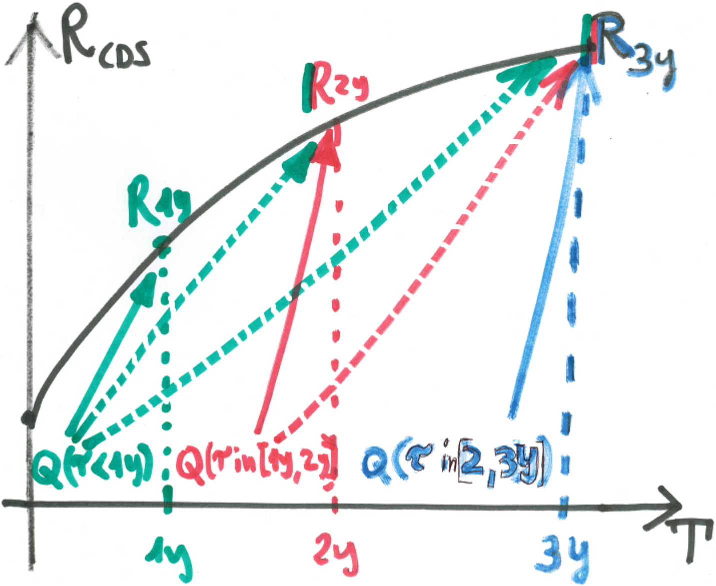
(Receiver) CDS Model-independent formulas

We may also use that $d_t\mathbb{Q}(\tau > t) = d_t(1 - \mathbb{Q}(\tau \le t)) = -d_t\,\mathbb{Q}(\tau \le t)$. We have

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \mathbb{Q}(\tau \le \cdot)) = -\mathrm{L}_{\mathrm{GD}}\left[\int_{T_a}^{T_b} P(0,t)\, d_t \boxed{\mathbb{Q}(\tau \le t)}\right] +$$

$$R\left[\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) d_t \boxed{\mathbb{Q}(\tau \le t)} + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \ge T_i)}\right]$$

The integrals in the survival probabilities given in the above formulas can be valued as Stieltjes integrals in the survival probabilities themselves, and can easily be approximated numerically by summations through Riemann-Stieltjes sums, considering a low enough discretization time step.



R_{CDS}
R_{1y}
R_{2y}
R_{3y}
$Q(\tau<1y)$
$Q(\tau$ in $[1y,2y]$
$Q(\tau$ in $[2,3y]$
1y
2y
3y
T

CDS Model-independent formulas

The market quotes, at time 0, the fair $R = R_{0,b}^{\text{mkt MID}}(0)$ coming from bid and ask quotes for this fair R.

This fair R equates the two legs for a set of CDS with initial protection time $T_a = 0$ and final protection time $T_b \in \{1y, 2y, 3y, 4y, 5y, 6y.7y, 8y, 9y, 10y\}$, although often only a subset of the maturities $\{1y, 3y, 5y, 7y, 10y\}$ is available.

Solve then

$$\mathrm{CDS}_{0,b}(t, R_{0,b}^{\mathrm{mktMID}}(0), \mathrm{L}_{\mathrm{GD}}; \mathbb{Q}(\tau > \cdot)) = 0$$

in portions of $\mathbb{Q}(\tau > \cdot)$ starting from $T_b = 1y$, finding the market implied survival $\{\mathbb{Q}(\tau \geq t), t \leq 1y\}$; plugging this into the $T_b = 2y$ CDS legs formulas, and then solving the same equation with $T_b = 2y$, we find the market implied survival $\{\mathbb{Q}(\tau \geq t), t \in (1y, 2y]\}$, and so on up to $T_b = 10y$.

CDS Model-independent formulas

This is a way to strip survival (or equivalently default) probabilities from CDS quotes in a model independent way. No need to assume an intensity or a structural model for default here.

However, the market in doing the above stripping typically resorts to intensities (also called hazard rates), assuming existence of intensities associated with the default time.

We will refer to the method just highlighted as **”CDS stripping”.**

CDS and Defaultable Bonds: Intensity Models

In intensity models the random default time τ is assumed to be exponentially distributed.

A strictly positive stochastic process $t \mapsto \lambda_t$ called *default intensity* (or hazard rate) is given for the bond issuer or the CDS reference name.

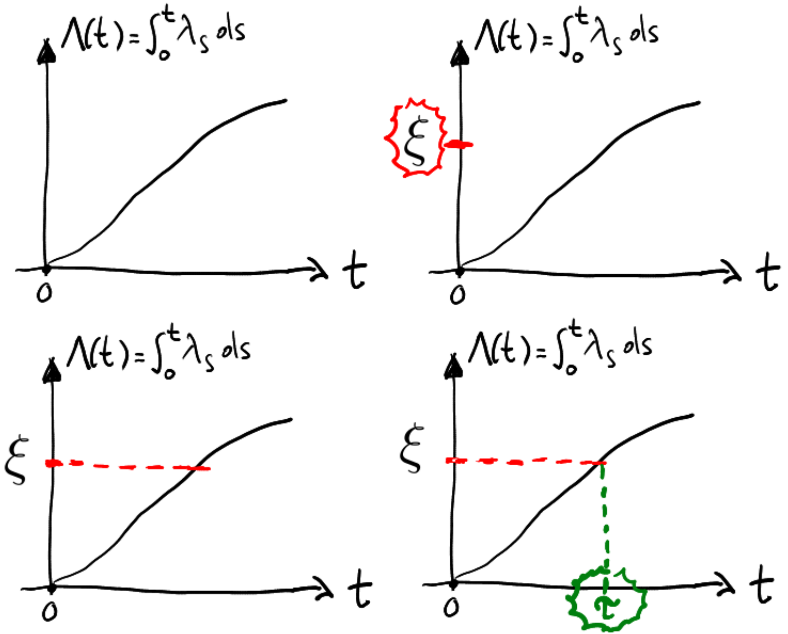
The *cumulated intensity* (or hazard function) is the process $t \mapsto \int_0^t \lambda_s \, ds =: \Lambda_t$. Since λ is positive, Λ is increasing in time.

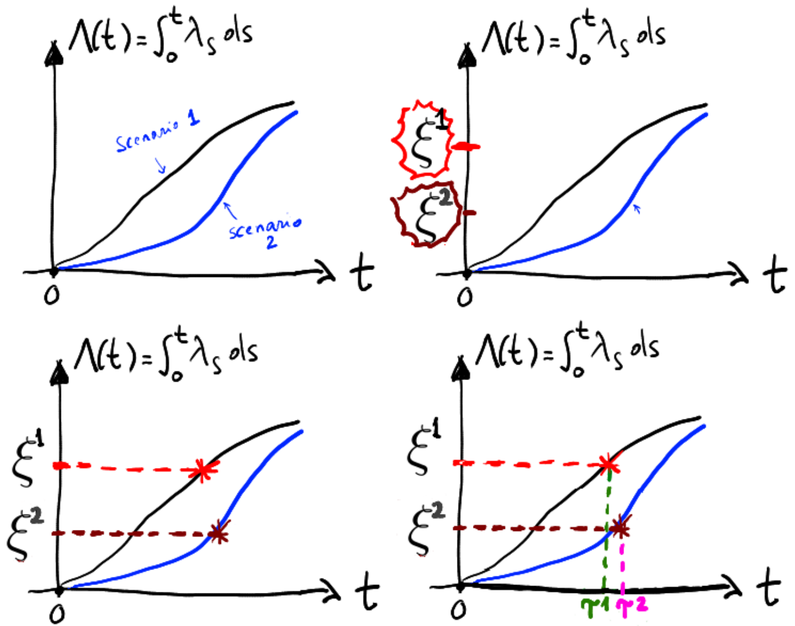
The default time is defined as the inverse of the cumulative intensity on an exponential random variable ξ with mean 1 and independent of λ

$$\tau = \Lambda^{-1}(\xi).$$

Recall that

$$\mathbb{Q}(\xi > u) = e^{-u}, \;\; \mathbb{Q}(\xi < u) = 1 - e^{-u}, \;\; \mathbb{E}(\xi) = 1.$$


$\Lambda(t) = \int_0^t \lambda_s \, ds$
t
0
$\Lambda(t) = \int_0^t \lambda_s \, ds$
ξ
t
0
$\Lambda(t) = \int_0^t \lambda_s \, ds$
ξ
t
0
$\Lambda(t) = \int_0^t \lambda_s \, ds$
ξ
τ
t
0



$\Lambda(t) = \int_0^t \lambda_s \, ds$
Scenario 1
scenario 2
ξ^1
ξ^2
τ^1
τ^2
0
t

CDS and Defaultable Bonds: Intensity Models

A few calculations: Probability of surviving time t:

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\Lambda^{-1}(\xi) > t) = \mathbb{Q}(\xi > \Lambda(t)) = \rightarrow$$

Let's use the tower property of conditional expectation and the fact that Λ is independent of ξ:

$$\rightarrow = \mathbb{E}[\mathbb{Q}(\xi > \Lambda(t)|\Lambda(t))] = \mathbb{E}[e^{-\Lambda(t)}] = \mathbb{E}[e^{-\int_0^t \lambda_s \, ds}]$$

This looks exactly like a bond price if we replace r by λ!

CDS and Defaultable Bonds: Intensity Models

Let's price a defaultable zero coupon bond with zero recovery. Assume that ξ is also independent of r.

$$\bar{P}(0,T) = \mathbb{E}[D(0,T)1_{\{\tau>T\}}] = \mathbb{E}[e^{-\int_0^T r_s\, ds}1_{\{\Lambda^{-1}(\xi)>T\}}] =$$

$$= \mathbb{E}[e^{-\int_0^T r_s\, ds}1_{\{\xi>\Lambda(T)\}}] = \mathbb{E}[\mathbb{E}\{e^{-\int_0^T r_s\, ds}1_{\{\xi>\Lambda(T)\}}|\Lambda, r\}]$$

$$= \mathbb{E}[e^{-\int_0^T r_s\, ds}\mathbb{E}\{1_{\{\xi>\Lambda(T)\}}|\Lambda, r\}]$$

$$= \mathbb{E}[e^{-\int_0^T r_s\, ds}\mathbb{Q}\{\xi > \Lambda(T)|\Lambda\}] = \mathbb{E}[e^{-\int_0^T r_s\, ds}e^{-\Lambda(T)}] =$$

$$= \mathbb{E}[e^{-\int_0^T r_s\, ds - \int_0^T \lambda_s\, ds} = \mathbb{E}[e^{-\int_0^T \boxed{(r_s+\lambda_s)}\, ds}]$$

So the price of a defaultable bond is like the price of a default-free bond *where the risk free discount short rate r has been replaced by r plus a spread λ.*

CDS and Defaultable Bonds: Intensity Models

This is why in intensity models, the intensity is interpreted as a credit spread.

Because of properties of the exponential random variable, one can also prove that

$$\mathbb{Q}(\tau \in [t, t + dt) | \tau > t, \text{"}\lambda[0, t]\text{"}) = \lambda_t \, dt$$

and the intensity $\lambda_t \, dt$ is also a local probability of defaulting around t.

So:
λ is an instantaneous credit spread or local default probability

ξ is pure jump to default risk

Intensity models and Interest Rate Models

As is now clear, the exponential structure of τ in intensity models makes the modeling of credit risk very similar to interest rate models.

The spread/intensity λ behaves exactly like an interest rate in discounting

Then it is possible to use a lot of techniques from interest rate modeling (short rate models for r, first choice seen earlier) for credit as well.

Intensity: Constant, time dependent or stochastic

- Constant λ_t: in this case $\lambda_t = \gamma$ for a deterministic constant credit spread (intensity);
- Time dependent deterministic intensity λ_t: in this case $\lambda_t = \gamma(t)$ for a deterministic curve in time $\gamma(t)$. This is a model with a term structure of credit spreads but without credit spread volatility.
- Time dependent and stochastic intensity λ_t: in this case λ_t is a full stochastic process. This allows us to model the term structue of credit spreads but also their volatility.

The case with constant intensity $\lambda_t = \gamma$: CDS

Assume as an approximation that the CDS premium leg pays continuously.

Instead of paying $(T_i - T_{i-1})R$ at T_i as the standard CDS, given that there has been no default before T_i, we approximate this premium leg by assuming that it pays "$dt\ R$" in $[t, t + dt)$ it there has been no default before $t + dt$.

The case with constant intensity $\lambda_t = \gamma$: CDS

This amounts to replace the original pricing formula of a CDS (receiver case, spot CDS with $T_a = 0 = \text{today}$)

$$\mathrm{CDS}_{0,b}(0, R, \mathrm{L_{GD}}; \mathbb{Q}(\tau > \cdot)) = R\left[-\int_0^{T_b} P(0,t)(t - T_{\beta(t)-1})d_t\mathbb{Q}(\tau \geq t)\right.$$

$$\left.+\sum_{i=1}^{b} P(0,T_i)\alpha_i\mathbb{Q}(\tau \geq T_i)\right] + \mathrm{L_{GD}}\left[\int_0^{T_b} P(0,t)\, d_t\, \mathbb{Q}(\tau \geq t)\right]$$

with (accrual term vanishes because payments continuous now)

$$R\int_0^{T_b} P(0,t)\mathbb{Q}(\tau \geq t)dt + \mathrm{L_{GD}}\int_0^{T_b} P(0,t)\, d_t\mathbb{Q}(\tau \geq t)$$

The case with constant intensity $\lambda_t = \gamma$: CDS

If the intensity is a constant γ we have

$$\mathbb{Q}(\tau > t) = e^{-\gamma t}, \quad d_t\mathbb{Q}(\tau > t) = -\gamma e^{-\gamma t} dt = -\gamma \mathbb{Q}(\tau > t) dt,$$

and the receiver CDS price we have seen earlier becomes

$$\mathrm{CDS}_{0,b}(t, R, \mathrm{L_{GD}}; \mathbb{Q}(\tau > \cdot)) = -\mathrm{L_{GD}} \left[\int_0^{T_b} P(0,t)\gamma\mathbb{Q}(\tau \geq t) dt \right]$$

$$+R \left[\int_0^{T_b} P(0,t)\mathbb{Q}(\tau \geq t) dt \right]$$

If we insert the market CDS rate $R = R_{0,b}^{\text{mkt MID}}(0)$ in the premium leg, then the CDS present value should be zero. Solve

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L_{GD}}; \mathbb{Q}(\tau > \cdot)) = 0 \quad \text{in} \quad R$$

to obtain

$$\boxed{\gamma = \frac{R_{0,b}^{\text{mkt MID}}(0)}{\mathrm{L_{GD}}}}$$

The case with constant intensity $\lambda_t = \gamma$: CDS

from which we see that also **the CDS premium rate *R* is indeed a sort of CREDIT SPREAD, or INTENSITY.**

We can play with this formula with a few examples.

CDS of FIAT trades at 300bps for 5y, with recovery 0.3

What is a quick rough calcul for the risk neutral probability that FIAT survives 10 years?

$$\boxed{\gamma = \frac{R_{0,b}^{\text{mkt FIAT}}(0)}{\text{L}_{\text{GD}}}} = \frac{300/10000}{1-0.3} = 4.29\%$$

The case with constant intensity $\lambda_t = \gamma$: CDS

Survive 10 years:

$$\mathbb{Q}(\tau > 10y) = \exp(-\gamma 10) = \exp(-0.0429 * 10) = 65.1\%$$

Default between 3 and 5 years:

$$\mathbb{Q}(\tau > 3y) - \mathbb{Q}(\tau > 5y) = \exp(-\gamma 3) - \exp(-\gamma 5)$$

$$= \exp(-0.0429 * 3) - \exp(-0.0429 * 5) = 7.2\%$$

If R_{CDS} goes up and REC remains the same, γ goes up and survival probabilities go down (default probs go up)

If REC goes up and R_{CDS} remains the same, LGD goes down and γ goes up - default probabilities go up

The case with time dependent intensity $\lambda_t = \gamma(t)$: CDS

We consider now **deterministic time-varying** intensity $\gamma(t)$, which we assume to be a positive and piecewise continuous function. We define

$$\Gamma(t) := \int_0^t \gamma(u)du,$$

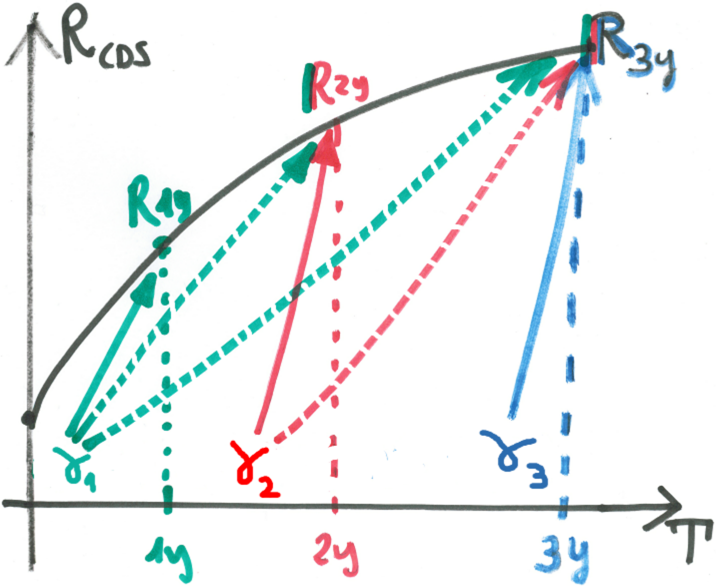
the **cumulated intensity, cumulated hazard rate,** or also **Hazard function**.

From the exponential assumption, we have easily

$$\mathbb{Q}\{s < \tau \le t\} = \mathbb{Q}\{s < \Gamma^{-1}(\xi) \le t\} = \mathbb{Q}\{\Gamma(s) < \xi \le \Gamma(t)\} =$$

$$= \mathbb{Q}\{\xi > \Gamma(s)\} - \mathbb{Q}\{\xi > \Gamma(t)\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)) \text{ i.e.}$$

"prob of default between s and t is "$e^{-\int_0^s \gamma(u)du} - e^{-\int_0^t \gamma(u)du} \approx \int_s^t \gamma(u)du$"
(where the final approximation is good ONLY for small exponents).



R_{CDS}
R_{1y}
R_{2y}
R_{3y}
γ_1
γ_2
γ_3
1y
2y
3y
T

CDS Calibration and Implied Hazard Rates/Intensities

Reduced form models are the models that are most commonly used in the market to infer implied default probabilities from market quotes.

Market instruments from which these probabilities are drawn are especially CDS and Bonds.

We just implement the stripping algorithm sketched earlier for ”CDS stripping”, but now taking into account that the probabilities are expressed as exponentials of the deterministic intensity γ, that is assumed to be piecewise constant.

By adding iteratively CDS with longer and longer maturities, at each step we will strip the new part of the intensity $\gamma(t)$ associated with the last added CDS, while keeping the previous values of γ, for earlier times, that were used to fit CDS with shorter maturities.

A Case Study of CDS stripping: Lehman Brothers

Here we show an intensity model with piecewise constant λ obtained by CDS stripping.

We also show the AT1P structural / firm value model by Brigo et al (2004-2010). This will not be subject for this course, but in case of interest, for details on AT1P see

`http://arxiv.org/abs/0912.3028`
`http://arxiv.org/abs/0912.3031`
`http://arxiv.org/abs/0912.4404`

Otherwise ignore the AT1P and σ_i parts of the tables.

- **August 23, 2007**: Lehman announces that it is going to shut one of its home lending units (*BNC Mortgage*) and lay off 1,200 employees. The bank says it would take a $52 million charge to third-quarter earnings.
- **March 18, 2008**: Lehman announces better than expected first-quarter results (but profits have more than halved).
- **June 9, 2008**: Lehman confirms the booking of a $2.8 billion loss and announces plans to raise $6 billion in fresh capital by selling stock. Lehman shares lose more than 9% in afternoon trade.
- **June 12, 2008**: Lehman shakes up its management; its chief operating officer and president, and its chief financial officer are removed from their posts.
- **August 28, 2008**: Lehman prepares to lay off 1,500 people. The Lehman executives have been knocking on doors all over the world seeking a capital infusion.
- **September 9, 2008**: Lehman shares fall 45%.
- **September 14, 2008**: Lehman files for bankruptcy protection and hurtles toward liquidation after it failed to find a buyer.

Lehman Brothers CDS Calibration: July 10th, 2007

On the left part of this Table we report the values of the quoted CDS spreads before the beginning of the crisis. We see that the spreads are very low. In the middle of Table 1 we have the results of the exact calibration obtained using a *piecewise constant* intensity model.

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
10 Jul 2007			100.0%		100.0%
1y	16	0.267%	99.7%	29.2%	99.7%
3y	29	0.601%	98.5%	14.0%	98.5%
5y	45	1.217%	96.2%	14.5%	96.1%
7y	50	1.096%	94.1%	12.0%	94.1%
10y	58	1.407%	90.2%	12.7%	90.2%

Table: Results of calibration for July 10th, 2007.

Lehman Brothers CDS Calibration: June 12th, 2008

We are in the middle of the crisis. We see that the CDS spreads R_i have increased with respect to the previous case, but are not very high, indicating the fact that the market is aware of the difficulties suffered by Lehman but thinks that it can come out of the crisis. Notice that now the term structure of both R and *intensities* is inverted. This is typical of names in crisis

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
12 Jun 2008			100.0%		100.0%
1y	397	6.563%	93.6%	45.0%	93.5%
3y	315	4.440%	85.7%	21.9%	85.6%
5y	277	3.411%	80.0%	18.6%	79.9%
7y	258	3.207%	75.1%	18.1%	75.0%
10y	240	2.907%	68.8%	17.5%	68.7%

Table: Results of calibration for June 12th, 2008.

Lehman Brothers CDS Calibration: Sept 12th, 2008

In this Table we report the results of the calibration on September 12th, 2008, just before Lehman's default. We see that the spreads are now very high, corresponding to lower survival probability and higher intensities than before.

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
12 Sep 2008			100.0%		100.0%
1y	1437	23.260%	79.2%	62.2%	78.4%
3y	902	9.248%	65.9%	30.8%	65.5%
5y	710	5.245%	59.3%	24.3%	59.1%
7y	636	5.947%	52.7%	26.9%	52.5%
10y	588	6.422%	43.4%	29.5%	43.4%

Table: Results of calibration for September 12th, 2008.

Until the end, rating agencies maintained a good rating for Lehman[5]

In our [S&P] view, Lehman [...] had adequate liquidity relative to reasonably severe and foreseeable temporary stresses. [...] We believe the downfall of Lehman reflected escalating fears that led to a loss of confidence – ultimately becoming a real threat to Lehmans viability in a way that fundamental credit analysis could not have anticipated with greater levels of certainty

If we check from published transition matrices what has been the probability that a S&P "A" rated name defaulted we have

$$\mathbb{P}(A \text{ rated name defaults within 1y}) = 0 \text{ in } 2005/6/7, \text{ and } 0.38\% \text{ in } 2008$$

Compare with CDS market: $\mathbb{Q}$ default prob in 1y: 21%.
0.38% vs 21%. **Huge risk premium between $\mathbb{P}$ and $\mathbb{Q}$ for Lehman.**
This is a regular feature: market implied $\mathbb{Q}$ default probabilities are always larger than fundamental history-based ones under $\mathbb{P}$.

[5] `http://ww2.cfo.com/banking-capital-markets/2008/09/ /rating-itself-sp-defends-lehmans-a/` accessed Sept 9 2014

CDS-Bond Basis (funding liquidity proxy)

The above stripping method could allow us to obtain intensities both from bonds (Z-spread) $\gamma^b(t)$ and CDS (fair spread) $\gamma^c(t)$.

Since Bonds are funded instruments and CDS are not, the CDS-bond basis is considered to be an indicator of funding liquidity

$$\ell(t) = \gamma^c(t) - \gamma^b(t).$$

The basis has been both "$+$" and "$-$" through history. Traders may set up basis trades if convinced arbitrage opportunities are showing up.

- Bond funding cost: $\ell \downarrow$
- CDS counterparty risk: $\ell \downarrow$
- Shorting credit: Easier buying CDS protection than shorting bonds. CDS more attractive and default leg more expensive $\ell \uparrow$.
- CDS protect from more general defaults than bonds and have cheapest do deliver advantages when buying protection, as one delivers a less valuable bond in exchange for face value: $\ell \uparrow$.

Stochastic Intensity. The CIR++ model

We have seen in detail CDS calibration in presence of **deterministic** and **time varying** intensity or hazard rates, $\gamma(t)dt = \mathbb{Q}\{\tau \in dt | \tau > t\}$

As explained, this accounts for credit spread structure but not for **volatility**.

The latter is obtained moving to stochastic intensity (Cox process). The deterministic function $t \mapsto \gamma(t)$ is replaced by a stochastic process $t \mapsto \lambda(t) = \lambda_t$. The Hazard function $\Gamma(t) = \int_0^t \gamma(u)du$ is replaced by the Hazard process (or cumulated intensity) $\Lambda(t) = \int_0^t \lambda(u)du$.

CIR++ stochastic intensity λ

We model the stochastic intensity as follows: consider

$$\lambda_t = y_t + \psi(t;\beta)\ , \quad t \geq 0,$$

where the intensity has a random component y and a deterministic component ψ to fit the CDS term structure. For y we take a Jump-CIR model

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t + dJ_t,\ \beta = (\kappa,\mu,\nu,y_0),\ 2\kappa\mu > \nu^2.$$

Jumps are taken themselves independent of anything else, with exponential arrival times with intensity η and exponential jump size with a given parameter.

In this course we will focus on the case with no jumps J, see B and El-Bachir (2006) or B and M (2006) for the case with jumps.

CIR++ stochastic intensity λ.

Calibrating Implied Default Probabilities

With no jumps, y follows a noncentral chi-square distribution; Very important: $y > 0$ as must be for an intensity model (Vasicek would not work). This is the CIR++ model we have seen earlier for interest rates.

About the parameters of CIR:

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t$$

κ: speed of mean reversion
μ: long term mean reversion level
ν: volatility.

CIR++ stochastic intensity λ. I

Calibrating Implied Default Probabilities

$$E[\lambda_t] = \lambda_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$$

$$\mathrm{VAR}(\lambda_t) = \lambda_0 \frac{\nu^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa t})^2$$

After a long time the process reaches (asymptotically) a stationary distribution around the mean μ and with a corridor of variance $\mu\nu^2/2\kappa$.
The largest κ, the fastest the process converges to the stationary state.
So, ceteris paribus, increasing κ kills the volatility of the credit spread.
The largest μ, the highest the long term mean, so the model will tend to higher spreads in the future in average.
The largest ν, the largest the volatility. Notice however that κ and ν fight each other as far as the influence on volatility is concerned.

CIR++ stochastic intensity λ. II

Calibrating Implied Default Probabilities

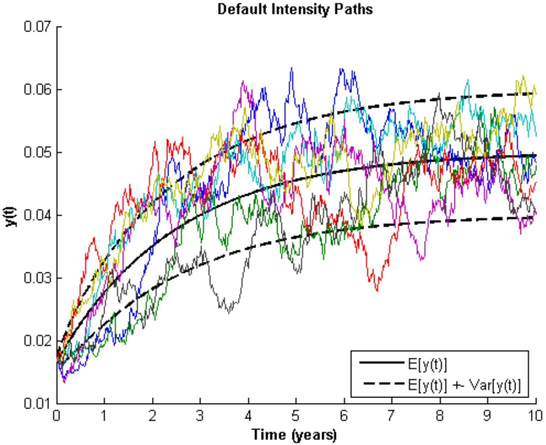


Figure: $y0 = 0.0165, \kappa = 0.4, \mu = 0.05, \nu = 0.04$

EXERCISE: The CIR model

Assume we are given a stochastic intensity process of CIR type,

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW(t)$$

where y_0, κ, μ, ν are positive constants. W is a brownian motion under the risk neutral measure.

a) Increasing κ increases or decreases randomness in the intensity? And ν?

b) The mean of the intensity at future times is affected by k? And by ν?

c) What happens to mean of the intensity when time grows to infinity?

d) Is it true that, because of mean reversion, the variance of the intensity goes to zero (no randomness left) when time grows to infinity?

e) Can you compute a rough approximation of the percentage volatility in the intensity?

EXERCISE: The CIR model

f) Suppose that $y_0 = 400bps = 0.04$, $\kappa = 0.3$, $\nu = 0.001$ and $\mu = 400bps$. Can you guess the behaviour of the future random trajectories of the stochastic intensity after time 0?
g) Can you guess the spread of a CDS with 10y maturity with the above stochastic intensity when the recovery is 0.35?

EXERCISE Solutions. I

a) We can refer to the formulas for the mean and variance of y_T in a CIR model as seen from time 0, at a given T. The formula for the variance is known to be (see for Example Brigo and Mercurio (2006))

$$\mathrm{VAR}(y_T) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2$$

whereas the mean is

$$E[y_T] = y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

We can see that for k becoming large the variance becomes small, since the exponentials decrease in k and the division by k gives a small value for large k. In the limit

$$\lim_{\kappa \to +\infty} \mathrm{VAR}(y_T) = 0$$

so that for very large κ there is no randomness left.

EXERCISE Solutions. II

We can instead see that VAR(y_T) is proportional to ν^2, so that if ν increases randomness increases, as is obvious from $\nu\sqrt{y_t}$ being the instantaneous volatility in the process y.

b) As the mean is

$$E[y_T] = y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

we clearly see that this is impacted by κ (indeed, ”speed of mean reversion”) and by μ clearly (”long term mean”) but not by the instantaneous volatility parameter ν.
c) As T goes to infinity, we get for the mean

$$\lim_{T \to +\infty} y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T}) = \mu$$

so that the mean tends to μ (this is why μ is called ”long term mean”).

EXERCISE Solutions. III

d) In the limit where time goes to infinity we get, for the variance

$$\lim_{T\to+\infty}[y_0\frac{\nu^2}{\kappa}(e^{-\kappa T}-e^{-2\kappa T})+\mu\frac{\nu^2}{2\kappa}(1-e^{-\kappa T})^2]=\mu\frac{\nu^2}{2\kappa}$$

So this does not go to zero. Indeed, mean reversion here implies that as time goes to infinite the mean tends to μ and the variance to the constant value $\mu\frac{\nu^2}{2\kappa}$, but not to zero.

EXERCISE Solutions. IV

e) Rough approximations of the percentage volatilities in the intensity would be as follows. The instantaneous variance in dy_t, conditional on the information up to t, is (remember that $VAR(dW(t)) = dt$)

$$VAR(dy_t) = \nu^2 y_t dt$$

The percentage variance is

$$VAR\left(\frac{dy_t}{y_t}\right) = \frac{\nu^2 y_t}{y_t^2} dt = \frac{\nu^2}{y_t} dt$$

and is state dependent, as it depends on y_t. We may replace y_t with either its initial value y_0 or with the long term mean μ, both known. The two rough percentage volatilities estimates will then be, for $dt = 1$,

$$\sqrt{\frac{\nu^2}{y_0}} = \frac{\nu}{\sqrt{y_0}}, \quad \sqrt{\frac{\nu^2}{\mu}} = \frac{\nu}{\sqrt{\mu}}$$

EXERCISE Solutions. V

These however do not take into account the important impact of κ in the overall volatility of finite (as opposed to instantaneous) credit spreads and are therefore relatively useless.

EXERCISE Solutions. VI

f) First we check if the positivity condition is met.

$$2\kappa\mu = 2 \cdot 0.3 \cdot 0.04 = 0.024; \quad \nu^2 = 0.001^2 = 0.000001$$

hence $2\kappa\mu > \nu^2$ and trajectories are positive. Then we observe that the variance is very small: Take $T = 5y$,

$$\mathrm{VAR}(y_T) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \theta \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2 \approx 0.0000006.$$

Take the standard deviation, given by the square root of the variance:

$$\mathrm{STDEV}(y_T) \approx \sqrt{0.0000006} = 0.00077.$$

which is much smaller of the level 0.04 at which the intensity refers both in terms of initial value and long term mean. Therefore there is almost no randomness in the system as the variance is very small compared to the initial point and the long term mean.

EXERCISE Solutions. VII

Hence there is almost no randomness, and since the initial condition y_0 is the same as the long term mean $\mu_0 = 0.04$, the intensity will behave as if it had the value 0.04 all the time. All future trajectories will be very close to the constant value 0.04.

g) In a constant intensity model the CDS spread can be approximated by

$$y = \frac{R_{CDS}}{1 - REC} \Rightarrow R_{CDS} = y(1 - REC) = 0.04(1 - 0.35) = 260bps$$

CIR++ stochastic intensity λ. I

Calibrating Implied Default Probabilities

For restrictions on the β's that keep ψ and hence λ positive, **as is required in intensity models**, we may use the results in B. and M. (2001) or (2006). We will often use the hazard process $\Lambda(t) = \int_0^t \lambda_s ds$, and also $Y(t) = \int_0^t y_s ds$ and $\Psi(t,\beta) = \int_0^t \psi(s,\beta)ds$.

If we can read from the market some implied risk-neutral default probabilities, and associate to them implied hazard functions Γ^{Mkt} (as we have done in the Lehman example), we may wish our stochastic intensity model to agree with them. By recalling that survival probabilities look exactly like bonds formulas in short rate models for r, we see that our model agrees with the market if

$$\exp(-\Gamma^{\text{Mkt}}(t)) = \exp\left(-\Psi(t,\beta)\right)\mathbb{E}[e^{-\int_0^t y_s ds}]$$

CIR++ stochastic intensity λ. II

Calibrating Implied Default Probabilities

IMPORTANT 1: This is possible only if λ is strictly positive;
IMPORTANT 2: It is fundamental, if we aim at calibrating default probabilities, that the last expected value can be computed analytically. **The only known diffusion model used in interest rates satisfying both constraints is CIR++**

CIR++ stochastic intensity λ

Calibrating Implied Default Probabilities

$$\exp(-\Gamma^{\text{Mkt}}(t)) = \mathbb{Q}\{\tau > t\} = \exp\left(-\Psi(t,\beta)\right) \mathbb{E}[e^{-\int_0^t y_s ds}]$$

Now notice that $\mathbb{E}[e^{-\int_0^t y_s ds}]$ is simply the bond price for a CIR interest rate model with short rate given by y, so that it is known analytically. We denote it by $P^y(0,t,y_0;\beta)$.

Similarly to the interest-rate case, λ is calibrated to the market implied hazard function Γ^{Mkt} if we set

$$\Psi(t,\beta) := \Gamma^{\text{Mkt}}(t) + \ln(P^y(0,t,y_0;\beta))$$

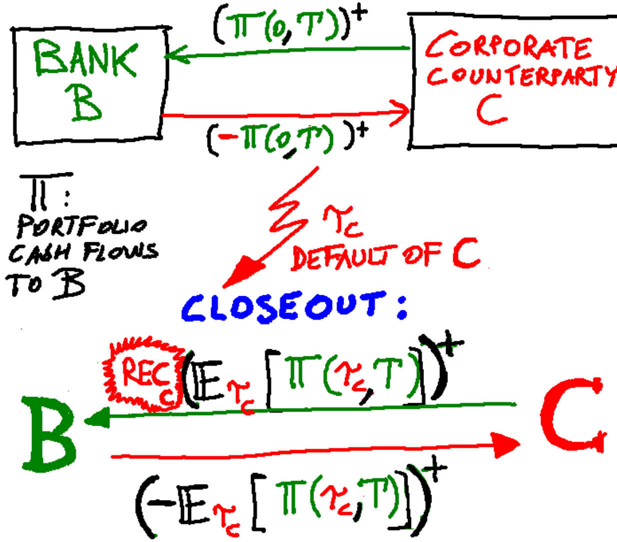
where we choose the parameters β in order to have a positive function ψ, by resorting to the condition seen earlier.

This concludes our introduction to Defaultable Bonds, CDS, credit spreads and intensity models.

We now turn to using such tools in one of the problems the industry is facing right now:

Pricing of counterparty credit risk, leading to the notion of Credit Valuation Adjustment (CVA)

Context



BANK B
CORPORATE COUNTERPARTY C
$(\Pi(0,T))^+$
$(-\Pi(0,T))^+$
Π : PORTFOLIO CASH FLOWS TO B
τ_C DEFAULT OF C
CLOSEOUT :
$REC_C (\mathbb{E}_{\tau_C}[\Pi(\tau_C,T)])^+$
B
C
$(-\mathbb{E}_{\tau_C}[\Pi(\tau_C,T)])^+$

Q & A: What is Counterparty Credit Risk?

Q What is counterparty risk in general?

A *The risk taken on by an entity entering an OTC contract with a counterparty having a relevant default probability. As such, the counterparty might not respect its payment obligations.*

> *The counterparty credit risk is defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flows. An economic loss would occur if the transactions or portfolio of transactions with the counterparty has a positive economic value at the time of default.*
>
> [Basel II, Annex IV, 2/A]

Q & A: Credit VaR and CVA

Q What is the difference between Credit VaR and CVA?

A *They are both related to credit risk.*

- *Credit VaR is a Value at Risk type measure, a Risk Measure. it measures a potential loss due to counterparty default.*
- *CVA is a price, it stands for Credit Valuation Adjustment and is a price adjustment. CVA is obtained by pricing the counterparty risk component of a deal, similarly to how one would price a credit derivative.*

Q & A: Credit VaR and CVA

Q What is the difference in practical use?

A *Credit VaR answers the question:*

- *"How much can I lose of this portfolio, within (say) one year, at a confidence level of 99%, due to default risk and exposure?"*
- *CVA instead answers the question:*
  "How much discount do I get on the price of this deal due to the fact that you, my counterparty, can default? I would trade this product with a default free party. To trade it with you, who are default risky, I require a discount."

Clearly, a price needs to be more precise than a risk measure, so the techniques will be different.

Q & A: Credit VaR and CVA

Q Different? Are the methodologies for Credit VaR and CVA not similar?

A *There are analogies but CVA needs to be more precise in general. Also, Credit VaR should use statistics under the physical measure whereas CVA should use statistics under the pricing measure*

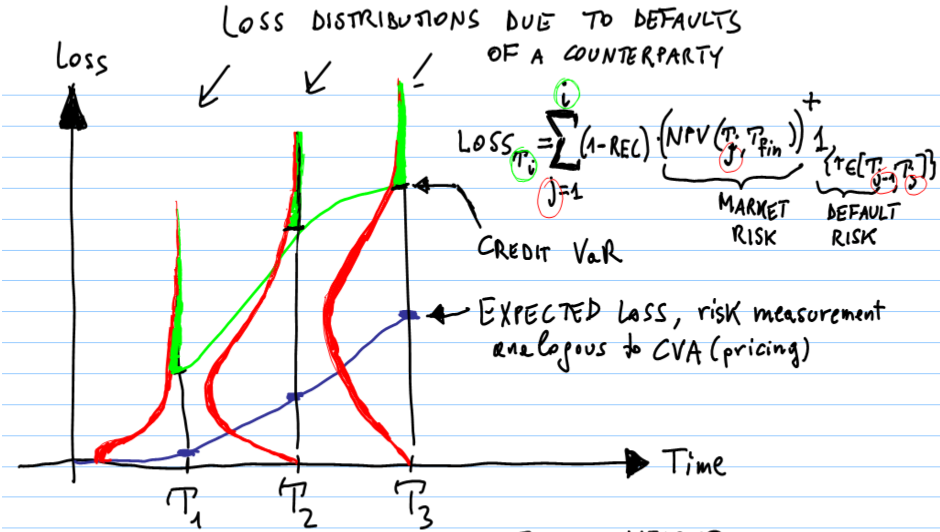
Q What are the regulatory bodies involved?

A *There are many, for Credit VaR type measures it is mostly Basel II and now III, whereas for CVA we have IAS, FASB and ISDA. But the picture is now blurring since Basel III is quite interested in CVA too*

Q What is the focus of this presentation?

A *We will focus on CVA.*

Q & A: Credit VaR and CVA



LOSSES ARE BUILT UNDER THE HISTORICAL MEASURE.

Q & A: CVA and Model Risk, WWR

Q What impacts counterparty risk CVA?

A *The OTC contract's underlying volatility, the correlation between the underlying and default of the counterparty, and the counterparty credit spreads volatility.*

Q Is it model dependent?

A *It is highly model dependent even if the original portfolio without counterparty risk was not. There is a lot of model risk.*

Q What about *wrong way risk*?

A *The amplified risk when the reference underlying and the counterparty are strongly correlated in the wrong direction.*

Q & A: Collateral

Q What is collateral?

A *It is a guarantee (liquid and secure asset, cash) that is deposited in a collateral account in favour of the investor party facing the exposure. If the depositing counterparty defaults, thus not being able to fulfill payments associated to the above mentioned exposure, Collateral can be used by the investor to offset its loss.*

Q & A: Netting

Q What is netting?

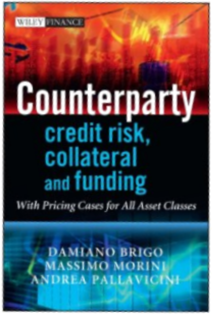
A *This is the agreement to net all positions towards a counterparty in the event of the counterparty default. This way positions with negative PV can be offset by positions with positive PV and counterparty risk is reduced. This has to do with the option on a sum being smaller than the sum of the options. CVA is typically computed on netting sets.*

For an introductory dialogue on Counterparty Risk see

CVA Q&A

D. Brigo (2012). Counterparty Risk Q&A: Credit VaR, CVA, DVA, Closeout, Netting, Collateral, Re-hypothecation, Wrong Way Risk, Basel, Funding, and Margin Lending. SSRN.com and arXiv.org.

Check also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Counterparty
credit risk,
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With Pricing Cases for All Asset Classes
DAMIANO BRIGO
MASSIMO MORINI
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AND HEDGING, CVA, MBS,
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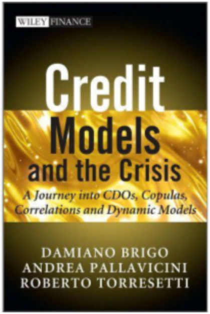
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Credit
Models
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A Journey into CDOs, Copulas,
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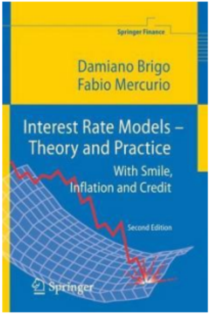

Springer Finance
Damiano Brigo
Fabio Mercurio
Interest Rate Models –
Theory and Practice
With Smile,
Inflation and Credit
Second Edition
Springer

General Notation

- We will call “Bank” or sometimes the ”investor” the party interested in the counterparty adjustment. This is denoted by “B”
- We will call “counterparty” the party with whom the Bank is trading, and whose default may affect negatively the Bank. This is denoted by “C”.
- “1” will be used for the underlying name/risk factor(s) of the contract
- The counterparty’s default time is denoted with τ_C and the recovery rate for unsecured claims with $\mathrm{R{\scriptstyle EC}}_C$ (we often use $\mathrm{L{\scriptstyle GD}}_C := 1 - \mathrm{R{\scriptstyle EC}}_C$).
- $\Pi_B(t, T)$ is the discounted payout without default risk seen by ‘B’ (sum of all future cash flows between t and T, discounted back at t). $\Pi_C(t, T) = -\Pi_B(t, T)$ is the same quantity but seen from the point of view of ‘C’. When we omit the index B or C we mean ‘B’.

Examples of products Π

If "B" enters an interest rate swap where "B" pays fixed K and receives from "C" LIBOR L with tenor $T_\alpha, T_{\alpha+1}, \dots, T_\beta$, then the payout is written, as we have seen earlier, as

$$\Pi(0, T_\beta) = \sum_{i=\alpha+1}^{\beta} D(0, T_i)(T_i - T_{i-1})(L(T_{i-1}, T_i) - K).$$

The majority of the instruments that are subject to Counterparty risk is given by Interest Rate Swaps.

General Notation

- We define $NPV_B(t,T) = \mathbb{E}_t[\Pi(t,T)]$. When T is clear from the context we omit it and write NPV(t).
- $$\Pi(s,t) + D(s,t)\Pi(t,u) = \Pi(s,u)$$
- $$\mathbb{E}_0[D(0,u)NPV(u,T)] = \mathbb{E}_0[D(0,u)\mathbb{E}_u[\Pi(u,T)]] =$$
$$= \mathbb{E}_0[D(0,u)\Pi(u,T)] = NPV(0,T) - \mathbb{E}_0[\Pi(0,u)]$$
$$= NPV(0,T) - NPV(0,u)$$

Unilateral counterparty risk

We now look into unilateral counterparty risk.

This is a situation where counterparty risk pricing is computed by assuming that only the counterparty can default, whereas the investor or bank doing the calculation is assumed to be default free.

Hence we will only consider here the default time τ_C of the counterparty. We will address the bilateral case later on.

The mechanics of Evaluating unilateral counterparty risk

payoff under counterparty default risk

- counterparty defaults after final maturity
  - original payoff of the instrument
- counterparty defaults before final maturity
  - all cash flows before default
    $\oplus$ recovery of the residual NPV at default if positive
    $\ominus$ Total residual NPV at default if negative

General Formulation under Asymmetry

$$\Pi_B^D(t,T) = \mathbf{1}_{\tau_C > T}\Pi_B(t,T)$$
$$+\mathbf{1}_{t<\tau_C\le T}\left[\Pi_B(t,\tau_C) + D(t,\tau_C)\left(REC_C\left(\mathrm{NPV}_B(\tau_C)\right)^+ - \left(-\mathrm{NPV}_B(\tau_C)\right)^+\right.\right.$$

This last expression is the general payoff seen from the point of view of 'B' (Π_B, NPV_B) under unilateral counterparty default risk. Indeed,

1. if there is no early default, this expression reduces to first term on the right hand side, which is the payoff of a default-free claim.
2. In case of early default of the counterparty, the payments due before default occurs are received (second term)
3. and then if the residual net present value is positive only the recovery value of the counterparty REC_C is received (third term),
4. whereas if it is negative it is paid in full by the investor/ Bank (fourth term).

General Formulation under Asymmetry

If one simplifies the cash flows and takes the risk neutral expectation, one obtains the fundamental formula for the valuation of counterparty risk when the investor/ Bank B is default free:

$$\mathbb{E}_t\left\{\Pi_B^D(t,T)\right\} =$$
$$\mathbf{1}_{\{\tau_C>t\}}\mathbb{E}_t\left\{\Pi_B(t,T)\right\} - \mathbb{E}_t\left\{\mathrm{LGD_C}\mathbf{1}_{\{t<\tau_C\leq T\}}\mathrm{D}(t,\tau_C)\left[\mathrm{NPV_B}(\tau_C)\right]^+\right\} \quad (*)$$

- First term : Value without counterparty risk.
- Second term : Unilateral Counterparty Valuation Adjustment
- $\mathrm{NPV}(\tau_C) = \mathbb{E}_{\tau_C}\left[\Pi(\tau_C, T)\right]$ is the value of the transaction on the counterparty default date. LGD = 1 - REC_counterparty.

$$\mathrm{UCVA}_0 = \mathbb{E}_t\left\{\mathrm{LGD_C}\mathbf{1}_{\{t<\tau_C\leq T\}}\mathrm{D}(t,\tau_C)\left[\mathrm{NPV_B}(\tau_C)\right]^+\right\}$$

Proof of the formula

In the proof we omit indices: $\tau = \tau_C$, REC=REC$_C$, LGD=LGD$_C$, NPV=NPV$_B$, $\Pi = \Pi_B$. The proof is obtained easily putting together the following steps. Since

$$1_{\{\tau>t\}}\Pi(t,T) = 1_{\{\tau>T\}}\Pi(t,T) + 1_{\{t<\tau\leq T\}}\Pi(t,T)$$

we can rewrite the terms inside the expectation in the right hand side of the simplified formula (*) as

$$\begin{aligned}
& \mathbf{1}_{\{\tau>t\}}\Pi(t,T) - \left\{LGD\mathbf{1}_{\{t<\tau\leq T\}}D(t,\tau)\,[\mathrm{NPV}(\tau)]^+\right\} \\
& = 1_{\{\tau>T\}}\Pi(t,T) + 1_{\{t<\tau\leq T\}}\Pi(t,T) \\
+ \quad & \{(\mathrm{Rec}-1)[1_{\{t<\tau\leq T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+]\} \\
= \quad & 1_{\{\tau>T\}}\Pi(t,T) + 1_{\{t<\tau\leq T\}}\Pi(t,T) \\
+ \quad & \mathrm{Rec}\;\; 1_{\{t<\tau\leq T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+ - 1_{\{t<\tau\leq T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+
\end{aligned}$$

Conditional on the information at τ the second and the fourth terms are equal to

Proof (cont'd)

$$
\begin{aligned}
& E_\tau[1_{\{t<\tau\leq T\}}\Pi(t,T) - 1_{\{t<\tau\leq T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+] \\
= \; & E_\tau[1_{\{t<\tau\leq T}\}[\Pi(t,\tau) + D(t,\tau)\Pi(\tau,T) - D(t,\tau)(E_\tau[\Pi(\tau,T)])^+]] \\
= \; & 1_{\{t<\tau\leq T\}}[\Pi(t,\tau) + D(t,\tau)E_\tau[\Pi(\tau,T)] - D(t,\tau)(E_\tau[\Pi(\tau,T)])^+] \\
= \; & 1_{\{t<\tau\leq T\}}[\Pi(t,\tau) - D(t,\tau)(E_\tau[\Pi(\tau,T)])^-] \\
= \; & 1_{\{t<\tau\leq T\}}[\Pi(t,\tau) - D(t,\tau)(E_\tau[-\Pi(\tau,T)])^+] \\
= \; & 1_{\{t<\tau\leq T\}}[\Pi(t,\tau) - D(t,\tau)(-\mathrm{NPV}(\tau))^+]
\end{aligned}
$$

since

$$
1_{\{t<\tau\leq T\}}\Pi(t,T) = 1_{\{t<\tau\leq T\}}\{\Pi(t,\tau) + D(t,\tau)\Pi(\tau,T)\}
$$

and $f = f^+ - f^- = f^+ - (-f)^+$.

Proof (cont'd)

Then we can see that after conditioning the whole expression of the original long payoff on the information at time τ and substituting the second and the fourth terms just derived above, the expected value with respect to $\mathcal{F}_t$ coincides exactly with the one in our simplified formula (*) by the properties of iterated expectations by which $\mathbb{E}_t[X] = \mathbb{E}_t[\mathbb{E}_\tau[X]]$.

What we can observe

- Including counterparty risk in the valuation of an otherwise default-free derivative $\Longrightarrow$ credit (hybrid) derivative.

- The inclusion of counterparty risk adds a level of optionality to the payoff.
  In particular, model independent products become model dependent also in the underlying market.
  $\Longrightarrow$ **Counterparty Risk analysis incorporates an opinion about the underlying market dynamics and volatility.**

The point of view of the counterparty “C”

The deal from the point of view of ‘C’, while staying in a world where only ‘C” may default.

$$\Pi_C^D(t,T) = \mathbf{1}_{\tau_C > T}\Pi_C(t,T)$$
$$+\mathbf{1}_{t<\tau_C\leq T}\left[\Pi_C(t,\tau_C) + D(t,\tau_C)\left((\mathrm{NPV}_C(\tau_C))^+ - REC_C\,(-\mathrm{NPV}_C(\tau_C))^+\right.\right.$$

This last expression is the general payoff seen from the point of view of ‘C’ (Π_C, NPV_C) under unilateral counterparty default risk. Indeed,

1. if there is no early default, this expression reduces to first term on the right hand side, which is the payoff of a default-free claim.
2. In case of early default of the counterparty ‘C”, the payments due before default occurs go through (second term)
3. and then if the residual net present value is positive to the defaulted ‘C’, it is received in full from ‘B’ (third term),
4. whereas if it is negative, only the recovery fraction REC_C it is paid to ‘B’ (fourth term).

The point of view of the counterparty "C"

The above formula simplifies to

$$\mathbb{E}_t\left\{\Pi_C^D(t,T)\right\} = \mathbf{1}_{\tau_C > t}\mathbb{E}_t\left\{\Pi_C(t,T)\right\} + \mathbb{E}_t\left\{\mathrm{LGD_C}\mathbf{1}_{t<\tau_C\le T}D(t,\tau_C)\left[-\mathrm{NPV_C}(\tau_C)\right]^+\right\}$$

and the adjustment term with respect to the risk free price $\mathbb{E}_t\left\{\Pi_C(t,T)\right\}$ is called

UNILATERAL DEBIT VALUATION ADJUSTMENT

$$\mathrm{UDVA}_C(t) = \mathbb{E}_t\left\{\mathrm{LGD_C}\mathbf{1}_{\{t<\tau_C\le T\}}D(t,\tau_C)\left[-\mathrm{NPV_C}(\tau_C)\right]^+\right\}$$

We note that $\mathrm{UDVA}_C = \mathrm{UCVA}_B$.
Notice also that in this universe $\mathrm{UDVA}_B = \mathrm{UCVA}_C = 0$.

Including the investor/ Bank default or not?

Often the investor, when computing a counterparty risk adjustment, considers itself to be default-free. This can be either a unrealistic assumption or an approximation for the case when the counterparty has a much higher default probability than the investor.

If this assumption is made when no party is actually default-free, the unilateral valuation adjustment is asymmetric: if “C” were to consider itself as default free and “B” as counterparty, and if “C” computed the counterparty risk adjustment, this would not be the opposite of the one computed by “B” in the straight case.

Also, the total NPV including counterparty risk is similarly asymmetric, in that the total value of the position to “B” is not the opposite of the total value of the position to “C”. There is no *cash conservation.*

Including the investor/ Bank default or not?

We get back symmetry if we allow for default of the investor/ Bank in computing counterparty risk. This also results in an adjustment that is cheaper to the counterparty “C”.

The counterparty “C” may then be willing to ask the investor/ Bank “B” to include the investor default event into the model, when the Counterparty risk adjustment is computed by the investor

The case of symmetric counterparty risk

Suppose now that we allow for both parties to default. Counterparty risk adjustment allowing for default of "B"?
"B" : the investor; "C": the counterparty;
("1": the underlying name/risk factor of the contract).
τ_B, τ_C: default times of "B" and "C". T: final maturity
We consider the following events, forming a partition

Four events ordering the default times

$$\begin{aligned} \mathcal{A} &= \{\tau_B \le \tau_C \le T\} \qquad E = \{T \le \tau_B \le \tau_C\} \\ \mathcal{B} &= \{\tau_B \le T \le \tau_C\} \qquad F = \{T \le \tau_C \le \tau_B\} \\ \mathcal{C} &= \{\tau_C \le \tau_B \le T\} \\ \mathcal{D} &= \{\tau_C \le T \le \tau_B\} \end{aligned}$$

Define $\mathrm{NPV}_{\{B,C\}}(t) := \mathbb{E}_t[\Pi_{\{B,C\}}(t,T)]$, and recall $\Pi_B = -\Pi_C$.

The case of symmetric counterparty risk

$$\Pi_B^D(t,T) = \mathbf{1}_{E\cup F}\Pi_B(t,T)$$
$$+\mathbf{1}_{\mathcal{C}\cup\mathcal{D}}\left[\Pi_B(t,\tau_C) + D(t,\tau_C)\left(REC_C\left(\mathrm{NPV}_B(\tau_C)\right)^+ - \left(-\mathrm{NPV}_B(\tau_C)\right)^+\right)\right]$$
$$+\mathbf{1}_{\mathcal{A}\cup\mathcal{B}}\left[\Pi_B(t,\tau_B) + D(t,\tau_B)\left(\left(\mathrm{NPV}_B(\tau_B)\right)^+ - REC_B\left(-\mathrm{NPV}_B(\tau_B)\right)^+\right)\right]$$

1. If no early default $\Rightarrow$ payoff of a default-free claim (1st term).
2. In case of early default of the counterparty, the payments due before default occurs are received (second term),
3. and then if the residual net present value is positive only the recovery value of the counterparty REC_C is received (third term),
4. whereas if negative, it is paid in full by the investor/ Bank (4th term).
5. In case of early default of the investor, the payments due before default occurs are received (fifth term),
6. and then if the residual net present value is positive it is paid in full by the counterparty to the investor/ Bank (sixth term),
7. whereas if it is negative only the recovery value of the investor/ Bank REC_B is paid to the counterparty (seventh term).

The case of symmetric counterparty risk

$$\mathbb{E}_t\left\{\Pi_B^D(t,T)\right\} = \mathbb{E}_t\left\{\Pi_B(t,T)\right\} + \mathrm{DVA}_B(t) - \mathrm{CVA}_B(t)$$

$$\mathrm{DVA}_B(t) = \mathbb{E}_\mathrm{t}\left\{\mathrm{LGD_B}\cdot \mathbf{1}(\mathrm{t} < \tau^{1\mathrm{st}} = \tau_\mathrm{B} < \mathrm{T})\cdot \mathrm{D}(\mathrm{t},\tau_\mathrm{B})\cdot[-\mathrm{NPV_B}(\tau_\mathrm{B})]^+\right\}$$

$$\mathrm{CVA}_B(t) = \mathbb{E}_\mathrm{t}\left\{\mathrm{LGD_C}\cdot \mathbf{1}(\mathrm{t} < \tau^{1\mathrm{st}} = \tau_\mathrm{C} < \mathrm{T})\cdot \mathrm{D}(\mathrm{t},\tau_\mathrm{C})\cdot[\mathrm{NPV_B}(\tau_\mathrm{C})]^+\right\}$$

$$1(A\cup B) = 1(t<\tau^{1st} = \tau_B < T),\;\; 1(C\cup D) = 1(t<\tau^{1st} = \tau_C < T)$$

- Obtained simplifying the previous formula and taking expectation.
- 2nd term : adj due to scenarios $\tau_B < \tau_C$. This is positive to the investor/ Bank B and is called ”Debit Valuation Adjustment” (DVA)
- 3d term : Counterparty risk adj due to scenarios $\tau_C < \tau_B$
- Bilateral Valuation Adjustment as seen from B: $\mathrm{BVA}_B = \mathrm{DVA}_B - \mathrm{CVA}_B$.
- If computed from the opposite point of view of “C” having counterparty “B”, $\mathrm{BVA}_C = -\mathrm{BVA}_B$. Symmetry.

CVA, DVA: Closeout

$$\textcolor{red}{\bar{V}_t} = \mathbb{E}_t\left\{\Pi_B^D(t,T)\right\} = \overbrace{\mathbb{E}_t\left\{\Pi_B(t,T)\right\}}^{V_t^0} + \mathrm{DVA}_B(t) - \mathrm{CVA}_B(t)$$

$$\mathrm{DVA}_B(t) = \mathbb{E}_t\left\{\mathrm{LGD_B}\cdot \mathbf{1}(t < \tau^{1st} = \tau_B < T)\cdot D(t,\tau_B)\cdot \left[-\underbrace{\mathrm{NPV}_B(\tau_B)}_{V^0_{\tau_B}\text{ or }\textcolor{red}{\bar{V}_{\tau_B}?}}\right]^+\right.$$

$$\mathrm{CVA}_B(t) = \mathbb{E}_t\left\{\mathrm{LGD_C}\cdot \mathbf{1}(t < \tau^{1st} = \tau_C < T)\cdot D(t,\tau_C)\cdot \left[\underbrace{\mathrm{NPV}_B(\tau_C)}_{V^0_{\tau_C}\text{ or }\textcolor{red}{\bar{V}_{\tau_C}?}}\right]^+\right\}$$

V^0 risk free closeout (much easier but discontinuity), $\bar{V}$ replacement closeout - recursive problem but more continuous. More in a minute.

The case of symmetric counterparty risk

Strange consequences of the formula new mid term, i.e. DVA

- credit quality of investor WORSENS $\Rightarrow$ books POSITIVE MARK TO MKT
- credit quality of investor IMPROVES $\Rightarrow$ books NEGATIVE MARK TO MKT
- Citigroup in its press release on the first quarter revenues of 2009 reported a *positive* mark to market due to its *worsened* credit quality: “Revenues also included [...] a net 2.5$ billion positive CVA on derivative positions, excluding monolines, mainly due to the widening of Citi’s CDS spreads”

The case of symmetric counterparty risk: DVA?

`October 18, 2011, 3:59 PM ET, WSJ. Goldman Sachs Hedges Its Way to Less Volatile Earnings.`

Goldman's DVA gains in the third quarter totaled $450 million [...] That amount is comparatively smaller than the $1.9 billion in DVA gains that J.P. Morgan Chase and Citigroup each recorded for the third quarter. Bank of America reported $1.7 billion of DVA gains in its investment bank. Analysts estimated that Morgan Stanley will record $1.5 billion of net DVA gains when it reports earnings on Wednesday [...]

Is DVA real? **DVA Hedging**. Buying back bonds? Proxying?

DVA hedge? One should sell protection on oneself, impossible, unless one buys back bonds that he had issued earlier. Very Difficult.
Most times: proxying. Instead of selling protection on oneself, one sells protection on a number of names that one thinks are highly correlated to oneself.

The case of symmetric counterparty risk: DVA?

Again from the WSJ article above:

[...] Goldman Sachs CFO David Viniar said Tuesday that the company attempts to hedge [DVA] using a basket of different financials.
A Goldman spokesman confirmed that the company did this by selling CDS on a range of financial firms. [...] Goldman wouldn't say what specific financials were in the basket, but Viniar confirmed [...] that the basket contained 'a peer group.'

This can approximately hedge the spread risk of DVA, but not the jump to default risk. Merrill hedging DVA risk by selling protection on Lehman would not have been a good idea. Worsens systemic risk.

DVA or no DVA? Accounting VS Capital Requirements

NO DVA: Basel III, page 37, July 2011 release

This CVA loss is calculated without taking into account any offsetting debit valuation adjustments which have been deducted from capital under paragraph 75.

YES DVA: FAS 157

Because nonperformance risk (the risk that the obligation will not be fulfilled) includes the reporting entitys credit risk, the reporting entity should consider the effect of its credit risk (credit standing) on the fair value of the liability in all periods in which the liability is measured at fair value under other accounting pronouncements FAS 157 (see also IAS 39)

DVA or no DVA? Accounting VS Capital Requirements

Stefan Walter says:

”The potential for perverse incentives resulting from profit being linked to decreasing creditworthiness means capital requirements cannot recognise it, says Stefan Walter, *secretary-general of the Basel Committee*: The main reason for not recognising DVA as an offset is that it would be inconsistent with the overarching supervisory prudence principle under which we do not give credit for increases in regulatory capital arising from a deterioration in the firms own credit quality.”

The case of symmetric counterparty risk: DVA?

When allowing for the investor to default: symmetry

- DVA: One more term with respect to the unilateral case.
- depending on credit spreads and correlations, the total adjustment to be subtracted (CVA-DVA) can now be either positive or negative. In the unilateral case it can only be positive.
- Ignoring the symmetry is clearly more expensive for the counterparty and cheaper for the investor.
- Hedging DVA is difficult. Hedging “by peers” ignores jump to default risk
- We assume the unilateral case in most of the numerical presentations
- WE TAKE THE POINT OF VIEW OF ‘B” from now on, so we omit the subscript ‘B’. We denote the counterparty as ‘C”.

CVA, DVA: A useful derivation in view of funding

- Immersion hypothesis for credit risk: work under default-free filtration $\mathcal{F}_t$ as much as possible.
- Assume conditional independence of defaults: spreads λ's may be correlated, but jump to defaults ξ's will be independent.

Conditional independence of defaults I

Recall that we are assuming

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau_i \leq u\}, u \leq t)$$

with i indexing all the default times in the system. Working under $\mathcal{F}$-immersion usually means that the risks in the basic cash flows Π are assumed not to be credit sensitive but to depend only on the filtration $\mathcal{F}$ of pre-default or default-free market information, eg default free interest rate swaps portfolio.

Conditional independence of defaults II

We are also assuming default times to be $\mathcal{F}$-conditionally independent:

$$\text{if} \quad \tau_B = \Lambda_B^{-1}(\xi_B), \quad \tau_C = \Lambda_C^{-1}(\xi_C),$$

then this means assuming that ξ_B and ξ_C are independent. Intensities $\lambda_B(t)$ and $\lambda_C(t)$ are taken $\mathcal{F}_t$ adapted (& can be correlated) and

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\min(\tau_B, \tau_C) > t) = \mathbb{Q}(\tau_B > t \cap \tau_C > t) =$$

We use the tower property + independence of ξ's on each other and $\mathcal{F}$:

$$= \mathbb{E}[\mathbb{Q}(\tau_B > t \ \cap \ \tau_C > t | \mathcal{F}_t)] = \mathbb{E}[\mathbb{Q}(\tau_B > t | \mathcal{F}_t)\mathbb{Q}(\tau_C > t | \mathcal{F}_t)] =$$

$$= \mathbb{E}[e^{-\Lambda_B(t)} e^{-\Lambda_C(t)}] = \mathbb{E}[e^{-\Lambda_B(t) - \Lambda_C(t)}] = \mathbb{E}[e^{-\int_0^t (\lambda_B(s) + \lambda_C(s))ds}]$$

Similarly, one can show the first to default time τ intensity λ

$$\text{is} \quad \mathbb{Q}(\tau \in [t, t+dt) | \tau > t, \mathcal{F}_t) = \lambda_t \, dt = (\lambda_B(t) + \lambda_C(t))dt.$$

Conditional independence of defaults III

Summing up:
Whenever we use the immersion hypothesis, meaning that we switch filtration from $\mathcal{G}$ to $\mathcal{F}$, we assume the ξ to be conditionally independent and the basic cash flows $\Pi(s,t)$ to be $\mathcal{F}_t$ adapted for all $s \le t$.

Switching to the filtration $\mathcal{F}$ typically transforms indicators such as $1_{\{\tau>t\}}$ into their $\mathcal{F}$ expectations $e^{-\int_0^t (\lambda_B(s)+\lambda_C(s))ds}$. This is often collected in the discount term $D(0,t;r)$ that becomes $D(0,t;r+\lambda)$.

$$D(0,t;r)1_{\{\tau>t\}} = e^{-\int_0^t r_s ds}1_{\{\tau>t\}} \text{ goes } e^{-\int_0^t r_s ds}e^{-\int_0^t \lambda_s ds} = D(0,t;r+\lambda)$$

The switching also transforms $1_{\{\tau\in dt\}}$ into $\lambda_t e^{-\int_0^t \lambda_s ds}dt$.

We now present the calculation of CVA and DVA under immersion. Here V will denote either V^0 or $\bar{V}$.

CVA, DVA: A useful derivation in view of funding

$$\mathrm{CVA}_B(t) = \mathbb{E}_t\left\{\mathrm{LGD_C}\cdot\mathbf{1}(t<\tau^{1st}=\tau_C<T)\cdot D(t,\tau_C)\cdot[V(\tau_C)]^+\right\}$$

$$= \mathbb{E}_t\left\{LGD_C\int_t^T 1_{\{\tau^{1st}\in du\}}1_{\{\tau_B>u\}}D(t,u)\,[V(u)]^+\right\}$$

$$= LGD_C\int_t^T \mathbb{E}_t\left\{1_{\{\tau_C\in du\}}1_{\{\tau_B>u\}}D(t,u)\,(V(u))^+\right\}$$

$$= LGD_C\int_t^T \mathbb{E}_t\left\{\mathbb{E}_u\left[1_{\{\tau_C\in du\}}1_{\{\tau_B>u\}}D(t,u)\,(V(u))^+\,|\mathcal{F}_{u+du}\right]\right\}$$

$$= LGD_C\int_t^T \mathbb{E}_t\left\{D(t,u)\,(V(u))^+\,\mathbb{E}_u\left[1_{\{\tau_C\in du\}}1_{\{\tau_B>u+du\}}|\mathcal{F}_{u+du}\right]\right\} = \ldots$$

$$\Big(\mathbb{E}_u\left[1_{\{\tau_C\in du\}}1_{\{\tau_B>u+du\}}|\mathcal{F}_{u+du}\right] = \mathbb{E}_u\left[1_{\{\tau_C\in du\}}|\mathcal{F}\right]\mathbb{E}_u\left[1_{\{\tau_B>u+du\}}|\mathcal{F}\right]$$

$$= \lambda_C(u)du\, e^{\left(-\int_t^u\lambda_C(s)ds\right)}e^{\left(-\int_t^u\lambda_B(s)ds\right)} = \lambda_C(u)du\, e^{\left(-\int_t^u(\lambda_C(s)+\lambda_B(s))ds\right.}$$

$$= \lambda_C(u)e^{-\int_t^u\lambda(s)ds}\,du\Big) = \mathbb{E}_t\left\{\mathrm{LGD}_C\int_t^T D(t,u;r+\lambda)\lambda_C(u)\,(V(u))^+\,du\right\}$$

CVA, DVA: A useful derivation in view of funding

$$\mathrm{CVA}_B(t) = \mathbb{E}_t\left\{\int_t^T D(t,u;r+\lambda)\mathrm{L}_{\mathrm{GD}_C}\,\lambda_C(u)\,(V(u))^+\,du\right\}$$

$$\mathrm{DVA}_B(t) = \mathbb{E}_t\left\{\int_t^T D(t,u;r+\lambda)\mathrm{L}_{\mathrm{GD}_B}\,\lambda_B(u)\,(-V(u))^+\,du\right\}$$

and we will see later that (without collateral and under the Reduced Borrowing Benefit case) Funding Cost and Benefit Adjustments (FCA, FBA) are (notice the formal analogies, used in industry)

$$\mathrm{FCA}_B(t) = \mathbb{E}_t\left\{\int_t^T D(t,u;r+\lambda)\mathrm{L}_{\mathrm{GD}_B}\,\lambda_B(u)\,(V(u))^+\,du\right\}$$

$$\mathrm{FBA}_B(t) = \mathbb{E}_t\left\{\int_t^T D(t,u;r+\lambda)\mathrm{L}_{\mathrm{GD}_B}\,\lambda_B(u)\,(-V(u))^+\,du\right\} = \mathrm{DVA}_B(t$$

Closeout: Replication (ISDA?) VS Risk Free

When we computed the bilateral adjustment formula from

$$\begin{aligned}
&\Pi_B^D(t,T) = \mathbf{1}_{E\cup F}\Pi_B(t,T)\\
&+\mathbf{1}_{C\cup D}\left[\Pi_B(t,\tau_C) + D(t,\tau_C)\left(REC_C\,(\mathrm{NPV}_B(\tau_C))^+ - (-\mathrm{NPV}_B(\tau_C))^+\right)\right]\\
&+\mathbf{1}_{A\cup B}\left[\Pi_B(t,\tau_B) + D(t,\tau_B)\left((-\mathrm{NPV}_C(\tau_B))^+ - REC_B\,(\mathrm{NPV}_C(\tau_B))^+\right)\right]
\end{aligned}$$

(where we now substituted $\mathrm{NPV}_B = -\mathrm{NPV}_C$ in the last two terms) we used the risk free NPV upon the first default, to close the deal. But what if upon default of the first entity, the deal needs to be valued by taking into account the credit quality of the surviving party? What if we make the substitutions

$$\mathrm{NPV}_B(\tau_C) \to \mathrm{NPV}_B(\tau_C) + \mathrm{UDVA}_B(\tau_C)$$

$$\mathrm{NPV}_C(\tau_B) \to \mathrm{NPV}_C(\tau_B) + \mathrm{UDVA}_C(\tau_B)?$$

Closeout: Replication (ISDA?) VS Risk Free

ISDA (2009) Close-out Amount Protocol.

”In determining a Close-out Amount, the Determining Party may consider any relevant information, including, [...] quotations (either firm or indicative) for replacement transactions supplied by one or more third parties that **may take into account the creditworthiness of the Determining Party** at the time the quotation is provided”

This makes valuation more continuous: upon default we still price including the DVA, as we were doing before default.

Closeout: Substitution (ISDA?) VS Risk Free

The final formula with subsitution closeout is quite complicated:

$$
\begin{aligned}
&\Pi_B^D(t,T) = \mathbf{1}_{E\cup F}\Pi_B(t,T) \\
&+\mathbf{1}_{C\cup D}\Big[\Pi_B(t,\tau_C) + D(t,\tau_C) \\
&\cdot\left(REC_C\left(\mathrm{NPV}_B(\tau_C)+\mathrm{UDVA}_B(\tau_C)\right)^+ - \left(-\mathrm{NPV}_B(\tau_C)-\mathrm{UDVA}_B(\tau_C)\right)^+\right) \\
&+\mathbf{1}_{A\cup B}\Big[\Pi_B(t,\tau_B) + D(t,\tau_B) \\
&\cdot\left(\left(-\mathrm{NPV}_C(\tau_B)-\mathrm{UDVA}_C(\tau_B)\right)^+ - REC_B\left(\mathrm{NPV}_C(\tau_B)+\mathrm{UDVA}_C(\tau_B)\right)^+\right)
\end{aligned}
$$

Closeout: Substitution (ISDA?) VS Risk Free

Brigo and Morini (2010)

We analyze the Risk Free closeout formula in Comparison with the Replication Closeout formula for a Zero coupon bond when:
1. Default of 'B' and 'C" are independent
2. Default of 'B' and 'C" are co-monotonic

Suppose 'B' (the lender) holds the bond,
and 'C' (the borrower) will pay the notional 1 at maturity T.

The risk free price of the bond at time 0 to 'B' is denoted by $P(0, T)$.

Closeout: Replication (ISDA?) VS Risk Free

Suppose 'B' (the lender) holds the bond, and 'C' (the borrower) will pay the notional 1 at maturity T.
The risk free price of the bond at time 0 to 'B' is denoted by $P(0, T)$.
If we assume deterministic interest rates, the above formulas reduce to

$$P^{D,Repl}(0,T) = P(0,T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \leq T)]$$

$$\begin{aligned} P^{D,Free}(0,T) &= P(0,T)[\mathbb{Q}(\tau_C > T) + \mathbb{Q}(\tau_B < \tau_C < T) \\ &\quad + REC_C \mathbb{Q}(\tau_C \leq \min(\tau_B, T))] \\ &= P(0,T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \leq T) + LGD_C \mathbb{Q}(\tau_B < \tau_C < T)] \end{aligned}$$

Risk Free Closeout and Credit Risk of the Lender

The adjusted price of the bond DEPENDS ON THE CREDIT RISK OF THE LENDER 'B' IF WE USE THE RISK FREE CLOSEOUT. This is counterintuitive and undesirable.

Closeout: Replication (ISDA?) VS Risk Free

Co-Monotonic Case

If we assume the default of B and C to be co-monotonic, and the spread of the lender ‘B” to be larger, we have that the lender ‘B” defaults first in ALL SCENARIOS (e.g. ‘C’ is a subsidiary of ‘B’, or a company whose well being is completely driven by ‘B’: ‘C’ is a trye factory whose only client is car producer ‘B”). In this case

$$P^{D,Repl}(0,T) = P(0,T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \leq T)]$$
$$P^{D,Free}(0,T) = P(0,T)[\mathbb{Q}(\tau_C > T) + \mathbb{Q}(\tau_C < T)] = P(0,T)$$

Risk free closeout is correct. Either ‘B” does not default, and then ‘C” does not default either, or if ‘B” defaults, at that precise time C is solvent, and B recovers the whole payment. Credit risk of ‘C” should not impact the deal.

Closeout: Substitution (ISDA?) VS Risk Free

Contagion. What happens at default of the Lender

$$P^{D,Subs}(t,T) = P(t,T)[\mathbb{Q}_t(\tau_C > T) + REC_C \mathbb{Q}_t(\tau_C \leq T)]$$

$$P^{D,Free}(t,T) = P^{D,Subs}(t,T) + P(t,T) LGD_C \mathbb{Q}_t(\tau_B < \tau_C < T)$$

We focus on two cases:

- τ_B and τ_C are independent. Take $t < T$.

$$\mathbb{Q}_{t-\Delta t}(\tau_B < \tau_C < T) \mapsto \quad \{\tau_B = t\} \quad \mapsto \mathbb{Q}_{t+\Delta t}(\tau_C < T)$$

  and this effect can be quite sizeable.

- τ_B and τ_C are comonotonic. Take an example where $\tau_B = t < T$ implies $\tau_C = u < T$ with $u > t$. Then

$$\mathbb{Q}_{t-\Delta t}(\tau_C > T) \mapsto \quad \{\tau_B = t, \tau_C = u\} \quad \mapsto 0$$

$$\mathbb{Q}_{t-\Delta t}(\tau_C \leq T) \mapsto \quad \{\tau_B = t, \tau_C = u\} \quad \mapsto 1$$

$$\mathbb{Q}_{t-\Delta t}(\tau_B < \tau_C < T) \mapsto \quad \{\tau_B = t, \tau_C = u\} \quad \mapsto 1$$

Closeout: Substitution (ISDA?) VS Risk Free

Let us put the pieces together:

- τ_B and τ_C are independent. Take $t < T$.

$$P^{D,Subs}(t-\Delta t, T) \mapsto \quad \{\tau_B = t\} \mapsto \quad \text{no change}$$

$$P^{D,Free}(t-\Delta t, T) \mapsto \quad \{\tau_B = t\} \mapsto \quad \text{add } \mathbb{Q}_{t-\Delta t}(\tau_B > \tau_C, \tau_C < T)$$

  and this effect can be quite sizeable.

- τ_B and τ_C are comonotonic. Take an example where $\tau_B = t < T$ implies $\tau_C = u < T$ with $u > t$. Then

$$P^{D,Subs}(t-\Delta t, T) \mapsto \quad \{\tau_B = t\} \mapsto \quad \text{subtract } X$$

$$X = LGD_C P(t,T)\mathbb{Q}_{t-\Delta t}(\tau_C > T)$$

$$P^{D,Free}(t-\Delta t, T) \mapsto \quad \{\tau_B = t\} \mapsto \quad \text{no change}$$

Closeout: Replication (ISDA?) VS Risk Free

The independence case: Contagion with Risk Free closeout

The Risk Free closeout shows that *upon default of the lender*, the mark to market to the lender itself jumps up, or equivalently **the mark to market to the borrower jumps down**. The effect can be quite dramatic.
The Replication closeout instead shows no such contagion, as the mark to market does not change upon default of the lender.

The co-monotonic case: Contagion with Replication closeout

The Risk Free closeout behaves nicely in the co-monotonic case, and there is no change upon default of the lender.
Instead the Replication closeout shows that *upon default of the lender* the mark to market to the lender jumps down, or equivalently **the mark to market to the borrower jumps up**.

Closeout: Replication (ISDA?) VS Risk Free

Impact of an early default of the Lender

Dependence→ Closeout↓	independence	co-monotonicity
Risk Free	Negatively affects Borrower	No contagion
Replication	No contagion	Further Negatively affects Lender

For a numerical case study and more details see Brigo and Morini (2010, 2011).

A simplified formula without τ^{1st} for bilateral VA

- The simplified formula is only a simplified representation of bilateral risk and ignores that upon the first default closeout proceedings are started, thus involving a degree of double counting
- It is attractive because it allows for the construction of a bilateral counterparty risk pricing system based only on a unilateral one.
- The correct formula involves default dependence between the two parties through τ^{1st} and allows no such incremental construction
- A simplified bilateral formula is possible also in case of substitution closeout, but it turns out to be identical to the simplified formula of the risk free closeout case.
- We analyze the impact of default dependence between investor ‘B’ and counterparty ‘C’ on the difference between the two formulas by looking at a zero coupon bond and at an equity forward.

A simplified formula without τ^{1st} for bilateral VA

One can show easily that the difference between the full correct formula and the simplified formula is

$$\begin{aligned} & E_0[1_{\{\tau_B<\tau_C<T\}} LGD_C D(0,\tau_C)(E_{\tau_C}(\Pi(\tau_C,T)))^+] \\ - \quad & E_0[1_{\{\tau_C<\tau_B<T\}} LGD_B D(0,\tau_B)(-E_{\tau_B}(\Pi(\tau_B,T)))^+]. \end{aligned} \tag{28}$$

A simplified formula without τ^{1st}: The case of a Zero Coupon Bond

We work under deterministic interest rates. We consider $P(t, T)$ held by 'B" (lender) who will receive the notional 1 from 'C"(borrower) at final maturity T if there has been no default of 'C".

The difference between the correct bilateral formula and the simplified one is, under risk free closeout,

$$LGD_C P(0, T)\mathbb{Q}(\tau_B < \tau_C < T).$$

The case with substitution closeout is instead trivial and the difference is null. For a bond, the simplified formula coincides with the full substitution closeout formula.

Therefore the difference above is the same difference between risk free closeout and substitution closeout formulas, and has been examined earlier, also in terms of contagion.

A simplified formula without τ^{1st}: The case of an Equity forward

In this case the payoff at maturity time T is given by $S_T - K$

where S_T is the price of the underlying equity at time T and K the strike price of the forward contract (typically $K = S_0$, 'at the money', or $K = S_0/P(0,T)$, 'at the money forward').

We compute the difference D^{BC} between the correct bilateral risk free closeout formula and the simplified one.

A simplified formula without τ^{1st}: The case of an Equity forward

$D^{BC} := A_1 - A_2$, where

$$A_1 = E_0 \left\{ 1_{\{\tau_B < \tau_C < T\}} LGD_C D(0, \tau_C)(S_{\tau_C} - P(\tau_C, T)K)^+ \right\}$$
$$A_2 = E_0 \left\{ 1_{\{\tau_C < \tau_B < T\}} LGD_B D(0, \tau_B)(P(\tau_B, T)K - S_{\tau_B})^+ \right\}$$

The worst cases will be the ones where the terms A_1 and A_2 do not compensate. For example assume there is a high probability that $\tau_B < \tau_C$ and that the forward contract is deep in the money. In such case A_1 will be large and A_2 will be small.

Similarly, a case where $\tau_C < \tau_B$ is very likely and where the forward is deep out of the money will lead to a large A_2 and to a small A_1.

However, we show with a numerical example that even when the forward is at the money the difference can be relevant. For more details see Brigo and Buescu (2011).

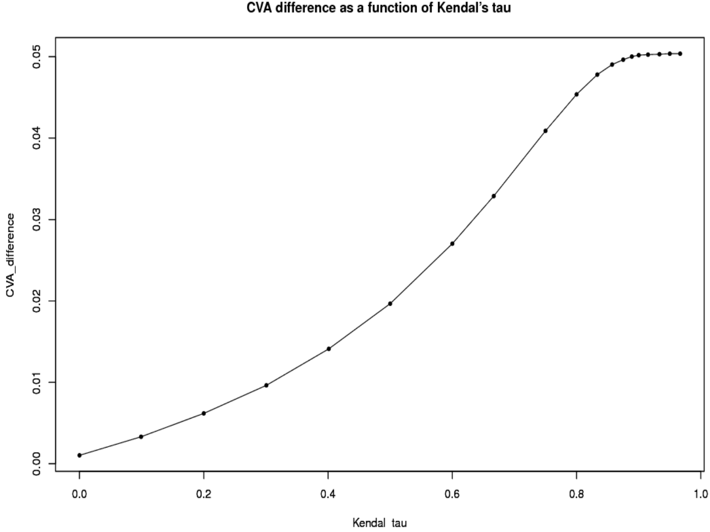


Figure: D^{BC} plotted against Kendall's tau between τ_B and τ_C, all other quantities being equal: $S_0 = 1$, $T = 5$, $\sigma = 0.4$, $K = 1$, $\lambda_B = 0.1$, $\lambda_C = 0.05$.

PAYOFF RISK

The exact payout corresponding with the Credit and Debit valuation adjustment is not clear.

- DVA or not?
- Which Closeout?
- First to default risk or not?
- How are collateral and funding accounted for exactly?

Worse than model risk: Payout risk. WHICH PAYOUT?

At a recent industry panel (WBS) on CVA it was stated that 5 banks might compute CVA in 15 different ways.

Methodology

1. Assumption: The *Bank/investor* enters a transaction with a *counterparty* and, when dealing with Unilateral Risk, the investor considers itself default free.
   Note : All the payoffs seen from the point of view of the *investor*.
2. We model and calibrate the default time of the *counterparty* using a stochastic intensity default model, except in the equity case where we will use a firm value model.
3. We model the transaction underlying and estimate the deal NPV at default.
4. We allow for the counterparty default time and the contract underlying to be correlated.
5. We start however from the case when such correlation can be neglected.

Approximation: Default Bucketing

General Formulation

1. Model (underlying) to estimate the NPV of the transaction.
2. Simulations are run allowing for correlation between the credit and underlying models, to determine the counterparty default time and the underlying deal NPV respectively.

Approximated Formulation under default bucketing

$$
\begin{aligned}
\mathbb{E}_0\Pi^D(0,T) &:= \mathbb{E}_0\Pi(0,T) - \mathrm{L{\small GD}}\,\mathbb{E}_0[1_{\{\tau<T_b\}}\, D(0,\tau)(\mathbb{E}_\tau\Pi(\tau,T))^+] \\
&= \mathbb{E}_0\Pi(0,T) - \mathrm{L{\small GD}}\,\mathbb{E}_0[(\sum_{j=1}^{b} 1\{\tau\in(T_{j-1},T_j]\})\, D(0,\tau)(\mathbb{E}_\tau\Pi(\tau,T))^+] \\
&= \mathbb{E}_0\Pi(0,T) - \mathrm{L{\small GD}}\sum_{j=1}^{b}\mathbb{E}_0[1\{\tau\in(T_{j-1},T_j]\}\, D(0,\tau)(\mathbb{E}_\tau\Pi(\tau,T))^+] \\
&\approx \mathbb{E}_0\Pi(0,T) - \mathrm{L{\small GD}}\sum_{j=1}^{b}\mathbb{E}_0[1\{\tau\in(T_{j-1},T_j]\}\, D(0,T_j)(\mathbb{E}_{T_j}\Pi(T_j,T))^+]
\end{aligned}
$$

Approximation: Default Bucketing and Independence

1. In this formulation defaults are bucketed but we still need a joint model for τ and the underlying Π including their correlation.
2. Option model for Π is implicitly needed in τ scenarios.

Approximated Formulation under independence (and 0 correlation)

$$\mathbb{E}_0 \Pi^D(0,T) := \mathbb{E}_0 \Pi(0,T)$$

$$-\text{L}_{\text{GD}} \sum_{j=1}^{b} \boxed{\mathbb{Q}\{\tau \in (T_{j-1}, T_j]\}} \mathbb{E}_0[D(0,T_j)(\mathbb{E}_{T_j}\Pi(T_j,T))^+]$$

1. In this formulation defaults are bucketed and only survival probabilities are needed (no default model).
2. Option model is STILL needed for the underlying of Π.

Ctrparty default model: CIR++ stochastic intensity

If we cannot assume independence, we need a default model.
Counterparty instantaneous credit spread: $\lambda(t) = y(t) + \psi(t;\beta)$

1. $y(t)$ is a CIR process with possible jumps

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW_t^y + dJ_t, \;\; \tau_C = \Lambda^{-1}(\xi), \;\; \Lambda(T) = \int_0^T \lambda(s)ds$$

2. $\psi(t;\beta)$ is the shift that matches a given CDS curve
3. ξ is standard exponential independent of all brownian driven stochastic processes
4. In CDS calibration we assume deterministic interest rates.
5. Calibration : Closed form Fitting of model survival probabilities to counterparty CDS quotes
6. B and El Bachir (2010) (Mathematical Finance) show that this model with jumps has closed form solutions for CDS options.

Literature on CVA across asset classes

Impact of dynamics, volatilities, correlations, wrong way risk

- **Interest Rate Swaps and Derivatives Portfolios** (Brigo Masetti (2005), Brigo Pallavicini 2007, 2008, Brigo Capponi P. Papatheodorou 2011, B. C. P. P. 2012 with collateral and gap risk)
- **Commodities swaps (Oil)** (Brigo and Bakkar 2009)
- **Credit: CDS on a reference credit** (Brigo and Chourdakis 2009, B. C. Pallavicini 2012 Mathematical Finance)
- **Equity Return Swaps** (Brigo and Tarenghi 2004, B. T. Morini 2011)
- Equity uses AT1P firm value model of B. and T. (2004) (barrier options with time-inhomogeneous GBM) and extensions (random barriers for risk of fraud).

Further asset classes are studied in the literature. For example see Biffis et al (2011) for CVA on **longevity swaps.**

Interest Rate and Commodities swaps and derivatives

We now examine UCVA with WWR for:

- Interest Rate Swaps and Derivatives Portfolios
- Commodities swaps

Interest rate swaps are the vast majority of market contracts on which CVA is computed.

Interest Rates Swap Case

Formulation for IRS under independence (no correlation)

$$\mathrm{IRS}^D(t,K) = \mathrm{IRS}(t.K)$$
$$-\mathrm{L{\small GD}} \sum_{i=a+1}^{b-1} \mathbb{Q}\{\tau \in (T_{i-1}, T_i]\}\, \mathrm{S{\small WAPTION}}_{i,b}(t; K, S_{i,b}(t), \sigma_{i,b})$$

Modeling Approach with corr.

Gaussian 2-factor G2++ short-rate $r(t)$ model:
$r(t) = x(t) + z(t) + \varphi(t;\alpha), r(0) = r_0$

$dx(t) = -ax(t)dt + \sigma dW_x$
$dz(t) = -bz(t)dt + \eta dW_z$

$dW_x dW_z = \rho_{x,z} dt$

$\alpha = [r_0, a, b, \sigma, \eta, \rho_{1,2}]$

$dW_x dW_y = \rho_{x,y} dt, dW_z dW_y = \rho_{z,y} dt$

Calibration

- The function $\varphi(\cdot;\alpha)$ is deterministic and is used to calibrate the initial curve observed in the market.
- We use swaptions and zero curve data to calibrate the model.
- The r factors x and z and the intensity are taken to be correlated.

Interest Rates Swap Case

Total Correlation Counterparty default / rates

$$\bar{\rho} = \mathrm{Corr}(dr_t, d\lambda_t) = \frac{\sigma\rho_{x,y} + \eta\rho_{z,y}}{\sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho_{x,z}}\sqrt{1 + \frac{2\beta\gamma^2}{\nu^2 y_t}}}.$$

where β is the intensity of arrival of λ jumps and γ is the mean of the exponentially distributed jump sizes.

Without jumps ($\beta = 0$)

$$\bar{\rho} = \mathrm{Corr}(dr_t, d\lambda_t) = \frac{\sigma\rho_{x,y} + \eta\rho_{z,y}}{\sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho_{x,z}}}.$$

IRS: Case Study

1) Single Interest Rate Swaps (IRS)

At-the-money fix-receiver forward interest-rate-swap (IRS) paying on the EUR market.

The IRS's fixed legs pay annually a 30E/360 strike rate, while the floating legs pay LIBOR twice per year.

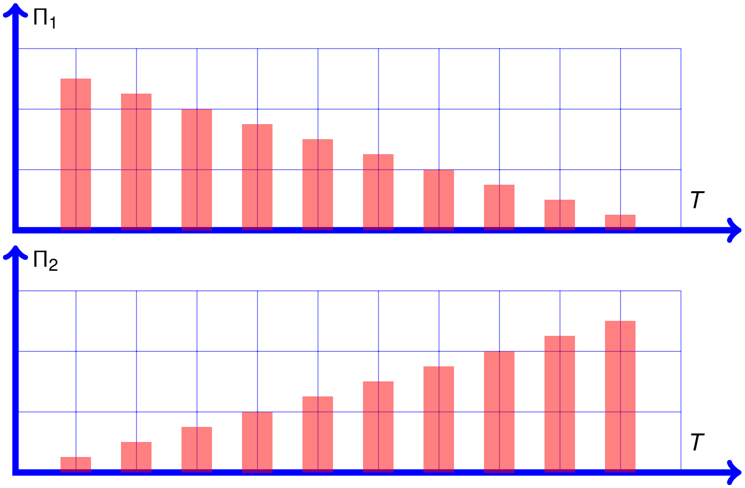
2) Netted portfolios of IRS.

- Portfolios of at-the-money IRS either with different starting dates or with different maturities.

1. (Π1) annually spaced dates $\{T_i : i = 0 \dots N\}$, T_0 two business days from trade date; portfolio of swaps maturing at each T_i, with $i > 0$, all starting at T_0.
2. (Π2) portfolio of swaps starting at each T_i all maturing at T_N.

Can also do exotics (Ratchets, CMS spreads, Bermudan)

IRS Case Study: Payment schedules



IRS Results

Counterparty risk price for netted receiver IRS portfolios Π1 and Π2 and simple IRS (maturity 10Y). Every IRS, constituting the portfolios, has unit notional and is at equilibrium. Prices are in bps.

λ	correlation $\bar{\rho}$	Π1	Π2	IRS
3%	-1	-140	-294	-36
	0	-84	-190	-22
	1	-47	-115	-13
5%	-1	-181	-377	-46
	0	-132	-290	-34
	1	-99	-227	-26
7%	-1	-218	-447	-54
	0	-173	-369	-44
	1	-143	-316	-37

Compare with ”Basel 2” deduced adjustments

Basel 2, under the ”Internal Model Method”, models wrong way risk by means of a 1.4 multiplying factor to be applied to the zero correlation case, even if banks have the option to compute their own estimate of the multiplier, which can never go below 1.2 anyway.

Is this confirmed by our model?

$$(140 - 84)/84 \approx 66\% > 40\%$$

$$(54 - 44)/44 \approx 23\% < 40\%$$

So this really depends on the portfolio and on the situation.

A bilateral example and correlation risk

Finally, in the bilateral case for Receiver IRS, 10y maturity, high risk counterparty and mid risk investor, we notice that depending on the correlations

$$\bar{\rho}_0 = \mathrm{Corr}(dr_t, d\lambda_t^0), \quad \bar{\rho}_2 = \mathrm{Corr}(dr_t, d\lambda_t^2), \quad \rho_{0,2}^{Copula} = 0$$

the DVA - CVA or Bilateral CVA does change sign, and in particular for portfolios Π1 and IRS the sign of the adjustment follows the sign of the correlations.

$\bar{\rho}_2$	$\bar{\rho}_0$	**Π1**	**Π2**	**10×IRS**
-60%	**0%**	-117(7)	-382(12)	-237(16)
-40%	**0%**	-74(6)	-297(11)	-138(15)
-20%	**0%**	-32(6)	-210(10)	-40(14)
0%	**0%**	-1(5)	-148(9)	31(13)
20%	**0%**	24(5)	-96(9)	87(12)
40%	**0%**	44(4)	-50(8)	131(11)
60%	**0%**	57(4)	-22(7)	159(11)

Payer vs Receiver

- Counterparty Risk (CR) has a relevant impact on interest-rate payoffs prices and, in turn, correlation between interest-rates and default (intensity) has a relevant impact on the CR adjustment.
- The (positive) CR adjustment to be subtracted from the default free price **decreases with correlation for receiver payoffs**. Natural: If default intensities increase, with high positive correlation their correlated interest rates will increase more than with low correlation, and thus a receiver swaption embedded in the adjustment decreases more, reducing the adjustment.
- The adjustment for payer payoffs increases with correlation.

Further Stylized Facts

- As the default probability implied by the counterparty CDS increases, the size of the adjustment increases as well, but the impact of correlation on it decreases.
- Financially reasonable: Given large default probabilities for the counterparty, fine details on the dynamics such as the correlation with interest rates become less relevant
- **The conclusion is that we should take into account interest-rate/ default correlation in valuing CR interest-rate payoffs.**
- In the bilateral case correlation risk can cause the adjustment to change sign

Exotics

For examples on exotics, including Bermudan Swaptions and CMS spread Options, see

Papers with Exotics and Bilateral Risk

- Brigo, D., and Pallavicini, A. (2007). Counterparty Risk under Correlation between Default and Interest Rates. In: Miller, J., Edelman, D., and Appleby, J. (Editors), Numerical Methods for Finance, Chapman Hall.
- Brigo, D., Pallavicini, A., and Papatheodorou, V. (2009). Bilateral counterparty risk valuation for interest-rate products: impact of volatilities and correlations. Available at Defaultrisk.com, SSRN and arXiv

Commodities and WWR

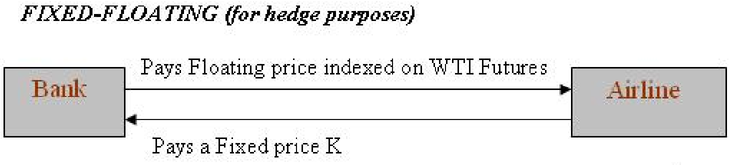
The correlation between interest rates dr_t (LIBOR, OIS) and credit intensities $d\lambda_t$, if measured historically, if often quite small in absolute value. Hence interest rates are a case where including correlation is good for stress tests and conservative hedging of CVA, but a number of market participant think that CVA can be computed by assuming zero correlations.

Whether one agrees or not, there are other asset classes on which CVA can be computed and where there is agreement on the necessity of including correlation in CVA pricing. We provide an example: Oil swaps traded with an airline.

It's natural to think that the future credit quality of the airline will be correlated with prices of oil.

Commodities: Futures, Forwards and Swaps

- **Forward**: OTC contract to buy a commodity to be delivered at a maturity date T at a price specified today. The cash/commodity exchange happens at time T.
- **Future**: Listed Contract to buy a commodity to be delivered at a maturity date T. Each day between today and T margins are called and there are payments to adjust the position.
- **Commodity Swap: Oil Example**:



Commodities: Modeling Approach

Schwartz-Smith Model

$$
\begin{aligned}
ln(S_t) &= x_t + l_t + \varphi(t) \\
dx_t &= -kx_t dt + \sigma_x dW_x \\
dl_t &= \mu dt + \sigma_l dW_l \\
dW_x \, dW_l &= \rho_{x,l} dt
\end{aligned}
$$

Variables

S_t: Spot oil price;
x_t, l_t: short and long term components of S_t;
This can be re-cast in a classic convenience yield model

Correlation with credit

$$
\begin{aligned}
dW_x \, dW_y &= \rho_{x,y} dt, \\
dW_l \, dW_y &= \rho_{l,y} dt
\end{aligned}
$$

Calibration

φ: defined to exactly fit the oil forward curve.
Dynamic parameters k, μ, σ, ρ are calibrated to At the money implied volatilities on Futures options.

Commodities

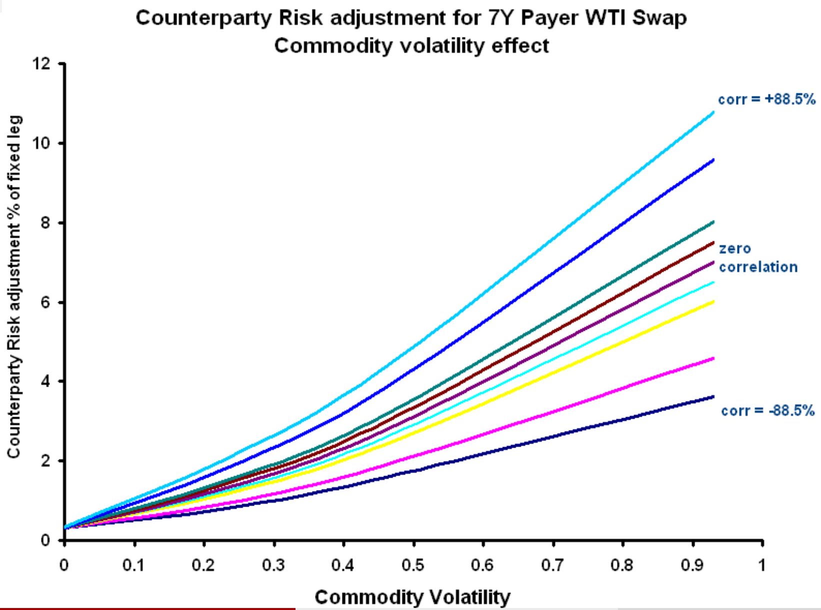
Total correlation Commodities - Counterparty default

$$\bar{\rho} = \text{corr}(d\lambda_t,\ dS_t) = \frac{\sigma_x \rho_{x,y} + \sigma_L \rho_{L,y}}{\sqrt{\sigma_x^2 + \sigma_L^2 + 2\rho_{x,L}\sigma_x \sigma_L}}$$

We assumed no jumps in the intensity

We show the counterparty risk CVA computed by the AIRLINE on the BANK. This is because after 2008 a number of bank's credit quality deteriorated and an airline might have checked CVA on the bank with whom the swap was negotiated.

Commodities: Commodity Volatility Effect



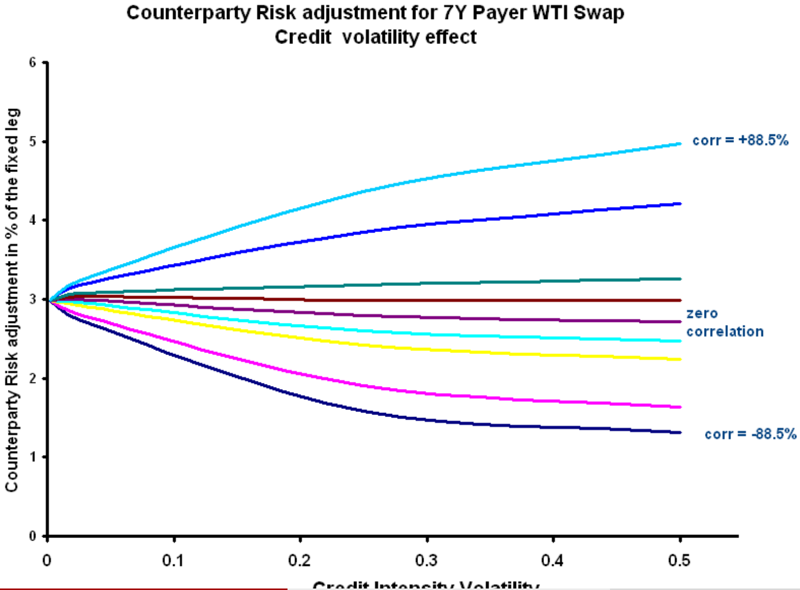
Commodities: Commodity Volatility Effect

Notice: In this example where CVA is calculated by the AIRLINE, positive correlation implies larger CVA.

This is natural: if the Bank credit spread widens, and the bank default becomes more likely, with positive correlation also OIL goes up.

Now CVA computed by the airline is an option, with maturity the default of the bank=counterparty, on the residual value of a Payer swap. As the price of OIL will go up at default due to the positive correlation above, the *payer* oil-swap will move in-the-money and the OIL option embedded in CVA will become more in-the-money, so that CVA will increase.

Commodities: Credit Volatility Effect



Commodities[1] : Credit volatility effect

$\bar{\rho}$	intensity volatility ν_R	0.025	0.25	0.50
-88.5	Payer adj	2.742	1.584	1.307
	Receiver adj	1.878	2.546	3.066
-63.2	Payer adj	2.813	1.902	1.63
	Receiver adj	1.858	2.282	2.632
-25.3	Payer adj	2.92	2.419	2.238
	Receiver adj	1.813	1.911	2.0242
-12.6	Payer adj	2.96	2.602	2.471
	Receiver adj	1.802	1.792	1.863
0	Payer adj	2.999	2.79	2.719
	Receiver adj	1.79	1.676	1.691
+12.6	Payer adj	3.036	2.985	2.981
	Receiver adj	1.775	1.562	1.527
+25.3	Payer adj	3.071	3.184	3.258
	Receiver adj	1.758	1.45	1.371
+63.2	Payer adj	3.184	3.852	4.205
	Receiver adj	1.717	1.154	0.977
+88.5	Payer adj	3.229	4.368	4.973
	Receiver adj	1.664	0.988	0.798

Fixed Leg Price maturity 7Y: 7345.39 USD for a notional of 1 Barrel per Month

[1] adjusment expressed as % of the fixed leg price

Commodities[2] : Commodity volatility effect

$\bar{\rho}$	Commodity spot volatility σ_S	0.0005	0.232	0.46	0.93
-88.5	Payer adj	0.322	0.795	1.584	3.607
	Receiver adj	0	1.268	2.546	4.495
-63.2	Payer adj	0.322	0.94	1.902	4.577
	Receiver adj	0	1.165	2.282	4.137
-25.3	Payer adj	0.323	1.164	2.419	6.015
	Receiver adj	0	0.977	1.911	3.527
-12.6	Payer adj	0.323	1.246	2.602	6.508
	Receiver adj	0	0.917	1.792	3.325
0	Payer adj	0.324	1.332	2.79	6.999
	Receiver adj	0	0.857	1.676	3.115
+12.6	Payer adj	0.324	1.422	2.985	7.501
	Receiver adj	0	0.799	1.562	2.907
+25.3	Payer adj	0.324	1.516	3.184	8.011
	Receiver adj	0	0.742	1.45	2.702
+63.2	Payer adj	0.325	1.818	3.8525	9.581
	Receiver adj	0	0.573	1.154	2.107
+88.5	Payer adj	0.326	2.05	4.368	10.771
	Receiver adj	0	0.457	0.988	1.715

Fixed Leg Price maturity 7Y: 7345.39 USD for a notional of 1 Barrel per Month

[2] adjusment expressed as % of the fixed leg price

Wrong Way Risk?

Basel 2, under the "Internal Model Method", models wrong way risk by means of a 1.4 multiplying factor to be applied to the zero correlation case, even if banks have the option to compute their own estimate of the multiplier, which can never go below 1.2 anyway.
What did we get in our cases? Two examples:

$$(4.973 - 2.719)/2.719 = 82\% >> 40\%$$

$$(1.878 - 1.79)/1.79 \approx 5\% << 20\%$$

Collateral Management and Gap Risk I

Collateral (CSA) is considered to be the solution to counterparty risk.

Periodically, the position is re-valued (”marked to market”) and a quantity related to the change in value is posted on the collateral account from the party who is penalized by the change in value.

This way, the collateral account, at the periodic dates, contains an amount that is close to the actual value of the portfolio and if one counterparty were to default, the amount would be used by the surviving party as a guarantee (and viceversa).
Gap Risk is the residual risk that is left due to the fact that the realingment is only periodical. If the market were to move a lot between two realigning (”margining”) dates, a significant loss would still be faced.

Folklore: Collateral completely kills CVA and gap risk is negligible.

Collateral Management and Gap Risk I

Folklore: Collateral completely kills CVA and gap risk is negligible.

We are going to show that there are cases where this is not the case at all (Brigo, Capponi and Pallavicini 2012, Mathematical Finance)

- Risk-neutral evaluation of counterparty risk in presence of collateral management can be a difficult task, due to the complexity of clauses.
- Only few papers in the literature deal with it. Among them we cite Cherubini (2005), Alavian *et al.* (2008), Yi (2009), Assefa *et al.* (2009), Brigo et al (2011) and citations therein.
- Example: Collateralized bilateral CVA for a netted portfolio of IRS with 10y maturity and 1y coupon tenor for different default-time correlations with (and without) collateral re-hypothecation. See Brigo, Capponi, Pallavicini and Papatheodorou (2011)

Collateral Management and Gap Risk II

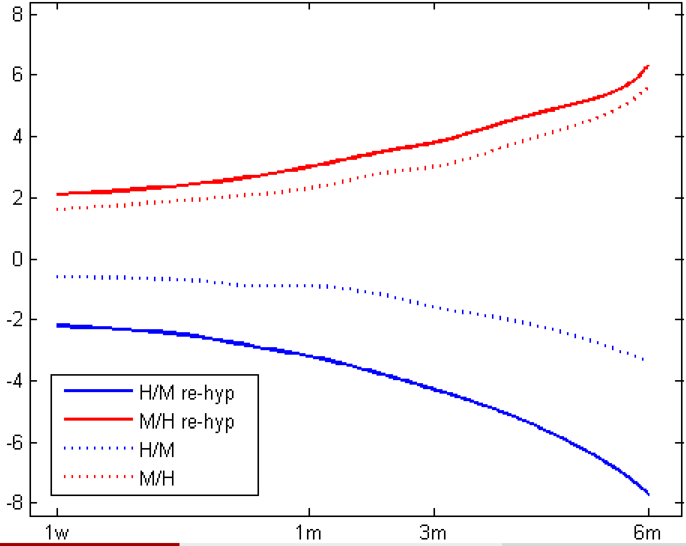


Figure explanation

Bilateral valuation adjustment, margining and rehypotecation

The figure shows the BVA(DVA-CVA) for a ten-year IRS under collateralization through margining as a function of the update frequency δ with zero correlation between rates and counterparty spread, zero correlation between rates and investor spread, and zero correlation between the counterparty and the investor defaults. The model allows for nonzero correlations as well.

Continuous lines represent the re-hypothecation case, while **dotted lines** represent the opposite case. The *red line* represents an investor riskier than the counterparty, while the *blue line* represents an investor less risky than the counterparty. All values are in basis points.

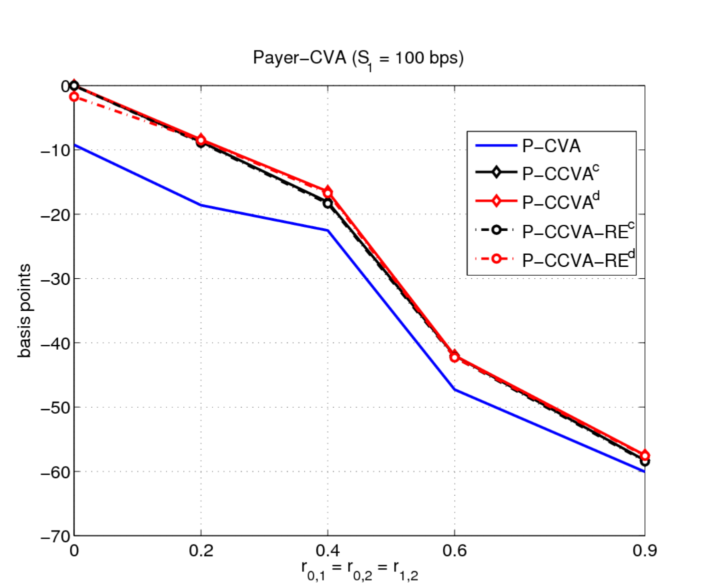
See the full paper by Brigo, Capponi, Pallavicini and Papatheodorou ‘Collateral Margining in Arbitrage-Free Counterparty Valuation Adjustment including Re-Hypotecation and Netting”
available at `http://arxiv.org/abs/1101.3926`

Figure explanation

From the fig, we see that the case of an investor riskier than the counterparty (M/H) leads to positive value for DVA-CVA, while the case of an investor less risky than the counterparty has the opposite behaviour. If we inspect the DVA and CVA terms as in the paper we see that when the investor is riskier the DVA part of the correction dominates, while when the investor is less risky the counterparty has the opposite behaviour.
Re-hypothecation enhances the absolute size of the correction, a reasonable behaviour, since, in such case, each party has a greater risk because of being unsecured on the collateral amount posted to the other party in case of default.

Let us now look at a case with more contagion: a CDS.

Collate


Payer−CVA (S_1 = 100 bps)
0
−10
−20
−30
−40
−50
−60
−70
basis points
0
0.2
0.4
0.6
0.9
$r_{0,1} = r_{0,2} = r_{1,2}$
P−CVA
P−CCVA^c
P−CCVA^d
P−CCVA−RE^c
P−CCVA−RE^d

Collateral Management and Gap Risk II

The figure refers to a payer CDS contract as underlying. See the full paper by Brigo, Capponi and Pallavicini (2011) for more cases.

If the investor holds a payer CDS, he is buying protection from the counterparty, i.e. he is a protection buyer.

We assume that the spread in the fixed leg of the CDS is 100 while the initial equilibrium spread is about 250.

Given that the payer CDS will be positive in most scenarios, when the investor defaults it is quite unlikely that the net present value be in favor of the counterparty.

We then expect the CVA term to be relevant, given that the related option will be mostly in the money. This is confirmed by our outputs.

Collateral Management and Gap Risk III

We see in the figure a relevant CVA component (part of the bilateral DVA - CVA) starting at 10 and ending up at 60 bps when under high correlation.

We also see that, for zero correlation, collateralization succeeds in completely removing CVA, which goes from 10 to 0 basis points.

However, collateralization seems to become less effective as default dependence grows, in that collateralized and uncollateralized CVA become closer and closer, and for high correlations we still get 60 basis points of CVA, even under collateralization.

The reason for this is the instantaneous default contagion that, under positive dependency, pushes up the intensity of the survived entities, as soon as there is a default of the counterparty.

Collateral Management and Gap Risk IV

Indeed, the term structure of the on-default survival probabilities (see paper) lies significantly below the one of the pre-default survival probabilities conditioned on $\mathcal{G}_{\tau-}$, especially for large default correlation.

The result is that the default leg of the CDS will increase in value due to contagion, and instantaneously the Payer CDS will be worth more. This will instantly increase the loss to the investor, and most of the CVA value will come from this jump.

Given the instantaneous nature of the jump, the value at default will be quite different from the value at the last date of collateral posting, before the jump, and this explains the limited effectiveness of collateral under significantly positive default dependence.

Collateral Management and Gap Risk V

The precise payout of residual CVA and DVA adjustment cash flows after collateralization will be introduced in the Funding Costs modeling part below, and will be called $\Pi_{CVAcoll}$ *and* $\Pi_{DVAcoll}$*. These are the terms that have been priced in the above examples.*

Q & A: Basel III and CVA Risk

Q What is happening with Basel III?

A *Basel noticed that during the crisis only one third of losses due to counterparty risk were due to actual defaults. The remaining losses have been due to CVA mark to market losses. Hence the pricing of counterparty risk has been twice as dangerous as the risk itself.*

Q Then we should ”risk-measure” CVA itself?

A *Indeed there is a lot of discussion around Value at Risk of CVA. This is not traditional credit VaR of course. It is something much more sophisticated. It is a percentile on future possible losses due to future averse movements of the PRICING of counterparty risk*

Q & A: Collateral and CVA. Gap Risk

Q But collateral should spare one the pains of this?

A *Collateral/CSA Margining is only an imperfect remedy to Counterparty risk, mostly due to Gap Risk, the risk of sudden mark to market changes and defaults between margining dates. This can be dramatic for assets that are subject to strong contagion under systemic risk. Also, we have re-hyphotecation, where collateral is not kept segregated as a guarantee. More generally, there are collateral disputes.*

Q & A: CVA Restructuring

Q So to manage CVA risk it is either Collateral (with the above caveats) or large capital requirements. Isn't this possibly creating a liquidity strain and depress the economy further?

A *A third possibility would be a macroeconomically healthy way of securitizing counterparty risk, a way recognized by regulators that banks could adopt to "buy" counterparty risk protection in the market. This is quite delicate. Floating margin lending, based on a notion of floating CVA, might be interesting from this point of view. More traditional fixed-premium cash CDO-type securitization mechanisms have failed so far.*

Q & A: CVA - Unilateral or Bilateral?

Basel II on **bilateral** counterparty risk:

> *Unlike a firm's exposure to credit risk through a loan, where the exposure to credit risk is unilateral and only the lending bank faces the risk of loss, the counterparty credit risk creates a bilateral risk of loss: the market value of the transaction can be positive or negative to either counterparty to the transaction.* [Basel II, Annex IV, 2/A]

Q & A: CVA - Unilateral or Bilateral?

Q When is valuation of counterparty risk CVA symmetric?

A *When we include the possibility that also the entity computing the counterparty risk adjustment may default, besides the counterparty itself.*

Q When is valuation of counterparty risk CVA asymmetric?

A *When the entity computing the counterparty risk adjustment considers itself default-free, and only the counterparty may default.*

Q Which one is computed usually for valuation adjustments?

A *Pre-crisis it used to be the asymmetric one; At the moment there is quite a debate*

Q & A: DVA

Q What happens in the symmetric case?

A *We have a new quantity called Debit Valuation Adjustment, or DVA.*

Q What is that?

A *It answers the question: "I recognize that I am default risky, so in trading this position with you, I accept to be charged more for this product than if I were default free, since you, my counterparty, are taking additional risk due to my possible default. DVA is then the increase in value I need to pay to enter this deal with you."*

Q & A: DVA

Q Looks like CVA seen from the other side. Is this why now pricing is symmetric?

A *Indeed, on side "B" you have*

$$DVA_{B,C} - CVA_{B,C}.$$

On the other side you have that $DVA_{B,C}$ *becomes* $CVA_{C,B}$ *and* $CVA_{B,C}$ *becomes* $DVA_{C,B}$*, so you have exactly*

$$-(DVA_{B,C} - CVA_{B,C}) = DVA_{C,B} - CVA_{C,B}.$$

Symmetry as in a swap. However, there are a few caveats

Q & A: DVA

Q Meaning?

A *DVA increases when the credit quality of the calculating entity worsens, because it becomes less likely that the calculating entity will have to repay its debt. However this is a profit that can only be realized by defaulting. Should it be accounted for?*

Q How would one hedge DVA?

A *One would have to sell protection of oneself (issue and then buy back bonds? Proxy Hedging?). Very difficult. Without hedging, is it really a price? However, it is from the other side, since it is CVA. Perspectival*

Q & A: DVA

Q Is this why you said DVA is debated?

A *Yes, regulators are fighting. FASB approved it. Basel does not recognize it, "perverse incentive". This makes CVA capital charges larger, since in future P&L simulations there will be no DVA balancing CVA.*

Q & A: First default

Q Does bilateral counterparty risk pricing, namely DVA - CVA, consider closeout? Namely, that at the **first** default the deal is liquidated or replaced?

A *Only if you take into account the first to default time in valuation. Correct CVA and DVA account for that. However first to default involves knowing the default "correlation" between the two entities in the deal. It may be difficult. Hence often the industry uses a formula ignoring first to default. This however involves double counting.*

Q & A: Closeout

Q And again on closeout, how is exactly the value of the residual deal computed at the closeout time?

A *You may have a risk free closeout, where the residual deal is priced at mid market without any residual credit risk, or you may have a replacement closeout, where the remaining deal is priced by taking into account the credit quality of the surviving party and of the party that replaces the defaulted one (so the new DVA - CVA at default).*

Q & A: Closeout

Q Does it make a big difference?

A *It does.*

- *Part of the market argues that if you are closing the deal at the closeout time, why should you worry about residual credit risk until the final maturity?*
- *On the other hand, if before the first default you were marking to market the deal including CVA and DVA, and all of a sudden at the first default you take CVA and DVA out, you create a discontinuity.*
- *It has been found that Risk Free closeout penalizes borrowers, whereas Replication closeout penalizes lenders, and the effect depends on the default correlation between parties.*

ISDA is not very assertive on closeout.

Q & A: CVA and Payout Risk

Q So we have:

- DVA Yes/No?
- First to Default time or not?
- Risk Free closeout or replication closeout?

It looks like not even the precise payout of CVA is clear, let alone model risk.

A *Yes, there is a lot of payout risk. In an interview to Risk Magainze, top tier 1 banks complained towards smaller banks by saying that the latter were more aggressive in CVA assumptions, thus taking clients that would otherwise work with the top tier 1. This aggressive pricing has been interpreted by tier 1 banks as using the cheapest form of CVA payout for the client.*
It has also been said that 5 banks may compute CVA in 15 different ways across functions and deals.

Inclusion of Funding Cost

When we work on CVA and DVA we are focusing on cash flows contingent on the first default, and on their valuation.

There are however other cash flows that are not related directly to the default event, but to the funding costs. We work on this now.

Inclusion of Funding Cost

When managing a trading position, one needs to obtain cash in order to do a number of operations:

- borrowing / lending cash to implement the replication strategy,
- possibly repo-lending or stock-lending the replication risky asset,
- borrowing cash to post collateral
- receiving interest on posted collateral
- paying interest on received collateral
- using received collateral to reduce borrowing from treasury
- borrowing to pay a closeout cash flow upon default

and so on. Where are such founds obtained from?

- Obtain cash from her Treasury department or in the market.
- receive cash as a consequence of being in the position.

All such flows need to be remunerated:

- if one is ”borrowing”, this will have a cost,
- and if one is ”lending”, this will provide revenues.

Introduction to Quant. Analysis of Funding Costs I

We now present an introduction to funding costs modeling. Motivation?

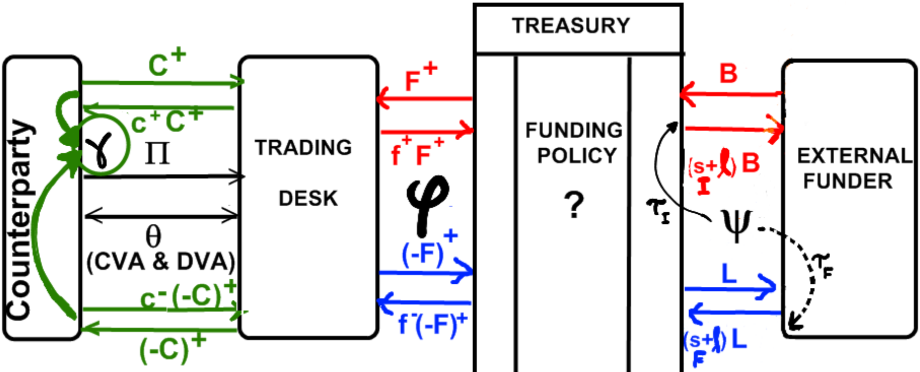
Funding Value Adjustment Proves Costly to J.P. Morgan's 4Q Results (Michael Rapoport, Wall St Journal, Jan 14, 2014)

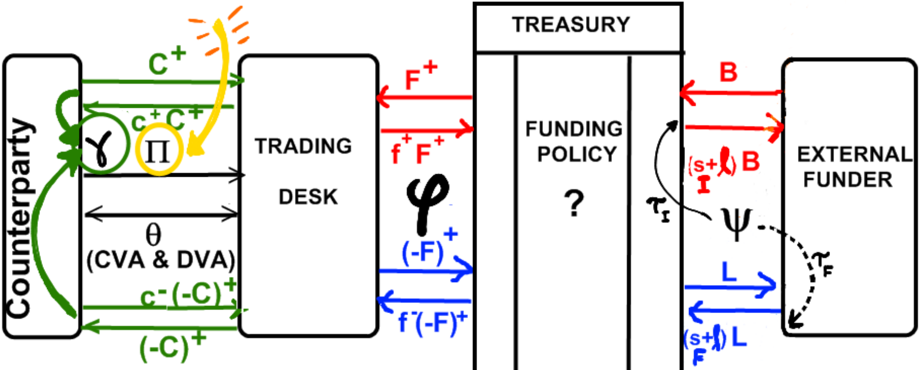
"[...] So what is a funding valuation adjustment, and why did it cost J.P. Morgan Chase $1.5 billion?

We now approach funding costs modeling by incorporating funding costs into valuation.

We start from scratch from the product cash flows and add collateralization, cost of collateral, CVA and DVA after collateral, and funding costs for collateral and for the replication of the product.

In the following τ_I denotes the default time of the investor / bank doing the calculation of the price (previously τ_B). "C": counterparty as before.


Counterparty
c+
c+C+
γ
Π
θ
(CVA & DVA)
c-(-C)+
(-C)+
TRADING
DESK
F+
f+F+
φ
(-F)+
f-(-F)+
TREASURY
FUNDING
POLICY
?
B
(s+λ) B
τI
Ψ
τF
L
(s+λ) L
EXTERNAL
FUNDER


Counterparty
c+
c+C+
γ
Π
θ
(CVA & DVA)
c-(-C)+
(-C)+
TRADING
DESK
F+
f+F+
φ
(-F)+
f-(-F)+
TREASURY
FUNDING
POLICY
?
B
(s+ℓ_I) B
τ_I
Ψ
τ_F
L
(s+ℓ_F) L
EXTERNAL
FUNDER

Basic Payout plus Credit and Collateral: Cash Flows

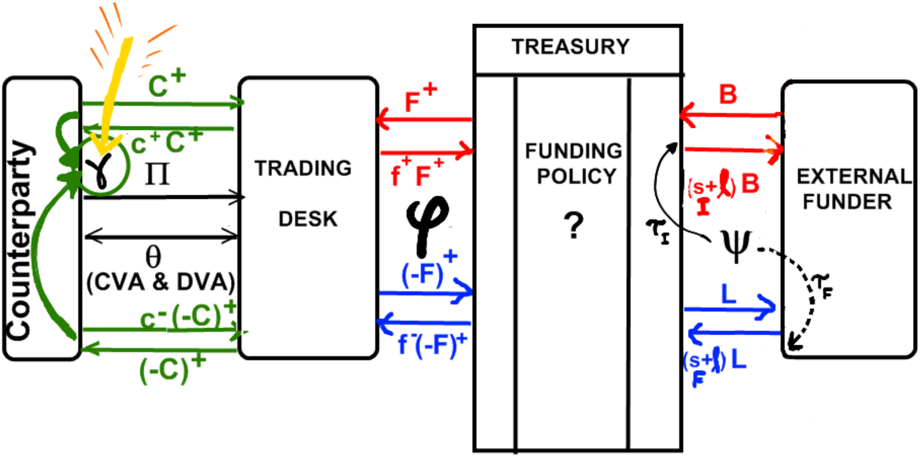
- We calculate prices by discounting cash-flows under the pricing measure. Collateral and funding are modeled as additional cashflows (as for CVA and DVA)
- We start from derivative's basic cash flows without credit, collateral of funding risks

$$\bar{V}_t \quad := \quad \mathbb{E}_t[\,\Pi(t, T \wedge \tau) + \ldots\,]$$

where

$\longrightarrow$ $\tau := \tau_C \wedge \tau_I$ is the first default time, and
$\longrightarrow$ $\Pi(t, u)$ is the sum of all payoff terms from t to u, discounted at t

Cash flows are stopped either at the first default or at portfolio's expiry if defaults happen later.



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Basic Payout plus Credit and Collateral: Cash Flows

- As second contribution we consider the collateralization procedure and we add its cash flows.

$$\bar{V}_t := \mathbb{E}_t[\,\Pi(t, T \wedge \tau)\,] + \mathbb{E}_t[\,\gamma(t, T \wedge \tau; C) + \dots]$$

where

$\longrightarrow$ C_t is the collateral account defined by the CSA,

$\longrightarrow$ $\gamma(t, u; C)$ are the collateral margining costs up to time u.

- The second expected value originates what is occasionally called Liquidity Valuation Adjustment (LVA) in simplified versions of this analysis. We will show this in detail later.
- If $C > 0$ collateral has been overall posted by the counterparty to protect us, and we have to pay interest c^+.
- If $C < 0$ we posted collateral for the counterparty (and we are remunerated at interest c^-).

Basic Payout plus Credit and Collateral: Cash Flows

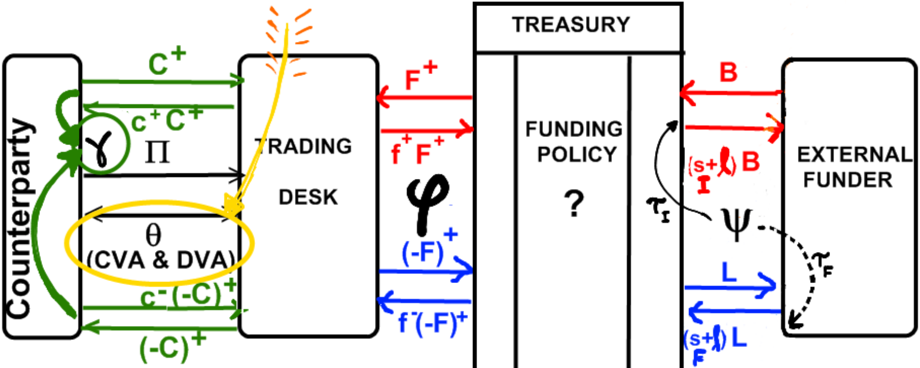
- The cash flows due to the margining procedure are

$$\gamma(t, u; C) = -\int_t^u D(t, s) C_s(\tilde{c}_s - r_s) ds$$

where the collateral accrual rates are given by

$$\tilde{c}_t := c_t^+ 1_{\{C_t > 0\}} + c_t^- 1_{\{C_t < 0\}}$$

Note that if the collateral rates in $\tilde{c}$ are both equal to the risk free rate, then this term is zero.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Close-Out: Trading-CVA/DVA under Collateral – I

- As third contribution we consider the cash flow happening at 1st default, and we have

$$
\begin{aligned}
\bar{V}_t \quad &:= \quad \mathbb{E}_t[\,\Pi(t, T \wedge \tau)\,] \\
&+ \quad \mathbb{E}_t[\,\gamma(t, T \wedge \tau; C)\,] \\
&+ \quad \mathbb{E}_t\left[\,1_{\{\tau < T\}} D(t,\tau)\theta_\tau(C,\varepsilon) + \ldots\right]
\end{aligned}
$$

  where

  $\longrightarrow$ ε_τ is the close-out amount, or residual value of the deal at default, which we called NPV earlier, and

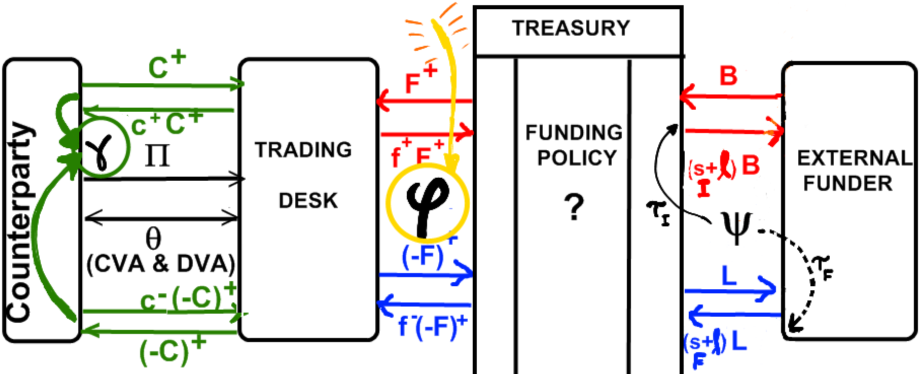
  $\longrightarrow$ $\theta_\tau(C,\varepsilon)$ is the on-default cash flow.

- θ_τ will contain collateral adjusted CVA and DVA payouts for the instument cash flows

Close-Out: Trading-CVA/DVA under Collateral – II

- We define θ_τ including the pre-default value of the collateral account since it is used by the close-out netting rule to reduce exposure
- In case of no collateral re-hypothecation (see full paper for all cases)

$$
\begin{aligned}
\theta_\tau(C,\varepsilon) &:= \varepsilon_\tau - 1_{\{\tau=\tau_C<\tau_I\}}\Pi_{\mathrm{CVAcoll}} + 1_{\{\tau=\tau_I<\tau_C\}}\Pi_{\mathrm{DVAcoll}} \\
\Pi_{\mathrm{CVAcoll}} &= \mathrm{L{\scriptstyle GD}}_C(\varepsilon_\tau^+ - C_{\tau^-}^+)^+ \\
\Pi_{\mathrm{DVAcoll}} &= \mathrm{L{\scriptstyle GD}}_I((-\varepsilon_\tau)^+ - (-C_{\tau^-})^+)^+
\end{aligned}
$$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Funding Costs of the Replication Strategy – I

- As fourth contribution we consider the cost of funding for the hedging procedures and we add the relevant cash flows.

$$\begin{aligned}\bar{V}_t &:= \mathbb{E}_t[\,\Pi(t, T\wedge\tau)\,] + \mathbb{E}_t\big[\,\gamma(t, T\wedge\tau; C) + 1_{\{\tau<T\}} D(t,\tau)\theta_\tau(C,\varepsilon)\,\big]\\ &+ \mathbb{E}_t[\,\varphi(t, T\wedge\tau; F, H)\,]\end{aligned}$$

  The last term, especially in simplified versions, is related to what is called FVA in the industry. We will point this out once we get rid of the rate r.

  $\longrightarrow$ F_t is the cash account for the replication of the trade,
  $\longrightarrow$ H_t is the risky-asset account in the replication,
  $\longrightarrow$ $\varphi(t, u; F, H)$ are the cash F and hedging H funding costs up to u.

- In classical Black Scholes on Equity, for a call option (no credit risk, no collateral, no funding costs),

$$\bar{V}_t^{\text{Call}} = \Delta_t S_t + \eta_t B_t =: H_t + F_t, \quad \tau = +\infty, \;\; C = \gamma = \varphi = 0.$$

Funding Costs of the Replication Strategy – II

- Continuously compounding format:

$$\begin{aligned} \varphi(t,u) &= \int_t^u D(t,s)(F_s + H_s)\left(r_s - \tilde{f}_s\right) ds \\ &- \int_t^u D(t,s) H_s \left(r_s - \tilde{h}_s\right) ds \\ \tilde{f}_t &:= f_t^+ 1_{\{F_t>0\}} + f_t^- 1_{\{F_t<0\}} \;\; \tilde{h}_t := h_t^+ 1_{\{H_t>0\}} + h_t^- 1_{\{H_t<0\}} \end{aligned}$$

- The expected value of φ is related to the so called FVA. If the treasury funding rates $\tilde{f}$ are same as asset lending/borrowing $\tilde{h}$

$$\varphi(t,u) = \int_t^u D(t,s) F_s \left(r_s - \tilde{f}_s\right) ds$$

- If further treasury borrows/lends at risk free $\tilde{f} = r \Rightarrow \varphi = FVA = 0$.

Funding Costs of the Replication Strategy – III

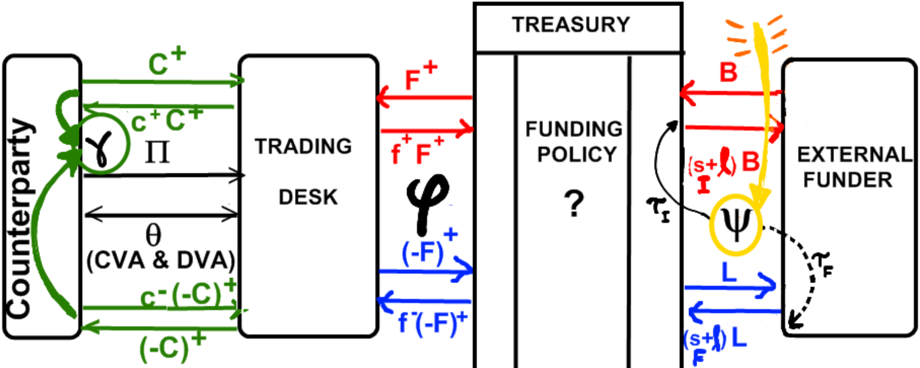
Replica: F cash & H risky asset. Cash is borrowed $F > 0$ from the treasury at an interest f^+ (cost) or is lent $F < 0$ at a rate f^- (revenue)

Risky asset position in replica is worth H. Cash needed to buy $H > 0$ is borrowed at interest f from treasury; in this case H can be used for asset lending (Repo for example) at a rate h^+ (revenue); etc ($H < 0$...)

Include default risk of funder and funded ψ, leading to CVA_F & DVA_F.

f^+ & f^- policy driven and related to λ_I, λ_C, more in a minute.

IMPORTANT: FVA coming from f^+ & f^- is largely offset by ψ terms valuation as we will see.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Default flows ψ for the Funding part I

$$
\begin{aligned}
\bar{V}_t &:= \mathbb{E}_t[\Pi(t, T \wedge \tau)] + \mathbb{E}_t\left[\gamma(t, T \wedge \tau; C) + 1_{\{\tau < T\}} D(t, \tau)\theta_\tau(C, \varepsilon)\right] \\
&+ \mathbb{E}_t[\varphi(t, T \wedge \tau; F, H)] + \mathbb{E}_t[\psi(t, \tau_F, \tau, T)]
\end{aligned}
$$

When our bank treasury is borrowing in the market from bank F, F charges our bank a CVA due to our credit risk. Seen from our bank, this charge is a DVA_F that makes the loan more expensive.

This means that if we fix the final notional, we will be able to borrow less than if we were default free. If we fix the amount borrowed now, we will have to repay more at the end. Overall the loan will be more expensive because of our bank credit risk. This is a cost.

Similarly, when our bank treasury lends externally, it measures a CVA_F on the loan due to the possibility that the borrower defaults. Loan is more remunerative due to upfront CVA_F charged to external Borrower (External Funder Benefit).

Default flows ψ for the Funding part II

We assume $\tilde{f} = \tilde{h}$ for simplicity, or $H = 0$.

IMPORTANT

We are adding the ψ treasury DVA-CVA term to our Equation but the Eq terms would ideally sit in different parts of the bank.

- The value of the ψ part is with the treasury,
- while the other parts are with the trading desk.
- We will shortly see the different ways the treasury may pass the cost/benefits in ψ to the desk
- This is controlled with the rates f^+ and f^- in the funding cost-benefit term φ through suitable credit spreads

Default flows ψ for the Funding part III

Total value of claim then includes cash flows from debit and credit risk in the funding strategy that are seen by the treasury:

$$\psi_{EFB}(t, \tau_F, \tau, T) = D(t,\tau)1_{\{\tau=\tau_I<T\}}\mathrm{L}_{\mathrm{GD}I}(F_\tau)^+$$

$$-D(t,\tau_F)1_{\{\tau\wedge\tau_F=\tau_F<T\}}\mathrm{L}_{\mathrm{GD}F}(-F_{\tau_F})^+$$

The first term on the right hand side is the funding DVA cash flow (leading to what is called occasionally DVA_2 or FDA, “Funding Debit Adjustment”). We will call the value of this cash flow DVA_F. This is triggered when our treasury is borrowing and defaults first, causing a loss to the external lender.

The second term on the right hand side is the funding credit valuation adjustment cash flow, that is triggered when our treasury is lending externally and the borrower defaults first. The value of this cash flow is called -CVA_F.

Default flows ψ for the Funding part IV

There is a possibly different definition for ψ.
If the treasury considers the desk as net borrowing, the lending of $(-F)^+$ will be considered not as a loan but as a reduction in borrowing.

In this sense there will be no CVA_F term now, since no lending is considered by the treasury.

In this case the cash flows of the credit adjustment for the funding part consist only of the debit adjustment part and are called Reduced Borrowing Benefit:

$$\psi_{RBB}(t, \tau_F, \tau, T) = D(t,\tau) 1_{\{\tau=\tau_I<T\}} \mathrm{L}_{\mathrm{GD}\,I}(F_\tau)^+$$

The two cases of External Funder Benefit (EFB) and Reduced Borrowing Benefit (RBB) will be discussed shortly also in connection with interest rates $\tilde{f}$.

$$(*) \quad \bar{V}_t = \mathbb{E}_t\big[\,\Pi(t, T\wedge\tau) + \gamma(t) + 1_{\{\tau<T\}} D(t,\tau)\theta_\tau + \varphi(t) + \psi(t,\tau_F,\tau)\,\big]$$

Can we interpret:

$\mathbb{E}_t\big[\,\Pi(t, T\wedge\tau) + 1_{\{\tau<T\}} D(t,\tau)\theta_\tau(C,\varepsilon)\,\big]$: RiskFree Price + DVA - CVA?

$\mathbb{E}_t[\,\gamma(t, T\wedge\tau) + \varphi(t, T\wedge\tau; F, H)\,]$: Funding adjustment LVA+FVA?

$\mathbb{E}_t[\,\psi(t,\tau_F,\tau,T)\,]$: Treasury CVA_F and DVA_F

Not really. This is not a decomposition. It is an equation. In fact since

$$\bar{V}_t = F_t + H_t + C_t \quad \text{(re–hypo)}$$

we see that the φ present value term depends on future $F_t = \bar{V}_t - H_t + C_t$ and generally the closeouts θ ψ, via ϵ, F and C, depend on future $\bar{V}$ too. All terms feed each other and there is no neat separation of risks. *Recursive pricing: Nonlinear PDE's / BSDEs* for $\bar{V}$

"FinalV = RiskFreeV (+ DVA?) - CVA + FVA" not possible...

... in theory. Approx and linearization in practice.

Funding incusive valuation equations I

- Substituting we obtain, under **IMMERSION** ($\pi_t \, dt = \Pi(t, t+dt)$)

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t,u;r+\lambda)[\pi_u + (r_u - \tilde{c}_u)C_u + \lambda_u \theta_u + \qquad \text{EQFund1}$$

$$+(r_u - \tilde{f}_u)(F_u + H_u) + (\tilde{h}_u - r_u)H_u + \lambda_u^I \mathrm{L}_{\mathrm{GD}_I}(F_u)^+ - \lambda_u^F \mathrm{L}_{\mathrm{GD}_F}(-F_u)^+]|\mathcal{F}_t\}du$$

- Set $Z_u = \lambda_u^I \mathrm{L}_{\mathrm{GD}_I}(F_u)^+ - \lambda_u^F \mathrm{L}_{\mathrm{GD}_F}(-F_u)^+$, the Treasury DVA-CVA term, and subtract $\epsilon = \bar{V}$, assuming replacement closeout, from θ, so as to isolate the Trading CVA and DVA terms. Use V=F+H+C

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t,u;r+\lambda)[\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + \qquad \text{EQFund2}$$

$$+(r_u - \tilde{f}_u + \lambda_u)\bar{V}_u + (\tilde{h}_u - r_u)H_u + Z_u]|\mathcal{F}_t\}du$$

Funding incusive valuation equations II

- Use Feynman Kac: we know that

$$\bar{V}_t = \mathbb{E}_t\left[\int_t^T D(t,u;\mu)[\alpha_u + \beta_u \bar{V}_u]du\right] = \mathbb{E}_t\left[\int_t^T D(t,u;\mu-\beta)\alpha_u du\right]$$

- Then from EQFund2 we have, absorbing λV in the discount:

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t,u;r)[\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + \quad \text{EQFund3}$$

$$+(r_u - \tilde{f}_u)\bar{V}_u + (\tilde{h}_u - r_u)H_u + Z_u]|\mathcal{F}_t\}du$$

Funding incusive valuation equations III

- or alternatively, absorbing the whole $(r - f + \lambda)V$

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t,u;\tilde{f})[\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + \qquad \text{EQFund4}$$

$$+(\tilde{h}_u - r_u)H_u + Z_u]|\mathcal{F}_t\}du$$

- Assuming $H = 0$ (rolled par swaps or, better, perfectly collateralized hedge with collateral incuded)

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t,u;\tilde{f})[\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + Z_u]|\mathcal{F}_t\}du \quad \text{EQFund4'}$$

Funding incusive valuation equations IV

- If $H \neq 0$, assume now generalized delta hedging (in vector sense)

$$H_u = S_u \frac{\partial \bar{V}(u, S)}{\partial S}$$

and use Feynam Kac again:

$$\bar{V}_t = \mathbb{E}^r \int_t^T D(t, u; \mu)[\alpha_u + m(u, S_u) \frac{\partial \bar{V}}{\partial S}] du = \mathbb{E}_t^{r+m} \left[\int_t^T D(t, u; \mu) \alpha_u du \right]$$

where in general E^m is a probability measure where S grows at rate m, ie with drift mS.

Funding incusive valuation equations V

- EqFund4 with delta hedging becomes ($(h-r)H = (h-r)\partial_S \bar{V}$)

$$\bar{V}_t = \int_t^T \mathbb{E}^h\{D(t,u;\tilde{f})[\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + Z_u]|\mathcal{F}_t\}du \quad \text{EQFund5}$$

- **This last equation depends only on market rates. There is no theoretical risk free rate or risk neutral measure in this Eq. Invariance Theorem: The pricing equation is invariant wrt the specification of the short rate r_t.**
- Recall: h are repo/stock lending rates for underlying risky assets,
- $(\theta_u - \bar{V}_u)$ are trading CVA and DVA after collateralization
- $(\tilde{f}_u - \tilde{c}_u)C_u$ is the cost of funding collateral with the treasury
- Z_u is the treasury CVA_F and DVA_F on the funding process
- NO Explicit funding term for the replica **as this has been absorbed in the discount curve and in the collateral cost**

Funding incusive valuation equations VI

- The last equation can be written as a semi-linear PDE or a BSDE
- As we explained, $\mathbb{E}^h$ is the expected value under a probability measure where the underlying assets evolve with a drift rate (return) of $\tilde{h}$. Remember that $\tilde{h}$ depends on H, and hence on V.
- Therefore the PRICING MEASURE DEPENDS ON THE FUTURE VALUES OF THE VERY PRICE V WE ARE COMPUTING. NONLINEAR EXPECTATION. THE PRICING MEASURE BECOMES DEAL DEPENDENT.
- Under the assumption $H = 0$ we can avoid the last Feynman Kac step and the deal dependent measure: we still price under the risk neutral measure ($\approx$ OIS) but the terms in EQFund4' bear the same description as EQFund5 we just commented.
- Notice that in EQFund5 or the simpler EQFund4' we DISCOUNT AT FUNDING directly. Some industry parties use this version and a funding discount curve.

Funding incusive valuation equations VII

Let's take a step back. Write EqFund1-2 more in detail.

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t,u;r+\lambda)[\pi_u + \lambda_u\theta_u + (r_u - \tilde{c}_u)C_u + \qquad \text{EQFund1'}$$

$$+(r_u - \tilde{f}_u)(\bar{V}_u - C_u) + (\tilde{h}_u - r_u)H_u + Z_u]|\mathcal{F}_t\}du$$

We can see easily (going back to $\mathcal{G}$) that

$$\int_t^T \mathbb{E}\{D(t,u;r+\lambda)[\pi_u + \lambda_u\bar{V}_u]\}du = V_t^0$$

and, given $\theta_u = \varepsilon_u - 1_{\{u=\tau_C<\tau_I\}}\Pi_{\mathrm{CVAcoll}}(u) + 1_{\{u=\tau_I<\tau_C\}}\Pi_{\mathrm{DVAcoll}}(u)$, under replacement closeout ($\varepsilon = \bar{V}$), rehypotecation and under $\mathcal{F}$ it is tempting to write EQFund1' as

Funding incusive valuation equations VIII

$\bar{V}$ = RiskFreePrice - CVA + DVA + LVA + FVA -CVA$_F$+ DVA$_F$

$$RiskFreePrice = V_t^0, \quad LVA = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)(r_u - \tilde{c}_u)C_u | \mathcal{F}_t \right\} du$$

$$-CVA = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\big[-\mathrm{LGD}_C \lambda_C(u)(\bar{V}_u - C_{u-})^+ \big] | \mathcal{F}_t \right\} du$$

$$DVA = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\big[\mathrm{LGD}_I \lambda_I(u)(-(\bar{V}_u - C_{u-}))^+ \big] | \mathcal{F}_t \right\} du$$

$$FVA = -\int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[(\tilde{f}_u - r_u)(\bar{V}_u - C_u) - (\tilde{h}_u - r_u)H_u \right] | \mathcal{F}_t \right\} du$$

$$-CVA_F = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[\mathrm{LGD}_F \lambda_F(u)(-(\bar{V}_u - C_u - H_u))^+ \right] | \mathcal{F}_t \right\} du$$

$$DVA_F = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[\mathrm{LGD}_I \lambda_I(u)(\bar{V}_u - C_u - H_u)^+ \right] | \mathcal{F}_t \right\} du$$

Funding incusive valuation equations IX

If we insist in applying these equations, rather than the r-independent EQFund5, then we need to find a proxy for r. This can be taken as the overnight rate (OIS discounting).

Further, if we assume that H_u is zero as it is perfectly collateralized and includes its collateral, then

$$FVA = -\int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[(\tilde{f}_u - r_u)(\bar{V}_u - C_u)\right]|\mathcal{F}_t \right\} du$$

Notice that when we are borrowing cash $F = V - C$, since usually $f > r$, FVA is negative and is a cost. Also LVA can be negative. Occasionally LVA and FVA are added together in a sort of total $\text{FVA}_{tot} = LVA + FVA$.

$$FVA_{tot} = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[-(\tilde{f}_u - r_u)\bar{V}_u + (\tilde{f}_u - \tilde{c}_u)C_u\right]|\mathcal{F}_t \right\} du$$

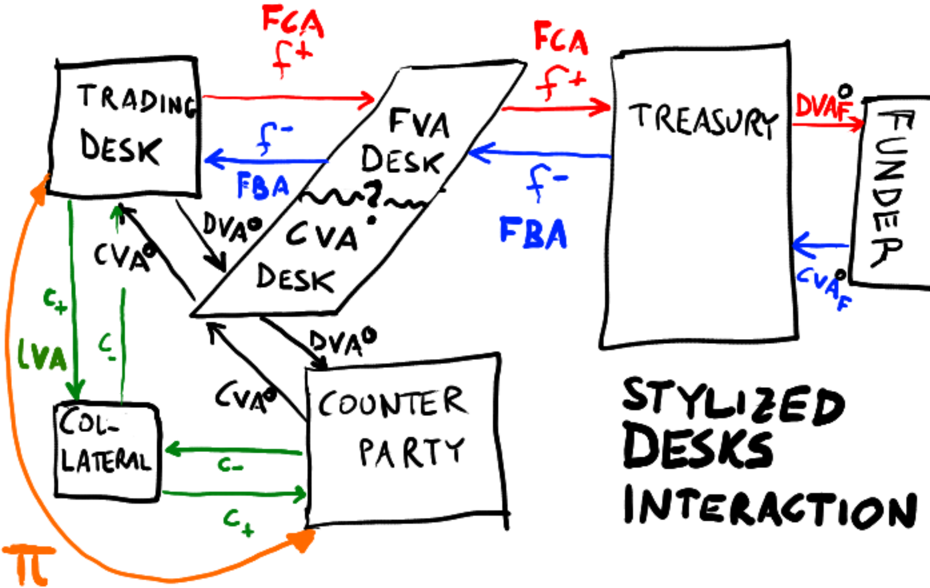
Funding incusive valuation equations X

Define $FVA = -FCA + FBA$ where $-FCA$ will be a Cost, and hence negative, while FBA will be a Benefit, hence positive.

$$FCA = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[(f_u^+ - r_u)(\bar{V}_u - C_u)^+\right] | \mathcal{F}_t \right\} du$$

$$FBA = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[(f_u^- - r_u)(-(\bar{V}_u - C_u))^+\right] | \mathcal{F}_t \right\} du$$

Notice the structural analogies with the expressions for CVA and DVA respectively.



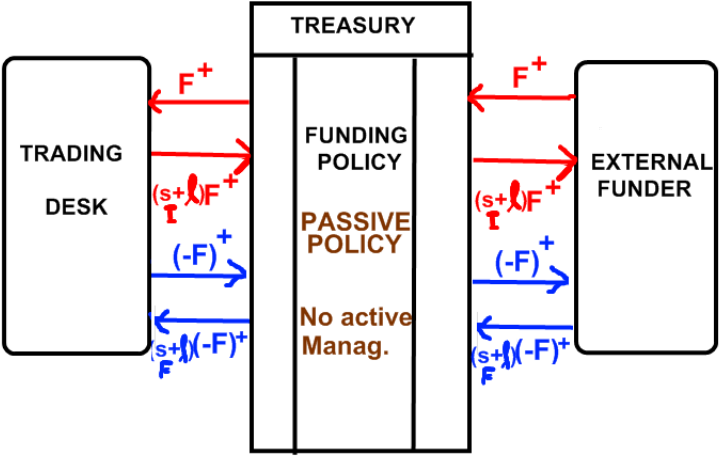
STYLIZED DESKS INTERACTION

Funding incusive valuation equations: EFB vs RBB

To further specify the equations we need to distinguish the assumptions on external lending by the treasury, and we will deal now separately with the two cases:

- External Funder Benefit (EFB)
- Reduced Borrower Benefit (RBB)

Funding incusive valuation equations: EFB case



Funding incusive valuation equations: EFB case

Assume that we use the EFB funding rates $\tilde{f}$ inclusive of credit risk, so that (set $s_{I,C,F} = \lambda_{I,C,F}\text{L}_{\text{GD}}{}_{I,C,F}$)
$f^+ = r + s_I + \ell^+, \quad f^- = r + s_F + \ell^-$

$$-FCA = -\int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)\left[(s_I+\ell^+)(\bar{V}_u - C_u)^+\right]|\mathcal{F}_t\right\}du$$

$$FBA = \int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)\left[(s_F+\ell^-)(-(\bar{V}_u - C_u))^+\right]|\mathcal{F}_t\right\}du$$

where FCA_ℓ is the part in ℓ^+, and FBA_ℓ is the part in ℓ^-. We see $-FCA =: -DVA_F - FCA_\ell, \quad FBA =: CVA_F + FBA_\ell$ where CVA_F and DVA_F are implicitly defined and coincide with the corresponding ψ valuation terms for treasury C/DVA's.
The presence of Credit Spreads in $\tilde{f}$ leads to components in FBA and FCA that offset the Treasury DVA$_F$ and CVA$_F$. Summing up:
$V = V_0 - CVA + LVA + DVA - FCA + FBA + DVA_F - CVA_F$ where

$$V_0 = \int_t^T \mathbb{E}\left\{D(t,u;r)\pi_u|\mathcal{F}_t\right\}du, \quad LVA = \int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)(r_u-\tilde{c}_u)C_u|\mathcal{F}_t\right\}du$$

$$-FCA(=-DVA_F-FCA_\ell) = -\int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)(s_I(u)+\ell^+(t))(\bar{V}_u-C_u)^+\Big]|\mathcal{F}_t\right\}du$$

$$FBA(=CVA_F+FBA_\ell) = \int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)\Big[(s_F+\ell^-)(-(\bar{V}_u-C_u))^+\Big]|\mathcal{F}_t\right\}du$$

$$-CVA = \int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)[-s_C(\bar{V}_u-C_{u-})^+]|\mathcal{F}_t\right\}du$$

$$DVA = \int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)[s_I(-(\bar{V}_u-C_{u-}))^+]|\mathcal{F}_t\right\}du$$

$$DVA_F = \int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)s_I(u)(\bar{V}_u-C_u)^+\Big]|\mathcal{F}_t\right\}du$$

$$-CVA_F = -\int_t^T \mathbb{E}\left\{D(t,u;r+\lambda)\Big[s_F(-(\bar{V}_u-C_u))^+\Big]|\mathcal{F}_t\right\}du$$

Double Counting: EFB Case

Summing up: $V = V_0(\textit{risk free}) +$

$$\underbrace{-CVA + DVA}_{\textit{Trading CVA DVA}} \quad \overbrace{+LVA}^{\textit{Coll cost \& benefit}} \quad \underbrace{-FCA + FBA}_{\textit{Replica funding cost \& benefit}} \quad + \overbrace{DVA_F - CVA_F}^{\textit{Funding CVA DVA}}$$

Remember also what we just found for FCA and FBA:

$$V = V_0 - CVA + DVA + LVA \quad \underbrace{-FCA}_{-DVA_F - FCA_\ell} \quad + \quad \underbrace{FBA}_{CVA_F + FBA_\ell} \quad + DVA_F - CVA_F$$

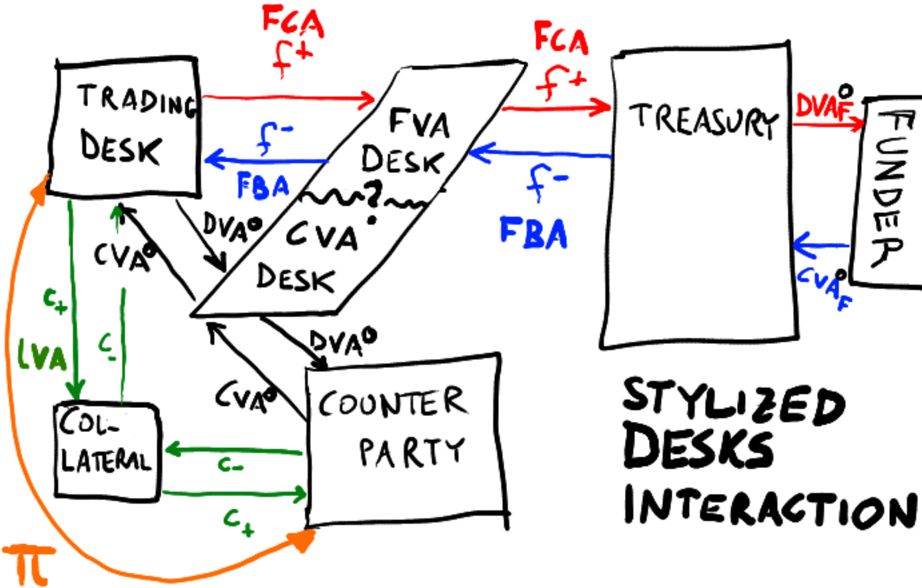
The blue and red terms are passed by the treasury to the desk so the total net value for the whole bank cancels

Double Counting: EFB Case

Keeping the full formula without simplifying

$$V = V_0 - CVA + DVA + LVA \quad \underbrace{-FCA}_{-DVA_F - FCA_\ell} \quad + \quad \underbrace{FBA}_{CVA_F + FBA_\ell} \quad + DVA_F - CVA_F$$

- If bases $\ell = 0$ then Funding costs are offset by the treasury CVA_F and DVA_F and "there are no funding costs" overall.
- However, for the trading desk (TDesk) there is still a cost $FCA = DVA_F + FCA_\ell$ to be paid to Treasury. This happens via the FVA desk if that exists, or via the CVA desk otherwise.
- TDesk also sees a benefit $FBA = CVA_F + FBA_\ell$ received from treasury via the FVA desk if existing, or CVA desk otherwise.
- Treasury pays DVA_F at time 0 to Funder, charging that as a cost FCA to Tdesk, and receives CVA_F at time 0 from funder, and passes that to the TDesk as benefit. All this via F/CVA desk
- CVA desk still deals with trading CVA and DVA


FCA
f+
TRADING DESK
FVA DESK
f-
FBA
?
CVA DESK
DVA°
CVA°
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LVA
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COL-LATERAL
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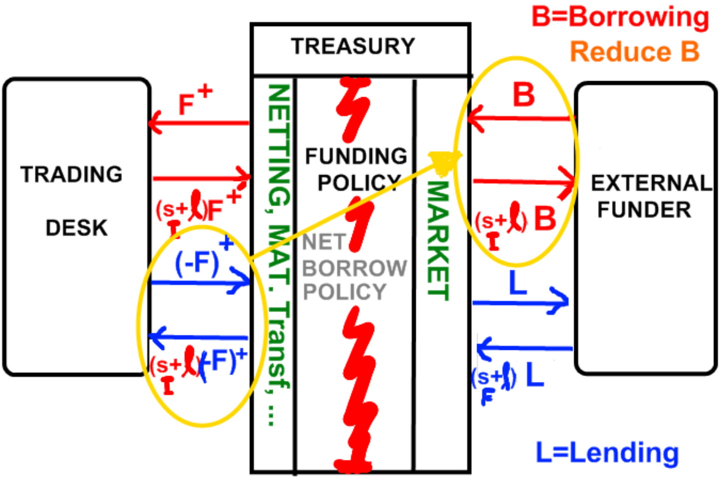
STYLIZED DESKS INTERACTION

Funding incusive valuation equations: RBB case



Funding incusive valuation equations: RBB case

We are now going to specialize the funding equations

$$FCA = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[(f_u^+ - r_u)(\bar{V}_u - C_u)^+ \right] |\mathcal{F}_t \right\} du$$

$$FBA = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[(f_u^- - r_u)(-(\bar{V}_u - C_u))^+ \right] |\mathcal{F}_t \right\} du$$

to the RBB case where

$$f^+ - r = s_I + \ell^+, \quad f^- - r = s_I + \ell^-.$$

We also take $\psi = \psi_{RBB}$ (no CVA_F part).

Funding incusive valuation equations: RBB case

The FCA term remains as in the EFB case.

However, notice what happens to FBA now, in the RBB case.

$$FBA = \int_t^T \mathbb{E}\left\{ D(t,u;r+\lambda)\left[(s_l+\ell^-)(-(\bar{V}_u - C_u))^+ \right] | \mathcal{F}_t \right\} du = DVA + FBA_\ell$$

We have that FBA includes a copy of the *trading* DVA

$$V_0 = \int_t^T \mathbb{E}\Big\{ D(t,u;r)\pi_u | \mathcal{F}_t \Big\} du, \quad LVA = \int_t^T \mathbb{E}\Big\{ D(t,u;r{+}\lambda)(r_u{-}\tilde{c}_u) C_u | \mathcal{F}_t \Big\} du$$

$$-FCA(= -DVA_F - FCA_\ell) = -\int_t^T \mathbb{E}\Big\{ D(t,u;r{+}\lambda)(s_l(u){+}\ell^+(t))(\bar{V}_u - C_u)^+ \Big] | \mathcal{F}_t \Big\} du$$

$$FBA(= \color{red}{DVA} + FBA_\ell) = \int_t^T \mathbb{E}\Big\{ D(t,u;r+\lambda)\Big[(\color{red}{s_l} + \ell^-)(-(\bar{V}_u - C_u))^+ \Big] | \mathcal{F}_t \Big\} du$$

$$-CVA = \int_t^T \mathbb{E}\Big\{ D(t,u;r+\lambda)\big[-s_C(\bar{V}_u - C_{u-})^+ \big] | \mathcal{F}_t \Big\} du$$

$$DVA = \int_t^T \mathbb{E}\Big\{ D(t,u;r+\lambda)\big[s_l(-(\bar{V}_u - C_{u-}))^+ \big] | \mathcal{F}_t \Big\} du$$

$$DVA_F = \int_t^T \mathbb{E}\Big\{ D(t,u;r+\lambda) s_l(u)(\bar{V}_u - C_u)^+ \Big] | \mathcal{F}_t \Big\} du$$

$-CVA_F = 0$; One of the two DVA must go.

Funding incusive valuation equations: RBB case I

$$V = V_0 - CVA + DVA + LVA \underbrace{-FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{DVA + FBA_\ell} + DVA_F$$

Now we no longer have exact offsetting terms. The DVA inside FBA will not be offset by a CVA_F. The problem is that the formula contains two identical DVA's.

Compare with the EFB case:

$$V = V_0 - CVA + DVA + LVA \underbrace{-FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{CVA_F + FBA_\ell} + DVA_F - CVA_F$$

Funding incusive valuation equations: RBB case II

When we compute the funding rate f^- we use our own $s_I = \lambda_I \mathrm{L}_{\mathrm{GD}_I}$ as a gain spread, based on the “reduced borrowing” argument.

But receiving back interest s_I as a benefit of reduced borrowing means we are in fact computing a rolling-DVA for F as $[t, t + dt)$ spans the whole trading interval. Since $F = V - C$, we are basically computing again the trading DVA by means of the funding rate f^-.

We are thus counting our own default risk twice on the *same* exposure scenario $(-(V - C))^+$. This is why, save for the basis term ℓ_-, we should take one of the two DVA’s out to avoid double counting.

Funding incusive valuation equations: RBB case III

We thus have two possible choices:

1: Privilege Credit Adjustments over Funding ones

$$V = V_0 - CVA + DVA + LVA \quad \underbrace{-FCA}_{-DVA_F - FCA_\ell} \quad + \quad \underbrace{FBA}_{\cancel{DVA} + FBA_\ell} \quad + DVA_F$$

Treasury is charged initially DVA_F, and charges this back to TDesk as part of FCA via FVADesk if $\exists$, else CVADesk.
For the reduced borrowing TDesk sees a benefit FBA_ℓ, obtained from treasury via FVADesk as a payment reduction, and TDesk is still charged DVA at time 0 and receives CVA at time 0 from counterparty via CVADesk. Overall (notice that if $\ell = 0$ there's no funding adjustment)

$$V = V_0 - CVA + DVA + LVA - FCA_\ell + FBA_\ell.$$

Funding incusive valuation equations: RBB case IV

2: Privilege Funding adjustments over the Credit ones

$$V = V_0 - CVA + \cancel{DVA} + LVA \quad \underbrace{-FCA}_{-DVA_F - FCA_\ell} \quad + \quad \underbrace{FBA}_{DVA + FBA_\ell} \quad + DVA_F$$

resulting in

$$V = V_0 - CVA + LVA - FCA_\ell + \underbrace{FBA}_{DVA + FBA_\ell}$$

Now DVA is managed by the FVA Desk. Notice that if liquidity basis $\ell = 0$ then $V = V_0 - CVA + LVA + \underbrace{FBA}_{DVA}$ and the only funding term is the benefit term given by trading DVA

Nonlinearities due to funding I

Aggregation–dependent and asymmetric valuation

Nonlinearity (having $\bar{V}$ on both sides of valuation equations) implies that valuation of a portfolio is aggregation dependent & is different for the 2 parties in a deal. In the classical pricing/ Black Scholes, if we have 2 or more derivatives in a portfolio we can price each separately & add up. Not with funding. More: Without funding, the price to one entity is minus the price to the other one. Not so with funding.

The classical transaction-independent arbitrage free price is lost, now the price depends on the specific entities trading the product and on their policies (λ, f, ℓ). Recall $\mathbb{E}^h$ and PDE coefficients depending on $\bar{V}$ nonlinearly.

Nonlinearities due to funding II

The end of Platonic pricing?

There is no Platonic measure $\mathbb{Q}$ in the sky to price all derivatives with an expectation where all assets have the risk free return r.
Now the pricing measure is product dependent, and every trade will have a specific measure. This is an implication of the PDE non-linearity.

Consistent global modeling across asset classes and risks

Once aggregation is set, funding valuation is non–separable. Holistic consistent modeling across trading desks & asset classes needed

In Theory: Nonlinearities due to funding

Here if we assume $\ell^+ \approx \ell^-$, and closeout term is the risk free price $V(\tau)$ rather than the replacement value $\bar{V}(\tau)$, then the problem becomes linear and is much more manageable. **In practice everyone in the industry assumes this and applies a posteriori corrections if needed.**

NVA

In the recent paper `http://ssrn.com/abstract=2430696` we introduce a Nonlinearity Valuation Adjustment (NVA), which analyzes the double counting involved in forcing linearization. Our numerical examples for simple call options show that NVA can easily reach 2 or 3% of the deal value even in relatively standard settings.

Price of Value?

Why should the client pay for our funding policy choices?

The funding adjusted ”price” is not a price in the conventional sense. We may use it for cost/profitability analysis or to pay our treasury, but can we charge it to a client?

Can the client charge us too as she has funding costs?

Price of Value?

Accessibility of valuation parameters

How can the client check our price is fair if she has no access to our funding policy (less transparent than credit standing) and vice versa?

It is more a ”value” than a ”price”.

Provocative question. Why do not we charge an Electricity Bill Valuation Adjustment (EBVA)?

Should funding costs be zero?

In a number of papers, Hull and White argued that there should be no funding costs.

They invoked the Modigliani Miller theorem. A folk version of the theorem is this:

"If market price processes follow random walks, and there are no

- taxes,
- bankruptcy costs,
- agency costs,
- asymmetric information

and if the market is efficient *then* the value of a firm does not depend on how the firm is financed.

Should funding costs be zero?

However the above assumptions do not hold in practice.

The very presence of liquidity bases ℓ violates the assumptions.

However we saw in the above calculations that if $\ell = 0$ then there are indeed no funding costs. For example, in the EFB framework

$$V = V_0 - CVA + DVA + LVA \underbrace{-FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{CVA_F + FBA_\ell} + DVA_F - CVA_F$$

we see that if $\ell = 0$ and $\tilde{c} = r$ we end up with

$$V = V_0 - CVA + DVA$$

and there are no funding costs indeed.

Should funding costs be zero?

So it is a matter of qualifying the assumptions in the Modigliani Miller theorem.

Market imperfections such as the bases ℓ, among others, may make the theorem not valid and hence funding costs become relevant. Information asymmetries on rates f are also important.

The framework above can be customized to trading via CCP, including initial margins costs.

See for example Brigo and Pallavicini (2014). Nonlinear consistent valuation of CCP cleared or CSA bilateral trades with initial margins under credit, funding and wrong-way risks. Journal of Financial Engineering 1 (1), 1–60.

Q & A. CCPs

Q And what about Central Counterparty Clearing houses (CCP's)?
A *CCPs are commercial entities that, ideally, would interpose themselves between the two parties in a trade.*

- *Each party will post collateral margins say daily, every time the mark to market goes against that party.*
- *Collateral will be held by the CCP as a guarantee for the other party.*
- *If a party in the deal defaults and the mark to market is in favour of the other party, then the surviving party will obtain the collateral from the CCP and will not be affected, in principle, by counterparty risk.*
- *Moreover, there is also an initial margin that is supposed to cover for additional risks like deteriorating quality of collateral, gap risk, wrong way risk, etc.*

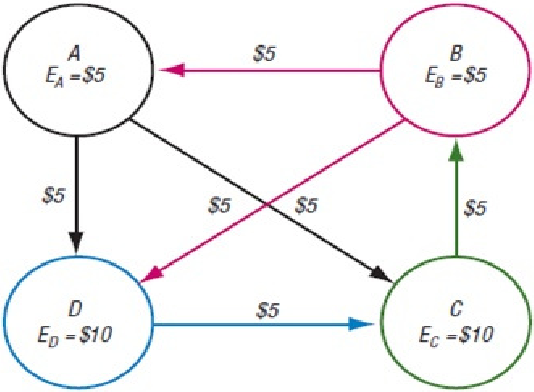


Figure: Bilateral trades and exposures without CCPs. Source: John Kiff.

http://shadowbankers.wordpress.com/2009/05/07/mitigating-counterparty-credit-risk-in-otc-markets-the-basics

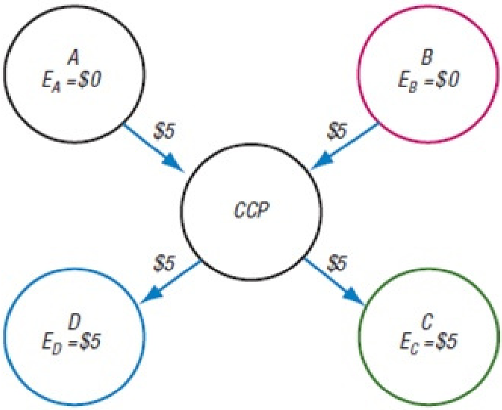


Figure: Bilateral trades and exposures with CCPs. Source: John Kiff.

http://shadowbankers.wordpress.com/2009/05/07/mitigating-counterparty-credit-risk-in-otc-markets-the-basics

Q & A: CCPs

Q It looks pretty safe. With the current regulation and law pushing firms to trade through central clearing, will all this analysis of credit, liquidity and funding risk be a moot point? Are CCP's going to be the end of CVA/DVA/FVA problems?

A *CCP's will reduce risk in many cases but are not a panacea. They also require daily margining, and one may question*

- *The pricing of the fees they apply*
- *The appropriateness of the initial margins and of overcollateralization buffers that are supposed to account for wrong way risk and collateral gap risk*
- *The default risk of CCPs themselves.*

Q & A: CCPs

Q So it is not that safe after all.

A *Valuation of the above points requires CVA type analytics, inclusive of collateral gap risk and wrong way risk, similar to those we discuss here. So unless one trusts blindly a specific clearing house, it will be still necessary to access CVA analytics and risk measures.*

Q & A: CCPs

Q So CCP's are not really a panacea. Other issues with CCPs?

A *The following points are worth keeping in mind:*[a]

- *CCPs are usually highly capitalised. All clearing members post collateral (asymmetric "CSA"). Initial margin means clearing members are overcollateralised all the time.*
- *TABB Group says extra collateral could be about 2 $ Trillion.*[b]
- *CCPs can default and did default. Defaulted ones - 1974: Caisse de Liquidation des Affaires en Marchandises; 1983: Kuala Lumpur Commodity Clearing House; 1987: Hong Kong Futures Exchange. The ones that were close to default- 1987: CME and OCC, US; 1999: BM&F, Brazil.*

[a] See for example Piron, Brigo (2012). Why collateral and CCPs can be bad for your wealth. SunGard's Adaptive White Paper.

[b] Rhode, W. (2011). European Credit and Rates Dealers 2011 – Capital, Clearing and Central Limit Order Books. TABB Group Research Report

Q & A: CCPs

Q And what about netting under CCPs? That should improve as everything goes through them.

A *Not so clear. A typical bank may have a quite large number of outstanding trades, making the netting clause quite material. With just one CCP for all asset classes across countries and continents, netting efficiency would certainly improve. However, in real life CCPs deal with specific asset classes or geographical areas, and this may even reduce netting efficiency compared to now.*

Q & A: CCPs

Q So CCPs could compete with each other?

A *Yes and one can be competitive in specific areas but hardly in all of them. Some CCPs will be profitable in specific asset classes and countries. They will deal mostly with standardised transactions. Even if CCPs could function across countries, bankruptcy laws can make collateral held in one place unusable to cover losses in other places.*[a]

[a] Singh, M. (2011) Making OTC Derivatives Safe - A Fresh Look. IMF paper

Q & A: CCPs

Q The geographical angle seems to be an issue, with no international law addressing how CCPs would connect through EMIR/CDR 4/Basel III and DFA. Is that right?

A *Indeed, there is currently "No legal construct to satisfy both Dodd Frank Act and EMIR and allow EU clients to access non-EU CCP's".*[a] *And there are also other conflicts in this respect. Where will CCPs be located and which countries will they serve? For example, the European Central Bank opposed LCH–Clearnet to work with Euro denominated deals because this CCP is not located in the Eurozone. This lead to a legal battle with LCH invoking the European Court of Justice.*

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[a] Wayne, H. (2012). Basel 3, Dodd Frank and EMIR. Citigroup Presentation.

Q & A: CCPs

Q But competition should be a good thing?

A *To compete CCPs may lower margin requirements, which would make them riskier, remember the above CCPs defaults? In the US, where the OTC derivatives market is going through slightly more than 10 large dealers and is largely concentrated among 5, we could have a conflict of interest. If CCPs end up incorporating most trades currently occurring OTC bilaterally, then CCPs could become "too big to fail".*[a]

Q So what should a bank do in modeling CCPs counterparty risk?

A *The CCP does not post collateral directly to the entities trading with it, as the collateral agreement is not symmetric. Hence, it is like CVA but computed without collateral. On top of that, one has the overcollateralization cost to lose. Hopefully, the default probability is low, making CVA small, bar strong contagion, gap risk and WWR*

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[a] Miller, R. S. (2011) Conflicts of interest in derivatives clearing. Congressional Research Service report.

Q & A: CCPs

Q It seems like even with CCPs one needs a strong analytical and numerical apparatus for pricing/hedging and risk.

A *Yes for all the reasons we illustrated, CCPs are not the end of CVA and its extensions. We need to consider and price/ risk-manage*

- *Checking initial margin charges across different CCPs to see which ones best reflect actual gap risk and contagion. This requires a strong pricing apparatus*
- *Computing counterparty risk associated with the default of the CCP itself*
- *Understanding quantitatively the consequences of the lack of coordination among CCPs across different countries and currencies.*

XVA Desk?

We now move to a general discussion on the CVA/FVA (XVA?) desk and of its role in the bank.

FVA Desk or CVA Desk, or both? XVA Desk?

First recall the role of the CVA Desk.

How do banks price and trade/hedge CVA?

The idea is to move Counterparty Risk management away from classic asset classes trading desks by creating a specific counterparty risk trading desk, or ”CVA desk”.

Under simplifying assumptions, this would allow ”classical” traders to work in a counterparty risk-free world in the same way as before the counterparty risk crisis exploded.

CVA Desks and ”Best practices”

What lead to CVA desks?

Roughly, CVA followed this historical path:

- Up to 1999/2000 no CVA. Banks manage counterparty risk through rough and static credit limits, based on exposure measurements (related to Credit VaR: Credit Metrics 1997).
- 2000-2007 CVA was introduced to assess the cost of counterparty credit risk. However, it would be charged upfront and would be managed mostly statically, with an insurance based approach.
- 2007 on, banks increasingly manage CVA dynamically. Banks become interested in CVA monitoring, in daily and even intraday CVA calculations, in real time CVA calculations and in more accurate CVA sensitivities, hedging and management.
- CVA explodes after 7[8] financials defaults occur in one month of 2008 (Fannie Mae, Freddie Mac, Washington Mutual, Lehman, [Merrill] and three Icelandic banks).

CVA Desks and ”Best practices”

CVA desk location in a bank

- Trading floor: PROS works with other trading desk, direct use of hedge trades (especially CDS).
  CONS: competition and political problems.
- Treasury: PROS since it involves credit policy, collateral, good for coordination with funding. DVA as funding benefit.
  CONS: interface w/ other desks needs to be managed carefully.
- Often CVA desk does systemically important operations for the bank. Should it be part of RISK / CRO? See how Goldman CVA desk may have saved the firm in the AIG case.[a] Nonprofit desk, runs a service.
- Considerable operational implications too for the bank functioning. COO?

[a] “How Goldman’s Counterparty Valuation Adjustment (CVA) Desk Saved The Firm From An AIG Blow Up” `http://www.zerohedge.com/`, accessed on Dec 1, 2014

CVA Desks and ”Best practices”

CVA desk and Classical Trading desks

The CVA desk charges classical trading desks a CVA fee in order to protect their trading activities from counterparty risk through hedging. This may happen also with collateral/CSA in place (Gap Risk, WWR, etc). The cost of implementing this hedge is the CVA fee the CVA desk charges to the classical trading desk. Often the hedge is performed via CDS trading.

CVA Desks and "Best practices"

CVA desk in the trasury department

Charging a fee is not easy and can make a lot of P&L sensitive traders nervous. That is one reason why some banks set the CVA desk in the treasury for example. Being outside the trading floor can avoid some "political" issues on P&L charges among traders.

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Furthermore, given that the treasury often controls collateral flows and funding policies, this would allow to coordinate CVA and FVA calculations and charges after collateral.

CVA Desks and "Best practices"

How the CVA desk helps other trading desks

The CVA desk[a] would free the classical traders from the need to:

- develop advanced credit models to be coupled with classical asset classes models (FX, equity, rates, commodities...);
- know the whole netting sets trading portfolios; traders would have to worry only about their specific deals and asset classes, as the CVA desk takes care of "options on whole portfolios" embedded in counterparty risk pricing and hedging;
- Hedge counterparty credit risk, which is very complicated.

[a] See for example "CVA Desk in the Bank Implementation", *Global Market Solutions* white paper

CVA Desks and "Best practices"

The CVA desk task looks quite difficult

The CVA desk has **little/no control** on inflowing trades, and has to:

- quote quickly to classical trading desks a "incremental CVA" for specific deals, mostly for pre-deal analysis with the client;
- For every classical trade that is done, the CVA desk needs to integrate the position into the existing netting sets and in the global CVA analysis in real time;
- related to pre-deal analysis, after the trade execution CVA desk needs to allocate CVA results for each trade ("marginal CVA")
- Manage the global CVA, and this is the core task: Hedge counterparty credit and classical risks, including credit-classical correlations (WWR), and check with the risk management department the repercussions on capital requirements.

CVA Desks and ”Best practices”

CVA Desks effectiveness if often questioned

Of course the idea of being able to relegate all CVA(/DVA/FVA) issues to a single specialized trading desk is a little delusional.

- WWR makes isolating CVA from other activities quite difficult.
- In particular WWR means that the idea of hedging CVA and the pure classical risks separately is not effective.
- CVA calculations may depend on the collateral policy, which does not depend on the CVA desk or even on the trading floor.
- We have seen FVA and CVA interact

In any case a CVA desk can have different levels of sophistication and effectiveness.

CVA Desks and "Best practices"

Classical traders opinions

Clearly, being P&L sensitive, the CVA desk role is rather delicate. There are mixed feelings.

- Because CVA is hard to hedge (especially the jump to default risk and WWR), occasionally classical traders feel that the CVA desk does not really hedge their counterparty risk effectively and question the validity of the CVA fees they pay to the CVA desk.
- Other traders are more optimistic and feel protected by the admittedly approximate hedges implemented by the CVA desk.
- There is also a psychological component of relief in delegating management of counterparty risk elsewhere.

Including FVA. XVA Desks?

As we have seen funding costs are now an important component of the valuation process, and FVA is calculated for the bank deals.

This may be charged internally to classical trading desks, who pay the FVA desk for the funding costs, and in turn charge the cost to clients externally.

XVA Desk

Both CVA and FVA reference collateral importantly, so they should be managed together, especially given analogies in these quantities, given DVA as funding benefit and given that one would like to avoid double counting.

Ideally, the XVA desk should immunize classical trading desk from credit risk and funding costs, using mirror trades that isolate those risks

XVA Desks?

XVA Desk and Mirror Trades

Isolating Credit Risk and Funding Costs away from traditional trading desks is made difficult by wrong way risk, where dependence makes all risks connected. One can manage this by assigning risk reserves to deal with wrong way risk losses.

One more difficulty is the little transparency on the bases ℓ. They depend on CDS-Bond basis & the bank funding policy: maturity transformation, netting of funding sets, fund transfer pricing policy, etc.

XVA Desks?

Cross Gammas

In this sense quantities that are helpful are cross gammas: sensitivities of computed values to joint shocks in credit and underlying risk factor, and possibly sensitivity to bases ℓ and underlying risk factors.

As own credit risk and the bases ℓ are difficult to hedge, a reserve is set in place for these risks.

Charging FVA to clients?

Charging FVA to Clients

From what we understand, *most of the banks we cited earlier charge FVA to clients*. The classical trading desk pays the funding costs to the FVA desk but then charges the FVA to the client. However, this is controversial. The client often has no transparency on our funding policy. Why should be pay for our choices? And what if the client decides to charge us her funding costs? Can this be done bilaterally given the lack of transparency?

We also debated the price vs value aspect of FVA earlier.

Possible objections to FVA charge are due to the Modigliani Miller theorem. We addressed these earlier via market imperfections and bases ℓ. *Banks are now satisfied with charging clients with FVA. Hence a bank that does not do that risks to be inconsistent with the market*.

FVA Desks?

FVA separate desk?

Some tier-2 banks are considering creating a FVA desk apart from the CVA Desk. However this is not a popular option with tier-1 banks and most banks are trying to incorporate the FVA function in the already existing CVA desk, that becomes a XVA desk. This is what may be happening with all the banks we mentioned earlier.

The reason is that the split between credit and fuding is not as clearcut as one may think. See our derivation of
CVA, DVA, LVA, FCA, FBA, CVA_F, DVA_F
and of all ways to recombine them.

All quantities are driven by s_I, s_C and ℓ^+, ℓ^-.

Recall also that in the full theory FVA and CVA are not really separable.

PART 4: RISK MEASURES

In this third part we look at the problem of risk measurement and management.

So far we discussed mostly valuation and hedging. This is important and is done under the risk neutral measure $\mathbb{Q}$, as we have seen earlier.

Risk Management however is partly based on historical estimation, and is interested in potential losses in the physical world, hence we need to go back to the historical/physical measure $\mathbb{P}$.

We introduce the two fundamental risk measures of Value at Risk (VaR) and Expected Shortfall (ES).

Risk Measures: A historical perspective I

This historical perspective is from Brian McHugh's review (2011)

This is an introduction into 'Risk Measures', particularly focusing on Value-at-Risk (VaR) and Expected Shortfall (ES) measures. A brief history of risk measures is given, along with a discussion of key contributions from various authors and practitioners.

What do we mean by ”Risk”? I

Risk is defined by the dictionary as ’a situation involving exposure to danger’. It is related to the randomness of uncertainty. Risk is also described as ’the possibilty of financial loss’ and this is the definition that will be discussed here.

Risk management, described by Kloman[6] as ’a discipline for living with the possibility that future events may cause adverse effects’, is of vital importance to the appropriate day to day running of financial institutions.

Here, downside risk (the probability of loss or less than expected returns) will be the focus of discussion as it is the most crucial area for risk managers. In particular, Value at Risk (VaR) and Expected Shortfall (ES) methodologies of measuring risk will be analysed.

What do we mean by ”Risk”? II

The question that comes to mind is where does this risk come from, and of course there is no single answer.

Risk can be created by a great number of sources, both directly and indirectly, it propagates from government policies, war, inflation, technological innovations, natural phenomena, and many others.

There are a number of risks faced by financial institutions everyday, these include market risk, credit risk, operational risk, liquidity risk, and model risk.

What do we mean by ”Risk”? III

- Market risk includes the unexpected moves in the underlying of the financial assets (stock prices, interest rates, fx rates...)
- Credit risk propogates from the creditworthiness of a counterparty in a contract and the possibility of losses caused by its default.
- Operational risk: possibility of losses occured by internal processes, people, and systems or from other sources externally.
- Liquidity risk stems from the inability, in some cases, to buy or sell financial instruments in sufficient time as to minimise losses.
- Model risk: inaccurate use of valuation and pricing models, for instance inaccurate distributions or unrealistic assumptions. Negative interest rates? (eg Vasicek, Hull White), Models with thin tails instead of fat tails? Bad future volatility structures? Unrealistic correlation patterns? (see discussion on LMM above).

What do we mean by ”Risk”? IV

- Finally, all such risks may interact in complex ways and their mutual dependence and contagion is a key aspect of modern research. As these risks are not really completely separable, this classification is purely indicative and not substantial.

[6] H. F. KLOMAN (1990), *Risk Management Agonistes*, Risk Analysis 10:201-205.

A brief history of VaR and Expected Shorfall I

The origins of VaR and risk measures can be traced back as far 1922 to capital requirements the New York Stock Exchange imposed on member firms according to Holton[7].

However, Markowitz's seminal paper 'Porfolio Theory' (1952), which developed a means of selecting portfolios based on an optimization of return given a certain level of risk, was the first convincing if stylized and simplistic method of measuring risk. His idea was to focus portfolio choices around this measurement.

A brief history of VaR and Expected Shorfall II

Risk management methodologies really took off from this point and over the next couple of decades new ideas, such as the Sharpe Ratio, the Capital Asset Pricing Model (CAPM) and Arbitrage Pricing Theory (APT), were being proposed and implemented.

Along with this came the introduction of the Black-Scholes option-pricing model in 1973, which lead to a great expansion of the options market, and by the early 1980s a market for over-the-counter (OTC) contracts had formed.

The related theory had important precursors in Bachelier (1900) and de Finetti (1931)[8]

A brief history of VaR and Expected Shorfall III

Perhaps the greatest consequence of the financial innovations of the 1970s and 1980s was the proliferation of leverage, and with these new financial instruments, opportunities for leverage abounded.

Think of an interest rate swap that is at the money: it costs nothing to enter this swap even on a huge notional, and yet this may lead to very large losses in the future.

Similarly for credit default swaps, oil swaps, and a number of other derivatives.

A brief history of VaR and Expected Shorfall IV

Along with academic innovation came technological advances. Information technology companies like Reuters, Telerate, and Bloomberg started compiling databases of historical prices that could be used in valuation techniques.

Financial instruments could be valued quicker with new hi-tech methods such as the Monte Carlo pricing for complex derivatives, and thus trades were being made quicker.

We have now reached super-human speed with high frequency trading, so debated that the EU is considering banning it.

However, in addition to all these innovations and advances came catastrophes in the financial world such as:

A brief history of VaR and Expected Shorfall V

- The Barings Bank collapse of 1995, which was solely due to the fraudulent dealings of one of its traders.
- Metallgesellschaft lost \$1.3 billion by entering into long term oil contracts in 1993.
- Long-Term Capital Management's near collapse in 1998 and subsequent bailout overseen by the Federal Reserve. *Somewhat ironically, members of LTCM's board of directors included Scholes and Merton.*

For more information on these financial disasters and others see Jorion (2007).[9] Organizations were now more than ever increasingly in need for a single risk measure that could be applied consistently across asset categories in hope that financial disasters such as these could be prevented. However, even this wouldn't be enough, as the Lehman collapse of 2008 has shown. We'll discuss why later.

A brief history of VaR and Expected Shorfall VI

Q: "What is Basel?"

A: "A city in Europe? Perhaps switzerland?"

The Basel Committee on Banking Supervision was central to the introduction and implementation of VaR on a worldwide scale. The Committee itself does not possess any overall supervising authority, but rather gives standards, guidelines, and recommendations for individual national authorities to undertake.

The first Basel Accord of 1988 on Banking Supervision attempted to set an international minimum capital standard, however, according to McNeil at al.[10] this accord took an approach which was fairly coarse and measured risk in an insufficiently differentiated way.

A brief history of VaR and Expected Shorfall VII

The G-30 (consultative group on international economic and monetary affairs) report in 1993 titled 'Derivatives: Practices and Principles' addressed the growing problem of risk management in great detail.

It was created with help from J.P. Morgans' RiskMetrics system, which measured the firm's risk daily.

The report gave recommendations that portfolios be marked-to-market daily and that risk be assessed with both VaR and stress testing.

While the G-30 Report focused on derivatives, most of its recommendations were applicable to the risks associated with other traded instruments.

For this reason, the report largely came to define the new risk management of the 1990s and set the an industry-wide standard.

A brief history of VaR and Expected Shorfall VIII

The report is also interesting, as it may be the first published document to use the word ”value-at-risk”.

Expected shortfall (ES) is a seemingly more recent risk measure, however, Rappoport (1993) [11] mentions a new approach called Average Shortfall in J.P. Morgan’s Fixed Income Research Technical Document, which first noted application of the theory of Expected Shortfall in finance.

The later paper of Artzner et al. (1999)[12] introduces four properties for measures of risk and calls the measures satisfying these properties as ’coherent’.

While such ”coherent” risk measures become ill defined in presence of liquidity risk (especially the proportionality assumption), this was the catalyst for the need of a new ’coherent’ risk measure.

A brief history of VaR and Expected Shorfall IX

As ES was practically the only operationally manageable coherent risk measure, ES was proposed as a coherent alternative to VaR.

---

[7] G. A. HOLTON (2002), working paper. *History of Value-at-Risk: 1922-1998*.

[8] Pressacco, F., and Ziani, L. (2010). Bruno de Finetti forerunner of modern finance. In: Convegno di studi su Economia e Incertezza, Trieste, 23 ottobre 2009, Trieste, EUT Edizioni Universit di Trieste, 2010, pp. 65-84.

[9] P. JORION *Value at Risk: The New Benchmark for Managing Financial Risk 3rd ed.* McGraw-Hill.

[10] A. MCNEIL, R. FREY AND P. EMBRECHTS (2005), *Quantitative Risk Management*, Princeton University Press.

[11] P. RAPPOPORT (1993), *A New Approach: Average Shortfall*, J.P. Morgan Fixed Income Research Technical Document.

[12] P. ARTZNER, F. DELBAEN, J. EBER AND D. HEATH , *Coherent Measures of Risk*, Mathematical Finance Vol.9 No.3.

Value at Risk I

Value at risk (VaR) is a single, summary, statistical measure of possible portfolio losses. It aggregates all of the risks in a portfolio into a single number suitable for use in the boardroom, reporting to regulators, or disclosure in an annual report, and it is the most widely used risk measure in financial institutions according to McNeil et al.

In addition to this, VaR estimates not only serve as a summary statistic, but are also often used as a tool to manage and control risk with institutions changing their market exposure to maintain their VaR at a prespecified level.

The theory behind VaR is quite simplistic, actually too simplistic: VaR is defined as

the loss level that will not be exceeded with a certain confidence level over a certain period of time.

Value at Risk II

Again, this is related to the idea of downside risk, which measures the likelihood that a financial instrument or portfolio will lose value.

Downside risk can be measured by quantiles, which are the basis of the mathematics behind VaR. We now introduce a formal definition of VaR.

Value at Risk III

VaR is related to the potential loss on our portfolio, due to downside risk, over the time horizon H. Define this loss L_H as the difference between the value of the portfolio today (time 0) and in the future H.

$$L_H = \text{Portfolio}_0 - \text{Portfolio}_H.$$

Consistently with earlier notation, we may call $\Pi(t, T)$ the sum of all future cash flows in $[t, T]$, discounted back at t, for our portfolio. These are random cash flows and not yet prices. Price of the portfolio at t is

$$\text{Portfolio}_t = \mathbb{E}_t^{\mathbb{Q}}[\Pi(t, T)].$$

T is usually the final maturity of the portfolio, and typically $H << T$.

Value at Risk IV

For example, if the portfolio is just an interest rate swap where we pay fixed K and receive LIBOR L with tenor $T_\alpha, T_{\alpha+1}, \ldots, T_\beta$, then the payout is written, as we have seen earlier, for $t \leq T_\alpha$, as

$$\Pi(t, T_\beta) = \sum_{i=\alpha+1}^{\beta} D(t, T_i)(T_i - T_{i-1})(L(T_{i-1}, T_i) - K).$$

Value at Risk V

$\text{VaR}_{H,\alpha}$ with horizon H and confidence level α is defined as that number such that

$$\mathbb{P}[L_H < \text{VaR}_{H,\alpha}] = \alpha$$

or,

$$\boxed{\mathbb{P}[\mathbb{E}_0^{\mathbb{Q}}[\Pi(0,T)] - \mathbb{E}_H^{\mathbb{Q}}[\Pi(H,T)] < \text{VaR}_{H,\alpha}] = \alpha}$$

so that our loss at time H is smaller than $\text{VaR}_{H,\alpha}$ with $\mathbb{P}$-probability α.

In other terms, it is that level of loss over a time T that we will not exceed with $\mathbb{P}$-probability α. It is the α $\mathbb{P}$-percentile of the loss distribution over T.

From this last equation, notice the interplay of the two probability measures.

Value at Risk VI

From the dialogue by Brigo (2011). ”Counterparty Risk FAQ: Credit VaR, PFE, CVA, DVA, Closeout, Netting, Collateral, Re-hypothecation, WWR, Basel, Funding, CCDS and Margin Lending”. See also the book by Brigo, Morini and Pallavicini: ”Credit, Collateral and Funding”, Wiley, March 2013.

A: VaR is calculated through a simulation of the basic financial variables underlying the portfolio under the historical probability measure, commonly referred as $\mathbb{P}$, up to the risk horizon H. At the risk horizon, the portfolio is priced in every simulated scenario of the basic financial variables, including defaults, obtaining a number of scenarios for the portfolio value at the risk horizon.

Value at Risk VII

- Q: So if the risk horizon H is one year, we obtain a number of scenarios for what will be the value of the portfolio in one year, based on the eveolution of the underlying market variables and on the possible default of the counterparties.
- A: Precisely. A distribution of the losses of the portfolio is built based on these scenarios of portfolio values. When we say ”priced” we mean to say that the discounted future cash flows of the portfolio after the risk horizon are averaged conditional on each scenario at the risk horizon but under another probability measure, the Pricing measure, or Risk Neutral measure, or Equivalent Martingale Measure if you want to go technical, commonly referred as $\mathbb{Q}$.
- Q: Not so clear... [Looks confused]

Value at Risk VIII

A: [Sighing] All right, suppose your portfolio has a call option on equity, traded with a Corporate client, with a final maturity of two years. Suppose for simplicity there is no interest rate risk, so discounting is deterministic. To get the Var, roughly, you simulate the underlying equity under the P measure up to one year, and obtain a number of scenarios for the underlying equity in one year.

Q: Ok. We simulate under P because we want the risk statistics of the portfolio in the real world, under the physical probability measure, and not under the so called pricing measure Q.

Value at Risk IX

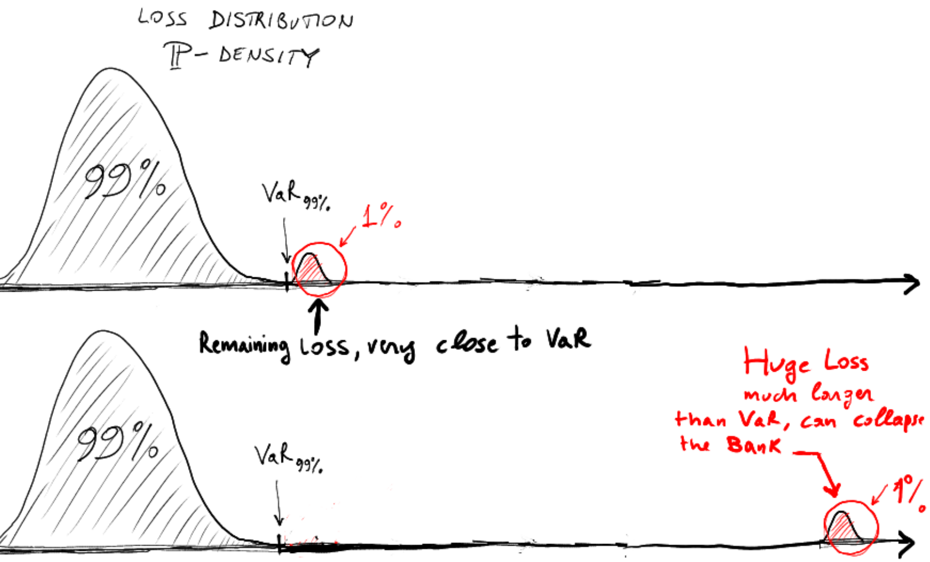
A: That's right. And then in each scenario at one year, we price the call option over the remaining year using for example a Black Scholes formula. But this price is like taking the expected value of the call option payoff in two years, conditional on each scenario for the underlying equity in one year. Because this is pricing, this expected value will be taken under the pricing measure Q, not P. This gives the Black Scholes formula if the underlying equity follows a geometric brownian motion under Q.

VaR drawbacks and Expected Shortfall I

As we explained in the introduction to risk measures, VaR has a number of drawbacks. We list two of them now, starting from the most relevant.

VaR drawback 1: VaR does not take into account the tail structure beyond the percentile.

Consider the following two cases.


LOSS DISTRIBUTION
P- DENSITY
99%
$VaR_{99\%}$
1%
Remaining Loss, very close to VaR
99%
$VaR_{99\%}$
Huge Loss
much larger
than VaR, can collapse
the Bank
1%

VaR drawbacks and Expected Shortfall I

From the picture above we see that we may have two situations where the VaR is the same but where the risks in the tail are dramatically different.

In the first case, the VaR singles out a 99% percentile, after which a slightly larger loss follows with 1% probability mass. The bank may be happy to know the 99% percentile in this case and to base its risk decision on that.

In the second case, the VaR singles out the same 99% percentile, after which an enormously much larger loss concentration follows with probability 1%. For example, this is now so large to easily collapse the bank. Would the bank be happy to ignore this potential huge and devastating loss, even if it has a small 1% probability?

VaR drawbacks and Expected Shortfall II

Probably not, and in this second case the bank would not base its risk analysis on VaR at 99%.

The VaR at 99% does not capture this difference in the two distributions, and if the bank does not explore the tail structure, it cannot know the real situation.

The most dangerous situation is the bank computing VaR and thinking it is in the first situation when it is actually in the second one.

VaR drawbacks and Expected Shortfall III

VaR drawback 2: VaR is not sub-additive on portfolios.
Suppose we have two portfolios P_1 and P_2, and a third portfolio $P = P_1 + P_2$ that is given by the two earlier portfolios together. VaR at a given confidence level and horizon would be sub-additive if

$$\text{VaR}(P_1 + P_2) \leq \text{VaR}(P_1) + \text{VaR}(P_2) \quad \text{(VaR subadditivity. Is it true?)}$$

ie the risk of the total portfolio is smaller than the sum of the risks of its sub-portfolios (benefits of diversification, among other things).

However, *this is not true*. It may happen that

$$\text{VaR}(P_1 + P_2) > \text{VaR}(P_1) + \text{VaR}(P_2) \quad \text{in some cases.}$$

While such cases are usually difficult to see in practice, it is worth keeping this in mind.

VaR drawbacks and Expected Shortfall IV

As a remedy to this sub-additivity problem (and only partly to the first drawback) Expected Shortfall (ES) has been introduced.

ES requires to compute VaR first, and then takes the expected value on the TAIL of the loss distribution for values larger than VaR, conditional on the loss being larger than Value at Risk.

ES is sub-additive (solves drawback 2).

ES looks at the tail after VaR, but only in expectation, without analyzing the tail structure carefully. Hence, it is only a partial solution to drawback 1.

VaR drawbacks and Expected Shortfall V

Recalling that we defined the loss L_H as the difference between the value of the portfolio today (time 0) and in the future H.

$$L_H = \text{Portfolio}_0 - \text{Portfolio}_H,$$

ES for this portfolio at a confidence level α and a risk horizon H is

$$\text{ES}_{H,\alpha} = \mathbb{E}^{\mathbb{P}}[L_H | L_H > \text{VaR}_{H,\alpha}]$$

By definition, ES is always larger than the corresponding VaR.

Be aware of the fact that ES has several other names, and there are other risk measures that are defined very similarly. Names you may hear are:
Conditional value at risk (CVaR), average value at risk (AVaR), and expected tail loss (ETL).

Value at Risk and ES: An example I

- PORTFOLIO: ZERO COUPON BOND and CALL OPTION ON A STOCK.
- FIRST ASSET of the PORTFOLIO: Zero coupon bond with Maturity $T = 2$ years and notional $N_B = 1000$.
- RISK FACTOR r: The BOND value is driven by interest rates. We choose a short rate model for r_t for this risk factor.
- Short term interest rate under the physical measure $\mathbb{P}$:

$$dr_t = k_r(\bar{\theta} - r_t)dt + \sigma_r d\bar{Z}_t,$$

$$r_0 = 0.01, k_r = 0.1, \bar{\theta} = 0.1, \sigma_r = 0.004.$$

Value at Risk and ES: An example II

- Short term interest rate under the pricing measure $\mathbb{Q}$:

$$dr_t = k_r(\theta - r_t)dt + \sigma_r dZ_t,$$

$$\theta = 0.05.$$

- We need both the $\mathbb{P}$ and $\mathbb{Q}$ dynamics of the risk factor: We will simulate r up to the risk horizon H using the $\mathbb{P}$ dynamics, hence $\bar{\theta}$. Then, to compute the price of the bond and equity call option at the different scenarios at time H we will use the $\mathbb{Q}$ dynamics, hence θ.
- SECOND ASSET of the PORTFOLIO: Call option on equity S with strike $K_c = 100$. Call option maturity: 2 years.
- RISK FACTOR S: The EQUITY CALL OPTION is driven by equity stock S_t. We choose a Black Scholes type model for the stock price S (but careful about the $\mathbb{Q}$ drift...)

Value at Risk and ES: An example III

- Equity under the physical measure: $dS_t = \mu S_t dt + \sigma_S S_t d\bar{W}_t$, $S_0 = 100$, $\mu = 0.09$, $\sigma_S = 0.2$.
- Equity under the pricing measure: $dS_t = r_t S_t dt + \sigma_S S_t dW_t$, with r_t the short term stochastic process given above. *Here the drift r_t is imposed by no-arbitrage!*
- We need both the $\mathbb{P}$ and $\mathbb{Q}$ dynamics of the risk factor: We will simulate S up to the risk horizon H using the $\mathbb{P}$ dynamics, hence μ. Then, to compute the price of the call option at the different scenarios at time H we will use the $\mathbb{Q}$ dynamics, hence r, where the drift in dr is θ.
- So

$$\Pi(t,T) = D(t,2y)N_B + D(t,2y)(S_{2y} - K_c)^+ =$$

$$= \exp\left(-\int_t^{2y} r_s ds\right) N_B + \exp\left(-\int_t^{2y} r_s\, ds\right)(S_{2y} - K_c)^+$$

Value at Risk and ES: An example IV

- TYPE OF RISK MEASURE
- VaR holding period: $H = 1y$.
- Confidence level: 99%
- ES holding period: $H = 1y$.
- Confidence level: 99%

Value at Risk and ES: An example V

- So, our loss is: $L_{1y} = \text{Portfolio}_0 - \text{Portfolio}_{1y}$, or

$$L_{1y} = \mathbb{E}_0^{\mathbb{Q}} \left[\exp\left(-\int_0^{2y} r_s ds \right) N_B + \exp\left(-\int_0^{2y} r_s \, ds \right) (S_{2y} - K_c)^+ \right]$$

$$-\mathbb{E}_{1y}^{\mathbb{Q}} \left[\exp\left(-\int_{1y}^{2y} r_s ds \right) N_B + \exp\left(-\int_{1y}^{2y} r_s \, ds \right) (S_{2y} - K_c)^+ \right]$$

- IMPORTANT: Notice that the risk factor r_t appears also in the DRIFT (local mean) of S under $\mathbb{Q}$, so that S and r need to be simulated consistently and jointly.

Value at Risk and ES: An example VI

- Correlation

$$\mathrm{corr}(dr, dS) = \rho \qquad (dZ_t dW_t = \rho dt),$$

  we try three cases:

- $\rho = 0$
- $\rho = -1$
- $\rho = 1$

Value at Risk and ES: An example VII

Results: One million paths in R

- $\rho = -1$: $VaR = 13.07$ $ES = 14.38$
- $\rho = 0$: $VaR = 9.34$ $ES = 10.85$
- $\rho = +1$: $VaR = -0.99$ $ES = -0.93$

Let's look at the three cases in detail

Value at Risk and ES: An example VIII

$$\rho = -1 : \quad VaR = 13.07 \quad ES = 14.38$$

Here there is totally negative correlation.

Remember that when interest rates go up in 1y, bonds go down: if r increases the zero coupon bond P decreases. This is also confirmed by the formula for the Vasicek zero coupon bond price $P(t, T) = A(t, T) \exp(-B(t, T) r_t)$ (recall $A > 0$ and $B > 0$): This is a decreasing function of r since it is an exponential with a negative exponent.

Totally negative correlation between r and S means that when r goes up (same as P goes down) S goes down and viceversa. Then we can write

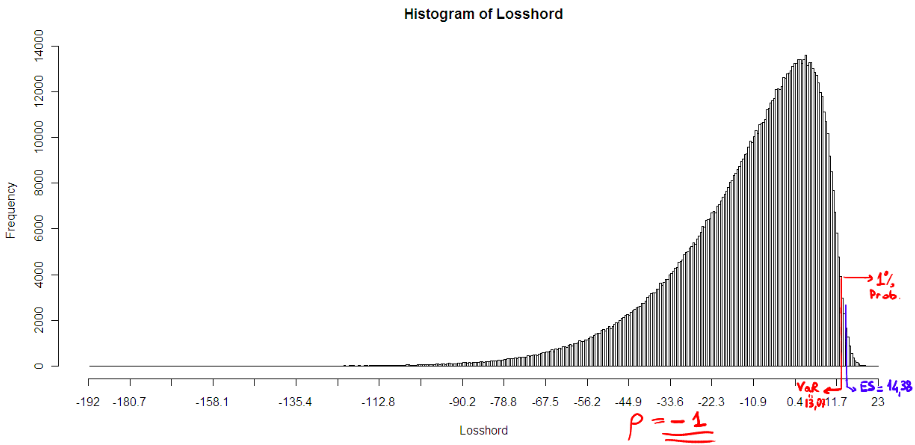
$\boxed{r \uparrow \text{(equivalently } P \downarrow) \Rightarrow S \downarrow}$ and $\boxed{r \downarrow \text{(equivalently } P \uparrow) \Rightarrow S \uparrow}$.

Value at Risk and ES: An example IX

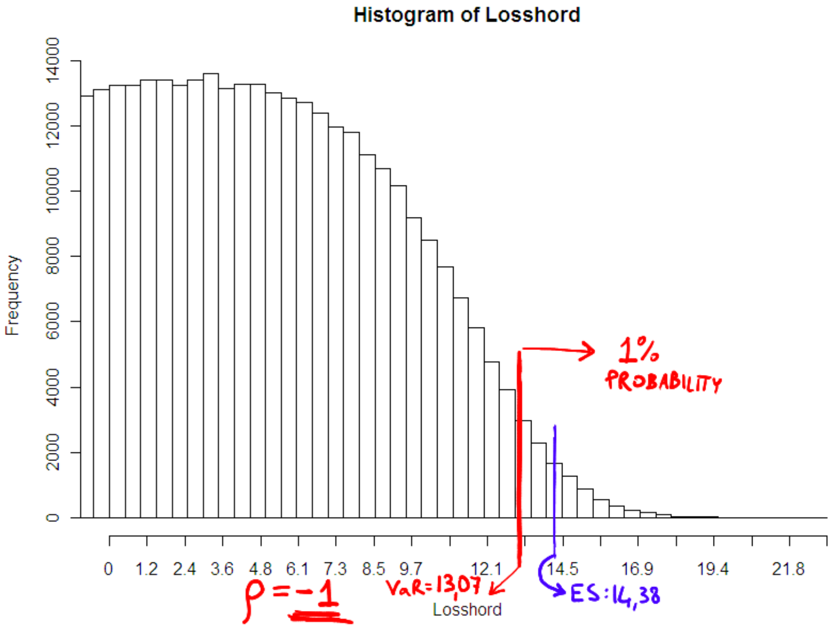
$\boxed{r \uparrow \text{(equivalently } P \downarrow) \Rightarrow S \downarrow}$ and $\boxed{r \downarrow \text{(equivalently } P \uparrow) \Rightarrow S \uparrow}$.

Hence, in our portfolio, when the bond goes down a lot in one year, leading to an important loss, the stock goes into the same way due to the correlation and there is an important loss also in the equity call option, which is monotonically increasing in S. Same for the gain, in the other direction. Hence the correlation links losses (gains respectively) from the bond to losses (gains) from the stock and the effects compound by happening, statistically, in the same scenarios.

Then *the loss distribution will be more spread out and the percentiles will be larger*, as shown by our VaR, which is the largest in this case. We show the situation in a couple of plots:


Histogram of Losshord
Frequency
0
2000
4000
6000
8000
10000
12000
14000
-192
-180.7
-158.1
-135.4
-112.8
-90.2
-78.8
-67.5
-56.2
-44.9
-33.6
-22.3
-10.9
0.4
11.7
23
Losshord
1% Prob.
VaR 13,01
ES = 14,38

$\rho = -1$

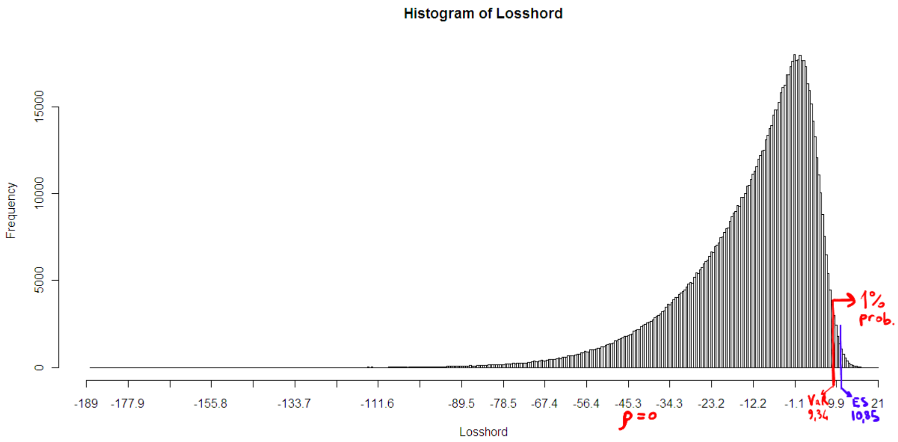

Histogram of Losshord
Frequency
0
2000
4000
6000
8000
10000
12000
14000
0
1.2
2.4
3.6
4.8
6.1
7.3
8.5
9.7
12.1
14.5
16.9
19.4
21.8
Losshord
1%
PROBABILITY
VaR=13,07
ES:14,38
ρ=−1

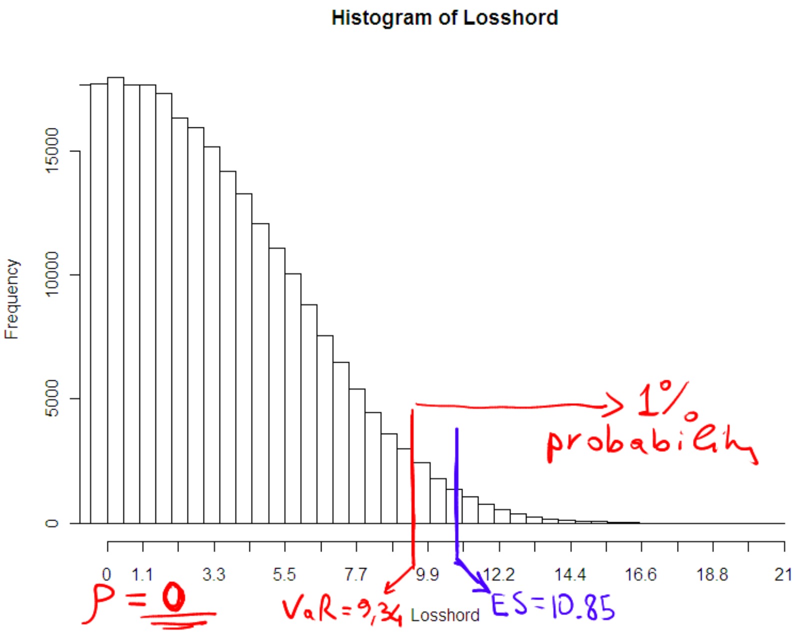
Value at Risk: An example I

Let's look at the second case:

$$\rho = 0: \quad VaR = 9.34 \quad ES = 10.85$$

Here there is no correlation (and since the shocks are jointly gaussian, this means independence). Hence there is no link between the direction of P and the direction of S. As a consequence losses are less extreme than in the previous case because bad scenarios in r and S happen independently and don't combine.


Histogram of Losshord
Frequency
0
5000
10000
15000
-189
-177.9
-155.8
-133.7
-111.6
-89.5
-78.5
-67.4
-56.4
-45.3
-34.3
-23.2
-12.2
-1.1
9.9
21
Losshord
1% prob.
VaR 9,34
ES 10,85
ρ=0


Histogram of Losshord
Frequency
0
5000
10000
15000
0
1.1
3.3
5.5
7.7
9.9
12.2
14.4
16.6
18.8
21
Losshord
1%
probability
P = 0
VaR = 9,34
ES = 10.85

Value at Risk: An example I

We may see that now that extreme losses or gains do not happen together, the distribution is less spread out. Compare to the plots for the case $\rho = -1$ and this is clear. The tail stops earlier and so does the VaR percentile. VaR is smaller here.

Value at Risk: An example I

Let's look at the third case:

$$\rho = 1 : \quad VaR = -0.99 \quad ES = -0.93$$

Here there is total positive correlation. This means that when r goes up (equivalently P goes down), leading to a loss in the Bond portfolio, S goes up too, leading to a gain in the Call option.

In the opposite case, when S goes down, leading to a loss in the equity call option, then r goes down (equivalently P goes up) and we have a gain in the bond portfolio.

It is then obvious that we have here the less risky situation: when we lose on one of the two assets we gain from the other one, so that our losses will be always reduced compared to the other two cases. Indeed we may see that from the plot.

Value at Risk: An example II

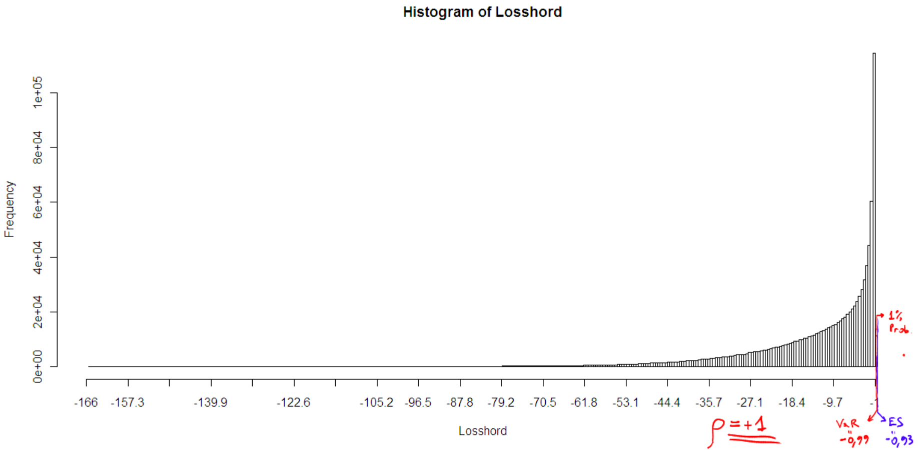
This holds to the extent that actually VaR and ES are negative and near zero, meaning that the worst we risk - according to these measures - is to make a small gain instead of a big one. But no actual losses.

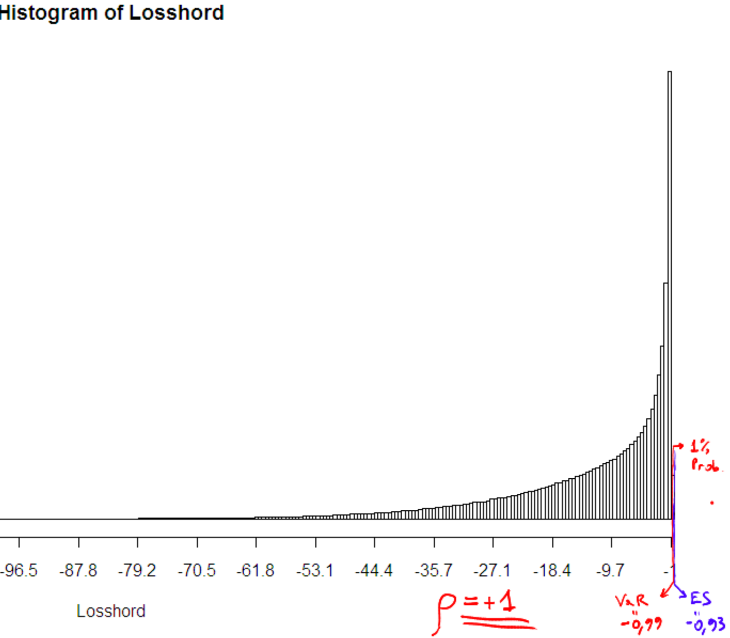
With a similar notation as before, we may now write

$$\boxed{r \uparrow \text{(equivalently } P \downarrow) \Rightarrow S \uparrow} \text{ and } \boxed{r \downarrow \text{(equivalently } P \uparrow) \Rightarrow S \downarrow}$$

showing clearly that P and S move into opposite directions.

Since the two assets move into opposite directions, the portfolio values will be much more concentrated near zero than in the previous cases. This is confirmed by the plot.


Histogram of Losshord
Frequency
0e+00
2e+04
4e+04
6e+04
8e+04
1e+05
-166
-157.3
-139.9
-122.6
-105.2
-96.5
-87.8
-79.2
-70.5
-61.8
-53.1
-44.4
-35.7
-27.1
-18.4
-9.7
Losshord
1% Prob.
ρ = +1
VaR = -0,99
ES = -0,93


Histogram of Losshord
-96.5
-87.8
-79.2
-70.5
-61.8
-53.1
-44.4
-35.7
-27.1
-18.4
-9.7
-1
Losshord
1%
Prob.
VaR
-0,99
ES
-0,93
$\rho = +1$

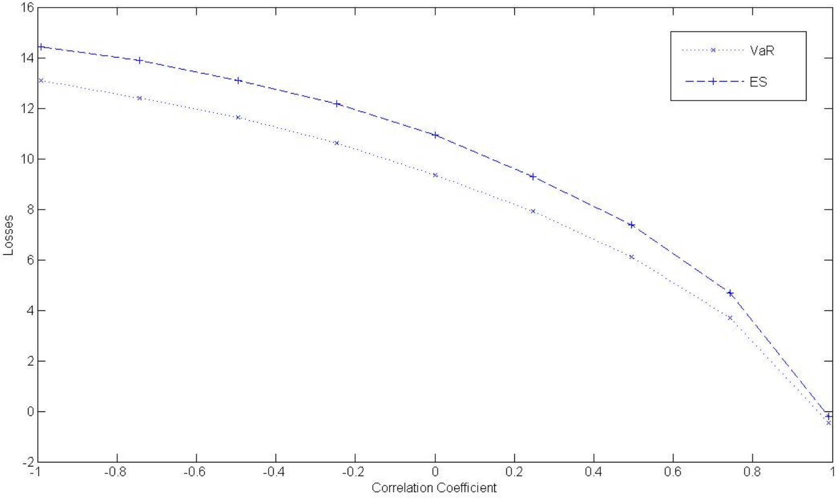
Value at Risk: An example I

So we have seen

- $\rho = -1: \quad VaR = 13.07 \quad ES = 14.38$
- $\rho = 0: \quad VaR = 9.34 \quad ES = 10.85$
- $\rho = +1: \quad VaR = -0.99 \quad ES = -0.93$

In this case the correlation between the two risk factors, interest rates and equity, r and S, plays a crucial role.
The next plot shows the impact of all possible values of correlation, ie correlation sensitivity for VaR and ES

Correlation sensitivity VaR and ES



Value at Risk: An example I

More generally, volatilities, correlation, dynamics and statistical dependencies have a very important impact on risk.

For very large portfolios it is difficult to obtain intuition on why some risk patterns are observed, as there are too many assets and parameters.

A rigorous quantitative analysis of risks is fundamental to have a safe result. However, the assumptions underlying the analysis need to be kept in mind and stress-tested

VaR type measures have also been applied to credit risk, leading to the Credit VaR measure we briefly discussed in the comparison with CVA in the credit part of this course.

Value at Risk: An example II

VaR of CVA itself is now one of the topical areas in the industry. This is not Credit VaR, but the VaR coming from the possible loss due to future adverse movements of CVA over a given risk horizon. Basel III is quite concerned with this.

Current research is focused on extending risk measures to properly include liquidity risk, see for example Brigo and Nordio ”Liquidity adjusted risk measures”.

PART 5: INTRODUCTION TO OPTIMAL EXECUTION AND ALGORITHMIC TRADING

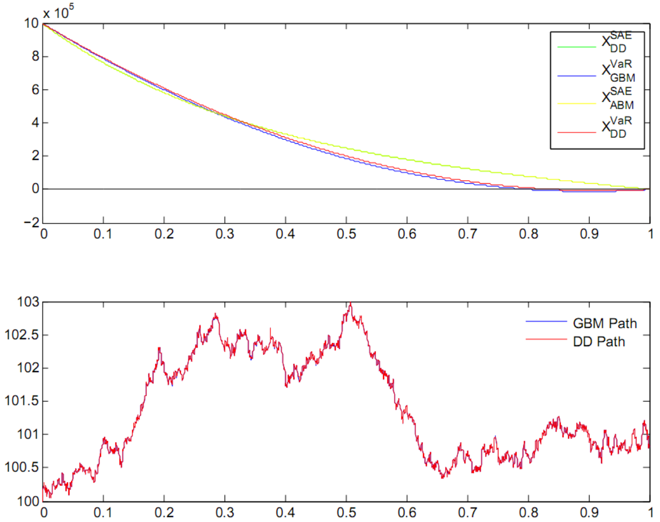
Market fragmentation: shares (or other contracts of interest) are held in relatively small numbers by a large number of market participants.

A LARGE order may require more shares than any participant has/is willing to transact at any one time. (Plus: it may send an alert signal to the market, resulting in a big price change)

Partition a large trade into smaller trades so as to minimize the effect of market impact.

Eg: Sell X shares by the time T while minimizing cost and risk in the execution. Costs are defined in terms of instantaneous market impact and permanent market impact. Risk may be defined in different ways.

An Example: Selling 1 million shares, with initial price 100 each, over 1 day, minimizing cost and risk



Optimal Trade Execution: Context I

Mid-price ABM. Almgren and Chriss (1999, 2000) (AC) combine expected execution cost and execution risk. Execution costs are increasing linearly in the trading rate, whereas the risk is given by the variance.

The advantage: a closed-form analytical solution. This solution is static and is usually in the class of the Volume Weighted Average Price (VWAP) solutions. Disadvantage: extremely stylized dynamics

Mid-price GBM. Gatheral and Schied (2011) solve the problem under the more realistic assumption of a geometric Brownian motion (GBM) but under a different risk (VaR), since variance as Risk is not tractable for GBM.

Optimal Trade Execution: Context II

GBM with Variance as risk. Numerical approach in Forsyth (2009a, 2009b). **The cost-risk efficient frontier is almost the same under ABM and GBM: the static solution with ABM is not very suboptimal.**

Summing up:

- ABM with Variance Risk Criterion : Tractable (AC)
- GBM with Variance Risk Criterion : Not tractable (F)
- GBM with Value at Risk (future loss percentile) : Tractable (GS)
- GBM with Expected shortfall (mean after percent) : Tractable (GS)

Optimal Trade Execution: Context III

Brigo and Di Graziano (2014) extend the Gatheral and Schied (2011) analytic result of the last two cases to a more general dynamics, namely the displaced diffusion (DD) model.

- DD with VaR or ES : Tractable
- DD with Squared Asset Expectation (SAE) Risk: Semitractable

How does the optimal execution problem solution change when we change dynamics and criteria? Extend analysis to DD & SAE

Optimal Trade Execution: Variables I

- The initial time is 0, the final time is T, and usually $t \in [0, T]$;
- S_t is the unaffected pre-impact share price at time t;
- $x(t)$: shares left to be sold at time t; Assumed absolutely continuous and adapted;
- $x(0) = X$ (sell X shares in total), $x(T) = 0$ (all shares sold by T).
- Selling shares impacts the shares price. Affected share price:

$$\widetilde{S}_t = S_t + \eta\, \dot{x}(t) + \gamma(x(t) - x_0)$$

- $\eta\, \dot{x}(t)$ is the instantaneous price impact of trading $dx(t) = \dot{x}(t)dt$ shares in $[t, t + dt)$ and only affects the $[t, t + dt)$ order.
- $-\gamma(x(0) - x_t)$ represents the permanent impact price that has been accumulating over $[0, t]$ by all transactions up to t.
- Cost and Risk of x: $C(x) := \int_0^T \widetilde{S}_t \, dx(t)$; $R(x)$? (several poss.)
- Find x that minimizes $\mathbb{E}[C(x) + L\, R(x)]$ with L leverage parameter

Optimal Trade Execution: Modelling Framework I

Following AC & especially G, BDiG assume that the trader's number of shares follows an absolutely continuous and adapted trajectory

$$t \mapsto x(t), \quad x(0) = X, \quad x(T) = 0.$$

Given this trading path, the price at which the transaction occurs is

$$\widetilde{S}_t = S_t + \eta\, \dot{x}(t) - \gamma(x_0 - x(t))$$

where η and γ are constants and S is the process for the unaffected stock price level.

Here η is a instantaneous impact parameter: when η increases, the trading activity will affect $\tilde{S}$ more and costs will increase.

while γ is a cumulative impact parameter: when γ increases, the impacted price $\tilde{S}$ decreases.

Optimal Trade Execution: Modelling Framework II

The problem formulation is almost completely specified:

- Need to postulate a Stochastic process for S (and hence $\tilde{S}$).
- Also need to decide on the risk function $R(x)$.

Optimal Trade Execution: The DD Dynamics I

Assume zero interest rates. Consider now the DD. Unaffected stock price S: given a GBM Y_t with volatility parameter σ, ie $dY_t = \sigma Y_t dW_t,\ (Y_0 = S_0 - K)$, define

$$S_t = K + Y_t \text{ or equivalently } dS_t = \sigma(S_t - K)dW_t,\ S_0 \text{ (DD)}$$

S is a shifted GBM. The shift is a constant K and the GBM is Y_t.

DD mimicks features of GBM and ABM in a single model.

$$\sigma(S_t - K) = \sigma S_t(1 - K/S_t)$$

$$\sigma S_t(1 - K/S_t) \approx \sigma S_t \text{ for } S_t >> K \quad (GBM).$$

$$\sigma(S_t - K) \approx \sigma(-K) \text{ for } S_t \approx 0 \quad (ABM).$$

This is usually summarized by practioners by saying that ”DD behaves like ABM for small stock values and as GBM for large ones”.

Optimal Trade Execution: The DD Dynamics II

DD also allows to set a minimum allowed value for the share price.

This is set at K, the shift parameter.

Formalizing the Optimal Execution problem for DD I

Summing up:

Trade execution strategy: absolutely continuous adapted stochastic process $t \mapsto x(t)$ for the amount of shares still to be sold at time T.

Aim to complete the sale of X shares by time T, so that

$$x(0) = X, \quad x(T) = 0.$$

The trade execution problem is finding the x as above that minimizes an objective function based on **costs** and **risk** terms.

Optimal Execution problem for DD: Cost and Risk I

The instantaneous cost of the strategy is the cost of buying $dx(t) = \dot{x}(t)dt$ shares at time $[t, t + dt)$ at the impacted stock price $\widetilde{S}_t$, spending $\widetilde{S}_t \dot{x}(t)dt$.

Costs arising from the strategy are, following Gatheral and Schied (2011)

$$
\begin{aligned}
C(x) := \int_0^T \widetilde{S}_t \, dx(t) &= \int_0^T [S_t + \eta \, \dot{x}(t) + \gamma(x(t) - x_0)]\dot{x}(t)dt \\
&= -XS_0 - \int_0^T x(t)dS_t + \eta \int_0^T (\dot{x}(t))^2 dt + \gamma X^2/2
\end{aligned}
$$

after using an integration by parts. This calculation does not yet involve the specific dynamics of S, and is general.

Optimal Execution problem for DD: Cost and Risk II

To have tractability with GBM, Gatheral and Schied adopt VaR or ES as **Risk** criteria, rather than Variance as in Almgren and Chriss.

Let $\nu_{\alpha,t,h}$ be the Value at Risk measure computed at time t, for the position, for a given confidence level α over a time horizon h.

$$\mathbb{P}\{S_t - S_{t+h} \leq \nu_{\alpha,t,h} | \mathcal{F}_t\} = \alpha.$$

If at t we have $x(t)$ shares with price S_t, the time t VR measure for a risk horizon h under DD dynamics at confidence level α would be

$$\begin{aligned}
\nu_t[x(t)(S_t - S_{t+h})] &= x(t)\nu_t[(S_t - S_{t+h})] \\
= x(t)\nu_t[(Y_t + K - Y_{t+h} - K)] &= x(t)\nu_t[(Y_t - Y_{t+h})] \\
= x(t)\nu_t[Y_t(1 - \exp(-\sigma^2 h/2 &+ \sigma(W_{t+h} - W_t)))] \\
= x(t)Y_t q_\alpha[1 - \exp(-\sigma^2 h/2 &+ \sigma\sqrt{h}Z)] = \\
= x(t)Y_t[1 - \exp(-\sigma^2 h/2 + \sigma\sqrt{h}q_{1-\alpha}(Z))] &=: \tilde{\lambda}_\alpha x(t)(S_t - K).
\end{aligned}$$

Optimal Execution problem for DD: Cost and Risk III

where Z is a standard normal, where we have used the homogeneity of VaR, and where $q_\alpha(X)$ is the α quantile of the distribution of X. This is the VaR measure for the instantaneous position at time t. If we average VaR over the life of the strategy we obtain the risk criterion

$$R^{\mathrm{VaR}_\alpha}(x) := \tilde{\lambda} \int_0^T x(t)(S_t - K)dt.$$

The expected shortfall risk criteria is of the same form, but with a different λ parameter:

$$m_\alpha = \frac{1}{1-\alpha}\left[\Phi\left(\frac{\ln(1-\tilde{\lambda}_\alpha) + \sigma^2 h/2}{\sigma\sqrt{h}}\right) - \Phi\left(\frac{\ln(1-\tilde{\lambda}_\alpha) - \sigma^2 h/2}{\sigma\sqrt{h}}\right)\right]$$

Optimal Execution problem for DD: Cost and Risk IV

We are now ready to define our criterion to be minimized. We put together costs and risks in a single criterion

$$\mathbb{E}\left[C(x) + L\, R(x)\right] = \mathbb{E}\left[C(x) + L\bar{\lambda}\int_0^T x(t)(S_t - K)dt\right].$$

In this criterion:

- L is a cost/risk leverage parameter that measures risk aversion in executing the order.
- $L = 0$ we only look at costs, whereas for large L risk dominates costs.
- $\bar{\lambda}$ has a precise endogenous expression depending on whether we are using the VaR or ES risk function

Optimal Execution problem for DD: Cost and Risk V

$$\mathbb{E}\left[C(x) + L\, R(x)\right] = -XS_0 + \gamma X^2/2 + \eta \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\frac{\bar{\lambda}}{\eta}\int_0^T x(t)(S_t - K)dt\right]$$

$$= -XS_0 + \gamma X^2/2 + \eta \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\check{\lambda}\int_0^T x(t)Y_t dt\right]$$

where we have set $\check{\lambda} = \bar{\lambda}/\eta$.
The problem is now finding

$$x^* = \operatorname{arginf}_x \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\frac{\bar{\lambda}}{\eta}\int_0^T x(t)Y_t dt\right].$$

Optimal Execution: Solution for DD I

The problem of finding:

$$x^* = \text{arginf}_x \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\frac{\bar{\lambda}}{\eta}\int_0^T x(t) Y_t dt\right].$$

has been solved by Gatheral and Schied (2011). If Y is our true unaffected underlying stock, the criterion we have is the same as the criterion for a GBM Y, and this has been solved in Gatheral (2011). Indeed, Theorem 1 in Gatheral and Schied (2011) provides the solution and we may substitute back $Y = S - K$ to obtain the following

Optimal Execution: Solution for DD II

Theorem

Optimal execution strategy for a displaced diffusion. *The unique optimal trade execution strategy attaining the infimum above is:*

$$x_t^* = \frac{T-t}{T}\left[X - \frac{\bar{\lambda}}{\eta} L \frac{T}{4} \int_0^t (S_u - K)du\right]$$

Furthermore, the value of the minimization problem is given by

$$\mathbb{E}\left[\int_0^T \dot{x}^*(t)^2 dt + \check{\lambda} L \int_0^T x^*(t)(S_t - K)dt\right]$$

$$= \frac{X^2}{T} + \frac{L\check{\lambda}TX(S_0 - K)}{2} - \frac{(L\check{\lambda})^2}{8\sigma^6}(S_0 - K)^2\left(e^{\sigma^2 T} - 1 - \sigma^2 T - \frac{\sigma^4 T^2}{2}\right)$$

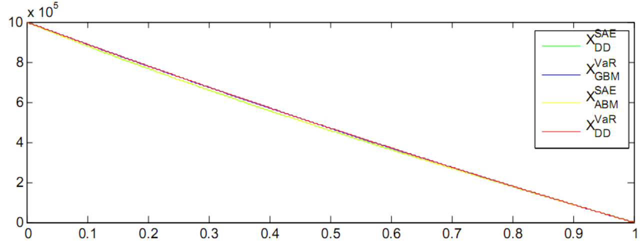
Optimal Execution: Solution for DD III

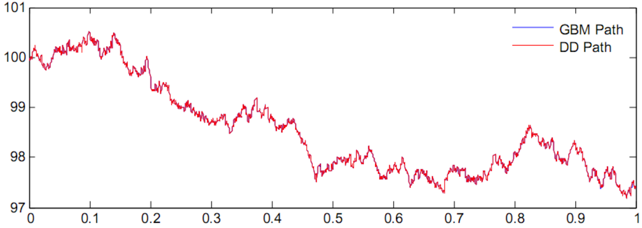
Cost, no risk

$$x_t^* = \frac{T-t}{T}\left[X - \frac{\bar{\lambda}}{\eta} L \frac{T}{4}\int_0^t (S_u - K)du\right]$$

- If $L = 0$ or if instantaneous impact η is very large compared to other parameters, then Risk is not there and we only minimize cost. This leads to
$$x_t^* \approx \frac{T-t}{T} X$$
This is a line in t. Some solutions are very close to this. Note also that this solution is not just adapted, but deterministic.
- Hence the optimal adapted solution with $L = 0$ is also deterministic.

Optimal Execution: Solution for DD IV





Optimal Execution: Solution for DD V

- **Cost and risk**

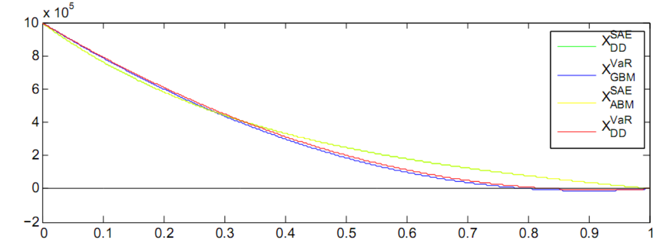
$$x_t^* = \frac{T-t}{T}\left[X - \frac{\bar{\lambda}}{\eta} L \frac{T}{4} \int_0^t (S_u - K) du\right]$$

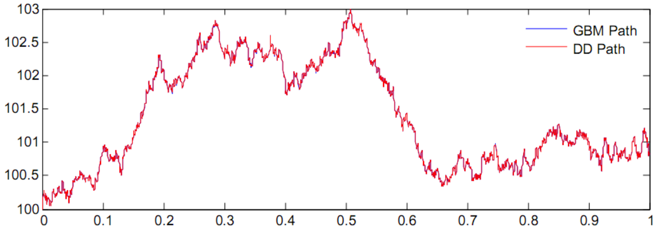
If L is not negligible and η is not extremely large, then a relevant component in the solution is $(1/t)\int_0^t S_u du$ which is the average shares price in time up to t. For example, if S were constantly equal to its initial value S_0 we would have

$$x_t^* = \frac{T-t}{T}\left[X - \frac{\bar{\lambda}}{\eta} L \frac{T}{4} (S_0 - K)\, t\right]$$

that looks like a convex (quadratic) function of t.

Optimal Execution: Solution for DD VI





A different risk criterion: SAE I

SAE. In order to test the robustness of the optimal strategy, introduce an alternative "squared-asset expectation" (SAE) risk criteria

$$R^{SAE}(x) \equiv \lambda \int_0^T x^2(t)\sigma^2 E[S_t^2]dt \quad (R^{AC} \equiv \lambda \int_0^T x^2(t)\sigma^2 S_0^2 dt \text{ for ABM})$$

This goes back towards considering variance as a measure of risk.

λ here is exogenous rather than endogenous.

The optimal execution strategy can be derived by solving the optimisation problem ($g(t) = \mathbb{E}[S_t^2]$)

$$x^* = \text{arginf}_x \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\lambda \int_0^T \sigma^2 x^2(t) g(t) dt\right].$$

A different risk criterion: SAE II

Based on calculus of variations the optimal solution needs to satisfy the ODE:

$$\ddot{x}(t) = k^2 g(t) x(t), \quad k = \sigma\sqrt{L\lambda}$$

with initial and terminal conditions are given by $x(0) = X$ and $x(T) = 0$ respectively. A solution to the boundary value problem above could be found using standard numerical routines. BDiG derive an approximation based on series expansion in the paper.

Important: The SAE criterion makes the optimal adapted solution a **deterministic one**, since the solution of the above ODE is clearly deterministic. This is not surprising given that the key element in the risk criterion, namely $g(t)$, is deterministic with SAE.

Comparing risk criteria I

To compare optimal strategies under SAE risk and VaR/ES, set the exogenous λ in SAE in a way that makes the comparison sensible ("equalizing the λ's").

A possible way is to check what happens for the VWAP solution

$$x_0(t) = X\frac{T-t}{T}$$

and match the risk functions corresponding to this solution.

$$\mathbb{E}_0\left[\int_0^T \check{\lambda}\; x_0(t)\;(S_t - K)dt\right] = \mathbb{E}_0\left[\int_0^T \lambda_{DD}^{SAE}\; x_0^2(t)\;\sigma^2\mathbb{E}_0[S_t^2]dt\right]$$

Comparing risk criteria II

in the DD case, while using the expansion $e^{\sigma^2 t} \approx 1 + \sigma^2 t$, leads to

$$\lambda_{DD}^{SAE} = \frac{\check{\lambda}}{2X\sigma^2} \frac{1}{(S_0 - K)\left(\frac{\sigma^2 T}{12} + \frac{S_0^2}{3(S_0 - K)^2}\right)}.$$

Set also

$$\lambda_{ABM}^{SAE} := \lambda_{DD}^{SAE} \ .$$

The graphs below compare the optimal solutions of the following dynamics/risk criterion combinations:
ABM+SAE, DD+SAE, GBM+VaR and DD+VaR.

(with λ's equalized).

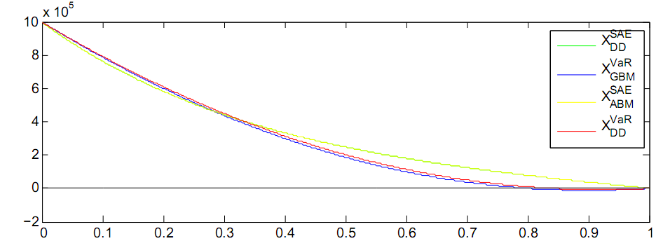
Model Comparison: $S_0 = 100$, $X = 10^6$, $T = 1$day, $\sigma_{1y} = 0.3(\sigma_{1d} = 0.0189)$, $L = 100$

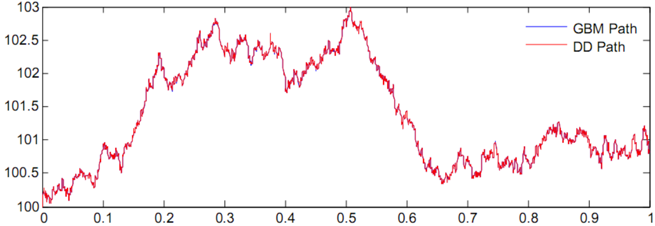
- The time horizon for the execution is set to one day (realistic).
- All models start at the same S_0. The volatility of the asset is $\sigma = 30\%$.
- Absolute vol in ABM is σS_0.
- To ensure that the integrated volatility of the DD and GBM models are roughly of the same order of magnitude we rescale the instantaneous volatility of the DD:

$$\sigma^{DD}(S_0 - K) = \sigma^{GBM} S_0.$$

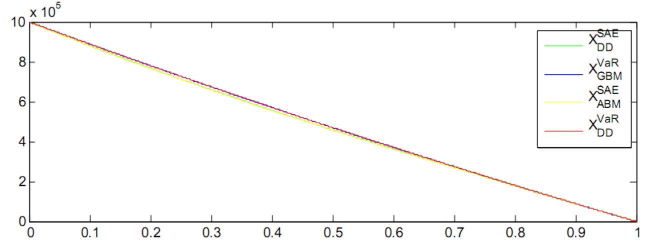
- Most importantly the cost/risk parameter L is relatively low at 100.

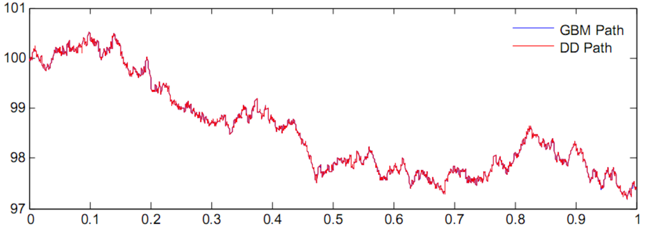
$S_0 = 100$, $X = 10^6$, $T = 1$day, $\sigma_{1d} = 0.0189$, $L = 100$. $K = 5$ in DD. $\eta = 20^{-6}$: impact of instantaneous sale of 1 million units is \$2 per asset (2% of mid price). Faster optimal execution for VaR model vs SAE.



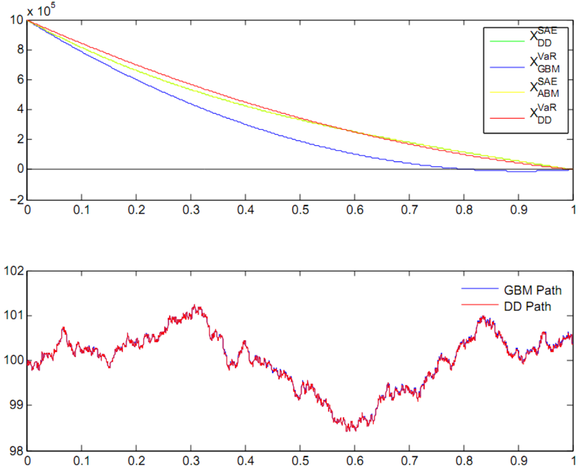


Same parameters, but η is increased $\times$ 10 to $\eta = 20^{-5}$ (relatively illiquid security). Price impact dominates over the risk component and the execution strategy is very close to linear

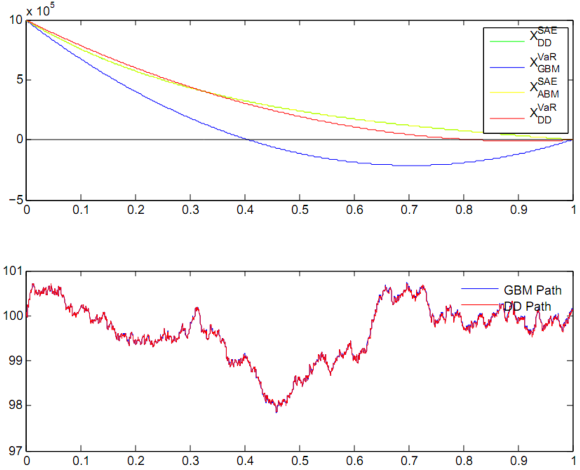




Same parameters but increased DD shift $K = 50$. Optimal policy faster for GBM+VaR compared to DD+VaR. In the 1st case the risk function depends on S_t whereas in the second case it depends on $S_t - K$, which is roughly half (but instantaneous volatility of the DD is rescaled for comparison)



Same parameters but $K = 50$ and η half its previous value, namely 10^{-6}. Then GBM+VaR solution is not monotone as no smooth constraint at zero has been imposed on x_t. Note also that a similar behaviour may occur with the DD+VaR combination.



Conclusions

- The optimal trade execution problem is solved under the Value-at-risk or Expected shortfall risk criteria when the underlying unaffected stock price follows a displaced diffusion model.
- From the examples above, it emerges that for high levels of risk aversion and relatively low impact, the optimal execution may be significantly different for the linear solution.
- Optimal policy may exhibit convexity
- High impact η or low risk aversion L may lead to almost VWAP / linear solutions
- Different combinations of asset dynamics and risk function give rise to different solutions in these cases, although this is partly due to the specific choice of parameters mapping between different models and of the subjective parameter L.
- Investigation with less stylized and more data driven dynamics for S is needed.

Finale

"Essentially, all models are wrong, but some are useful".
Prof. G.E.P Box

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A final note on the exam. Not all the course material has been explained in detail in class. If in doubt on what is examinable, a good strategy is to look at the final exercise set. If a part of the course is not needed to solve the exercise set, then it is not examinable.

Thank you for your attention.

Exam

For the exam, the exercise set should be studied very carefully.

The following slides of the course, in the present set, are examinable. In the following, ”RO” stands for ”Read Once”, and it is meant to be helpful for your general culture, job interviews etc, but not necessary for the exam.