Lecture 5 American Options

An American option is a contract between two parties made at a certain time t such that the buyer of the contract has the right, but not the obligation, to exercise the option at any time τ with $t \le \tau \le T$. If the option is exercised at τ, then the seller pays the buyer an amount $Y(\tau) \ge 0$. For instance, $Y(\tau) = (S(\tau) - c)^+$ for an American call option and $Y(\tau) = (c - S(\tau))^+$ for an American put option based on a single stock. One can identify an American option by its payoff process $Y^A = \{Y(t),\ t = 0, 1, \ldots, T\}$. American options enjoy the additional flexibility — possibility of exercising earlier than T — compared to their European option counterparts. What is the value $V^A(t)$ of an American option?

5.1 A special case: "American = European"

Since the holder of an American option can always choose not to exercise the option until time T, $V^A(t) \ge V(t)$ where $V(t)$ is the time t value of the European option with the payoff $Y = Y(T)$. Nevertheless, there are situations where the two value processes coincide.

Proposition 5.1 *Consider an American option Y^A and the corresponding European option with time T value $Y = Y(T)$. If $V(t) \ge Y(t)$ for all t, then $V(t) = V^A(t)$ for all t, and it is optimal to wait until time T to exercise.*

Proof For the holder of an American option, exercising at t only ends up with payoff $Y(t)$, while selling the corresponding European option (or shorting the portfolio which replicates the European option) would guarantee you a time t payoff $V(t)$. Hence the option should not be exercised at t. Since t is arbitrary, it is optimal to wait until T to decide whether to exercise.

Consider the American call option with $Y(t) = (S(t) - c)^+$ at each t where $c = 2.05$ in the example given in Lecture 2. Proposition 5.1 applies to this case. See Figure 5.1.

Note: "American calls = European calls"

Yes, the quote is really true, i.e. American calls have the same values as their European counterparts in the simple set-up given in this section, thus there should be no earlier exercises. This result will be proved in Section 5.2 following Theorem 5.1. Briefly, it is based on the fact that $\{Y^*(\tau)\}$ is a submartingale.

A stochastic process $X = \{X(t),\ t = 0, 1, \ldots, T\}$ is called a Q-*supermartingale* under a probability measure Q on Ω and with respect to $\mathbb{F}$, if the conditional expectation

$$E_Q\ (X(t) \mid \mathcal{F}_{t-1}) \le X(t-1) \quad \forall\, t = 1, \ldots, T;$$

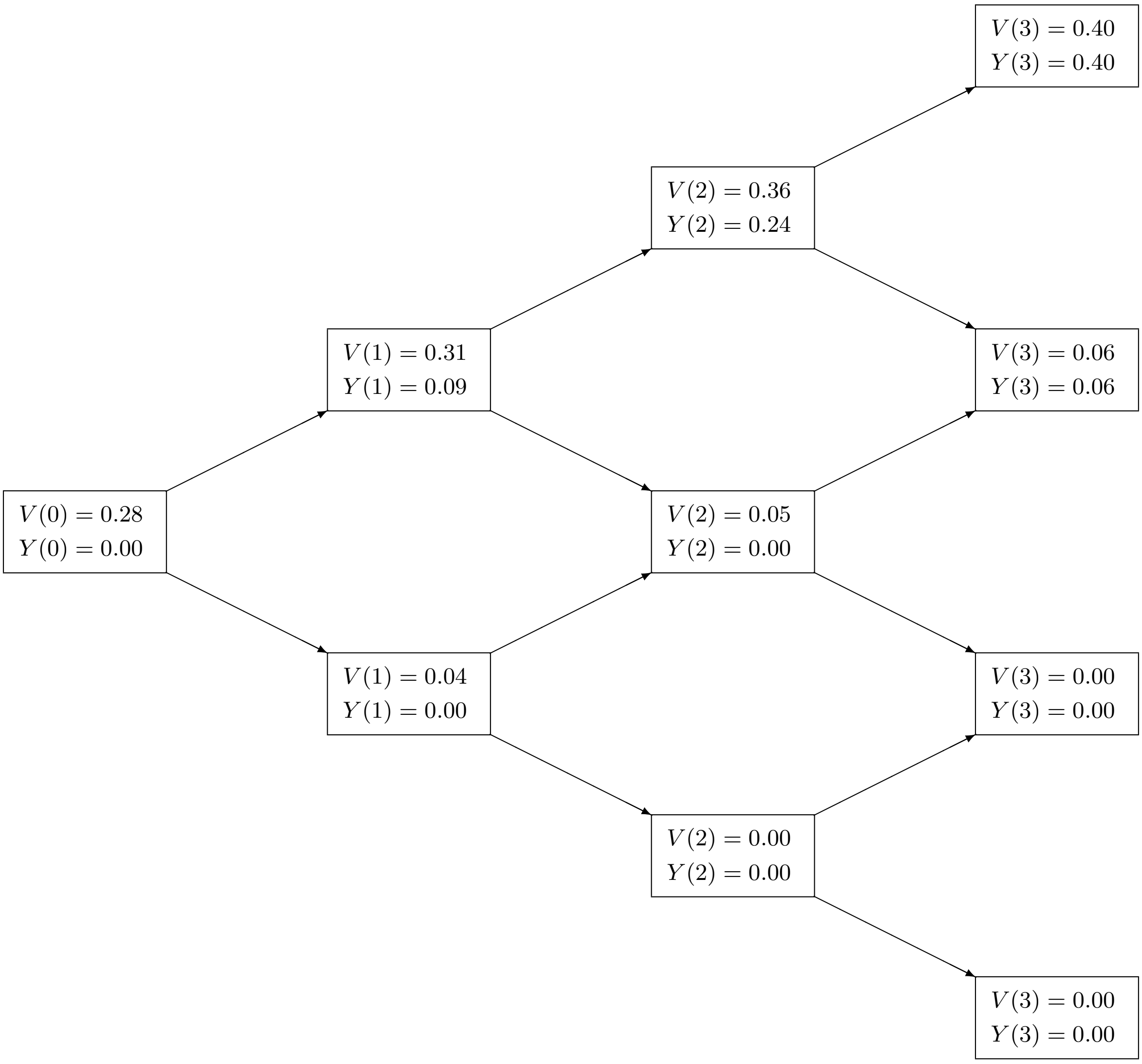

Figure 5.1: Exercise at t or T?

On the other hand, X is called *Q-submartingale* if

$$E_Q\ (X(t) \mid \mathcal{F}_{t-1}) \geq X(t-1) \quad \forall\, t = 1, \ldots, T.$$

All martingales are both supermartingales and submartingales, but not vice versa. Recall that the discounted value process of a European option is a Q-martingale under an EMM Q. It turns out that the discounted value process of an American option is a Q-supermartingale.

5.2 Optimal stopping

Section 5.1 is not a typical case. You may check the American put option in the same example. For instance, when $S(2) = 1.69$, an immediate exercise gives you the payoff $2.05 - 1.69 = 0.36$, compared to the value of the corresponding European put:
$1.06^{-1}\left[\frac{14}{15}\,(2.05 - 1.81) + \frac{1}{15}\,(2.05 - 1.56)\right] = 0.24 < 0.36$. Hence postponing the exercise decision until T is suboptimal (why?).

To study when it is optimal to exercise an American option and evaluate the option, we need to introduce stopping times.

A stopping time τ is a random variable taking values in the set $\{0, 1, \ldots, T; \infty\}$ such that for every $t \leq T$, the event $\{\tau = t\} \in \mathcal{F}_t$, i.e. the information on whether $\{\tau = t\}$ occurs is available at time t. As a simple example, suppose the stock price $S(0) = 2$, then $\tau_1 = \min\{t:\ S(t) > 2.1\}$ is a stopping time, but $\tau_2 = \max\{t:\ S(t) > 2.1\}$ is not a stopping time. We allow stopping times to take the value ∞ in order to represent some events of interest that never occur up to time T.

An American option Y^A is said to be marketable if for every stopping time $\tau \leq T$ the contingent claim $Y(\tau)$ can be replicated. Here is a basic result for American option pricing.

Theorem 5.1 *For an EMM Q, define a stochastic process $Z = \{Z(t),\ t = 0, 1, \ldots, T\}$ iteratively via the dynamic programming equations*

$$\begin{cases} Z(T) &= Y(T) \\ Z(t) &= \max\ \{\ Y(t),\ E_Q\ [Z(t+1)B(t)/B(t+1) \mid \mathcal{F}_t]\ \}, \quad t \leq T-1. \end{cases} \tag{5.1}$$

Then

(a) *For each t,*

$$Z(t) = \max_{\tau}\ E_Q\ [Y(\tau)B(t)/B(\tau) \mid \mathcal{F}_t], \tag{5.2}$$

where the maximum is over all stopping times $t \leq \tau \leq T$.

(b) *The maximum on the RHS of (5.2) is attained by the stopping time*

$$\tau(t) = \min\ \{t' \geq t:\ Z(t') = Y(t')\}. \tag{5.3}$$

(c) *The discounted version Z^* of Z is the smallest Q-supermartingale satisfying*

$$Z(t) \geq Y(t) \quad \forall t. \tag{5.4}$$

(Z is called the Snell envelope of Y^A.)

(d) *For a marketable American option Y^A, its value process is given by*

$$V^A(t) = Z(t) \quad \forall t, \tag{5.5}$$

and the optimal (early) exercise strategy at time t is given by the stopping time $\tau(t)$.

Proof

For (a) and (b), use backward induction. (5.2) and (5.3) clearly hold for $t = T$. Suppose (5.2) holds for t, then

$$\begin{aligned}
& Z(t-1) \\
= \ & \max\ \{Y(t-1),\ E_Q[Z(t)B(t-1)/B(t) \mid \mathcal{F}_{t-1}]\} \\
= \ & \max\ \{Y(t-1),\ E_Q\{\max_{\tau \geq t}\ E_Q[Y(\tau)B(t)/B(\tau) \mid \mathcal{F}_t]\ B(t-1)/B(t) \mid \mathcal{F}_{t-1}\}\} \\
\geq \ & \max\ \{Y(t-1),\ E_Q\{E_Q[Y(\tau)B(t)/B(\tau) \mid \mathcal{F}_t]\ B(t-1)/B(t) \mid \mathcal{F}_{t-1}\}\} \\
= \ & \max\ \{Y(t-1),\ E_Q[Y(\tau)B(t-1)/B(\tau) \mid \mathcal{F}_{t-1}]\}
\end{aligned}$$

for any stopping time $\tau \geq t$. Hence

$$\begin{aligned}
& Z(t-1) \\
\geq \ & \max\ \{Y(t-1),\ \max_{\tau \geq t}\ E_Q[Y(\tau)B(t-1)/B(\tau) \mid \mathcal{F}_{t-1}]\} \\
\geq \ & \max_{\tau \geq t-1}\ E_Q[Y(\tau)B(t-1)/B(\tau) \mid \mathcal{F}_{t-1}] \quad \text{(why?)}.
\end{aligned}$$

On the other hand, assuming (5.3) for t leads to

$$\begin{aligned}
& Z(t-1) \\
= \ & \max\ \{Y(t-1),\ E_Q[Z(t)B(t-1)/B(t) \mid \mathcal{F}_{t-1}]\} \\
= \ & \max\ \{Y(t-1),\ E_Q\{E_Q[Y(\tau(t))B(t)/B(\tau(t)) \mid \mathcal{F}_t]\ B(t-1)/B(t) \mid \mathcal{F}_{t-1}\}\} \\
= \ & \max\ \{Y(t-1),\ E_Q[Y(\tau(t))B(t-1)/B(\tau(t)) \mid \mathcal{F}_{t-1}]\} \\
= \ & E_Q[Y(\tau(t-1))B(t-1)/B(\tau(t-1)) \mid \mathcal{F}_{t-1}] \quad \text{(why?)} \\
\leq \ & \max_{\tau \geq t-1}\ E_Q[Y(\tau)B(t-1)/B(\tau) \mid \mathcal{F}_{t-1}].
\end{aligned}$$

Therefore, (5.2) and (5.3) have been verified for $t-1$.

For (c), it follows from (5.1) that Z^* is a Q-supermartingale and $Z(t) \geq Y(t)$ for all t. Suppose U is another process such that U^* is a Q-supermartingale and $U(t) \geq Y(t)$ for all t. Then

$$U(t-1) \geq \max\{Y(t-1),\ E_Q[U(t)B(t-1)/B(t) \mid \mathcal{F}_{t-1}]\} \quad t = 1, \ldots, T-1. \tag{5.6}$$

Starting from $U(T) \geq Y(T) = Z(T)$ and working backwards iteratively in (5.6) and (5.1) will lead to $U(t) \geq Z(t)$ for all t.

For (d), we use an arbitrage argument (*hedging*) as follows.

Suppose $V^A(t) > Z(t)$. Then one can sell the option for $V^A(t)$ and take a portfolio replicating $Y(\tau(t))$ at the cost $Z(t)$ and invest $V^A(t) - Z(t)$ in the bank account. Later, if the buyer exercises the option at some time $\tau \leq \tau(t)$, you liquidate the portfolio, collect $Z(\tau)$ and pay the buyer $Y(\tau)$. These transactions guarantee you a positive profit. On the other hand, if the buyer does not exercise by $\xi = \tau(t) < T$, then you repeat this process: take a portfolio replicating $Y(\tau(\xi))$ at the cost $E_Q[Z(\xi+1)B(\xi)/B(\xi+1) \mid \mathcal{F}_\xi]$, which is at most $Z(\xi) = Y(\xi)$ by (5.1). As before, if the buyer exercises at some time $\tau \leq \tau(\xi)$, then the value of the portfolio will be enough to cover the payoff $Y(\tau)$. If the buyer does not exercise by $\tau(\xi)$, then you repeat the process once again, etc. The basic fact is that you always have enough money in the portfolio to cover the needed payoff, and you overall profit will be at least $V^A(t) - Z(t) > 0$.

For the opposite case $V^A(t) < Z(t)$, you can reverse the strategy: buy the option for $V^A(t)$, take the negative of the previous portfolio, collect $Z(t)$ and invest the difference $Z(t) - V^A(t)$ in the bank account. Later you exercise the option at time $\tau(t)$ and liquidate the replicating portfolio at the same time. Since $V(\tau(t)) = Y(\tau(t))$, the amount you collect from the option seller is exactly equal to your liability on the portfolio. In the mean time, you have $[Z(t) - V^A(t)]\ B(\tau(t))/B(t) > 0$ in your bank account.

Therefore, there would be an arbitrage opportunity if $V^A(t) \neq Z(t)$. Moreover, (5.3) specifies an optimal exercise strategy for the American option buyer, because other strategies would run the possible risk of exercising when $Z(\tau) > Y(\tau)$ at some time τ. In that case, the buyer would sacrifice the amount $Z(\tau) - Y(\tau) > 0$.

We have thus far completed the proof of Theorem 5.1.

Exercise 5.1 Construct a binomial tree for the American put option values and the optimal exercise times in the example given in Lecture 2.

Now we discuss some general conditions under which an American option is the same as its European counterpart.

Proposition 5.2 *If $\{Y^*(\tau)\}$ is a Q-submartingale for a marketable American option Y^A, then for every $t = 0, 1, \ldots, T$, the optimal exercise strategy is just $\tau(t) = T$, and $V^A(t) = V(t)$, where $V(t)$*

is the time t value of the European option with terminal payoff $Y(T)$.

Proof

$$
\begin{aligned}
& V^A(t) \\
= \; & E_Q[Y(\tau(t))B(t)/B(\tau(t)) \mid \mathcal{F}_t] \\
= \; & \sum_{s=t}^{T} E_Q\left[Y(s)B(t)/B(s)\ I_{\{\tau(t)=s\}} \,\middle|\, \mathcal{F}_t\right] \\
\leq \; & \sum_{s=t}^{T} E_Q\left\{E_Q\left[\ Y(T)B(t)/B(T)\ I_{\{\tau(t)=s\}} \,\middle|\, \mathcal{F}_s\right] \,\middle|\, \mathcal{F}_t\right\} \\
= \; & E_Q[Y(T)B(t)/B(T) \mid \mathcal{F}_t] \\
= \; & V(t).
\end{aligned}
$$

Corollary 5.1 *In the given set-up, there should be no exercise earlier than T for an American call option.*

Proof To check $\{Y^*(t)\}$ is a submartingale, note that for every $t = 1, \ldots, T$,

$$
\begin{aligned}
& E_Q[(S^*(t) - c\ /B(t))^+ \mid \mathcal{F}_{t-1}] \\
\geq \; & E_Q[S^*(t) - c\ /B(t) \mid \mathcal{F}_{t-1}] \\
= \; & S^*(t-1) - c\ E_Q[1/B(t) \mid \mathcal{F}_{t-1}] \\
\geq \; & S^*(t-1) - c\ /B(t-1).
\end{aligned}
$$

Since $E_Q[(S^*(t) - c\ /B(t))^+ \mid \mathcal{F}_{t-1}] \geq 0$, we have $E_Q[(S^*(t) - c\ /B(t))^+ \mid \mathcal{F}_{t-1}] \geq [S^*(t-1) - c\ /B(t-1)]^+$.

Note: It is a crucial part of the proof of Corollary 5.1 that S^* is a Q-martingale, which is not the case in the next section.

5.3 Options on a dividend-paying stock

In a more realistic market, some dividend-paying stocks may issue cash payments, called *dividends*, to shareholders on a periodic basis. A dividend is referred to as the reduction in the stock price on the *ex-dividend date*. There may be several ex-dividend dates in a stock, and possibly many different forms of dividends. For illustration, we consider the binomial tree in Lecture 2 again, where $T = 3$ is the only ex-dividend date and the dividend is issued as a constant *yield* λ of the stock. This means the shareholder will receive a dividend payment at T which amounts to either $\lambda uS(T-1)$ or $\lambda dS(T-1)$ according to the stock fluctuation. In the meantime, the ex-dividend stock price at time T will be either $(1-\lambda)uS(T-1)$ or $(1-\lambda)dS(T-1)$. This corresponds to the traditional assumption that the stock price declines on the ex-dividend date by the dividend amount. One can easily complete this modified binomial tree.

Various options (calls, puts, and others; European or American) on dividend-paying stocks can be priced virtually in the same way as before, except that the exercise payoff is identified as $[(1-\lambda)uS(T-1)-c]^+$ or $[(1-\lambda)dS(T-1)-c]^+$ for calls, etc. In this situation, the methodology using binomial trees enjoys its great flexibility.

This example also shows that on a dividend-paying stock, an American call need not have the same value as its European counterpart. Corollary 5.1 does not hold for American call options on dividend-paying stocks. To see this, consider the inequality

$$\begin{aligned} & (1+r)[S(T-1)-c]^+ \\ > \quad & q\ [(1-\lambda)uS(T-1)-c]^+ \ + \ (1-q)\ [(1-\lambda)dS(T-1)-c]^+, \end{aligned} \tag{5.7}$$

which amounts to $1.06\ (2.29-2.05) > \frac{14}{15}[(1-\lambda)2.45-2.05]^+ + \frac{1}{15}[(1-\lambda)2.11-2.05]^+$ when $S(2)=2.29$, thus $\lambda > 0.05$ leads to exercising at $t=2$.

Another simple form of dividend is "known dollar dividend" (see Hull's book p354).