Short rate simulation using Hull-White model

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Let $r(t)$ be the short rate. The Hull-White (one factor) interest rate model describes $r(t)$ by the following stochastic equation.

$$\begin{aligned} dr &= (\theta(t) - ar)dt + \sigma dW \quad \text{Continuous version} && (1) \\ \delta r &= (\theta(t) - ar)\delta t + \sigma Z\sqrt{\delta t} \quad \text{Discrete version} && (2) \\ r(t+\delta t) &= r(t) + (\theta(t) - ar)\delta t + \sigma Z\sqrt{\delta t} && (3) \end{aligned}$$

where

1. $\theta(t) = \dfrac{dF(0,t)}{dt} + aF(0,t) + \dfrac{\sigma^2}{2a}(1 - e^{-2at})$ and $F(0,t)$ is the initial forward curve.
2. Z is a random variable from the standard normal distribution.
3. a and σ (volatility) are parameters. These are usually estimated from market data.

Note that the relation between the initial zero curve $R(t)$ and initial forward curve $F(0,t)$ is given by (please see [1, Section 5.5])

$$F(0,t) = R + t\frac{dR}{dt} \tag{4}$$

Hence we have the following approximation when δt is small.

$$F(0,t) \approx R(t+\delta t) + t\frac{R(t+\delta t) - R(t)}{\delta t} \tag{5}$$

If we know $R(t)$, then we know $F(0,t)$.

Suppose a and σ, and the zero curve $R(t)$ are given. Then $\theta(t)$ is known. Lets fix a step size δt. We now describe an algorithm to generate $r(t)$ inductively using (3), for $t = 0, \delta t, 2\delta t, 3\delta t, \ldots$.

1. From (3),

$$\begin{aligned} r(t+\delta t) &= r(t) + (\theta(t) - ar)\delta t + \sigma Z\sqrt{\delta t} && (6) \\ &= r(t) + \theta(t)\delta t - ar(t)\delta t + \sigma Z\sqrt{\delta t} && (7) \end{aligned}$$

   We approximate $\theta(t)\delta t$ as follows.

$$\theta(t)\delta t = F(0, t+\delta t) - F(0,t) + aF(0,t)\delta t + \frac{\sigma^2}{2a}(1 - e^{-2at})\delta t \tag{8}$$

   where $F(0,t)$ is computed using (5).

2. Let $r(0) = R(0)$.
3. Draw a standard normal random varaible Z_1. By (7)

$$r(\delta t) = r(0 + \delta t) = r(0) + \theta(0)\delta t - ar(0)\delta t + \sigma Z_1\sqrt{\delta t}$$

4. Draw a standard normal random varaible Z_2. By By (7)

$$r(2\delta t) = r(\delta t + \delta t) = r(\delta t) + \theta(\delta t)\delta t - ar(\delta t)\delta t + \sigma Z_2\sqrt{\delta t}$$

Carry on like this, we get $r(t)$ for $t = 0, \delta t, 2\delta t, 3\delta t, \ldots$.

It is known that $r(t)$ has a normal distrbution with the following mean and variance.

$$\begin{aligned} E(r(t)) &= \frac{\theta}{a} + \left(r(0) - \frac{\theta}{a}\right) e^{-at} \\ Var(r(t)) &= \frac{\sigma^2}{2a}(1 - e^{-2at}) \end{aligned}$$

References

[1] John C. Hull, *Options, Futures and Other Derivatives*, Prentice Hall, Fifth Edition