Counterparty Credit Risk

Kandidatprojekt

by

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1 Introduction

For years, it has been common practise in the financial industry to price derivative portfolios without taking the counterparty credit quality into account. However, the true portfolio value must incorporate the possibility of losses due to default of the counterparty. In recent times this topic gained strong attention, fueled by the bankruptcy of Lehman Brothers in 2008.

The adjustment of the value of a “default-free” portfolio to the possibility of counterparty default is known as credit value adjustment (CVA). The computation of such CVA can be quite complex. One way to simplify the computation is to assume independence between the default probabilities of the counterparty and the exposure of a dealer to the counterparty.

A model for handling such dependence between the default probabilities and the exposure in a computationally feasible way was proposed in [HW12]. In their paper, Hull and White also give a numerical example and report the impact on credit value adjustment when using their model, compared with a model in which default probabilities and exposure are independent. The purpose of this project is to implement their model and replicate the numerical values which they report.

The organization of the paper is as follows: We begin in chapter 2 by giving a short introduction to credit value adjustment, and generalize the [HW12] model description in a way that is inspired from [ZP07]. Moreover we give a very brief introduction to hazard rates from [La98]. In chapter 3, we describe how to implement the numerical example for the model in [HW12], and explain how to calculate the credit value adjustment as well as its sensitivities. The numerical results from this implementation will be reported and discussed. Finally, chapter 4 summarizes our results.

2 Counterparty Credit Risk

Counterparty credit risk is the risk that the counterparty to a financial contract will default prior to the expiration of the contract and will not be able to make all the payments required by the contract. This type of risk can only occur with over-the-counter (OTC) derivatives, as the cash-flows of exchange-traded derivatives are guaranteed by the exchange.

2.1 Basics and Credit Value Adjustment

Suppose a derivative dealer holds a set of n derivatives with one of its counterparties. Let $V_i(t)$ be the value of contract i (to the dealer) at time t, and T be the longest maturity outstanding of these contracts, and

$$W(t) := \sum_{i=1}^{n} V_i(t)$$

be the value of portfolio at time t. The *counterparty-level exposure* is the maximum potential loss that can occur if the counterparty defaults. With no collateralization, the net exposure[1] at time t is

$$E_{NC}(t) := \max\{W(t), 0\}.$$

In order to mitigate this risk, firms typically use collateral agreements. Such an agreement might be done by setting a treshold K, which means that should the net value $W(t)$ move above K, the difference between the portfolio value and the treshold will be posted as collateral $C(t)$. If only the counterparty is required post collateral,[2] then

$$C(t) := \max\{W(t) - K, 0\}. \tag{1}$$

The *default unwind date* is the time when the dealer is able to either replace the transactions he has with the counterparty or unwind the hedges he has for those transactions. If a default occurs in practice, there usually is a *cure period*, which is the period of time prior to the default unwind date during which the counterparty ceases to post collateral and fails to return excess collateral.

The cure period can be thought of as two components. The first component is the period of time that elapses between the counterparty failing to post collateral or return excess collateral and the dealer declaring an early termination. The second

[1] Contracts without netting agreements are also possible. However, we will only consider the case in which there exists a netting agreement between the counterparties during this paper.

[2] If on the other hand only the dealer is required to post collateral, we can simply write $-C(t)$.

is the time that elapses between the dealer declaring an early termination and the default unwind date.

Under this type of collateralization agreement (which we assume for the rest of the paper) and a cure period length of c, the net exposure at a default unwind time t is

$$E_C(t) := \max\{W(t) - C(t-c), 0\}.$$

It is not explicitly stated in [HW12], but for $l < 0$ we set $W(l) = 0$, so from (1) it holds for all $t \in [0, c)$ that

$$C(t-c) \quad = \quad \max\{-K, 0\},$$

which becomes particularly important when the treshold is negative.

Whenever a default occurs, a fraction of the outstanding exposure is usually recovered through bankruptcy procedures. This fraction is referred to as *recovery rate* R, which we assume to be constant.

Denoting the default unwind time by τ, the discounted loss to the dealer is

$$L^* := (1-R)\mathrm{PV}\left(E_C\left(\tau\right)\right)\mathbb{1}_{\{\tau \leq T\}},$$

where PV is the present value operator.

The *Credit Value Adjustment* (CVA) is the expected value of the discounted losses unter the risk-neutral market measure Q :

$$\mathrm{CVA} := \mathbb{E}^Q\left[L^*\right] = (1-R)\int_0^T \mathbb{E}^Q\left[\mathrm{PV}(E_C(t)|\tau = t)\right] d\mathrm{PD}(0,t), \tag{2}$$

where $\mathrm{PD}(s,t)$ is the risk-neutral probability of counterparty default between times s and t.

In the case of independence between the default probabilities PD and the dealer's exposure to the counterparty $\mathrm{PV}(E_C(t))$, equation (2) simplifies to

$$\mathrm{CVA} = (1-R)\int_0^T \mathbb{E}^Q\left[\mathrm{PV}(E_C(t))\right] d\mathrm{PD}(0,t).$$

2.2 Wrong Way Counterparty Credit Risk

In the previous section we assumed independence between the default probabilities of the counterparty and the exposure of the dealer to it.

But consider a scenario in which the the dealer enters a swap with an oil producer where the dealer receives a fixed amount and pays the floating crude oil price. A

low oil price increases the value of the swap and hence the counterparty exposure increases, but also reduces the revenue of the oil producer which in turn tends to increase credit spread as the default scenario gets more likely.
The assumption of independence between those variables does not seem to be a very realistic one. We will therefore relax this assumption, and allow dependence between them.
We say there is *wrong-way risk* to the dealer if an increase in exposure positively correlates with a higher probability of default of the counterparty. A negative correlation will be referred to as *right-way risk*. In order to avoid lengthy texts, we refer to such dependence as wrong-way risk in the remaining paper.
A way of modelling the wrong-way risk default probabilities is by using Cox processes, which are a generalization of Poisson processes.
First recall that an (inhomogeneous) Poisson process $N(t)$ with $N_0 = 0$ and deterministic hazard function $h(t)$ satifies

$$\mathbb{P}[N_t = 0] = \exp\left(- \int_0^t h(s)ds \right).$$

We will use such a Poisson process as a model to simulate defaults of the counterparty. This allows us to write the probability of no default occuring from time 0 up to time t as

$$\text{PS(0,t)} = \exp\left(- \int_0^t h(s)s \right).$$

Equivalently, the probability of a default occuring between 0 and t is

$$\text{PD(0,t)} = 1 - \exp\left(- \int_0^t h(s)s \right). \tag{3}$$

The hazard rate can be interpreted as the survival rates of the counterparty within a short time interval, conditional on no earlier default. In other words, given that the counterparty has survived up to time t and given the history of the portfolio value $W(t)$ up to time t, then

$$PS(t, t + dt) = h(t)dt + o(dt)$$

for some small $dt > 0$.
In order to model dependence on other random variables, the hazard function need to be generalized in a way allows for it to be stochastic. This can achieved by using

Cox processes.
A Cox process is a stochastic process in which the hazard function $h(t)(\omega)$ is stochastic, but in such a way that if we condition on a particular realization ω, the process becomes an inhomogeneous Poisson process with hazard function $h(t)(\omega)$.
The source randomness could be the stochastic interest rate on riskless debt, stock prices or other variables deemed relevant for predicting the likelihood of default. In this paper we assume that the only source of randomness are the portfolio values, i.e.

$$h(t) = f(W(t)). \tag{4}$$

In general, the default probabilities can be obtained from the term structure of credit default swap (CDS) spreads $s(t)$, which are quoted in the market. A standard CDS is a contract where the buyer purchases credit protection of a reference entity from another party, to cover the losses due to a credit event of the reference entity. In our case, this reference entitity is the counterparty. To pay for this protection, the buyer makes a regular stream of payments, depending on the credit spread, to the seller of the CDS. This payment is known as the *premium leg.* Suppose a buyer wants to buy maturity T credit protection. Assume for a moment an interest rate of 0, and that a continuous and constant stream of payments $s(T)$ is made (up to the occurence of a credit event) for credit protection. The expected premium $p_{prem}(T)$ paid is

$$p_{prem}(T) = \mathbb{E}^Q \left[s(T) \int_0^T PS(0,t)dt \right].$$

The payment of the seller in case of a default is known as the *protection leg.* The expected value for this protection $p_{prot}(T)$ is

$$p_{prot}(T) = \mathbb{E}^Q \left[(1-R) \int_0^T h(t)PS(0,t)dt \right].$$

In order to avoid arbitrage, the credit spread $s(T)$ needs to be in such a way that the expected payments of the premium and protection leg are equal, i.e.

$$\mathbb{E}^Q \left[s(T) \int_0^T PS(0,t)dt \right] = \mathbb{E}^Q \left[(1-R) \int_0^T h(t)PS(0,t)dt \right]$$

or equivalently[3]

$$\frac{s(T)}{1-R} = \frac{\mathbb{E}^Q\left[\int_0^T h(t)PS(0,t)dt\right]}{\mathbb{E}^Q\left[\int_0^T PS(0,t)dt\right]}. \tag{5}$$

In the case of constant hazard rates $h(t) \equiv h$, this condition simplifies to

$$\frac{s(T)}{1-R} = h. \tag{6}$$

When hazard rates change deterministically over time, the average hazard rate $\bar{h}$ from 0 to T can be used as an approximation instead in equation (6).
Since credit spreads for different maturites are observable, this gives us the possibility to calculate the hazard rate (up to some uncertainty about the recovery rate) and thereby the default probabilities. In our setup, independence between the hazard rates and the exposure imply deterministic hazard rates. The default probabilities can thus directly be calculated by inserting (6) into (3). One obtains

$$\text{PD(0,T)} \approx 1 - \exp\left(-\frac{s(T)T}{1-R}\right) \tag{7}$$

However in the case of wrong-way/right-way risk, the hazard rates are not deterministic, but stochastic. In this case, the expected value (under the Q-martingale) of the hazard rate default probabilities in equation (3) need to match the default probabilies that are inferred from credit spreads at time T as in equation (7), i.e.

$$\mathbb{E}^Q\left[\exp\left(-\int_0^T h(t)dt\right)\right] = \exp\left(-\frac{s(T)T}{1-R}\right).$$

Note that this equation imposes a condition on the functional form f. Specifically, it needs to hold that

$$\mathbb{E}^Q\left[\exp\left(-\int_0^T f(W(t))dt\right)\right] = \exp\left(-\frac{s(T)T}{1-R}\right). \tag{8}$$

We will later explain how this can be achieved for a specific functional form.

[3] Here one sees that the assumption of a deterministic and constant recovery rate is not as innocent as it initially appears to be.

3 An Example of Wrong Way Risk

3.1 Assumptions

Assume that we want to calculate the CVA of a portfolio that consists of a single one-year forward foreign exchange transaction X with initial exchange rate x_0, delivery exchange rate K_0 and Q-dynamics

$$\begin{aligned} dX(t) &= X_t \alpha_X dt + X_t \sigma_X dW_t^Q \\ dB_d(t) &= r_d B_d dt \\ dB_f(t) &= r_f B_f dt \end{aligned}$$

where r_d and r_f are the domestic respective foreign short rates, B_d and B_f are the corresponding riskless asset prices, W^Q is a Q-Brownian Motion, σ_X is the volatility of the exchange rate and $X(t)$ is the spot exchange rate at time t. The default-free market value of such a forward contract at time t is[4]

$$W(t) = e^{-r_d(1-t)} \mathbb{E}_t^Q [X(1) - K_0],$$

where

$$\mathbb{E}_t^Q [X(1) - K_0] = X(t) e^{(r_d - r_f)(1-t)} - K_0.$$

We assume the functional form of the hazard rates is given by

$$f(W(t)) = \exp\left(a(t) + bW(t)\right), \tag{9}$$

where b is a constant parameter and $a(t)$ is a deterministic function of time. Here the dealer is exposed to wrong-way risk if $b > 0$, whereas a negative value of b implies right-way risk exposure.

Moreover assume that there exists a collateral agreement between the dealer and the counterparty, which requires (only) the counterparty to post collateral if the portfolio value exceeds a treshold K and a cure period length of c. For simplicity we assume a constant spread rate for all maturities, i.e. $s(t) \equiv s$.

3.2 CVA Simulation

In order to calculate the CVA, we simulate the portfolio value at a discrete set of dates. For this, consider an equidistant partition $0 = t_0 < t_1 < ... < t_N = 1$ with

[4]See [Bj09] Proposition 17.2.

$\Delta t := t_1$ and $t_i^* := \frac{t_{i-1}+t_i}{2}$.
A simulation $X^j(t_i^*)$ of the foreign exchange rate $X(t_i^*)$ can iteratively be computed by simulating a sequence of independent standard normal random variables $(Y_i^j)_{(i,j)}$ and then letting

$$\begin{aligned} X^j(t_i^*) &= X^j(t_{i-1}^*)\exp\left((r_d - r_f - 0.5\sigma_X^2)(t_i^* - t_{i-1}^*) + \sigma_X\sqrt{t_i^* - t_{i-1}^*}Y_i^j\right), \qquad i \in \{1, ...N\} \\ X^j(t_0) &= x_0. \end{aligned}$$

By $W^j(t_i^*)$ and $\mathrm{PV}(E_C^j(t_i^*))$ we denote the corresponding simulated values of the portfolio value $W(t_i^*)$ and the discounted exposure $\mathrm{PV}(E_C(t_i^*))$.
The credit value adjustment can then be approximated under the risk-neutral measure by

$$\begin{aligned} CVA &= (1-R)\int_0^1 \mathbb{E}^Q\left[\mathrm{PV}(E_C(t)|\tau = t)\right] d\mathrm{PD}(0,t) \\ &\approx (1-R)\sum_{i=1}^N \mathbb{E}^Q\left[\mathrm{PV}(E_C(t_i^*)|\tau \in [t_{i-1}, t_i])\right]\mathrm{PD}(t_{i-1}, t_i) \\ &= (1-R)\sum_{i=1}^N \mathbb{E}^Q\left[\mathrm{PV}(E_C(t_i^*))(\exp(-\int_0^{t_{i-1}} f(W(t))dt) - \exp(-\int_0^{t_i} f(W(t))dt))\right] \\ &\approx (1-R)\sum_{i=1}^N \frac{1}{n}\sum_{j=1}^n \mathrm{PV}(E_C^j(t_i^*))(\exp(-\Delta t\sum_{l=1}^{i-1}\underbrace{f(W^j(t_l^*))}_{=:h_l^j}) - \exp(-\Delta t\sum_{l=1}^{i} f(W^j(t_l^*)))) \\ &=: CVA^b \end{aligned}$$

where the last approximation holds due to law of large numbers (with n simulations) and a discretization of the integrated hazard rates

$$\int_0^{t_k} f(W(t))dt \approx \sum_{l=1}^k f(W(t_l^*))\Delta t.$$

The superscript b in CVA^b is used to indicate which b from equation (9) we are using for the calculation.
In the case where exposure and default probabilities are independent ($b = 0$), we do not need the use of hazard rates due to equation (7). In this case, the CVA value can directly be calculated by

$$CVA^0 = (1-R)\sum_{i=1}^N \frac{1}{n}\sum_{j=1}^n \mathrm{PV}\left(E_C^j(t_i^*)\right)\left(\exp\left(-\frac{st_{i-1}}{1-R}\right) - \exp\left(-\frac{st}{1-R}\right)\right).$$

Note that the present value operator in our example becomes

$$PV(E_C(t_i^*)) = e^{-r_d t_i^*} E_C(t_i^*).$$

3.2.1 Calibration of the functional form

The computation of h_i^j is a bit trickier, as we have not yet defined $a(t)$ in equation (9). From equation (8) we require that

$$\exp\left(-\frac{st_i}{1-R}\right) = \mathbb{E}^Q\left[\exp\left(-\int_0^{t_i} h(t)dt\right)\right] \approx \frac{1}{n}\sum_{j=1}^{n} \exp\left(-\Delta t \sum_{l=1}^{i} h_l^j\right). \tag{10}$$

The process of adjusting $a(t)$ such that (10) holds is called *calibration* of the functional form. We only need to determine the values for $a(t)$ at time $(t_i^*)_i$, since we will take value $h(t_i^*)$ when discretizing the integrated hazard rates.
The idea for solving this problem is that we can iteratively determine $a(t_i^*)$ by solving (10) numerically. The pseudocode below illustrates how this can be achieved.

Algorithm 1 Calibration of $a(t_i^*)$

Input: $(h_l^j)_{j,l}, (W_i^j)_j \qquad l \in \{1, ..., i-1\}, j \in \{1, ..., n\}$
Output: $a(t_i^*)$

$S \leftarrow \exp\left(-\frac{st_i}{1-R}\right)$

$F(x) \leftarrow \frac{1}{n}\sum_{j=1}^{n} x + b \cdot W_i^j + \Delta t \sum_{l=1}^{i-1} h_l^j$

Find x^* **such that**

$F(x^*) - S = 0$

end Find

$a(t_i^*) \leftarrow x^*$

return $a(t_i^*)$

The algorithm typically requires some manually defined interval for x^* to be solved within. In the numerical analysis we do in the next chapter, the interval $[-20, 10]$ does the job. Changes in the parameters of the model might require a different interval choice.

3.3 CVA Sensitivities

A way of measuring the impact on CVA of a change in one if its parameters, e.g. a small change Δs in the spread, is by using the second order Taylor approximation

$$\Delta CVA(s) = \frac{\partial CVA}{\partial s}\Delta s + \frac{\partial^2 CVA}{\partial s^2}(\Delta s)^2.$$

The change in the FX-rate can be defined analogously.
The sensitivities $\frac{\partial CVA}{\partial x}$ and $\frac{\partial^2 CVA}{\partial x^2}$ are very useful when hedging changes in CVA arising due to changes in the underlying parameter x.
In the following we will discuss how to calculate these sensitivities in our setup.

3.3.1 Spread Sensitivities

For the case in which the default probabilities of the counterparty are independent from the exposure to the counterparty, we can calculate the spread sensitivities exactly. We have

$$\begin{aligned}
\Delta_s^0 &:= \frac{\partial CVA^0}{\partial s} \\
&= \frac{\partial}{\partial s}(1-R)\sum_{i=1}^{N}\frac{1}{n}\sum_{j=1}^{n}\mathrm{PV}\left(E_C^j(t_i^*)\right)\left(\exp\left(-\frac{st_{i-1}}{1-R}\right)-\exp\left(-\frac{st_i}{1-R}\right)\right) \\
&= \sum_{i=1}^{N}\left(t_i\exp\left(-\frac{st_i}{1-R}\right)-t_{i-1}\exp\left(\frac{st_{i-1}}{1-R}\right)\right)\frac{1}{n}\sum_{j=1}^{n}\mathrm{PV}\left(E_C^j(t_i^*)\right)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_s^0 &:= \frac{\partial^2 CVA^0}{\partial s^2} \\
&= \frac{\partial^2}{\partial s^2}(1-R)\sum_{i=1}^{N}\frac{1}{n}\sum_{j=1}^{n}\mathrm{PV}\left(E_C^j(t_i^*)\right)\left(\exp\left(-\frac{st_{i-1}}{1-R}\right)-\exp\left(-\frac{st_i}{1-R}\right)\right) \\
&= \frac{1}{1-R}\sum_{i=1}^{N}\left(t_{i-1}^2\exp\left(-\frac{st_{i-1}}{1-R}\right)-t_i^2\exp\left(-\frac{st_i}{1-R}\right)\right)\frac{1}{n}\sum_{j=1}^{n}\mathrm{PV}\left(E_C^j(t_i^*)\right).
\end{aligned}$$

In order to calculate the spread sensitivities when wrong-way (right-way) risk exists ($b \neq 0$), recall that for any $f \in C^2$ by definition

$$\frac{\partial f}{\partial x}(x_0) = \lim_{\epsilon\to 0}\frac{f(x_0+\epsilon)-f(x_0)}{\epsilon} = \lim_{\epsilon\to 0}\frac{f(x_0+\epsilon)-f(x_0-\epsilon)}{2\epsilon} \tag{11}$$

and that the second partial derivative can be written as

$$\frac{\partial^2 f}{\partial x^2}(x_0)=\lim_{\epsilon\to 0}\frac{f(x_0+\epsilon)-2f(x_0)+f(x_0-\epsilon)}{\epsilon^2}. \tag{12}$$

For small $\epsilon > 0$, we can approximate the wrong-way spread sensitivites (which we denote by Δ_s^b and Γ_s^b) by calculating the wrong way credit value adjustment CVA_{s+}^b for a spread rate for $s+\epsilon$, and then use the approximation

$$\Delta_s^b := \frac{CVA_{s+}^b - CVA^b}{\epsilon}. \tag{13}$$

By letting CVA_{s-}^b be the calculated wrong way credit risk for a spread rate $s-\epsilon$, the Gamma with respect to spread is analogously approximated by

$$\Gamma_s^b := \frac{CVA_{s+}^b - 2CVA^b + CVA_{s-}^b}{\epsilon^2}. \tag{14}$$

Note that can use the same simulations of the portfolio value to calculate CVA_{s+}^b and CVA_{s-}^b as we did for CVA^b - only the hazard rates need to be calculated again.

3.3.2 Exchange Rate Sensitivities

In order to estimate the impact of a change in the exchange rate, we will proceed in a similar manner as when calculating the spread sensitivities. One difference is that in case of $b=0$, we are not able to calculate the sensitivities in a similar fashionable way as we did for the spread sensitivities. We therefore have to rely on using the approximations of equations (11) and (12) for both the independent case as well as the case in which wrong-way (right-way) risk exists.

In order to use these approximations, we reuse the same sampled standard normal random variables Y_i^j which were used to generate the samples for exchange rate $X^j(t_i^*)$ (the simulation j of $X(t_i^*)$ with starting value $X_0 = x_0$). Let $x_0^+ := x_0+\epsilon$ and $X_+^j(t_i^*)$ be the corresponding sampled exchange rate with a starting value $X_+^j(0) = x_0^+$ using Y_i^j, and let $X_-^j(t_i^*)$ be defined analogously. The exchange rate sensitivities can then be approximated by

$$\Delta_x^b := \frac{CVA_{x+}^b - CVA_{x-}^b}{2\epsilon} \tag{15}$$

and

$$\Gamma_x^b := \frac{CVA_{x+}^b - 2CVA^b + CVA_{x-}^b}{\epsilon^2},$$

where CVA_{x+}^b and CVA_{x-}^b are the CVA calculated by using the simulated exchange

rates X_+^j and X_-^j.
The procedure of calculating exchange rate sensitivities is particularly time-consuming when dependence between default probabilities and the portfolio exists, as a change in the portfolio value requires a recalibration of the hazard rates. However, in the case of $b = 0$ the default probabilties are independent of the exposure. This reduces the computation time, as a change in the portfolio values does not affect the default probabilities.
The values for the initial exchange rate x_0 and the spread rate s are usually of two different order of magnitudes, so it is appropriate to use different ϵ in the calculation of the spread sensitivities than the one used in the calculation of the exchange rate sensitivities. We will use the notation ϵ_s for the ϵ in the calculation of the spread rate sensitivities, and ϵ_x for the ϵ in the calculation of the exchange rate sensitivities.

3.4 Numerical results

So far we have described how to calculate the CVA. In this section we provide some numerical estimates for the impact and wrong-way or right-way risk on CVA and its Greeks. For this, consider the example from the previous chapter to buy a foreign exchange currency with principal of \$100 million and the following parameters:

$$\begin{aligned} x_0 &= K_0 = 1.0 \\ r_d &= r_f = 0.05 \\ \sigma_X &= 0.15 \\ s &= 0.0125 \\ R &= 0.4. \end{aligned}$$

For the calculation of the CVA and sensitivities, the Monte Carlo method described in the previous section is being used with $n = 5000$ simulations and $N = 100$ discretization steps. The independent case and the wrong-way or right-way risk case are calculated by using the same simulations of the portfolio values. These values are then used to calculate the ratios

$$\frac{CVA^b}{CVA^0}, \frac{\Delta_s^b}{\Delta_s^0}, \frac{\Gamma_s^b}{\Gamma_s^0}, \frac{\Delta_x^b}{\Delta_x^0}, \frac{\Gamma_x^b}{\Gamma_x^0}$$

per million US-\$, thus measuring the impact of wrong-way (or right-way) risk.
Due to the nature of Monte Carlo simulations, there is some inherent statistical uncertainty in these ratios. Therefore 100 of such ratios are calculated, and the mean value is being reported in the tables. Moreover we provide a lower bound which 95 % of our sampled ratios exceed, as well as an upper bound of which 95 %

fall below.
For the calculation of the sensitivities we use $\epsilon_s = 0.000000015$ (roughly 0.1 ‰ of s) and $\epsilon_x = 0.002$ (0.2 % of x_0) as parameters for ϵ in equations (13) and (15).
The choice of ϵ_s and ϵ_x requires a bit of sensibility. A slightly higher value for ϵ_s yields significantly different mean values for the calculation of the impact on the ratio of Γ_s, albeit reducing the variance. On the other hand, a slightly lower value increases the variance of the sample values of changes in the ratio for Γ_s to the point that these values become not very meaningful. The same holds to a lesser extent also true for changes in the value of ϵ_x with respect to changes in Γ_x.
Four different collateral arrangements are considered: no collateral, a threshold of \$10 million, a threshold of zero, and a negative treshold of \$5 million. For the latter three we assume a cure period c of each 15 days.
Tables 1 to 4 report results for the same cases as in [HW12]: b is 0.03 and the dealer's position on the FX contract is long; b is 0.03 and the dealer's position is short; b is –0.03 and the dealer's position is long; and b is –0.03 and the dealer's position is short.
At first it might seem unnecessary to report that many cases: should not the impact of wrong-way risk be the just the opposite of right-way risk? From a qualitative point of view it is obvious that wrong-way affects the impact differently, but that there also is a quantitatively assymetric difference is not as obvious. After all, the calibration ensures that the expected value of the hazard rates always have the same expected value, regardless of wrong-way or right-way risk.
The reason for the asymmetric impact is, that when calibrating the the hazard rates of e.g. wrong-way risk, even though high values of $W^j(t)$ enter in the functional form exponentially, these values are "factored down" (by a factor of 0.03), whereas they enter linearly with a factor of 1 when multiplying them with the corresponding portfolio value in order to calculate the CVA value. This means that relatively high hazard rate values are multiplied with high values of the exposure, thus amplifying the impact of changes in CVA.
Moreover note the asymmetric impact between being in a long and short position on the forward contract. The differences here are due to the market value of the contract following a lognormal distribution, which is not symmetric. Therefore a dealer who buys the forward contract is affected by wrong-way (or right-way) risk in an asymmetrical way, compared with being the seller of the forward contract.
Numerical results of our simulation are reported in the tables 1 to 4.

Table 1: Impact of wrong-way risk on CVA and sensitivities for a long forward contract to buy 100 million units of a foreign currency in one year.

Impact of b=0.03 in % per $mm on:	No Collateral			$K = 10$ $c = 15$			$K = 0$ $c = 15$			$K = -5$ $c = 15$		
	Lower	Mean	Upper	Lower	Mean	Upper	Lower	Mean	Upper	Lower	Mean	Upper
CVA	53.3	55.2	57.5	39.9	40.9	41.7	35.3	36.2	37.1	54.2	56.5	59.2
[HW12] value		54.8			41.7			37.3			53.5	
Delta wrt FX Rate	31.0	31.8	32.8	13.9	14.5	15.1	9.2	9.6	9.9	38.3	40.1	42.1
[HW12] value		32.0			15.6			12.8			39.3	
Gamma wrt FX	1.9	2.3	2.7	-24.0	-22.0	-20.3	15.0	15.8	16.6	8.9	23.6	62.4
[HW12] value		2.6			-25.4			17.7			-0.7	
Delta wrt Spread	52.7	54.5	56.7	39.6	40.5	41.4	35.0	35.9	36.8	53.8	56.2	58.6
[HW12] value		53.8			41.2			36.8			52.8	
Gamma wrt Spread	123.2	170.5	225.2	83.7	118.8	153.8	77.0	113.5	151.9	88.2	151.4	222.9
[HW12] value		181.8			124.3			122.8			184.3	

Table 2: Impact of wrong-way risk on CVA and sensitivities for a short forward contract to sell 100 million units of a foreign currency in one year.

Impact of b=0.03 in % per $mm on:	No Collateral			$K = 10$ $c = 15$			$K = 0$ $c = 15$			$K = -5$ $c = 15$		
	Lower	Mean	Upper	Lower	Mean	Upper	Lower	Mean	Upper	Lower	Mean	Upper
CVA	39.7	40.8	42.0	32.9	33.6	34.3	26.2	26.7	27.2	25.1	25.7	26.4
[HW12] value		40.5			34.0			27.6			28.9	
Delta wrt FX Rate	15.8	16.2	16.8	6.6	7.2	7.8	-2.9	-2.6	-2.2	-182.0	-132.8	-99.7
[HW12] value		16.2			7.7			-1.9			-341.9	
Gamma wrt FX	-6.7	-6.4	-5.9	-20.4	-19.0	-17.8	14.4	15.1	16.1	-77.4	-12.7	49.8
[HW12] value		-7.0			-21.4			16.4			26.5	
Delta wrt Spread	39.4	40.2	41.2	32.7	33.3	34.1	26.0	26.6	27.1	25.1	25.7	26.4
[HW12] value		40.0			33.7			27.4			28.8	
Gamma wrt Spread	88.6	128.0	169.8	59.0	86.4	115.0	47.5	84.7	112.9	-8.3	58.3	125.9
[HW12] value		114.8			91.0			77.0			70.7	

Table 3: Impact of right-way risk on CVA and sensitivities for a long forward contract to buy 100 million units of a foreign currency in one year.

Impact of b=-0.03 in % per $mm on:	No Collateral			$K=10$ $c=15$			$K=0$ $c=15$			$K=-5$ $c=15$		
	Lower	Mean	Upper	Lower	Mean	Upper	Lower	Mean	Upper	Lower	Mean	Upper
CVA	-38.9	-37.4	-36.8	-32.8	-32.4	-31.9	-29.1	-28.7	-28.3	-37.3	-36.6	-35.9
[HW12] value		-37.5			-32.7			-29.1			-35.7	
Delta wrt FX Rate	-26.9	-26.5	-26.1	-18.6	-18.3	-17.8	-12.5	-12.2	-11.9	-29.2	-28.4	-27.6
[HW12] value		-26.7			-18.8			-14.8			-35.7	
Gamma wrt FX	-7.9	-7.4	-7.0	8.6	10.3	12.2	-15.8	-15.3	-14.1	-27.0	-19.5	-8.3
[HW12] value		-8.2			11.7			-16.0			6.2	
Delta wrt Spread	-37.6	-37.1	-36.6	-32.6	-32.6	-31.7	-28.9	-28.5	-28.1	-37.1	-36.4	-35.8
[HW12] value		-37.2			-32.5			-28.9			-35.6	
Gamma wrt Spread	-50.7	-37.4	-23.7	-46.8	-32.6	-18.4	-39.1	-24.4	-5.3	-68.9	-36.9	-6.6
[HW12] value		-79.2			-74.5			-72.1			-77.3	

Table 4: Impact of right-way risk on CVA and sensitivities for a short forward contract to sell 100 million units of a foreign currency in one year.

Impact of b=-0.03 in % per $mm on:	No Collateral			$K=10$ $c=15$			$K=0$ $c=15$			$K=-5$ $c=15$		
	Lower	Mean	Upper	Lower	Mean	Upper	Lower	Mean	Upper	Lower	Mean	Upper
CVA	-34.5	-33.9	-33.4	-31.2	-30.7	-30.1	-26.0	-25.5	-25.0	-25.9	-25.4	-24.9
[HW12] value		-33.9			-30.8			-25.9			-26.9	
Delta wrt FX Rate	-19.7	-19.2	-18.8	-13.9	-13.4	-12.9	-4.0	-3.6	-3.0	58.2	90.4	133.8
[HW12] value		-19.3			-13.6			-4.9			209.1	
Gamma wrt FX	0.3	0.8	1.2	11.3	12.8	14.5	-15.9	-15.3	-14.5	-106.8	-16.6	93.4
[HW12] value		0.9			14.4			-16.7			-37.5	
Delta wrt Spread	-34.2	-33.6	-33.1	-31.0	-30.5	-29.9	-25.8	-25.4	-24.9	-25.8	-25.3	-24.8
[HW12] value		-33.6			-30.6			-25.7			-26.7	
Gamma wrt Spread	-50.6	-33.2	-17.8	-45.1	-31.5	-15.7	-41.7	-19.2	0.1	-64.2	-22.3	16.7
[HW12] value		-78.8			-75.5			-71.3			-69.0	

In the cases of no collateral agreements as well as $K = 10$ and $K = 0$, our calculated values match the values reported in [HW12] quite well. For $K = -5$ only the CVA value is somewhat similar, whereas the sensitivity ratios in some cases differ substantially. One explanation for this is, that if the impact on the CVA is not reasonably close to the [HW12] values, this difference is amplified when comparing CVA values that are calculated by small changes in s and x_0.
Moreover if we take a look at the boundaries, we see that they are relatively tight whenever our values are close to the [HW12] values. As the boundaries widen, so do the differences between the [HW12] and our values. The biggest variance is clearly the impact on the Γ_s values.
Another point is the actual choice of ϵ_x and ϵ_s. As mentioned, small changes in these parameters have a high impact on the sensitivity ratios, especially the Γ-values. In the table below we illustrate the effect of a small change in ϵ_s on the Γ_s-ratios for the case $-b = 0.03$, $K = 0$ and being the seller of the contract. Besides ϵ_s, all the parameters are the same as above.

Table 5: Impact of right-way risk on the Gamma sensitivity of a long forward contract with respect to spread for small changes in ϵ_s.

Impact of b=-0.03 in % per \$mm on Gamma wrt Spread for:	$K = 0$ $c = 15$		
	Lower	Mean	Upper
$\epsilon_s = 0.000000005$	-157.5	3.4	168.2
$\epsilon_s = 0.00000001$	-32.4	9.4	53.8
$\epsilon_s = 0.000000015$	-39.1	-24.4	-5.3
$\epsilon_s = 0.00000002$	-42.0	-32.4	-21.3
$\epsilon_s = 0.00000005$	-45.0	-43.4	-41.5
[HW12] value		-72.1	

A small increase in the ϵ_s parameter changes the mean value significantly, while reducing the variance. Nevertheless, these values remain quite different than the ones in [HW12].
We suggest that such differences can be explained by rounding errors, rather than by statistical "extremes" in the simulation of the exchange rate. Even though we used an error tolerance of 10^{-30} when calibrating the functional form, small rounding errors can still occur when calculating CVA^b_{s+} and CVA^b_{s-} values in equation (14).

However, when multiplying the difference with

$$\frac{1}{\epsilon_s^2} = \frac{1}{0.000000015^2} \approx 4 \cdot 10^{15},$$

the initially small rounding errors may become significant. An educated guess is, that the rounding error is amplified due to the floating point representation of our numerical values, by which the absolute representation error of “large” values is larger than the absolute representation error for “small” values.

4 Summary

The purpose of this project was to implement the hazard rate model for wrong-way/right-way risk as proposed in [HW12] and replicate the numerical values reported in tables 1 to 4.
The model was implemented and the numerical values were replicated to a large extent. Additionally, intervals for the impact of such risk on CVA and its sensitivities were provided.
In general, our computed values matched the ones in [HW12] quite well when the boundaries were tight. When they were wide, the numerical results from [HW12] were not matched that good and sometimes even outside the boundaries which we calculated.
We found that the impact on the Γ-values with respect to spread were comparatively hard to replicate, partly due to these values having a structurally high variance, regardless of the specific collateral agreement. In such a case, variance reduction methods might be used to give better estimates.
The calculations turned out to be computationally intensive, considering the simplicity of our portfolio consisting of a single forward contract compared to more complex portfolios and the simplicity of the model that was used.

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