

# Credit Risk Modeling and CDS Valuation

Wayne Fang   Jordane Giuly   Xin Qiu  
Xiaoli Yan   Daoyuan Zhou   Yujia Zhu

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## **Abstract**

The goal of this paper is to determine the Incremental Risk Charge (IRC) and the Comprehensive Risk Measure (CRM) of a portfolio consisting of credit derivatives and tranches. More specifically, we implement different methods to calibrate default intensity models, backtest our IRC calculations over historical data, and focus our attention on a basket of Credit Default Swaps (CDS).

# 1 Introduction

The 2008 financial crisis has led to heavy regulatory measures in the credit derivative market. Not only do banks have to determine accurately the mark-to-market value of their portfolios and understand precisely their risks, they also have to provide regulatory institutions detailed reports about their projected risks.

## 1.1 IRC-CRM

With internal models missing major market (and credit) risks during the recent financial crisis, the Basel Committee has suggested new capital charges for trading books.

$$Capital = (m_c + b) * VaR + VaR(specific) \quad (1)$$

In the original market risk capital formula as shown in Equation 1,  $VaR$  is the standard Value-at-Risk measure, based on 99% 10-day loss;  $m_c$  is a model-based multiplier, and  $b$  is an additional factor, depending on VaR backtesting excesses.

$$Capital = (m_c + b) * VaR + (m_s + b) * (StressedVaR) + IRC + \max\{CRM, Floor\} + SC \quad (2)$$

Now, in the new model as shown in Equation 2,  $StressedVaR$  is VaR calibrated to the financial crisis data;  $IRC$  is an incremental charge for default and migration risks for non-securitized products;  $CRM$  is an incremental charge for correlation trading portfolios;  $Floor$  is calculated as  $\alpha = 8\%$  times capital charge for specific risk according to the modified standardized measurement method for the correlation book;  $SC$  is standardized charge on securitization exposures (not covered by CRM), comparable to the banking book.

CRM (Comprehensive Risk Measure) measures the risks of correlation instruments and their hedges (including CDS), but without re-securitization positions, while IRC (Incremental Risk Charge) measures the risks of flow products bonds and CDS (if not part of CRM). On the other hand, the simulated risks in CRM include default and migration and all price risks (multiple defaults, credit spread volatility, volatility of implied correlations, basis risks, and recovery rate), while those in IRC include credit rating migrations and default.

## 1.2 Literature Review

Merton (1974) proposes a simple model of the firm that provides a way of relating credit risk to the capital structure of the firm. Merton's (1974) model has always been a popular method to assess credit risk. In this model, it is assumed that a company has a zero-coupon debt that will become due at a promised time  $T$ . The equity of the company is a call option on the assets of the company with maturity  $T$  and a strike price equal to the promised debt payment. If the value of its assets is less than the debt repayment at time  $T$ , the company defaults. Merton's model requires the current value of the company's assets, the volatility of the company's assets, the outstanding debt, and the debt maturity as inputs. The model can be used to estimate either the risk-neutral probability that the company will default or the credit spread on the debt. Merton's copula approach is however problematic because first of all, the choice of copula and the parameters of the chosen copula are usually difficult to interpret, and results are very sensitive to the chosen copula family and parameters; secondly, as argued in Duffie [2004], the standard copula approach does not offer a stochastic model for correlated credit spreads, which is important for realism and indeed necessary for valuing options on CDOs and CDS portfolios. Duffie and Garleanu (2001) address the risk analysis and valuation of collateralized debt obligations (CDOs). They illustrate the effects of correlation for the market valuation in a simple jump-diffusion setting for correlated default intensities. They adopt a pre-intensity model, where each obligor's default time has some pre-intensity process. In the multi-issuer model, they hold constant the default-risk model of each underlying obligor and find that the mean-reversion rate, diffusive volatility, and mean jump size are common to the underlying pair of independent basic affine processes.

Though this intensity model proves to be very useful in single-name CDS markets, it's often argued that intensity-based models are inappropriate for portfolio credit risk modeling due to its limitation to generate correlations among single names.

Mortensen (2006) introduces a semi-analytical valuation method in a multi-variate intensity-based model, which shows that the intensity models can be both tractable and able to generate realistic correlations. Default intensities are modeled as correlated affine jump-diffusions decomposed into common and idiosyncratic parts. The model is similar to the model proposed by Duffie and Garleanu (2001) but offers two extensions. First, a more flexi-

ble specification of the default intensities is proposed, allowing credit quality and correlation to be chosen independently. Second, heterogeneous default probabilities are allowed in the semi-analytical solution, whereas the analytical method of Duffie and Garleanu (2001) requires homogeneity. This is a very important improvement since in practice single-name CDS spreads often vary by several hundred basis points within the benchmark credit indices.

### 1.3 Roadmap

This paper presents the calculation of the Incremental Risk Charge and Comprehensive Risk Measure of a portfolio consisting of several CDS contracts. We consider a 1-year time horizon for the determination of our IRC, and focus our attention on the default risk of the single names.

The first part of this report introduces the CDS contract and the default intensity model. In the second part, we show different methods for the calibration of a single-name intensity. The third part presents our IRC calculation for several quantiles with backtesting results. Finally, the fourth part introduces methods to calibrate the default intensities of a portfolio of CDS, taking into account correlation between the single names.

We use the following CDS contracts: Ally Financial Inc. (ALLY), Bank of America (BOA), Citigroup (CITI), and Seagate Technology PLC (STX). The words in parentheses are how we refer to these specific CDS contracts throughout this paper. From these four contracts, we use the CDS spreads quoted on a daily basis from June 2008 to June 2009. We choose these contracts because they have defaulted before around the same time period, during which the global economy went through a significant crisis. It would thus be interesting to observe their behavior.

## 2 The Credit Default Swap

### 2.1 Overview of a CDS

Credit default swaps (CDS) transfer the credit risk of a reference entity from one party to another. A basic CDS contract involves two parties that agree to a contract that terminates at either maturity or credit event, whichever occurs earlier. The latter involves the default of the reference entity that is unable to meet its obligations on the loan. To receive default insurance, the protection buyer makes a regular stream of payments, the premium leg, to the protection seller until the earlier of maturity and credit event. If a credit event occurs before maturity, the protection seller pays the protection buyer the default leg, which is the difference between face value and the recovery value of the reference entity. The basic structure of a CDS contract is depicted in Figure 1.

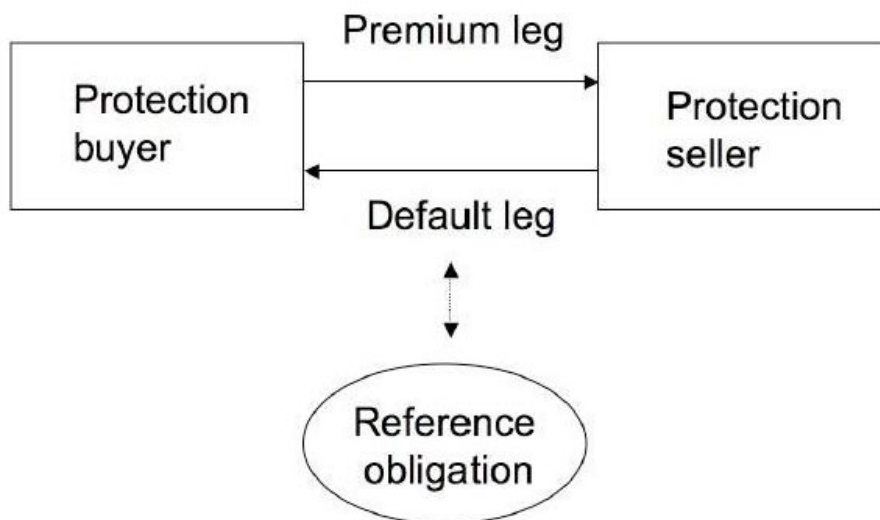


Figure 1: This figure shows the mechanics of credit default swaps.

## 2.2 Example

Suppose a protection buyer purchases a 5-year protection on a company with CDS spread of 300bp and face value of protection of \$10 million. The protection buyer thus makes quarterly payments equal to  $\$10 \text{ million} \times 0.03 \times 0.25 = \$75,000$ . Assume that after a while the reference entity suffers a credit event with a recovery price of \$45 per \$100 of face value. The cash flow due to this event can be described as follows:

- The protection buyer is compensated by the protection seller for the loss on the face value of the asset. This is equal to  $\$10 \text{ million} \times (100\% - 45\%) = \$5.5 \text{ million}$ .
- The protection buyer pays the accrued premium from the previous premium payment date to the time of the credit event. For example, if the credit event happens after a month, then the protection buyer pays  $\$10 \text{ million} \times 0.03 \times 1/12 = \$18,750$  for the amount of premium accrued.

## 2.3 Mark-To-Market Value

A CDS can be valued by using a term structure of default swap spreads, a recovery rate, and a model. Consider an investor who buys a 5-year protection on a company at a default swap spread of 60bp and then wants to value the position one year later. On that date the 4-year credit default swap is quoted in the market as 170bp. Given this example, the current value of the position can be calculated by subtracting the expected present value of the 4-year premium leg at 60bp from the current market value of the remaining 4-year protection. This would give the mark-to-market (MTM) value. Since the mark-to-market value of a new default swap is zero, this implies that the expected present value of the 4-year premium leg at 170bp is equal to the current market value of the remaining 4-year protection. Using this fact, the MTM can be calculated by subtracting the expected present value of the 4-year premium leg at 60bp from the expected present value of the 4-year premium leg at 170bp. Letting DV01 be the expected present value of 1bp paid on the premium leg until default or maturity, which one comes first, the MTM is thus  $[(170\text{bp} \times \text{DV01}) - (60\text{bp} \times \text{DV01})] = 110\text{bp} \times \text{DV01}$ .

In general, the present value of a position initially traded at time  $t_0$  with spread  $S(t_0, t_N)$ , maturity  $t_N$ , and which has been offset at valuation time  $t_v$

with a position traded at spread  $S(t_V, t_N)$  can be calculated as the following for a long protection position:

$$MTM(t_V, t_N) = [S(t_V, t_N) - S(t_0, t_N)]DV01(t_V, t_N)$$

The calculation of DV01 requires a model involving survival probabilities because the riskiness of the premium payment depends on the survival probability of the reference entity. This will be described in the next sections.

## 2.4 Spread

Assume that a CDS has annual swap spread  $S$ , premium payment dates  $(t_m)$ , and maturity  $T$ . Let  $R_t(T)$  be the pre-default value of a unit recovery payment at  $T^1 \leq T$ ,  $C_m$  be the day count fraction for period  $m$ , and  $V_t(T)$  be the risky DV01, the pre-default value of a stream of premium payments until  $\min(T^1, T)$  plus any accruals. The protection seller pays the default loss  $\ell^1$  at  $T^1 \leq T$ . This has pre-default value  $D_t = \ell R_t(T)$ . The protection buyer pays the swap spread  $SC_m$  at each  $t_m < T^1$  plus any accruals (assuming 0 points upfront) as depicted in Figure 2. This has pre-default value  $P_t(S) = SV_t(T)$ .

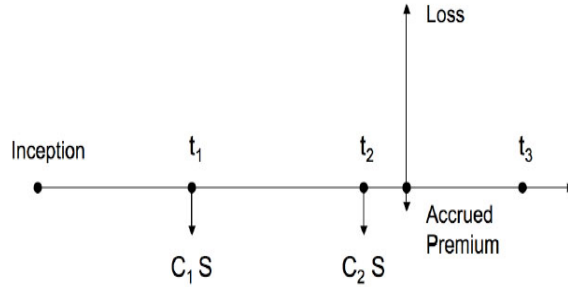


Figure 2: This figure shows the cash flows to the protection buyer.

The fair spread  $S$  sets the values of the default and the premium legs equal to each other. Thus, its value at inception date  $t$  is the solution  $S = S_t(T)$  to the equation  $D_t = P_t(S)$ :

$$S_t(T) = \ell R_t(T) / V_t(T) \tag{3}$$



### 3 Market Calibration

Now that the CDS spread is clearly defined, we put our attention on pricing a CDS and calibrating the underlying model to the actual data.

Let us consider a CDS contract over a single name. In the rest of our paper, our risk analysis will cover the default risk only. Therefore, this single name has a probability of defaulting that evolves over time.

We first introduce the default intensity that is an unobservable process, which guides the default probability estimation. Then, we present our work of the calibration of this process.

#### 3.1 Default Intensity and Survival Probability

The default intensity process can be defined as follows: It is the instantaneous probability that the single-name CDS will default between  $t$  and  $t + dt$ , conditioning on the fact that it has not defaulted before. Thus, if we call  $\tau$  the default time of our reference entity, we have:

$$\lambda_t = \mathbb{P}(t \leq \tau < t + dt | \tau > t)$$

where  $\lambda_t$  is a stochastic process. The survival probability until time  $s$ , given that the name has not defaulted before  $t$  can be expressed based on this default intensity:

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \mathbb{E} \left[ \exp \left( - \int_t^s \lambda_u du \right) \right] \quad (4)$$

In the rest of our report, we will use a square-root process (also called CIR process) to model the default intensity of a single name. Therefore, the dynamics of the default probability process is given by the following stochastic differential equation:

$$d\lambda_t = \kappa(c - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t$$

where  $W_t$  is a standard Brownian motion.

Using a CIR process for the default intensity presents several advantages. First of all, the process is almost surely positive if the Feller condition is

satisfied over the parameters:

$$2\kappa c \geq \sigma^2$$

The intensity indeed has to be positive.

Then, under a CIR process, the expression of the survival probability presented in Equation 4 has a closed form:

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \exp(a(t, s) + b(t, s)\lambda_t) \quad (5)$$

where, for  $\gamma = \frac{1}{2}\sqrt{\kappa^2 + 2\sigma^2}$ , we have:

$$b(t, s) = \frac{-\sinh(\gamma(s-t))}{\gamma \cosh(\gamma(s-t)) + 0.5\kappa \sinh(\gamma(s-t))}$$

$$a(t, s) = \frac{2\kappa c}{\sigma^2} \log \frac{\gamma \exp(0.5\kappa(s-t))}{\gamma \cosh(\gamma(s-t)) + 0.5\kappa \sinh(\gamma(s-t))}$$

## 3.2 Calibration On the Term Structure

Now that the model for a single name is set up, we need to calibrate the parameters  $\theta = (\kappa, c, \sigma, \lambda_0)$  of the CIR default intensity process. Because this process is not observable on the market, we need to calibrate it directly over some data. However, CDS spreads are quoted on the market for different maturities.

### 3.2.1 Expected Values for the Legs

As we have seen, Equation 3 links the CDS spread with the expected values of the default and survival legs. Furthermore, we have the following expressions to determine these expected values:

$$R_t(T) = e^{-r(T-t)}\mathbb{P}(\tau \leq T | \mathcal{F}_t) + r \int_t^T e^{-r(s-t)}\mathbb{P}(\tau \leq s | \mathcal{F}_t)ds$$

$$V_t(T) = \int_t^T e^{-r(s-t)}\mathbb{P}(\tau > s | \mathcal{F}_t)ds$$

Thus, because  $\mathbb{P}(\tau \leq T | \mathcal{F}_t)$  has a closed form expression, depending on the parameters  $\theta$  only, we can write  $R$  and  $V$  as functions of  $\theta$  as well:

$$R_t(T) = R_t(T; \theta)$$

$$V_t(T) = V_t(T; \theta)$$

### 3.2.2 Least Squares Estimation

For a given single name, we retrieve from Bloomberg the term structure of the CDS spreads, i.e. the quotes  $S(T_i)$  where  $T_i$  is equal to 1 year, 3 years, 5 years and 10 years. We then use a least squares method to determine the optimal set of parameters  $\theta$ :

$$\arg \min_{\theta} \sum_i [S(T_i) V_0(T_i; \theta) - \ell R_0(T_i; \theta)]^2 \quad (6)$$

### 3.2.3 Results

The blue curves in Figure 3 show the results of this calibration method over the term structure.

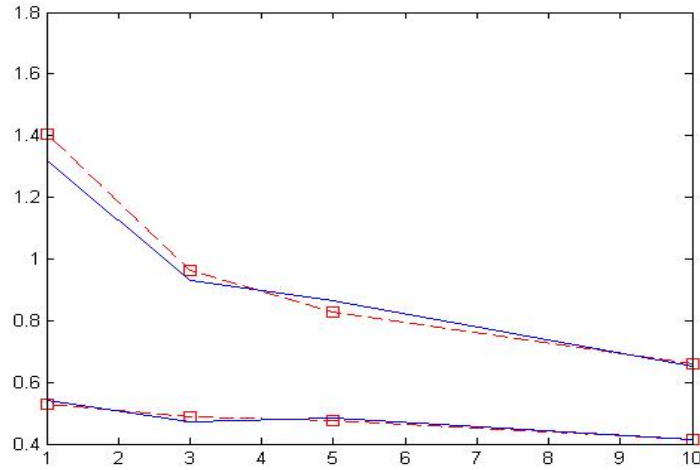


Figure 3: This figure shows the blue curves as the fitted CDS spreads with the calibrated parameters and the red curves as the actual CDS spreads. The top two sets of curves are from the 1-year ALLY CDS contract while the bottom two sets of curves are from the 1-year BOA CDS contract.

### 3.3 Calibration Over the Time Series

The calibration over the term structure has the advantage to be computationally fast. However, it is hard to capture correlation between two single names with only their term structure. We thus introduce a maximum likelihood estimation over the spread time series to calibrate the intensity.

#### 3.3.1 Objective

Given a default intensity model with parameters  $\theta$ , we have risk neutral pricing formulas that link the intensity  $\lambda$ , the parameters  $\theta$ , and the spread  $S$  with the following relationship:

$$S = f(\lambda; \theta)$$

Indeed, in the expression for the legs  $R$  and  $V$ , the unique dependence on the model comes from the survival probability. Due to the closed form presented in Equation 5, this probability can be expressed as a function of  $\lambda$  and  $\theta$ . We thus want to determine the  $\theta$  that maximizes the actual probability that the observed spreads come from the model driven by the  $\theta$ .

#### 3.3.2 Maximum Likelihood Function

The actual measure density of the spread observations  $S = S_i : i = 0, 1, \dots, n$  satisfies the following based on a Markov-chain assumption:

$$\begin{aligned} h_S(s_0, \dots, s_n | \theta) &= h(s_n | s_{n-1}, \dots, s_0, \theta) \dots h(s_1 | s_0, \theta) h(s_0 | \theta) \\ &= h(s_n | s_{n-1}, \theta) \dots h(s_1 | s_0, \theta) h(s_0 | \theta) \end{aligned}$$

Therefore, the log-likelihood function of the observed spreads  $S$  is

$$\begin{aligned} L(\theta; S) &= \log[h_S(S_0, \dots, S_n | \theta)] \\ &\approx \sum_{i=1}^n \log[h(S_i | S_{i-1}, \theta)] \end{aligned}$$

We do not have any expression for  $h_S$ , thus we introduce a change of measure, to get back to the default intensity process:

$$\begin{aligned} h(s|S_{i-1}, \theta) &= g(f^{-1}(s; \theta)|f^{-1}(S_{i-1}; \theta), \theta) \partial_s f^{-1}(s) \\ &= \frac{g(f^{-1}(s; \theta)|f^{-1}(S_{i-1}; \theta), \theta)}{f'(f^{-1}(s; \theta); \theta)} \end{aligned}$$

where  $g(\cdot|\lambda_{i-1}, \theta)$  is the actual measure density of  $\lambda_i$  given  $\lambda_{i-1}$

Finally, because  $\lambda$  is a CIR process, we have a closed form expression for the function  $g$ :

$$g(x|\lambda_{i-1}, \theta) = de^{-v+dx} \left( \frac{dx}{v} \right)^{\frac{q}{2}} I_q(2\sqrt{vdx})$$

where  $I_q$  is the modified Bessel function of the first kind of order  $q$ ,

$$\begin{aligned} d &= \frac{2\tilde{\kappa}}{\sigma^2(1 - e^{-\tilde{\kappa}\Delta t})} \\ v &= de^{-\tilde{\kappa}\Delta t} \lambda_{i-1} \\ q &= \frac{2\tilde{\kappa}\tilde{c}}{\sigma^2} - 1 \end{aligned}$$

and  $\Delta t$  is the time between two observations.

Here,  $\tilde{\kappa}$  and  $\tilde{c}$  are the actual parameters of the default intensity. They are linked to the risk neutral parameters by the relations:  $\tilde{\kappa} = \kappa - \eta$  and  $\tilde{c} = \frac{c}{1-\eta}$ , where  $\eta$  is a new parameter to estimate.

### 3.3.3 Approximation and Numerical Results

In order to reduce computational cost, we make an approximation to calculate  $f^{-1}$ . Near  $\lambda = 0$ , we approximate  $f(\lambda; \theta)$  by  $f'(0)\lambda + f(0)$ , and for  $\lambda$  far from 0, we use a linear approximation of  $f$ . Thus,  $f$  is approximated by two straight lines, and  $f^{-1}(s; \theta)$  can be determined without solving numerically an equation of the type  $f(\lambda; \theta) = 0$ .

Figure 4 shows the results of this calibration over the time series through the use of MLE.

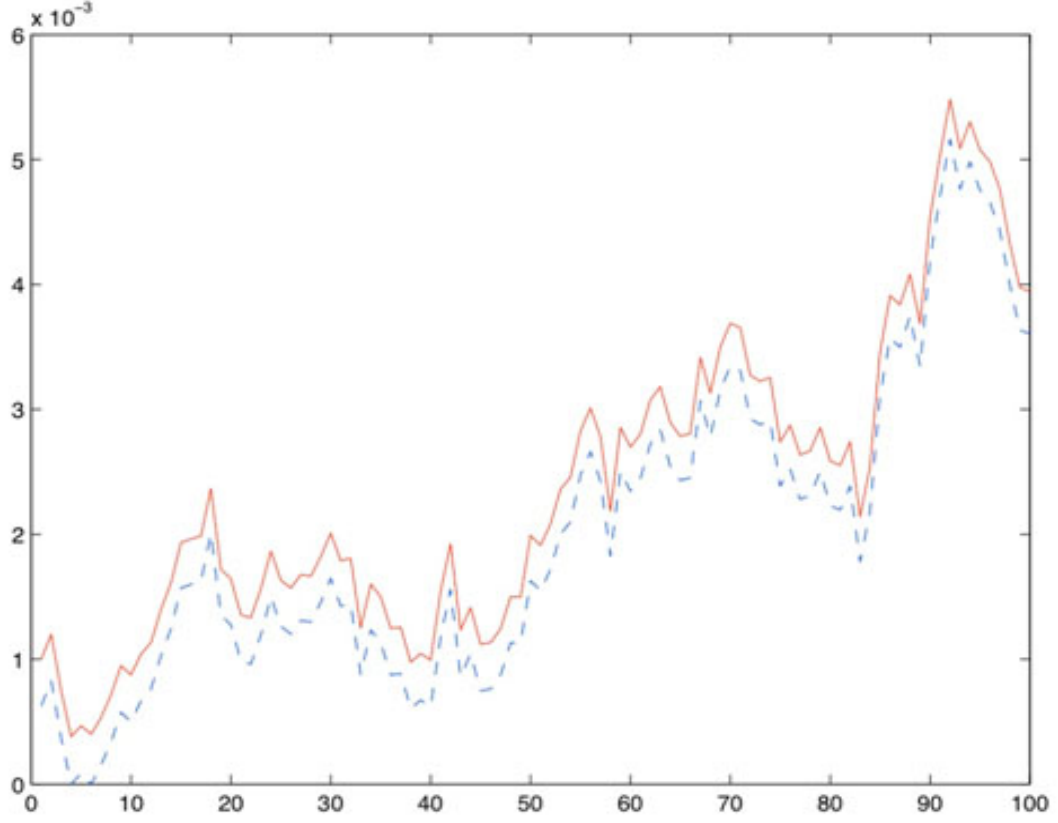


Figure 4: This figure shows the MLE calibration. The red line is the target default intensity time series. The blue line is obtained by applying  $f(., \hat{\theta})$  to the spread time series.

Given a set of parameters  $\theta_0$ , we simulate a path for the default intensity, using a discretization scheme for the SDE. Then, we derive the spread time series by applying  $f(., \theta_0)$  to the default intensity time series. Finally, starting from different  $\theta$ s, we run the MLE algorithm, and come up with the optimal parameters  $\hat{\theta}$ .

## 4 IRC Calculation and Backtest Results

We presented in the previous section two ways to calibrate our default intensity model over real data. In this section, we first show how we calculate the IRC for a single name, at a given date, for a one-year horizon time. To determine the IRC of a given contract, we first need to compute a price distribution for each given date, at the horizon time. To do so, we simulate various paths for the intensity, derive a default time from the path, and calculate the contract price. Once the IRC is determined, we then present several ways to backtest our IRC calculations.

### 4.1 Default Time

Given a path for the default intensity process, the first default time of the underlying can be easily derived. Indeed, the integration from time 0 to  $\tau$ , the default time, of the default intensity process follows an exponential law with parameter 1:

$$S^1 = \int_0^\tau \lambda_s ds$$

Figure 5 shows the simulation of several default times.  $S^k$  as shown on the vertical axis are the arrival times of a standard Poisson process. These arrival times are exponentially distributed with parameter 1. Default times map  $S^k$  to  $T^k$  based on the path of the default square-root intensity model with the calibrated parameters.

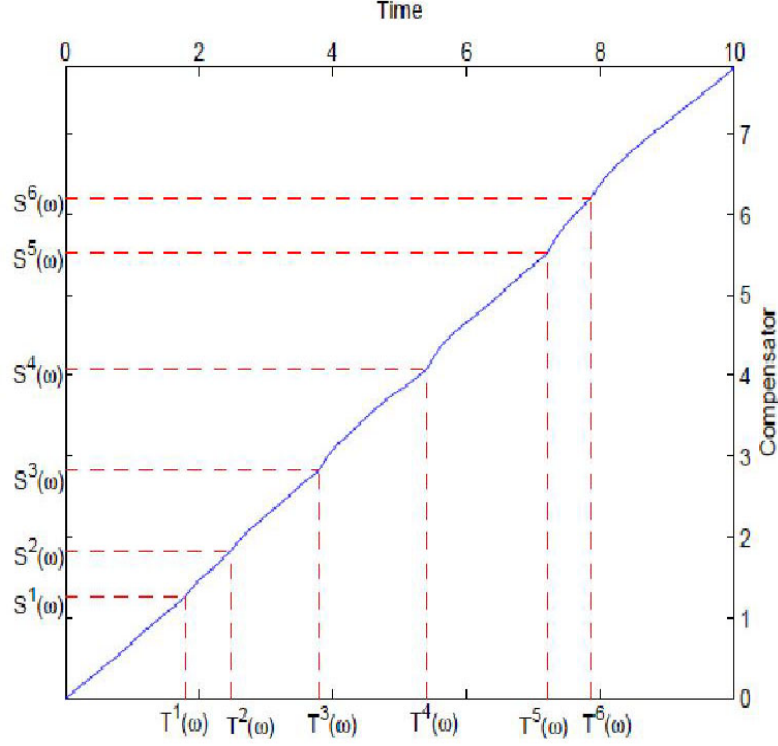


Figure 5: This figure shows consecutive default times.

## 4.2 Value of Payments

Given a path for our default intensity, and a default time  $\tau$ , two possibilities can occur:

- If  $\tau < H$ : The reference entity defaults before the horizon  $H$ . In that case, a protection seller gets the payoff

$$D_H(T) = S_0(T) \int_0^\tau e^{r(H-s)} ds - l e^{r(H-\tau)}$$



- If  $\tau > H$ : The protection seller gets the payoff

$$S_H(T) = S_0(T) \int_0^H e^{r(H-s)} ds - (S_H(T) - S_0(T)) DV01_H(T)$$

Therefore, the cumulative payments (prices) to the seller of a CDS contract at  $H$  is

$$P_H(T) = \mathbb{1}\{\tau < H\} D_H(T) + \mathbb{1}\{\tau > H\} S_H(T)$$

### 4.3 Incremental Risk Charge

The default time is simulated 1000 times for each of the one hundred days (daily quotes) of a contract, which gives the distribution of cumulative payments (prices) to the seller exactly one year later. The 1-year IRC for this price distribution is calculated by using the smallest number  $V$  such that the probability of a loss greater than  $V$  is no more than the loss tolerance. The 1-year IRC is determined for one hundred days of each single contract, with four contracts in total.

An example of this price distribution is shown in Figure 6. In this figure, the horizontal axis is the price while the vertical axis is the frequency of the price based on the simulation. The red CDS contract (1-year ALLY) has a steeper term structure with a higher probability of default, but to compensate for this risk, the spread is higher. On the other hand, the blue CDS contract (1-year BOA) has a flatter term structure with a lower probability of default and thus lower spread.

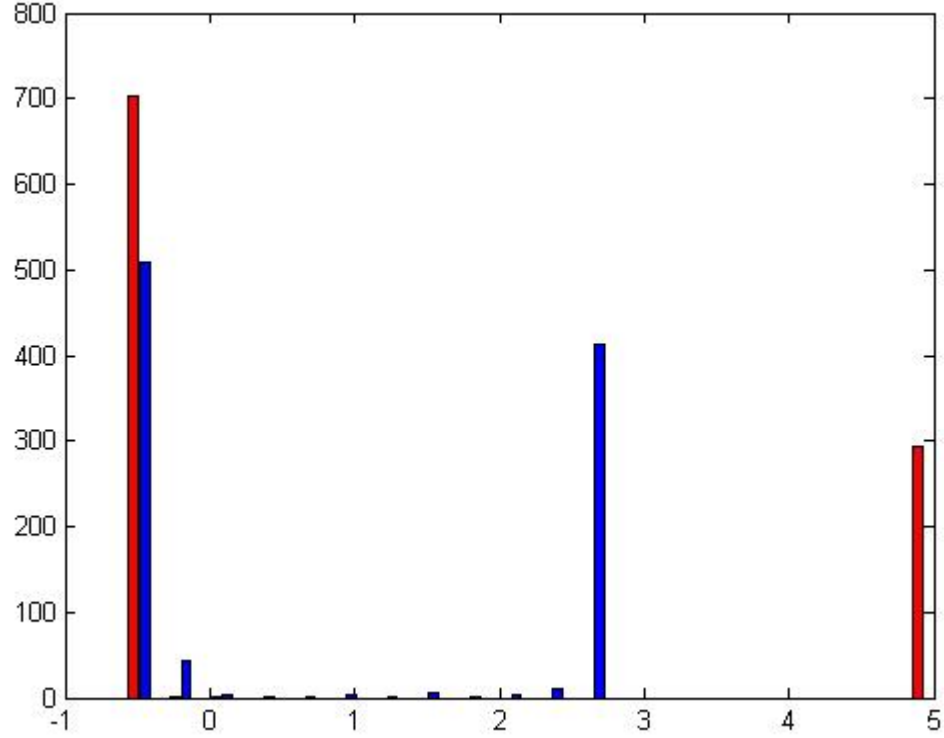


Figure 6: This figure shows the price distribution for two different CDS contracts from the protection seller's perspective. The red represents the 1-year ALLY CDS contract while the blue represents the 1-year BOA CDS contract.

#### 4.4 Backtesting

The model should be backtested appropriately to evaluate the quality of the estimates. This involves a statistical procedure that compares the actual profits and losses to the corresponding IRC estimates. For example, if a daily IRC has a confidence level of 99%, we expect an exception to occur once every 100 days on average. This type of test that looks at the frequency of exceptions over a specified time interval is known as unconditional coverage. However, a good backtest not only looks at this, but also takes into account when the exceptions occur. More specifically, a good IRC model not only produces the correct amount of exceptions, but also produces exceptions that

are evenly spread over time so that they are independent of each other. This type of test that looks at the clustering of exceptions over a specified time interval is known as conditional coverage. For unconditional coverage, we use the Kupiec POF Test (proportion of failures). For conditional coverage, we use the Christoffersen Interval Forecast Test.

#### 4.4.1 Kupiec POF Test

The Kupiec test looks for the consistency between the number of exceptions and the confidence level. Under the null hypothesis that the model is correct, the Kupiec test tells us that the probability of getting  $x$  exceptions from  $T$  observations follows a binomial distribution:

$$f(x|T, p) = \binom{T}{x} p^x (1 - p)^{T-x}$$

where  $p$  is the probability of an exception for a given confidence level. The null hypothesis for the Kupiec test is

$$H_0 : p = \hat{p} = \frac{x}{T}$$

The Kupiec test is conducted as a likelihood-ratio (LR) test with the following test statistic:

$$LR_{POF} = -2 \ln \left( \frac{(1 - p)^{T-x} p^x}{[1 - (\frac{x}{T})]^{T-x} (\frac{x}{T})^x} \right)$$

The null hypothesis is rejected and the model deemed inaccurate if the value of the  $LR_{POF}$  statistic exceeds the critical value of the  $\chi^2$  distribution with one degree of freedom.

#### 4.4.2 Christoffersen Interval Forecast Test

Christoffersen extends the Kupiec test to include a separate statistic for independence of exceptions. The test defines an indicator variable  $I$  that gets a value of 1 if IRC is exceeded and a value of 0 otherwise. Then it defines  $n_{ij}$  as the number of days when condition  $j$  occurred assuming that condition  $i$  occurred on the previous day. For example, if  $I_{t-1} = 1$  and  $I_t = 0$  then  $n_{10}$  gets incremented. Furthermore, let  $\pi_i$  be the probability of observing an exception conditional on state  $i$  on the previous day:

$$\pi_0 = \frac{n_{01}}{n_{00} + n_{01}}, \quad \pi_1 = \frac{n_{11}}{n_{10} + n_{11}}, \quad \pi = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}}$$

The Christoffersen test for independence of exceptions is conducted as a likelihood-ratio (LR) test with the following test statistic:

$$LR_{ind} = -2\ln \left( \frac{(1 - \pi)^{n_{00}+n_{10}} \pi^{n_{01}+n_{11}}}{(1 - \pi_0)^{n_{00}} \pi_0^{n_{01}} (1 - \pi_1)^{n_{10}} \pi_1^{n_{11}}} \right)$$

This independence statistic can be combined with the Kupiec statistic to get a joint test that evaluates the correct failure rate and independence of exceptions (conditional coverage):

$$LR_{cc} = LR_{POF} + LR_{ind}$$

The null hypothesis is rejected and the model deemed inaccurate if the value of the  $LR_{cc}$  statistic exceeds the critical value of the  $\chi^2$  distribution with two degrees of freedom. If the Christoffersen test fails, the reason may be inaccurate coverage, clustered exceptions, or even both. The failure can be pinpointed by calculating  $LR_{POF}$  and  $LR_{ind}$  separately and using the  $\chi^2$  distribution with one degree of freedom as the critical value for both statistics.

#### 4.4.3 Results

The results of the backtests are shown in Figure 7. Since our data do not have many violations (some contracts do not have any violations at all for certain IRCs), many of the test statistics end up being NaN in MATLAB. In regards to all the test statistics that have numerical values, the  $LR_{POF}$  exceeds the critical value of the  $\chi^2$  distribution with one degree of freedom and the  $LR_{CC}$  exceeds the critical value of the  $\chi^2$  distribution with two degrees of freedom. Thus, all the backtest outcomes are rejected, and the model is deemed inaccurate.

	Success Rate	LR <sub>Pof</sub>	LR <sub>Ind</sub>	LR <sub>Cc</sub>	Critical Value $\chi^2(1)$	Critical Value $\chi^2(2)$	Test Outcome
ALLY 99% IRC	1.0000	NaN	NaN	NaN	6.64	9.21	Reject
ALLY 95% IRC	1.0000	NaN	NaN	NaN	3.84	5.99	Reject
ALLY 90% IRC	1.0000	NaN	NaN	NaN	2.71	4.61	Reject
ALLY 85% IRC	1.0000	NaN	NaN	NaN	2.07	3.79	Reject
BOA 99% IRC	1.0000	NaN	NaN	NaN	6.64	9.21	Reject
BOA 95% IRC	1.0000	NaN	NaN	NaN	3.84	5.99	Reject
BOA 90% IRC	1.0000	NaN	NaN	NaN	2.71	4.61	Reject
BOA 85% IRC	1.0000	NaN	NaN	NaN	2.07	3.79	Reject
CITI 99% IRC	1.0000	NaN	NaN	NaN	6.64	9.21	Reject
CITI 95% IRC	1.0000	NaN	NaN	NaN	3.84	5.99	Reject
CITI 90% IRC	1.0000	NaN	NaN	NaN	2.71	4.61	Reject
CITI 85% IRC	0.9950	111.7703	NaN	NaN	2.07	3.79	Reject
STX 99% IRC	1.0000	NaN	NaN	NaN	6.64	9.21	Reject
STX 95% IRC	1.0000	NaN	NaN	NaN	3.84	5.99	Reject
STX 90% IRC	0.9900	75.0650	NaN	NaN	2.71	4.61	Reject
STX 85% IRC	0.9400	31.7020	128.4193	160.1213	2.07	3.79	Reject

Figure 7: This figure shows the backtest results.

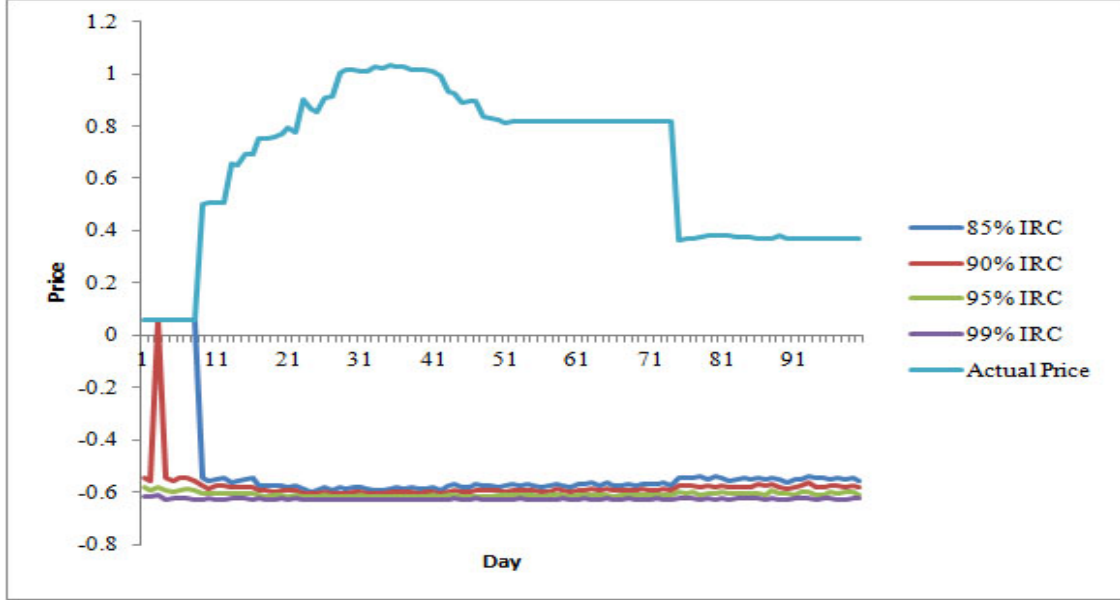


Figure 8: This figure shows the 1-year STX CDS contract with its 1-year IRCs and actual prices used for backtesting.

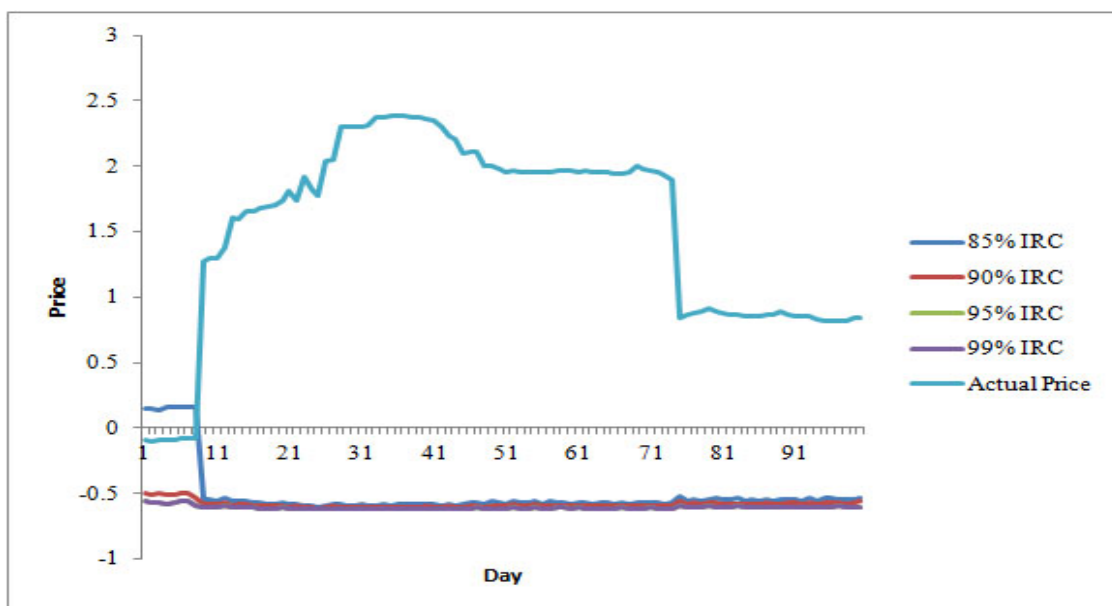


Figure 9: This figure shows the 3-year STX CDS contract with its 1-year IRCs and actual prices used for backtesting.

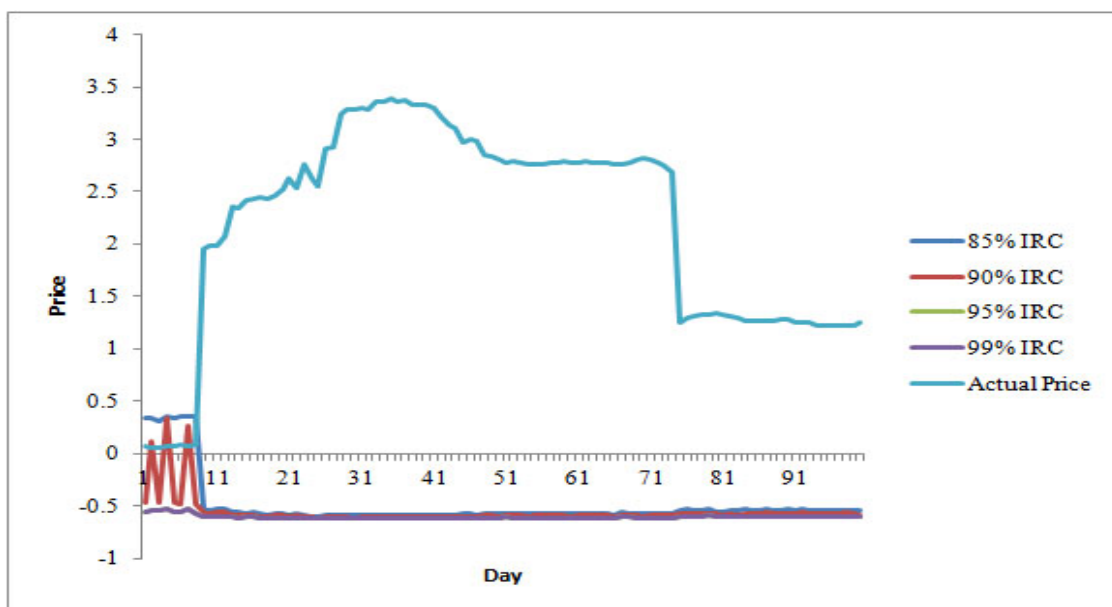


Figure 10: This figure shows the 5-year STX CDS contract with its 1-year IRCs and actual prices used for backtesting.

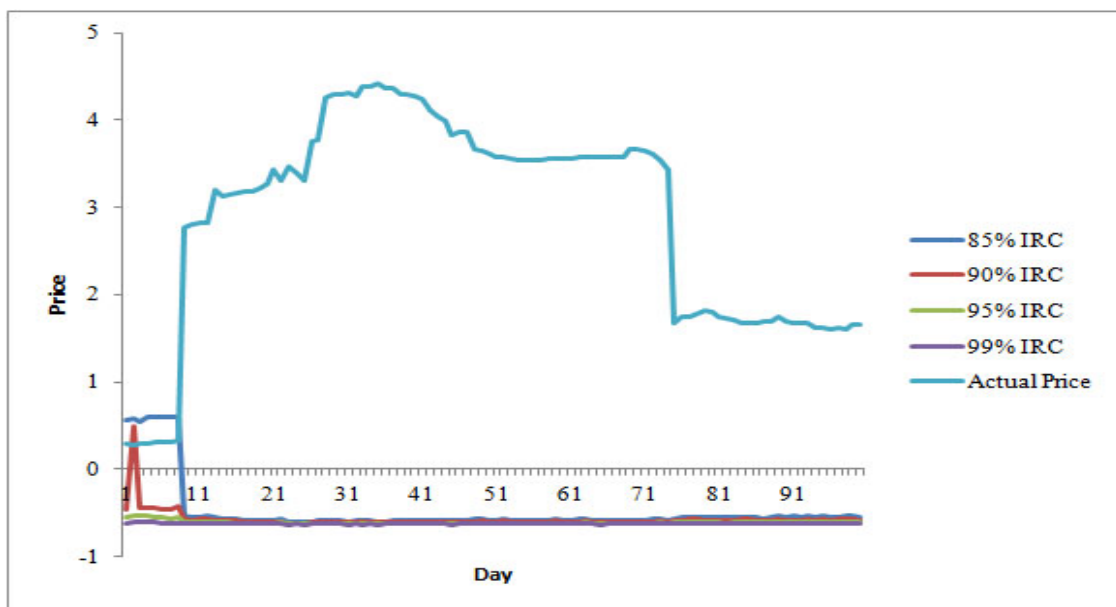


Figure 11: This figure shows the 10-year STX CDS contract with its 1-year IRCs and actual prices used for backtesting.

## 5 Portfolio of Several CDS contracts

Now that we are able to backtest and simulate our IRC on a single CDS contract, we will focus our attention on a basket of CDS.

Let us consider a portfolio of 20 to 50 names, i.e. the names that compose the index iTraxx 50, and hold a basket of 5-year CDS on those names. Each CDS has its own default intensity. We could calibrate those intensities with the previous method, i.e. on the term structure of each contract. However, if we do so, we do not capture any correlation between the names. Indeed, the fact that a name has defaulted should impact the intensities of the other names constituting the portfolio.

### 5.1 The Common Factor Model

We now adopt a common factor model, where each intensity is the sum of a common factor and a specific process, both CIR processes.

$$\lambda_i = \alpha_i X_c + X_i \quad (7)$$

where  $i = 1 \dots n$ ,  $X_c$  is the common factor, and  $X_i$  is the specific process.

Let us call by  $\theta_c$  and  $\theta_i$  the parameters of  $X_c$  and  $X_i$ . Because we want to keep a CIR dynamic for the default intensity  $\lambda_i$ , we need to impose the following relations between the parameters:

$$\alpha_i \kappa_c = \kappa_i$$

$$\alpha_i \sigma_c = \sigma_i$$

This model has an important advantage because we still have a closed form for the survival probability of each name.

### 5.2 MLE on Two Single Names

The next question is how to calibrate this model. Term structure calibration does not enable us to capture correlation between the names and the market. Thus, we need to work on a time series analysis and implement maximum likelihood estimator.



Let us consider  $n$  names. Name  $i$  has the observed spread time series  $(S_t^i)_t$ , and the default intensity  $\lambda_i$ . The CIR process  $\lambda_i$  has the following parameters:  $(\alpha_i \kappa_c, c_c + c_i, \alpha_i \sigma_c, \eta_i)$ , and the new log-likelihood function that we are trying to minimize is the following:

$$L(\theta; S) = \sum_{i=1}^n \sum_t \log[h(S_t^i | S_{t-1}^i; \theta_c, \theta_i)] \quad (8)$$

We manage to implement a generic function that can take an arbitrary number of spread time series, and return the maximum likelihood parameters for the corresponding intensities. Figure 12 exemplifies the results of this function for two names. The red curves are the targeted intensities. The blue and green curves are the results of the MLE on both names, using different starting points. The two black curves show the separate MLE of these single names, using the function presented in part 3. Our function is robust and consistent with the individual MLE in part 3.

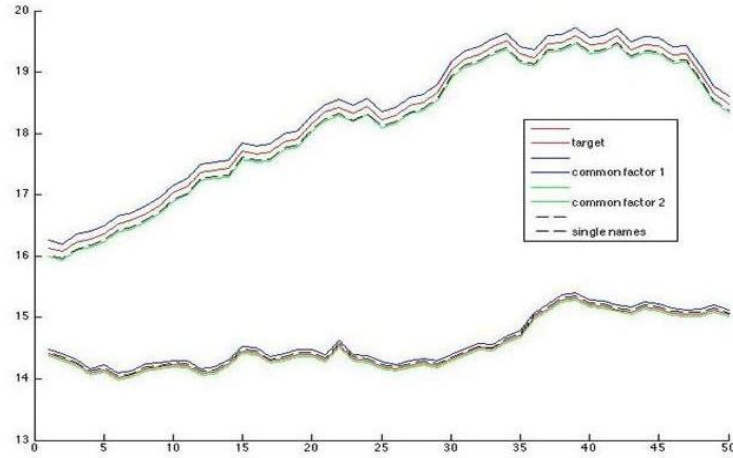


Figure 12: This figure shows the MLE of two single names with common factor model.

### 5.3 Index Calibration

Here we cite the semi-analytical approach adopted by Allan Mortensen (2006). Here we assume that the common factor follows

$$dY(t) = \kappa_Y[\theta_Y - Y(t)]dt + \sigma_Y\sqrt{Y(t)}dW_Y(t) + dJ_Y(t)$$

and the idiosyncratic components follow

$$dX_i(t) = \kappa_i[\theta_i - X_i(t)]dt + \sigma_i\sqrt{X_i(t)}dW_i(t) + dJ_i(t)$$

In short, we shall denote these as

$$Y \sim AJD(Y(0), \kappa_Y, \theta_Y, \sigma_Y, l_Y, \mu_Y)$$

$$X_i \sim AJD(X_i(0), \kappa_i, \theta_i, \sigma_i, l_i, \mu_i)$$

Note that a scaled AJD is an AJD with unchanged jump intensity but scaled jump size. The drift and diffusion parameters are scaled as usual in the CIR case, and thus

$$a_i Y \sim AJD(a_i Y(0), \kappa_Y, a_i \theta_Y, \sqrt{a_i} \sigma_Y, l_Y, a_i \mu_Y)$$

With this model specification, we can move on to calibrate the index tranches and each single name in the pool with a semi-analytical approach. The general approach is as follows:

1. Fix the parameters for common factors first, and for a hypothetical average portfolio with  $a_{avg} = 1$ , calibrate the initial level,  $Y(0) + X_{avg}(0)$ , and the mean-reversion level,  $\theta_Y + \theta_{avg}$ , to fit the 1-year and 5-year spreads of the average CDS curve.
2. For each underlying name, calibrate  $X_i(0)$  and  $\theta_i$  to fit the name-specific 1-year and 5-year CDS spreads.
3. With all the parameters, obtain the joint default intensity of the pool, based on which we can obtain the theoretical tranche spreads given initial parameters for common factor.
4. Adjust those parameters to fit the theoretical tranche spreads to the market quotes.

The major difficulty involved in this procedure is the time cost. However, this is indeed a very clear model to capture the correlations among the single-name CDS that make up the index.

## 6 Conclusion

In this project, we manage to apply a stochastic model to the default intensity in order to calculate the IRC of single name CDS contracts. In the credit derivative pricing, the default intensity is the process that drives the survival probabilities of the underlying asset, and leads to risk neutral pricing. Using a CIR model presents the advantage of capturing some market volatility and also some correlation since a common factor is introduced.

The first task is to calibrate such a model to observable data, the spread quotations. We achieve this part by computing two different methods: a least squares method over the term structure and a maximum likelihood estimation over the time series. These two methods prove to be robust, but very time consuming.

Once the models are calibrated, we calculate the IRC for single name CDS contracts, at a one-year horizon time. Due to a high computation cost and a lack of default events in the collected data, our backtesting fails to validate our models. However, we are convinced that further tests on a bigger dataset would bring satisfactory results.

We finally present an extension to this topic by applying our calibration methods over a portfolio of CDS contracts. Whereas the MLE can be easily generalized, further studies would be required to include tranche quotations into the calibration.

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