TITLE OF THESIS IN CAPITAL LETTERS

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APPROVED BY:

Title and Name of Supervisor (Thesis Supervisor)	....
Title and Name of Examiner	....
Title and Name of Examiner	....

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ABSTRACT

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Bond portfolio management and interest rate risk quantification is an important field of practice in finance. In practice there exist different types of bonds are issued from a variety of sources including the treasury banks, municipalities and corporations. In this thesis only government bonds, i.e. zero-coupon bonds that are assumed to be non-defaultable. To analyse bond dynamics, a governing interest rate model should be used. In this thesis two popular interest rate models, Vasicek Model and LIBOR Market Model, are analyzed in a practical framework for the final aim of using for bond portfolio management problem. For each model stochastic dynamics, parameter estimation, bond pricing, and interest rate simulation are introduced.

In the last part of this thesis Markowitz's Modern Portfolio Theory is introduced and adapted for bond portfolio selection problem. The traditional mean / variance problem and its modified version, mean / VaR problem, are solved for both model. We use the term structure models to estimate expected returns, return variances, covariances and value-at-risk of different bonds. For all implementations we use R, which is a programming language for statistical computing.

ÖZET

BÜYÜK HARFLERLE TEZİN TÜRKÇE ADI

Türkçe tez özetini buraya yazınız.

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LIST OF SYMBOLS/ABBREVIATIONS

a_{ij}	Description of a_{ij}
α	Description of α
DA	Description of abbreviation

1. INTRODUCTION

Bond portfolio management is an important field of study in finance for bonds constitute a considerable part of banks' and institutional investors' asset allocations. Even though bonds (especially government bonds) are typically and erroneously considered as riskless income securities, investors entertain several risks factors that may even cause them to lose large amounts of money. These risk factors can be classified in three major groups: interest rate risk, credit risk and liquidity risk. Interest rate risk is a market risk, which is a potential value loss caused by the volatility in interest rates. For a bond holder, bond value and interest rates are inversely proportional; a rise in rates will decrease the price of bond, and vice versa. Credit risk, more specifically credit default risk, refers to the loss of creditor when the obligator cannot make the payments required by the contract. Credit risk has always been an important aspect of corporate bond management but it was not taken into consideration for government bonds until the late-2000s recession when many countries have faced sovereign risk. Liquidity risk arises when there is a difficulty of selling the security in the market and a potential value loss occurs due to lack of cash. Although credit and liquidity risks are topics that should be treated carefully within bond portfolio management, in this thesis we will be solely interested in interest rate risk, and there is no harm in doing this since these risk factors are commonly investigated independently by financial institutions.

As a common practice, bond portfolio managers use interest rate immunization strategies against interest rate risk. First immunization strategies that are based on Macaulay's duration definition are introduced by Samuelson in 1945 [1] and Redington in 1952 [2]. Then in 1971 Fisher and Weil [3] developed the traditional theory of immunization that is still used as one of the main risk management strategies. Essentially duration is a one-dimensional risk measure whereas the value of a bond portfolio depends on the whole term structure movement consisting of different rates with different maturities, hence a multi-dimensional variable. Also duration comes with compelling assumptions that shifts in yield curve are parallel and small. Thence traditional immunization does not allow proper risk management for bond portfolios. However, the above-mentioned assumptions can be relaxed. Large shifts can be expounded by the introduction of convexity term and non-parallel yield curve changes can be obtained by the introduction of term structure models.

In this thesis we investigate the application of modern portfolio theory as a framework for bond portfolio management. Modern portfolio theory, introduced by Markowitz in 1952 [4], is developed for stock portfolio management that aims to allocate assets by minimizing risk per unit of expected risk premium in a mean-variance framework. Although modern portfolio theory is considered as the main tool for solving portfolio selection problems, it is scarcely adapted and used for bond portfolio management. The first reason for the delayed effort for bond portfolio application is related to the historical fact that interest rates were not as volatile as they have been over the past decades; a portfolio approach was considered redundant. Second, there were technical problems in implementation: large number of parameters are required to setup the problem and parameters cannot be simply estimated by historical estimation that relies on stationary moments assumption. These problems became beatable by the introduction of term structure models for interest rates. Term structure models are ideal candidates to implement for bond portfolio selection problems for several reasons. These models are developed based on the specific dynamics of interest rates and they can reflect economic and financial state of market. This is why, they are the main tools for pricing and hedging of interest rate derivatives in practice. Additionally, these models provide solutions for expected return and covariance structure eliminating exigency for historical estimation of parameters. Furthermore, these parameters are expressed as functions of time and hence the diminishing time to maturity property of bonds can easily be handled. Some important researches about the term structure models application for bond portfolio management in literature are conducted by Wilhelm [5], Sorensen [6], Korn and Kraft [7], and Puhle [8]. One of the aims of this thesis is to explain and analyze this bond portfolio problem with applications of Vasicek Model as a short rate model example (where we broadly follow Puhle's work), and LIBOR Market Model as a high-dimensional market model example.

One of the shortcomings of mean-variance framework is that it implies an equal probability assignment for positive and negative returns. Nevertheless it is a well known fact that most of the assets and the portfolios that they constitute do not possess symmetrical return distributions, just like for a setting where interest rate dynamics are governed by standard term structure models (including Vasicek and LIBOR Market Models), there is no known distribution expressing portfolios' expected returns. Therefore mean-variance approach can be misleading. It can be more sensible to quantify risk with a measure that can relate any non-normality in expected returns. For this reason we will use a modified version of classical Markowitz problem where the downside risk (loss) is expressed as the Value-at-Risk (VaR) of the portfolio. Like what we did in the classical mean-variance problem, we will again implement Vasicek and LIBOR market models to describe rate dynamics.

The thesis' organization is a follows: Chapter 2 gives general information and definitions about interest rates, bonds, and interest rate derivatives. Short rate dynamics, bond pricing and calibration methods for Vasicek Model is introduced in Chapter 3 ***Missing part, should be decided for the rest.***

2. INTEREST RATES AND BASIC INSTRUMENTS

In this chapter some frequently used concepts in interest rate theory that will be used throughout the thesis are described briefly.

2.1. Interest Rates and Bonds

Interest is defined by Oxford Dictionary as *the money paid regularly at a particular rate for the use of money lent, or for delaying the repayment of a debt.* It is a compensation for lost opportunities that could have been made with the loaned asset or for the risk of losing some part or all of the loan. Also interest rate implies the time value of money; the worth of 1 unit of currency will not be the same in future. Interest rates do change over time and prediction of their future values is an important issue for financial players. Mathematical interest rate models try to represent the future evolution of interest rates as stochastic processes. These models then can be used for purposes such as pricing, hedging, and risk management. Interest rates that interest rate models deal with can be conveniently seperated in two categories: government rates implied by government issued bonds and interbank rates which are the rates of interest charged on short-term loans made between banks. However the mathematical modelling in literature and in this thesis uses both of these domains in a unified manner. Interest rates can be mathematically defined in different ways. In this thesis zero-coupon bond is the starting point for interest rate definitions.

Definition 2.1. *Zero-Coupon Bond: A zero-coupon (discount) bond with face value 1 and maturity T guarantees to its holder the payment of one unit of currency at maturity.*

The value of the contract at time t is denoted by the stochastic process $P(t, T)$ defined on $[t, T]$. In a market where accrued interest is positive, the bond is bought at a value less than its face value, i.e. $0 < P(t, T) < 1 \quad \forall t < T$, implying that some interest will be gained over the time period $[t, T]$. Futhermore, in an arbitrage-free bond market, the value of the bond must be non-increasing w.r.t. maturity, i.e. $T_i \leq T_j \Leftrightarrow P(t, T_i) \geq P(t, T_j)$.

Definition 2.2. *Simply compounded spot interest rate: Suppose an amount $P(t,T)$ is invested at a simply-compounding spot interest rate during time period $[t,T]$ which accrues to unity at time T. Let this rate to be denoted as $L(t,T)$. It is defined by:*

$$(1 + L(t,T)(T-t))\, P(t,T) = 1 \tag{2.1}$$

Equivalently:

$$L(t,T) = \frac{1 - P(t,T)}{(T-t)\, P(t,T)} \tag{2.2}$$

Zero-coupon bond price in a simply compounding interest rate framework is hence given by

$$P(t,T) = \frac{1}{(1 + L(t,T)(T-t))} \tag{2.3}$$

Definition 2.3. *Continuously compounded spot interest rate: Suppose an amount $P(t,T)$ is invested at a continuously compounding spot interest rate during time period $[t,T]$ which accrues to unity at time T. Let this rate to be denoted as $R(t,T)$. It is defined by:*

$$e^{R(t,T)(T-t)} P(t,T) = 1 \tag{2.4}$$

Equivalently:

$$R(t,T) = \frac{\ln P(t,T)}{(T-t)} \tag{2.5}$$

Zero-coupon bond price in a continuously compounded interest rate framework is hence given by:

$$P(t,T) = e^{-R(t,T)(T-t)} \tag{2.6}$$

Definition 2.4. *Instantaneous short interest rate: Instantaneous short rate is defined as the limit of continuously compounding spot rate $R(t,T)$ or simply compounding spot rate $L[t,T]$ as T approaches to t^+. The short rate $r(t)$ is given by*

$$r(t) = \lim_{T \searrow t^+} R(t,T) = \lim_{T \searrow t^+} L(t,T) = \left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t} \tag{2.7}$$

Definition 2.5. *Zero-Coupon Yield Curve: Zero-coupon yield curve (term structure of interest rates) is the graph of the function*

$$T \to L(t,T), \quad T > t$$

for simply compunding or,

$$T \to R(t,T), \quad T > t$$

for continously compounding.

It presents the yields of similar-quality bonds against their maturities. Typically yield curves take three shapes: normal, inverted, and flat. An example for each type is given in Figure 2.1.

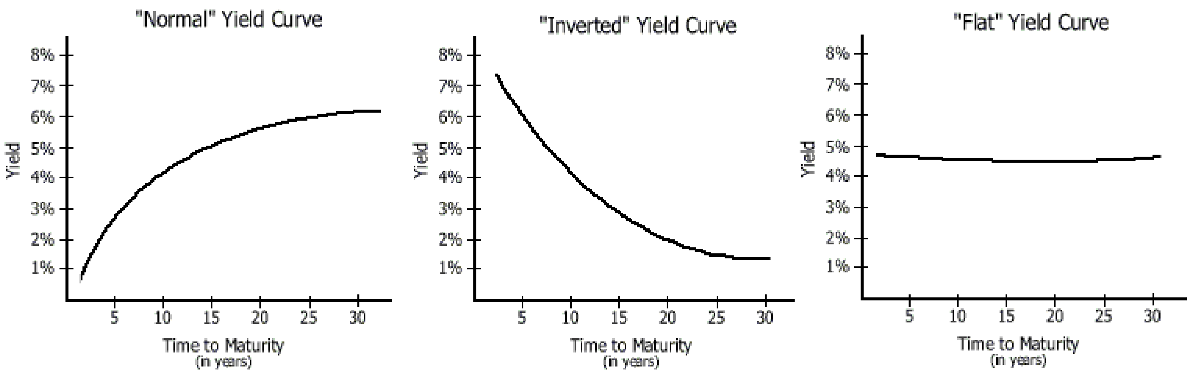

Figure 2.1. Normal, Inverted, and Flat yield curves.

Definition 2.6. *Bank Account: Bank account can be defined as an account that can be used either to lend or borrow money at the instantaneous rate* $r(t)$*, denoting the interest rate at which money accrues when being continuously re-invested. Let 1 unit of currency is put in such a bank account at* $t = 0$*, and the bank account follows the following differential equation:*

$$dB(t) = r_t B(t)dt,$$

then it will grow to $B(t)$ *at time* t *given by:*

$$B(t) = \exp\left(\int_0^t r(\tau)d\tau\right) \tag{2.8}$$

As the main idea of interest rate modelling is to express evolution of interest rates as stochastic processes, the interest rate term hence the bank account process will be stochastic. The bank account serves to associate different time values of money, and enables the setting up of a discount factor.

Definition 2.7. *Discount Factor: Discount factor is the time* t *value of a 1 unit currency at time* $T \geq t$ *with respect to the dynamics of the bank account given in* *definition 2.6. It is denoted by $D(t,T)$ and is given by:*

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(- \int_t^T r(\tau) d\tau \right) \tag{2.9}$$

Like the bank account process, discount factors are stochastic processes.

2.2. Forward rates

Forward rates: Forward rate is the future expected interest rate that can be derived from the current yield curve, i.e. the future yield on a bond. It is the fair rate of a forward rate agreement (FRA) with expiry T_1 and maturity T_2 at time t. It can be defined in the following way: At time t, let the prices of bonds maturing at T_1 and T_2, $T_1 < T_2$, be $P(t,T_1)$ and $P(t,T_2)$ respectively. The forward rate active during period $[T_1,T_2]$ is the rate that would make the following two investment strategies equal:

1. Buy $\frac{1}{P(t,T_1)}$ unit of bond $P(t,T_1)$, collect the face value at T_1, reinvest all the money in bond $P(T_1,T_2)$, collect the face value at T_2.
2. $\frac{1}{P(t,T_2)}$ unit of bond $P(t,T_2)$, collect the face value at T_2.

Forward rate can be characterized depending on the compounding method.

Definition 2.8. *Simply compounded forward rate: Equating the two investment strategies described previously gives:*

$$\frac{1}{P(t,T_1)P(T_1,T_2)} = \frac{1}{P(t,T_2)} \tag{2.10}$$

where first and second strategies are given at left and right hand side respectively. By equation (2.12),

$$P(T_1,T_2) = \frac{1}{1 + L(T_1,T_2)(T_2 - T_1)} \tag{2.11}$$

The rate $L(T_1, T_2)$, which is unknown at the current time t, will be represented by the forward rate $F(t, T_1, T_2)$. More precisely:

$$P(t, T_1, T_2) = \frac{1}{1 + F(t, T_1, T_2)(T_2 - T_1)} \tag{2.12}$$

Replacing this equation in equation (2.10) and solving for $F(t, T_1, T_2)$ gives:

$$F(t; T_1, T_2) = \frac{1}{(T_2 - T_1)} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right) \tag{2.13}$$

Definition 2.9. *Continuously compounded forward rate: Following the similar steps of definition 2.8, the solution to the following:*

$$P(t, T_1, T_2) = \frac{1}{e^{R(t,T_1,T_2)(T_2 - T_1)}} \tag{2.14}$$

gives the forward rate $R(t, T_1, T_2)$ by:

$$R(t, T_1, T_2) = \frac{\ln P(t, T_1) - nP(t, T_2)}{(T_2 - T_1)} \tag{2.15}$$

Definition 2.10. *Instantaneous forward rate: Instantaneous forward rate is defined as the limit of continuously compounding forward rate $R(t, T_1, T_2)$ or simply compounding forward rate $L(t, T_1, T_2)$ as T_2 approaches to T_1^+. The rate $f(t)$ is given by*

$$f(t) = \lim_{T_2 \searrow T_1^+} R(t, T_1, T_2) = \lim_{T_2 \searrow T_1^+} L(t, T_1, T_2) = -\left. \frac{\partial \ln P(t, T)}{\partial T} \right|_{T_2 = T_1} \tag{2.16}$$

2.3. Interest rate derivatives

In this section we will introduce interest rate derivatives that will be used in this thesis, predominantly for calibration purposes. We will be also making some derivative pricing examples to check the sanity of our models and codes before using them directly for bond portfolio optimization problems. These derivatives are caps and floors, interest rate swaps, and swaptions.

2.3.1. Caps and Caplets

A caplet is a call option on a forward interest rate, where the buyer has the right but not the obligation to enter a contract that enables him to borrow money from the issuer at a pre-defined level, namely caplet (strike) rate, and lend money to the issuer at the spot rate observed at exercise date. Note that the forward rate on which the contract is written becomes spot rate at exercise date. Obviously, the holder exercises the caplet if the realised spot rate is higher than strike rate. Analogously, a floorlet is a put option on a forward interest rate, where the buyer has the right but not the obligation to enter a contract that enables him to lend money from the issuer at strike rate, and borrow money to the issuer at the spot rate. Similarly, the holder exercises the floorlet if the realised spot rate is lower than strike rate. Caplets and floorlets are not actually available in the markets, they are traded in the form of caps and floors. Calpets (floorlets) are analogous to European call (put) options; and caps (floors) can be tought as a series of European call (put) options on forward interest rates that protects the buyer from large increases (decreases) in interest rates.

Consider a caplet on the i^{th} forward interest rate; the caplet matures (exercised) at time T_i and applies for the period $[T_i, T_{i+1}]$. Suppose that the caplet is written on a loan amount of A and the caplet rate is K. Then the payoff received at time T_{i+1} is

$$V(T_{i+1}) = A(L(T_i, T_i, T_{i+1}) - K)^+(T_{i+1} - T_i) \tag{2.17}$$

The time t value of the caplet is obtained by multiplying this caplet payoff with its corresponding discount factor $D\,(t, T_{i+1})$:

$$V(t) = A(L(T_i, T_i, T_{i+1}) - K)^+(T_{i+1} - T_i)D(t, T_{i+1}) \tag{2.18}$$

The time t value of a cap written on forward rates $i \in \{\alpha, \ldots, \beta - 1\}$ is just the summation of their corresponding caplets' discounted values:

$$V(t) = A \sum_{i=\alpha}^{\beta-1} (L(T_i, T_i, T_{i+1}) - K)^+ (T_{i+1} - T_i) D(t, T_{i+1}) \tag{2.19}$$

Formulae for floors and floorlets can be trivially obtained by changing the payoff term $(L(t, T_i, T_{i+1}) - K)^+$ to $(K - L(t, T_i, T_{i+1}))^+$ in caps and caplets' equations.

2.3.2. Interest Rate Swap

Interest rate swap (IRS) is an exchange of a stream of fixed interest payments for a stream of floating interest payments. The fixed leg denotes a stream of fixed payments, and the rates (swap rate) are specified at the beginning of the contract. Although these rates can be set to different value, generally they are fixed to a single value and this will be the case for the swaps discussed throughout this thesis. The floating leg consists of a stream of varying payments associated with a benchmark interest rate, for example LIBOR or EURIBOR. A swap where the holder receives floating payments while paying fixed payments is called a payer swap and the contrary is called receiver swap.

Consider a tenor structure $T_i,\ i = 0, 1, \ldots, \alpha, \alpha+1, \ldots, \beta$. For a swap with tenor $T_\beta - T_\alpha$, reset dates for the contract are $T_\alpha, \ldots, T_{\beta-1}$, and the payments are made on $T_{\alpha+1}, \ldots, T_\beta$. For a notional value A and swap rate K, the fixed payment made at T_i is:

$$A\,(T_i - T_{i-1})\,K$$

and the floating payment is

$$A\,(T_i - T_{i-1})\,L(T_{i-1}, T_{i-1}, T_i)$$

. For a payer swap, the payoff at T_i is:

$$A\,(T_i - T_{i-1})\,(L(T_{i-1}, T_{i-1}, T_i) - K)$$

The time t discounted value of a payer swap can be expressed as:

$$V(t) = A \sum_{i=\alpha+1}^{\beta} D\,(t, T_i)\,(T_i - T_{i-1})\,(L(T_{i-1}, T_{i-1}, T_i) - K)\,, \tag{2.20}$$

or as:

$$V(t) = A\,D\,(t, T_\alpha) \sum_{i=\alpha+1}^{\beta} P\,(T_\alpha, T_i)\,(T_i - T_{i-1})\,(L(T_{i-1}, T_{i-1}, T_i) - K)\,. \tag{2.21}$$

Payoff for the receiver swap for the same setting is just given by the same formula with an opposite sign.

Definition 2.11. *Par swap rate: Par swap rate (forward swap rate) denoted by $SR_{\alpha,\beta}(t)$ corresponding to $T_\beta - T_\alpha$ swap is the value of fixed swap rate that makes the contract fair at time t, i.e. $V(t) = 0$. Setting equation 2.21 to 0 and solving for K gives:*

$$K = SR_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum\limits_{i=\alpha+1}^{\beta} \delta P(t, T_i)} \tag{2.22}$$

2.3.3. European Swaption

A eurapean payer swaption is an option granting its owner the right but not the obligation to enter into an underlying $T_\beta - T_\alpha$ swap at T_α, the swaption maturity. Time t value of a swaption with a notional A and swap rate K is given by

$$V(t) = A\,D\,(t, T_\alpha) \left(\sum_{i=\alpha+1}^{\beta} P\,(T_\alpha, T_i)\,(T_i - T_{i-1})\,(L(T_{i-1}, T_{i-1}, T_i) - K) \right)^+ \tag{2.23}$$

Receiver version for the same setting is just given by a sign change in swap payoff term. Another version of the time t discounted payer swaption payoff can be given in terms of par swap rate given by equation 2.22 as:

$$V(t) = A\,D\,(t, T_\alpha)\,\left(SR_{\alpha,\beta}(T_\alpha) - K\right)^+ \sum_{i=\alpha+1}^{\beta} P\,(T_\alpha, T_i)\,(T_i - T_{i-1}) \tag{2.24}$$

The participants of swaption market are generally large corporations, financial institutions, hedge funds, and banks wanting protection from rising (falling) interest rates by buying payer (receiver) swaptions. The swaption market is an over-the-counter market, trading occurs directly between two parties. Also swaptions have less standard structure compared to products traded on the exchange such as stocks and futures.

3. Vasicek Model

Vasicek Model is a one-factor short rate model describing the instantaneous interest rate movements. The interest rate follows an Ornstein-Uhlenbeck mean-reverting process, under the risk neutral measure, defined by the stochastic differential equation

$$dr_t = \beta(\mu - r_t)dt + \sigma dW_t, \quad r(0) = r_0 \tag{3.1}$$

where μ is the mean reversion level, β is the reversion speed and σ is the volatility of the short rate, and W_t the Wiener process. In Vasicek Model, mean reversion is achieved by the drift term: when the short rate r_t is above μ, the drift term tends to pull r_t downward and when r_t is below μ, the drift term tends to push r_t upward. This is the typical martingale modeling for the Vasicek model that can be found in literature. Actually the model implicitly assumes that the market price of the risk, generally denoted by λ, is equal to zero. However, in an arbitrage free market there exist a market price of risk process that is common for all assets in the market. Since the described model's parameters cannot take the required information about the actual market data (market price of risk), theoretical prices differ from the market bond prices and hence allow arbitrage to occur. This type of models are also called equilibrium models. Nevertheless, one can always be interested in fitting the model to current market data for essential practical applications such as risk management. Hull and White [9] extended the standard Vasicek model by allowing a time dependent functional form of market price of risk term, which solves this calibration problem. In this thesis, instead of a time dependent form we assume a constant λ, which is the generally used and investigated case in literature. By this setting, we can easily model the standard Vasicek model in objective measure (real world) dynamics. We can make the change of risk-neutral measure to real-world measure by setting,

$$dW_t^0 = dW_t + \lambda r_t dt$$

which leads to,

$$dr_t = (\beta\mu - (\beta + \lambda\sigma)\, r_t)\, dt + \sigma dW_t^0, \quad r(0) = r_0 \tag{3.2}$$

3.1. Analytical pricing of zero coupon bond

Arbitrage-free prices of zero-coupon bonds can be derived with the following pricing equation [8]

$$P(t,T) = E_t \left[\exp\left(-\int_t^T f(u,u)du + \sum_{i=1}^{d} \int_t^T \lambda_i(u)dz_i(u) - \sum_{i=1}^{d} \frac{1}{2} \int_t^T \lambda_i(u)^2 du \right) \right] \tag{3.3}$$

where f is functional form of the instantaneous forward rate curve, s is volatility function of the instantaneous forward rates, λ is market price of risk, and d is the number of governing Brownian motions. In order to price zero coupon bonds in Vasicek model, these functions should be specified. First of all, the general formula for short rate process is given by

$$f(t,t) = f(0,t) + \sum_{i=1}^{d} \int_0^t s_i(u,t) \left(\left(\int_u^t s_i(u,s)ds - \lambda_i(u) \right) \right) du + \sum_{i=1}^{d} \int_0^t s_i(u,t)dz_i(u) \tag{3.4}$$

For Vasicek Model, $d = 1$, since there is one governing Brownian motion. Also we assumed that the market price of interest rate risk is constant. Futhermore the forward rate volatilities are assumed to be of the following form

$$s_1(t,T) = s(t,T) = \sigma_r e^{-\beta(T-t)} \tag{3.5}$$

and the initial instantaneous forward rate curve is given by

$$f(0,T) = \mu + e^{-\beta T}(f(0,0) - \mu) + \lambda \frac{\sigma_r}{\beta} \left(1 - e^{-\beta T}\right) - \frac{\sigma_r^2}{2\beta^2}\left(1 - e^{-\beta T}\right)^2 \tag{3.6}$$

Now that all the required specifications are given, solution of equation (3.1) for $s \leq t$ is given by

$$r(t) = r(s)e^{-\beta(t-s)} + \mu\left(1 - e^{-\beta(t-s)}\right) + \sigma \int\limits_s^t e^{-\beta(t-u)} dW(u) \tag{3.7}$$

It can be shown that $r(t)$ is conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given by

$$E_s[r(t)] = r(s)e^{-\beta(t-s)} + \mu(1 - e^{-\beta(t-s)}) \tag{3.8}$$

$$\mathrm{var}_s(r(t)) = \sigma^2 \left(\frac{1 - e^{-2\beta(t-s)}}{2\beta}\right) \tag{3.9}$$

Finally analytic solution for zero coupon bond price obtained by equation (3.3) is

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)} \tag{3.10}$$

where

$$\begin{aligned} A(t,T) = R(\infty)\left(\tfrac{1}{\beta}(1 - e^{-\beta(T-t)}) - (T-t)\right) \\ -\tfrac{\sigma^2}{4\beta^3}(1 - e^{-\beta(T-t)})^2 \end{aligned} \tag{3.11}$$

with

$$R(\infty) = \left(\mu + \lambda\frac{\sigma}{\beta} - \frac{1}{2}\frac{\sigma^2}{\beta^2}\right)$$

and

$$B(t,T) = \frac{1}{\beta}(1 - \exp(-\beta(T-t))) \tag{3.12}$$

To get the yield curve implied by Vasicek model, we can equate equation 3.10 to the zero-coupon bond price formula for continuously compounding interest rate given by equation 2.6. Solving for $R(t,T)$ gives:

$$R(t,T) = \frac{r(t)B(t,T) - A(t,T)}{(T-t)} \tag{3.13}$$

3.2. Discrete approximation of zero coupon bond price by the exact simulation of short rate

Zero coupon bond price can be approximated by the exact simulation of short rate process. To simulate the exact process r_t at times $0 = t_0 < t_1 < \ldots < t_n$ the following recursion is used:

$$r_{t_{i+1}} = e^{-\beta(t_{i+1}-t_i)} r_{t_i} + \mu(1 - e^{-\beta(t_{i+1}-t_i)}) + \sigma\sqrt{\frac{1}{2\beta}(1 - e^{-2\beta(t_{i+1}-t_i)})} Z_{i+1} \tag{3.14}$$

where Z is a vector of iid. standard normal variates. Within the dynamics of a short rate model, arbitrage-free time t price of a contingent claim can be calculated by taking the expectation of its payoff at T discounted by the short rate process over the period $[t,T]$ [10]. For a zero-coupon bond with unit face value we have:

$$P(t,T) = E_t\left[e^{-\int_t^T r_s ds}\right] \tag{3.15}$$

To calculate this expectation, first we will generate n short rate paths with m time steps and for each path we will take approximate integral given in equation 3.15 by the following summation:

$$-\int_t^T r_s ds \cong -\sum_{i=1}^{m} \mathrm{r}_{t_i} \frac{T-t}{m}$$

Exponentiating this for all paths and taking their average gives the simulated zero-coupon bond price.

3.3. Calibration and Historical Estimation

The problem of fitting interest rate models can generally be treated with two different approaches, historical estimation and calibration. One can decide which one to use depending on the availability of data: historical estimation uses historical time series to estimate parameters using statistical methods whereas calibration requires current data to determine the parameters. As for historical estimation, maximum likelihood method can be used. As pointed out in the previous section, there is no hope for the yield curve to be predicted by Vasicek model to match some given observed spot yield curve since the model is completely determined by the choice of the five parameters, $r(0), \beta, \mu, \lambda$, and σ. However, one may try to fit the model for example by defining a least-squares minimization problem, minimizing the sum of squared deviations between market yields and the yields produced by the Vasicek model with the above parameters as problem variables. Another possible approach can be calibration to interest rate derivatives such as caps and swaptions. Again parameters minimizing the sum of squared deviations between market cap / swaption prices and the ones produced by the Vasicek model can be sought. As a matter of fact, it would be advantageous to use a combination of these methods for both consistency with historical data and veracious reflection of market's current situation.

3.3.1. Historical Estimation

Vasicek model describes the instantaneous spot interest rate movement, which cannot be directly observed in the market. Typical approach is to use the smallest term rate that can be actually observed as the proxy for instantaneous rate. The available spot and historical data generally consist of yield curves based on treasury securities or some reference rate such as LIBOR and EURIBOR rates. We can decide on which data to use according to our purpose for using Vasicek model. Later in this thesis we will be dealing with zero coupon (government) bond portfolios and evolution of interest rates we will be described by Vasicek model. Thus it is very natural to choose to calibrate the model to yield curves derived from treasury securities. On the other hand, one might also be interested in calibrating the model to LIBOR curves, for example if floating rate notes depending on LIBOR rates are to be investigated. In the following example, daily yields on actively traded non-inflation-indexed issues adjusted to constant maturities data from 02.01.2002 to 05.06.2011 are used as the calibration input. The maturities are 1M, 3M, 6M and 1Y, 2Y, 3Y, 5Y, 7Y, 10Y. 1 month rate will serve as the short rate surrogate and its graph is given in figure 3.1.

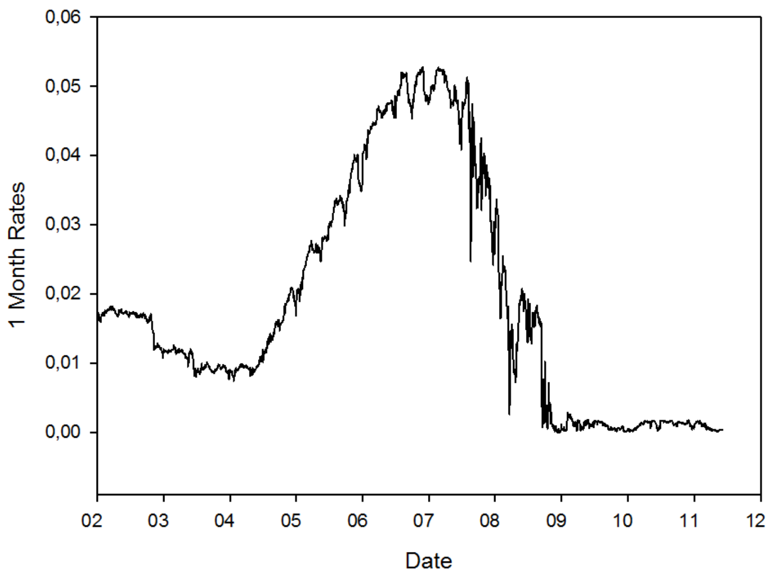

Figure 3.1. US 1-month maturity daily yields for 02.01.2002-05.06.2011

With the constant risk premium assumption the real world dynamics given in equation 3.7 becomes:

$$dr_t = \beta \left(\mu - \frac{\lambda\sigma}{\beta} - r_t \right) dt + \sigma dW_t^0, \quad r(0) = r_0 \tag{3.16}$$

Let's rewrite this equation as

$$dr_t = (b - ar_t)\, dt + \sigma dW_t^0, \quad r(0) = r_0 \tag{3.17}$$

with the following substitutions:

$$b = \beta\left(\mu - \frac{\lambda\sigma}{\beta}\right) \quad a = \beta$$

The parameters for this form can be estimated by maximum likelihood estimation (MLE) technique and it is very straightforward for the transitional distribution of short rate conditional on $\mathcal{F}_s$ can be explicitly solved. Closed-form maximum likelihood estimates for functions of the parameters of the Vasicek model are given in Brigo and Mercurio [10] as:

$$\begin{aligned} \hat{\alpha} &= \frac{n\sum_{i=1}^{n} r_i r_{i-1} - \sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i-1}}{n\sum_{i=1}^{n} r_{i-1}^2 - \left(\sum_{i=1}^{n} r_{i-1}\right)^2} \\ \hat{\theta} &= \frac{\sum_{i}^{n} (r_i - \hat{\alpha} r_{i-1})}{n(1-\hat{\alpha})} \\ \hat{V}^2 &= \frac{1}{n}\sum_{i=1}^{n} \left[r_i - \hat{\alpha} r_{i-1} - \hat{\beta}(1-\hat{\alpha})\right]^2 \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} \alpha &= \exp\left(-adt\right) \\ \theta &= \frac{b}{a} \\ V^2 &= \frac{\sigma^2}{2a}\left(1 - \exp\left(-2adt\right)\right) \end{aligned} \tag{3.19}$$

and dt is the time step between observed rate data. Note that estimation technique provides direct estimates for β and σ given as:

$$\begin{aligned} \hat{\beta} &= \frac{-\log(\hat{\alpha})}{dt} \\ \hat{\sigma} &= \sqrt{\frac{2\log(\hat{\alpha})\hat{V}^2}{dt(\hat{\alpha}^2-1)}} \end{aligned}$$

but μ and λ parameters are estimated in a combined way as:

$$\hat{\mu} - \frac{\hat{\lambda}\hat{\sigma}}{\hat{\beta}} = \hat{\theta}$$

After this point, how to decide upon $\hat{\mu}$ and $\hat{\lambda}$ values becomes an open ended question. A solution can be achieved by importing $\hat{\mu}$ acquired from a calibration to market prices and solve for $\hat{\lambda}$ from historical estimation. But while combining these two different calibration techniques one should be aware of that market prices describe the risk neutral measure whereas historical data describe objective measure. We want to propose the following solution where we do not call for market prices. Let's start by assuming that $\lambda = 0$, i.e., the real world measure is equal to the risk-neutral measure where the investors are assumed to be non risk-averse. Under this assumption the estimated parameters for our example data are

$$\hat{\mu} = 0.01120524 \quad \hat{\beta} = 0.2474551 \quad \hat{\sigma} = 0.0133463.$$

If the zero risk premium assumption is true then we can expect that Vasicek yield curve described by the estimated parameters would be in accordance with the historical yields. To check this, we will compare the Vasicek yield with average historical yields and spot yield curve. Their plots are given in figure 3.2.

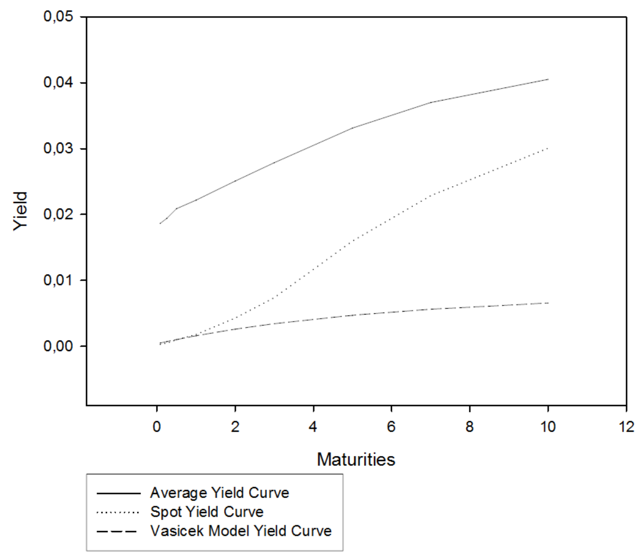

Figure 3.2. Vasicek Model curve and average yields curve for 02.01.2002-05.06.2011

As can be seen from the figure curves do not fit. The average yield and spot curves look like a normal yield curve, a curve where yields rise as maturity lengthens because investors price the risk arising from the uncertainty about the future rates into the yield curve by demanding higher yields for maturities further into the future. On the other hand, Vasicek model curve is slightly increasing but almost flat; it does not capture properly the associated risk premium. Also one should take notice of the gap between average and spot curves which is due to the low interest rates observed after the sharp downfall starting from year 2008.

Now we will assume a positive price of the risk and estimate its value as the one that best fits Vasicek curve to the average historical yields. The following minimization problem searches for the λ that fits Vasice yield curve to historical yield curves at every observed day.

$$\min_{\lambda} \sum_{i \in \mathcal{T}} \sum_{j \in \mathcal{M}} \left(Y_i^{Market}(0,j) - Y_i^{Vasicek}(0,j) \right)^2$$
$$s.t.\ \hat{\mu} - \tfrac{\hat{\lambda}\hat{\sigma}}{\hat{\beta}} = \hat{\theta}$$

where $\mathcal{T}$ is the set of all historical dates and $\mathcal{M}$ is the set of all maturities.

For our example this optimization problem gives:

$$\hat{\mu} = 0.03366797 \quad \hat{\lambda} = 0.415157$$

Note that $\hat{\beta}$ and $\hat{\sigma}$ stay the same. The graph of the Vasicek yield curve with the new parameter set along with average and spot curves are given in figure 3.3.

With the addition of non-zero risk premium parameter, now the Vasicek model yield curve looks more like a normal yield curve. It is below the average yield curve since its initial short rate parameter $r(0)$ is at a low level compared to the past. Also it is above the spot yield curve because mean reversion level and risk premium parameters overestimates curret market expectations.

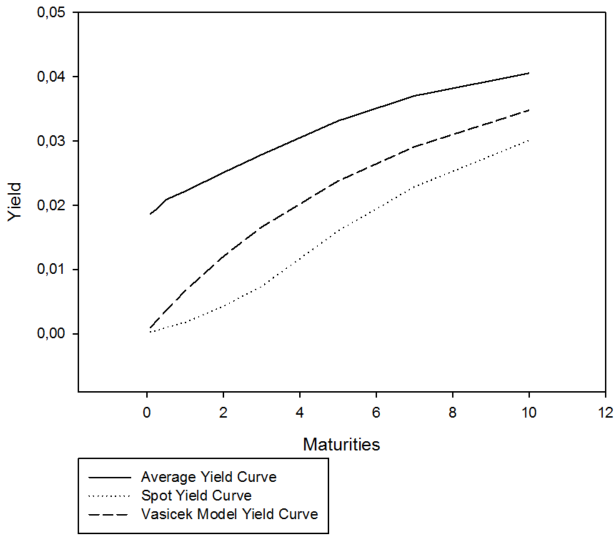

Figure 3.3. Vasicek Model curve and average yields curve for 02.01.2002-05.06.2011

3.3.2. Calibration to Spot Market Data

Practitioners may prefer to calibrate their models based on the current market prices since they imply future expectations about the market. If the data for calibration securities are available, calibration to current market prices can be considerably useful especially for pricing derivative instruments. Zero coupon bonds, bond options, caps & floors, swap, and swaptions are the possible securities that can be used for calibration purposes. Unfortunately for Vasicek model (actually for single factor models in general) there is no hope for a calibration to spot data for the problem is underdeterminate. Under the risk neutral measure Vasicek model is determined by the choice of 3 free variables μ, β, and σ and there would be infinitely many solutions for a traditional least squares minimization problem. A first step to overcome this problem can be choosing to use volatility term derived from historical calibration since volatility does not change with the change of measure, it should be same under the real world measure and risk neutral measure. Another possible solution can be defining upper and lower bounds for the range of parameters.

4. LIBOR Market Model Theory

In this chapter we present the most famous member of interest rate market models, the LIBOR Market Model (LMM). After their introduction, market models for interest rates became very popular among the researchers and practitioners. What makes market models interesting is that they allow the modelling of quantities that are directly observable in the market. Contrary to short rate models such as Vasicek model and instantaneous forward rate models such as Heath-Jarrow-Morton model which are dealing with theoretical quantities, LMM models the evolution of a set of forward (LIBOR) rates that are actually observable in the market and whose volatilities are bound up with traded securities. The other famous member of the market models is the Swap Market Model (SMM) which models the dynamics of the forward swap rates. An important feature about these models is that the LMM can be used to price caps/floors, and SMM can be used to price European swaptions in accordance with the Black's caps/floors and swaptions formulas respectively. This equivalance relationship renders these models very useful tools since the market standard for the pricing of caps/floors and swaptions is based on Black's formulation.

It is also worthwhile to mention, without going into details, the mathematical inconsistency between these two models. LMM is based on the log-normality assumption of the forward (LIBOR) rates whereas SMM is based on the log-normality assumption of the swap rates; nevertheless the two assumptions cannot coexist [10]. To check this, suppose forward LIBOR rates are assumed to be log-normally distributed under their related forward measure. By a change of measure forward rate dynamics can be written under the swap measure and with the help of Ito's formula swap rate dynamics can be produced. Distribution of swap rates would not turn out to be log-normal. The same is also true for the other way around. Consequently if one chooses LMM (SMM) as its framework, the model would be automatically calibrated to caps (swaptions) but not to the swaptions (caps). In this thesis, we will be only interested in LMM because it will constitute a sufficient example for our purpose of using a market model for a study of bond portfolio optimization. Also forward rates seem to be more characteristic and illustrative than swap rates.

LIBOR Market Model is introduced and developed by Miltersen, Sandmann and Sondermann [11], Brace, Gatarek and Musiela [12], Jamshidian [13] and Musiela and Rutkowski [14]. It is also refered as BGM (Brace, Gatarek, Musiela) model or known by the name of the other authors that first published it, but it may be more appropriate to name it Log-normal Forward LIBOR Market Model, since as this name implies, it models forward LIBOR rates under the log-normality assumption. LMM is a high-dimensional model describing the motion of interest rate curve as a (possibly dependent) movement of finite number of forward rates. Their motion is governed simultaneouly by a multi-dimentional brownian motion under a common measure. In literature different variations of the original model can be found, here we will represent the standard LMM as represented in [10] and [15].

LIBOR or the London Inter Bank Offered Rate is a benchmark rate based on the interest rates at which selected banks in London borrow or lend money to each other at London interbank lending market. It is the primary benchmark designating short term interest rates around the world. It's calculated every business day in 10 currencies and 15 terms, ranging from overnight to one year.

4.1. LIBOR Market Model Dynamics

One of the cornerstone feature of LMM is that it models forward rates over discrete tenor quantities [10]. Before starting explaining model's dynamics, definitons of dicretized time structure and forward rates should be given.

4.1.1. Tenor Structure

Musiela and Rutkowski [14] developed a discrete-tenor formulation based on finitely many bonds. Let $t = 0$ be the current time. Consider a set of discretized dates representing (generally equally spaced) year fractions, each associated with a bond's maturity. This tenor structure is represented by:

$$0 = T_0 < T_1 < \cdots < T_n$$

For simplicity, in this thesis it is assumed that the tenor spacing is the same for each $[T_i, T_{i+1}]$ pair and equal to δ.

4.1.2. Forward LIBOR Rate (Simply-Compounded)

A forward LIBOR rate $L(t, T_i, T_{i+1})$ is the rate that is alive in interval $[t, T_i]$ and becomes equal to the spot rate denoted by $L[T_i, T_{i+1}]$ at time T_i. In this notation T_i is called reset date and and the period $[T_i, T_{i+1}]$ is called as the rate's tenor. It is defined as

$$1 + (T_{i+1} - T_i)L(t, T_i, T_{i+1}) = \frac{P(t, T_i)}{P(t, T_{i+1})} \tag{4.1}$$

i.e.

$$L_i(t) = L(t, T_i, T_{i+1}) = \frac{P(t, T_i) - P(t, T_{i+1})}{(T_{i+1} - T_i)P(t, T_{i+1})} \tag{4.2}$$

From now on, the generic LIBOR forward rate will be expressed as $L_i(t) = L(t, T_i, T_{i+1})$, both notation will be used interchangeably. In Definition 2.2 simply compounded forward rates are defined as functions of zero coupon bonds. These zero coupon bonds theoretically represent government treasury bonds and can be observed in the corresponding currency yield curve. By a common acknowledgement, these zero coupon bonds are assumed to be risk free since the debt payer (government) is considered as non-defaultable. On the other hand, LIBOR rates are not risk free; although they represent the lending-borrowing rates between World's leading banks, there is always a credit risk and this risk is reflected as a risk premium in the rates. For this reasons defining forward LIBOR rates in terms of zero coupon bonds may create an ambiguity. Actually, $P(t, T)$ terms appearing in the above equations stand for so-called LIBOR zero coupon bonds, which are not real securities traded in the market but rather a way of expressing the LIBOR yield curve. For simplicity this difference is almost always ignored in literature and so it is in this thesis.

In LMM, forward rates dynamic is expressed as

$$\frac{dL_i(t)}{L_i(t)} = \mu_i^{\mathbb{P}}(t)dt + \sigma_i(t)dW_i^{\mathbb{P}}(t) \quad for\, i = 0, \ldots, n-1, under\, \mathbb{P} \tag{4.3}$$

where $\sigma_i(t)$ are bounded deterministic d-dimensional row vectors and $W^{\mathbb{P}}$ is a standard d-dimensional Brownian motion under some probability measure $\mathbb{P}$, instantaneously correlated with

$$dW_i^{\mathbb{P}}(t)dW_j^{\mathbb{P}}(t) = \rho_{i,j}(t)dt \tag{4.4}$$

where $\rho_{i,j}(t)$ denotes correlation matrix. Initial condition for the process is given by the current yield curve:

$$L_i(0) = L(0, T_i, T_{i+1}) = \frac{P(0, T_{i+1}) - P(0, T_i)}{(T_{i+1} - T_i)P(0, T_{i+1})}$$

In an arbitrage free market the price of a tradable asset discounted by any numeraire (a reference asset) is a martingale under the measure corresponding to this numeraire [10]. Forward LIBOR rates defined in equation 4.2 are not tradable assets; one cannot buy some amount of forward LIBOR rate from the market. Let's rearrange this equation and write the following:

$$L(t, T_i, T_{i+1})P(t, T_{i+1}) = \frac{P(t, T_i) - P(t, T_{i+1})}{(T_{i+1} - T_i)} \tag{4.5}$$

Now the left hand side of this equation is a tradable asset; one can buy $L(t, T_i, T_{i+1})$ amount of $P(t, T_{i+1})$ from the market. Also one can freely choose the zero coupon bond $P(t, T_{i+1})$ as its reference asset. Dividing the tradable asset $L(t, T_i, T_{i+1})P(t, T_{i+1})$ by the numeraire $P(t, T_{i+1})$ gives $L(t, T_i, T_{i+1})$ which is a martingale process under the measure corresponding to zero coupon bond $P(t, T_{i+1})$, $\mathbb{Q}^{P(t,T_{i+1})}$. Therefore the dynamics of $L(t, T_i, T_{i+1})$ should be driftless under $\mathbb{Q}^{P(t,T_{i+1})}$ and are given by the following stochastic differential equation

$$dL_i(t) = L_i(t)\sigma_i(t)dW_i^{\mathbb{Q}^{P(t,T_{i+1})}}(t) \quad for\, i = 0, \ldots, n-1, under\ \mathbb{Q}^{P(t,T_{i+1})} \tag{4.6}$$

This observation directly leads to Black'76 formula which is used to price caplets whose payoffs depend only on a single forward rate.

4.1.3. Black's Formula for Caplets

The market standard for caplet pricing is Black's formula, which is an extention of the famous derivative pricing tool presented by Black & Scholes in 1973. Black & Scholes formula is based on the assumption that stock prices follow geometric Brownian motion with constant drift and volatility, i.e., the stock price path follows a log-normal random walk. The log-normal framework of LMM inevitably leads to the Black's formula, with some minor modifications. Let $V_i(t)$ denote time t value of a caplet with reset date T_i and payment date T_{i+1}. Then from equation (2.17),

$$V_i(T_{i+1}) = (L_i(T_i) - K)^+\delta$$

From the definition of risk natural pricing $V_i(t)$ is given as:

$$V_i(t) = N(t)E^N\left[\frac{(L_i(T_i) - K)^+\delta}{N(T_{i+1})}\right]$$

There is no restriction about numéraire selection, however as it is pointed out in section 4.2 choosing the equivalent martingale measure with the zero-coupon bond process $P(t, T_{i+1})$ as the numéraire would result with a driftless $L_i(T_i)$ process. This leads to:

$$V_i(t) = P(t, T_{i+1})E^{\mathbb{Q}^{P(t,T_{i+1})}}\left[\frac{(L_i(T_i) - K)^+\delta}{P(T_{i+1}, T_{i+1})}\right]$$

Rearranging this equation by noting that $P(T_{i+1}, T_{i+1}) = 1$ and δ is constant yields to:

$$V_i(t) = P(t, T_{i+1})\delta E^{\mathbb{Q}^{P(t,T_{i+1})}}\left[(L_i(T_i) - K)^+\right] \tag{4.7}$$

To evaluate this expectation, distribution function of forward rate $L_i(T_i)$ under $\mathbb{Q}^{P(t,T_{i+1})}$ must be known. Under $\mathbb{Q}^{P(t,T_{i+1})}$, stochastic differential equation describing the LIBOR forward rate process of $L_i(t)$ becomes a geometric Brownian motion given by:

$$dL_i(t) = \sigma_i(t)L_i(t)dW_i^{P(t,T_{i+1})}(t) \tag{4.8}$$

where $W_i^{P(t,T_{i+1})}$ denotes the Brownian motion under the measure $\mathbb{Q}^{P(t,T_{i+1})}$. Here the derivation for the solution of this sde is omitted, for a detailed derivation see pages 200-202 of [10]. The log-normal distribution of $L_i(t)$ is given by:

$$L_i(T_i) \sim \ln N(L_i(0) - \frac{1}{2}\upsilon_i^2,\ \upsilon_i^2)$$

where

$$\upsilon_i^2 = \int_0^{T_i} \sigma_i^2(t)dt$$

Once the distribution is obtained, the following analogy can be built with Black & Scholes formula to get Black's formula for caplets:

- underlying asset: $L_i(t)$
- volatility of returns of the underlying asset: $\upsilon_i(t)$
- strike level: K
- maturity: T_i
- spot price of the underlying asset: $L_i(0)$
- risk free rate: 0

Hence Black's formula for caplets is:

$$V(0, T_i, \upsilon_i) = P(0, T_{i+1})\delta[L(0, T_i, T_{i+1})N(d_1) - KN(d_2)]$$

where

$$N(d_1) = \frac{\ln\left(\frac{L(0,T_i,T_{i+1})}{K}\right) + \frac{\upsilon_i^2}{2}}{\upsilon_i}$$

and

$$N(d_2) = \frac{\ln\left(\frac{L(0,T_i,T_{i+1})}{K}\right) - \frac{\upsilon_i^2}{2}}{\upsilon_i}$$

4.1.4. Equivalence between LMM and Blacks caplet prices

Pay attention that in Black & Scholes formula the volatility term is constant, but in Black's formula for caplets, it is defined as the integral of the the instantaneous volatility at time t for the LIBOR forward-rate $L_i(t)$ along the period $[0, T_i]$. Let say there is an average "Black volatility" for caplet maturing at T_i that is given as:

$$\sigma_{i,Black}^2 = \frac{\int_0^{T_i} \sigma_i^2(t)dt}{T_i - 0} \tag{4.9}$$

If this $\sigma_{i,Black}^2$ is used as the volatility input of the Black's formula, the caplet price will be equal to the one given by LMM that uses the instantaneous volatility function appearing on equation (4.9) [10]. This volatility is called implied Black volatility of a caplet (volatility implied by the caplet's market price).

4.2. Drift Term

For non trivial pricing cases where the payoff of an instrument depends on more than one forward LIBOR rate, evolution of all corresponding forward LIBOR rates should be derived. Unfortunatelly, there isn't any numéraire that can make all forward rate processes martingales. We saw that the only forward rate that would be driftless is the one that has the same maturity date as the zero coupon bond chosen as the numéraire. Let a numéraire N is fixed and assume that there exists a corresponding equivalent martingale measure $\mathbb{Q}^N$ such that N-relative prices are martingales. In general terms, with such a measure choice, forward rates' dynamic equation will become:

$$\frac{dL_i(t)}{L_i(t)} = \mu_i^{\mathbb{Q}^N}(t)dt + \sigma_i(t)dW_i^{\mathbb{Q}^N}(t) \quad for\, i = 0, \ldots, n-1 \tag{4.10}$$

Drift functions corresponding to each forward rate process are then derived by the use of Girsanov's change of measure theorem.

In this thesis the two mostly discussed measures in literature, terminal and spot measures are presented.

4.2.1. Terminal Measure

If $P(t, T_n)$ is selected as the numéraire, then the equivalent measure $\mathbb{Q}^P(t, T_n)$ is called the terminal measure. Note that under terminal measure the last forward LIBOR rate process $L(t, T_{n-1}, T_n)$ is a martingale, i.e. driftless. With this choice of numéraire the process becomes:

$$\frac{dL_i(t)}{L_i(t)} = \mu_i^{\mathbb{Q}^{P(t,T_n)}}(t)dt + \sigma_i(t)dW_i^{\mathbb{Q}^{P(t,T_n)}}(t) \quad for\, i = 0, \ldots, n-1 \tag{4.11}$$

. It can be shown that the corresponding drift function is [14]

$$\mu_i^{\mathbb{Q}^{P(t,T_n)}}(t) = -\sum_{\substack{j \geq i+1 \\ j \leq n-1}} \frac{\delta_j L_j(t)\sigma_i(t)\sigma_j(t)\rho_{i,j}(t)}{1 + \delta_j L_j(t)} \tag{4.12}$$

Note that for $i = n-1$, drift function corresponding to $L(t, T_{n-1}, T_n)$ is equal to zero. Under terminal measure, current price ($t = 0$) of a derivarite that gives a payoff at time T_k is given by:

$$V(0) = P(0,T_n)E\left[payoff(T_k)\prod_{i=k}^{n-1}(1+\delta L_i(T_k))\right] \tag{4.13}$$

.

4.2.2. Spot Measure

Jamshidian [13] proposes a numéraire, creating a so-called rolling forward risk-neutral world: a discrete savings account which represents the investment of 1 unit money at time T_0 in a T_1-bond, with a reinvestment of collected face value at time T_1 in a T_2-bond, and so on is selected as numéraire. It is given as:

$$N_{spot}(t) = \frac{P_{\eta(t)}(t)}{P_1(0)}\prod_{i=1}^{\eta(t)-1}\frac{P_i(T_i)}{P_{i+1}(T_i)} \tag{4.14}$$

The measure corresponding to this numéraire is called spot measure. It can be shown that [13] the corresponding drift function is

$$\mu_i^{\mathbb{Q}^{N_{spot}}}(t) = \sum_{j=\eta(t)}^{i}\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\sigma_i(t)\sigma_j(t)\rho_{i,j}(t) \tag{4.15}$$

Under spot measure, current price ($t = 0$) of a derivarite that gives a payoff at time T_k is given by:

$$V(0) = E\left[payoff(T_k)\prod_{i=0}^{k-1}\frac{1}{1+\delta L_i(T_i)}\right] \tag{4.16}$$

4.3. Discretization of dynamics of forward rates

Since the dynamics of forward LIBOR rates given in equation (4.3) cannot be expressed as a known distributional process, numerical methods can be used for their evaluation. Monte Carlo simulation is a widely used method for generating forward LIBOR rates. With Monte Carlo method, forward rates are simulated not as a continuous time process but instead with their discrete approximations. These approximate quantities are defined over a finite set of date, commonly equally seperated with Δt. For the sake of simplicity and practical purposes, in this thesis it is assumed that δ is an integer multiple of Δt; in other words every T_i on tenor structure coincides with a simulated date. The standard approach is to discretize the log of the forward rates, as this ensures that the forward rates remain positive. Applying Ito's lemma with $f(L_i(t)) = \log(L_i(t))$ where $L_i(t)$ is given by equation 4.3 gives

$$d\log(L_i(t)) = \mu_i^{\mathbb{P}}(t) - \frac{1}{2}\sigma_i^2(t))dt + \sigma_i(t)dW_i^{\mathbb{P}}(t). \tag{4.17}$$

The simplest and the most popular discretization scheme that can be applied to approximate the process given in equation 4.17 is the Euler scheme. The corresponding Euler scheme of equation 4.17 is given by:

$$\log(L_i(t+\Delta t)) = \log(L_i(t)) + (\mu_i(t) - \frac{1}{2}\sigma_i^2(t))\Delta t + \sigma_i(t)\Delta W_i(t) \tag{4.18}$$

Taking the exponential of this equation gives the Euler scheme for forward LIBOR rates

$$L_i(t+\Delta t) = L_i(t)\ \exp\left[(\mu_i(t) - \frac{1}{2}\sigma_i^2(t))\Delta t + \sigma_i(t)\Delta W_i(t)\right] \tag{4.19}$$

The Euler approximation assumes that the forward rates and volatility terms are are approximately constant between $[T_i; T_{i+1}]$ and equal to their value at the start of the interval.

The main flaw of this discretization is that it is not arbitrage-free since discrete deflated bond prices calculated with the generated forward rates are not positive martingales over the finite set of date they are defined [16]. Therefore, it is worthwhile to mention the method introduced by Glasserman & Zhao [16] which ensures arbitrage-free discretization.

4.3.1. Arbitrage-free discretization of lognormal forward Libor rates

Discretization of continuous time LMM for simulation purposes causes the lost of its two important properties: securing arbitrage- free bonds and pricing caps and swaptions in accord with Black's formula. Glasserman and Zhao [16] introduce new algorithms which preserve the mentioned continuous-time formulation properties even after the discretization. They transform the Libor rates to positive martingales, discretize the martingales, and then recover the Libor rates from these discretized variables, rather than discretizing the rates themselves.

<u>4.3.1.1. Terminal Measure.</u> Differences of deflated bond prices are selected as appropriate martingales. Let

$$X_n(t) = X_n(t) \prod_{i=n+1}^{N} (1 + \delta L_i(t)) = \frac{1}{\delta}(D_n(t) - D_{n+1}(t)) \tag{4.20}$$

Then each X_n is a martingale and satisfies

$$\frac{dX_n}{X_n} = \left(\sigma_n + \sum_{j=n+1}^{N} \frac{\delta X_j \sigma_j}{1 + \delta X_j + \cdots + \delta X_N} \right) dW \tag{4.21}$$

and

$$L_n = \frac{X_n}{1 + \delta X_{n+1} + \cdots + \delta X_N} \tag{4.22}$$

.

Euler discretization of X_n is given by

$$X_i(t+\Delta t) = X_i(t)\exp(-\frac{1}{2}{\sigma_{X_n}}^2(t))\Delta t + \sigma_i(t)\Delta W_i(t)) \tag{4.23}$$

with

$$\sigma_{X_n} = \sigma_n + \sum_{j=n+1}^{N} \frac{\delta X_j \sigma_j}{1+\delta X_j + \cdots + \delta X_N} \tag{4.24}$$

<u>4.3.1.2. Spot Measure.</u> Following martingale variables are used for discretizing spot Libor measure

$$V_n(t) = \left(1 - \frac{1}{1+\delta L_n(t)}\right)\prod_{i=1}^{n-1}\frac{1}{1+\delta L_i(t)}, V_{N+1} = \prod_{i=1}^{N}\frac{1}{1+\delta L_i(t)} \tag{4.25}$$

Then each V_n is a martingale and satisfies

$$\frac{dV_n}{V_n} = \left[\left(\frac{V_n + V_{n-1} + \ldots + V_1 - 1}{V_{n-1} + \ldots + V_1 - 1}\right)\sigma_n + \sum_{i=\eta}^{n-1}\left(\frac{V_i}{V_{i-1} + \ldots + V_1 - 1}\right)\sigma_i\right] dW \tag{4.26}$$

and

$$L_n = \frac{V_n}{\delta(1 - V_n - \ldots - V_1)} \tag{4.27}$$

Euler discretization of V_n is given by

$$V_i(t+\Delta t) = V_i(t)\exp(-\frac{1}{2}{\sigma_{V_n}}^2(t))\Delta t + \sigma_i(t)\Delta W_i(t)) \tag{4.28}$$

with

$$\sigma_{V_n} = \phi\left(\frac{V_n + V_{n-1} + \ldots + V_1 - 1}{V_{n-1} + \ldots + V_1 - 1}\right)\sigma_n + \sum_{i=\eta}^{n-1}\phi\left(\frac{V_i}{V_{i-1} + \ldots + V_1 - 1}\right)\sigma_i \tag{4.29}$$

where

$$\phi(x) = \min\left\{1, x^+\right\}$$

4.4. Volatility Structure

In this section, the two widely used approaches for specifying the volatility structure are described, following closely Brigo and Mercurio [10]. First approach is based on the assumption that forward rates have deterministic piecewise-constant instantaneous volatilities. This is a sensible assumption since interest rates are not determined for periods shorter than $\delta = [T_i, T_{i+1}]$ in LMM. Second approach assumes that volatilities are described by a parametric equation. This approach has the advantage that volatilities can be defined at any time point t, so this gives a freedom when a discretization with small times steps are performed. Also, it can become practically useful in case of lacking data, making calibration hard or impossible. In such a case, an intuitive volatility expectation can easliy be deployed to the model.

4.4.1. Piecewise-constant instantaneous volatility

In the most general terms, the whole matrix of piecewise-constant instantaneous volatilities can be presented as follows:

$$\Sigma = \begin{bmatrix} \sigma_{T_1,T_0} & \text{Dead} & \text{Dead} & \ldots & \text{Dead} \\ \sigma_{T_2,T_0} & \sigma_{T_2,T_1} & \text{Dead} & \ldots & \text{Dead} \\ \sigma_{T_3,T_0} & \sigma_{T_3,T_1} & \sigma_{T_3,T_2} & \ldots & \text{Dead} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_{T_{n-1},T_0} & \sigma_{T_{n-1},T_1} & \sigma_{T_{n-1},T_2} & \ldots & \sigma_{T_{n-1},T_{n-2}} \end{bmatrix} \tag{4.30}$$

where $\sigma_{i,j}$ is the constant volatility of forward rate i for period $(T_j, T_{j+1}]$. Once a forward rate is reset, its volatility becomes "dead" for the proceeding periods. Estimation of the above matrix is done by calibration methods that will be explained in chapter 5. While some of the methods can reproduce this matrix, some may require further assumptions regarding different factors influencing the volatilities. Practically there are three factors considered in most formulations:

- T, forward rate maturity
- $(T-t)$, forward rate time to maturity
- t, calender time

Let ϕ, φ and γ be functions of T, $(T-t)$ and t respectively. Volatility matrix can be respresented in the terms of these functions as:

$$\Sigma = \begin{bmatrix} \phi_1\varphi_1\gamma_0 & \text{Dead} & \text{Dead} & \dots & \text{Dead} \\ \phi_2\varphi_2\gamma_0 & \phi_2\varphi_1\gamma_1 & \text{Dead} & \dots & \text{Dead} \\ \phi_3\varphi_3\gamma_0 & \phi_3\varphi_2\gamma_1 & \phi_3\varphi_1\gamma_2 & \dots & \text{Dead} \\ \dots & \dots & \dots & \dots & \dots \\ \phi_{n-1}\varphi_{n-1}\gamma_0 & \phi_{n-1}\varphi_{n-2}\gamma_1 & \phi_{n-1}\varphi_{n-3}\gamma_2 & \dots & \phi_{n-1}\varphi_1\gamma_{n-2} \end{bmatrix} \tag{4.31}$$

One of the most popular assumptions about the volatilities is that they can be considered as time homogenous. In this case, time to maturity of the forward rates is the only influencing factor. For example, $L_5(T_2)$ will have the same volatility as $L_3(T_0)$ and $L_7(T_4)$. The volatility matrix can be constructed by setting $\phi = 1$ and $\gamma = 1$ as follows:

$$\Sigma = \begin{bmatrix} \varphi_1 & \text{Dead} & \text{Dead} & \dots & \text{Dead} \\ \varphi_2 & \varphi_1 & \text{Dead} & \dots & \text{Dead} \\ \varphi_3 & \varphi_2 & \varphi_1 & \dots & \text{Dead} \\ \dots & \dots & \dots & \dots & \dots \\ \varphi_{n-1} & \varphi_{n-2} & \varphi_{n-3} & \dots & \varphi_1 \end{bmatrix} \tag{4.32}$$

The second widely used assumption is that the volatilities depend solely on the maturity of the underlying forward rate. In this case, volatility of a certain forward rate is the same throughout its life horizon. The volatility matrix can be constructed by setting $\varphi = 1$ and $\gamma = 1$ as follows:

$$\Sigma = \begin{bmatrix} \phi_1 & \text{Dead} & \text{Dead} & \dots & \text{Dead} \\ \phi_2 & \phi_2 & \text{Dead} & \dots & \text{Dead} \\ \phi_3 & \phi_3 & \phi_3 & \dots & \text{Dead} \\ \dots & \dots & \dots & \dots & \dots \\ \phi_{n-1} & \phi_{n-1} & \phi_{n-1} & \dots & \phi_{n-1} \end{bmatrix} \tag{4.33}$$

4.4.2. Continuous parametric form of instantaneous volatility

Another popular way to express the volatility structure is to use functional forms that can be financially meaningfull and computationally feasible. These functions should also be able to conserve the characteristic shape of the interest rate term structure. Their flexibility for calibration to interest rate derivatives is another important aspect to be considered. Again it is convenient to form a function that can reflect the basic influencing factors introduced in section (4.4.1): forward rate maturity, forward rate time to maturity, and calender time. This function can be represented as the product of three functions, each depending one of the factors stated above:

$$\sigma_i(t) = f(T_i)g(T_i - t)h(t) \tag{4.34}$$

In this section, function forms proposed by Rebonato [17] are presented and their financial implications are briefly discussed.

4.4.2.1. Function depending on forward rate maturity, $f(T_i)$. Function depending on forward rate maturity serves as a smoothing function that forces $\sigma_i(t)$ to fit current term structure of volatility and hence assures correct pricing of current market caplets.

It can be given as:

$$f(T_i) = 1 + \upsilon_i, \quad 1 \leq i \leq n-1 \tag{4.35}$$

where i is the i^th forward rate with maturity T_i.

4.4.2.2. Function depending on forward rate time to maturity, $g(T_i - t)$. Empirical observations show that term structure of volatilities generally resemmeble the graph shown in figure : it is increasing in short term, making a hump at the mid-range and finally monotonically decreasing in the long term. Rebonato [17] claims that this shape can be reflected by the time homogenous component of $\sigma_i(t)$. Moreover, function's parametric form should comply with observable market data and be financially intuitive. Finally calculation of variance and covariance terms requires integration of the square of this function hence integration of the square of this function by means of a simple analytic solution is desirable. Rebonato [17] proposes the following function:

$$g(T_i - t) = (a + b(T_i - t)) \exp(-c(T_i - t)) + d \tag{4.36}$$

where a, b, c and d are completely free parameters that one may choose in order to specify the short, mid, and long term behaviour of the term structure. However while deciding on parameter values, the following conditions should be satistied for the sake of financial intuition sanity [17].

- $a + d > 0$
- $d > 0$
- $c > 0$

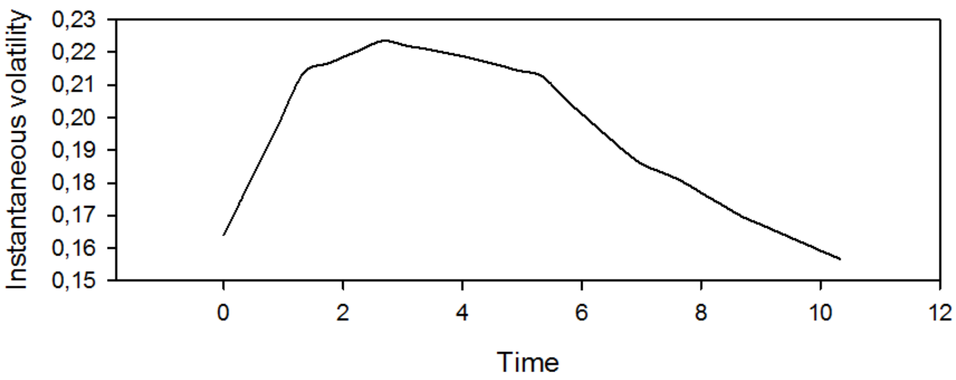

Figure 4.1. An example possible shape of instantaneous volatility

4.4.2.3. Function depending on calender time, $h(t)$. While determining this function, there is a room for subjectivity since there are no clear financially intuitive conditions that must be satisfied. But its flexibility should be in a balance: too much relaxation may lead to capturing of unwanted market noise and too much rigidity would make difficult to get the effect of time dependence. Rebonato [17] proposes a function that is a linear combination of sine functions multiplied by an exponential decay factor:

$$h(t) = \left(\sum_{i=1}^{N} \varepsilon_i \sin\left(\frac{t\pi i}{M} + \varepsilon_{i+1} \right) \right) \exp(-\varepsilon_{N+1} t) \tag{4.37}$$

where N is the number of free parameters, recommended to be as small as 2 or 3, and M is the maturity of the longest caplet available. Generally $h(t)$ is set to one to avoid noise in the calibration.

5. Calibration

In this chapter we focus on the calibration of LIBOR Market Model. One of the reasons why forward LIBOR market models are popular as interest rate models lies in their success of reproducing market prices for simple interest rate derivatives by proper calibration. In chapter 4 we saw that the dynamics of forward LIBOR rates are defined in terms of two crucial parameters: volatility and covariance among rates. Generally the aim of LMM calibration is to estimate these parameters with the information implied by market so that model prices of derivatives match their market values as close as possible. Volatility and correlation structures can aslo be determined so that a match with historical data is achieved or the practitioner's foresight about the future rate movements is captured, however in this thesis we will get into their details.

The standard calibration instruments are caps and swaptions since they are heavily traded plain vanilla options and they can be priced by market standard closed form formulations.

5.1. Calibration to Caps

Calibrating LMM to caps (and floors) is one of the frequently used approaches in practice and it is widely mentioned in literature. The main advantage of this calibration technique is that it is very simple and intuitive. The disadvantage is that caps do not provide information about the correlation structure. First we show how caplet volatilities are stripped from cap volatilities with the aid of Black'76 formula and then how forward rate volatilities are calibrated to stripped caplet volatilities according to their definion of being non-parametric or parametric.

5.1.1. Stripping caplet volatilities from cap quotes

In real markets, caplets are not traded directly but they are traded in the form of caps. For this reason, the best data one can get from market consists of cap prices. Since by definition, caps can be split additively in caplets, i.e. cap prices can be calculated by summing its constituent caplets, a bootstrapping algorithm can be used to price caplets separately.

Three small remarks would be appropriate before starting the explanation of the algorithm: First, when a cap's volatility is mentioned one should understand that this volatility is valid throughout the cap's life, in other words as if all the caplets forming this cap has this unique volatility.

Second, caps generally have yearly maturities of 1 to 20 years. But a significant part of caps (US Dolar Caps) have reset dates at every three months; i.e. they consist of caplets which are active for three month periods. For bootstrapping algorithm to work, first one should perform interpolation to estimate cap volatilities for each quarter of the years. Generally there are two approaches: linear and cubic interpolation. Linear interpolation is commonly used in market practice but higher order polynomial interpolations such as cubic and quartic are becoming more popular for they result with more smooth curves.

Third, one may also have to deal with missing data for maturities shorter than the shortest maturity quoted. Generally, and in this thesis, data is missing for maturities shorter than 1 year. One solution to solve this shortcoming can be assigning the same volatility for all caplets of the shortest maturity cap (possibly the same volatility of the cap itself); or one could perform extrapolation.

Let there be n cap in market, each with consecutive yearly maturity $0 = T_0 < T_1 < T_2 < \ldots < T_n$, consisting of caplets with quarterly maturities. In such a case to be able to perform stripping it should be assumed that there are virtual caps with quarterly maturities, and while deciding on these virtual caps's volatilities, the interpolation/extrapolation approaches mentioned before can be used. In this framework, the first cap (actually it is a virtual cap) consist only of a single caplet, it resets at 3^{rd} month and the payment is made at 6^{th} month. The second cap (also a virtual one) consist of two caplets; they reset at 3^{rd} and 6^{th} month and their payments are made at 6^{th} and 9^{th} month respectively. The third cap (this is the first actually quoted cap in this example), consist of three caplets; they reset at 3^{rd}, 6^{th} and 9^{th} month and their payments are made at 6^{th}, 9^{th}, and 12^{th} month respectively. Composition of the rest of the real and virtual caps follows the same logic. In the first iteration, volatility of the first caplet is found from the following equation:

$$V_{6m}^{Cap}(t) = V_{6m}^{Caplet}(t)$$

This implies that the first caplet volatility is equal to the first cap's volatility: $\sigma_{6m}^{Caplet}(t) = \sigma_{6m}^{Cap}(t)$. In the next iteration, volatility of the second caplet is found from the following equation:

$$V_{9m}^{Cap}(t) = V_{6m}^{Caplet}(t) + V_{9m}^{Caplet}(t)$$

by solving it with respect to $\sigma_{9m}^{Caplet}(t)$. This can be done by inverting the Black'76 function with respect to $\upsilon_i(t)$, or by using some numerical solver. The stripping algorithm continues in the same way.

As a numerical example, we will apply this procedure to market quoted cap volatilities for the date 06 June 2011 given in table 5.1. Table 5.2 shows actual and virtual cap volatilities along with their corresponding stripped caplet volatilities. In getting second and fourth columns of table 5.2 it is assumed that for maturities less than 1 year volatilities are flat and equal to first cap's volatility and quarterly volatilities are calculated using linear interpolation. The plot comparing cap and stripped caplet volatilities is given in figure 5.1.

Table 5.1. 10 Year Cap Volatility Data Example

Maturity (year)	**Cap Volatility**	**ATM Strike**
1	0,6088	0,0039
2	0,7379	0,0068
3	0,7422	0,0109
4	0,602	0,0152
5	0,4982	0,0193
6	0,4335	0,0228
7	0,3897	0,0258
8	0,3591	0,0282
9	0,3368	0,0301
10	0,3199	0,0317

5.1.2. Non-Parametric Calibration to Caps

The aim of non-parametric calibration to caps is to obtain volatility structure under the assumption that instantaneous volatilities are piecewise-constant as described in section 4.4.1 using the stripped caplet volatilities. We will illustrate how this is performed for the two popular cases mentioned in section 4.4.1, the time homogeneity assumption and dependence on underlying maturity assumption.

<u>5.1.2.1. Volatilities depending on the time to maturity of the forward rates.</u> As pointed out in section 4.4, one of the most frequently used assumption is the time homogeneity of forward rate volatilities. Forward rate volatilities are assumed to be dependent only on their time to maturities. This assumption implicates that significant variations at volatility structure are not anticipated in the future. At the end of calibration, the volatility matrix will be in the form that is given in equation 4.32. Now that caplet volatilities are obtained as described in section 5.1.1, it is not difficult to get forward

Table 5.2. Caplet volatilities stripped from cap volatilities.

Maturity (year)	**Linearly interpolated or extrapolated volatility**	**Caplet volatility**	**Maturity (year)**	**Linearly interpolated or extrapolated volatility**	**Caplet volatility**
0.5	0.549	0.549	5.25	0.482	0.353
0.75	0.577	0.604	5.5	0.466	0.33
1	0.609	0.657	5.75	0.45	0.306
1.25	0.641	0.706	6	0.434	0.283
1.5	0.673	0.752	6.25	0.423	0.314
1.75	0.706	0.799	6.5	0.412	0.297
2	0.738	0.846	6.75	0.401	0.281
2.25	0.739	0.743	7	0.39	0.265
2.5	0.74	0.744	7.25	0.382	0.289
2.75	0.741	0.746	7.5	0.374	0.277
3	0.742	0.747	7.75	0.367	0.265
3.25	0.727	0.653	8	0.359	0.253
3.5	0.672	0.399	8.25	0.354	0.272
3.75	0.637	0.447	8.5	0.348	0.263
4	0.602	0.4	8.75	0.342	0.254
4.25	0.576	0.414	9	0.337	0.245
4.5	0.55	0.379	9.25	0.333	0.26
4.75	0.524	0.343	9.5	0.328	0.253
5	0.498	0.307	9.75	0.324	0.246

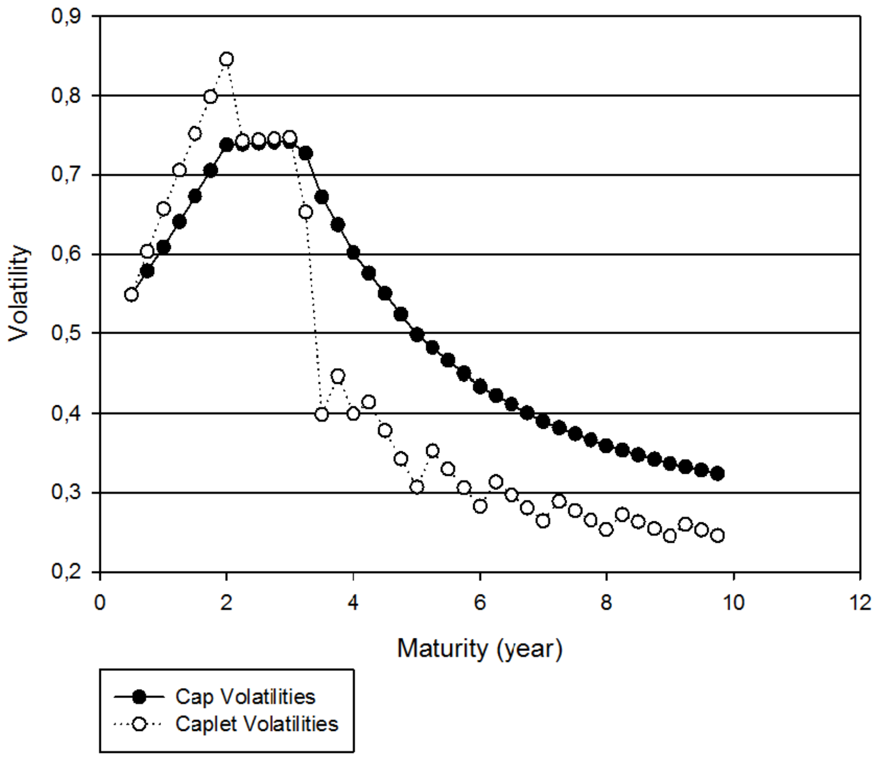

Figure 5.1. Cap and Stripped Caplet Volatilities.

rate volatilities with the help of eqauation (4.9). Since forward rate volatilities are assumed to be piecewise constant, integration at the right hand side of the equation (4.9) becomes a summation as follows:

$$\sigma_i^{2Caplet} = \frac{\sum_{j=1}^{n-1} \bar{\sigma}_i^2(T_j)\delta_j}{T_i - 0} \tag{5.1}$$

In the first step, the above equality for the first caplet, i.e. $i = 1$, becomes:

$$\sigma_1^{2Caplet} = \frac{\bar{\sigma}_1^2(T_1)\delta_1}{T_1}$$

and since $\delta_1 = T_1$, $\bar{\sigma}_1(T_1) = \sqrt{\sigma_1^{2Caplet}}$. This volatility is not only valid for the forward LIBOR rate $L_1(0)$, but also for all the rates having one period before their maturity. These are $L_2(1), L_3(2), \ldots, L_{n-1}(n-2)$.

In the next step, equation 5.1 is solved for $i = 2$ in terms of $\bar{\sigma}_2(T_1)$; volatility for the forward rate forward LIBOR rate $L_2(0)$. Note that this can be done for $\bar{\sigma}_2(T_2)$ has already been found in the first step. Then likewise, this volatility is assigned to all forward LIBOR rates that have two periods before their maturity, $L_3(1), L_4(2), \ldots, L_{n-1}(n-3)$.

The procedure continues similarly for the next steps until finally $L_{n-1}(1)$ is obtained.

The results of the first four step for our numerical example is given in the following matrix:

$$\begin{bmatrix} 0.549 & \text{Dead} & \text{Dead} & \ldots & \ldots \\ 0.654 & 0.549 & \text{Dead} & \ldots & \ldots \\ 0.753 & 0.654 & 0.549 & \ldots & \ldots \\ 0.835 & 0.753 & 0.654 & 0.549 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

It is worth mentioning the most important shortcoming of this method. Note that at each step there is a quadratic function to be solved. It is very likely that one can end up with taking square root of a negative value. The reason is time homogeneity assumption requires $\sigma_i^{2\,Caplet}(T_i - 0) \leq \sigma_{i+1}^{2\;\;Caplet}(T_{i+1} - 0),\ \forall i = 1, \ldots, n-2$ and this condition is generally not satisfied. In such a case, one possible solution can be replacing the negative value with zero. This will help algorithm to be carried on however it will make impossible recovery of observed caplet prices.

5.1.2.2. Volatilities depending on the maturity of the forward rates. Under this assumption, the instantaneous volatilities are uniquely forward rate specific. What makes this assumption appealing is that fitting of an exogenous volatlity term structure is immediate. On the other hand, it has the shortcoming that each forward rate has a single constant volatility all along their lifetime. Therefore forward rates that have the same residual term to maturity will have different volatilities. As a result, shape of volatilities' term structure will change as time goes on.

Again recall equation (5.1). Since volatilities depend only on forward rate maturities, a forward rate's volatility is the same for all its lifetime, i.e.

$$\bar{\sigma}_i^2(T_j) = \bar{\sigma}_i^2,\ \forall i = 1, \ldots, n-1$$

Then equation (5.1) simple becomes

$$\begin{aligned} \sigma_i^2 &= \frac{\sum_{j=1}^{n-1} \bar{\sigma}_i^2 \delta_j}{T_i - 0} \\ &= \frac{\bar{\sigma}_i^2 T_i}{T_i - 0} \\ &= \bar{\sigma}_i^2 \end{aligned} \tag{5.2}$$

Simply assigning each $i^t h$ caplet volatility as the volatilifty of the forward rate maturing at T_i completes the calibration process.

The results of the first four step for our numerical example is given in the following matrix:

$$\begin{bmatrix} 0.549 & \text{Dead} & \text{Dead} & \ldots & \ldots \\ 0.604 & 0.604 & \text{Dead} & \ldots & \ldots \\ 0.657 & 0.657 & 0.657 & \ldots & \ldots \\ 0.706 & 0.706 & 0.706 & 0.706 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

5.1.3. Parametric Calibration of Volatility Structure to Caps

Parametric calibration to caps requires implied caplet volatilities as input, thus the bootstraping procedure for extracting caplet volatilities described in section 5.1.1 should be performed before starting. Recall that the most general functional form for forward rate volatilities is given by equation (4.34). Calibration starts with the estimation on the part depending on time to maturity, $g(T_i - t)$. Since we are dealing with caplet volatilities, equation (4.9) should be satisfied:

$$\begin{aligned} \sigma^2_{i,Black} &= \frac{\int\limits_0^{T_i} \sigma_i^2(t)du}{T_i - 0} \\ &= \frac{\int\limits_0^{T_i} \left|(a + b(T_i - t))\exp(-c(T_i - t)) + d\right| dt}{T_i - 0} \end{aligned} \tag{5.3}$$

The unconstrained optimization problem minimizing differences between theoretical and market caplet volatilities is expressed as:

$$\min \sqrt{\sum_{i=1}^{n-1} \left(\sigma^2_{i,Black} - \int\limits_0^{T_i} \left|(a + b(T_i - t))\exp(-c(T_i - t)) + d\right| dt \right)^2}$$

Performing optimization with initial parameter values $a, b, c, d = 0.5$ for the example cap data given in table 5.1returns the following solution: $a = 1.1855, b = -2.1705, c = 0.6114$, and $d = 0.0077$. This result are in accord with our emprical expectations $a + d > 0, c > 0$, and $d > 0$. A comparison of caplet volatilities calculated using the functional form with optimization results as parameter values and stripped caplet volatilities is given in figure 5.2.

5.3.

As can be seen from figure (5.2) fitting to todays volatility term structure is not perfect. The forward rate depending function $f(T_i)$ will play the role of smoothing

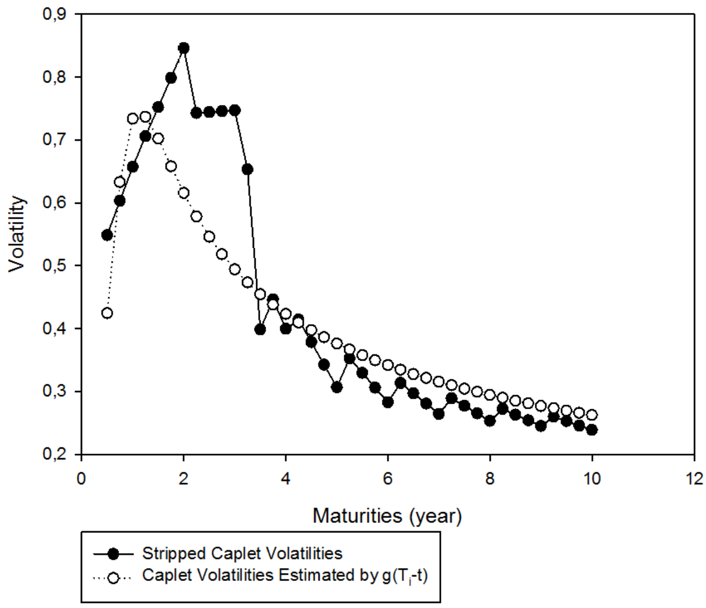

Figure 5.2. Stripped Caplet Volatilities vs. Estimated Caplet Volatilities by $g(T_i - t)$

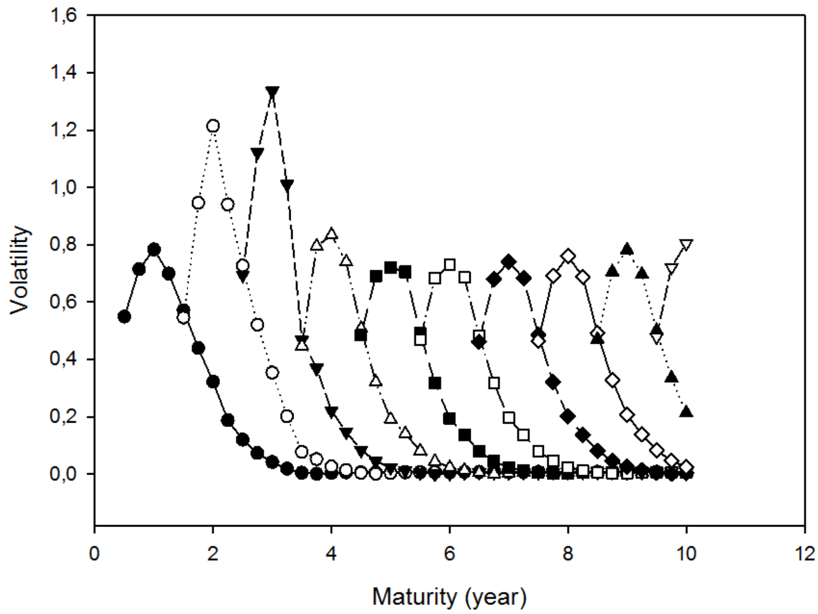

Figure 5.3. Shape of volatility term structure in time

function that ensures perfect pricing of todays market caplets in this calibration process. Now that the parameters' values of $g(T_i - t)$ are specified, the next minimization problem is to be performed:

$$\min \sqrt{\sum_{i=1}^{n-1} \left[\left(\sigma^2_{i,Black} - \int_0^{T_i} \left| (\hat{a} + \hat{b}(T_i - t)) \exp(-\hat{c}(T_i - t)) + \hat{d} \right| dt \right) (1 + \upsilon_i) \right]}$$

Since perfect fitting to caplet volatilities is achieved with this second minimization problem, estimation of the last part of equation 4.34, $h(t)$ is generally done by assigning value one. At the end of calibration, the movement of volatility term structure for our example described by parametric $\sigma_i(t)$ is shown in figure

5.2. Non-Parametric Calibration of Volatility Structure to Swaptions

In section 5.1.2 we showed that cap prices are sufficient for calibrating LMM volatility structure, at least if the derivatives to be priced are forward rate options. Equivalance of caplet prices given by Black'76 formula and LMM model allowed model calibration to implied caplet volatilities retrieved by inverting Black'76 formula. However, a big portion of the interest rate derivatives have the swap rate as underlying, so it is naturally desirable to provide swaption data for calibration. Unfortunately there is no known closed-form solution for swaption pricing in LMM framework. Nevertheless, a Black formula extention for swaptions, proposed by Andersen and Andreasen [18], is used in market practice. Theoratically it is impossible to price both forward LIBOR rate and swap options with Black formula since forward LIBOR and swap rate cannot be lognormal simultaneously [19]. But Andersen and Andreasen [18] show that swap rate can be approximated by a lognormal process since in practice forward rate curve movements are generally parallel ($L_i(t)$ are strongly correlated) and the ratio of volatility functions can be considered as constant. With the Black formula as the swaption pricing tool, Dariusz Gatarek [20, 21, 22] develops the calibration method called the separated approach, which enables calibration of the LMM to full set of swaptions traded in the market.

Recall the value of the payer swaption at maturity T_α given by equation 2.21. Let the term

$$A_{\alpha,\beta}(t) = \delta \sum_{i=\alpha+1}^{\beta} P(t,T_i)$$

denote the annuity of the underlying swap. Let's select annuity term as the pricing numéraire and denote the corresponding measure as $\mathbb{Q}^{A_{\alpha,\beta}(t)}$. $^{A_{\alpha,\beta}(t)}$ is known as the forward swap measure for the swaption [13]. Applying Ito's lemma to equilibrium swap rate given in equation (**??**) gives:

$$\begin{aligned} dSR_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} \frac{\partial SR_{\alpha,\beta}(t)}{\partial L_{i-1}(t)} L_{i-1}(t)\sigma_i(t)dW(t)^{\mathbb{Q}^{A_{\alpha,\beta}(t)}} \\ &= SR_{\alpha,\beta}(t) \sum_{i=\alpha+1}^{\beta} R^i_{\alpha,\beta}(t)\sigma_i(t)dW(t)^{Q^{A_{\alpha,\beta}(t)}} \end{aligned} \tag{5.4}$$

where

$$R^i_{\alpha,\beta}(t) = \frac{\delta L_{i+1}(t)}{1+\delta L_{i+1}(t)} \frac{P(t,T_\alpha)A_{i-1,\beta}(t) + P(t,T_\beta)A_{\alpha,i-1}(t)}{(P(t,T_\alpha) - P(t,T_\beta))A_{\alpha,\beta}(t)}$$

Under the assumption that equilibrium swap rate process is a positive martingale with respect to the measure $\mathbb{Q}^{A_{\alpha,\beta}(t)}$, the dynamics can be express as Brownian motion given by:

$$dSR_{\alpha,\beta}(t) = SR_{\alpha,\beta}(t)\sigma_{\alpha,\beta}(t)dW(t)^{Q^{A_{\alpha,\beta}(t)}} \tag{5.5}$$

From 5.4 and 5.5:

$$\sigma_{\alpha,\beta}(t) = \sum_{j=\alpha+1}^{\beta} R^j_{\alpha,\beta}(t)\sigma_j(t)$$

Gatarek [21] makes the assumption that $\frac{P(t,T_i)}{\sum_{i=\alpha+1}^{\beta} P(t,T_i)}$ is independent of $L_j(t)$ and get the following approximation for $R^i_{\alpha,\beta}(t)$:

$$R^i_{\alpha,\beta}(t) \cong \frac{P(t,T_{i-1}) - P(t,T_i)}{P(t,T_\alpha) - P(t,T_\beta)} \tag{5.6}$$

Hence swap volatilities can be expressen as weighted average of forward LIBOR rate volatilities:

$$\sigma_{\alpha,\beta}(t) = \frac{\sum\limits_{j=\alpha+1}^{\beta} P(t,T_j)L_{j-1}(t)\sigma_j(t)}{\sum\limits_{j=\alpha+1}^{\beta} P(t,T_j)L_{j-1}(t)}$$

Let Φ^i denote a $n-1 \times n-1$ covariance matrix of LMM whose entries are given by

$$\varphi^i_{jk} = \int\limits_0^{T_i} \sigma_j(t).\sigma_k(t)dt \tag{5.7}$$

Again using parallel movements of interest rates assumption $R^i_{\alpha,\beta}(t)$ is approximated by $R^i_{\alpha,\beta}(0)$. Finally

$$T_j \underset{market}{\sigma^2_{\alpha,\beta}} = \int\limits_0^{T_j} \sigma^2_{\alpha,\beta}(t)dt \cong \sum_{k=j+1}^{\beta}\sum_{l=j+1}^{\beta} R^l_{j,\beta}(0)\varphi^j_{lk}R^k_{j,\beta}(0) \tag{5.8}$$

where $\underset{market}{\sigma^2_{\alpha,\beta}}$ denotes market swaption volatility. ATM swaption volatility matrix market data quoted on 6 June 2011 is given in table 5.5 in which rows represent option maturities and columns represent length of the underlying swap. Market data lacks for 6, 8 ,and 9 year maturities so they are linearly interpolated for calibration purpose. We also need the LIBOR yield curve for the same date to calculate the discount factors (virtual zero coupon bond prices) appearing in equation 5.6. LIBOR yield curve for 06 June 2011 is given in figure 5.4 and the corresponding zero-coupon bond prices are given in table 5.4.

Now let's examine covariance matrix of LMM (Φ^i) and market ATM swaption

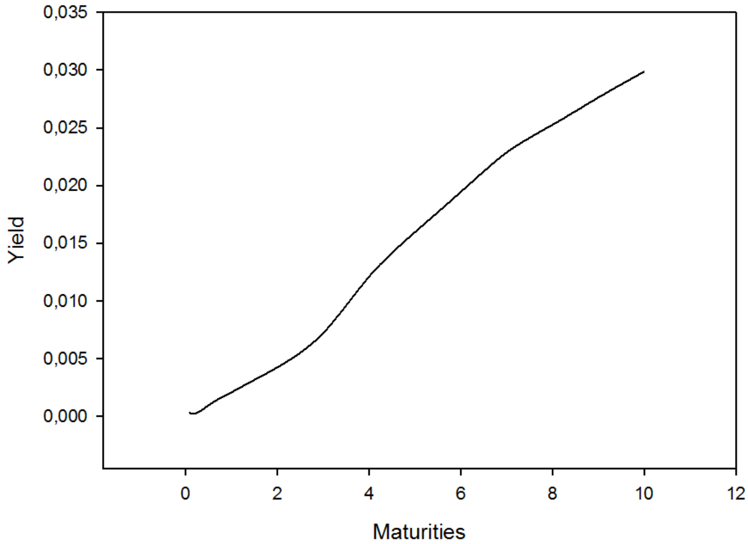

Figure 5.4. LIBOR yield curve for 06 June 2011.

Table 5.3. Forward LIBOR rates for 06 June 2011.

$L_0(0)$	0.0021
$L_1(0)$	0.0065
$L_2(0)$	0.0133
L_3	0.0271
L_4	0.0319
L_5	0.0375
L_6	0.0443
L_7	0.043
L_8	0.0476
L_9	0.0514
L_{10}	0.0617

Table 5.4. Zero-coupon bond prices for 06 June 2011

P(0,0.25)	0.999918
P(0,0.5)	0.999505
P(0,1)	0.997892
P(0,2)	0.991496
P(0,3)	0.978514
P(0,4)	0.952676
P(0,5)	0.923255
P(0,6)	0.889852
P(0,7)	0.852127
P(0,8)	0.81703
P(0,9)	0.779908
P(0,10)	0.741782

Table 5.5. Market data of ATM swaption volatilities, 06 June 2011

Swaption maturity (year)	Underlying swap length (year)									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.815	0.648	0.523	0.44	0.394	0.356	0.331	0.313	0.3	0.29
2	0.575	0.461	0.396	0.355	0.33	0.309	0.294	0.283	0.274	0.267
3	0.411	0.355	0.322	0.301	0.288	0.276	0.266	0.259	0.252	0.247
4	0.317	0.293	0.278	0.268	0.259	0.252	0.246	0.241	0.236	0.233
5	0.274	0.262	0.253	2.47	0.242	0.237	0.232	0.228	0.225	0.222
6	0.253	0.245	0.238	1.345	0.23	0.226	0.222	0.219	0.217	0.214
7	0.232	0.227	0.223	0.22	0.217	0.214	0.212	0.209	0.208	0.206
8	0.221	0.217	0.214	0.212	0.209	0.207	0.205	0.203	0.202	0.2
9	0.211	0.208	0.205	0.203	0.202	0.2	0.199	0.197	0.196	0.195
10	0.2	0.198	0.196	0.195	0.194	0.193	0.192	0.191	0.19	0.189

volatility matrix more closely. For every $i \in \{1, \ldots, n-1\}$, Φ^i containts $\frac{i(i+1)}{2}$, i.e. number of terms in the lower triangular matrix given in equation (4.30), free variables. This makes a total number of

$$\sum_{i=1}^{n-1} \frac{i(i+1)}{2} = \frac{n^3 + 5n}{6}$$

free variables. ATM swaption market data provides only $\frac{(n-1)(n-2)}{2}$ volatilities, excluding volatility terms for period $[T_0, T_1]$, which then should be extracted from market cap data. As a result, a total of $\frac{(n)(n-1)}{2}$ volatility term is available. Therefore some assumptions should be made regarding the parametrization of Φ^i. Longstaff, Santa-Clara, and Schwartz [23] propose fitting a time-homogeneous string model to the set of market prices for swaptions and caps. They make the assumption that the covariance between $\frac{dF_i(t)}{F_i(t)}$ and $\frac{dF_j(t)}{F_j(t)}$ is completely time homogeneous, i.e. it depends only on $T_i - t$

and $T_j - t$. To apply this assumption Gatarek [20] expresses Φ^i as:

$$\Phi^i = \Lambda_i \Phi$$

where Λ_i are some positive numbers and Φ is a covariance matrix. Equation (5.8) then becomes:

$$T_j \underset{market}{\sigma^2_{\alpha,\beta}} = \Lambda_i \sum_{k=j+1}^{\beta} \sum_{l=j+1}^{\beta} R^l_{j,\beta}(0) \varphi_{lk} R^k_{j,\beta}(0) \tag{5.9}$$

Assigning $\Lambda_i = T_i$ leads to the Longstaff-Schwartz-Santa Clara's time-homogenous string model. There are other possible functions that can be specified for $\Lambda_i = T_i$. Gatarek [22] lists a few possible choices:

- $\Lambda_i = e^{T_i}$
- $\Lambda_i = \sqrt{e^{T_i}}$
- $\Lambda_i = T_i - \ln T_i$
- $\Lambda_i = \sqrt{T_i - \ln T_i}$
- $\Lambda_i = 1$

Diagonal entries of Φ are calculated with the following formula:

$$\varphi_{ii} = \frac{T_i \sigma^2_{\alpha,\beta}}{\Lambda_i} \tag{5.10}$$

Off diagonal entries are calculated with the following formula:

$$\varphi_{k,N-1} = \frac{T_k \sigma^2_{k,N} - \Lambda_k \left(\sum_{l=k+1}^{N} \sum_{i=k+1}^{N} R^i_{k,N}(0) \varphi_{i-1,l-1} R^l_{k,N}(0) - 2R^{k+1}_{k,N}(0) \varphi_{k,N-1} R^N_{k,N}(0) \right)}{2\Lambda_k R^{k+1}_{k,N}(0) R^N_{k,N}(0)} \tag{5.11}$$

for $k = 1, \ldots, m$ and $N = k+2, \ldots$ The three dimentional array $R^k_{i,j}(0)$ in equation (5.11) is defined as:

$$R^k_{i,j}(0) = \frac{P(0, T_{k-1}) - P(0, T_k)}{P(0, T_i) - P(0, T_j)}$$

The recursive algorithm for calculating 5.11 is given in [22]. With time homogeneity assumption, i.e. $\Lambda_i = T_i$, variance-covariance matrix Φ corresponding to example market data given in table 5.5 is given in table 5.6.

Table 5.6. Variance-covariance matrix corresponding to example market data given in table 5.5

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.664	0.45	0.414	0.29	0.378	0.139	0.222	0.227	0.253	0.237
2	0.45	0.331	0.226	0.18	0.163	0.185	0.082	0.126	0.165	0.076
3	0.414	0.226	0.169	0.122	0.103	0.097	0.097	0.084	0.07	0.071
4	0.29	0.18	0.122	0.1	0.086	0.081	0.08	0.068	0.066	0.055
5	0.378	0.163	0.103	0.086	0.075	0.068	0.064	0.066	0.065	0.052
6	0.139	0.185	0.097	0.081	0.068	0.064	0.06	0.059	0.06	0.055
7	0.222	0.082	0.097	0.08	0.064	0.06	0.054	0.052	0.051	0.051
8	0.227	0.126	0.084	0.068	0.066	0.059	0.052	0.049	0.048	0.046
9	0.253	0.165	0.07	0.066	0.065	0.06	0.051	0.048	0.044	0.043
10	0.237	0.076	0.071	0.055	0.052	0.055	0.051	0.046	0.043	0.04

By definition covariance matrix must be positive-semidefinite. The one that is found by equations (5.10) and (5.11) does not necessarily satisfy this condition. A modified covariance matrix can be formed by removing eigenvectors of ϕ associated with negative eigenvalues. Multiplying eigenvectors associated with positive eigenvalues by squared root of these eigenvalues returns a new matrix. Multiplying this matrix by its transpose gives a positive-semidefinite matrix. Modified matrix derived from the one given in table (5.6) is given in table (5.7).

Now forward LIBOR rate volatilities can be determined using equation (5.7). Remember that instantaneous volatilities were assumed to be piecewise constant. With

Table 5.7. Modified variance-covariance matrix corresponding to example market data given in table 5.5

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.758	0.437	0.366	0.268	0.308	0.166	0.202	0.204	0.217	0.197
2	0.437	0.356	0.223	0.177	0.169	0.162	0.099	0.126	0.152	0.09
3	0.366	0.223	0.198	0.136	0.14	0.091	0.101	0.098	0.097	0.089
4	0.268	0.177	0.136	0.11	0.102	0.079	0.078	0.074	0.079	0.065
5	0.308	0.169	0.14	0.102	0.132	0.055	0.076	0.081	0.091	0.08
6	0.166	0.162	0.091	0.079	0.055	0.089	0.044	0.053	0.062	0.036
7	0.202	0.099	0.101	0.078	0.076	0.044	0.07	0.055	0.049	0.061
8	0.204	0.126	0.098	0.074	0.081	0.053	0.055	0.057	0.061	0.054
9	0.217	0.152	0.097	0.079	0.091	0.062	0.049	0.061	0.076	0.052
10	0.197	0.09	0.089	0.065	0.08	0.036	0.061	0.054	0.052	0.064

this assumption we can write:

$$\begin{aligned} \underset{Modified}{\varphi^i_{jk}} = \Lambda_i \underset{Modified}{\varphi_{j,k}} &= \int_0^{T_i} \sigma_j(.).\sigma_k(.)dt \\ &= (\sigma_j(.).\sigma_k(.)^T)T_i \end{aligned} \tag{5.12}$$

Again with time homogeneity assumption this equality reduces to:

$$\underset{Modified}{\varphi_{j,k}} = (\sigma_j(.).\sigma_k(.)^T)$$

$\sigma_j(.)$ representing i^th column of matrix 4.30. Forward LIBOR rate volatility matrix can then be derived following the next algorithm.

The resulting forward LIBOR rate volatilities is given in table 5.8. Result shows that our model is calibrated with five factors. Gatarek [21] shows that two factors can describe the model up to ten years and three factors up to twenty years. Further, he claims that in most cases a two factor calibration is satisfactory since all the swaptions are not liquid.

Moreover, Gatarek [22] adds an optimization algorithm to the seperated approach discussed above. The aim is to determine Λ_i values which results with the minimum root mean squared error of the differences between theoretical and market swaption volatilities. Optimization forces covariance matrix Φ to be positive-semidefinite, i.e. searches for Λ_i values that would result with as many non-negative eigenvalues as possible.

Now that we have determined the set of instantaneous volatilities we can calculate the correlation between the rates using the following formula:

$$cor\left(i,j\right)=\frac{\sum_{k=1}^{n}\sigma_i\left(T_k\right)\sigma_j\left(T_k\right)}{\sqrt{\sum_{k=1}^{n}\sigma_i^2(T_k)}\sqrt{\sum_{k=1}^{n}\sigma_j^2(T_k)}} \tag{5.13}$$

. The resulting correlation matrix of forward rates is given in table 5.9.

Table 5.8. Forward LIBOR rate volatilities calibrated to example market data given in table 5.5

Forward rates	Years									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
L_1	0.871	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead
L_2	0.501	0.323	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead
L_3	0.42	0.039	0.142	Dead	Dead	Dead	Dead	Dead	Dead	Dead
L_4	0.308	0.07	0.029	0.099	Dead	Dead	Dead	Dead	Dead	Dead
L_5	0.354	-0.028	-0.054	-0.035	0.041	Dead	Dead	Dead	Dead	Dead
L_6	0.191	0.204	0.019	0.055	-0.087	0	Dead	Dead	Dead	Dead
L_7	0.232	-0.054	0.037	0.089	-0.059	0	0	Dead	Dead	Dead
L_8	0.234	0.027	-0.013	0.009	-0.036	0	0	0	Dead	Dead
L_9	0.249	0.085	-0.077	-0.016	0.004	0	0	0	0	Dead
L_{10}	0.226	-0.072	-0.024	0.009	-0.082	0	0	0	0	0

Table 5.9. Forward LIBOR rate correlations calibrated to example market data given in table 5.5

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
L_1	1	0.84	0.944	0.927	0.975	0.64	0.88	0.98	0.907	0.947
L_2	0.84	1	0.841	0.893	0.777	0.908	0.629	0.885	0.93	0.633
L_3	0.944	0.841	1	0.921	0.866	0.684	0.857	0.917	0.793	0.835
L_4	0.927	0.893	0.921	1	0.846	0.798	0.885	0.939	0.864	0.817
L_5	0.975	0.777	0.866	0.846	1	0.511	0.795	0.934	0.909	0.958
L_6	0.64	0.908	0.684	0.798	0.511	1	0.56	0.752	0.759	0.401
L_7	0.88	0.629	0.857	0.885	0.795	0.56	1	0.878	0.672	0.894
L_8	0.98	0.885	0.917	0.939	0.934	0.752	0.878	1	0.934	0.901
L_9	0.907	0.93	0.793	0.864	0.909	0.759	0.672	0.934	1	0.791
L_{10}	0.947	0.633	0.835	0.817	0.958	0.401	0.894	0.901	0.791	1

6. Pricing of Interest Rate Derivatives with LMM

As previously pointed out in section 4.3, the complexity of the Libor Market Model processes obligates usage of numerical methods for their generation. If we want to use Libor Market Model for pricing interest rate derivatives or calculating future bond prices we can make use of Monte Carlo simulations, a numerical method that is widely used for market models. So far we introduced the general framework of Libor Market Model and showed how it is calibrated to market data. In this chapter our aim is to illustrate simulation implementation for this model. We will be pricing caplets and swaptions with LMM calibrated to market data and compare the results with traditional market pricing tools.

6.1. Monte Carlo Implementation

Monte Carlo implementation basically consists of 3 main parts: generation of the forward rates, calculation of derivative's payoff, and calculation of discount factor. For the generation of forward LIBOR rates paths, first of all a discretization scheme and a measure should be decided upon. Let's assume that we decided to use the exponential of the Euler discretization of $log(L_i)$ given by equation (4.19) and terminal measure with the corresponding drift function given by equation (4.12). Then the tenor structure and the step size of iteration should be specified. Tenor dates are typically specified by settlement dates and step size $(t_{k+1} - t_k)$ is selected such that tenor dates are divisible by them. In this way we get the following scheme:

$$L_i(t_{k+1}) = L_i(t_k)\exp\left[\left(-\sum_{\substack{j\geq i+1\\ j\leq n-1}} \frac{\delta_j L_j(t_k)\sigma_i(t_k)\sigma_j(t_k)\rho_{i,j}(t_k)}{1+\delta_j L_j(t_k)} - \frac{1}{2}\sigma_i^2(t_k)\right)(t_{k+1}-t_k) + \sigma_i(t_k)\left(W_i(t_{k+1}) - W_i(t_{k+1})\right)\right] \tag{6.1}$$

From the definition of Brownian Motion, the increments $W_i(t_{k+1}) - W_i(t_k)$ are normally distributed random numbers with means 0 and variance $\sqrt{(t_{k+1} - t_k)}$ correlated according to equation (4.4). These correlated Brownian Motions can be easily generated as follows

$$\begin{pmatrix} W_1(t_{k+1}) \\ W_2(t_{k+1}) \\ \vdots \\ W_{n-1}(t_{k+1}) \end{pmatrix} - \begin{pmatrix} W_1(t_k) \\ W_2(t_k) \\ \vdots \\ W_{n-1}(t_k) \end{pmatrix} = \mathbb{Q}\sqrt{(t_{k+1} - t_k)} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{pmatrix}$$

where $\mathbb{Q}$ is the lower triangular matrix obtained by Cholesky decomposition of the correlation matrix $\mathbb{R}$ satisfying $\mathbb{R} = \mathbb{Q}\mathbb{Q}^T$. , we will consider semi-annual forward LIBOR rates $L_i(t)$ with $T_i \in \{0, 0.5, 1, 1.5, 2\}$ and $t \in \{0, 0.25, 0.5, \ldots, 2\}$. Just for the sake of simplicity of this illustration, let's assume that the initial term structure and volatility structure are both flat and their values are given as $L_i(0) = 0.05$, $\sigma_i(t) = 0.1$. Similarly, let's specify the correlation matrix with the parametric function $\rho_{i,j}(t) = \rho_{i,j} = \exp\left(-a\left|(T_i - T_j)\right|\right)$, and with $a = 1$. A simulated path with the described setup is given in table 6.1. Each column represents the term structure of LIBOR rates (yield curve) at simulation dates, bold ones coinciding with tenor dates. Notice how yield curve gets shorter and shorter at each settlement date. We can make a

Table 6.1. A sample path of LIBOR rates

t	**0**	0.25	**0.5**	0.75	**1**	1.25	**1.5**	1.75	**2**
$L_0(t)$	**0.0500**	-	**-**	-	**-**	-	**-**	-	**-**
$L_1(t)$	**0.0500**	0.0526	**0.0543**	-	**-**	-	**-**	-	**-**
$L_2(t)$	**0.0500**	0.0556	**0.0570**	0.0522	**0.0496**	-	**-**	-	**-**
$L_3(t)$	**0.0500**	0.0524	**0.0536**	0.0496	**0.0466**	0.0450	**0.0428**	-	**-**
$L_4(t)$	**0.0500**	0.0519	**0.0528**	0.0522	**0.0535**	0.0522	**0.0489**	0.0478	**0.0513**

quick observation about the effect of correlation by generating a new path using the same setup and the same independent random variables but with a different a value for correlation function. Setting a to an arbitrarily large number, say 10, makes correlation matrix almost identical, hence Brownian motions governing forward rates becomes almost independent. One can observe by looking at table 6.2 that the term structure is far from being flat along the whole time horizon for the rates moves independently.

Another useful illustration can be done by creating an almost rank-1 correlation matrix,

Table 6.2. A sample path of LIBOR rates with low correlation

t	**0**	0.25	**0.5**	0.75	**1**	1.25	**1.5**	1.75	**2**
$L_0(t)$	**0.0500**	-	**-**	-	**-**	-	**-**	-	**-**
$L_1(t)$	**0.0500**	0.0526	**0.0543**	-	**-**	-	**-**	-	**-**
$L_2(t)$	**0.0500**	0.0550	**0.0553**	0.0492	**0.0484**	-	**-**	-	**-**
$L_3(t)$	**0.0500**	0.0489	**0.0494**	0.0479	**0.0459**	0.0460	**0.0441**	-	**-**
$L_4(t)$	**0.0500**	0.0505	**0.0507**	0.0529	**0.0572**	0.0570	**0.0546**	0.0514	**0.0523**

whose off-diagonal entries are very close to 1. This extreme case can easily be obtained for our usual parametric form by setting a to an arbitrarily small number close to 0, say 0.01. Again let's generate a new path using the same setup and the same independent random variables. One can observe by looking at table 6.3 that the term structure conserves its flat shape along the time horizon since the rates are highly correlated.

Table 6.3. A sample path of LIBOR rates with high correlation

t	**0**	0.25	**0.5**	0.75	**1**	1.25	**1.5**	1.75	**2**
$L_0(t)$	**0.0500**	-	**-**	-	**-**	-	**-**	-	**-**
$L_1(t)$	**0.0500**	0.0526	**0.0543**	-	**-**	-	**-**	-	**-**
$L_2(t)$	**0.0500**	0.0531	**0.0548**	0.0546	**0.0512**	-	**-**	-	**-**
$L_3(t)$	**0.0500**	0.0529	**0.0547**	0.0544	**0.0508**	0.0474	**0.0487**	-	**-**
$L_4(t)$	**0.0500**	0.0530	**0.0548**	0.0547	**0.0515**	0.0481	**0.0492**	0.0566	**0.0617**

Now that the forward rates are generated, derivate payoffs and their time zero values can be calculated. Let's start with a series of caplets example. Assume that there are 4 successive caplets with maturities of $0.5, 1, 1.5, 2$ years and payment dates of $1, 1.5, 2, 2.5$ years respectively. Let's assume that caplets are ATM, i.e., $K = 0.05$. Caplet payoffs received at payment dates are found by using equation (2.17). Finally we want to find discounted payoffs, and especially we are interested in current time, i.e. $t = 0$ values. Using equation (4.13) we obtain the results given in table 6.4.

Table 6.4. Add caption

t	0	0.25	**0.5**	0.75	**1**	1.25	**1.5**	1.75	**2**
$L_0(t)$	0.0500	-	-	-	-	-	-	-	-
$L_0(t)$	0.0500	0.0526	**0.0543**	-	-	-	-	-	-
$L_0(t)$	0.0500	0.0556	0.0570	0.0522	**0.0496**	-	-	-	-
$L_0(t)$	0.0500	0.0524	0.0536	0.0496	0.0466	0.0450	**0.0428**	-	-
$L_0(t)$	0.0500	0.0519	0.0528	0.0522	0.0535	0.0522	0.0489	0.0478	**0.0513**
Payoff			**0.0021**		**0.0000**		**0**		**0.0007**
$\prod_{i=k}^{n-1} (1 + \delta L_i(T_k))$			**1.1105**		**1.0839**		**1.0516**		**1.0267**
$P(0,2)$			**0.8839**		**0.8839**		**0.8839**		**0.8839**
$V(0)$			**0.0021**		**0.0000**		**0.0000**		**0.0006**

As a second example, let's assume that we have a swaption that starts at $t = 0.5$ and covers three semi-annual periods, i.e., the payment dates are at $t = 1$, $t = 1.5$, and $t = 2$. Let's assume that strike rates K are identical and equal to 0.05. Swaption payoff at exercice date and its time zero value can be found using equation (2.17) and (4.13) respectively. For our example path given in table 6.1, swap rate, annuity, and discount terms appearing in payoff equations are as follows: Up to now

Table 6.5. Add caption

$A(T_\alpha) = \sum_{i=\alpha+1}^{\beta} \delta P_i(T_\alpha)$	1.3515
$SR_{\alpha,\beta}(T_\alpha) = \frac{1-P_\beta(T_\alpha)}{\sum_{i=\alpha+1}^{\beta} \delta P_i(T_\alpha)}$	0.1080
$V(T_\alpha) = (SR_{\alpha,\beta}(T_\alpha) - K)^+ A(T_\alpha)$	0.0918
P $(0.5, 2)$	0.8540
P $(0, 2)$	0.8839
V (0)	0.0811

we have obtained numéraire rebased caplet and swaption prices for a single sample path. If N (say 10,000) many number of prices are generated by the same way, it can be conveniently assumed that every single price is an element of this sequence of independent random variables, all following the same distribution. Then expected value of prices can be approximated by the strong law of large numbers by

$$E[V(0)] \simeq \frac{V(0)^1 + V(0)^2 + \ldots + V(0)^N}{N}$$

A Monte Carlo simulation outcome must contain an error bound for the difference between the expected and the actual value. The Central Limit Theorem states that the distribution of the sample mean converges to the normal distribution. Thus for a sample size of N and confidence level of α we have

$$1 - \alpha = P\left(\Phi^{-1}(\frac{\alpha}{2}) \leq \frac{E[V(0)] - V(0)}{s/\sqrt{N}} \leq \Phi^{-1}(\frac{1-\alpha}{2})\right)$$

where s is the sample standart deviation and and $\Phi^{-1}(.)$ is the quantile function of the standard normal distribution.

6.1.1. Numerical Comparison

The purpose of this section is to show the results of caplet and swaption prices we obtain by the Monte Carlo implementation for standard Euler discretization of forward rates and for the arbitrage-free discretization of suitable martingales introduced by Glasserman and Zhao. We will compare these results with the prices given by Black formulae. The volatility input for Black's caplet formula is easy to get due to the equivalance between LMM and Blacks caplet prices showed in section 4.1.4. The same is not valid for Black's swaption formula since the similar equivalance exists between Black's swaption prices and SMM, not LMM. There are several approximation methods to compute swaption prices analytically within the LIBOR Market Model framework developed by Brace [24, 25], Andersen and Andreasen [18], Rebonato [26], and Hull and White [27]. The common market practice for swaption pricing is to simulate swap rates using lognormal forward swap model or use directly Black's formula for swaptions along with Black swaption volatility, which is equivalent to SMM. The main idea of the approximation method that we will use in this section, Hull-White's formula, is to compute a similar value to Black swaption volatility in terms of LIBOR Market

Model parameters. Once this Black-like volatility is computed, it can directly be used in Black's swaption formula. For detailed explaination of this approximation and for some others see [10].

The example data of 06 June 2011 and its calibration results obtained in section 5 will be used as the base scenario. For caplets, the volatility structure is selected as the one given by non-parametric calibration to implied caps volatilities where forward rate volatilities depend on the maturity of the forward rates. Correlation structure is irrelevant for caplets since their expectations do not involve more than one forward rate [10]. For swaptions, volatility and correlations structures are calibrated according to non-parametric calibration to implied swaptions volatilities.

The following four tables show monte carlo expectations, upper-lowe bounds of 95% confidence interval for each discretization and measure choice, and Black's formula for caplets having maturities 0.5, 1, ..., 9.5.

Table 6.6. Caplet prices with different maturities given by Monte Carlo simulation using Euler scheme & terminal measure, and Black's formula.

Maturity	MC Price	St. Error	Lower B.	Upper B.	Black Price
0.5	1.63E-04	1.88E-06	1.61E-04	1.65E-04	1.62E-04
1	4.11E-04	5.77E-06	4.05E-04	4.17E-04	4.09E-04
1.5	7.58E-04	1.29E-05	7.45E-04	7.71E-04	7.54E-04
2	1.27E-03	2.97E-05	1.24E-03	1.30E-03	1.29E-03
2.5	1.57E-03	3.39E-05	1.53E-03	1.60E-03	1.59E-03
3	2.29E-03	6.16E-05	2.23E-03	2.35E-03	2.30E-03
3.5	1.72E-03	2.59E-05	1.69E-03	1.75E-03	1.72E-03
4	2.10E-03	3.30E-05	2.07E-03	2.14E-03	2.12E-03
4.5	2.41E-03	3.78E-05	2.38E-03	2.45E-03	2.40E-03
5	2.26E-03	3.21E-05	2.23E-03	2.29E-03	2.27E-03
5.5	2.72E-03	4.13E-05	2.68E-03	2.77E-03	2.77E-03
6	2.70E-03	3.89E-05	2.66E-03	2.74E-03	2.68E-03
6.5	3.18E-03	4.81E-05	3.13E-03	3.23E-03	3.12E-03
7	3.00E-03	4.32E-05	2.96E-03	3.05E-03	3.01E-03
7.5	3.33E-03	5.04E-05	3.28E-03	3.38E-03	3.37E-03
8	3.28E-03	4.81E-05	3.23E-03	3.33E-03	3.29E-03
8.5	3.60E-03	5.51E-05	3.54E-03	3.65E-03	3.62E-03
9	3.61E-03	5.38E-05	3.55E-03	3.66E-03	3.57E-03
9.5	3.92E-03	6.04E-05	3.85E-03	3.98E-03	3.87E-03

Table 6.7. Caplet prices with different maturities given by Monte Carlo simulation using Euler scheme & spot measure, and Black's formula.

Maturity	MC Price	St. Error	Lower B.	Upper B.	Black Price
0.5	1.63E-04	1.87E-06	1.61E-04	1.65E-04	1.62E-04
1	4.09E-04	5.63E-06	4.03E-04	4.14E-04	4.09E-04
1.5	7.55E-04	1.28E-05	7.43E-04	7.68E-04	7.54E-04
2	1.29E-03	2.86E-05	1.26E-03	1.32E-03	1.29E-03
2.5	1.60E-03	3.46E-05	1.56E-03	1.63E-03	1.59E-03
3	2.26E-03	5.14E-05	2.21E-03	2.31E-03	2.30E-03
3.5	1.72E-03	2.55E-05	1.69E-03	1.74E-03	1.72E-03
4	2.10E-03	3.20E-05	2.07E-03	2.14E-03	2.12E-03
4.5	2.40E-03	3.67E-05	2.37E-03	2.44E-03	2.40E-03
5	2.27E-03	3.19E-05	2.23E-03	2.30E-03	2.27E-03
5.5	2.79E-03	4.18E-05	2.74E-03	2.83E-03	2.77E-03
6	2.63E-03	3.68E-05	2.59E-03	2.67E-03	2.68E-03
6.5	3.13E-03	4.63E-05	3.08E-03	3.18E-03	3.12E-03
7	3.00E-03	4.23E-05	2.95E-03	3.04E-03	3.01E-03
7.5	3.37E-03	4.99E-05	3.32E-03	3.42E-03	3.37E-03
8	3.30E-03	4.72E-05	3.26E-03	3.35E-03	3.29E-03
8.5	3.60E-03	5.28E-05	3.55E-03	3.66E-03	3.62E-03
9	3.56E-03	5.13E-05	3.51E-03	3.62E-03	3.57E-03
9.5	3.82E-03	5.62E-05	3.77E-03	3.88E-03	3.87E-03

Table 6.8. Caplet prices with different maturities given by Monte Carlo simulation using Glasserman scheme & terminal measure, and Black's formula.

Maturity	MC Price	St. Error	Lower B.	Upper B.	Black Price
0.5	1.77E-04	2.10E-06	1.75E-04	1.79E-04	1.62E-04
1	4.42E-04	6.35E-06	4.36E-04	4.49E-04	4.09E-04
1.5	7.93E-04	1.42E-05	7.79E-04	8.07E-04	7.54E-04
2	1.36E-03	3.18E-05	1.33E-03	1.40E-03	1.29E-03
2.5	1.64E-03	3.70E-05	1.60E-03	1.68E-03	1.59E-03
3	2.38E-03	5.97E-05	2.32E-03	2.44E-03	2.30E-03
3.5	1.88E-03	2.97E-05	1.85E-03	1.90E-03	1.72E-03
4	2.32E-03	3.79E-05	2.28E-03	2.35E-03	2.12E-03
4.5	2.57E-03	4.24E-05	2.53E-03	2.62E-03	2.40E-03
5	2.48E-03	3.75E-05	2.44E-03	2.52E-03	2.27E-03
5.5	3.01E-03	4.78E-05	2.97E-03	3.06E-03	2.77E-03
6	2.93E-03	4.45E-05	2.89E-03	2.98E-03	2.68E-03
6.5	3.34E-03	5.28E-05	3.29E-03	3.40E-03	3.12E-03
7	3.24E-03	4.85E-05	3.19E-03	3.29E-03	3.01E-03
7.5	3.61E-03	5.73E-05	3.55E-03	3.67E-03	3.37E-03
8	3.45E-03	5.11E-05	3.40E-03	3.50E-03	3.29E-03
8.5	3.75E-03	5.87E-05	3.69E-03	3.81E-03	3.62E-03
9	3.60E-03	5.33E-05	3.54E-03	3.65E-03	3.57E-03
9.5	3.90E-03	6.02E-05	3.84E-03	3.96E-03	3.87E-03

Table 6.9. Caplet prices with different maturities given by Monte Carlo simulation using Glasserman scheme & spot measure, and Black's formula.

Maturity	MC Price	St. Error	Lower B.	Upper B.	Black Price
0.5	1.62E-04	1.87E-06	1.60E-04	1.64E-04	1.62E-04
1	4.12E-04	5.70E-06	4.06E-04	4.18E-04	4.09E-04
1.5	7.59E-04	1.31E-05	7.46E-04	7.72E-04	7.54E-04
2	1.25E-03	2.91E-05	1.22E-03	1.28E-03	1.29E-03
2.5	1.58E-03	3.34E-05	1.54E-03	1.61E-03	1.59E-03
3	2.31E-03	5.52E-05	2.26E-03	2.37E-03	2.30E-03
3.5	1.68E-03	2.50E-05	1.66E-03	1.71E-03	1.72E-03
4	2.11E-03	3.22E-05	2.08E-03	2.14E-03	2.12E-03
4.5	2.35E-03	3.61E-05	2.32E-03	2.39E-03	2.40E-03
5	2.20E-03	3.15E-05	2.17E-03	2.24E-03	2.27E-03
5.5	2.71E-03	4.13E-05	2.67E-03	2.75E-03	2.77E-03
6	2.62E-03	3.67E-05	2.59E-03	2.66E-03	2.68E-03
6.5	3.01E-03	4.43E-05	2.96E-03	3.05E-03	3.12E-03
7	2.90E-03	4.04E-05	2.86E-03	2.94E-03	3.01E-03
7.5	3.22E-03	4.62E-05	3.17E-03	3.26E-03	3.37E-03
8	3.07E-03	4.32E-05	3.03E-03	3.12E-03	3.29E-03
8.5	3.39E-03	4.96E-05	3.34E-03	3.44E-03	3.62E-03
9	3.30E-03	4.66E-05	3.25E-03	3.35E-03	3.57E-03
9.5	3.59E-03	5.15E-05	3.54E-03	3.64E-03	3.87E-03

The following four tables show monte carlo expectations and upper-lower bounds of 95% confidence interval for european swaptions having different maturities (rows) and tenors (columns) for each discretization and measure choice. The fifth table shows the results of Hull - White approximation.

Swaption maturity (year)		Underlying swap length (year)									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
	1	2.74E-04	1.31E-03	7.34E-03	1.66E-02	2.87E-02	4.41E-02	5.92E-02	7.69E-02	9.67E-02	9.88E-02
	2	1.67E-03	8.69E-03	1.82E-02	3.04E-02	4.57E-02	6.09E-02	7.86E-02	9.85E-02	1.06E-01	0
	3	7.28E-03	1.68E-02	2.89E-02	4.42E-02	5.94E-02	7.70E-02	9.69E-02	1.07E-01	0	0
	4	9.33E-03	2.13E-02	3.65E-02	5.14E-02	6.89E-02	8.85E-02	1.01E-01	0	0	0
	5	1.18E-02	2.68E-02	4.15E-02	5.86E-02	7.80E-02	9.26E-02	0	0	0	0
	6	1.47E-02	2.91E-02	4.60E-02	6.50E-02	8.07E-02	0	0	0	0	0
	7	1.41E-02	3.06E-02	4.92E-02	6.65E-02	0	0	0	0	0	0
	8	1.61E-02	3.44E-02	5.27E-02	0	0	0	0	0	0	0
	9	1.78E-02	4.17E-02	0	0	0	0	0	0	0	0
	10	2.14E-02	0	0	0	0	0	0	0	0	0

Table 6.10. Add caption

Swaption maturity (year)	Underlying swap length (year)									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	2.01E-05	5.25E-05	1.34E-04	1.99E-04	2.79E-04	3.25E-04	3.86E-04	4.66E-04	5.56E-04	6.03E-04
2	4.32E-05	1.11E-04	1.72E-04	2.46E-04	3.05E-04	3.60E-04	4.39E-04	5.33E-04	5.81E-04	0
3	7.28E-05	1.31E-04	2.05E-04	2.58E-04	3.16E-04	3.93E-04	4.83E-04	5.32E-04	0	0
4	6.37E-05	1.37E-04	1.91E-04	2.47E-04	3.22E-04	4.11E-04	4.63E-04	0	0	0
5	8.23E-05	1.30E-04	1.83E-04	2.56E-04	3.42E-04	3.92E-04	0	0	0	0
6	7.94E-05	1.21E-04	1.85E-04	2.64E-04	3.17E-04	0	0	0	0	0
7	6.73E-05	1.29E-04	2.04E-04	2.70E-04	0	0	0	0	0	0
8	6.76E-05	1.51E-04	2.34E-04	0	0	0	0	0	0	0
9	8.50E-05	1.60E-04	0	0	0	0	0	0	0	0
10	9.27E-05	0	0	0	0	0	0	0	0	0

Table 6.11. Add caption

Swaption maturity (year)	Underlying swap length (year)									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	3.27E-04	1.53E-03	8.39E-03	1.86E-02	3.16E-02	4.73E-02	6.22E-02	7.87E-02	9.65E-02	1.01E-01
2	1.94E-03	9.91E-03	2.03E-02	3.33E-02	4.91E-02	6.39E-02	8.05E-02	9.83E-02	1.08E-01	0
3	8.30E-03	1.88E-02	3.17E-02	4.75E-02	6.23E-02	7.89E-02	9.67E-02	1.09E-01	0	0
4	1.04E-02	2.34E-02	3.92E-02	5.40E-02	7.06E-02	8.84E-02	1.02E-01	0	0	0
5	1.30E-02	2.87E-02	4.36E-02	6.02E-02	7.80E-02	9.31E-02	0	0	0	0
6	1.58E-02	3.06E-02	4.72E-02	6.50E-02	8.07E-02	0	0	0	0	0
7	1.49E-02	3.14E-02	4.92E-02	6.62E-02	0	0	0	0	0	0
8	1.66E-02	3.44E-02	5.21E-02	0	0	0	0	0	0	0
9	1.78E-02	4.26E-02	0	0	0	0	0	0	0	0
10	2.20E-02	0	0	0	0	0	0	0	0	0

Table 6.12. Add caption

Swaption maturity (year)	Underlying swap length (year)									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	2.50E-05	6.09E-05	1.51E-04	2.17E-04	2.94E-04	3.28E-04	3.75E-04	4.34E-04	4.94E-04	5.60E-04
2	4.94E-05	1.23E-04	1.85E-04	2.57E-04	3.05E-04	3.47E-04	4.05E-04	4.68E-04	5.28E-04	0
3	8.07E-05	1.41E-04	2.14E-04	2.56E-04	3.02E-04	3.59E-04	4.21E-04	4.74E-04	0	0
4	6.79E-05	1.41E-04	1.88E-04	2.34E-04	2.91E-04	3.54E-04	4.06E-04	0	0	0
5	8.59E-05	1.29E-04	1.73E-04	2.32E-04	2.95E-04	3.40E-04	0	0	0	0
6	8.10E-05	1.16E-04	1.67E-04	2.26E-04	2.69E-04	0	0	0	0	0
7	6.63E-05	1.18E-04	1.76E-04	2.22E-04	0	0	0	0	0	0
8	6.13E-05	1.31E-04	2.01E-04	0	0	0	0	0	0	0
9	7.53E-05	1.40E-04	0	0	0	0	0	0	0	0
10	8.43E-05	0	0	0	0	0	0	0	0	0

Table 6.13. Add caption

		Underlying swap length (year)									
Swaption maturity (year)		**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
	1	2.54E-04	1.27E-03	7.23E-03	1.64E-02	2.85E-02	4.38E-02	5.89E-02	7.65E-02	9.63E-02	9.83E-02
	2	1.66E-03	8.61E-03	1.80E-02	3.01E-02	4.55E-02	6.06E-02	7.82E-02	9.81E-02	1.05E-01	0
	3	7.24E-03	1.67E-02	2.87E-02	4.41E-02	5.91E-02	7.67E-02	9.65E-02	1.07E-01	0	0
	4	9.30E-03	2.12E-02	3.64E-02	5.13E-02	6.87E-02	8.82E-02	1.01E-01	0	0	0
	5	1.17E-02	2.67E-02	4.13E-02	5.84E-02	7.78E-02	9.23E-02	0	0	0	0
	6	1.47E-02	2.91E-02	4.59E-02	6.48E-02	8.05E-02	0	0	0	0	0
	7	1.41E-02	3.05E-02	4.91E-02	6.62E-02	0	0	0	0	0	0
	8	1.61E-02	3.43E-02	5.25E-02	0	0	0	0	0	0	0
	9	1.78E-02	4.16E-02	0	0	0	0	0	0	0	0
	10	2.13E-02	0	0	0	0	0	0	0	0	0

Table 6.14. Add caption

Swaption maturity (year)		Underlying swap length (year)									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
	1	1.83E-05	4.98E-05	1.31E-04	1.97E-04	2.75E-04	3.20E-04	3.81E-04	4.61E-04	5.50E-04	5.97E-04
	2	4.21E-05	1.09E-04	1.70E-04	2.43E-04	3.00E-04	3.56E-04	4.35E-04	5.27E-04	5.75E-04	0
	3	7.36E-05	1.32E-04	2.04E-04	2.55E-04	3.14E-04	3.91E-04	4.79E-04	5.27E-04	0	0
	4	6.36E-05	1.35E-04	1.88E-04	2.45E-04	3.19E-04	4.06E-04	4.59E-04	0	0	0
	5	8.17E-05	1.28E-04	1.81E-04	2.54E-04	3.39E-04	3.89E-04	0	0	0	0
	6	8.06E-05	1.22E-04	1.86E-04	2.63E-04	3.15E-04	0	0	0	0	0
	7	6.79E-05	1.30E-04	2.03E-04	2.69E-04	0	0	0	0	0	0
	8	6.77E-05	1.49E-04	2.31E-04	0	0	0	0	0	0	0
	9	8.34E-05	1.60E-04	0	0	0	0	0	0	0	0
	10	9.29E-05	0	0	0	0	0	0	0	0	0

Table 6.15. Add caption

Swaption maturity (year)	Underlying swap length (year): 1	2	3	4	5	6	7	8	9	10
1	3.50E-04	1.59E-03	8.43E-03	1.86E-02	3.15E-02	4.72E-02	6.20E-02	7.85E-02	9.61E-02	1.01E-01
2	1.97E-03	9.93E-03	2.04E-02	3.33E-02	4.90E-02	6.38E-02	8.02E-02	9.78E-02	1.07E-01	0
3	8.28E-03	1.87E-02	3.16E-02	4.73E-02	6.21E-02	7.86E-02	9.62E-02	1.08E-01	0	0
4	1.05E-02	2.34E-02	3.91E-02	5.39E-02	7.03E-02	8.80E-02	1.02E-01	0	0	0
5	1.29E-02	2.86E-02	4.34E-02	5.99E-02	7.75E-02	9.26E-02	0	0	0	0
6	1.57E-02	3.05E-02	4.70E-02	6.46E-02	8.02E-02	9.66E-02	0	0	0	0
7	1.48E-02	3.12E-02	4.89E-02	7.79E-02	9.55E-02	0	0	0	0	0
8	1.64E-02	3.41E-02	6.07E-02	0	0	0	0	0	0	0
9	1.76E-02	4.23E-02	0	0	0	0	0	0	0	0
10	2.19E-02	0	0	0	0	0	0	0	0	0

Table 6.16. Add caption

Swaption maturity (year)		**Underlying swap length (year)**									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
	1	2.46E-05	6.19E-05	1.54E-04	2.21E-04	3.00E-04	3.36E-04	3.85E-04	4.43E-04	5.04E-04	5.73E-04
	2	5.00E-05	1.25E-04	1.89E-04	2.63E-04	3.12E-04	3.55E-04	4.13E-04	4.77E-04	5.40E-04	0
	3	8.29E-05	1.44E-04	2.19E-04	2.63E-04	3.10E-04	3.68E-04	4.30E-04	4.85E-04	0	0
	4	6.97E-05	1.45E-04	1.93E-04	2.41E-04	2.98E-04	3.62E-04	4.15E-04	0	0	0
	5	8.75E-05	1.32E-04	1.78E-04	2.36E-04	3.00E-04	3.46E-04	0	0	0	0
	6	8.04E-05	1.17E-04	1.69E-04	2.29E-04	2.74E-04	0	0	0	0	0
	7	6.63E-05	1.19E-04	1.78E-04	2.32E-04	0	0	0	0	0	0
	8	6.10E-05	1.31E-04	1.82E-04	0	0	0	0	0	0	0
	9	7.50E-05	1.41E-04	0	0	0	0	0	0	0	0
	10	8.40E-05	0	0	0	0	0	0	0	0	0

Table 6.17. Add caption

		Underlying swap length (year)									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Swaption maturity (year)	**1**	3.36E-04	1.57E-03	8.49E-03	1.87E-02	3.17E-02	4.74E-02	6.23E-02	7.89E-02	9.67E-02	1.01E-01
	2	1.97E-03	9.98E-03	2.05E-02	3.34E-02	4.92E-02	6.41E-02	8.06E-02	9.84E-02	1.08E-01	0
	3	8.35E-03	1.88E-02	3.18E-02	4.76E-02	6.25E-02	7.90E-02	9.68E-02	1.09E-01	0	0
	4	1.05E-02	2.35E-02	3.93E-02	5.41E-02	7.07E-02	8.85E-02	1.02E-01	0	0	0
	5	1.30E-02	2.88E-02	4.36E-02	6.02E-02	7.80E-02	9.31E-02	0	0	0	0
	6	1.58E-02	3.07E-02	4.72E-02	6.50E-02	9.26E-02	0	0	0	0	0
	7	1.49E-02	3.14E-02	4.92E-02	0	0	0	0	0	0	0
	8	1.66E-02	3.43E-02	6.11E-02	0	0	0	0	0	0	0
	9	1.78E-02	4.26E-02	0	0	0	0	0	0	0	0
	10	2.20E-02	0	0	0	0	0	0	0	0	0

7. Static Bond Portfolio Management

In this chapter, static government bond portfolio selection through optimization is described with general approach of Markovitz's Modern portfolio theory [4]. First we will define the bond selection problem as a classical mean-variance optimization problem. Then we will set a new portfolio optimization problem by considering a modified version of the Markovitz model where the portfolio's variance is replaced with the portfolio's Value-at-Risk (Var). We will use the two term structure models that we have reviewed, Vasicek as an example of short rate model, and LIBOR Market Model as an example of high dimensional market model, to estimate expected future values of different bonds, their variance-covariance structures and VaR values that are the main parameters for both of the optimization problems.

7.1. Modern Portfolio Theory

Modern portfolio theory introduced by Markowitz in 1952 is considered as the first systematical approach for portfolio selection and still used as an essential tool by practitioners. It is essentially developed for allocation of traded stocks and there are numerous researches in literature about its performance for equity portfolio applications. Nevertheless there are few examples of its extensions for bond portfolio management. Complications of this application can be summarized by two arguments [28]. First, the volatility of interest rates was not a primary concern a few decades ago as they were relatively stable. But now it would be inadmissible to neglect substantial risks in bond investments that can be caused by possible changes in interest rates. Although the problem of low risk diversification stemming from the high correlation between bonds still exists, it is sensible to rely on the high number of different maturity bonds to benefit from this effect. Second, mean and variance-covariance parameters estimated for stocks do not change over time or position, hence are stationary. On the other hand, since bonds have fixed maturities and fixed par values at these maturities, statistical methods such as simple historical estimation fail to provide necessary parameters. Hopefully, introduction of term structure models opened the door for researches to adapte the modern portfolio theory for bond portfolio selection purposes. These models bring a solution for the latter problem of parameter estimaton for they can reflect the time varying moments of bonds and capture their fixed maturity structure. In 1992 Wilhelm [5] introduced a bond portfolio selection model using dynamic term structure of Cox/Ingersoll/Ross model [29]. In 1999 Sorensen studied dynamic portfolio strategies of an investor in continuous-time using the one-factor Vasicek model. In 2002, Korn and Kraft [7] studied a pure bond portfolio selection problem where interest rates follow either the Vasicek or the Cox/Ingersoll/Ross term structure model. Korn and Koziol [28] applied the Markowitz approach of portfolio selection to government bond portfolios. Multi-factor Vasicek type term structure models were used to estimate expected returns, return variances and covariances of different bonds. Puhle derived the Heath/Jarrow/Morton [30] term structure framework and two special cases Vasicek and the Hull/White models [9] using the stochastic discount factor pricing methodology.

Modern portfolio theory is an investment theory that aims to maximize the expected return of the portfolio in the future for an allowed maximum risk level, represented by portfolio variance. Alternatively, the objective can be expressed as minimizing the risk for a desired expected portfolio return. In order to achieve this aim, MPT assigns appropriate weights for each asset and thus creates the opportunity for a diversification of the risk [4]. Diversification can reduce portfolio variance even below the variance level of the least risky asset that it is in the portfolio. The effect of diversification is enhanced when the assets are negatively correlated, although diversification even can be achieved for uncorrelated and -to some extent- positively correlated assets.

In MPT, it is assumed that the investor is concerned only about expected future return and variance of the portfolio. Another assumption is that the portfolio is set at the beginning of the investment horizon and stays intact along it, hence comes the term static portfolio optimization.

The standard optimization problem described by MPT can be given in general terms as follows:

$$\begin{aligned} \min \operatorname{var}(W_T) \\ s.t. E[W_T] = \bar{W}_T \\ \sum_{i=1}^{n} w_i P_i = W_0 \end{aligned} \tag{7.1}$$

where W_0 is the initial, W_T is the terminal and $\bar{W}_T$ is the desired terminal wealth of the portfolio. P_i's are the initial prices of each asset i (can be any asset, stock, bond etc...) and w_i's represent amount of assets is in the portfolio. An additional non-negativity constraint can be imposed on w_i's, forbidding the short selling of all or some of the assets. Modern portfolio theory also lays the groundwork for various portfolio selection models through the incorporation of additional constraints and modified objective functions [31]. But in this thesis we will stick with the standart form as it is adequate for our purposes.

It should be noted that the normality assumption for stock returns in classical Markowitz model is not explicitely imposed on the problem given in equation 7.5 and we particularly say nothing about the distribution of terminal wealth of portfolio. Actually we will see in the forthcoming sections that the term structure models that are investigated in this thsis would not yield a known probability distribution for future portfolio wealth.

MPT problem is defined for a given level of risk fixed by the investor and the solution is optimal (or so called efficient) with respect to this particular risk level. The optimum allocations would be different if another risk level is considered, i.e. there will be other optimum portfolios for different risk levels. Among all the possible portfolio combinations, the set of all optimal portfolios constitutes the efficient frontier. It is always beneficial to draw the efficient frontier for it allows investors to understand how a portfolio's expected returns vary with the amount of risk taken.

7.2. Extention of MPT: optimal mean / Value-at-Risk portfolio

The classical MPT theory and its application to equity portfolios come with the assumption that returns are normally distributed, i.e. portfolio return distribution can be described by means and variances of assets. On the other hand, empirical results points out that return distributions are actually fat tailed [32]. The probabilities for extreme losses (or gains) can be underestimated by normality assumption. This oriented researchers to replace variance with other risk measures that can capture non-normality of returns. Some of this works can be found in [33], [34], and [35]. From the beginning of 1990's, the urge to control and regulate financial risks led to the utilization of value at risk as the generally accepted risk measure. Its diffusion gained speed by the release of JP Morgan's VaR estimation system called RiskMetrics [36] in October 1994. In April 1995 Basle Committee on Banking Supervision announced in the proposal "Supervisory Treatment of Market Risks" that the banks are to use their value at risk estimations as the basis for calculation of capital adequacy requirements. Value at risk is nothing but the upper percentile of the loss distribution and it denotes the worst expected loss over a fix time period within a confidence interval. More formally, the $(1-\alpha)\%$ VaR value of the loss for the the time period $[0, T]$ is given by:

$$\Pr(VaR_{\alpha,T} \leq Loss()) = \alpha. \tag{7.2}$$

Despite its popularity taking root from its ease of representing and interpreting extreme losses, value at risk do not possess desirable mathematical properties. First of all, it is not sub-additive for non-uniform distributions, i.e. VaR of a portfolio is greater or equal than the sum VaRs of each individual assets in that portfolio [37]. Second, mean/VaR optimization problem is nonconvex and can display many local minima [38]. Furthermore VaR function becomes non-smooth when Monte Carlo simulation is used for valuation of assets. For examples of researches focusing on measuring VaR, see [39], [40], [37] and for optimization problems featuring VaR, see [41] , [42], [43]. Consider the portfolio with initial and at the end of investment hoziron values as W_0 and W_T respectively. The value at risk of this portfolio is the $(1-\alpha)$ quantile of the loss distribution for period $[0, T]$ is

$$\Pr\left(\mathrm{VaR}_{\alpha,T} \leq W_0 - W_T\right) = \alpha, \tag{7.3}$$

or equivalently

$$\mathrm{VaR}_{\alpha,T} = \inf\left\{x \mid \Pr\left(W_0 - W_T \geq x\right) < \alpha\right\}. \tag{7.4}$$

The investor's aim is to construct the potfolio with the minimum value at risk at an α treshhold for a desired final portfolio value. The optimization problem described by MPT can easily be modified by setting the objective function as the VaR of the portfolio for the end of the investment horizon. Thus the problem is expressed in general terms as follows:

$$\begin{aligned} \min \mathrm{VaR}_{\alpha,T} \\ s.t. E[W_T] = \bar{W}_T \\ \sum_{i=1}^{n} w_i P_i = W_0 \end{aligned} \tag{7.5}$$

7.3. Bond Portfolio Optimization with MPT approach

Throughtout this section, Puble's work is followed, for more detailed read sections 4.2.1-3 of [8]. Although theoratically MPT approach can be applied to all tradable assets, some adaptations must be done according to asset type, as it is the case for bond portfolios. First of all, bonds have definite maturities; if the investment horizon is later than some of the bonds' maturities, the cash flow arising from these bonds should be reinvested by some means or other. Puhle [8] derivations and results follow the assumption that they are re-invested on the fix spot rate $R(t, T)$ from their maturity t until the end of the investment horizon T. It is a sensible assumption for static mean variance framework since this strategy do not require anticipation of decisions after t. But in a real world application the portfolio manager would wish to rebalance or reconstruct its portfolio especially when a cash flow occurs. To model this, one should start thinking about using a dynamic selection model but we can restrict ourselves by only considering cases where the investment horizon is shorter or equal than the shortest maturing bond, making the problem static.

We start by considering an equally spaced tenor structure $0 = t_0 < t_1 < \ldots < t_m = T < \ldots < t_n = \tau$. Let's assume that the investment horizon is $[0, T]$ and one can invest into zero-coupon bonds having maturities $\{t_m, t_{m+1}, \ldots, t_n\}$. The initial wealth of the portfolio is expressed as:

$$W_0 = \sum_{i=m}^{n} w_i P(0, t_i) \tag{7.6}$$

where w_i denotes the amount of the initial wealth allocated for i^{th} bond having the price $P(0, t_i)$. At the end of investment horizon the terminal wealth of the portfolio would be:

$$W_T = \sum_{i=m}^{n} w_i P(T, t_i) \tag{7.7}$$

According to MPT the investor wants to obtain a certain desired value for the portfolio's expected return. Taking expected value of the above equation gives:

$$E\left[W_T\right] = \sum_{i=m}^{n} w_i E\left[P(T, t_i)\right] \tag{7.8}$$

For the above desired final wealth, investor's main objective is to minimize the total final portfolio variance, which is expressed as:

$$\operatorname{var}(W_T) = \sum_{i=m+1}^{n} w_i^2 \operatorname{var}\left(P(T, t_i)\right) + \sum_{i=m+1}^{n} \sum_{\substack{j=m+1 \\ j \neq i}}^{n} w_i w_j \operatorname{cov}\left(P(T, t_i), P(T, t_j)\right) \tag{7.9}$$

Portfolio variance equation can also be written in matrix multiplication form:

$$\operatorname{var}(W_T) = wCw^T \tag{7.10}$$

where C is the so-called covariance matrix having $\text{var}\,(P(T,t_i))$ terms at diagonal and $\text{cov}\,(P(T,t_i),P(T,t_j))$ terms at non-diagonal entries, and w is the vector of weights. Pay attention that C is a $n-m\times n-m$ matrix and w is a vector of length $n-m$, whereas the portfolio may consist of $n-m+1$ bonds. This is due to the fact that holding a bond having maturity date equal to the end of the investment horizon is a risk-free investment. Equations (7.6),(7.8) and (7.9) provide the necessary input parameters for the optimization problem described at (7.5).

We will forbid the short selling of bonds in our applications. The reason for this is the unrealistic portfolio compositions resulting from highly correlated bond prices when short selling is allowed.

7.4. One-Factor Vasicek Model Application

In this section we will see how classical mean / variance problem and its extension, mean / VaR problem, can be used for bond portfolio selection purpose when the interest rate is assumed to follow the dynamics of one-factor Vasicek Model. We will see how the classical model inputs, bonds' mean and variance-covariance parameters can be estimated with closed form formulation and also with simulation. As for the VaR measure, we will first try to express portfolio VaR by a closed form approximation using Fenton-Wilkinson method. Then we will demonstrate how the VaR is computed by simulation.

7.4.1. Mean / Variance Problem

Closed form expressions for the expectation and variance / covariance terms apprearing in equations (7.8) and (7.9) are very easy to derive. Recall that under Vasicek Model, zero coupon bond price is given by equation (3.10) where the short rate $r(t)$ is normally distributed with mean and variance given in equations (3.8) and (3.9) respectively. Note that if X is normally distributed random variable with mean μ and variance σ^2, then a linear transform $aX + b$ is also normally distributed:

$$aX + b \sim N(a\mu + b, a^2\sigma^2)$$

By this transformation, the distribution of the term at the exponent in equation (3.10) is given by:

$$\begin{aligned} A(t,T) - B(t,T)r(t) \sim N(&-B(t,T)E_t[r(T)] \\ &+A(t,T), B(t,T)^2\mathrm{var}_t(r(T))) \end{aligned} \tag{7.11}$$

Thus zero-coupon bond prices are distributed log-normally with the following parameters:

$$P(t,T) \sim \ln N(-B(t,T)E_t[r(T)] + A(t,T), B(t,T)^2\mathrm{var}_t(r(T))) \tag{7.12}$$

Using mean, variance, and covariance formulae for log-normal distribution gives us the following eqauations:

$$E\left[P(t,T)\right] = \exp\left(A(t,T) - B(t,T)E[r(t)] + \tfrac{1}{2}B(t,T)^2\mathrm{var}(r(t))\right) \tag{7.13}$$

$$\mathrm{var}\left(P(t,T)\right) = E[P(t,T)]^2\left(\exp(B(t,T)^2\mathrm{var}(r(t)) - 1\right) \tag{7.14}$$

$$\mathrm{cov}\left(P(t,T), P(\tau,T)\right) = E\left[P(t,T)\right]E\left[P(\tau,T)\right](\exp(B(t,T)B(\tau,T)\mathrm{cov}(r(t), r(\tau)) - 1) \tag{7.15}$$

The required parameters for the problem can also be obtained by simulation. Sample short rate paths are generated by exact simulation formula given in equation (3.14) and sample bond prices are calculated for each path with Vasicek general bond price formula that is given in equation (3.10). Simulation's output is a matrix where rows and columns represent paths and bond prices respectively. Taking simple average of each column will give expected values appearing in equation (7.8). Covariance matrix is also easily obtained by computing the covariances between the columns.

Now we will show an example of bond portfolio selection problem for Vasicek Model. Now we will revisit the example bond portfolio selection problem discussed in section . Let's assume that an investor wants to construct a portfolio with an investmant horizon of 5 years. Let's assume that there are ten zero coupon bonds in the market with maturities 1 year, 2 years,..., 10 years. As we have mentioned before, the investor would prefer to include only the bonds having maturities greater or equal than its planned investment horizon to not to be concerned with early cash flows. To get the efficient frontier, first the minimum variance portfolio is sought without any restriction imposed on the return. The resulting minimum variance portfolio's return is then gradually increased and imposed as the desired return level on the problem that is to be resolved several times until the maximum return portfolio is reached.

As Vasicek Model parameters, we will use the estimations obtained by historical calibration in section 3.3.1. Both closed form formulae and simulation can be used to get the required inputs for the optimization problem. To test the applicability of both approaches, let's compare their results for future bond prices and variance-covariance matrices. As can be seen from the following tables the results are nearly the same but obviously closed form formulation should be prefered for computational efficiency.

Zero coupon bond weights, portfolio returns, and portfolio deviations for some of the optimum portfolios of our example when short selling is not allowed are given in table 7.5. Remark the shift in weights from the risk-free asset $P(5,5)$ towards the most risky and most paying bond $P(5,10)$. In Vasicek application we observe that optimum portfolios includes at most 2 - 4 adjacent zero coupon bonds. The reason for this is the high correlation between bonds that are non-linear functions of perfectly correlated spot interest rates defined by one-factor models [8].

Table 7.1. Vasicek model parameter calibrated to 06 June 2011.

Parameter	Value
Latest Maturity	10 years
Investment Horizon	5 years
$r(0)$	0.0003
β	0.033668
μ	0.247455
σ	0.013346
λ	0.415157

Table 7.2. Future bond prices calculated by closed form formulation and simulation

	Closed Form Formulae	Simulation Mean	Simulation Error Bound
P(5,6)	0.972883	0.972867	5.68E-05
P(5,7)	0.940876	0.940849	1.02E-04
P(5,8)	0.905704	0.905669	1.38E-04
P(5,9)	0.868693	0.868652	1.65E-04
P(5,10)	0.830841	0.830797	1.85E-04

Table 7.3. Covariance matrix obtained by closed form formulation.

	P(5,6)	P(5,7)	P(5,8)	P(5,9)	P(5,10)
P(5,6)	0.0002449	0.000422	0.000545	0.000627	0.000677
P(5,7)	0.0004217	0.000726	0.000939	0.00108	0.001167
P(5,8)	0.000545	0.000939	0.001213	0.001396	0.001508
P(5,9)	0.0006269	0.00108	0.001396	0.001605	0.001735
P(5,10)	0.0006773	0.001167	0.001508	0.001735	0.001875

Table 7.4. Covariance matrix obtained by simulation.

	P(5,6)	P(5,7)	P(5,8)	P(5,9)	P(5,10)
P(5,6)	0.000246	0.000422	0.000545	0.000627	0.000677
P(5,7)	0.000422	0.000727	0.000939	0.001082	0.001168
P(5,8)	0.000545	0.000940	0.001213	0.001398	0.001510
P(5,9)	0.000627	0.001081	0.001396	0.001607	0.001738
P(5,10)	0.000678	0.001169	0.001508	0.001738	0.001878

Table 7.5. Zero-coupon bond weights for short-sale constrained portfolios

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**sd(W)**
1.000	0.000	0.000	0.000	0.000	0.000	1.126072	0
0.671	0.329	0.000	0.000	0.000	0.000	1.131072	0.00604
0.343	0.657	0.000	0.000	0.000	0.000	1.136072	0.012064
0.014	0.986	0.000	0.000	0.000	0.000	1.141072	0.018091
0.000	0.603	0.397	0.000	0.000	0.000	1.146072	0.024189
0.000	0.187	0.813	0.000	0.000	0.000	1.151072	0.030291
0.000	0.000	0.710	0.290	0.000	0.000	1.156072	0.036425
0.000	0.000	0.183	0.817	0.000	0.000	1.161072	0.042585
0.000	0.000	0.000	0.562	0.438	0.000	1.166072	0.048776
0.000	0.000	0.000	0.000	0.862	0.138	1.171072	0.054989
0.000	0.000	0.000	0.000	0.000	1.000	1.176072	0.061233

7.4.2. Mean / VaR Problem

To setup the mean / VaR problem, again we need future expected value of the portofolio and it can be found by means described for mean / variance problem in the previous subsection. Now, instead of portfolio variance, we need to express portfolio value-at-risk for the investment horizon. Final value of the portfolio is a linear combination of lognormally distributed correlated random variables, $P(T,t)$'s, with nonidentical means and standard deviations. Since there is no known distribution expressing sum statistics of multiple lognormal random variables, we ought to use approximation methods and Monte Carlo simulation. The common point of lognormal sum approximations is that they express the sum distribuiton lognormal as well. Although it seems best to rely on Monte Carlo simulation, under certain conditions approximation methods may become attractive standing out with their fast results. One can find several examples of implementations for different lognormal approximation methods and their performance in literature. We decided to use Fenton-Wilkinson approximation method because it is very straightforward and easy to implement for our case.

<u>7.4.2.1. Fenton-Wilkinson (FW) approximation.</u> Let's assume that the sum of N lognormal random variables X_i with means μ_i, standard deviations σ_i and correlated with coefficients r_{ij} can be approximated as

$$\sum_{i=1}^{N} X_i \cong \exp(Z) \tag{7.16}$$

where Z is a normal random variable. Fenton-Wilkinson method [44] derives the mean (μ_z) and the standard deviation (σ_z) of Z by matching the first and the second moments of the both sides of equation 7.16. Hence mean and standard deviation are given by [45]

$$\mu_z = 2\ln u_1 - \frac{1}{2}\ln u_2 \tag{7.17}$$

$$\sigma_z = \sqrt{\ln u_2 - 2\ln u_1} \tag{7.18}$$

where

$$u_1 = \sum_{i=1}^{N} \exp(\mu_i + \frac{\sigma_i^2}{2}) \tag{7.19}$$

$$u_2 = \sum_{i=1}^{N} \exp(2\mu_i + 2\sigma_i^2) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \exp(\mu_i + \mu_j + \frac{1}{2}(\sigma_i^2 + \sigma_j^2 + 2r_{ij}\sigma_i\sigma_j) \tag{7.20}$$

With the help of this approximation, we can express future values of our bond portfolio by a single lognormal distribution and get closed form loss and VaR functions.

$$W_T = \sum_{i=T+1}^{\tau} X_i = \sum_{i=T+1}^{\tau} w_i P(T,i) \tag{7.21}$$

where

$$P(T,i) \sim \ln N(-B(T,i)E_{t=0}[r(T)] + A(T,i),\ B(T,i)^2 \mathrm{var}_{t=0}(r(T))) \tag{7.22}$$

Then mean and standard deviation of X_i for $i = 0, \ldots, n-1$ are

$$\begin{aligned} \mu_i &= -B(T,i)E_{t=0}[r(T)] + A(T,i) + \ln(w_i) \\ \sigma_i &= B(T,i)\sqrt{\mathrm{var}_{t=0}(r(T))} \end{aligned} \tag{7.23}$$

The terms $2r_{ij}\sigma_i\sigma_j$ appearing in equation (7.20) are calculated through equations (**??**),(7.15) and (**??**).

The loss one makes by holding a portfolio is given by

$$\begin{aligned} Loss &= W_0 - W_T \\ &= W_0 - \exp(\mu_{W_T} + \sigma_{W_T} z), \quad z \sim N(0,1) \end{aligned} \tag{7.24}$$

VaR is defined as the percentile of the loss distribution over a fix time period. Pay attention that both of the loss functions defined at (**??**) and (**??**) are strictly increasing functions of $z \sim N(0,1)$. Let $f(z) = Loss(z)$, then f^{-1} exists such that

$$\begin{aligned}
&\Pr(VaR_{\alpha,T} \leq Loss(z)) = \alpha \Leftrightarrow \Pr(a \leq z) = \alpha \\
&\Pr(a \leq z) = \alpha \Leftrightarrow \Phi^{-1}(\alpha) = a \\
&VaR_{\alpha,T} = f(a) \\
&\qquad = Loss(a) \\
&\qquad = Loss(\Phi^{-1}(\alpha))
\end{aligned} \tag{7.25}$$

where $\Phi^{-1}()$ is the quantile function of the standard normal distribution.

When we do not make any assumption about the distribution function of the portfolio value, to estime VaR, we can use Monte Carlo simulation and apply order statistic. We simulate T year ahead daily short rates and calculate bond prices $P(T,i)$. Finally, we compute the losses for each simulated path by $\sum_{i=T+1}^{\tau} w_i P(0,i) - \sum_{i=T+1}^{\tau} w_i P(T,i)$ and then we compute the VaR as $(1-\alpha)$ percent quantile value of these losses.

<u>7.4.2.2. Comparison of Fenton-Wilkinson and Simulation.</u> In literature, there are several works showing how well Fenton-Wilkinson and other approximation methods perform in particular frameworks, especially in signal processing and wireless communication areas. Also summation of lognormal variables for Asian option pricing is commonly discussed. In this section, we will try to assess Fenton-Wilkinson approximation's performance in evaluating value-at-risk of a bond portfolio under Vasicek model, with different initial parameter sets. Let's consider again the scenario given in table 7.1. Asssume that our investment horizon is 5 years and we have a set of available bonds with maturities 6 ,..., 10 years. Further assume that initial wealth is \$1 and investment amounts for each bond are equal. In our experiments first we will select a parameter, change its value gradually while keeping other parameters constant, calculate the corresponding value-at-risk value by using both Fenton-Wilkinson approximation and simulation. VaR values calculated by simulation will be considered as the true values. The results are given in the following figures.

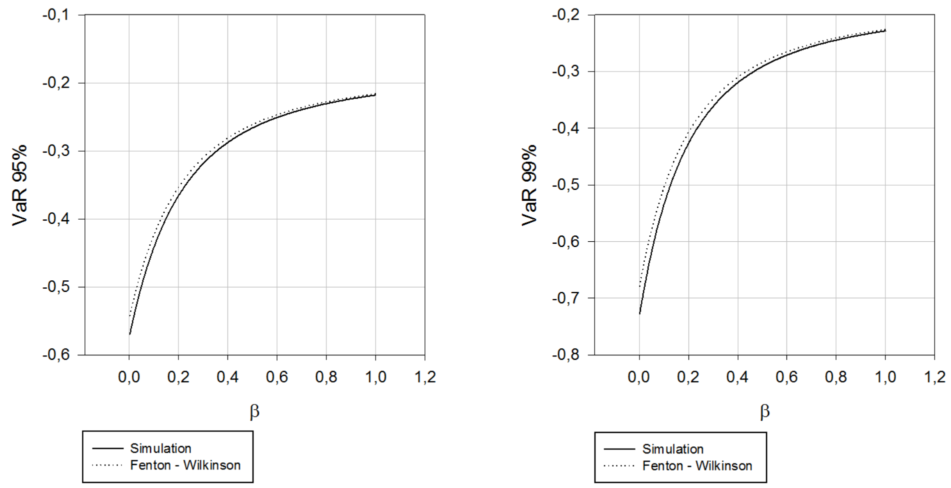

Figure 7.1. VaR comparison of Fenton-Wilkinson and Simulation methods with varying β

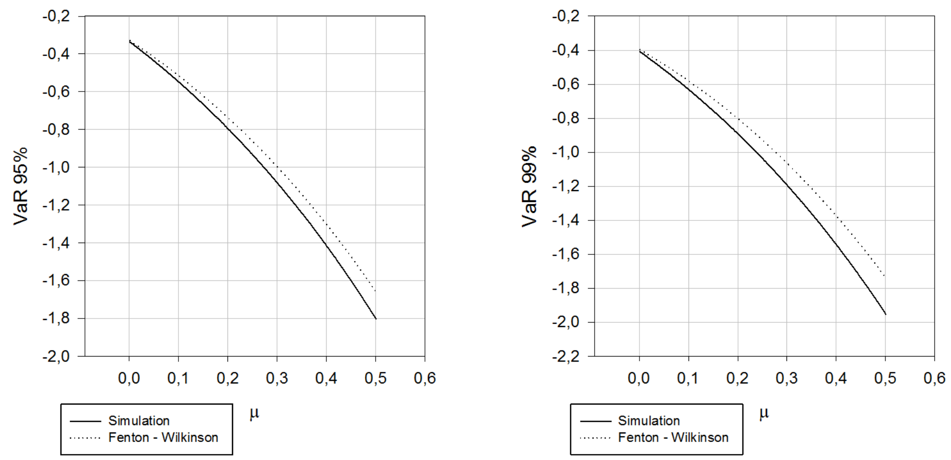

Figure 7.2. VaR comparison of Fenton-Wilkinson and Simulation methods with varying μ

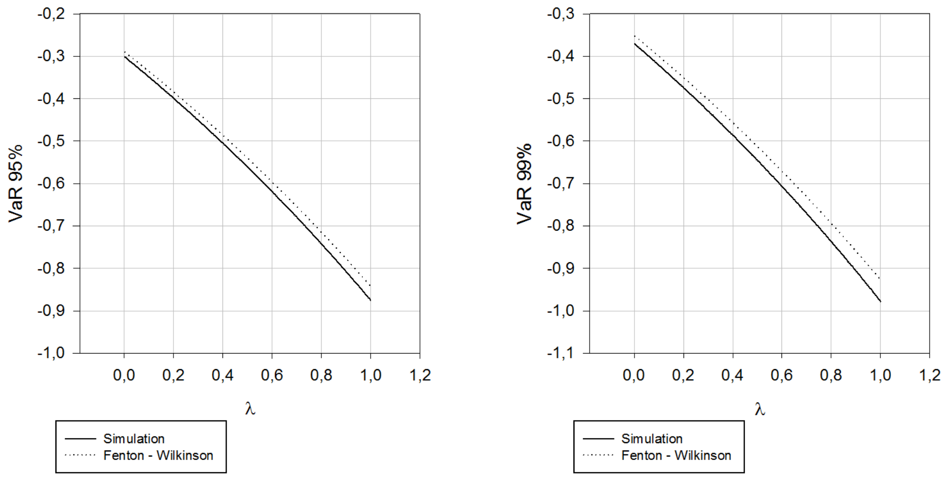

Figure 7.3. VaR comparison of Fenton-Wilkinson and Simulation methods with varying λ

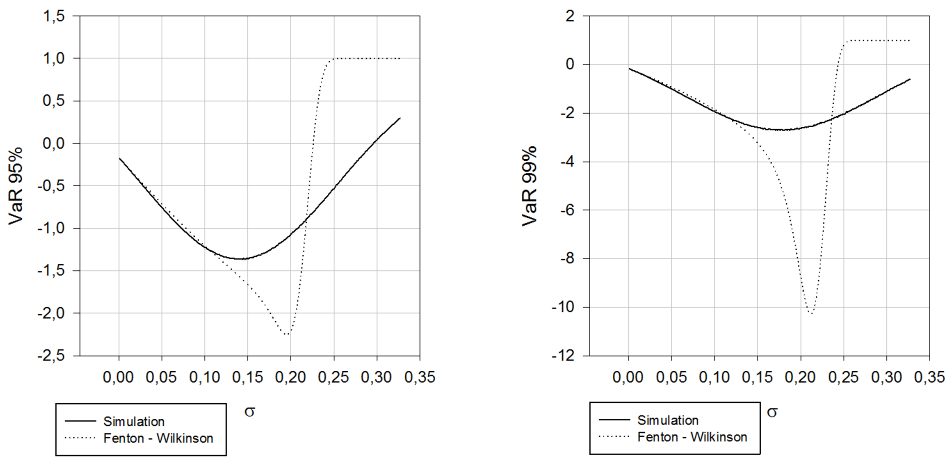

Figure 7.4. VaR comparison of Fenton-Wilkinson and Simulation methods with varying σ

From first figure we can deduce that for higher values of mean reversion rate approximation method results converges to the ones of simulation. Second and third figures indicate that as the long term mean reversion level or market price of risk increase error between approximation and simulation grows. But within a reasonable region of expected mean reversion level and market price of risk for our scenario, approximation method can be considered to behave quite well. Also approximation overestimates the VaR constantly no matter how the above parameters are changed. Final figure clearly shows that for small short rate volatility values approximation method can prove useful however for our scenario it deviates extensively after a level of 10%. It can be concluded that under certain conditions it may be acceptable to use Fenton-Wilkinson method instead of simulation, but it will not be advised since our exact simulation can be done very fast with today's availability of computational power.

Now that we saw how portfolio value-at-risk can be calculated we will revisit the example bond portfolio selection problem discussed in section; portfolio VaR is set as the new objective function. Again we assume that an investor wants to construct a portfolio with an investmant horizon of 5 years and that there are ten zero coupon bonds in the market with maturities 1 year, 2 years,..., 10 years from which the investor would prefer to include only the bonds having maturities greater or equal than its planned investment horizon. To get the efficient frontier, first the minimum value-at-risk portfolio is sought without any restriction imposed on the return. The resulting minimum value-at-risk portfolio's return is then gradually increased and imposed as the desired return level on the problem that is to be resolved several times until the maximum return portfolio is reached.

We solve the problem by expressing VaR by using Fenton Wilkinson approximation and simulation. Zero coupon bond weights, portfolio returns, and portfolio VaR for some of the optimum portfolios of our example when short selling is not allowed for both methods are given in tables 7.6 and 7.7.

(MISSING COMMENT)

Table 7.6. Zero-coupon bond weights (Fenton Wilkinson Approximation)

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**VaR**
1.00	0.00	0.00	0.00	0.00	0.00	1.126066	-0.11195241
0.74	0.18	0.08	0.00	0.00	0.00	1.131066	-0.10645718
0.48	0.35	0.17	0.00	0.00	0.00	1.136066	-0.10100519
0.21	0.54	0.25	0.00	0.00	0.00	1.141066	-0.09559829
0.00	0.71	0.17	0.13	0.00	0.00	1.146066	-0.09056089
0.00	0.40	0.33	0.27	0.00	0.00	1.151066	-0.08545246
0.00	0.09	0.50	0.41	0.00	0.00	1.156066	-0.08005977
0.00	0.00	0.40	0.32	0.28	0.00	1.161066	-0.07583364
0.00	0.00	0.04	0.47	0.49	0.00	1.166066	-0.07016819
0.00	0.00	0.00	0.21	0.38	0.41	1.171066	-0.06638414
0.00	0.00	0.00	0.00	0.00	1.00	1.176066	-0.05517476

Table 7.7. Zero-coupon bond weights (Monte Carlo Simulation)

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**VaR**
1.00	0.00	0.00	0.00	0.00	0.00	1.126066	-0.11195241
0.67	0.32	0.00	0.00	0.00	0.00	1.131066	-0.10645751
0.35	0.65	0.00	0.00	0.00	0.00	1.136066	-0.10096207
0.02	0.97	0.00	0.00	0.00	0.00	1.141067	-0.09546695
0.00	0.61	0.39	0.00	0.00	0.00	1.146067	-0.08994458
0.00	0.20	0.79	0.01	0.00	0.00	1.151068	-0.08441852
0.00	0.00	0.72	0.28	0.00	0.00	1.156068	-0.07886348
0.00	0.00	0.20	0.80	0.00	0.00	1.161068	-0.07328034
0.00	0.00	0.00	0.58	0.41	0.00	1.166068	-0.06763214
0.00	0.00	0.00	0.00	0.90	0.10	1.171068	-0.06192462
0.00	0.00	0.00	0.00	0.00	1.00	1.176068	-0.0557486

7.5. LIBOR Market Model Application

In this section we will see how classical mean / variance problem and its extension, mean / VaR problem, can be used for bond portfolio selection purpose when the interest rate term structure is assumed to follow the dynamics of LIBOR Market Model. We will see how the classical model inputs, bonds' mean and variance-covariance parameters, and portfolio value at risk can be estimated with simulation.

From equation (4.2), at a tenor date T_i the price of a bond with a maturity date T_n, $n > i$, is given by

$$P(T_i, T_n) = \prod_{j=i}^{n-1} \frac{1}{1 + \delta L_j(T_i)} \tag{7.26}$$

The bond price is a function of forward LIBOR rates, hence their joint distribution is involved in the calculation. Since at most only one these rates can be lognormal (driftless) under a certain numéraire, there is no analytical solution giving future bond prices or covariance structure. Hence the required parameters will be obtained by simulation. In order to generate sample forward LIBOR rates paths we will use the discretizations introduces in section 4.3. A simulation run returns the generated yield curves for dates corresponding to each bond maturity. Sample bond prices for each simulation run can then be calculated using equation (7.26). A $n \times M - 1$ matrix is obtained for n repetitions and M maturity dates (note that $P(t, T) = 1$ for $t = T$, independent of the short rate). Taking simple average of each column will give expected values appearing in equation (7.8). Covariance matrix is also easily obtained by computing the covariances between the columns.

7.5.1. Mean / Variance Problem

e will solve the same example problem described for mean / variance Vasicek application in section 7.4.1. Volatility and correlation structures are calibrated according to non-parametric calibration to implied swaptions volatilities for 06 June 2011 which are summarized in table 7.8. We will solve the problem for each combination of discretization scheme and measure choice. Required parameters for the optimization problem are obtained by Monte Carlo simulation with 10^5 sample size; the resulting expected future bond prices are given in table 7.9 and estimated variance-covariance structures are given in tables 7.10, 7.11, 7.12, and 7.13.

Table 7.8. LIBOR market model parameter values for example scenario

Parameter	Value
Latest Maturity	10 years
Investment Horizon	5 years
$L(0)$	Table 5.3
$\sigma_i(t)$	Table 5.5
$\rho_{i,j}$	Table 5.9

Table 7.9. Expected future bond prices under different schemes and measures

	Euler Scheme & Terminal Measure	Euler Scheme & Spot Measure	Arbitrage Free Scheme & Terminal Measure	Arbitrage Free Scheme & Spot Measure
P(5,6)	0.9642585	0.9635366	0.9640845	0.9641448
P(5,7)	0.9237318	0.9223101	0.923584	0.9236792
P(5,8)	0.8860779	0.8841095	0.8857373	0.8862287
P(5,9)	0.8461788	0.8436351	0.8457514	0.846836
P(5,10)	0.8051806	0.802026	0.8047084	0.806422

Table 7.10. Covariance matrix for Euler Scheme & Terminal Measure

	P(5,6)	P(5,7)	P(5,8)	P(5,9)	P(5,10)
P(5,6)	0.000164	0.000198	0.00027	0.000362	0.000458
P(5,7)	0.000198	0.000356	0.000449	0.000585	0.000738
P(5,8)	0.00027	0.000449	0.000622	0.000809	0.000992
P(5,9)	0.000362	0.000585	0.000809	0.00106	0.001307
P(5,10)	0.000458	0.000738	0.000992	0.001307	0.001644

Table 7.11. Covariance matrix for Euler Scheme & Spot Measure

	P(5,6)	P(5,7)	P(5,8)	P(5,9)	P(5,10)
P(5,6)	0.000164	0.000198	0.00027	0.000361	0.000454
P(5,7)	0.000198	0.000361	0.000455	0.00059	0.000742
P(5,8)	0.00027	0.000455	0.000629	0.000815	0.000994
P(5,9)	0.000361	0.00059	0.000815	0.001065	0.001307
P(5,10)	0.000454	0.000742	0.000994	0.001307	0.001638

Table 7.12. Covariance matrix for Arbitrage Free Scheme & Terminal Measure

	P(5,6)	P(5,7)	P(5,8)	P(5,9)	P(5,10)
P(5,6)	0.000169	0.000188	0.000255	0.000346	0.000442
P(5,7)	0.000188	0.000335	0.000413	0.000542	0.00069
P(5,8)	0.000255	0.000413	0.000572	0.00075	0.000922
P(5,9)	0.000346	0.000542	0.00075	0.00099	0.001225
P(5,10)	0.000442	0.00069	0.000922	0.001225	0.001549

Table 7.13. Covariance matrix for Arbitrage Free Scheme & Spot Measure

	P(5,6)	P(5,7)	P(5,8)	P(5,9)	P(5,10)
P(5,6)	0.000159	0.000201	0.000271	0.000354	0.000443
P(5,7)	0.000201	0.000363	0.000459	0.000587	0.000735
P(5,8)	0.000271	0.000459	0.000628	0.000804	0.000983
P(5,9)	0.000354	0.000587	0.000804	0.001036	0.001273
P(5,10)	0.000443	0.000735	0.000983	0.001273	0.00159

Zero coupon bond weights, portfolio returns, and portfolio deviations for some of the optimum portfolios of our example when short selling is not allowed are given in tables 7.14, 7.15, 7.16, and 7.17. Quite naturally we observe the shift from the riskless 5 year bond to the most risky and high return promising 10 year bond in all cases. Diversification is not achieved in this case due to high correlation among the bonds.

Table 7.14. Zero-coupon bond weights of efficient short-sale constrained portfolios for Euler Scheme & Terminal Measure

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**sd(W)**
1.00	0.00	0.00	0.00	0.00	0.00	1.085408	0.00
0.86	0.00	0.00	0.14	0.00	0.00	1.085708	0.00442
0.71	0.00	0.00	0.29	0.00	0.00	1.086008	0.008817
0.57	0.00	0.00	0.43	0.00	0.00	1.086308	0.013219
0.42	0.00	0.00	0.58	0.00	0.00	1.086608	0.017622
0.28	0.00	0.00	0.72	0.00	0.00	1.086908	0.022025
0.14	0.00	0.00	0.86	0.00	0.00	1.087208	0.026429
0.00	0.00	0.00	0.98	0.02	0.00	1.087508	0.030854
0.00	0.00	0.00	0.55	0.45	0.00	1.087808	0.035681
0.00	0.00	0.00	0.11	0.89	0.00	1.088108	0.04056
0.00	0.00	0.00	0.00	0.70	0.30	1.088408	0.045587
0.00	0.00	0.00	0.00	0.00	1.00	1.088708	0.049348

Table 7.15. Zero-coupon bond weights of efficient short-sale constrained portfolios for Euler Scheme & Spot Measure

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**sd(W)**
1.00	0.00	0.00	0.00	0.00	0.00	1.085408	0.00
0.83	0.01	0.16	0.00	0.00	0.00	1.085708	0.00474
0.74	0.04	0.22	0.00	0.00	0.00	1.086008	0.008338
0.50	0.09	0.41	0.00	0.00	0.00	1.086308	0.01235
0.43	0.00	0.57	0.00	0.00	0.00	1.086608	0.016364
0.29	0.00	0.71	0.00	0.00	0.00	1.086908	0.020378
0.15	0.00	0.85	0.00	0.00	0.00	1.087208	0.024392
0.00	0.00	0.35	0.64	0.00	0.01	1.087508	0.028406
0.00	0.00	0.17	0.63	0.00	0.20	1.087808	0.032421
0.00	0.00	0.00	0.61	0.00	0.39	1.088108	0.036436
0.00	0.00	0.00	0.35	0.00	0.65	1.088408	0.04045
0.00	0.00	0.00	0.00	0.00	1.00	1.088708	0.044465

Table 7.16. Zero-coupon bond weights of efficient short-sale constrained portfolios for Arbitrage Free Scheme & Terminal Measure

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**sd(W)**
1.00	0.00	0.00	0.00	0.00	0.00	1.085408	0.00
0.87	0.00	0.11	0.02	0.00	0.00	1.085708	0.002998
0.74	0.00	0.22	0.04	0.00	0.00	1.086008	0.005967
0.61	0.00	0.33	0.06	0.00	0.00	1.086308	0.008943
0.48	0.00	0.44	0.08	0.00	0.00	1.086608	0.01192
0.22	0.00	0.67	0.11	0.00	0.00	1.086908	0.017876
0.00	0.00	0.72	0.28	0.00	0.00	1.087208	0.023863
0.00	0.00	0.65	0.24	0.00	0.11	1.087508	0.027092
0.00	0.00	0.60	0.17	0.00	0.23	1.087808	0.030361
0.00	0.00	0.50	0.03	0.00	0.47	1.088108	0.036972
0.00	0.00	0.32	0.00	0.00	0.68	1.088408	0.043652
0.00	0.00	0.11	0.00	0.00	0.89	1.088708	0.050386
0.00	0.00	0.00	0.00	0.00	1.00	1.089008	0.051061

Table 7.17. Zero-coupon bond weights of efficient short-sale constrained portfolios for Arbitrage Free Scheme & Spot Measure

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**sd(W)**
1.00	0.00	0.00	0.00	0.00	0.00	1.085408	0.00
0.90	0.00	0.00	0.00	0.00	0.10	1.085708	0.005365
0.80	0.00	0.00	0.00	0.00	0.20	1.086008	0.010712
0.70	0.00	0.00	0.00	0.00	0.30	1.086308	0.016064
0.60	0.00	0.00	0.00	0.00	0.40	1.086608	0.021416
0.50	0.00	0.00	0.00	0.00	0.50	1.086908	0.026768
0.40	0.00	0.00	0.00	0.00	0.60	1.087208	0.032121
0.30	0.00	0.00	0.00	0.00	0.70	1.087508	0.037474
0.21	0.00	0.00	0.00	0.00	0.79	1.087808	0.042827
0.11	0.00	0.00	0.00	0.00	0.89	1.088108	0.04818
0.01	0.00	0.00	0.00	0.00	0.99	1.088408	0.053533

7.5.2. Mean / VaR Problem

We will solve the same example problem described for mean / VaR Vasicek application in section 7.4.2. Volatility and correlation structures are calibrated according to non-parametric calibration to implied swaptions volatilities for 06 June 2011 which are summarized in table 7.8. We will solve the problem for each combination of discretization scheme and measure choice. We will use the same generated paths in section **??** and hence the expected future bond prices are the same as given in table 7.9. Losses for each simulated path are calculated by $\sum_{i=T+1}^{\tau} w_i P(0,i) - \sum_{i=T+1}^{\tau} w_i P(T,i)$ and VaR is the $(1-\alpha)$ percent quantile value of these losses. Zero coupon bond weights, portfolio returns, and portfolio VaR for some of the optimum portfolios of our example when short selling is not allowed are given in tables 7.18, 7.19, 7.20, and 7.21. Quite naturally we observe the shift from the riskless 5 year bond to the most risky and high return promising 10 year bond in all cases. Diversification is not achieved in this case due to high correlation among the bonds.

Table 7.18. Add caption

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**VaR**
1.00	0.00	0.00	0.00	0.00	0.00	1.09	-0.08
0.86	0.00	0.03	0.03	0.02	0.05	1.085909	-0.07092
0.72	0.01	0.07	0.10	0.00	0.10	1.086409	-0.0632
0.62	0.02	0.11	0.00	0.02	0.22	1.086913	-0.05538
0.48	0.01	0.09	0.19	0.02	0.22	1.087414	-0.04764
0.30	0.01	0.17	0.26	0.02	0.24	1.087915	-0.03994
0.16	0.00	0.22	0.32	0.00	0.29	1.088417	-0.03222
0.05	0.00	0.23	0.34	0.03	0.35	1.088918	-0.02451
0.00	0.00	0.15	0.30	0.07	0.47	1.089421	-0.01628
0.00	0.00	0.00	0.00	0.26	0.74	1.089917	-0.00879
0.00	0.00	0.00	0.00	0.00	1.00	1.090418	-0.00102

Table 7.19. Add caption

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**VaR**
1.00	0.00	0.00	0.00	0.00	0.00	1.09	-0.08
0.72	0.09	0.00	0.01	0.03	0.14	1.085909	-0.07256
0.62	0.13	0.01	0.00	0.00	0.23	1.086409	-0.06484
0.55	0.11	0.03	0.01	0.03	0.27	1.086913	-0.05702
0.55	0.05	0.01	0.00	0.00	0.39	1.087414	-0.04928
0.23	0.27	0.02	0.06	0.12	0.30	1.087915	-0.04158
0.47	0.00	0.00	0.00	0.00	0.53	1.088417	-0.03386
0.24	0.10	0.03	0.01	0.08	0.53	1.088918	-0.01792
0.13	0.02	0.08	0.15	0.15	0.47	1.089421	-0.01043
0.20	0.03	0.00	0.01	0.00	0.76	1.089917	-0.00265
0.08	0.03	0.03	0.00	0.14	0.72	1.090418	0.005091
0.09	0.00	0.00	0.00	0.00	0.91	1.090919	0.012788
0.01	0.00	0.00	0.00	0.00	0.99	1.091422	0.022281

Table 7.20. Add caption

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**VaR**
0.87	0.00	0.11	0.02	0.00	0.00	1.09	-0.08
0.61	0.00	0.33	0.06	0.00	0.00	1.085909	-0.07226
0.35	0.00	0.55	0.10	0.00	0.00	1.086409	-0.05831
0.09	0.00	0.78	0.13	0.00	0.00	1.086913	-0.05272
0.00	0.00	0.65	0.24	0.00	0.11	1.087414	-0.04839
0.00	0.00	0.55	0.10	0.00	0.35	1.087915	-0.03888
0.00	0.00	0.42	0.00	0.00	0.58	1.088417	-0.02866
0.00	0.00	0.22	0.00	0.00	0.78	1.088918	-0.0139
0.00	0.00	0.01	0.00	0.00	0.99	1.089421	-0.00715

Table 7.21. Add caption

w5	**w6**	**w7**	**w8**	**w9**	**w10**	**E[W]**	**VaR**
1.00	0.00	0.00	0.00	0.00	0.00	1.09	-0.08
0.78	0.03	0.00	0.02	0.03	0.14	1.085909	-0.08181
0.67	0.00	0.06	0.07	0.00	0.20	1.086409	-0.0683
0.56	0.00	0.08	0.11	0.00	0.25	1.086913	-0.0648
0.52	0.00	0.02	0.15	0.00	0.31	1.087414	-0.05541
0.35	0.00	0.00	0.23	0.00	0.42	1.087915	-0.04222
0.08	0.00	0.00	0.09	0.16	0.67	1.088417	-0.03881
0.00	0.00	0.00	0.04	0.15	0.81	1.088918	-0.02421
0.00	0.00	0.00	0.00	0.09	0.91	1.089421	-0.02008
0.00	0.00	0.00	0.00	0.00	1.00	1.089917	-0.00567

8. CONCLUSION

In this thesis, we studied Vasicek and LIBOR Market models and their applications for static zero-coupon bond portfolio selection problem. First we explained and illustrated model dynamics, calibration, bond pricing for both models. Then we introduced static portfolio optimization problem and discussed how it can be applied for forming efficient zero-coupon bond portfolios. In order to evalute the risk, we used variance as the touchstone risk measure and value-at-risk as a more modernistic one.

We estimated Vasicek model parameters from historical data using maximum likelihood estimation. Being a member of equilibrium models, Vasicek model is inconsistent with the current market term structure and it cannot be calibrated to the spot yield curve. To overcome this inconsistency, we attempted a second calibration step where we implicate a constant risk premium and try to fit the model yield curve to average historical yield curves. Our aim in this effort of fitting the model to market data is to make a coherent comparison with LIBOR Market model, which can be automatically calibrated to market, while solving bond portfolio optimization problems. Although LIBOR Market model is very flexible in defining volatility and correlation structures, its calibration is not that straightforward. There are many completed and ongoing studies investigating efficient calibration techniques, but the results are often controversial. In the thesis we are not particularly concerned with LIBOR Market's calibration topic, so we just used some frequently used and mentioned methods in literature to be able to make a complete implementation of the model.

Bond portfolio optimization problem deals with the future value of the portfolio and its risk in terms of variance or value-at-risk. This requires the calculation of future bond prices. With Vasicek model, bond pricing either by analytical formulations or simulation can be performed very easily and fast. Simulation of LIBOR Market model, on the other hand, is much more slower. We performed simulation for it is still a commonly used numerical method for LIBOR Market model, even though it is computationally expensive and slowly converging.

To solve the quadratic mean / variance problems we used the "solve.QP" function included in "quadprog" package of R. Problem is quickly solved once the parameters which are simply today's and future bond prices, and covariance between future bond prices are calculated. To solve the mean / VaR problem, which has a non-linear and non-smooth (when simulation is used) objective function, we used the "DEoptim" function included in "DEoptim" package of R. The problem can be solved in a short time when VaR is expressed in a closed form. We approximated a closed form VaR expression for Vasicek application. But it takes a lot more time when we use simulation because at each iteration losses are calculated for each sample path and they are sorted to get the sample quantile corresponding to the given probability.

For Vasicek model application, we observed that efficient frontier obtained by solving mean / VaR problem when simulation is used has similar weight distribution with mean / variance problem. This is due to a near symmetrical portfolio return distribution function for our example. The same is not valid for LIBOR Market model application; efficient portfolios do not exhibit parallelism.

To answer the question of which model to use should be probed elaborately with more emprical research. We cound not conduct such a work due to lack of over the counter market data, especially necessary for LIBOR Market model calibration.

APPENDIX A: APPLICATION

The appendices start here.

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