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Pricing and hedging interest rate caps

With the LIBOR, Hull-White, and G2++ interest rate models - Evidence from the Danish market

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Executive summary

This thesis investigates the pricing and hedging accuracy of alternative interest rate models. The models include the LIBOR market model, the Hull-White model and the G2++ model. Based on market data of Danish interest rate caps this thesis provides empirical evidence on the relative performance of the interest rate models and, moreover, investigates how the instantaneous volatility in the LIBOR market model should be specified and whether an additional stochastic factor improves the performance of the Hull-White model. The LIBOR market model is represented using three different volatility parameterizations counting the Rebonato, Exponential and Constant volatility parameterizations. The G2++ model represents the two-factor Hull-White model.

Inspired by the works of Gupta & Subrahmanyam (2005) the models are compared based on their ability to fit current market prices, price interest rate caps out-of-sample and hedge interest rate caps. The hedging accuracy is measured using a replicating portfolio strategy based on zero-coupon bonds.

The results of the calibration to market prices show that the most accurate in-sample estimation is achieved with the LIBOR market model which is able to exactly fit market prices of interest rate caps. In addition, when pricing interest rate caps out-of-sample the LIBOR market model is more accurate at predicting future prices of interest rate caps compared to the Gaussian models (Hull-White and G2++). However, when measuring the hedging ability the results indicate that the Hull-White and the G2++ model provide more accurate hedging of interest rate caps than the LIBOR market model but on average the difference between the models is less than 1 basis point for a one-day rebalancing interval.

Compared to the Hull-White model the two-factor model (G2++) provides more accurate calibration to interest rate caps as well as more accurate out-of-sample pricing. Furthermore, when hedging interest rate caps the G2++ model produces smaller errors compared to the Hull-White model. The results also show that the Hull-White model is consistently over-hedging short maturity caps.

In relation to the LIBOR market model this thesis finds that specifying instantaneous volatility using the Rebonato parameterization produces the most accurate in-sample estimation and out-of-sample pricing of interest rate caps. The ability of the Rebonato parameterization to fit both humped and decreasing shapes of the term structure of volatilities leads to more accurate pricing compared to the Exponential and Constant volatility parameterizations. However, when measuring the hedging performance the results are inconclusive.

The investigative work in this thesis involves a number of methodological considerations and the results are tested for robustness towards calibration method and bias in sample data. Two criteria are applied for model calibration; standard least squares and percentage least squares. The results show superior calibration using the standard criterion.

Conclusively, the empirical evidence of out-of-sample pricing and hedging are in accordance with Gupta & Subrahmanyam (2005) who find that a LIBOR market model provides more accurate pricing and, moreover, that more efficient hedging is achieved with multifactor interest rate models

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1. Introduction

The market for over-the-counter (OTC) interest rate options including interest rate caps and floors has grown over the past decade and is today among the largest and most liquid option markets. By the end of June 2009 the global market accounted for about 48.513 billions US dollars in outstanding notional principal and around 1.414 billions US dollars in gross market values (BIS Quarterly Review, December 2009, Table 4).

The large volume of the market for interest rate options makes the subject of pricing interest rate options interesting and relevant. For instance, the accurate pricing is relevant to risk managers using Value-at-Risk (VaR) methods and option dealers who trade the instruments. In relation to VaR pricing accuracy of alternative models is important because the VaR measures depend on models in order to estimate future option prices. Similarly, to option dealers accurate pricing is important when choosing between alternative models, however, it is also important to get the hedging correct since it will reduce or eliminate future uncertainty.

The pricing of interest rate options is based on the area of dynamic term structure modelling. Interest rate caps and floors can be priced using a variety of different models but one model in particular (the LIBOR market model) has gained popularity due to an ability to encompass well-established market formulas (Brigo & Mercurio 2006, p. 195). However, following easier access to data from OTC interest rate option markets research focusing on comparative investigations and empirical tests of dynamic term structure models has gained attention in recent works. E.g. Gupta & Subrahmanyam (2005) compare the performance of a number of different interest rate models based on data from US caps and floors markets. Gupta & Subrahmanyam find that pricing accuracy can be achieved within a one-factor framework by choosing a model to fit the skew of the underlying interest rate probability distribution and specifically they find that a one-factor LIBOR market model in particular produces accurate results. However, excellent pricing accuracy does not necessarily mean that the model also will deliver good hedging results because pricing accuracy and hedging accuracy each relate to different assumptions of the model. Therefore Gupta & Subrahmanyam also investigate the models in relation to their hedging accuracy. For term structure models with continuously compounded rates as underlying modelling objects, they find that the hedging of interest rate options clearly improves when adding a second stochastic factor to the models. Moreover, when compared to fitting the skew of the underlying distribution the additional factor is more important.

Pricing accuracy refers to *out-of-sample* pricing[1] which is the ability of a model to price options accurately, conditional on current term structure. Out-of-sample pricing corresponds to measuring a model's ability to capture information from market data and translate the information into accurate option prices. Contrary, hedging accuracy (*hedging errors*) reflects the dynamic performance of a given model. I.e. hedging errors measure

[1] See sections 5.2 and 7.1.

the ability of a model to capture underlying movements in the term structure in the future, after the model has been initially calibrated to fit current market data.

In accordance with the research of Gupta & Subrahmanyam I find it interesting to investigate the pricing and hedging performance of dynamic term structure models and to investigate whether further evidence can be found based on data from other markets. Therefore the focus of this thesis will be on the implementation of different dynamic term structure models in order to conduct empirical comparisons based on market data. The attention will be on interest rate caps as these are the instruments for which market data is available. Interest rate caps are plain vanilla types of financial instruments and depending on the analytical tractability of the dynamic term structure models some of them provide closed form solutions for interest rate caps. Hence, it is possible to implement models without the use of numerical simulation techniques (e.g. finite difference methods or Monte Carlo simulation).

Classic distributional assumptions of interest rate models include the normal (Gaussian) distribution and the log-normal distribution. The Hull-White model and the LIBOR market model represent the assumption of normality and log-normality, respectively, and are commonly known and applied in practice (Brigo & Mercurio 2006, "Talking to the traders" pp. 935-950). The Hull-White and the LIBOR market models are also implemented and investigated in the analysis of Gupta & Subrahmanyam. Both models can be initially fitted to market prices of interest rate caps but an exact fit is somewhat more difficult to obtain with the Hull-White model (Rebonato 2004 p. 692). On the other hand, the LIBOR market model provides a theoretical no-arbitrage framework for Black's formula which for many years has been (and still is) the market standard formula for quoting interest rate caps. The link between the LIBOR market model and Black's formula enables the model to exactly fit market observable prices of interest rate caps. This characteristic has also made the LIBOR market model a popular tool for pricing more exotic financial instruments because the model is able to use all available market information.

However, the LIBOR market model depends on the specification of instantaneous volatility (and correlation) of the forward rates, i.e. the volatility functions influence the model's evolution of future interest rates and, hence, the pricing of derivatives within the model. In principle this means that the user of the model has to specify the instantaneous volatility. Given different specifications of the instantaneous volatility the user will obtain different models. The choice of instantaneous volatility within the LIBOR market model is therefore interesting because the pricing and hedging performance can be compared given alternative volatility specifications.

The aim of this thesis is to examine the pricing and hedging accuracy of alternative term structure models given different assumptions of volatility or different number of driving factors. The focus is twofold as this thesis will be comparing the performance of different term structure models as well as studying the effects of making changes within the models. The following sections present the problem statement and the thesis structure.

1.1 Problem statement

In this thesis I will investigate and compare the pricing and hedging accuracy of models which represent two different classical distributional assumptions: The LIBOR market model and the Hull-White model representing the log-normal and the normal distribution, respectively.

It is commonly assumed that interest rates follow a log-normal distribution suggesting the use of models with this underlying assumption. This assumption can be confirmed empirically by two means. First, according to Bakshi, Cao and Chen (1997 p. 2005) the out-of-sample pricing errors of the alternative models can be compared in order to obtain an indication of model misspecification. Second, comparing the hedging accuracy of the models will reveal the relative ability of the models to capture the market dynamics of interest rate movements. This leads to the first question of this thesis:

When pricing and hedging interest rate caps in the Danish market which type of model provides the better pricing and hedging accuracy – Gaussian or lognormal models?

Changes within the models are investigated alongside the comparable study. The choice between models is a central issue to any dealer, but improving the accuracy within the models is just as important and can possibly alter the conclusion of the comparable study. Additional factors can be added to traditional one-factor short-rate models in order to present a more realistic evolution of the term structure. Short rate models only containing one driving factor are known to imply perfect positive correlation among forward rates. However, this does not coincide with empirical observations of term structure behaviour. On the other hand, the LIBOR market model does not imply perfect positive correlation among forward rates, but the dynamics of the forward rates depend on the specification of the instantaneous volatility which is decided by the user of the model.

Therefore, in relation to the Hull-White model I will investigate the following question:

Do multifactor interest rate models provide more accurate hedging and pricing of interest rate caps and floors than their single factor equivalents?

Regarding the LIBOR market model the question is which volatility specification should be applied. A linear-exponential formulation proposed by Rebonato[2] is often suggested. Taking the Rebonato specification as a benchmark I will investigate the effects on the pricing and hedging accuracy within the LIBOR market model when applying two simple specifications of volatility:

Given alternative parameterizations of instantaneous volatility - how should the instantaneous volatility be specified in order to improve the hedging and pricing performance of the LIBOR market model?

[2] Originally proposed in an article by Rebonato in 1999 but the specification is referred widely in the literature e.g. see Rebonato (2002).

1.2 Methodology

The methodology in this thesis builds on the approaches applied in the literature investigating the empirical performance of dynamic term structure models. This thesis is mainly inspired by Gupta & Subrahmanyam (2005) and Amin & Morton (1994). Hence, this thesis takes on an empirical approach in order to investigate and conclude on the research questions of the problem statement. The theoretical part of this thesis aims to give the reader an introduction to the area of dynamic term structure modelling and to give an understanding of the models which are applied in the empirical study. Moreover, relevant theory of interest rates and interest rate options are defined in accordance with financial theory in order to establish the fundamentals necessary for conducting the empirical investigation of the models.

The empirical study is based on the experimental designs which are explained in details in chapter 7. The empirical study involves a number of methodological decisions concerning subjects such as the composition of the sample data, criteria and methods for calibration of model prices to market prices, criteria for comparing model performance of pricing and hedging, and alternative approaches to hedging interest rate caps. The implications of using alternative criteria and methods for calibration and measurement of model accuracy will be discussed. The considerations concerning the specific decisions made in relation to above-mentioned subjects are discussed in the relevant chapters.

Empirical studies are based on specific data samples and over time data will represent different economic environments, hereby a given empirical study is subject to the uncertainty that the results are only valid for a specific dataset. Hence, in order to conclude on the comparative performance of the models two sample periods are applied in order to test the robustness of the results towards different datasets.

Several applications have been applied in order to conduct the empirical study. The models and the empirical framework are implemented using MATLAB, regressions are implemented using SAS, and data are obtained using Bloomberg and DataStream. An overview of the relevant code is provided in appendix C.

1.3 Thesis structure

This thesis is structured in two main parts. The first part consists of the chapters 2, 3 and 4 and establishes the theoretical underpinnings necessary to implement the interest rate models in the empirical study.

- Chapter 2 *Dynamic term structure models* introduces the area of dynamic term structure modelling and provides an overview of different approaches developed to modelling the dynamics of the term structure of interest rates.
- Chapter 3 *Fundamentals* establishes the basic theory of interest rates and interest rate options relevant to the empirical study, describing LIBOR rates, interest rate swaps, and the structure and pricing of interest rate caplets and caps. Moreover, chapter 3 focuses on relevant market practice concerning caplets and caps.
- Chapter 4 *The models* introduces the interest rate models which will be evaluated in the empirical study. The LIBOR market, Hull-White and the G2++ models are presented with focus on model characteristics and the pricing of interest rate caps. Given the importance of model calibration in order to conduct the empirical study, chapter 4 also covers calibration examples for each model demonstrating the calibration of the models to market data of interest rate caps. Moreover, chapter 4 presents the three parametric types of instantaneous volatility functions implemented in the LIBOR market model.

The second part of this thesis contains the empirical study and consists of the chapters 5 to 9.

- Chapter 5 *Empirical studies on dynamics term structure models* reviews the results of similar empirical studies on interest rate models and discusses the criteria applied in order to measure the comparative performance of interest rate models.
- Chapter 6 *Hedging interest rate caps* concentrates on different approaches to hedging and defines how hedging of interest rate caps are implemented.
- Chapter 7 *Experimental designs* explains in detail the frameworks applied in order to compare the pricing and hedging accuracy of the different interest rate models. The experimental designs are fundamental to the empirical study.
- Chapter 8 *The data* presents the sample data used in the empirical study and discuss the assumptions and data adjustments alongside. Moreover, chapter 8 present summary statistics and stylized facts of the data applied.
- Chapter 9 *Empirical results* present and discuss the results of the empirical study. Section 9.1 concentrates on model calibration and compares the results of model implementation discussing the in-sample estimation and the parameter stability of the interest rate models. Section 9.2 evaluates the results of out-of-sample pricing accuracy and section 9.3 evaluates the results of testing hedging performance.

Finally, chapter 10 *Conclusion* sums up and concludes the thesis and chapter 11 *Future research* suggests alternative research areas.

1.4 Delimitations

The area of dynamic term structure modelling covers a wide range of different interest rate models representing different characteristics and assumptions. Interest rate models which take basis in the normal and lognormal distributions alone account for several models. Hence, a wide range of models cannot be covered within the scope of this thesis and I therefore limit the selection of models to include the LIBOR market model, Hull-White model and G2++ model.

Moreover, this thesis concentrates on the basic versions of said models, therefore, later developed extension such smile-modelling and models with stochastic volatility, credit risk or jumps are not considered.

The aim of this thesis is to conduct a comparative study of the pricing and hedging abilities of the interest rate models based on market data. Therefore, this thesis will focus on the implementation of the models rather than derivation of the models and model specific pricing formulae. Such derivations are available in a number of different textbooks concentrating on interest rate term structure modelling. Thus, the presentation of the different interest rate models focuses on the relevant mathematics which is applied in the empirical study.

The empirical research is based on market data of Danish interest rate caps indexed on the 6 month CIBOR rate. In practice, the valuation of interest rate derivatives requires the use of specific day count conventions to calculate year fractions. However, this issue is not addressed and all year fractions are assumed constant.

2 Dynamic term structure models

Many financial instruments have uncertain future payments which cannot be priced by means of simple discounting of cash flows. Instead the pricing can be done using dynamic interest rate models which are able to generate and assign probabilities to scenarios for future term structures and by that enabling us to price instruments with non-deterministic cash flows. Since the development of the first dynamic interest rate models several alternative models have been developed. In this chapter I will provide an overview of the different categories within the area and introduce some of the most commonly known models.

Broadly speaking we have three different categories of models: short-rate models, forward rate models and market models.

2.1 Short-rate models

Vasicek (1977) and Cox, Ingersoll and Ross (1985) developed some of the first short-rate models. These models have good analytical properties because simple formulas for bond and some option prices can be derived from them.

In general terms, short-rate models assume that the development of the short rate can be characterized by a stochastic process:

$$\mathrm{d}r(t) = \mu\big(t, r(t)\big)\mathrm{d}t + \sigma\big(t, r(t)\big)\mathrm{d}W(t) \tag{2.1}$$

Where μ and σ are the drift and volatility functions of the short rate and W is a one dimensional Brownian motion under the risk neutral measure $\mathbb{Q}$.

Short-rate models therefore implicitly assume that all information about future interest rates is contained in the current instantaneous short-term rate, and that all prices of default free bonds can be represented as functions of the short-term rate and time only.

The functional form of μ and σ determines the behaviour of the short rate r. Different functional forms of μ and σ will lead to different short-rate models from which the term structure of interest rates can be derived.[3] For instance, define a process with mean-reverting drift $\mu(t, r(t)) = k(\theta - r(t))$ and constant volatility, $\sigma(t, r(t)) = \sigma$ this leads to the Vasicek model.

In the Vasicek model the drift and volatility functions are constants since they only depend on the level of the short rate $r(t)$. Therefore the Vasicek model is time-homogenous. A drawback of time homogeneous models is that the initial term structure not necessarily matches the observed market term structure (Brigo & Mercurio 2006 p. 57) implying that observed market prices of bonds cannot be replicated.

[3] Cairns (2004 chap. 4) and Brigo & Mercurio (2006 chap. 3) each provides an overview of well known short-rate models and their analytical properties.

The issue of a poor fit to the initial term structure can be dealt with by making the long term level of the short rate θ time dependant. This assumption leads to the Hull-White model. In order to make the initial term structure observed in the market a part of the model it can be shown that (Brigo & Mercurio, 2006, p. 73)

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial t} + kf^M(0,t) + \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right) \tag{2.2}$$

where $f^M(0,t)$ is the market instantaneous forward rate at time 0 for the maturity t. This approach enables us to remake the initial term structure within the model by using the initial term structure as input to the model.

In accordance with the Vasicek model the short-rate $r(t)$ is normally distributed in the Hull-White model. The Gaussian property of the interest rate implies that the short rate $r(t)$ can be negative with positive probability (Brigo & Mercurio, 2006, p. 59). Negative interest rates are rarely observed and the possibility of the model generating negative interest rates is therefore considered a drawback of the Vasicek and Gaussian models in general.[4]

The drawback of negative interest rates have been addressed by other models such as the Cox, Ingersoll and Ross (CIR) model in which they propose a square-root process for the short-rate or by the Black-Karasinski model in which the short-rate is assumed to be log-normally distributed.

2.2 Forward rate models: The HJM framework

Heath, Jarrow and Morton (1992) derived a framework for forward rate models in which the dynamics of forward rates and bond prices are related. Instead of describing the evolution of a single quantity as with the short-rate models, Heath, Jarrow and Morton (HJM) models describe the evolution of interest rates with the use of forward rates. Forward rates are considered fundamental building blocks of the term structure. Hence, by starting with a model for the forward rate term structure, the process of yield curve fitting is naturally contained in the model because forward rates are known today. Compared with the calibration of short-rate models this is an advantage of the HJM framework.

HJM models assume that the dynamics of forward rates can be described by a stochastic differential equation

$$\mathrm{d}f(t,T) = \mu(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W(t)\,, \tag{2.3}$$

where $r(t,T)$ is the instantaneous forward rate at time t for the future point in time $T > t$, and μ and σ are the drift and volatility, respectively, related to the instantaneous forward rate.

[4] Nevertheless a few occurrences of negative nominal interest rates have been observed. Switzerland in the 70'ies and Japan in the 90'ies. Economically it seems unlikely because if interest rates turn negative one would expect the owner to hold cash and not to keep money in the bank.

Within the HJM framework the assumption of no-arbitrage leads to a relationship between the drift and the volatility which state that the drift term is completely specified by the volatility

$$\mu(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)\mathrm{d}s \tag{2.4}$$

Thereby the evolution of the forward rate process is described entirely through volatility functions. Substituting equation (2.4) into the forward rate dynamics (2.3) we get

$$\mathrm{d}r(t,T) = \left(\sigma(t,T)\int_t^T \sigma(t,s)\mathrm{d}s\right)\mathrm{d}t + \sigma(t,T)\mathrm{d}W(t)\,, \tag{2.5}$$

which is the arbitrage free dynamics of the forward rate curve under the risk neutral measure $\mathbb{Q}$.

The HJM framework is very general and indeed it is possible to express many of the short rate models as HJM models. In order to obtain a particular model we need to specify the initial forward rate term structure together with a volatility structure for the forward rates. The drift is then determined once the volatility is specified. However, as pointed out by Brigo & Mercurio (2006 p. 184) only a few volatility specifications imply these HJM models which mean that often burdensome procedures are needed to price interest rate derivatives. Yet some specifications of the volatility are known for which the short-rate models can be derived within the HJM framework.

For instance, if the model is specified with constant volatility $\sigma(t,T) = \sigma$, we obtain the Ho-Lee model since each increment ($\mathrm{d}W$) will shift all points on the forward term structure by an equal amount ($\sigma \cdot \mathrm{d}W$). This corresponds to a parallel shift of the forward rate term structure which is known from the Ho-Lee model.

Another example which lead to a well-known model is where the volatility is defined as $\sigma(t,T) = \sigma(t)\exp\left[-\int_t^T \theta(s)\mathrm{d}s\right]$ where the volatility $\sigma(t)$ is constant and $\sigma(t) > 0$, and where θ is the level of mean reversion. This specification leads to the Hull-White model (Continuous time finance, Lecture notes April 2009).

Moreover, the volatility can also be proportional. For instance, $\sigma(t,T) = \sigma(t,T)r(t,T)$ for some deterministic $\sigma(t,T)$ depending only on t and T. Thereby the distribution of the forward rate will be approximately lognormal. The only problem is that this choice is inadmissible as it produces forward rates that will explode within finite time (Glasserman 2004 p. 154).

2.3 Market models

The latest generation of interest rate models are the market models also known as LIBOR market models. The development of these models is mainly contributed to the works of Miltersen, Sandmann & Sondermann (1997) and Brace, Gatarek & Musiela (1997). Their idea was to model discrete forward rates as opposed to modelling instantaneous spot- or forward rates. Hereby they could take into account the observable market interest rates such as LIBOR forward rates.

Miltersen et al. only model the evolution of a finite number of forward rates and specifically they assume that simple interest rates over a fixed finite period of time are log-normally distributed. Based on these assumptions Miltersen et al. find that closed form solutions for caps and floors can be derived. Furthermore, they show that the assumption of log-normally distributed simple interest rates is consistent with the arbitrage free term structure theory of HJM. Moreover, the closed form solutions are equivalent to Black's caplet pricing formula, hence, implying a relationship between market quoted volatilities of interest rate caps and floors and the instantaneous volatilities of forward rates in the LIBOR market model (Brigo & Mercurio, 2006, p. 195) which enables exact calibration of the LIBOR market model to market observable prices of interest rate caps and floors.

Similar to HJM models the instantaneous volatility structure determines the drift of the forward rates in the LIBOR market model. Moreover, as mentioned by Rebonato (2004 p. 700) the (exact) calibration to caplet prices can be obtained in an infinite number of ways. I.e. option traders have to specify the volatility on the basis of criteria other than the recovery of market prices. I will return to the subject of instantaneous volatility and calibration related to the LIBOR market model in chapter 4.

3 Fundamentals

This chapter introduces the fundamental theory of LIBOR rates, interest rate swaps, and interest rate caps which will be applied later in the empirical study. Moreover, this chapter describes useful knowledge of market practice related to interest rate caps and discusses advantages and disadvantages in relation to alternative methods for zero-coupon interest rate curve construction.

3.1 LIBOR rates

LIBOR rates are simply-compounded spot interest rates which are quoted on a daily basis as the average of the deposit rates offered by a group of major banks.[5] LIBOR rates are regarded as the benchmark rates of the main money-market and are quoted for maturities ranging from 1 week to 12 months. The mathematical definition of a simply-compounded spot interest rate at the current time t and with the maturity time T, is given by the expression:

$$L(t,T) = \frac{1 - P(t,T)}{\tau(t,T)\, P(t,T)} \tag{3.1}$$

The term $P(t,T)$ denotes the time t value of a zero-coupon bond with maturity at time T. $\tau(t,T)$ denotes the time between t and T measured in years. Hence, a LIBOR rate can be interpreted as the return an investor would achieve for making a deposit over a given time interval $[t,T]$. Similarly, the simply-compounded forward rate can be interpreted as the return an investor would achieve for making a deposit over a given future time interval. A simply-compounded forward rate observed at current time t and valid over the future time interval $[T,S]$, where T is the expiry date, S the maturity date, and $T < S$, is defined as:

$$F(t,T,S) = \frac{P(t,T) - P(t,S)}{\tau(T,S)\, P(t,S)} \tag{3.2}$$

For time $t = T$, the forward rate becomes the LIBOR spot rate $F(T,T,S) = L(T,S)$.

3.2 Interest rate swaps

An interest rate swap (IRS) corresponds to a collection of forward rate agreements (FRA contracts) cf. appendix A.1. The most common type of interest rate swap is the payer IRS in which the buyer of the IRS pays a fixed interest rate and in return receives floating interest rate payments. Hence, the value of an interest rate swap corresponds to the sum of a series of FRA contracts. Denoting $\mathbb{T} = \{T_\alpha, \dots, T_\beta\}$ as a set of reset and payment dates where T_α is the first reset date of the floating interest rate, $T_{\alpha+1}$ is the first payment date, and T_β is the maturity date and the last payment date, and denoting τ_i as the time between

[5] The term LIBOR is an abbreviation of London Interbank Offer Rate. Interbank rates are quoted in several different market places. For instance, the LIBOR rates are derived from the quotations of the major banks in London and the CIBOR rates are the specific Danish interbank rates derived from the quotations of major Danish banks. Danish LIBOR rates are quoted daily and published on www.nationalbanken.dk.

T_i and T_{i-1} measured in years. The price of an interest rate swap with notional amount 1 can be expressed as:

$$\mathbf{IRS}(t, \mathbb{T}, \tau, K) = \sum_{i=\alpha+1}^{\beta} \mathbf{FRA}(t, T_{i-1}, T_i, \tau_i, K) \tag{3.3}$$

Interest rate swaps are described in greater details in Brigo & Mercurio (2006 pp. 13-16). In accordance with the FRA contract the level of the fixed rate K (the swap rate) for which the value of the interest rate swap is equal to zero, corresponds to the fair value of the swap contract. The fair value level of the swap rate is also referred to as the par forward swap rate, $S_{\alpha,\beta}(t)$. The par forward swap rate at time $t = 0$ is defined as:

$$S_{\alpha,\beta}(0) = \frac{P(0, T_\alpha) - P(0, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)} \tag{3.4}$$

3.3 Zero-coupon term structure

The pricing of interest rate caps requires the use of a zero-coupon yield curve in order to obtain the implicit forward rates and to discount future values. With interest rate caps traded OTC between corporate actors the zero-coupon yield curve should be derived based on instruments which resemble the risk associated with the corporate buyers and sellers. Commonly money and swap market instruments are used in order to reflect the market risk.[6] Common market practice is to construct the zero-coupon yield curve via bootstrapping deposit rates (LIBOR rates) and standardized swap rates. E.g. see Ron (2000) for a practical guide to constructing zero-coupon yield curves.

Discount factors implied by simply-compounded deposit rates can be obtained by rewriting equation (3.1) to the following formula:

$$P(t, T) = \frac{1}{\tau(t, T)\, L(t, T)} \tag{3.5}$$

In order to obtain discount factors from swap-market instruments the bootstrapping procedure begins with the shortest term interest rate swap and continues for increasing maturities. Wilmott (2006 p. 257) outline the procedure for bootstrapping discount factors from market quoted swap rates. Wilmott applies the following formula:

$$P(t, T_\beta) = \frac{1 - S_{\alpha,\beta}(t) \sum_{i=\alpha+1}^{\beta-1} \tau_i\, P(t, T_i)}{1 + S_{\alpha,\beta}(t) \tau_\beta} \tag{3.6}$$

The first $\beta - 1$ discount factors are found by bootstrapping prior instruments including the deposit rates. Usually the swap market only quotes the interest rate swaps for integer maturities and given that the quoted swap rates have semi or quarterly annual payments,

[6] The less risky instruments in the Treasury bond market would not adequately reflect the risk associated with interest rates caps. Contrary, swap and money market instruments are considered related products in the sense that OTC traded interest rate caps are indexed on LIBOR rates and are issued by the same corporate actors.

discount factors for intermediate points are required in order to apply the bootstrapping procedure in equation (3.6). Without market quoted prices on the intermediate dates (semi or quarterly annual date points) discount factors on the intermediate dates must be obtained by an interpolation scheme. Several interpolation approaches exist in order to obtain missing data points e.g. see Ron (2000) or Hagan and West (2006) for a discussion of alternative interpolation schemes, however, as pointed out in by Ron (2000), and Hagan and West (2006) there is no single correct way to interpolate to obtain missing data points. It depends on the purpose of the pricing which method may apply better.

3.4 Interest rate options: Caplets and caps

An interest rate cap is a collection of caplets and a caplet is an option which at a given time T_i will pay the difference between the actual interest rate and the strike rate K of the caplet. Often the underlying interest rate is a LIBOR rate which we can denote L (cf. equation (3.1)), based on the level of the underlying rate the caplet payoff X at the settlement time T_i is given by the expression:

$$X_{T_i} = N \cdot \tau_i \cdot \max\{0, L(T_{i-1}, T_i) - K\} \tag{3.7}$$

Where K is the strike level (also referred to as the cap/caplet rate), N is the notional amount of the contract, $L(T_{i-1}, T_i)$ the LIBOR rate as defined in equation (3.1) on which the contract is written, and τ_i is the length of the time interval $[T_{i-1}, T_i]$ (the year fraction) for which the payment takes place. From the payoff structure it can be seen that the caplet provides protection against rising interest rates because the option pays off when the underlying interest rate increases to a level above the strike rate. Hence, a caplet is similar to a European call option written on the underlying LIBOR rate.

Caps consist of portfolios of almost identical caplets. The caplets only differ in the reset dates, i.e. a cap is a collection of European call options providing protection against rising interest rates over a certain future time period. I have illustrated the structure of a cap in figure 3.1 below.

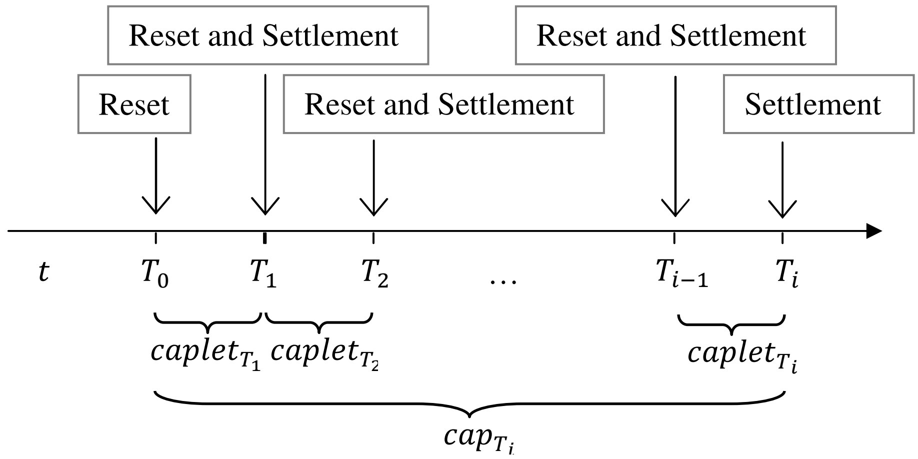

***Figure 3.1**: Graphical illustration of the payment structure of an interest rate cap. The figure is inspired by Rebonato (2002 p. 66).*

A cap can be decomposed additively into the single caplets that make up the cap thus the pricing of caps consists of the sum of a series of caplets (Brigo & Mercurio 2006 p. 17; MF2 notes 2008 p. 20).

3.4.1 Black's formula

It is common market practice to price interest rate caps using Black's formula for caplets. According to Brigo & Mercurio (2006 p. 17) the price of a cap can be expressed as a sum of Black's formulas:

$$\mathbf{Cap}^{Black} = N \sum_{i=\alpha+1}^{\beta} P(0,T_i)\, \tau_i\, Bl(K, F(0,T_{i-1},T_i), \sigma_{\alpha,\beta}, 1) \tag{3.8}$$

Where N is the notional amount, $P(0,T_i)$ is the price at time 0 of a zero-coupon bond maturing at time T_i and $F(0,T_{i-1},T_i)$ is the simply compounded LIBOR forward rate at time 0 for the future period $[T_{i-1},T_i]$ as cf. equation (3.2). $\sigma_{\alpha,\beta}$ is the cap volatility. $Bl(\cdot)$ denotes Black's formula for the price of a single caplet and Φ denotes the standard cumulative normal distribution. The price of a single caplet is given by:

$$Bl(K,F,\sigma_{\alpha,\beta}) = F\Phi\left(d_1(K,F,\sigma_{\alpha,\beta})\right) - K\Phi\left(d_2(K,F,\sigma_{\alpha,\beta})\right) \tag{3.9}$$

Where d_1 and d_2 is defined by the expression:

$$d_{1,2}(K,F,\sigma_{\alpha,\beta}) = \frac{\ln(F/K) \pm 0{,}5 \cdot (\sigma_{\alpha,\beta}\sqrt{T_{i-1}})^2}{\sigma_{\alpha,\beta}\sqrt{T_{i-1}}} \tag{3.10}$$

The case in which the interest rate cap only has one payment date $(\alpha + 1 = \beta)$ the interest rate cap collapses into a single caplet and the at-the-money caplet strike rate is $K = F(0,T_\alpha,T_{\alpha+1})$ equal to the forward rate. Consider a cap with the payment times

$T_{\alpha+1}, \dots, T_\beta$, and the associated year fractions $\tau_{\alpha+1}, \dots, \tau_\beta$. Given a series of caplets the ATM cap strike rate is defined by:

$$K = K_{ATM} = S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_\beta)} \tag{3.11}$$

I.e. the interest rate cap is at-the-money if the strike rate is equal to the corresponding forward swap rate. Hence, the ATM cap strike is the average of the forward rates covering the reset and settlement periods of the cap, i.e. the cap strike rate is the average strike rate of the individual caplet strike rates.

3.4.2 Market practice

The use of Black's formula for pricing caplets and thus interest rate caps has been standard market practice for years. Even before the development of a consistent no-arbitrage framework to support the log-normal assumption of Black's formula, traders used Blacks formula for pricing caps. In the very beginning when traders were first to price interest rate caps a variant of the Black-Sholes formula were used in order to price caplets, in which the stock price was substituted with interest rates. This approach is also known as the "yield trick" (MF2 notes 2008 p. 20).

The market quotes interest rate caps in terms of the *implied Black volatility* and not in terms of the nominal value of the cap. I.e. the volatility parameter σ, entered into the Black cap formula (equation 3.8 to 3.10) account for how the right market cap price is obtained (Brigo & Mercurio 2006 p. 88). Different interpretations of the volatility parameter apply depending on whether it is the volatility of a cap or of a single caplet. According to Björk (2004 p. 370) implied Black volatilities are referred to as either *flat volatilities* or as *spot volatilities*.

In case of an interest rate cap the implied Black volatility is referred to as *implied flat volatility*. I.e. the volatility implied by the Black cap formula (3.8) given that the same volatility is applied for each caplet in the cap. In case of a single caplet the implied Black volatility is referred to as *implied spot volatility* which is the implied volatility of a single caplet corresponding to the volatility that has to be entered into Black's caplet formula (equation 3.9) in order to obtain the market price of the caplet. The difference between the flat volatility and the spot volatility is that the flat volatility imposes the same volatility for all caplets in the cap, implicitly assuming that all caplets share the same volatility (Brigo & Mercurio 2006 p. 88).

A cap volatility structure (flat volatilities) can be illustrated by plotting the cap volatility for e.g. at-the-money (ATM) caps across different maturities. This is illustrated in Brigo & Mercurio (2006 p. 18). Below I have shown two graphs, using data for ATM caps from the Danish market, to illustrate the different shapes of the observed market cap volatility structure. Figure 3.2.A displays recently observed cap volatilities in the Danish market and figure 3.2.B illustrates the cap volatilities two years ago.

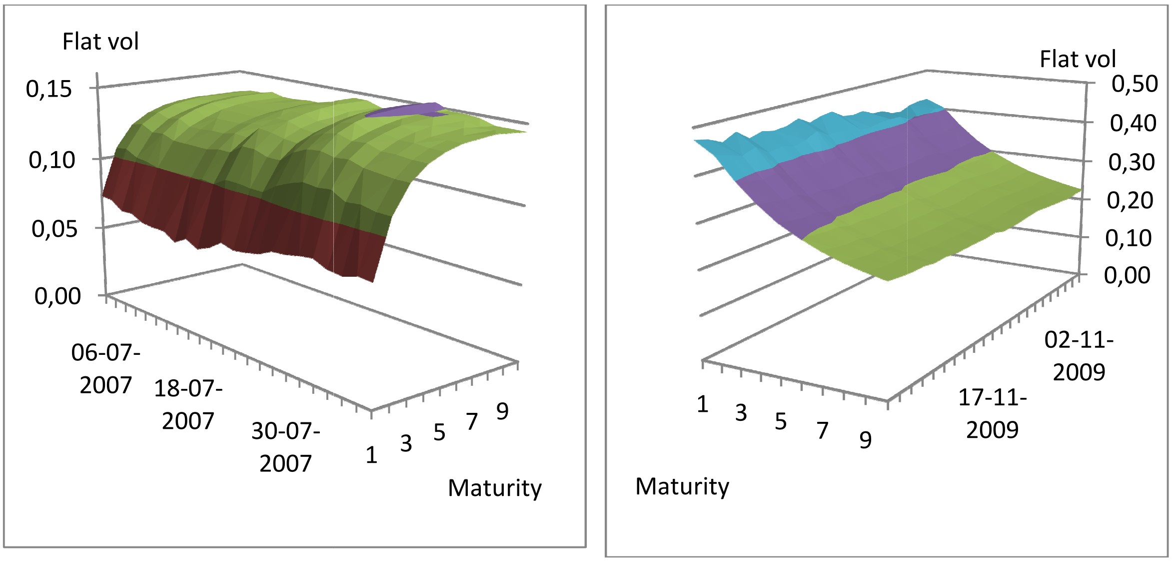

***Figure 3.2**: Development of the cap volatility structure (flat volatilities) over 30 day intervals. Two different periods in time for ATM caps indexed on 6 month CIBOR rates. The right figure illustrates recent volatility structure development and the figure to the left illustrates the development two years ago.*

The two figures indicate the diversity in the shape of the cap volatility structure, and show the possible differences in volatilities of caps as time passes. An alternative representation of the volatility structure is to plot the implied volatility of caplets (spot volatilities) with maturity, this plot is referred to as *the term structure of volatilities* (Brigo & Mercurio 2006 p. 88). Typically the shape of the term structure of volatilities is observed to be either humped or decreasing and the different shapes of the volatility structure clearly motivate the development of model able to capture a range of different shapes. I will return to the shapes of the term structure of volatilities and how it is linked to the instantaneous volatility of forward rates in the LIBOR market model in chapter 4.5.

4 The models

The different models for which I will investigate the pricing and hedging capabilities are presented in section 4.1 to 4.3. Section 4.1 covers the LIBOR market model, section 4.2 introduces the one-factor Hull-White model, and section 4.3 covers the G2++ model. The primary reference for this chapter is Brigo & Mercurio (2006 chap. 3 and 4).

4.1 The LIBOR market model

The LIBOR market model exists in several different variations which present the model with different modifications to the original framework. However, in this thesis I will focus on the standard LIBOR market model as presented in Brigo & Mercurio (2006 chap. 4).

The modelling objects in the LIBOR market model are simple forward rates following the LIBOR market convention. Therefore, the LIBOR market model only describes the evolution of a finite number of interest rates, and hence, a discrete tenor structure is required in order to model the term structure development in the LIBOR market model:

Consider a set of dates $\mathcal{E} = \{T_0, T_1, \dots, T_M\}$ for which the discrete tenor structure can be defined such that $0 < T_0 < T_1 < \cdots < T_M$. Let pairs of dates (T_{i-1}, T_i), $i \geq 0$ be the expiry-maturity dates for a set of discrete forward rates and define the tenor of the forward rate as $\tau_i = T_i - T_{i-1}$, $i = 0, \dots, M$. Furthermore, let t be the current time and define $T_{-1} \coloneqq 0$.

The LIBOR forward rate, cf. equation (3.2), is denoted as $F_i(t) = F(t, T_{i-1}, T_i)$. Figure 4.1 below illustrates the tenor structure and the LIBOR forward rates:

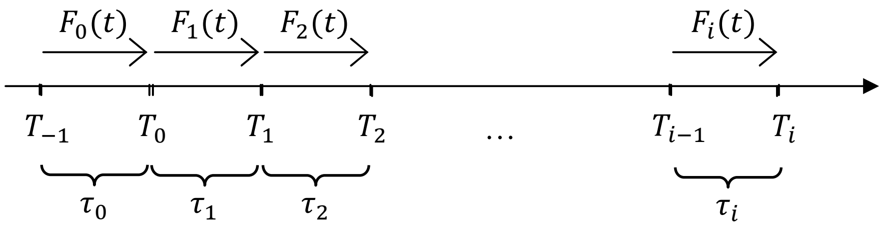

Figure 4.1: *Graphical illustration of the LIBOR forward rates at time* $t = T_{-1} = 0$.

The numerator of LIBOR forward rate in equation (3.2) can be viewed as a portfolio of two zero-coupon bonds for which the portfolio is long a bond with maturity at time T_{i-1} and short a bond expiring at time T_i. The function takes the denominator as the numeraire asset, and thereby we can consider the LIBOR forward rate as a function of tradable assets. This observation proves important as the LIBOR forward rate then can be considered a martingale under a specific forward measure which enables the derivation of a closed form solution for pricing caplets within the LIBOR market model. I will return to this in the following.

An advantage of the LIBOR market model is that it provides a closed form solution to pricing interest rate caplets which is equivalent to Black's formula. Moreover, the equivalent pricing formula implies a link between the implied volatilities of Black's formula for caplets and the instantaneous volatilities of forward rates in the LIBOR market model. The link between the volatilities enables the model to be exactly calibrated to market observable prices of interest rate caps and floors. The derivation of a closed form solution is closely related to the forward (adjusted) measure.[7]

4.1.1 The forward measure and the forward rate dynamics

Each of the simple LIBOR forward rates is assumed to be log-normally distributed. According to Miltersen et al. (1997 p. 413), the assumption of log-normality of the LIBOR forward rates restricts the modelling objects to be non-redundant i.e. the objects are restricted to interest rates that cannot replicate each other. The point is that we cannot consistently model LIBOR forward rates with 3 months tenor at the same time as LIBOR forward rates with 6 months tenor. The reason is that the sum of log-normally distributed variables is itself not log-normally distributed. Hence the tenor structure consists of a finite number M of LIBOR forward rates (the spanning set of interest rates) with the reset times $T_0, T_1, \dots, T_{M-1}$white equidistant time intervals between the resetting times.[8]

Using the forward measure the dynamics of the LIBOR forward rates can be described by a system of stochastic differential equations. The forward measure is denoted as Q^i. The forward measure is the measure associated with the numeraire $P(\cdot, T_i)$ which mean that Q^i is associated to the price of a zero-coupon bond with the same maturity date T_i as the maturity of the forward rate $F_i(t)$ (Brigo & Mercurio 2006 p. 208). The price of the numeraire at maturity is 1. The forward measure is also referred to as the settlement-linked measure.

Under the forward measure Q^i the settlement-linked forward rate $F_i(t)$ can be shown to be a martingale which means that the dynamics of the forward rate can be described by a driftless process under Q^i:

$$\mathrm{d}F_i(t) = \sigma_i(t) F_i(t) \mathrm{d}W_i^i(t)\,, t \leq T_{i-1} \tag{4.1}$$

where $W_i^i(t)$ is a standard Brownian motion under the associated forward measure corresponding to the ith component of the vector of Brownian motions and $\sigma_i(t)$ is the instantaneous volatility of the forward rate $F_i(t)$ at time t (Brigo & Mercurio, 2006, p. 208).[9]

Hence, each forward rate can be expressed as a martingale under the settlement-linked measure which means that a model in which the LIBOR forward rates evolve after driftless processes can be formulated. I.e. at any given point in time the probability

[7] See Brigo & Mercurio (2006 pp. 200-202) for a detailed derivation of Black's formula for caplets within the LIBOR market model.
[8] Modelling M forward rates corresponds to an assumption of $M + 1$ zero-coupon bonds.
[9] This version of the LIBOR market model is also referred to as the log-normal forward-LIBOR model.

distribution associated with the numeraire asset implies that the forward LIBOR rate is equal to the expected value of the future LIBOR spot rate, i.e.:

$$F_i(t) = E_t^{Q^i}[L(T_{i-1}, T_i)], t < T_{i-1} \tag{4.2}$$

The LIBOR spot rate is eventually the rate that will be used for settlement of the caplet contract. Thus, given that the forward rate (i.e. the expected value of the future LIBOR spot rate) is log-normally distributed under the associated forward measure, $\log(F_i(T_{i-1})) \sim N(F_i(t), \sigma^2_{T_{i-1},caplet} \cdot (T_{i-1} - t))$ (the volatility corresponds to equation 8.4 below), the price of a caplet at current time t would be given by Black's formula[10] (MF2 Lecture notes 2008 pp. 20-21) with the only difference being the volatility entering the caplet pricing formula which is defined as:

$$\sigma^2_{T_{i-1},caplet} = \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i^2(s) \mathrm{d}s \tag{4.3}$$

That is the Black volatility should equal the average variance of the forward rates also referred to as the root-mean-square volatility (Rebonato 2004 p. 684). The relation between Black volatilities and instantaneous volatilities in equation (4.3) is central to calibration the LIBOR market model and I will return to this in section 4.6.

Sometimes the pricing of instrument requires that all forward rates evolve under the same forward measure. In general, the dynamics of the forward rate $F_i(t)$ under a measure Q^j different from the settlement-linked measure Q^i will not be a driftless process. Therefore, in order to obtain the dynamics of the forward rates under one common measure the dynamics must be appropriately adjusted. This can be done by applying proposition (6.3.1) from Brigo & Mercurio 2006 p. 213 which states that:

The dynamics of forward rate $F_i(t)$ under the forward adjusted measure Q^j in the three cases $j < i$, $j = i$ and $j > i$ are,

$$\begin{aligned}
&j < i, t \le T_j: \ \mathrm{d}F_i(t) \\
&\qquad = \sigma_i(t)F_i(t) \sum_{k=j+1}^{i} \frac{\rho_{i,k}\tau_k\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)} \mathrm{d}t + \sigma_i(t)F_i(t)\,\mathrm{d}W_i(t) \\
&j = i, t \le T_{i-1}: \ \mathrm{d}F_i(t) = \sigma_i(t)F_i(t)\,\mathrm{d}W_i(t) \\
&j > i, t \le T_{i-1}: \ \mathrm{d}F_i(t) \\
&\qquad = -\sigma_i(t)F_i(t) \sum_{k=i+1}^{j} \frac{\rho_{i,k}\tau_k\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)} \mathrm{d}t + \sigma_i(t)F_i(t)\,\mathrm{d}W_i(t)
\end{aligned} \tag{4.4}$$

Where $W = W^j$ is a standard Brownian motion under the forward measure Q^j. In addition, the terminal measure (cf. appendix A.2) corresponds to pricing under a single common measure.

[10] Black's formula is defined in equation (3.8 – 3.10) in section 3.1.

4.1.2 Implementation

The previous sections of chapter 4 have described the basics of the LIBOR market model. This section will focus on the implementation of the LIBOR market model, and a useful way of describing the dynamics together with the inputs necessary to implement the model will be presented.

According to Rebonato (2002) a convenient way to specify the dynamics of the LIBOR forward rates is to write:

$$\frac{\mathrm{d}L_i(t)}{L_i(t)} = \mu_i(t, (F_1(t), \dots, F_M(t))\mathrm{d}t + \sigma_i(t)\sum_{j=1}^{n} b_{ij}\mathrm{d}W_j(t) \tag{4.5}$$

Where

$$b_{ij}(t) = \frac{\sigma_{ij}(t)}{\sqrt{\sum_{i=1}^{n} \sigma_{ij}^2(t)}} \tag{4.6}$$

The sum $\sum_{i=1}^{n} \sigma_{ij}^2(t) = \sigma_i^2(t)$ is equal to the total instantaneous volatility of the forward rate and each σ_{ij} is the volatility contribution to the ith forward rate from the jth Brownian motion. This notation is used as it allows for the number of Brownian motions driving the evolution of the forward rates to be less than the number of forward rates being modelled.[11] Defining the dynamics as in equation (4.5) above is useful because we are able to separate the volatility parameter and the correlation part. $\sigma_i(t)$ defines the total volatility of the ith forward rate and $b_{ij}(t)$ is related to the correlation coefficients of the forward rates by:

$$\boldsymbol{bb'} = \boldsymbol{\rho}$$

where $b_{ij}(t)$ is an element in the $M \times n$ matrix $\boldsymbol{b}$ (Rebonato 2002 pp. 71-72). Hence, it is possible to work with the instantaneous volatility and correlation separately when implementing the model.

Therefore together with a set of initial LIBOR forward rates $(F_1(0), \dots, F_1(0))$ the list of inputs necessary in order to implement the model contains:

1. Instantaneous volatilities of the LIBOR forward rates $\sigma_i(t)$ for $i = 1, \dots, M$
2. Instantaneous correlations given by the matrix $\boldsymbol{\rho}$

Section 4.1.3 specifies the instantaneous volatilities and an example of how to specify the instantaneous correlations is given in appendix A.3.

[11] E.g. principal component analysis can be applied in order to find the 'driving' factors. See Rebonato (2002).

4.1.3 Instantaneous volatility

The instantaneous volatility is commonly represented using the approach of *piecewise-constant volatilities* or by using an approach in which the volatilities are represented by a *parametric form.* Piecewise-constant volatilities imply that for short intervals of time the volatility of a forward rate is constant, the instantaneous volatilities can thus be organized in a matrix. Using a parametric form imply that the volatility is represented by a function of time, maturity and a set of parameters (Brigo & Mercurio 2005 pp. 210-213). In the following I will proceed using the latter approach. The aim is to obtain flexible specifications that are able to reflect observable features of the market behaviour of volatilities and which at the same time are financially meaningful.

It is important to distinguish between the instantaneous volatilities of the forward rates and the corresponding Black implied volatilities (the spot volatilities). As mentioned previously in equation (4.3) the relationship between Black caplet volatilities and instantaneous forward rate volatilities is given by:

$$\sigma^2_{T_{i-1},Caplet} = \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i^2(s)\mathrm{d}s \tag{4.7}$$

The recovery of Black caplet prices within the LIBOR market model requires the instantaneous volatilities to be deterministic functions of time. Limited only by the relation in equation (4.3) this gives us infinity of possible specification to choose from (Rebonato 2002 p. 67 and p. 150 proposition 6.1). Notice that the Black implied volatility is the root-mean-square instantaneous volatility and that the term structure of volatilities can be represented using instantaneous volatilities.

4.1.3.1 Market observable features

As indicated in section 3.4.2 the market behaviour of interest rate cap volatilities change over time. The features of the market behaviour can be studied by observing the term structure of volatilities. According to Rebonato (2002) and figure 3.2 in section 3.4.2 previous observations suggests that the Black implied volatilities of caplets plotted with maturity is either a

1. humped shape or
2. monotonically decreasing

According to Rebonato (2002 pp. 153-156) the humped shape can be explained by the trading patterns of different forward and futures contracts together with the influence of the central banks on the short rate. Given “normal” market conditions Rebonato argues that a higher level of transparency exists as central banks usually indicate their views on the rate before making changes. Thus actions taken by the central banks tend to be expected by the market meaning a lower level of volatility for the very short end of the curve. This is also pointed out by Babbs & Webber (1997 p. 398) as a signalling effect related to the monetary authorities control over short rates. Furthermore, volatilities of forward rates with maturity in the long end of the curve seem to be driven by long term inflation expectations making the development of the long end less volatile (Rebonato 2004 p. 672). The effect of new financial information thus seems to have the largest impact on volatilities of forward rates in between. On the other hand in periods of high uncertainty the short end of the curve can be more volatile implying a decreasing volatility structure, an observation which corresponds to the development during the financial crisis. In the beginning of the financial crisis the short end volatilities increased and changed the shape of the volatility structure from humped to decreasing (see figure 8.2 and 8.3 in chapter 8).

In the first versions of the LIBOR market model the volatility of the various forward rates was often assumed to be constant and independent of time. Thus each forward rate would have the same volatility moving through calendar time. Besides the simplicity of the constant volatility specification the advantages are few. One drawback of time-independent constant volatility is that it imposes a certain evolution on the term structure of volatilities:[12] as time passes the volatility term structure will change by "cutting off" the head instead of the tail making the shape of volatility term structure change as time passes (Rebonato 2004 pp. 668-669; Brigo & Mercurio 2006 p. 230).

However, besides small intraday changes, historical observations of the term structure of volatilities suggests that the volatility curve tend to keep its shape for longer periods of time[13] thus it is often argued that the volatility specification should be *time homogeneous* i.e. the function should be able to reproduce the current shape of the volatility curve in the future (Brigo & Mercurio 2006 p. 228).

The idea of time-homogeneous volatility is that as time passes one after another the various forward rates will have the same remaining time to expiry. E.g. consider the forward rate $F_i(t)$ today. In one "period" of time this forward rate will correspond to the $F_{i-1}(t)$ forward rate in terms of time until expiry. Assuming time homogeneous volatility implies that we expect the forward rates to react similarly (in terms of the volatility) as time passes. Specifically, this means that forward rates with the same time to expiry react in the same way to the same Brownian shocks.

In order to make the term structure of volatilities remain the same shape as time passes, the volatility can be expressed as a function λ of the time until the forward rate resets i.e.:

$$\sigma_i(t) = \lambda(T_{i-1} - t) \tag{4.8}$$

4.1.3.2 Specifying instantaneous volatility

Several specifications of the instantaneous volatility of forward rates have been proposed throughout the literature. I have chosen to work with three different parametric forms of the instantaneous volatility. The parameterizations consist of two simple specifications

[12] An illustration is given in Brigo & Mercurio 2006 p. 231.
[13] The shape of the volatility curve does exhibit volatility. Recent developments have focused on developing models with stochastic volatility. This, however, is beyond the scope of this thesis.

and one which is commonly applied in the literature. The first specification is constant volatility given by the parameterization:

$$\sigma_i(t) = a \tag{4.9}$$

The volatility in equation (4.9) is constant across calendar time and maturities. The constant volatility specification is too simple to be able to fit the observed volatility structure. Nevertheless, by using a simple parameterization the model is less likely to fit noise in market data when calibrating the model, and this should result in more robust out-of-sample pricing and hedging (Christiansen & Hansen 2002 p. 75).

The second specification is the exponential volatility structure given by the parameterization

$$\sigma_i(t) = a \cdot e^{-c(T_{i-1}-t)} \tag{4.10}$$

The exponential parameterization is less simple and it is able to fit monotonically decreasing or increasing volatility structures (of the instantaneous volatility) but does not allow for humped shapes. Figure 4.2 below illustrates typical shapes of the instantaneous volatility with the exponential specification.

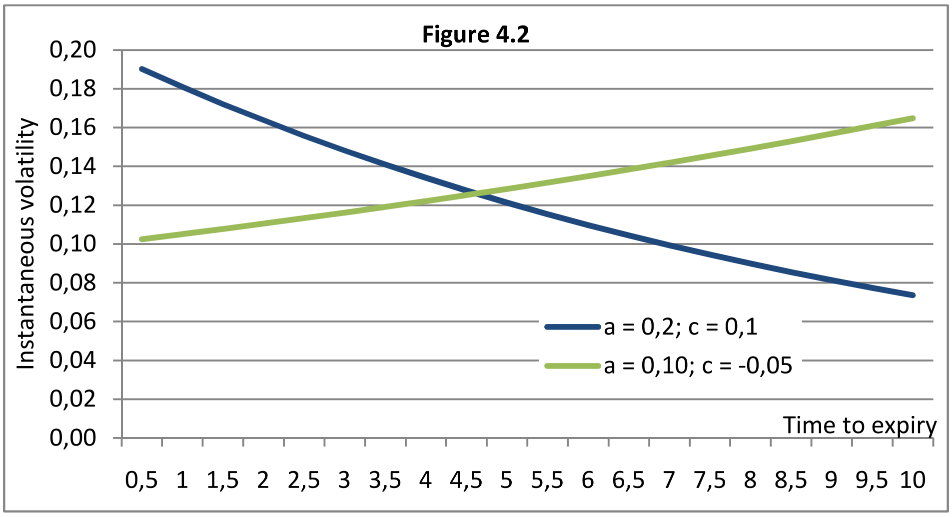

Figure 4.2*: Typical shapes of the instantaneous volatility as a function of time to maturity with the specification given in equation (4.10).*

The third specification was originally proposed by Rebonato and is often suggested in the literature (Brigo & Mercurio 2006 p. 212). The Rebonato parameterization has become popular due to the ability to fit humped shaped volatility structures often observed in practice. The specification is given by the functional form

$$\sigma_i(t) = [a(T_{i-1} - t) + b] \cdot e^{-c(T_{i-1}-t)} + d \tag{4.11}$$

The parameterization in equation (4.11) is able to capture shapes of the instantaneous volatility structure where the structure either has a hump or is monotonically decreasing. The parameters can be interpreted as follows: $a + d > 0$ ensures a positive volatility in the short end and it is linked to the volatility of the caplets with the shortest maturity, $d > 0$ is the long term level of volatility and is thus linked to the volatility of the caplets with the longest maturity. The volatility is assumed to converge asymptotically towards the long term level (Rebonato 2002 pp. 168-169).

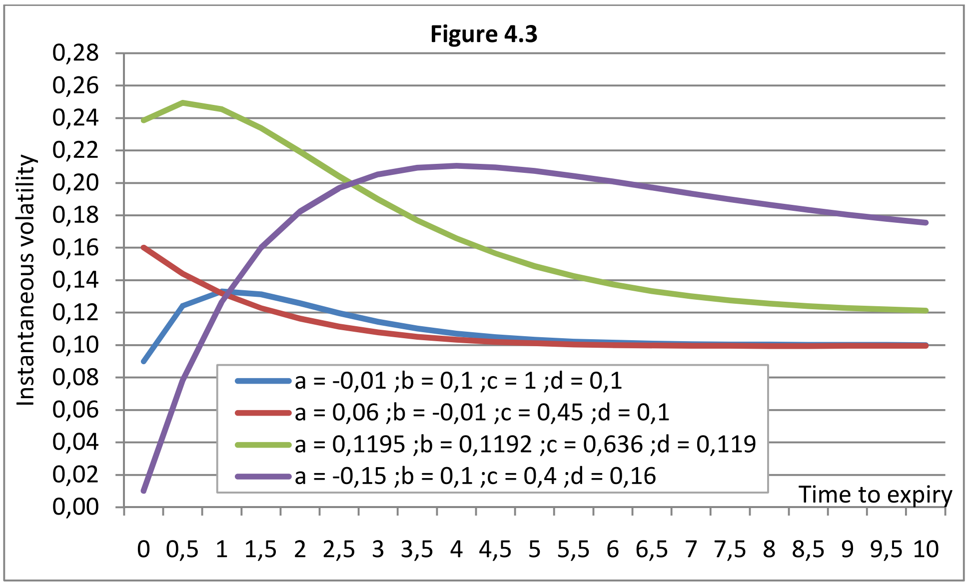

Figure 4.3: *Typical shapes of the instantaneous volatility as a function of time to maturity with the specification given in equation (4.11).*

4.1.4 Calibrating the LIBOR Market Model to interest rate caps

The intention of this chapter is to show how the LIBOR market model can be calibrated to market quoted cap volatilities. The purpose of the calibration is to capture information from market data and therefore the task is to make the model prices of interest rate caps match the market prices as close as possible.

Equation (4.3) defines the connection between the caplet volatilities of Black's formula and the instantaneous volatilities of the forward rates in the LIBOR market model. The link between implied and instantaneous volatilities enables the calibration of the volatility parameterizations to market observed volatilities and thus calibration to market prices. To ensure an exact match between market and model prices a two-step procedure can be applied. Brigo & Mercurio (2006 p. 225) collect the two-step procedure into one expression:

$$\sigma^2_{T_{i-1},caplet\_market} \cdot T_{i-1} = \varphi_i^2 \lambda^2 (T_{i-1} - t, \theta) \tag{4.12}$$

where $\lambda(\cdot)$ defines the volatility parameterization (equation (4.9) to (4.11)), and θ denotes the set of parameters used in the volatility specification. The first step of the calibration procedure involves fitting the parameterization i.e. the λ part of equation (4.12) which also is referred to as the time-homogeneous fit. This is done by applying a numerical iterative procedure, for instance least-squares, where the parameters θ are adjusted until the best fit is achieved. The second step then involves choosing φ accordingly such that an exact fit is achieved. φ is given by the expression:

$$\varphi_i^2 = \frac{\sigma^2_{T_{i-1},caplet\_market} \cdot T_{i-1}}{\lambda_i^2(T_{i-1} - t, \theta)} \tag{4.13}$$

The parameters θ are given by the first calibration step such that $\lambda^2(T_{i-1} - t, \theta)$ represents the caplet volatility implied by the parameterization (Brigo & Mercurio 2006 p. 225). Equation (4.13) ensures that differences between the time-homogeneous parameterization and the market volatilities are levelled out by φ.

4.1.4.1 Bootstrapping caplet volatilities

As mentioned in section 3.2 the market quoted volatilities available are typically the implied flat volatilities of caps. This poses a practical problem because the volatilities of the individual caplets are required in order to calibrate the different volatility parameterizations. Thus before calibrating the model I use a procedure to bootstrap the implied caplet volatilities from the implied cap volatilities. First, the price of an interest rate cap calculated with the same flat implied volatility for each caplet must be equal to the cap price when it is calculated using the different individual spot volatilities for each caplet. Second, as mentioned in chapter 3 an interest rate cap consists of a collection of caplets, however, the first cap only consists of one caplet i.e. the price and hence the volatility has to be identical for the cap and the caplet. I.e.:

$$\mathbf{Cap}_1^{MKT} = \mathbf{Cpl}_1 \;\Leftrightarrow\; \sigma_{\mathbf{Cap}_1^{MKT}} = \sigma_{\mathbf{Cpl}_1} \tag{4.14}$$

Hence, the volatility of the first caplet is known implicitly from the first cap. The information from the first cap can be used to find the volatility of the second caplet and henceforth by equating the price of the next cap with the sum of the previous caplets, $\mathbf{Cap}_2^{MKT} = \mathbf{Cpl}_1 + \mathbf{Cpl}_2$ for which the only unknown is the volatility of the second caplet.

4.1.4.2 An example of calibration

In this section I will give an example of how the different volatility parameterizations of the LIBOR market model in practice is calibrated to market prices of interest rate caps. The example is based on the last day in data sample two (see chapter 8). The data consists of discount factors and flat volatilities gathered on the 1st December 2009 for ATM quoted caps from the Danish market. The caps are indexed on the 6 month CIBOR rate. Therefore the length of the interval between the expiry-maturity dates $[T_{i-1}, T_i]$ should be six months. The data in the market is not quoted with 6 month intervals between the maturity years and therefore in order to obtain the discount factors and the flat volatilities on the missing times I use linear interpolation.

In order to obtain the term structure of caplet volatilities I apply caplet bootstrapping procedure described in section (4.1.5.1). After obtaining the flat volatilities for all 6 month time intervals I calculate the value of each cap in monetary terms by applying

Black's formula (equation 3.8). The caps are quoted at-the-money and the ATM strike rates are calculated by using equation (3.4), where the interest rate cap is ATM when the strike rate is equal to the forward swap rate covering the same period of time. After calculating the market prices in nominal terms, I can proceed with the bootstrapping procedure. The first cap collapses into a caplet as is only has a single payment thus the volatility is the same. The result from bootstrapping the caplet volatilities from market prices are shown in figure 4.4 below.

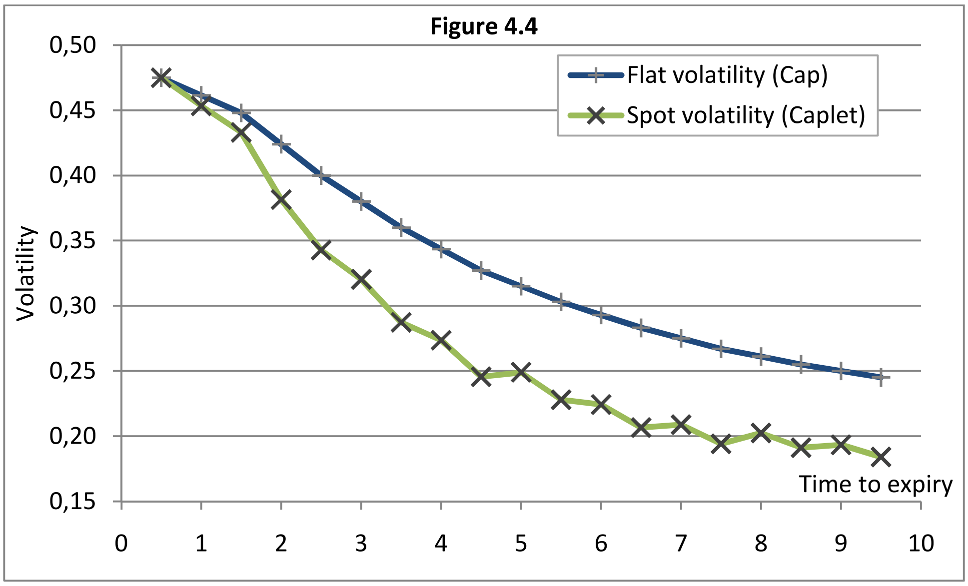

***Figure 4.4**: Results from bootstrapping caplet volatilities from market quoted cap volatilities on date 2009-12-01.*

The bootstrapping procedure provides the data necessary in order to calibrate the model. Figures 4.5 to 4.7 show the results of calibrating the different volatility parameterizations presented in equation (4.9 to 4.11) and the ability of the different specifications to fit market quoted volatilities. The graph on the left shows the time homogeneous fit and the graph on the right shows the correction following from equation (4.13).

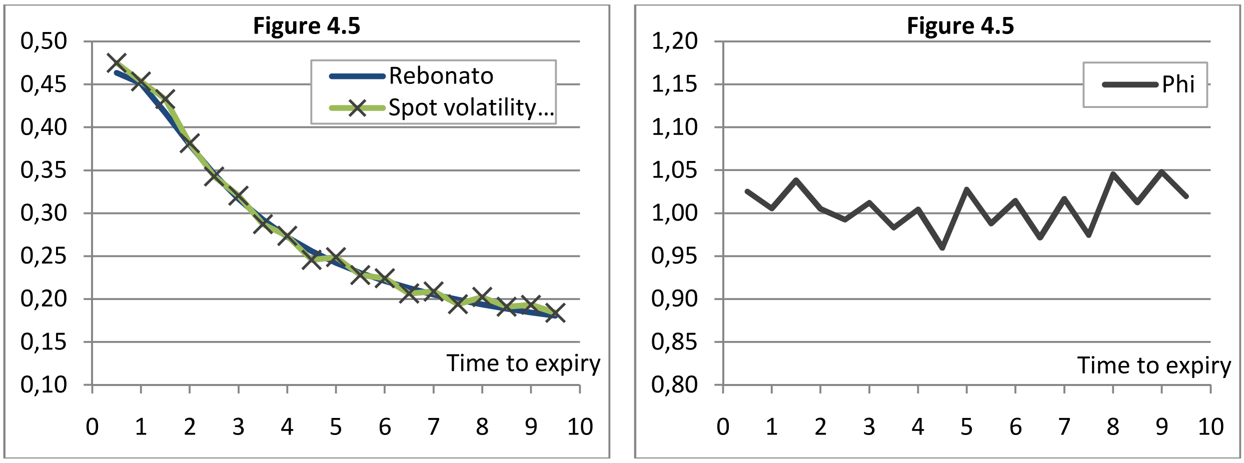

Figure 4.5: *Left: Calibration of the Rebonato parameterization (equation (4.11)) to spot volatilities on 2009-12-01. Right: Adjustment cf. equation (4.13).*

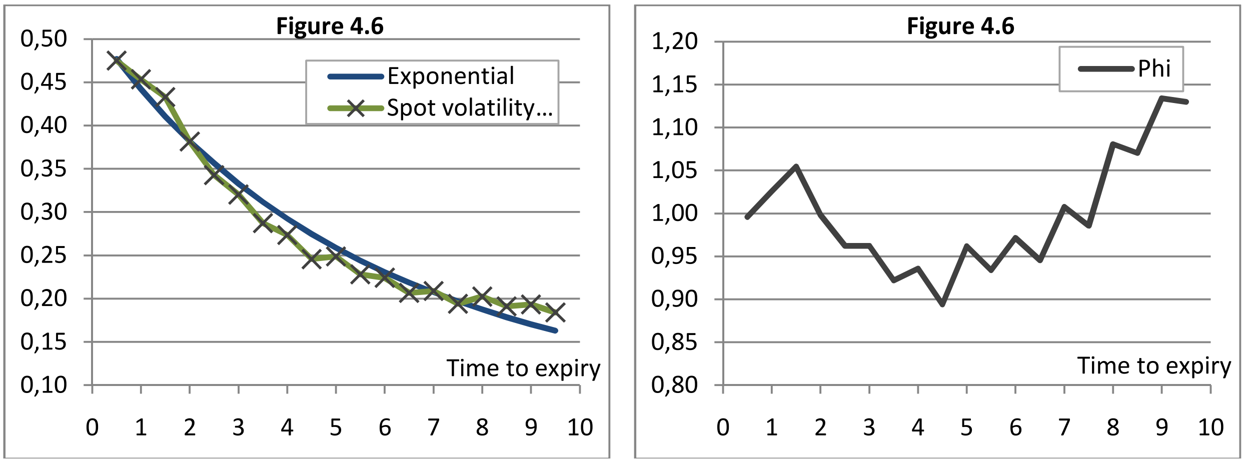

Figure 4.6: *Left: Calibration of the Exponential parameterization (equation (4.10)) to spot volatilities on 2009-12-01. Right: Adjustment cf. equation (4.13).*

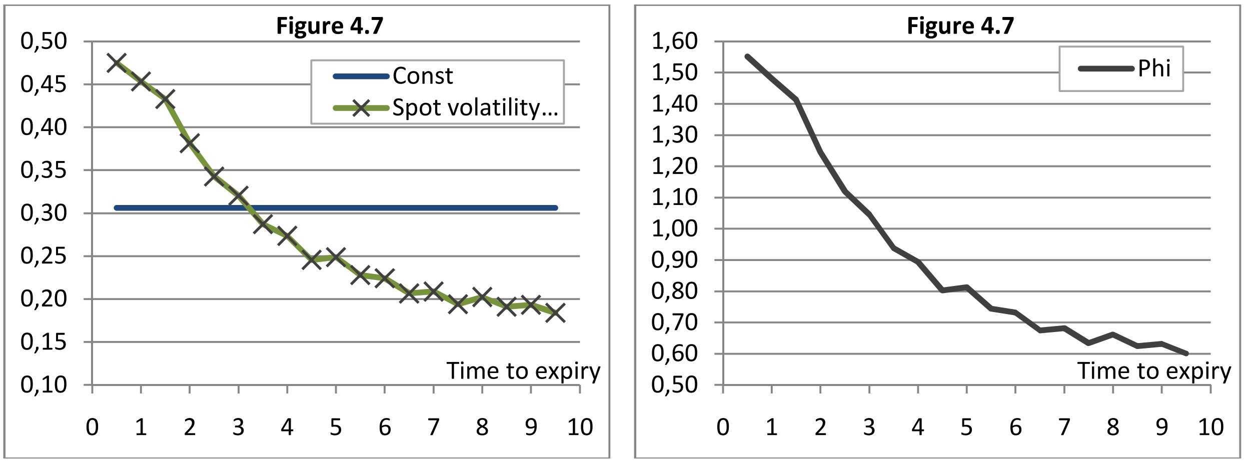

Figure 4.7: *Left: Calibration of the Constant volatility parameterization (equation (4.9)) to spot volatilities on 2009-12-01. Right: Adjustment cf. equation (4.13).*

4.2 The Hull-White model

In this chapter I focus on the one-factor Hull-White short-rate model. The model is presented along with relevant pricing formulae which will be applied later on in the empirical analysis (see part II starting with chapter 5).

The Hull-White model was developed based on the original Vasicek model hence the model is also referred to as the extended Vasicek model. Hull and White introduced a time varying mean reversion level which enables the model to exactly fit an exogenous term structure of interest rates.[14] This would imply that they assumed the evolution of the short rate, under the risk neutral measure $\mathbb{Q}$, to be described by the following mean reverting process:

$$\mathrm{d}r(t) = [\vartheta(t) - ar(t)]\mathrm{d}t + \sigma \mathrm{d}W(t) \tag{4.15}$$

Where the function $\vartheta(t)$ is a time varying function, and the parameter a is the mean reversion speed which controls the speed of adjustment towards the mean reversion level defined by $\vartheta(t)$. The parameter σ is the volatility of the short rate, and a and σ are positive constants. $W(t)$ is the random component, defined as a standard Brownian motion.

The mean reversion level $\vartheta(t)$ is a deterministic function of time and it is chosen such that the model exactly fits the current market term structure of interest rates, which implies that the function takes the form:

$$\vartheta(t) = \frac{\partial f^M(0,t)}{\partial T} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{4.16}$$

Where f^M denotes the market instantaneous forward rate at time zero.

The instantaneous short rate of the Hull-White model is normally distributed and hence the model belongs to the category of Gaussian interest rate models.[15] Usually, the advantage following from this property is analytical tractability, and the Hull-White model also includes closed-form solutions for pricing zero-coupon bonds and option on zero-coupon bonds. Contrary, the disadvantage of the short rate being normally distributed is the positive probability of the model producing negative interest rates. In practice, however, this probability is considered negligible (Brigo & Mercurio 2006 p. 74).

[14] Hull and White actually considered further time varying parameters in the Vasicek model besides time varying mean reversion. For instance, volatility which would enable an exact fit to a given term structure of volatilities e.g. see p. 72 in Brigo & Mercurio (2006). However, the focus in this thesis will be on the extension with time varying mean reversion.

[15] See p. 73 in Brigo & Mercurio (2006) for the distributional characteristics.

4.2.1 Pricing interest rate caps with Hull-White

Closed form solutions to pricing interest rate caps are available in the Hull-White model. However, the pricing require the analytical expression for pricing put option on zero-coupon bonds. According to Brigo & Mercurio (2006) an interest rate cap is equivalent to a portfolio of European put options on zero-coupon bonds (see pp. 40-41 in Brigo & Mercurio). Recall from section 3.4 equation (3.8) that interest rate caps are portfolios of caplets on the underlying interest rate and that the caplet payoff is:

$$X_{T_i} = N \cdot \tau_i \cdot \max\{0, L(T_{i-1}, T_i) - K\}$$

The expected value of the caplet payoff at time T_{i-1} is equal to the payoff discounted one period between time T_{i-1} and T_i, that is $X_{T_i} \cdot P(T_{i-1}, T_i)$. Using the definition of the LIBOR forward rate in equation (3.2) the caplet payoff can be rewritten as:

$$\begin{aligned} P(T_{i-1}, T_i) \cdot N \cdot \max\left\{0, \frac{1}{P(T_{i-1}, T_i)} - 1 - K\tau_i\right\} \\ = N \cdot \tau_i \max\{0, 1 - (1 + K\tau_i)P(T_{i-1}, T_i)\} \end{aligned}$$

Hence a caplet is equal to a put option on a zero coupon bond working on the notional amount $N(1 + K\tau_i)$ and with the strike rate of $(1 + K\tau_i)^{-1}$.

The Hull-White closed form solution for European put options on zero-coupon bonds is given by the expression:

$$\mathbf{ZBP}(t, T, S, X) = XP(t, T)\Phi(-\mathrm{h} + \sigma_{\mathrm{P}}) - P(t, S)\Phi(-\mathrm{h}) \tag{4.17}$$

Where

$$\sigma_P = \sigma \sqrt{\frac{1 - \exp\big(-2a(T - t)\big)}{2a}} B(T, S)$$

$$h = \frac{1}{\sigma_P} \ln\left(\frac{P(t, S)}{P(t, T)X}\right) + \frac{\sigma_P}{2}$$

Therefore pricing formula for interest rate caps is given by the expression:

$$\begin{aligned} \mathbf{Cap}(t, T, N, X) = N \sum_{i=1}^{n} (1 + X\tau_i)\mathbf{ZCP}(t, t_{i-1}, t_i, \frac{1}{1 + X\tau_i}) \Leftrightarrow \\ \mathbf{Cap}(t, T, N, X) = N \sum_{i=1}^{n} \big[P(t, t_{i-1})\Phi\big(-h_i + \sigma_P^i\big) - (1 \\ + X\tau_i)P(t, t_i)\Phi(-h_i))\big] \end{aligned} \tag{4.18}$$

Where

$$\sigma_P^i = \sigma \cdot \sqrt{\frac{1 - \exp\big(-2a(t_{i-1} - t)\big)}{2a}} \cdot B(t_{i-1}, t_i)$$

and

$$h_i = \frac{1}{\sigma_P^i} \ln \frac{P(t,t_i)(1+X\tau_i)}{P(t,t_{i-1})} + \frac{\sigma_P^i}{2}$$

The formulae presented for pricing caps in the Hull-White model are applied in the analysis. The parameter values can be determined implicitly by calibrating the model to market observed option prices.

4.2.2 Calibrating the Hull-White model to interest rate caps

On a given date the implied parameters of the Hull-White model are obtained by minimizing the sum of squared residuals between model and market prices of interest rate caps.

$$SSR = \min \sum_{i=1}^{n} (\mathbf{Cap}_{i,HW} - \mathbf{Cap}_{i,mkt})^2 \tag{4.19}$$

Where n is the number of calibration instruments. In contrast to the LIBOR market model, there is no direct link between the term structure of market quoted volatilities and the interest rate cap (and caplet) volatilities in the short-rate models, and therefore a different approach is applied when calibrating the Hull-White model. According to Brigo & Mercurio (2006) p. 89 the built-in caplet volatility of a short rate model is a random variable; hence the volatility function is non-deterministic, meaning that caplet volatilities cannot be obtained from the short rate models using the relationship in equation (4.3).

However, it is possible to obtain an implied volatility term structure of the short-rate model after the model has been calibrated to market prices. This requires that the Hull-White caplet price, denoted $\mathbf{Cpl}_{HW}(\cdot)$, is calculated and that Black's formula is used in order to calculate the percentage Black volatility which will give the corresponding Hull-White model price by solving the following equation:

$$\mathbf{Cpl}_{HW}(\cdot) = \mathbf{Cpl}_{Black}(v_{T-caplet}^{Model}) \tag{4.20}$$

Thus by interpreting the Black volatility, $v_{T-caplet}^{Model}$, as the implied volatility of the short-rate model, deterministic volatilities can be obtained. This approach can be applied in order to graph a term structure of volatilities implied by the Hull-White model, and in order to enable comparison of different models fit to the market volatility term structure.

In the example below the Hull-White model is calibrated to market prices of interest rate caps observed on date 2009-12-01. Figure 4.9 shows the precision of the calibration measured as the difference between the model price and the market price (left) and as the fit to the implied market caplet cap volatilities (right). In general, the difference between the model and the market price is less than five basis points, except for the very short term caps for which the Hull-White price is around 4 basis points higher. This difference is well illustrated by the fit to the term structure of volatilities in figure 4.9, where the implied volatility of the Hull-White model is higher in the short end.

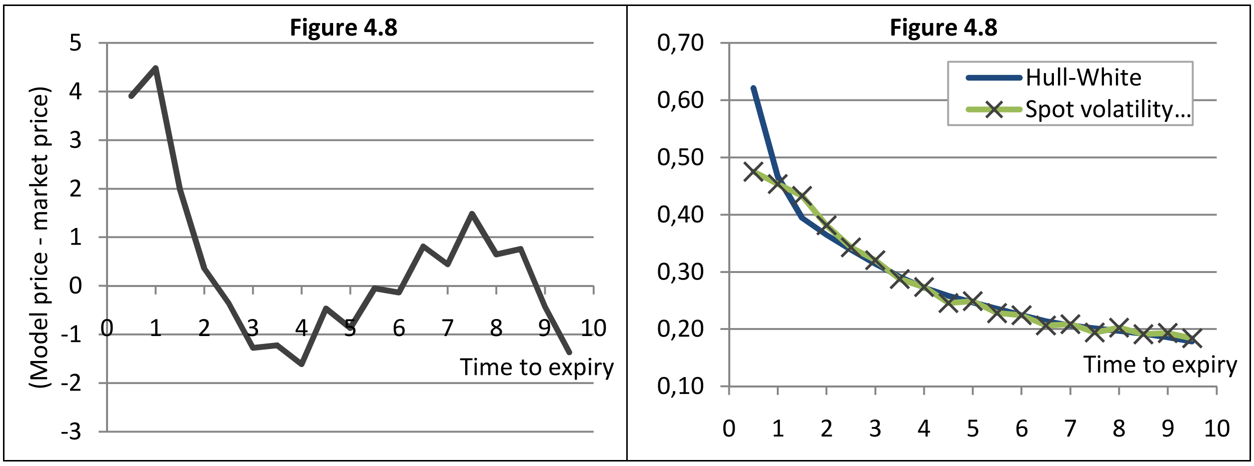

Figure 4.8*: The graph on the left displays the error (in basis points) between Hull-White cap prices and market cap prices when calibrated on date 2009-12-01. For the same date, the graph on the right shows the term structure of caplet volatilities implied by market data plotted with the term structure of caplet volatilities implied by the Hull-White cap prices. The Hull-White parameters are* $a = 0{,}1202$ *and* $\sigma = 0{,}0127$.

The model parameters are positive; however, according to Brigo & Mercurio (2006) p. 134, when calibrating the Hull-White model it is common to observe a negative parameter value for the speed of mean reversion (a), meaning that the model is diverging from the long term mean reversion level. This is observed for some dates when calibrating the model to the sample data in the empirical analysis, and is illustrated in the plot of calibrated Hull-White parameters over the sample periods in figure A.4.1 in appendix A.4. Negative model parameters are, however, a realisation of the applied data period.

4.3 The two-factor Gaussian model: G2++

A natural way to extend the one-factor Hull-White model is to add a second stochastic factor to the model. The two-factor Hull-White model adds a stochastic component to the long-term mean-reversion level to achieve a better description of the movements in the interest rate term structure. Compared to a one-factor model, the additional stochastic factor helps to explain the variability of interest rates more precisely, given that the model is able to model interest rates with non-perfect correlation.[16]

According to Brigo & Mercurio (2006) the two-factor Hull-White model is equivalent with the two-additive-factor Gaussian model (G2++) (see pp. 159-162). In G2++ the short-rate process is given by the sum of two correlated normally distributed factors and a deterministic function

[16] At least the model offers a more realistic setup for the description of interest rate movements as perfect correlations among interest rates are rarely observed.

which, similar to the one-factor Hull-White model, enables the model to fit exogenous term structures of discount factors. Closed form solutions of zero-coupon bond prices and options on bonds (i.e. also interest rate caps cf. section 4.2.1) are available in G2++ as well as in two-factor Hull-White, nevertheless, I have chosen to work with G2++ instead of the two-factor Hull-White model since G2++ provides less complicated formulas and hence is easier to implement.

In G2++ the dynamics of the short-rate, under the risk neutral measure $\mathbb{Q}$, is given by the sum of the two individual stochastic processes, $x(t)$ and $y(t)$, and a deterministic function $\varphi(t)$:

$$r(t) = x(t) + y(t) + \varphi(t) \tag{4.21}$$

The stochastic processes $x(t)$ and $y(t)$ are described by the expressions,

$$\mathrm{d}x(t) = -ax(t)\mathrm{d}t + \sigma \mathrm{d}W_1(t) \tag{4.22}$$

$$\mathrm{d}y(t) = -by(t)\mathrm{d}t + \eta \mathrm{d}W_2(t) \tag{4.23}$$

I.e. Ornstein-Uhlenbeck processes[17] with zero drift and for which (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ. The parameters a, b, σ, η are all positive constants and the instantaneous correlation factor is naturally given by the interval $-1 \leq \rho \leq 1$. The initial value of the instantaneous short-rate is set as $r(0) = r_0$, and the values of the random processes x and y at time $t = 0$ are both zero (Brigo & Mercurio 2006 pp. 142-143).

The function $\varphi(t)$ is chosen such that the model exactly fits observed zero coupon rates in the market, which is the case when $\varphi(T)$, for each maturity T, satisfies[18],

$$\begin{aligned} \varphi(T) = f^M(0,T) + \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2 + \frac{\eta^2}{2b^2}\left(1 - e^{-bT}\right)^2 \\ + \rho\frac{\sigma\eta}{ab}(1 - e^{-aT})\left(1 - e^{-bT}\right) \end{aligned} \tag{4.24}$$

A drawback of using G2++ compared to two-factor Hull-White is the loss of intuition on the interpretation of the two factors x and y. In two-factor Hull-White the factors are determined as the random components of the mean reversion level and the short-rate, respectively. In G2++ the interpretation is less clear, however, Brigo & Mercurio (2006 p. 161) have derived formulas for transforming parameters between the models, which can be used to interpret the parameters of G2++. Similar to the one-factor Hull-White model, the drawback of negative interest rates remains present in the G2++ model as a result of the normal distribution assumption of the short-rate.

[17] An Ornstein-Uhlenbeck process is a modification of the arithmetic Brownian motion which permits mean reversion (McDonald 2006 pp. 654-655).
[18] A proof is given in corollary 4.2.1 in Brigo & Mercurio (2006).

4.3.1 Pricing interest rate caps with the G2++ model

Cf. section 4.2.1 interest rate caplets are equivalent to put options on zero-coupon bonds. In G2++ the closed-form solution of the time t interest rate caplet paying $max[0, L(T_1, T_2) - X]\alpha(T_1, T_2)N$ at time T_2 is given by the formula:

$$\begin{aligned}\mathbf{Cpl}(t, T_1, T_2, N, X) &= -N\big(1 + X\alpha(T_1, T_2)\big)P(t, T_2) \\ &\quad\cdot \Phi\left(\frac{\ln\frac{P(t,T_1)}{\big(1+X\alpha(T_1,T_2)\big)P(t,T_2)}}{\Sigma(t,T_1,T_2)} - \frac{1}{2}\Sigma(t,T_1,T_2)\right) \\ &\quad + P(t,T_1)N\Phi\left(\frac{\ln\frac{P(t,T_1)}{\big(1+X\alpha(T_1,T_2)\big)P(t,T_2)}}{\Sigma(t,T_1,T_2)} + \frac{1}{2}\Sigma(t,T_1,T_2)\right)\end{aligned} \tag{4.25}$$

Where N is the notional amount, X is the strike rate, $L(T_1, T_2)$ the LIBOR rate and $\alpha(T_1, T_2)$ is the year fraction between times T_1 and T_2. Sigma is defined below in equation (4.27).

The pricing formula of an interest rate cap is achieved by defining $\mathbb{T} = \{T_0, T_1, \dots, T_n\}$ as the set of reset and payment dates and $\tau = \{\alpha(T_0, T_1), \alpha(T_1, T_2), \dots, \alpha(T_{n-1}, T_n)\}$ as the matching year fractions. The closed form solution to the cap price is then the following collection of caplets:

$$\mathbf{Cap}(t, \mathbb{T}, \tau, N, X) = \sum_{i=1}^{n} \mathbf{Cpl}\,(t, T_{i-1}, T_i, N, X) \tag{4.26}$$

Where Σ in equation (4.34) is the variance of the forward rate. Σ is given by the expression:

$$\begin{aligned}\Sigma(t, T_{i-1}, T_i) &= \frac{\sigma^2}{2a^3}\left[1 - e^{-a(T_i - T_{i-1})}\right]^2\left[1 - e^{-2a(T_i - t)}\right] \\ &\quad + \frac{\eta^2}{2b^3}\left[1 - e^{-b(T_i - T_{i-1})}\right]^2\left[1 - e^{-2b(T_i - t)}\right] \\ &\quad + 2\rho\frac{\sigma\eta}{ab(a+b)}\left[1 - e^{-a(T_i - t)}\right]\left[1 - e^{-b(T_i - t)}\right]\left[1 \right.\\ &\quad \left.- e^{-(a+b)(T_i - t)}\right]\end{aligned} \tag{4.27}$$

The formulae for pricing interest rate caps and caplets presented above are applied in the empirical analysis.

4.3.2 Calibrating G2++ to interest rate caps

Similar to the Hull-White model there is no direct link between the term structure of market quoted volatilities and the cap (and caplet) volatilities in the G2++ model. Therefore in order to calibrate G2++ equation (4.19) is applied. Analogous to the Hull-White model the G2++ model implied volatility structure can be obtained using equation (4.20).

Figure 4.9 (left) displays the difference between G2++ model prices and market prices on date 2009-12-01. Furthermore figure 4.9 (right) illustrates the fit to the implied caplet volatility structure. The differences between the model and the market prices are around five basis points and less for all maturities. Compared to the Hull-White calibration, the errors are very similar in magnitude, except for the short term caps for which G2++ provides a better fit to the market volatilities (cf. figure 4.8 compared to figure 4.8). However, with five parameters vis-à-vis two parameters in Hull-White the G2++ model is expected to provide a better fit to the market volatility structure.

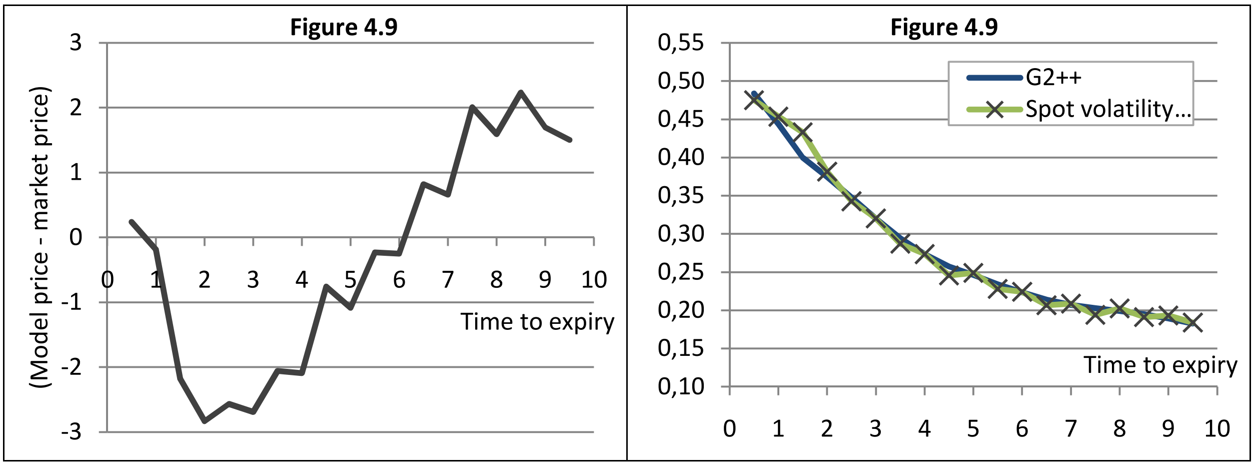

***Figure 4.9**: The graph on the left shows the error between G2++ cap prices and market cap prices when calibrated on 2009-12-01. For the same date, the graph on the right shows the term structure of caplet volatilities implied by market data plotted with the term structure of caplet volatilities implied by the G2++ cap prices. The G2++ parameters are:* $a = 0{,}1395$, $\sigma = 0{,}3619$, $b = 20{,}8584$, $\eta = 0{,}2949$ *and* $\rho = -0{,}7850$.

The parameter values are stated in the figure. According to Brigo & Mercurio (2006) the correlation parameter is often close to -1. Perfect negative correlation indicates a tendency of the model to degenerate into a single-factor model, i.e. the additional stochastic factor does not improve the term structure modelling. Another indication of G2++ degenerating into a one-factor model is when the parameters of mean reversion speed (a, b) and the volatility parameters (σ, η) are equal.

In the example above, the correlation parameter is $-0{,}7334$. Moreover, figure A.4.2 in appendix A.4 displays that the correlation is somewhat consistent to this level during the sample period. Contrary, the other parameters (a, b, σ, η) are less stable during sample period 1, however, the parameters do not converge towards the same values.

5 Empirical study

The second part of the thesis, the chapters 5 to 9, contains the empirical analysis. The aim is to investigate and answer the research questions of the problem statement by carrying out an empirical analysis based on the interest rate models presented in chapter 4. The implementation of the interest rate models builds on the theory described in Part I of this thesis and the design of the empirical analysis is inspired by the works of Gupta & Subrahmanyam (2005). The interest rate models are compared based on pricing and hedging abilities in the context of market data of Danish interest rate caps, and when possible the results are compared with similar studies.

Section 5.1 provides an overview of related studies and in section 5.2 I discuss the performance criteria which are applied in order to evaluate the relative performance of the interest rate models. The following chapters (6 to 8) explain the hedging strategy, the experimental designs, and describe the data applied in the empirical analysis. At last, I present and evaluate the results in chapter 9.1-9.3.

5.1 Empirical studies on dynamic term structure models

Amin & Morton (1994) were among the first to empirically investigate the performance of alternative interest rate models. Using time series of transaction prices of Eurodollar futures options they study the ability of different short rate models (using different volatility specifications within the HJM framework) to predict option prices with one-day lagged implied volatility functions. Amin & Morton (1994) test six different specifications and find that the most accurate prediction of market prices is achieved with a constant volatility specification (equivalent to the continuous-time Ho-Lee model). In a paper by Kuo & Paxson (2006) Amin & Morton's analysis is repeated,[19] however, in contrast to the results of Amin & Morton (1994), Kuo & Paxson's results show that an exponential decaying volatility (corresponding to the Vasicek and Hull-White short-rate models) performs better. Although the two studies are based on different data sets the results show that the empirical conclusions are not always without ambiguity.

A study similar to Amin & Morton's (1994) focussing on the LIBOR market model is conducted by Christiansen & Hansen (2002). Using market data of option prices on the 13-week US Treasury bill rate they compare the out-of-sample pricing over various timescales of three different volatility specifications; constant, affine, and exponential. Based on linear regressions Christiansen & Hansen (2002) cannot reject the hypothesis that the LIBOR market model significantly predicts option prices on the 13-week US Treasury bill rate over a one-day timescale. In general, the LIBOR market model predicts option prices well and only a small number of extreme observations cause the pricing errors. Furthermore, Christiansen & Hansen (2002) do not find evidence that any of the volatility specifications fares significantly better than the others. Diaz, Meneu & Navarro

[19] Kuo & Paxson apply a statistical method different from Amin & Morton's in order to derive their results, nevertheless, the purpose of their analysis remains the same.

(2009) also focus on the LIBOR market model and investigate the effects of using different deterministic volatility specifications. Using the Rebonato volatility parameterization as a benchmark and given two alternative calibration methods, Diaz et al. (2009) propose two new volatility specifications and compare the volatility specifications based on the in-sample performance (i.e. the ability to accurately replicate current market prices of interest rate caps). They find that the two new specifications provides a more accurate fit to current market prices of interest rate caps but also that the volatility specifications result in more instable parameter estimates due to a higher number of parameters to be fitted.

Other studies have focused on the empirical performance of the Cox, Ingersoll and Ross models. Jagannathan, Kaplin & Sun (2003) analyze and compare the empirical performance of multi-factor CIR models, investigating the ability of the models to fit market data of LIBOR and swap rates together with the ability to price interest rate options. In relation to option pricing performance Jagannathan et al. measure the in-sample pricing errors. They find that the models produce large errors relative to the market bid-ask spread, hence, concluding that the CIR models are not able to price consistently into the bid-ask spread; short maturity caps are overvalued and longer maturity caps are undervalued. Furthermore by regressing pricing errors on the slope of the yield curve, mis-specification of model parameters are identified as a source to pricing errors in the one-factor CIR model. The magnitude of mispricing is related to the magnitude of the slope of the yield curve meaning that the one-factor CIR model does not account for the dynamics of the slope. However, Jagannathan et al. only calibrate the models to interest rate data meaning that their approach differs from that of similar studies, such as Amin & Morton (1994) and Gupta & Subrahmanyam (2005), in which option prices are used for calibration. A study by Mannolini, Mari & Renò (2008) investigate the performance of the extended CIR model in pricing interest rate caps and floors. Similar to the analysis of Jagannathan et al. they price at-the-money caps based on model calibration to yield curve data and they compare the pricing errors with that of other short-rate models, among these the CIR model and the extended Vasicek model (Hull-White). Mannolini et al. find that the in-sample pricing of caps improves when the models are able to exactly fit current yield curves and a regression of the pricing errors on the yield curve slope reveals no evidence of mispricing related to modelling the yield curve slope in the extended CIR model.

Driessen, Klaasen & Melenberg (2003) investigate the pricing and hedging performance of alternative interest rate models. Focusing on multi-factor versus one-factor models Driessen et al. find that multi-factor models outperform one-factor models in hedging interest rate caps and swaptions. However, this result only holds when using factor-hedging; adopting a different hedging strategy (bucket hedging) which involves a large set of hedging instruments implies that one-factor models perform as well as multifactor models. Gupta & Subrahmanyam (2005) investigate the cross-sectional (across moneyness) and time-dimensional out-of-the sample pricing as well as the hedging ability of a range of different interest rate models, including short-rate, HJM and LIBOR market models. Their study is based on data from US caps and floors markets. Focusing on one-factor versus two-factor models, they find evidence that interest rate models including a second stochastic factor provides more accurate results when hedging interest rate caps and floors compared to the one-factor models that they nest. Contrary, the two-factor models only provide marginally better results in out-of-sample pricing. The smallest average pricing error is achieved with a one-factor LIBOR market model. A similar study by Kuo & Wang (2009) also find evidence that the hedging ability improves with the inclusion of a second stochastic factor. Kuo & Wang investigate the pricing and hedging of Euribor options across moneyness and maturities and compare model performance based on in-sample estimation, out-of-sample prediction and hedging criteria.

5.2 Criteria for comparing model performance

Three criteria are applied in order to compare the quality of the interest rate models in this study:

1. In-sample price estimation
2. Out-of-sample price estimation
3. Hedging accuracy

The criteria are commonly applied in the literature focusing on empirical investigation of option pricing models (see section 5.1).

In-sample price estimation is related to the ability of a model to capture market information of interest rates and volatility structures in order to replicate market prices. Accurate replication of market quoted option prices is an important feature of interest rate models. In practice, the models are used in relation to buying and selling market quoted derivatives as well as OTC derivatives hence, replication of option prices is regarded a validation of the interest rate models. Given that an interest rate model is not able to consistently replicate current market prices the model is of little use when estimating prices of instruments with different characteristics as well. A reasonable margin of error for consistent in-sample estimation would correspond to the market bid-ask spread. In addition to accurate replication of market prices, stability of the calibrated model parameters are also a desirable feature and should be evaluate in connection with the in-sample estimation when determining the quality of a model.

Pricing out of sample is commonly applied as a measure of the model misspecification. Out-of-sample pricing measure the accuracy of an interest rate models to predict future option prices over a given timescale and conditional on the term structure information but with a lagged model-implied parameters (Gupta & Subrahmanyam (2005); Christiansen & Hansen (2002)). Hence, out-of-sample pricing corresponds to a static test of the models which should indicate the degree of misspecification. The test is static in the sense that only a change in model parameters would cause the model to price the future option with an error. The idea is to obtain implied parameters from the calibration step and given that the model is well-specified it should be able to determine future prices of interest rate caps based on the implied parameters from the calibration. Model misspecification is then detected if the hypothesis that the model is able to predict future option prices can be rejected. Strictly speaking, pricing gout of sample with lagged parameters is not pure out-of-sample pricing because today's discount factors are used, however, this is done in order to eliminate effects of yield curve movements. The models can be considered useful in risk management if the model provides consistent out-of-sample pricing. A different approach, referred to as cross-sectional out-of-sample pricing, measure the ability of a model to price options of the same type written on the same underlying and at the same point in time but with a different strike level. The idea of the cross-sectional approach is to examine the ability of a model to value the options given the calibration to a single strike level and hereby measure how well the model captures volatility smiles.

According to Gupta & Subrahmanyam (2005) hedging accuracy is a measure of the dynamic performance of the models. Given the initial calibration to market prices the models are evaluated based on their ability to hedge future option prices. In contrast to out of sample pricing the hedging error of a model is exclusively based on the information available today, i.e. making hedging performance a measure of the consistency between the dynamics of the model and the dynamics of the underlying. Models with well specified dynamics should be able to hedge future option values regardless of the realization of the underlying. Thus testing models for their hedging accuracy corresponds to examine whether the dynamics of the interest rate embedded in the model is similar to the dynamics driving the actual economic environment. This can also be interpreted as testing the assumption of pricing with replicating portfolios. Given that the hedging is carried out across different strike levels it also becomes at measure of the ability to match a given cross section of future option prices (smile hedging). The deviation of the replicating portfolio from the realized values of the option prices is sensitive to the specification of the dynamics.

6 Hedging interest rate caps

In this section I will explain different approaches to hedging interest rate caps and explain the strategy applied when the hedging accuracy of the different interest rate models are compared in section 9.3.

Hedging refers to the concept of minimising risk, and in practice interest rate options are hedged with respect to several different sources of uncertainty (e.g. interest rate risk and volatility risk). However, the literature conceptually distinguish between two different types of hedging (see Rebonato (2004 pp. 673-674) and Gupta & Subrahmanyam (2005 p. 711)) often described as hedging within the model and hedging outside the model. Within the model hedging concentrates on offsetting risk-exposure related to price changes of the underlying asset (i.e. the random components of the model) which in the case of a one-factor interest rate model is the underlying spot or forward rate. Hedging within the model is also known as delta-hedging. On the other hand outside the model hedging refers to offsetting risk related to the sensitivity of the option price with respect to variations in inputs, which the model assumes constant. An example is the hedging of price sensitivity with respect to changes in volatility (Vega hedging).

According to Gupta & Subrahmanyam (2005) hedging outside-the-model is an inappropriate measure when comparing the hedging accuracy of alternative models. They argue that, given the assumption of constant parameters, price changes related to changes in the input parameters would have zero probability of occurring within the model itself and is therefore theoretically inconsistent with the models. The point is that offsetting risk due to changes in input parameters correspond to hedging with respect to a misspecification of the model parameters. Hence hedging outside-the-model is a measure of whether the model is misspecified (see also Björk 2004 p. 127). However, the purpose of comparing the hedging ability of the different interest rate models is to investigate how well the models are at capturing the true dynamics of interest rates movements. Therefore in line with Gupta & Subrahmanyam (2005) I will focus on within-the-model hedging when comparing the performance of the LIBOR, Hull-White and G2++ models.

6.1 Delta

In relation to interest rate options, delta is a measure the sensitivity of the option price with respect to changes in the underlying interest rate. In mathematical terms delta is defined as the partial derivative of the option price with respect to the underlying interest rate:

$$\Delta = \frac{\partial V}{\partial F} \tag{6.1}$$

In the context of a particular model the delta hedge ratio can sometimes be defined in closed form. However, this is not possible in all models implying that the measure must be approximated. A commonly applied approach is to approximate the delta ratio numerically by shifting the initial term structure and at the same time calculating the corresponding shifts in the option price. The ratio between the change in option price and the change in interest rates then measure the option's price sensitivity towards changes in the underlying:

$$\Delta = \frac{\partial V}{\partial F} \approx \frac{V_{up} - V_{down}}{F_{up} - F_{down}} \tag{6.2}$$

Where V_{up} and V_{down} denotes the option price calculated for corresponding *up* and *down* parallel shift of the forward interest rate curve.[20] The size of the shifts imposed on the forward interest rates is commonly set to one basis point. Naturally, the numerical approximation lead to estimation errors, however, for less complex options such as interest rate caplets and caps, for which the option function is a smooth curve of the underlying interest rate, the estimation error will be small (Skovmand 2008 p. 18).

The approach of numerically approximating the option deltas within different interest rate models is applied in the study by Gupta & Subrahmanyam (2005 p. 710).

6.2 Hedging strategies

Driessen et al. (2003) list two different approaches to hedging the delta risk of interest rate caps which they refer to as factor-hedging and bucket-hedging. Factor-hedging is related to the number of random components in a model and corresponds to hedging the option only with positions in a number of instruments equal to the number of random factors in the model. The same hedging instruments are used for hedging all options. According to Driessen et al. (2003 p. 656) theoretically the choice of hedging instrument is irrelevant because any instrument can be applied as long as the prices of the instruments can be related to the random components of the model. For instance, to hedge interest rate caps based on one-factor short-rate models, Driessen et al. (2003) choose a zero-coupon bond with 6 months time to maturity, and to hedge the same options based on two-factor models they choose a 6 month zero-coupon bond together with a 10 year zero-coupon bond.

Contrary, bucket hedging implies choosing hedging instruments with maturities matching the relevant reset and payment dates of the option being hedged. Hence, the hedging instruments are different for each option and independent of the number of stochastic factors in the model. E.g. in order to hedge a single caplet two hedging instruments are required with maturities matching the expiry and maturity dates of the caplet.

The hedge ratios in relation to the factor hedging can be derived by calculating the partial derivatives of the interest rate caps pricing formulae with respect to the underlying random components. However, by choosing zero-coupon bonds as the hedging instruments the hedge ratios in relation to bucket hedging can be read directly from the pricing formula for each of the models. This is the case since the price of a cap can be

[20] In practice the shift could take different shapes, e.g. a curvature shift or a steepening shift.

expressed in terms of put options on zero-coupon bonds (cf. chapter 4.2.1). The pricing formulae of caplets in the Gaussian models are of the form:

$$\mathbf{Cpl} = P(t,T_i) \cdot \big(-(1 + X\tau) \cdot \Phi(d_2) \big) + P(t,T_{i-1}) \cdot \big(\Phi(d_1) \big) \tag{6.3}$$

This is the replicating portfolio. In order to hedge an interest rate cap consisting of n caplets then require $n+1$ zero-coupon bonds. The same approach can be applied for hedging with the LIBOR market model by rewriting the caplet pricing formula (from equation 3.8) as:

$$\begin{aligned} \mathbf{Cpl} &= P(t,T_i) \cdot \big(\tau_i L_i(t) \Phi(d_1) - X\tau_i \Phi(d_2) \big) \\ &= P(t,T_i) \cdot \left(\left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1 \right) \Phi(d_1) - X\tau_i \Phi(d_2) \right) \\ &= P(t,T_{i-1}) \cdot \Phi(d_1) - P(t,T_i) \cdot \big(\Phi(d_1) + X\tau_i \Phi(d_2) \big) \end{aligned} \tag{6.4}$$

Hence each caplet can be hedged by holding a portfolio which is long a number of zero-coupon bonds maturing at the expiry date and short a number of zero-coupon bonds with maturity matching the maturity date of the caplet (e.g. see Miltersen et al. 1997 p.420). The hedge position of the cap is then the sum of the hedge positions for the individual caplets. Moreover, according to Lindewald (2004 pp. 49-52) the replicating portfolio strategy of non-exact calibrating parameterizations (cf. section 4.1.4) in the LIBOR market model, can be adjusted in order to keep the replicating portfolio self-financing. By replacing the amount $\Phi(d_1)$ of the bond $P(t,T_{i-1})$ with:

$$\frac{\mathbf{Cpl} - P(t,T_i) \cdot \big(\Phi(d_1) + X\tau_i \Phi(d_2) \big)}{P(t,T_{i-1})} \tag{6.5}$$

6.3 Hedging in practice

In practice interest rate caps are commonly hedged using series of Eurodollar futures contracts (Gupta & Subrahmanyam (2005 p. 710). A long position in a futures contract behaves opposite to that of a short position in a caplet with respect to changes in the interest rate term structure. The amount of futures contracts to buy in order to hedge the interest rate caplet is determined by estimating the sensitivity of the interest caplet to a change in the forward rate term structure relative to the sensitivity of the futures price related to the same shift in the forward rate term structure (cf. equation (6.2). Futures markets have a high degree of liquidity making the instruments suitable for hedging. The sum of the individual hedge positions of the caplets make up the hedge position of a cap.

7 Experimental designs

In this chapter I will describe the frameworks for testing the out-of-sample pricing accuracy and the hedging accuracy of alternative interest rate models. The methods are commonly applied in the literature for comparing model performance (see chapter 5.2). Overall I follow the approach applied in Gupta & Subrahmanyam (2005), however, when measuring the hedging ability a different strategy to hedging interest rate risk is applied.

7.1 Out of sample pricing accuracy

The pricing performance of the different interest rate models are examined using out-of-sample pricing errors. The models are compared on both time-dimensional (lagged model parameters) out of sample pricing and cross sectional (across moneyness) out of sample pricing.

Time dimensional

The procedure for computing the out of sample pricing errors basically consists of three steps covering calibration, pricing and error estimation. First, the models are calibrated to market prices in order to obtain the implied parameters of the models on a given date (e.g. obtain a and σ in the case of calibrating Hull-White). Second, at a future date the implied parameters from the calibration are used together with the current interest rate term structure to calculate the predicted model prices. Third, comparing predicted model prices with the observed market prices results in out of sample pricing errors. The test procedure can be summarised into an iterative procedure:

1. Back out the implied model parameters on date T by calibrating the model to market prices of interest rate caps at date T.
2. On the future date $(T + h)$ apply the same model together with the parameters obtained in step 1 and the interest rate term structure from the date $(T + h)$ to calculate the predicted option prices of the model on that date.
3. Compute the out of sample pricing errors by subtracting the market prices on date $(T + h)$ from the predicted model prices estimated in step 2.
4. Repeat step 1 to 3 for each option on each date in the dataset to compute the average out of sample pricing errors.

The out of sample timescale (h) is measured in business days. I will run the above procedure separately for each model (Hull-White, G2++ and LIBOR) based on market prices of ATM interest rate caps, the calibration procedure and the analytical pricing formulae presented in chapters 4.1 to 4.3. Furthermore, to investigate the consequence of increasing the out of sample period (h) I will repeat the procedure for different timescales of 1, 5 and 20 business days corresponding to an out of sample period of 1 day, 1 week and 1 month, respectively. The relative performance of the models is evaluated based on four different measures of average pricing error which are discussed in chapter 9.2.

Moreover, for each model I run a regression of market prices on the predicted model prices to study biases in the pricing errors. The regression is presented in appendix A.5.

Cross-sectional

To further compare the relative performance of the model I will also investigate the cross sectional pricing ability of the models based on pricing interest rate caps away-from-the-money. First, the models are calibrated to market prices of ATM interest rate caps and, second, applying the implied parameters from the ATM calibration the pricing errors across different categories of moneyness are estimated. The pricing errors are for options on the same date. Different volatilities across option moneyness (the volatility smile) will result in mispricing of options away from the money and the magnitude of these errors can be compared in order to evaluate model performance across moneyness and to compare biases across moneyness. The procedure for cross-sectional pricing accuracy can be summarized into the following steps:

1. Back out the implied model parameters on date T by calibrating the model to market prices of at-the-money interest rate caps at date T.
2. Apply the same model on the same date T to price away-from-the-money interest rate caps of all maturities.
3. Compute the cross-sectional pricing errors by subtracting the market prices on date T from the model prices estimated in step 2.
4. Repeat step 1 to 3 for each option on each date in the dataset to compute the average cross-sectional pricing errors.

Given that the results are based on specific methodological choices for instance in relation to calibration, the results may differ dependent on the adopted methodology. Therefore in order to test the robustness of the results I will run the tests based on different calibration criteria. This should validate the results with respect to changes in the adopted methodology. Furthermore the literature review indicated alternative conclusion given different sample periods, therefore to test the validity of the results based on the sample data I compare the performance during two different periods of the sample data.

7.2 Measuring the hedging accuracy

The hedging performance of different interest rate models is compared by examining the magnitude of the out-of-sample hedging errors. Contrary to the test of the models pricing accuracy, all parameters are lagged in the hedging analysis. In the following I will present the framework for the empirical tests of the models hedging performance.

The hedging analysis consists of two steps: First, construct a hedge based on a given model on date T. Second, examine how the hedge performs over a small time interval (h) subsequently. In details this means that I implement the following iterative procedure in order to evaluate the magnitude of the out-of-sample hedging errors:

1. Back out the implied model parameters on date T by calibrating the model to market prices of interest rate caps at date T. [21]
2. At date T; Construct the replicating portfolio of the model using the model's interest rate cap pricing formula together with current market information at date T of zero-coupon bonds, strike rates, and market prices of interest rate caps.
3. At a future date $(T+h)$ compute the hedging error by liquidating the hedge portfolio i.e. the difference between the market price of the cap being hedged and the value of the replicating portfolio on date $(T+h)$. The time interval h reflects a h-day rebalancing interval.
4. Repeat step 1 to 3 for each interest rate cap on each date in the dataset to compute the average out-of-sample hedging errors.

Similar to the pricing test the out of sample timescale (h) is measured in business days. I will run the above procedure separately for each model (Hull-White, G2++ and LIBOR) based on market prices of ATM interest rate caps, the calibration procedure and the analytical pricing formulae presented in chapters 4.1 to 4.3. The test is conducted for different sizes of the h-day rebalancing interval (1, 5 and 20 business days) to test for the effect of increasing the out-of-sample hedging period. The relative performance of the models is evaluated based on the magnitude of the average hedging errors, the average absolute hedging errors and the ratio of the variances between the hedged and the un-hedged positions. Furthermore, in order to test the relative performance of different volatility specifications within the LIBOR market model I also run the above hedging scheme for each of the volatility specifications in chapter 4.1.

The framework for testing the hedging performance of interest rate models is not without complications. The market data of interest rate caps are contracts quoted with specific maturity periods rather than specific maturity dates. Each day the market quoted prices of interest rate caps thus refer to prices of new contracts and not to the prices of the interest rate caps quoted the day before. Hence, the market price on date $T+h$ does not reflect the market price of the original interest rate cap for which a hedge (the replicating portfolio) was constructed on date T. The procedure for testing hedging performance is based on the assumption that the market prices reflect the price development of a specific interest rate cap. In order to overcome the inconsistency between market quoted data and the replicating portfolio, Gupta & Subrahmanyam (2005) construct price series of the interest rate caps for each day until the end of the rebalancing interval. To construct the prices they use the current interest rates and the current term structure of volatilities. The constructed price is then used as a substitute for the market price of the interest rate cap when estimating the hedging errors. However, the substituted price remains a model price and not a market price. An alternative to overcome this problem is to assume sufficiently small time intervals for the rebalancing period. Hence, the errors related to the inconsistency will be small. Given that the sample data consists of interest rate caps with maturities from 1 to 10 years being hedged for periods of one day up to one month, I choose the latter approach taking the market price as a substitute for the original price.

[21] Similar to step 1 in the procedure for out-of-sample pricing.

8 The data

In this chapter I will present the data to be used in the empirical analysis. The data consists of market observations from the Danish interest rate and interest rate options markets. However, to implement the interest rate models some data adjustments are required, and the necessary adjustments and assumption are explained and discussed alongside. Furthermore, summary statistics together with characteristics of the data during the sample periods are presented.

The data sample consists of two datasets. The first dataset contains daily observations of interest rate caps indexed on the Danish 6 month CIBOR rate. The interest rate cap prices are quoted as Black76 flat volatilities cf. section 3.4, and are quoted for 10 different maturities ranging from 1 year to 10 years. Furthermore, for each maturity the interest rate caps are quoted for 4 different cap strike rates; At-The-Money, 3%, 4%, and 5%, where the ATM strike level is defined as the forward swap rate with 6 months expiry and matching maturity (see equation 3.4). All prices are quoted as closing mid prices.

The second dataset consists of daily observations of zero-coupon interest rates based on money and swap market instruments. The zero-coupon rates are quoted for the maturities ranging from 6 months to 10 years with 6 month intervals, corresponding to the maturities of the market quoted interest rate caps and the intermediate dates. Similar to the cap quotes the prices are closing mid.

The dataset containing interest rate cap quotes is obtained from the Bloomberg professional services (the Bloomberg ticker codes are available in the attached dataset) and contains observations from a total of 1976 business days between the dates 2002-04-29 and 2009-12-01. However, the year 2006 only contains observation on a limited number of business days due to lack of trading in the market. In relation to the empirical analysis the lack of observations would either imply the use of irregular lengths of the out-of-sample pricing and hedging timescales or that the sample period should be divided into several smaller subsamples. To avoid the drawback of using irregular lengths of the timescale period in the out-of-sample pricing and hedging tests a number of dates are excluded from the original dataset. The dataset is divided into two sub-samples containing a pre-2006 period and a post-2006 period, respectively. Alternatively; the missing observations could be obtained by interpolation between dates with data. However, I prefer the approach chosen for which I avoid manufacturing further data. The second dataset, containing interest rates, is obtained from DataStream. The ticker codes are available in the attached dataset. This dataset contains observations on all dates observed in dataset one.

Although the datasets are obtained from different data providers (Bloomberg and Datastream) the data is from the same supplier, ICAP, which is a large OTC derivative broker specializing in interest rate derivatives. ICAP provides data to Bloomberg and Datastream of interest rate caps and interest rates. Hence, inconsistencies e.g. with respect to the exact timing of the observations are excluded.

8.1 Data adjustments

Given that the Danish interest rate caps are indexed on the 6 month CIBOR rate and therefore have semi-annual payments (i.e. semi-annual caplets), the implementation of the models requires the corresponding discount factors and cap volatilities on the intermediate dates which lies in between the integer year maturities.

Data adjustments of the interest rates are not required given that the zero-coupon rates provided by ICAP are available on all relevant payment and reset dates. However, similar to the interest rates, volatility data for all caplet expiry and maturity dates are required in order to enable the model calibration. I obtain the intermediate volatilities (flat volatilities) by linear interpolation of quoted implied volatilities (i.e. the two integer maturities 2 year and 3 year will be used to find 2.5 year implied volatility). In addition, calibration of Hull-White and the G2++ models as well as the caplet volatility bootstrap (for the calibration of the different LIBOR volatility parameterizations) requires the cap market premiums in nominal terms. The observed implied flat volatilities are converted into nominal values using Black's formula (3.8) and the prices of individual caplets are obtained by using the caplet bootstrapping procedure described in section 4.1.5.1.

8.2 Summary statistics

I have summarized the sample data in terms of nominal market values of the interest rate caps in table B.1, displaying mean, standard deviation, minimum and maximum across different strike rates and maturities for sample period 1 and 2, respectively. The table shows that the options have higher nominal values for increased maturity due to the increased number of underlying caplets (options) in the cap. Moreover, a higher standard deviation of the longer term caps is also apparent. Across strike rates, the interest rate caps generally display higher values for lower strike rates. The data from lower strike interest rate caps contain more ITM options and therefore on average have higher nominal values.

Figure 8.1 displays the average shape of the 6 month forward rate term structure during sample period 1 and sample period 2 and figure B.2 in appendix displays the behaviour of four zero-coupon interest rates between 2002-04-29 and 2009-12-01. Figure 8.1 shows that the average term structure of the forward rates is increasing in sample period 1. During sample period 2 the average term structure of forward rates is inverted in the short end (from 0.5 to 2 years).

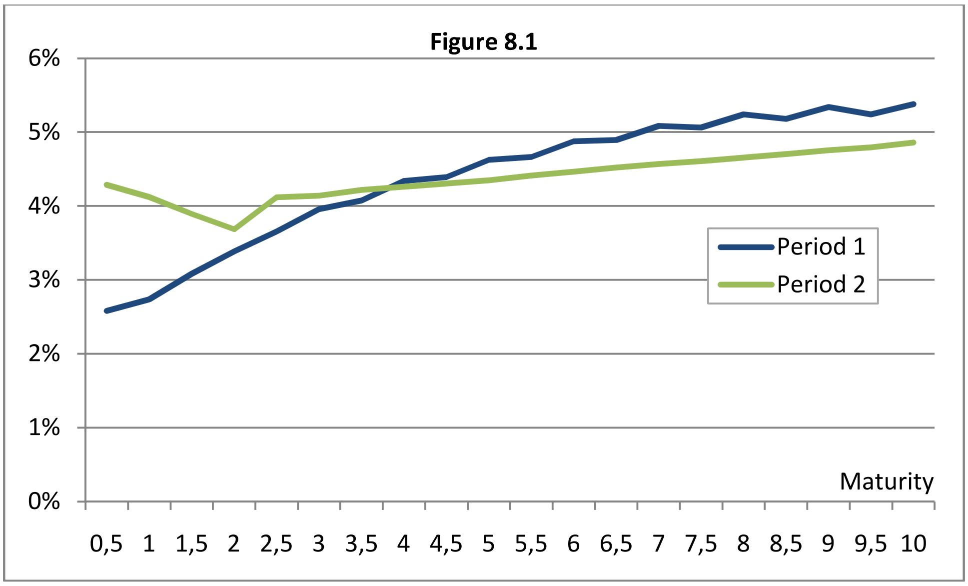

Figure 8.1: *Sample period 1 and 2 average term structure of 6 month forward interest rates (cf. equation 3.2). Sample period 1 covers 2002-04-29 to 2005-10-31 and sample period 2 covers 2007-02-01 to 2009-12-01.*

Figure 8.1 and B.2 in appendix display that the average shape of the interest rate term structure is upward sloping. During sample period 1 the zero-coupon interest rate term structure is upward sloping for all dates. During the second sample period at the end of 2008 the short term zero-coupon rates are higher than the longer term zero-coupon rates. The inverted term structures in sample period 2 are observed during the financial crisis in 2008. In 2009 the zero-coupon term structure is upward sloping again. Moreover, from figure B.2 in appendix it shows that zero-coupon interest rate term structure is flat during 2006. This period is missing many observations possibly caused by the flat term structure environment during 2006. A flat term structure could have had a decreasing effect on the number of trades due to expectations of non-increasing future interest rates.

The average implied volatility during the sample periods is displayed in figure 8.2. I.e. the mean values across all maturities. The average implied volatility shifts around the beginning of the financial crisis where the average implied volatility increases.

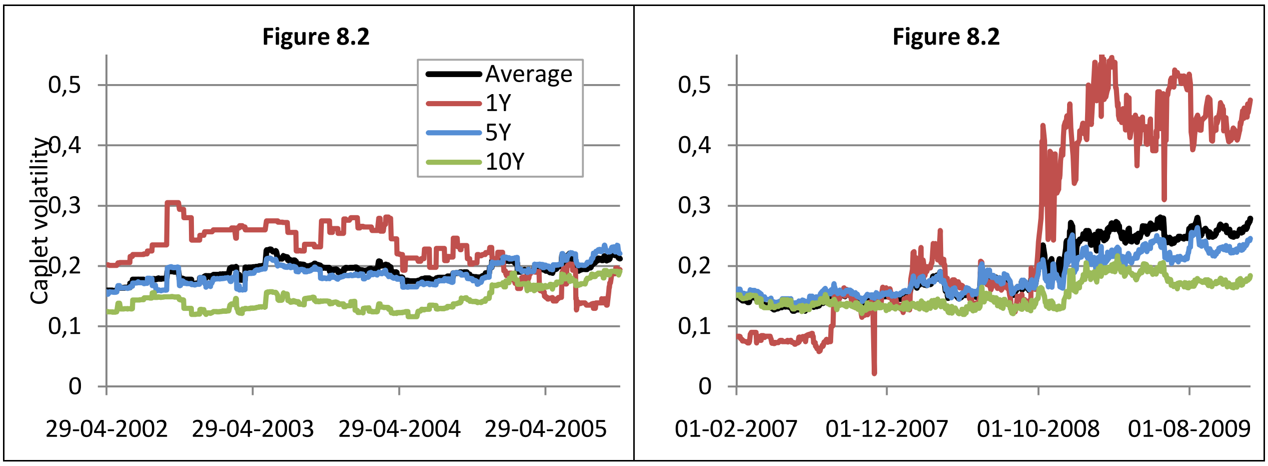

Figure 8.2: *Average caplet volatility (spot volatility) of ten ATM interest rate caps during sample period 1 (left) and sample period 2 (right).*

This period is covered by the second sub-sample. In general higher market premiums and higher standard deviation of cap market premiums are observed during the second sample reflecting the higher volatility in general. Figure 8.3 displays the average term structure of volatilities in the two sample periods. The average term structure of volatilities for each sample period (flat volatilities plotted with time to maturity) is displayed in figure 8.3. Sample period 1 displays a humped shape term structure of volatilities whereas the average term structure of volatilities during sample period 2 is decreasing.

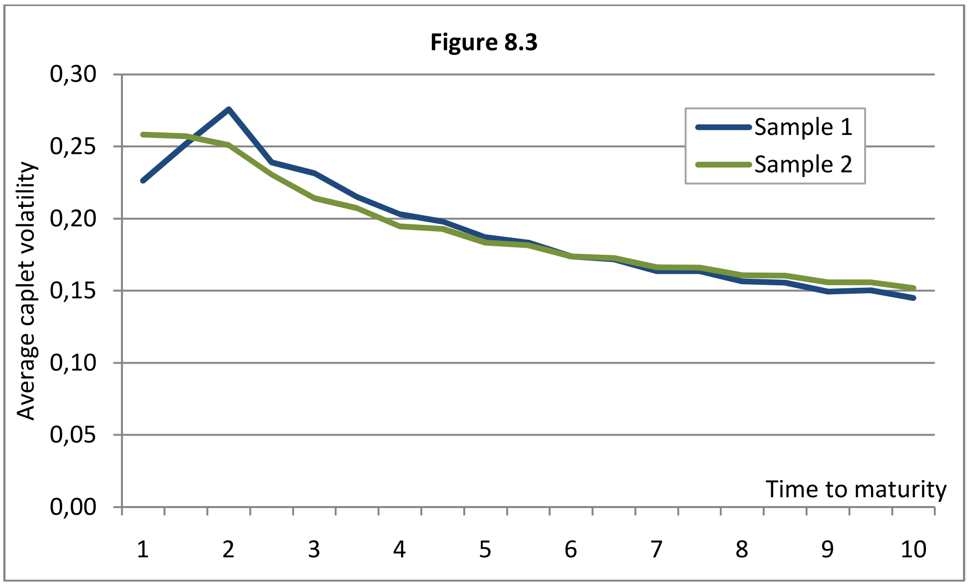

***Figure** 8.3: Average term structure of volatilities (spot volatilities) in sample period 1 and sample period 2, plotted with time to maturity.*

The data set contains caps quoted for 4 different strike levels. To ease the interpretation of the data in relation to the cross-sectional out-of-sample pricing, the data is categorized based on moneyness. I.e. the strike level of a single cap is considered relative to the term structure of interest rates. According to Brigo & Mercurio (2006 p. 18) the moneyness of an interest rate cap is defined relative to the ATM strike level, i.e.

$$moneyness = \frac{K}{K_{ATM}}$$

The caps are organized according to moneyness into five categories representing:

1. Deep in-the-money
2. In-the-money
3. At-the-money
4. Out-of-the-money
5. Deep out-of-the-money

9 Empirical results

This chapter presents the results from taking the models to the data. Section 9.1 compares the in-sample pricing accuracy of the models. Section 9.2 compares the out-of-sample pricing performance and finally section 9.3 reviews the hedging accuracy of the models.

9.1 Calibration

Using the calibration approaches described in chapter 4 the models are calibrated to market data of ATM interest rate caps in order to back out the implied model parameters on each day in the data sample. The parameters are used later in section 9.2 and 9.3 for testing the pricing and hedging accuracy of the models. In the calibration examples presented in chapter 4 I applied a standard criterion for calibration; minimizing the sum of squared errors between market and model prices, however, different criteria can be used. For instance a proportional criterion, minimizing the squared percentage pricing errors, is often applied (Fusai & Roncoroni 2008). The proportional criterion is applied in Diaz et al. (2009), who investigate the effects of using the standard approach versus the proportional approach when calibrating different volatility parameterization within the LIBOR market model. They find that when using squared percentage pricing errors the calibration results in a more precise fit to the short end of the volatility structure but also that the calibration results in larger errors on longer term caps when compared to the approach of squared pricing errors. There are pros and cons of each method. In general, the standard criterion assigns more weight to expensive options (resulting in a closer fit to longer term caps) whereas the proportional criterion provides a better fit to the short end. However, in cases where the option price is close to zero the proportional criterion can become instable (Fusai & Roncoroni (2008 pp. 358-359).

Hence, the standard and the proportional criteria offer a trade-off between a better fit to short term caps versus longer term caps. In the following I will apply both methods in order to compare the robustness towards calibration criteria of the different models. I will proceed in section 9.2 and 9.3 with the implied parameters from the approach which obtains the better results in reproducing market prices. Moreover, the other criterion will be applied in order to validate the results in the pricing and hedging tests as robust towards changes in the calibration method.

9.1.1 Comparison of in-sample fit

Table 9.1 displays the accuracy of the calibration in sample period 1 and sample period 2 of the Hull-White and the G2++ models and for each of the calibration criteria presented above. Exact replication of market prices is achieved with the LIBOR market model (according to equation 4.13). Therefore, I will only compare the performance of the two Gaussian models with respect to in-sample pricing accuracy. The G2++ model should provide lower errors compared to the Hull-White model since the G2++ model is equivalent to the two-factor Hull-White model and therefore encompass the single factor Hull-White model.

Table 9.1 shows that the two-factor model in general provides a better fit to market prices of interest rate caps. The average sum of squared residuals in period one, using the mean squared error as calibration criteria, is $1{,}49 \cdot 10^{-7}$ for the G2++ model versus $1{,}17 \cdot 10^{-6}$ for the Hull-White model. The pattern is consistent across sample periods and calibration criteria. The magnitude of the fitting errors is compared in nominal terms by calculating the average of the absolute errors between market prices and model prices (mean abs err). The magnitude is measured in basis points, i.e. per nominal value of DKK 10.000. Using mean squared error as calibration criteria, the G2++ model is calibrated with an average absolute error around 1 basis point and the Hull-White model is calibrated with an average absolute error around 2 basis points. I have not obtained an estimate of the average bid-ask spread for interest rate caps in the Danish market for either of the sample periods. According to traders there is too little activity in the Danish market for interest rate caps for them to deliver bid-ask quotes. This is of course a disadvantage of using data from the Danish market for interest rate caps. However, according to Gupta & Subrahmanyam (2005 p. 717) the average bid-ask spread of US caps and floors is around 2-3 basis points. Based on the US bid-ask spread the general fit of the models when using the MSE criterion is good. Given the lower liquidity of the Danish interest rate caps, however, the bid-ask spread is likely to be higher than 2-3 basis points.

Table 9.1

	Period	Mean squared error			
		SSR mean	SSR min	SSR max	Mean abs err (bp)
Hull-White	1	1,2E-06	5,7E-08	2,9E-06	2
G2++		1,5E-07	5,5E-09	2,0E-06	1
		Percentage mean squared error			
		SSR mean	SSR min	SSR max	Mean abs err (bp)
Hull-White		4,5E-05	1,7E-07	1,3E-03	11
G2++		1,7E-06	7,1E-09	3,6E-04	2
	Period	**Mean squared error**			
		SSR mean	SSR min	SSR max	Mean abs err (bp)
Hull-White	2	1,5E-06	1,4E-08	1,1E-05	2
G2++		6,3E-07	1,2E-09	8,0E-06	1
		Percentage mean squared error			
		SSR mean	SSR min	SSR max	Mean abs err (bp)
Hull-White		5,4E-04	3,1E-08	3,7E-03	26
G2++		8,3E-07	3,1E-09	1,2E-05	1

Sum of squared residuals(SSR) mean, minimum and maximum and mean absolute error, calculated for two different criteria when calibrating the Hull-White and G2++ models to market prices of ATM interest rate caps. The mean absolute error is measured in basis points.

When changing the calibration criterion to minimizing the percentage mean squared error, the mean absolute error of the Hull-White model increases from 2 to 11 basis points in sample period 1 and from 2 to 26 basis points in sample period 2. The errors of the G2++

model also increase when using the percentage mean squared error as calibration criteria, however, the errors increase much less compared to the Hull-White model (1 basis point and less). The relative performance between the models is robust with respect to calibration criteria. The most accurate calibration is achieved using the standard approach.

9.1.2 LIBOR volatility parameterizations

Table 9.2 compares the in-sample estimation of the different volatility parameterization (presented in section 4.1.4) within the LIBOR market model. The calibration is non-exact because the volatility is not adjusted using equation (4.13). Using mean squared errors, the Exponential and the Rebonato specifications provides a much better fit to current market prices compared to the Constant volatility specification, however, this result is not surprising given that the average shape of the caplet volatility structure is humped and decreasing (c.f. figure 8.3). The Rebonato parameterization provides the most accurate fit, which also is expected since this volatility specification nest the other two. However, the performance is only marginally better than the exponential volatility specification and at the cost of higher parameter instability (cf. figures in appendix B.5).

Table 9.2

Period 1:	**Mean squared error**			
	SSR mean	SSR min	SSR max	Mean abs err (bp)
Constant	1,25E-04	8,36E-06	3,54E-04	18
Exponential	1,62E-05	1,30E-07	9,85E-05	7
Rebonato	1,63E-06	2,54E-08	8,20E-05	2
	Percentage mean squared error			
	SSR mean	SSR min	SSR max	Mean abs err (bp)
Constant	1,05E-04	1,32E-05	3,02E-04	20
Exponential	1,18E-05	9,32E-08	1,87E-04	5
Rebonato	4,90E-05	4,13E-08	2,54E-04	11
Period 2:	**Mean squared error**			
	SSR mean	SSR min	SSR max	Mean abs err (bp)
Constant	3,08E-04	1,02E-06	3,09E-03	24
Exponential	3,39E-05	1,18E-07	1,04E-03	8
Rebonato	3,49E-05	4,60E-08	2,20E-03	7
	Percentage mean squared error			
	SSR mean	SSR min	SSR max	Mean abs err (bp)
Constant	1,80E-04	2,20E-06	4,47E-03	23
Exponential	5,28E-05	3,77E-08	1,77E-03	10
Rebonato	1,47E-05	7,91E-09	1,24E-03	3

Sum of squared residuals (SSR) mean, minimum and maximum and mean absolute error, for two different loss functions when calibrating the LIBOR market model with different volatility parameterizations to market prices of ATM interest rate caps.

In contrast to the calibration of Hull-White and G2++, the calibration based on percentage squared errors results in lower pricing errors when calibrating the LIBOR market model. However, the second criteria also results in more varying parameters in general, and this is particular the case for the Rebonato parameterization which takes four input parameters.

Given that the standard calibration criterion produces lower pricing errors for the Gaussian models and produce more stable model parameters in general, I have chosen to continue the out of sample pricing analysis and the hedging analysis based on this criterion.

9.1.3 A note on calibration

In this thesis all models are calibrated using the nonlinear least squares algorithm (lsqnonlin) in MATLAB. The lsqnonlin function enables the use of different numerical methods in order to solve nonlinear optimization problems, including Gauss-Newton, Levenberg-Marquardt and trust-region-reflective algorithms. Explaining the details of these approaches is beyond the scope of this thesis, however, common to the algorithms is the dependence on an initial guess of the solution and the use of gradient descent based search methods.[22] The Levenberg-Marquardt algorithm is applied in Amin & Morton (1994 pp. 152-153) and in Mannolini et al. (2008 pp. 393-394), however, because I obtain more accurate results when using the trust-region-reflective algorithm in MATLAB, I have chosen to apply this optimization algorithm in order to calibrate the models[23]. A common disadvantage of the optimization algorithms is that they belong to a class of local optimizers (Fusai & Roncoroni 2008 p. 360) which means that the optimization methods depends on an initial guess of the solution. This is a disadvantage in relation to calibrating the models because there is no guarantee that the final result corresponds to a global solution. Fusai & Roncoroni (2008 p. 359) point out that a gradient-based optimization algorithm is likely to return a local minimum dependent on the initial guess. Hence, the initial guess will influence the final results when using a local optimizer. In order to illustrate the influence of the initial guess on the final result I have calibrated the G2++ model using different initial values of the parameters. I assume the problem is biggest for the G2++ model since the calibration involves fitting five parameters. Table 9.3 illustrates the final parameters when calibrating with different initial guesses and figure 9.1 illustrates the fit to market volatilities.

[22] A detailed description of the lsqnonlin function in MATLAB is available at mathworks.com.
[23] Brigo & Mercurio (2006 pp. 166-167) calibrate the G2++ model using a global optimization algorithm known as simulated-annealing. The trade-off from using global versus local search algorithms is longer calibration times.

Table 9.3

	Guess 1	Result 1	Guess 2	Result 2
σ	0,19	0,11	0,19	0,21
η	0,3	0,28	0,19	0,26
a	0,2	0,11	0,42	0,34
b	0,42	0,35	0,42	0,41
ρ	-0,8	-0,79	-0,8	-0,67
SSR		**1,66E-06**		**2,24E-06**

Comparison of final parameter values of calibration with different initial guesses. Data from 2009-10-12.

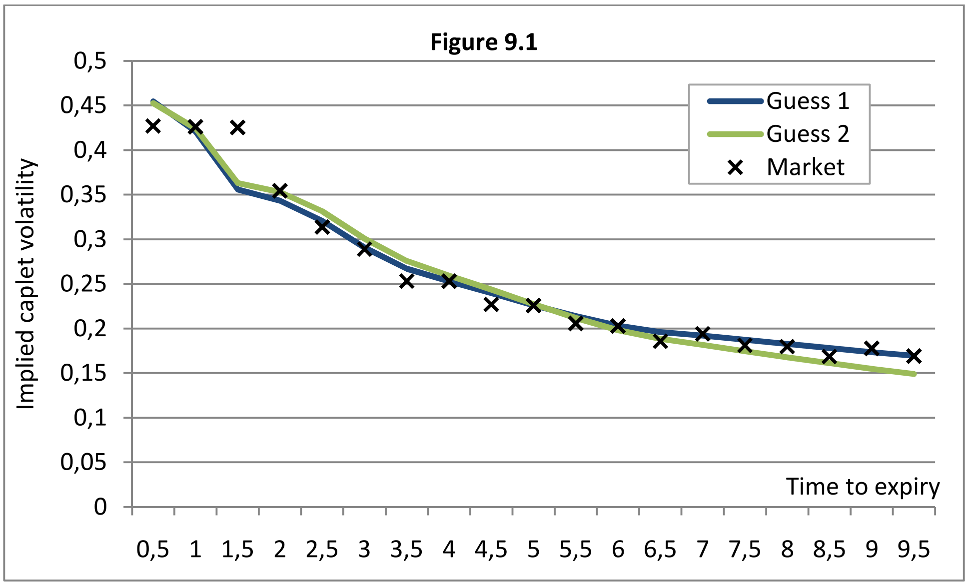

***Figure 9.1**: G2++ model implied volatilities fit to market volatilities. The curves are based on two different calibrations with different starting values. The initial values of the parameters are stated in table 9.3. The model is calibrated to market data on date 2009-10-12.*

9.1.4 Concluding remarks

The in-sample estimation shows that the G2++ model achieves a better fit to market prices of interest rate caps compared to the Hull-White model, however, at the cost of higher instability in parameters. All models are on average calibrated with an acceptable margin of error less than the assumed bid-ask spread.

Within the LIBOR market model a similar picture is shaped. On average the Rebonato parameterization presents the best fit to market prices but also presents higher parameter-instability due to a larger number of parameters to fit.

In general, the models achieve smaller in-sample errors in sample period 1. Moreover, a closer fit to market prices is achieved with the standard criterion, however, the proportional criterion ensure a better fit to the short term caps. With this in mind I continue the out-of-sample pricing and hedging tests, i.e. the result should be tested for their robustness towards calibration criteria and data period.

The optimization algorithm applied in order to calibrate the models poses a disadvantage, since the method does not guarantee a global solution.

9.2 Pricing accuracy

Based on the framework explained in chapter 7.1 this section investigates the pricing ability of the different interest rate models. I present results for the models accuracy in pricing interest rate caps out-of-sample over different timescales and cross-sectional across different strike rates (moneyness). Furthermore, I run a regression of market prices on model prices in order to determine the prediction ability of the models and to capture bias in the pricing errors.

In line with Amin & Morton (1994) and Gupta & Subrahmanyam (2005) I measure the out-of-sample pricing errors using four different sets of statistics:

1. Average pricing error
2. Average absolute pricing error
3. Average percentage pricing error
4. Average absolute percentage pricing error

The average and the average absolute pricing errors reflects information about the magnitude of the out-of-sample pricing errors and should be interpreted in relation to the size of the bid-ask spread related to interest rate caps. Opposite, the average percentage and the average absolute percentage pricing errors measure the error relative to the option price and thus point to the relative fitting performance of the models. The relative error measures are not biased towards the more expensive options.

9.2.1 Timescale out-of-sample pricing

Table 9.4 below presents the error statistics for each model when pricing at-the-money interest rate caps out-of-sample over 1 day and 1 week timescales.

The average and the average absolute errors are measured in basis points, i.e. the error is per a notional amount of DKK 10,000. With an average absolute error of 0.89 basis points in period 1 and 3.73 basis points in period 2 the LIBOR market model produce lower prediction errors than the Gaussian models. The Hull-White model produces the largest errors in both sample periods. In general, however, the models prediction errors are tolerable. Gupta & Subrahmanyam (2005) estimated the bid-ask spread in the US caps and floors markets to be around 2 basis points. The bid-ask spread is presumably higher in the Danish market, especially in the second sample period which covers the years from 2007 to 2009. The percentage errors of the LIBOR and G2++ models are small (less than 1%), indicating a small bias in predicting option prices. The bias is more pronounced for the Hull-White model, indicating larger bias in predicting option prices.

Table 9.4: Timescale out-of-sample pricing

1 day	Period	Average error	Average absolute error	Average percentage error	Average percentage absolute error
LIBOR		-0,06	0,89	-0,02%	0,39%
Hull-White	1	0,77	3,19	4,48%	5,22%
G2++		0,23	2,32	0,70%	1,72%
LIBOR		-0,18	3,73	0,00%	1,75%
Hull-White	2	-0,12	4,30	1,63%	4,67%
G2++		-0,17	3,74	-0,10%	2,64%

1 week	Period	Average error	Average absolute error	Average percentage error	Average percentage absolute error
LIBOR		-0,32	3,31	-0,08%	1,48%
Hull-White	1	1,00	5,67	4,65%	6,07%
G2++		0,46	5,03	0,86%	2,80%
LIBOR		-0,95	8,54	-0,25%	3,84%
Hull-White	2	-0,66	8,34	1,16%	5,91%
G2++		-0,68	7,97	-0,45%	4,11%

Average pricing errors based on out of sample pricing with 1 day and 1 week lagged parameters. All models are calibrated to ATM interest rate caps. The average error is defined as the average of the predicted model price minus the observed market price across the sample period.

In general, the Hull-White model is more inaccurate than the other models, nevertheless, the performance of the Hull-White model improves in the second sample period, where the percentage absolute error is 4,67% versus 5,22% in the first sample period. Neither the LIBOR nor the G2++ model improve going from sample period 1 to 2. The most accurate model in pricing out of sample is the LIBOR market model, indicating better pricing performance over a one-day timescale with the lognormal model. Furthermore, with an absolute average percentage error of 1,72% vis-à-vis 5,22% in sample period 1 and 2,64% vis-à-vis 4,67% in sample period 2, the Gaussian two-factor model displays lower pricing errors compared to the one-factor Hull-White model. The results for pricing out of sample are in line with the results of Gupta & Subrahmanyam (2005 p.718). In general, they find that the lognormal model is more accurate in pricing out of sample, and that two-factor models produce lower out-of-sample pricing errors than their one-factor equivalents. The magnitudes of the errors are marginally higher than the levels reported in Gupta & Subrahmanyam (2005 p. 718), which report an average absolute error of 2.3 basis points for the Hull-White model and 1.2 basis points for the LIBOR market model.

To investigate the effects of increasing the timescale for the out-of-sample pricing, I measure the pricing errors for an out-of-sample period of 5 business days, corresponding to a period of 1 week. The results are presented in table 9.4 above. The average errors increase when increasing the timescale for out-of-sample pricing. However, the comparative performance is similar to the results when pricing 1 day out-of-sample. The lognormal model is more accurate than the Gaussian models, except for the absolute error in sample two where the error is higher. However, the percentage errors remain lower indicating less bias. Therefore, increasing the timescale for the out-of-sample pricing does not alter the results. This is similar to the results in Gupta & Subrahmanyam (2005); the errors increase and the comparative performance of the models remains the same.

In order to test the prediction ability of the models a regression of the estimated model prices on market prices are conducted. The results are displayed in appendix A.5. The regression shows that the LIBOR market model is less misspecified compared to the Gaussian models.

9.2.2 Cross sectional pricing errors

The figures below display the results of pricing caps across different strike rates. The models are calibrated to data of ATM options and then used for pricing options with different strike rates on the same day in order to investigate patterns in relation to the cross sectional pricing accuracy of the models. The error statistics are categorized based on option moneyness (cf. p. 51). Patterns in the errors related to strike level and maturity. Figure 9.2 and B.3 in appendix B show the pattern of mispricing related to moneyness.

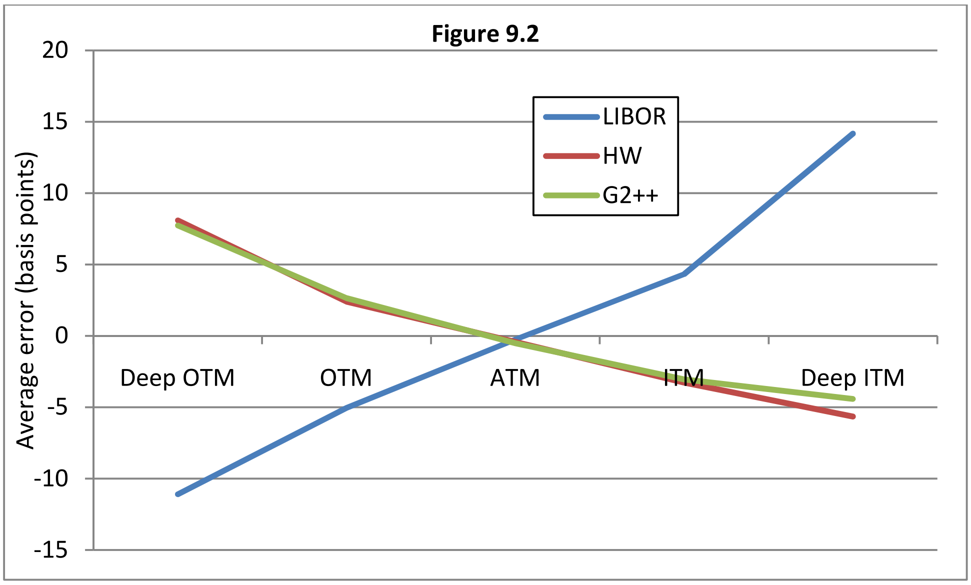

Figure 9.2*: Sample period 1 cross-sectional out-of-sample pricing errors. The models are calibrated to market prices of ATM caps and then implied parameters are used to estimate prices of away-from-the-money options. The average error is given by the difference between predicted model price and the observed market price.*

The LIBOR market model undervalues out-of-the-money interest rate caps and overvalues in-the-money caps. Contrary, the Gaussian models tend to overvalue out-of-the-money caps and undervalue in-the-money in sample period 1. In sample period 2, the LIBOR market model displays a pattern similar to the first sample period, whereas the Gaussian models tend to overvalue almost all options (see fig. B.3 in appendix). The observations are similar to the patterns of mispricing observed in Amin & Morton (1994

p. 163) where the exponential HJM volatility specification (corresponding to the Vasicek model) tends to cause overpricing for most options except far in-the-money options.

9.2.3 Pricing with different volatility specifications in LIBOR

Table 9.8 presents the error statistics when pricing one day out of sample with different volatility specifications in the LIBOR market model. Compared to the errors in table 9.4, the errors are significantly larger when comparing the performance of different volatility specifications within the LIBOR market model. This difference in the size of the errors is a result of the non-exact calibration of the volatility parameterizations as shown in figure 4.5 to 4.7. However, the purpose is to evaluate the comparative performance of the volatility parameterizations. The results in table 9.8 are similar to the results in table 9.2, which displays the in-sample performance of the models. The Exponential and Rebonato specifications produce lower errors than the constant volatility specification, and Rebonato produce marginally lower average absolute errors compared to the exponential volatility. The patterns are similar for both sample periods.

Table 9.5

	Period	Average error	Average absolute error	Average percentage error	Average percentage absolute error
Constant		2,99	18,30	-4,30%	9,05%
Exponential	1	2,50	6,74	0,72%	3,48%
Rebonato		0,69	2,48	0,54%	1,30%
Constant		6,86	24,39	-1,57%	10,66%
Exponential	2	-1,48	9,56	-0,17%	5,07%
Rebonato		2,36	8,79	0,58%	3,85%

Out of sample pricing with one day lagged parameters and different volatility parameterizations in the LIBOR market model. Calibration to ATM interest rate caps. The average error is defined as the average of the predicted model price minus the observed market price across the sample period.

9.2.4 Concluding remarks

The results of the timescale out-of-sample pricing show that the LIBOR market model is more accurate in pricing interest rate caps compared to the Gaussian models. Increasing the out-of-sample timescale implies larger errors for all models. However, the comparative performance between the models remains, indicating that more accurate pricing is achieved with a lognormal model compared to a Gaussian model. The cross-sectional out-of-sample pricing reveals a clear skew across moneyness in all models.

The two-factor Gaussian model produces lower errors than the one-factor model, hence, the results are in line with the results of Gupta & Subrahmanyam (2005).

Within the LIBOR market model the Rebonato volatility parameterization produces the lowest pricing errors. The results are consistent across sample periods indicating robustness towards the sample data period.

9.3 Hedging performance

Using the framework explained in section 7.2 this section investigates the hedging performance of the alternative interest rate models. With basis in a replicating portfolio strategy I present results for each of the models when hedging market prices of interest rate caps over different timescales.

9.3.1 Hedging ATM interest rate caps

Table 9.9 displays the average error and the average absolute error when hedging ATM caps over a timescale (rebalancing interval) of one day, one week and one month, respectively. The errors are calculated as the difference between the market price and the value of the replicating portfolio implied by each model, and is measured in basis points.

Table 9.6

	Period	1 day		1 week		1 month	
		Average error	Average abs error	Average error	Average abs error	Average error	Average abs error
LIBOR		0,75	6,61	3,01	15,74	11,97	37,22
Hull-White	1	-0,04	6,26	1,95	13,78	9,85	32,25
G2++		0,55	5,64	2,54	13,36	10,39	31,82
LIBOR		2,18	11,28	5,14	23,94	12,89	43,80
Hull-White	2	2,12	10,80	4,94	22,08	12,43	39,94
G2++		2,19	10,47	5,01	21,84	12,39	39,75

Hedging errors for ATM interest rate caps across 10 maturietes and for different timescale rebalancing periods. The hedging error is defined as the difference between the market value of the cap and the replicating portfolio, i.e. market price – hedge portfolio.

The difference between the average absolute errors of the best and the worst performing models are less than 1 basis point in each sample period, hence the difference between the hedging accuracy of the models is small. On average the two-factor Gaussian model (G2++) produce the lowest absolute hedging errors for all timescales and sample periods. However, in sample period 1 the G2++ model displays higher average hedging than the Hull-White model, especially for increasing rebalancing intervals. The average hedging error is positive, indicating that the models are under-hedging i.e. the value of the hedge position (the replicating portfolio) is less than the value of the interest rate cap being hedged, i.e. the hedge does not cover a short position in the interest rate cap. The LIBOR market model is, consistently, the least accurate model in hedging across all timescales and sample periods. Therefore, the hedging accuracy results indicate better hedging performance with Gaussian models compared to the lognormal model when hedging at-the-money interest rate caps. As for the two-factor model versus the one-factor model, the absolute hedging errors of G2++ are consistently below the errors of Hull-White, ranging from 0,62 basis points for the one day period to 0,72 basis points for the 1 month period. The relative performance of the models is consistent across sample period.

On average, there is little difference between the models performance. The difference between the average hedging errors is less than 2 basis points for all hedging intervals and the difference between average absolute hedging errors is less than 5 basis points. The patterns are similar for each sample period. The hedging errors increase with the length of the rebalancing interval.

Figure 9.3 shows the average hedging errors in sample period 1 plotted with time to maturity. The average hedging error is increasing with time to maturity of the interest rate cap for the LIBOR market model. The Hull-White and the G2++ models, however, each depends on the initial fit to market prices. The LIBOR market model increase almost linearly for each maturity. The Hull-White model displays clear over-hedging of short maturity caps, which could be caused by the poor fit to the short end of the volatility structure. Contrary, the G2++ model displays better hedging of short maturity caps.

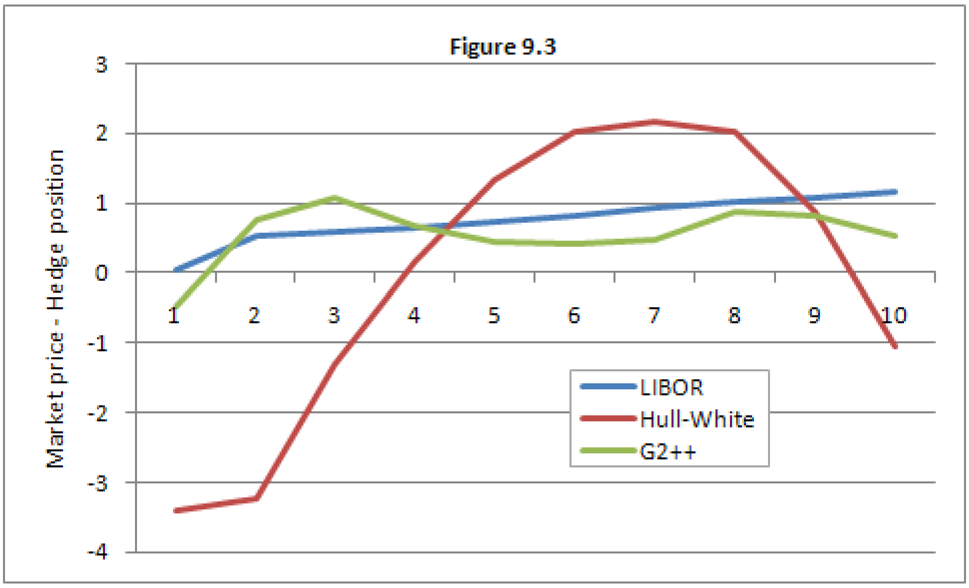

Figure 9.3*: Sample period 1: Average hedging errors of different maturity interest rate caps based on one-day out-of-sample hedging of ATM interest rate caps with the replicating portfolio from three models: The Hull-White (blue), G2++ (green), and the LIBOR market model (red). The hedging error is defined as the difference between the market price of the interest rate cap and the value of the replicating portfolio.*

Different aspects can have an influence on the increasing hedging error with time to maturity. First, longer term interest rate caps include more caplets and therefore imprecision could be scaled causing more inaccurate hedging with the increasing number of underlying caplets. Second, the hedging error could be increasing with the longer term caplets causing the larger hedging error for longer term caps. However, in order to conclude on these aspects an analysis based on caplets is required. A third aspect could be related to a lack of liquidity of longer maturity caps. Babbs & Webber (1994) argue that short-rate term structure models are not able to adequately describe the short-rate process due to monetary authorities influence on domestic short-term interest rates, for which monetary authorities control the rate setting behaviour and, moreover, the signalling of their rate setting behaviour.

Figure B.4 in appendix B displays the average hedging errors of the LIBOR, Hull-White and the G2++ models during sample period 2. The average errors of the G2++ and the LIBOR market model increase compared to sample period 1. The Hull-White model performs similar to the G2++ model in sample period 2. This could be caused by a better fit (on average) to short term caps.

9.3.2 Robustness test: different calibration strategy

Given the methodological approach applied in order to calibrate the models, the results of the hedging analysis can be questioned. This section investigates the robustness of the hedging results with respect to the calibration procedure. The proportional calibration criterion (cf. p. 52) allows more weight to less expensive options when calibrating the models. Therefore the proportional criterion contains more information about the short term caps.

Table 9.10 displays the results given that the hedging test is based on parameters obtained by proportional calibration to ATM interest rate caps. The results show that the Hull-White model is sensitive towards the calibration method applied. The hedging errors of the Hull-White model increase significantly compared to the errors shown in table 9.9. The LIBOR and the G2++ models are less sensitive given that the models both provide adequately fit to short end caps.

Table 9.7

		1 day		1 week		1 month	
	Period	Average error	Average abs error	Average error	Average abs error	Average error	Average abs error
LIBOR		0,75	6,61	3,01	15,74	11,97	37,22
Hull-White	1	0,19	12,83	2,17	17,36	10,01	32,59
G2++		0,50	5,96	2,49	13,49	10,38	31,87
LIBOR		2,18	11,28	5,14	23,94	12,89	43,80
Hull-White	2	22,92	32,10	25,35	40,67	31,14	54,35
G2++		2,29	10,51	5,11	21,89	12,51	39,84

Hedging errors for ATM interest rate caps across 10 maturietes and for different timescale rebalancing periods. The hedging error is defined as the difference between the market value of the cap and the replicating portfolio, i.e. market price – hedge portfolio.

Figure 9.4 shows the average hedging errors plotted with time to maturity of the interest rate cap being hedged. Compared to figure 9.3 the average hedging errors of the G2++ and the LIBOR market models are not sensitive to change in the calibration method. The Hull-White model, however, is significantly exposed towards changing the calibration method. Figure 9.4 shows that hedging accuracy of the Hull-White model improves for short term caps (1 and 2 years maturity), however, at the cost of increased errors for caps with maturities above two years.

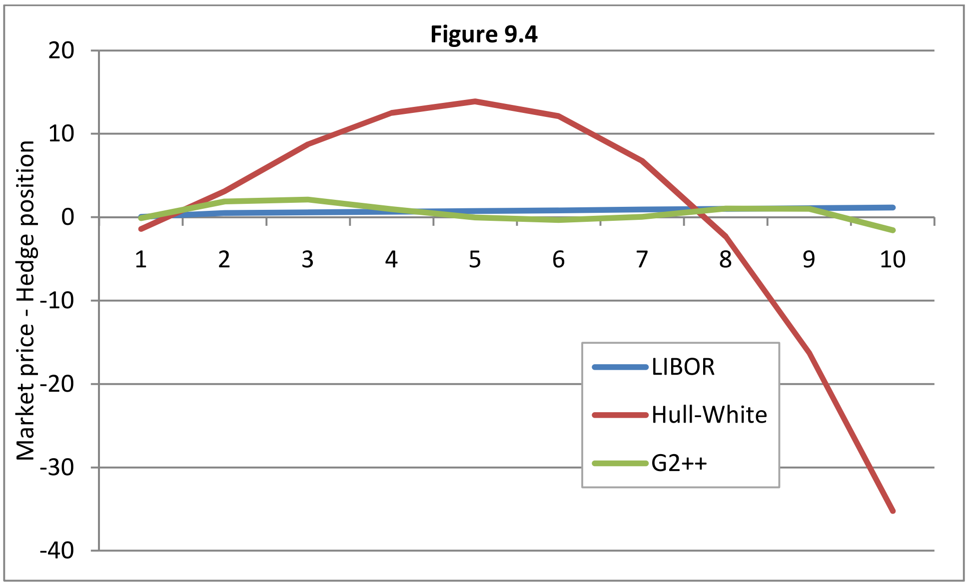

Figure 9.4*: Sample period 1, percentage least-squares calibration: Average hedging errors of different maturity interest rate caps based on one-day out-of-sample hedging of ATM interest rate caps with the replicating portfolio from three models: The Hull-White (blue), G2++ (green), and the LIBOR market model (red). The hedging error is defined as the difference between the market price of the interest rate cap and the value of the replicating portfolio.*

9.3.3 The LIBOR market model - which volatility function?

This section presents results for the hedging accuracy of the LIBOR market model given different volatility parameterizations. Table 9.8 displays the average error and the average absolute error when hedging ATM caps over a timescale (rebalancing interval) of one day, one week and one month, respectively. Again, the error is calculated as the difference between the market price and the value of the replicating portfolio, and is measured in basis points.

Table 9.8

	Period	1 day		1 week		1 month	
		Average error	Average abs error	Average error	Average abs error	Average error	Average abs error
Constant		0,74	6,26	2,94	14,95	11,66	35,50
Exponential	1	0,76	6,52	3,03	15,54	12,01	36,83
Rebonato		0,75	6,61	3,01	15,75	11,99	37,25
Constant		2,23	11,12	5,15	23,64	12,92	43,23
Exponential	2	2,17	11,26	5,13	23,86	12,91	43,61
Rebonato		2,18	11,22	5,12	23,80	12,89	43,62

Average hedging errors for different rebalancing periods. The errors are based on ATM interest rate caps across 10 maturietes. The hedging error is defined as the difference between the market value of the cap and the value of the replicating portfolio, i.e. (market price – hedge portfolio). The errors are in basis points.

The results reveal little difference between the hedging errors, less than 1 basis point for one-day one-week timescales.

9.3.4 Concluding remarks

The results of the hedging accuracy tests indicate more accurate hedging with the Gaussian models compared to the lognormal model. However, for a rebalancing interval of one-day the difference is less than 1 basis point between the model with the largest error and the model with the lowest error in both sample periods. Furthermore, the pattern of the hedging errors across time-to-maturity of the Gaussian models depends on the models initial fit to market prices. The Hull-White model exhibits a poor fit to short maturity caps which results in over-hedging of short maturity caps with the Hull-White model. The results of the hedging analysis supports previous results (Gupta & Subrahmanyam (2005)) showing that two-factor models improve the hedging accuracy relative to one-factor models. The G2++ model consistently produce lower errors than the Hull-White model, however, the results are sensitive towards applied calibration criteria.

The results from hedging interest rate caps using replicating portfolios based on different volatility parameterizations (within the LIBOR market model) leave the question open. Differences between errors are less than 1 basis point. Perhaps a different hedging strategy would reveal different results.

10 Conclusion

This thesis has investigated the pricing and hedging accuracy of alternative interest rate models. Based on data from the Danish market for interest rate caps the aim of this thesis was to provide further empirical evidence of the comparative performance of interest rate models which represents normal and lognormal distributional assumptions. The models include the LIBOR market model, the Hull-White model and the G2++ model. The G2++ model was included in order to investigate the effects of an additional stochastic factor on the short-rate model's pricing and hedging accuracy. In addition, the LIBOR market model was represented using three different volatility parameterizations (Rebonato, Exponential and Constant), in order to investigate how the volatility should be specified in the model.

Inspired by the works of Gupta & Subrahmanyam (2005) focussing on the empirical performance of interest rate models an experimental design was adapted in order to measure and compare the pricing and hedging ability of the models. The comparison was based on the models in-sample estimation, out-of sample pricing and hedging ability.

In relation to the in-sample estimation, the LIBOR market model encompasses the market standard formula (Black's formula) for pricing interest rate caplets, and, hence, is by construction able to exactly fit market prices of interest rate caps. The Gaussian models, on the other hand, are not exactly calibrated to market prices. The results of the in-sample estimation show that the G2++ model is calibrated to market prices of interest rate caps with a smaller error compared to the Hull-White model. Hence, the additional factor provides a more accurate calibration to market prices. However, both models are able to recover current market prices of interest rate caps with an average error below 2 basis points. Within the LIBOR market model the Rebonato parameterization produces the most accurate replication of current market prices compared to the Exponential and Constant parameterizations.

Furthermore, the in-sample estimation shows that the models containing a higher number of parameters, for instance G2++ compared to the Hull-White model or the Rebonato parameterization versus the simpler Exponential and Constant parameterizations, exhibit higher parameter-instability. Two criteria are applied in order to calibrate the models; standard least squares and percentage least squares. Due to the results from calibrating the models it can be concluded that the standard criterion provides the most accurate fit to market prices. The proportional criterion, however, ensures a better fit to the short-term options.

The pricing accuracy of the interest rate models was tested based on the out-of-sample pricing performance. The results show that the comparative performance of the models is in line with the results in Gupta & Subrahmanyam (2005). On average the LIBOR market model is more accurate than the Hull-White and G2++ models. Though, the result is sensitive towards change in sample data because the difference between the models performance is eliminated in the second sample period. In addition, the results also show that increasing the timescale from one day to one week implies increased out-of-sample pricing errors for all models and that the comparative performance of the models remains for increasing timescales. Therefore, in relation to the first question of the problem statement, the results indicate that a lognormal model provides better pricing accuracy compared to a Gaussian model when pricing at-the-money interest rate caps out-of-sample. The results of the cross-sectional out-of-sample pricing reveal a skew across moneyness in all models.

When measuring the pricing accuracy of the Hull-White model versus the G2++ the results show that the G2++ model provides lower pricing-errors compared to the one-factor Hull-White model. The absolute percentage errors are on average 2-3%-points higher in the Hull-White model. Hence, in line with Gupta & Subrahmanyam the results indicate improved pricing accuracy of multifactor interest rate models. In relation to the LIBOR market model the results of the out-of-sample pricing show that the Rebonato volatility parameterization produces the smallest pricing errors. Though, parameter instability follows when increasing the complexity of the parameterization.

The hedging ability of the interest rate models was measured using a replicating portfolio strategy based on zero-coupon bonds. The results indicate that the Gaussian models provide more accurate hedging of interest rate caps compared to the LIBOR market model and moreover that the G2++ model produces smaller hedging errors than the Hull-White model. In addition, the pattern of the hedging errors across time-to-maturity of the Gaussian models reveals that the magnitude of the hedging error depends on the initial fit to market prices. For short maturity caps the Hull-White model displays larger hedging errors because of a poor initial fit to market prices. Hence, the Hull-White model is over-hedging short maturity caps. However, given a rebalancing interval of one-day the difference between the average hedging errors is less than 1 basis point between the largest and the lowest hedging error in both sample periods. Moreover, when measuring the hedging performance based on a different calibration criterion (percentage least squares) the results are revealed sensitive towards the applied calibration criteria.

In relation to the LIBOR market model the difference between the average absolute hedging errors based on different volatility parameterizations are less than 1 basis point for one-day and one-week timescales. Hence, the results are inconclusive with respect to which volatility parameterization provides the more accurate hedging.

Conclusively, the results of the out-of-sample pricing and hedging arse in accordance with the results in Gupta & Subrahmanyam (2005) which show that a LIBOR market model provides more accurate pricing and moreover that more efficient hedging is achieved with multifactor interest rate models. The LIBOR market model provides more accurate pricing compared to the Gaussian models, and in relation to hedging the two-factor G2++ model is more accurate compared to the one-factor Hull-White model. However, the results in this thesis also reveal that in practice there is little difference between the models. In some cases the difference is too small to provide a clear answer of which model is more accurate than the other.

11 Future research

Investigating the pricing and hedging of alternative interest rate models involves a number of methodological decisions. Different aspects in relation to model calibration and hedging strategy have been encountered and the results in relation to calibration and hedging, in particular, attract interest for future research.

First, the calibration of the short-rate models in section 9.1 revealed that the final result is sensitive towards which criterion to apply and moreover that a number of algorithms involves the risk of obtaining local solutions. In this context I find it valuable to further investigate the impact of applying alternative algorithms on the accuracy of in-sample estimation, and in relation to pricing other interest rate derivatives. The subject is more relevant to the G2++ model compared to the Hull-White model due to a larger number of parameters in the G2++ model. The Hull-White model contains two parameters for which a grid search algorithm could be applied. The efficiency of applying a global search algorithm in order to calibrate the G2++ model could be investigated and compared to the efficiency of the local search algorithm applied in this thesis.

Second, the hedging approach of constructing a replicating portfolio using zero-coupon bonds is unlikely to be applied in practice. The results are inconclusive with respect to the LIBOR volatility parameterizations and when comparing alternative models the results reveal little distinction between the interest rate models. Hence, the results of investigating the hedging accuracy raise a question of how an alternative hedging strategy could be implemented and whether this would alter the conclusions. For instance, in line with the Driessen et al (2003) the hedging of interest rate options could be investigated using a factor-hedging approach which allows the models to hedge interest rate caps only by using a number of hedging instruments corresponding to the number of driving factors in the model. This will reduce the number of underlying instruments compared to the replicating portfolio approach and resemble the approach used in practice.

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Appendix

A.1 Forward rate agreements

According to Brigo & Mercurio (2006 pp. 11-12) a forward rate can also be defined by a forward rate agreement (FRA). A FRA is a contract which guarantees the holder an interest-rate payment at a given future time, S, and for a given future time interval $[T, S]$. At maturity S the holder of the FRA pays a fixed rate K and receives in return a floating payment based on the spot LIBOR rate $L(T, S)$ reset at time T. The value of the FRA contract is then equal to the difference between the value of the fixed and the floating payments. Based on a notional amount of 1 the value at time t of a FRA paying a fixed rate K at time S corresponds to:

$$\mathbf{FRA}(t, T, S, \tau(T, S), K) = [P(t, S)\tau(T, S)K - P(t, T) + P(t, S)]$$

The connection with the forward rate resembles the fair value of the contract, i.e. a fair quoted FRA contract corresponds to adjusting the fixed rate K to a level at which the value of the FRA contract is zero, K then corresponds to the forward interest rate $F(t, T, S)$ in equation (3.2).

A.2 The terminal measure

In cases, where we want to price other derivatives than the plain vanilla type of options such as caps and floors, it becomes necessary to conduct the pricing under a single forward measure. Consider the case for some exotic derivative where the payoff depends on the evolution of more than one forward rate, and of the forward rates at dates different from the maturity date of the contract. In this case the joint distribution of the forward rates under one common measure is required.

The terminal measure Q^M is the forward measure with the zero-coupon bond $P(t, T_M)$ as numeraire, where T_M (the terminal time) is the maturity date of the "last" forward rate. I.e. the terminal measure corresponds to the forward measure with the same maturity as the termination of the underlying contract. Only the forward rate with maturity date T_M has zero drift under this measure all other rates have to be adjusted according to equation (4.4) with the drifts defined as in the case where $M = j > i$. I.e. by substituting $j = M$ into equation (4.4) it is possible to write the evolution of the forward rates under the terminal measure Q^M as:

$$\mathrm{d}F_i(t) = -\sigma_i(t)F_i(t) \sum_{k=i+1}^{M} \frac{\rho_{i,k}\tau_k\sigma_k(t)F_k(t)}{1 + \tau_k F_k(t)} \mathrm{d}t + \sigma_i(t)F_i(t)\, \mathrm{d}W_i^M(t)$$

Where the sum is only defined for $i \leq M - 1$. I.e. there is no drift adjustment for the last forward rate F_M(t) (Björk 2004 p.375; Brigo & Mercurio 2006 p. 213).

A.3 Instantaneous correlation

According to Brigo & Mercurio (2006 p. 221) correlations among forward rates have no impact on the prices of interest rate caps because the individual caplets payoff only depends on the development of a single forward rate. Therefore the specification of the instantaneous correlation is of less importance to the main questions of this thesis. On the other hand, the correlation among forward rates matters for derivatives where the payoff is dependent on the joint dynamics of forward rates. Similar to the instantaneous volatility a function can be defined to describe the correlation among forward rates. For this purpose I will use a full rank parameterization, meaning that when modelling M forward rates the correlation matrix $\boldsymbol{\rho}$ is a $M \times M$ matrix.

There are three desirable qualities of the instantaneous correlations (Brigo & Mercurio 2006 chap. 6.9). First, typically positive correlations among all forward rates are preferred. Second, forward rates which are closer in time should correlate more than rates farther apart, meaning that the parameterization should exhibit decreasing correlations when moving away from the diagonal in the correlation matrix. Third, higher correlations among adjacent forward rates of larger tenor are preferred, meaning that the sub-diagonals of the correlation matrix should be increasing.

For the purpose of illustration I will look at one parameterization. Several specifications which meet the qualities described above have been proposed. In the example below I will look at a classic two-parameter parameterization in which the correlation between forward rate $F_i(t)$ and $F_j(t)$ is defined by the expression:

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\, e^{-\beta|i-j|}, \qquad \beta \geq 0$$

Where the parameter ρ_∞ asymptotically describes the correlation among the farthest rates modelled. The figure below illustrates the instantaneous correlations among $M = 15$ forward rates.

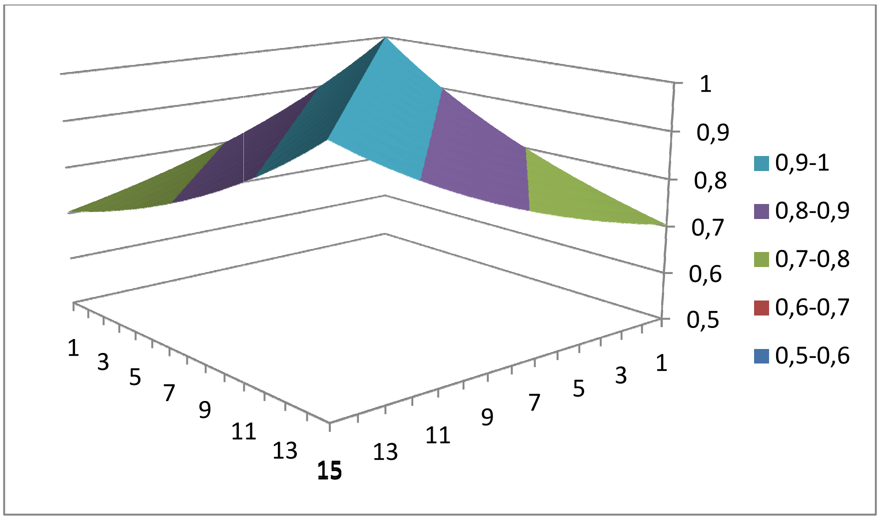


Figure*: Instantaneous correlations between $F_i(t)$ and $F_j(t)$ with the specification given above. The parameters are $\rho_\infty = 0{,}95$ and $\beta = 0{,}2$.*

A.4 Parameter stability in the Hull-White and G2++ models

In order to be consistent with the models assumptions, the time series of model parameters should be constant. Although the parameters are not directly comparable across different models, parameter stability illustrates the relative performance of the Hull-White and the G2++ models. Given the higher number of parameters in G2++ I expect the parameters of this model to be less stable than the parameters of the Hull-White model. Figure A.4.1 displays the time-series of Hull-White model parameters during sample period 1 and figure A.4.2 shows the time-series of G2++ parameters during the same period.

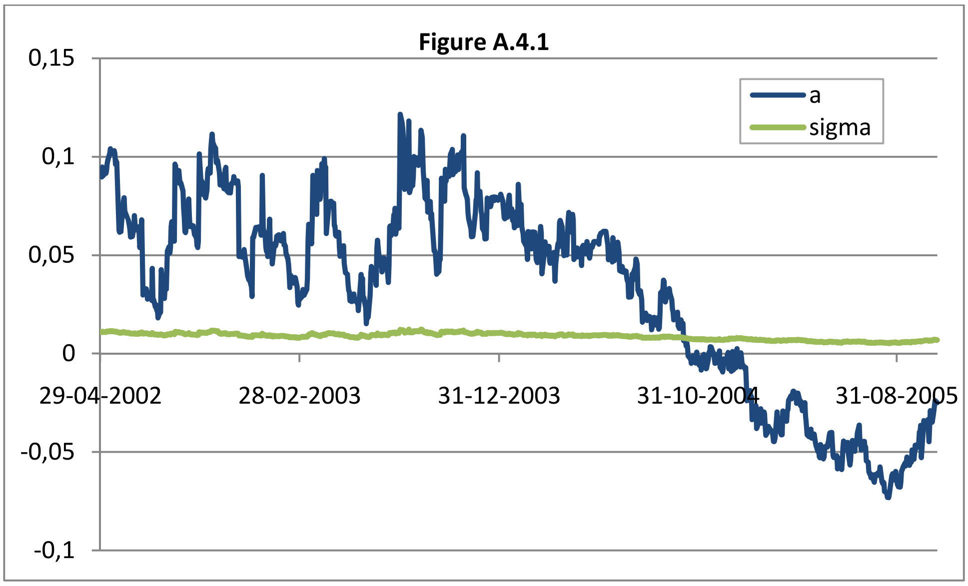


Figure A.4.1*: Hull-White model parameters during sample period 1 (from 2002-04-29 to 2005-10-31). The Hull-White model is calibrated using the standard criterion. The parameter 'sigma' exhibits a stable development during the period whereas the mean reversion speed (a) is volatile.*

The Hull-White volatility parameter (σ) and the correlation parameter in the G2++ model display a stable development, however, the other parameters are not stable over time. The G2++ parameters are less stable than the Hull-White parameters.

Table A.4.1 displays the mean and standard deviation of the model parameters for each period and for different calibration criteria. The standard deviation of the Hull-White model parameters increases in sample period 2 and increases when calibrating the model using the proportional criterion. The standard deviation of the G2++ model parameters increases from sample period 1 to sample period 2, when the model is calibrated using the standard criterion. Moreover, the standard deviation increases in general when using the proportional criterion. However, the result is inconclusive for the G2++ model when comparing the sample periods calibrated with the proportional criterion.

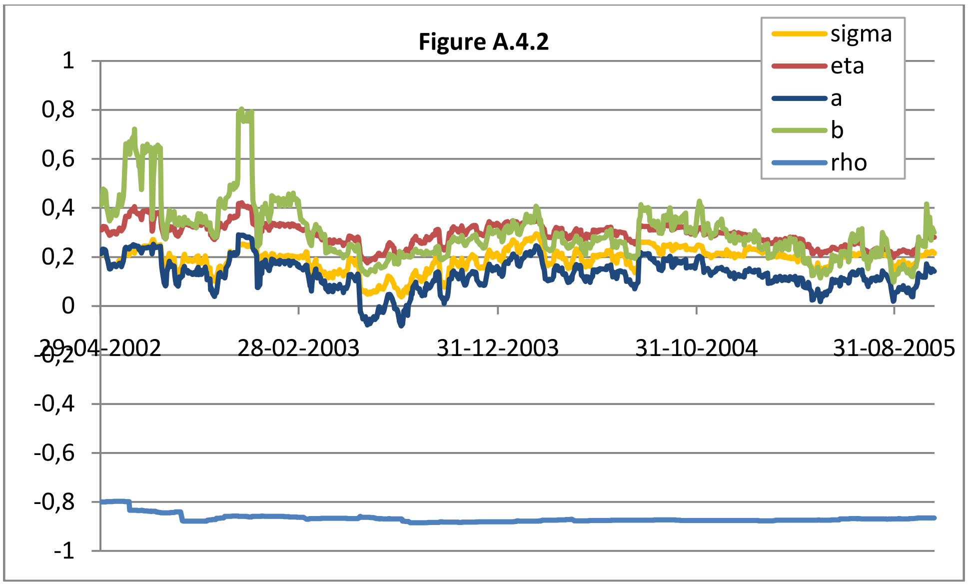


***Figure A.4.2**: G2++ model parameters in sample period 1 (from 2002-04-29 to 2005-10-31). The model is calibrated using the standard criterion.*

Table A.4.1

Period 1:		**Mean squared error**		**Percentage mean squared error**	
	Parameter	Mean	Std.dev	Mean	Std.dev
Hull-White	a	0,03217	0,04896	-0,05882	0,06667
	σ	0,00876	0,00171	0,00700	0,00175
		Mean	Std.dev	Mean	Std.dev
G2++	σ	0,18901	0,04828	0,09458	0,07218
	η	0,29165	0,04623	0,22486	0,09749
	a	0,12878	0,06559	-0,01028	0,18071
	b	0,30274	0,12366	0,31905	0,36262
	ρ	-0,86747	0,01700	-0,82834	0,05889
Period 2:		**Mean squared error**		**Percentage mean squared error**	
	Parameter	Mean	Std.dev	Mean	Std.dev
Hull-White	a	0,07410	0,09143	0,39454	1,86735
	σ	0,00951	0,00309	0,01644	0,04787
		Mean	Std.dev	Mean	Std.dev
G2++	σ	0,11984	0,04300	0,13480	0,03601
	η	0,28601	0,10496	0,29727	0,10045
	a	0,10843	0,07645	0,12684	0,06324
	b	0,54244	0,36463	0,52769	0,36135
	ρ	-0,75859	0,06301	-0,79718	0,00479

Mean and standard deviation of Hull-White and G2++ model parameters given two different calibration criteria and sample periods. The models are calibrated to market prices of ATM interest rate caps. Hull-White start parameters: a = 0.085, sigma = 0.05. G2++ start parameters: sigma = 0.19, eta = 0.3, a = 0.2, b = 0.42, rho = -0.8.

A.5 Test of prediction ability

In order to test the prediction ability of the models and to investigate possible systematic biases of the different models when pricing interest rate caps out-of-sample, I run the following cross-sectional regression:

$$Cap_{mkt,i} = \beta_0 + \beta_1 \cdot Cap_{model,i} + \varepsilon_i$$

Where $Cap_{mkt,i}$ is the market price of cap number i, and $Cap_{model,i}$ is the model price of the corresponding cap i, computed using the lagged model parameters. I.e. I run a simple linear regression of the observed market prices on the estimated model prices. The null hypothesis $\mathrm{H}_0: \beta_1 = 1$ will test the prediction ability of the models, given that the hypothesis cannot be rejected then it cannot be rejected that the model is accurate in predicting interest rate cap prices.[1] Furthermore, the slope parameter β_1 will reveal possible mispricing related to the time-to-maturity of the option price.

I run the regression using the one-day and five-day lagged parameters and for three different cap maturities. The results for a one-day timescale in sample period 1 are presented in table 9.6. The slope in the regression is close to one in almost all cases. However, the overall picture is inconclusive because the null hypothesis is both rejected and accepted. For each model the null hypothesis is rejected at least for one of the cap maturities, hence the null hypothesis cannot be accepted for all maturities within a single model.

The value of the slope parameter indicates that the Hull-White model overvalues short term interest rate caps in period one and that the G2++ model undervalues the short term caps. The regression based on longer time-to-maturity caps displays similar results for the 5 year and 10 year caps in period one and two. The regressions display high explanatory power. This is reflected in high values of $R\^2$, except for the Hull-White model regression based on 1 year caps. According to Amin & Morton (1994) high explanatory power is expected in regressions based on time series of option prices.

[1] See also Christiansen & Hansen (2001) for a similar regression based on different LIBOR market models or Amin & Morton (1994) for a regression based on different HJM models.

Table 9.6

1 year cap:	**$\beta 0$**	**$\beta 1$**	**R^2**	**t-stat**
LIBOR	0,00000	0,99881	0,992	-0,43
	0,00245	*0,00279*		
Hull-White	-0,00020	0,88039	0,575	-1,75
	0,00007	*0,06830*		
G2++	-0,00011	1,06071	0,819	1,76
	0,00003	*0,03440*		
5 year cap:	**$\beta 0$**	**$\beta 1$**	**R^2**	**t-stat**
LIBOR	0,00015	0,99366	0,995	-2,81
	0,00005	*0,00226*		
Hull-White	-0,00018	1,01008	0,989	3,15
	0,00007	*0,00320*		
G2++	0,00010	0,99475	0,990	-1,61
	0,00008	*0,00326*		
10 year cap:	**$\beta 0$**	**$\beta 1$**	**R^2**	**t-stat**
LIBOR	0,00087	0,98548	0,985	-3,68
	0,00023	*0,00395*		
Hull-White	-0,00017	0,99927	0,974	-0,14
	0,00032	*0,00533*		
G2++	0,00074	0,98642	0,976	-3,00
	0,00027	*0,00453*		

Regression results for running the regression in equation EQ for three different interest rate caps. The t-stat represents the t-statistics for the null hypothesis H0: $\beta 1 = 1$. The standard errors are reported in italic and are Newey-West autocorrelation and heteroscedasticity consistent.

Christiansen & Hansen (2001) tested the LIBOR market model based on data of US Treasury bill options and found that the LIBOR market model almost provides an exact prediction of market prices when based on one-day lagged volatility parameters. Amin & Morton (1994), however, tested several different HJM models and rejected the null hypothesis in nine out of twelve regressions.

Clearly, the results from the regressions above, based on data of ATM interest rate caps, indicate that the smallest out-of-sample pricing errors and hence the least model misspecification appears in the lognormal model.

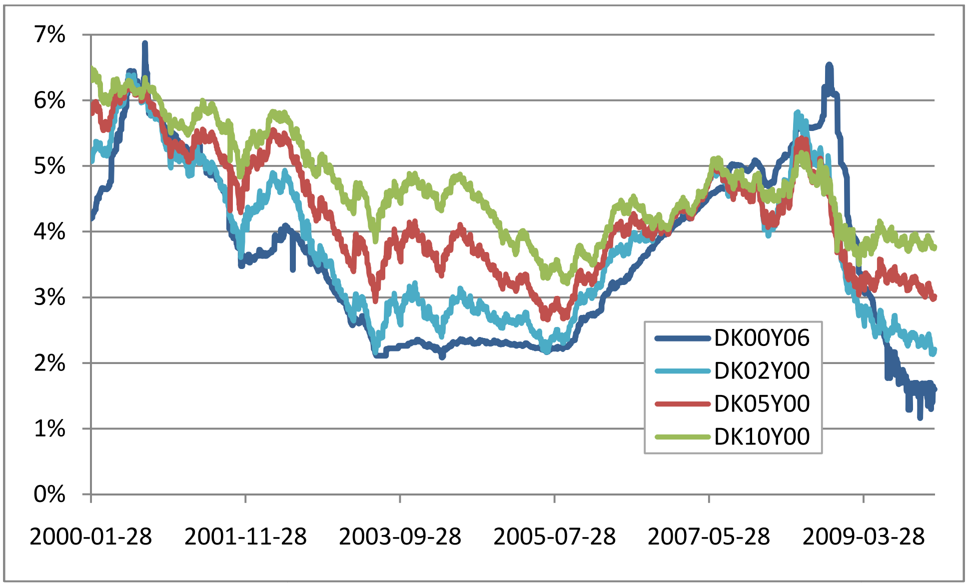
B.1 Summary statistics of interest rate caps

Table B.1: Summary statistics of market prices of interest rate caps

Period 1		Maturity									
	Strike	1 year	2 year	3 year	4 year	5 year	6 year	7 year	8 year	9 year	10 year
Mean	ATM	9	48	103	166	234	305	377	449	519	587
Std		2	11	19	26	32	36	38	40	40	40
Min		3	27	63	111	167	227	293	363	435	506
Max		14	187	239	300	363	430	500	569	631	692
Mean	3%	10	64	159	282	424	580	745	914	1.083	1.246
Std		20	63	108	152	192	228	260	287	311	331
Min		0	4	27	67	123	194	275	368	467	570
Max		101	323	527	767	1.000	1.230	1.446	1.643	1.842	2.022
Mean	4%	2	24	70	138	221	318	424	536	649	761
Std		8	33	63	94	124	153	179	202	222	239
Min		0	0	8	24	51	86	130	185	244	308
Max		53	232	347	481	641	805	960	1.100	1.247	1.377
Mean	5%	0	8	30	65	111	166	229	297	367	438
Std		2	13	29	48	67	85	103	118	132	143
Min		0	0	3	11	25	44	70	103	139	180
Max		17	151	211	288	373	465	563	660	750	834
Period 2		Maturity									
	Strike	1 year	2 year	3 year	4 year	5 year	6 year	7 year	8 year	9 year	10 year
Mean	ATM	13	57	106	160	216	274	333	393	453	513
Std		7	26	36	46	56	65	75	84	92	101
Min		1	25	54	89	129	172	218	264	313	360
Max		37	194	232	275	324	395	474	558	639	721
Mean	3%	65	168	285	408	534	664	797	931	1.068	1.205
Std		41	95	133	157	173	182	189	193	196	198
Min		0	26	85	150	235	333	436	547	663	789
Max		177	392	604	782	941	1.088	1.232	1.379	1.522	1.666
Mean	4%	29	77	136	200	269	343	422	505	590	679
Std		28	58	82	99	111	118	124	128	132	135
Min		0	7	31	60	100	146	201	264	344	426
Max		130	260	398	505	604	695	785	881	974	1.070
Mean	5%	8	23	47	76	109	148	190	236	285	338
Std		15	29	42	52	58	64	68	72	75	77
Min		0	2	11	23	41	64	94	125	161	196
Max		85	164	217	271	326	376	427	481	537	596

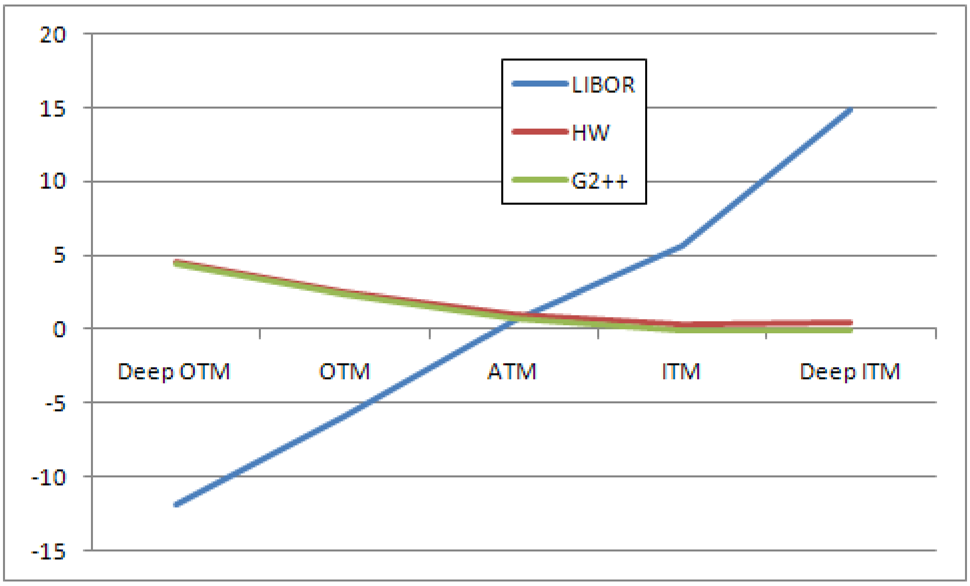
Summary statistics of market interest rate cap prices. Sample period 1 consists of 905 trading days and sample period 2 consists of a total of 733 trading days. Prices are expressed in basis points i.e. 1 basis point implies a cap price of DKK 1 for a notional principal of DKK 10.000. Zero implies a price less than 0.5 bp.

B.2 Danish zero-coupon interest rates



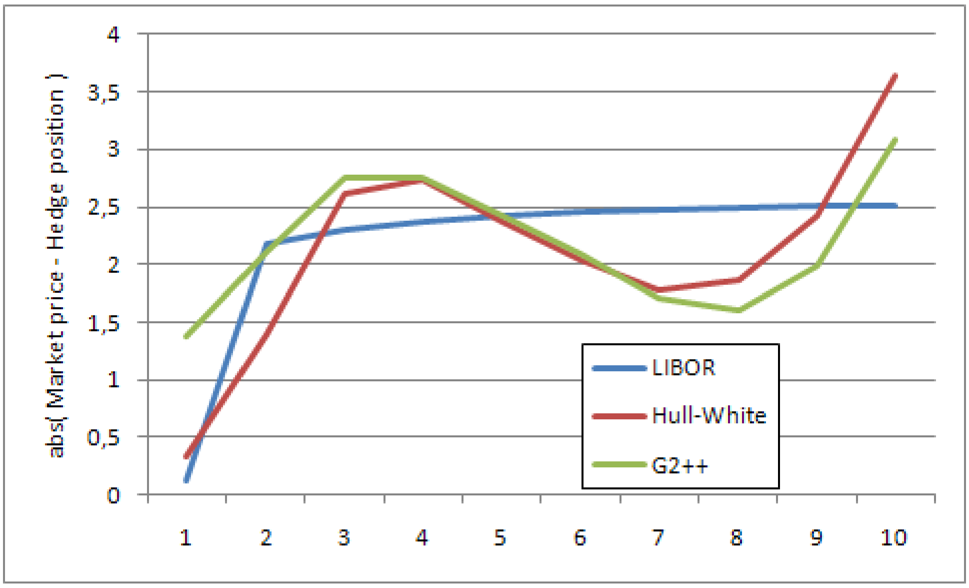
The development of the 6 month, 2 year, 5 year and 10 year time to maturity zero-coupon interest rates from date 29-04-2002 to date 01-12-2009.

B.3 Cross-sectional out-of-sample pricing: Sample period 2



Sample period 2 cross-sectional out-of-sample pricing errors. The models are calibrated to market prices of ATM caps and then implied parameters are used to estimate prices of away-from-the-money options. The average error is measured in basis points and is given by the difference between predicted model price and the observed market price.

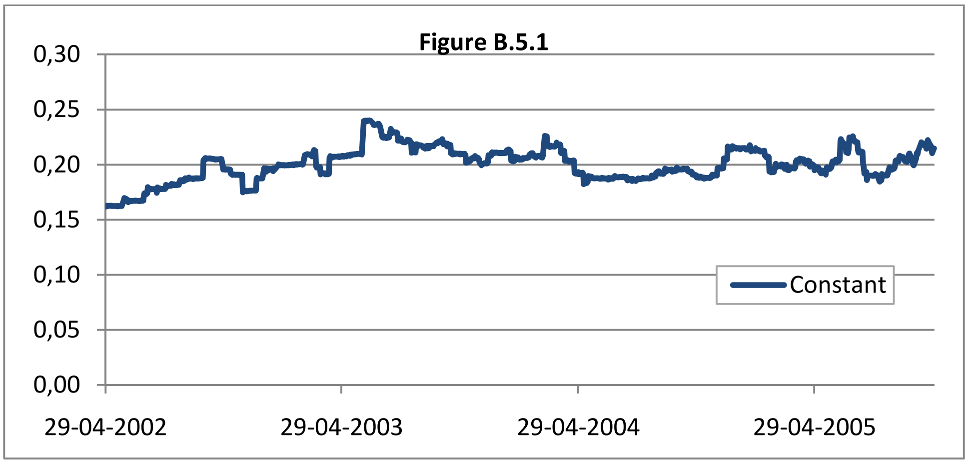
B.4 Sample period 2 average hedging errors

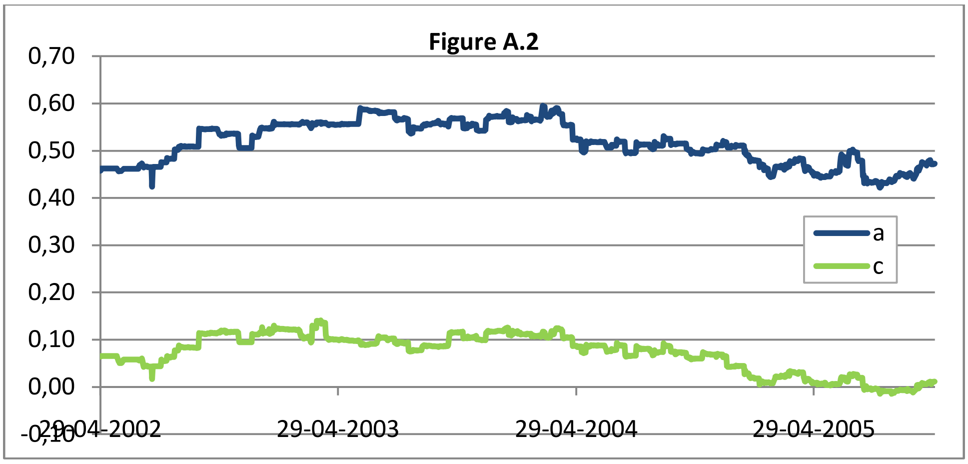


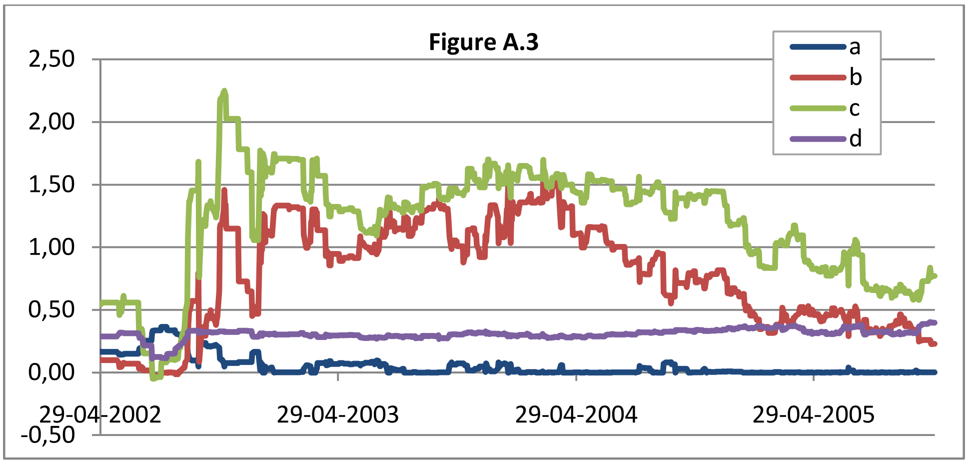
Sample period 2: Average hedging errors one-day out of sample for the Hull-White (green), G2++ (blue) and LIBOR market (red) models. Replicating portfolio hedge of 1 year maturity interest rate cap indexed on 6 month CIBOR. Hedging errors are defined as (market price – value of hedge position).

B.5 Parameter stability – LIBOR market model

The figures below display parameters values of the Constant (figure B.5.1), Exponential (figure B.5.2) and Rebonato (figure B.5.3) parameterizations during sample period 1.

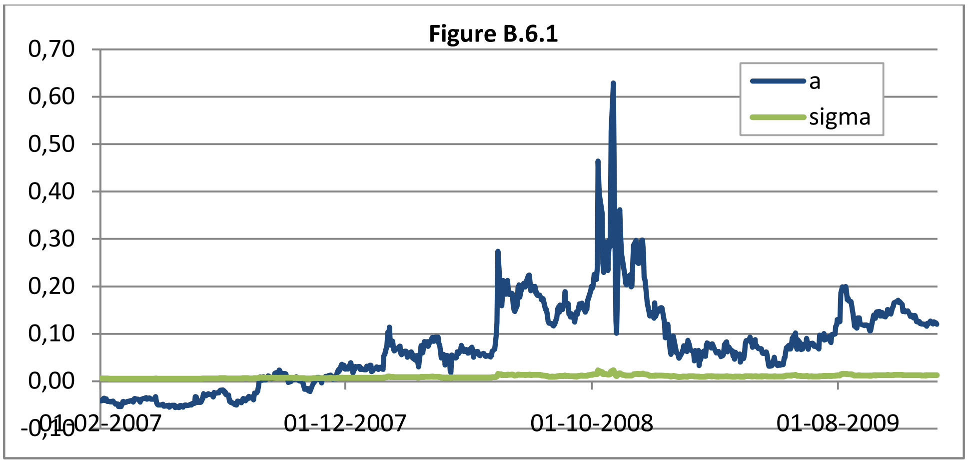






B.6 Parameter stability – Hull-White and G2++

The figures below display parameters values of the Hull-White model (figure B.6.1) and the G2++ model (figure B.6.2) during sample period 2 when calibrating to ATM interest rate caps using least-squares between model and market prices.



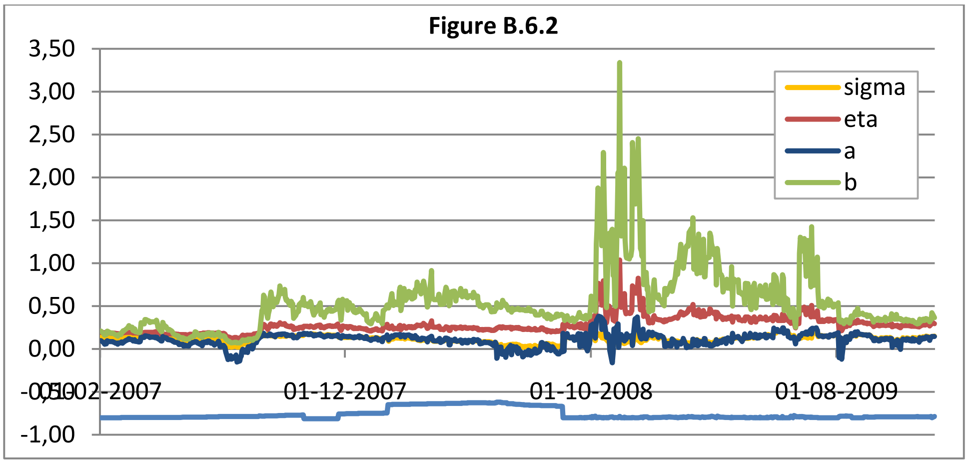


Figure B.6.2
3,50
3,00
2,50
2,00
1,50
1,00
0,50
0,00
-0,50
-1,00
01-02-2007
01-12-2007
01-10-2008
01-08-2009
sigma
eta
a
b

C MATLAB .m-files

C.1 Caplet.m

```
function [cpl] = Caplet(F, K, sigma, T, P, tau)
% Brigo & Mercurio (2006 p. 17 and p. 222)
% F = forward rate from time T_{i-1} to T_{i}
% K = strike rate
% sigma = T_{i-1} caplet volatility
% P = discount factor = P(0,T_{i})
% tau = year fraction
    v = sigma * sqrt(T);
    m = log(F / K);
    Nd1 = normcdf(((m + v^2 /2) / v), 0, 1);
    Nd2 = normcdf(((m - v^2 /2) / v), 0, 1);
    cpl = P * (F*Nd1 - K*Nd2) * tau;
end
```

C.2 Black76_Cap.m

```
function [capPrices] =...
    Black76_Cap(Volatilities, DiscountFactors, ForwardRates, Strikes)
% Black's formula (Brigo & Mercurio, 2006 p.17 and p.222)
% Volatilities = flat or spot volatilities
    [rows cols] = size(vols);
    capPrices = ones(rows, cols);
    cap = 0;
    tms = 0.5:0.5:9.5; % Reset times

    for i = 1:rows
        for j = 1:cols
            K = Strikes(i,j+1);
            for k = 1:j
                sigma = Volatilities(i,k);
                F = ForwardRates(i,k+1);
                P = DiscountFactors(i,k+1);
                T = tms(k);
                cpl = Caplet(F,K,sigma,T,P);
```

```
                cap = cap + cpl;
            end
            capPrices(i,j) = cap;
            cap = 0;
        end
    end
end
```

C.3 HW_cap.m

```
function [Prices] = HW_cap(parametre)
% Hull-White cap formula (Brigo & Mercurio 2006 p. 76)
    global gbl_DiscountfactorVector; % discount factors
    global gbl_StrikeVector; % cap strike levels
    a = parametre(1);
    sigma = parametre(2);
    tms = 0.5:0.5:10; % reset and payment times
    tau = 0.5; % constant year fraction
    Prices = ones(19, 1); % vector of cap prices

    for i = 1:19 % for 1 to 19 = the number of caplets in each cap
        Cap = 0;
        X = gbl_StrikeVector(i+1);
        for j = 1:i
            B = (1/a) * (1-exp(-a*(tau)));
            sigma_ip = sigma * sqrt((1-exp(-2*a*(tms(j)-0))) / (2*a)) *
B;
            h_i = (1/sigma_ip) * log(gbl_DiscountfactorVector(j+1)*...
                (1+X*tau)/gbl_DiscountfactorVector(j)) + sigma_ip/2;
            d1 = -h_i + sigma_ip;
            d2 = -h_i;
            Cpl = gbl_DiscountfactorVector(j)*normcdf(d1,0,1) -...
                (1+X*tau)*gbl_DiscountfactorVector(j+1)*normcdf(d2,0,1);
            Cap = Cap + Cpl;
        end
        Prices(i,1) = Cap;
    end
end
```

C.4 G2_cap.m

```
function [Prices] = G2_cap (parameters)
% G2++ interest rate cap formula (Brigo & Mercurio 2006 pp.156-157)
% The vectors of discount factors and strike rates
% are adjusted accordingly when calling this function from the
% shortRate_Calibration.m file.
    global C_P; % discount factors
    global C_X; % strike rates
    tms = 0.5:0.5:10; % caplet expiry and payment times
    tau = 0.5; % year fraction
    t = 0; % the time at which the price is calculated
    nom = 10000;% the nominal amount
    Prices = ones(19, 1);% vector of cap prices

    sigma = parameters(1);
    eta = parameters(2);
```

```
    a = parameters(3);
    b = parameters(4);
    rho = parameters(5);

    for i = 1:19
        Cap = 0;
        for j = 1:i
            Nx = nom * (1 + C_X(i+1) * tau);
            SIGMA = (sigma^2/(2*a^3)) * (1-exp(-a*(tms(j+1)-
tms(j))))^2*...
                (1-exp(-2*a*(tms(j)-t)))+ (eta^2/(2*b^3)) *...
                (1-exp(-b*(tms(j+1)-tms(j))))^2 * (1-exp(-2*b*(tms(j)-
t)))...
                + (2*rho*sigma*eta/(a*b*(a+b))) *...
                (1-exp(-a*(tms(j+1)-tms(j)))) *...
                (1-exp(-b*(tms(j+1)-tms(j)))) * (1-exp(-(a+b)*(tms(j)-
t)));

            d1 = log((nom * C_P(j))/(Nx * C_P(j+1))) / SIGMA - 0.5 *
SIGMA;
            d2 = log((nom * C_P(j))/(Nx * C_P(j+1))) / SIGMA + 0.5 *
SIGMA;
            Cpl = -Nx*C_P(j+1)*normcdf(d1,0,1) +
C_P(j)*nom*normcdf(d2,0,1);
            Cap = Cap + Cpl;
        end
        Prices(i,1) = Cap;
    end
end
```

C.5 ShortRate_Calibration.m

```
function [parameters] =...
    ShortRate_Calibration(DiscountFactors, Strikes, marketPrices, m, c)
% Calibration of the Hull-White and G2++ models using lsqnonlin
% m = model: 'HW' or 'G2'
% c = calibration criterion: 1 or 2
    global gbl_DiscountfactorVector;
    global gbl_StrikeVector;
    global gbl_market; % vector of cap premiums
    global gbl_model;
    gbl_model = m;
    global gbl_criterion;
    gbl_criterion = c;
    [rows cols] = size(DiscountFactors);
    opt = optimset('MaxIter', 100);

    switch m
        case 'HW'
            initialGuess = [0.085 0.05];
            lb = [-Inf 0.000000001];
            ub = [Inf Inf];
        case 'G2'
            initialGuess = [0.19 0.19 0.42 0.42 -0.8];
            lb = [-Inf -Inf -Inf -Inf -1];
            ub = [Inf Inf Inf Inf 1];
    end
```

```
    for i = 1:rows
        gbl_market = marketPrices(i,:);
        gbl_DiscountfactorVector = DiscountFactors(i,:);
        gbl_StrikeVector = Strikes(i,:);

        [x,resnorm,FVAL,Exitfalg,output] =...
            lsqnonlin(@shortRateCalibrationCriteria, initialGuess, lb,
ub, opt);

        parameters(i,:) = x; % calibrated parameter values
        initialGuess = x; % adjust the initial guess
    end
end
```

C.6 LMM _calibration.m

```
function [parameters] = LMM_calibration(CplVol, type, c)
% LIBOR market model calibration using lsqnonlin
% type: 1 = Constant, 2 = Exponential, 3 = Rebonato
    global gbl_market;
    global gbl_model;
    gbl_model = type;
    global gbl_criterion;
    gbl_criterion = c;
    [rows cols] = size(CplVol);

    % Define maximum number of iterations
    opt = optimset('MaxIter',2500,'MaxFunEvals', 2500);
    % Set initial guess, lower and upper boundary on function parameters
    switch type
        case 1 % Constant
            lb = -Inf;
            ub = Inf;
            initialGuess = 0.19;
        case 2 % Exponential
            lb = [-Inf -Inf];
            ub = [Inf Inf];
            initialGuess = [0.1 0.1];
        case 3 % Rebonato
            lb = [0 -Inf -Inf 0];
            ub = [Inf Inf Inf Inf];
            initialGuess = [0.06 -0.01 0.45 0.1];
        otherwise
    end

    for i = 1:rows
        gbl_market = CplVol(i,:);
        [x] = lsqnonlin(@LMM_Criteria, initialGuess, lb, ub, opt);
        parameters(i,:) = x;
        initialGuess = x;
    end
end
```

C.7 Calibration_Criteria.m

```
function [err] = Calibration_Criteria(parameters)
% standard least squares or percentage least squares
    global gbl_market; % market cap prices or caplet volatilities
    global gbl_model;
    global gbl_criterion;

    switch gbl_model
        case 1
            model = LMM_Const(parameters);
        case 2
            model = LMM_Exp(parameters);
        case 3
            model = LMM_Reb(parameters);
        case 'HW'
            model = HW_cap(parameters);
        case 'G2'
            model = G2_cap(parameters);
    end

    switch gbl_criterion
        case 1
            err = model - gbl_market;
        case 2
            err = (model - gbl_market) ./gbl_market;
    end
end
```

C.8 LMM _ConstantVolatility.m

```
function [vol] = LMM_ConstantVolatility(v)
% Constant volatility parameterization
    for i = 1:19
        vol(i) = v;
    end
end
```

C.9 LMM _ExponentialVolatility.m

```
function [vol] = LMM_Exp(parameters)
%% Exponential volatility parameterization cf. equation (4.3)
    a = parameters(1);
    c = parameters(2);
    t = 0;
    tms = 0.5:0.5:9.5; % For each expiry time
    for i = 1:19
        T = tms(i);
        Cplvol = 0.5 * a ^ 2 * (1 - exp(-2 * c * (T - t))) / c;
        vol(i) = Cplvol/T;
    end
end
```

C.10 LMM _RebonatoVolatility.m

```
function [vol] = LMM_RebonatoVolatility(vector)
%% Rebonato parameterization cf. equation (4.3)

    a = vector(1);
    b = vector(2);
    c = vector(3);
    d = vector(4);

    t = 0;
    tms = 0.5:0.5:9.5; % For each expiry time

    for i = 1:19
        T = tms(i);

        part1 = (1 / (4 * c ^ 3)) * exp(-2 * c * T) * (exp(2 * c * T)
...
            * (2 * c ^ 2 * (T - T) ^ 2 + 2 * c * (T - T) + 1) * b ^ 2
...
            - 2 * c * exp(c * T) * (4 * d * exp(c * T) * (-c * T + c * T
- 1)...
            + a * exp(c * T) * (-2 * c * T + 2 * c * T - 1)) * b ...
            + 2 * c ^ 2 * (exp(2 * c * T) * a ^ 2 + 4 * d * exp(c * (T +
T))...
            * a + 2 * c * d ^ 2 * exp(2 * c * T) * T));

        part2 = -( (1 / (4 * c ^ 3)) * exp(-2 * c * T)...
            * (exp(2 * c * t) * (2 * c ^ 2 * (T - t) ^ 2 +...
            2 * c * (T - t) + 1) * b ^ 2 - 2 * c * exp(c * t) * ...
            (4 * d * exp(c * T) * (-c * T + c * t - 1)...
            + a * exp(c * t) * (-2 * c * T + 2 * c * t - 1)) * b ...
            + 2 * c ^ 2 * (exp(2 * c * t) * a ^ 2 + 4 * d *...
            exp(c * (T + t)) * a + 2 * c * d ^ 2 * exp(2 * c * T) *
t)));

        Cplvol = part1 + part2;
        vol(i) = Cplvol/T;
    end
end
```

C.11 LMM _HedgeRatios.m

```
function [h1_vec h2_vec] =  LMM_HedgeRatios...
    (i,tms,cplVolMarket,cplVolModel, R_fwd, X, ZCB)
% i = number of caplets in the cap
     for j = 1:i
        T = tms(j);
        m = log(R_fwd(k,j+1) / X);
        v = cplVolModel(k,j) * sqrt(T);
        Nd1_2 = normcdf(((m + v^2 /2) / v),0,1);
        Nd2_2 = normcdf(((m - v^2 /2) / v),0,1);

        % duplicating portfolio hedge ratios:
        h1 = Nd1_2; % amount in bond P(t,T_i-1)
        h2 = -(h1 + X*0.5*Nd2_2); % amount in bond P(t,T_i)
```

```
        % alternative which account for non-exact fit:
        cpl = caplet(R_fwd(k,j+1),X,cplVolMarket(k,j),T,ZCB(k,j+1));
        h1 = (cpl - h2*ZCB(k,j+1) ) / ZCB(k,j);

        % Place them in a vector
        h1_vec(j) = h1; % short term bonds
        h2_vec(j) = h2; % the longer term bonds
    end
end
```

Note: Hull-White and G2++ hedge ratios are obtained directly from the HW_cap and G2_cap functions.