Pricing Formula for Geometric Asian Options

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1 Discrete Case

Consider a geometric Asian option with strike price K and maturity date T that initiates at time 0. Let S_t stand for the price process of its underlying asset, and r, the risk-free interest rate. In discrete monitored cases assume the price of the underlying asset is sampled $n+1$ times at time $t_0, t_1, t_2, \ldots, t_n$, where $t_0 = 0$ and $t_i - t_{i-1} = T/n \equiv \Delta t$ for all $i = 1, 2, \ldots, n$. In the Black-Scholes framework, the fair price of this geometric Asian option at time 0 is given by

$$C^{(n)} \equiv e^{-rT}\mathbb{E}\left[(G^{(n)} - K)^+\right],$$

where

$$G^{(n)} = \sqrt[n+1]{S_{t_0} S_{t_1} S_{t_1} \cdots S_{t_n}}.$$

Setting $X_i \equiv S_{t_i}/S_{t_{i-1}}, \forall i = 1, 2, \ldots, n$, we have

$$\ln X_i \sim N\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right),$$

and

$$\begin{aligned} G^{(n)} &= S_0 \sqrt[n+1]{X_1^n X_2^{n-1} X_3^{n-2} \cdots X_n} \\ &= S_0 e^{\left(\frac{n}{n+1}\right)\ln X_1 + \left(\frac{n-1}{n+1}\right)\ln X_2 + \left(\frac{n-2}{n+1}\right)\ln X_3 + \cdots + \left(\frac{1}{n+1}\right)\ln X_n}, \end{aligned}$$

where

$$\begin{aligned} &\left(\frac{n}{n+1}\right)\ln X_1 + \left(\frac{n-1}{n+1}\right)\ln X_2 + \left(\frac{n-2}{n+1}\right)\ln X_3 + \cdots + \left(\frac{1}{n+1}\right)\ln X_n \\ \sim\ & N\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t\left(\frac{1}{n+1}\sum_{k=1}^{n} k\right), \sigma^2 \Delta t\left(\frac{1}{(n+1)^2}\sum_{k=1}^{n} k^2\right)\right) \\ \sim\ & N\left(\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}, \frac{2n+1}{6(n+1)}\sigma^2 T\right). \end{aligned}$$

Substituting

$$\begin{aligned} A &= \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}, \\ B &= \sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T} \end{aligned}$$

into the general formula (see the notes on Merton's jump-diffusion model), we obtain

$$\begin{aligned} C^{(n)} &= e^{-rT}\left(S_0 e^{A+\frac{B^2}{2}} N\left(\frac{\ln\frac{S_0}{K}+A}{B}+B\right) - KN\left(\frac{\ln\frac{S_0}{K}+A}{B}\right)\right) \\ &= S_0 e^{-\left(r+\frac{n+2}{6(n+1)}\sigma^2\right)\frac{T}{2}} N\left(\frac{\ln\frac{S_0}{K}+\left(r+\frac{n-1}{6(n+1)}\sigma^2\right)\frac{T}{2}}{\sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T}}\right) - Ke^{-rT}N\left(\frac{\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}}{\sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T}}\right). \end{aligned}$$

2 Continuous Case

When n tends to infinity, the last expression tends to the pricing formula in continuous case

$$C = S_0 e^{-\left(r+\frac{\sigma^2}{6}\right)\frac{T}{2}} N\left(\frac{\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{6}\right)\frac{T}{2}}{\sigma\sqrt{T/3}}\right) - Ke^{-rT}N\left(\frac{\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}}{\sigma\sqrt{T/3}}\right).$$