Master Course Exercises
and Implicit Mock-up Exam
FINANCIAL ENGINEERING - BS1031
MSc in Risk Management and Financial Engineering

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IMPLICIT MOCK-UP EXAM

This set of exercises reflects in good measure what the students may expect at the exam.

The exam will consist in problems similar to those solved here.

Solving exactly all problems will lead to a total mark of 100.

Students are advised to study carefully these exercises and related solutions.

Most solutions will be discussed during tutorials.

Exercise + Solutions: Absence of Arbitrage in Black Scholes I

Consider the Black and Scholes basic economy given by a bank account and a stock, whose prices are given respectively by

$$dB_t = rB_t dt, \;\; B_0 = 1, \quad dS_t = \mu S_t dt + \sigma S_t dW_t, \; S_0 = 1$$

where r, μ, σ are positive constants and W is a brownian motion under the physical measure P.

Consider the limiting case where $\sigma = 0$, ie the stock has no volatility.

Prove that if $\mu \neq r$ then one has arbitrage.

Exercise + Solutions: Absence of Arbitrage in Black Scholes II

SOLUTION. To prove arbitrage we need to prove that there exists at least one arbitrage opportunity, ie a self-financing trading strategy (ϕ^B, ϕ^S) such that

$$\phi_0^B B_0 + \phi_0^S S_0 = 0, \;\; \phi_T^B B_T + \phi_T^S S_T > 0 \;\; a.s.$$

for some maturity $T > 0$.

Let us start for the case where $\mu > r$. We may easily establish an arbitrage opportunity as follows.
At time 0 we buy $\phi_0^S = 1$ unit of stock and short-sell 1 unit of bank account: $\phi_0^B = -1$.
This has cost zero as both position have a value of 1 at time 0:

$$\phi_0^B B_0 + \phi_0^S S_0 = (-1)\, 1 + 1 \cdot 1 = 0.$$

Exercise + Solutions: Absence of Arbitrage in Black Scholes III

Integrating the bank account and stock equations

$$dB_t = rB_t dt, \quad dS_t = \mu S_t dt$$

yields $B_t = e^{rt}$ and $S_t = e^{\mu t}$.
Since $\mu > r$, our strategy is just to wait, as S grows with μ and B with r.
So now our strategy is to hold onto the initial amounts without buying or selling anything.

$$\phi_t^B = -1, \ \phi_t^S = 1 \quad \text{for all} t, \quad 0 < t < T.$$

It is obviously self financing as this does not involve any external funding or any extraction of money from the accounts.
At maturity we have

Exercise + Solutions: Absence of Arbitrage in Black Scholes IV

$$\phi_T^B B_T + \phi_T^S S_T = (-1)\, e^{rT} + 1 \cdot e^{\mu T} > 0$$

since $\mu > r$.

If $\mu < r$ we do the opposite: at time zero we buy one unit of Bank account and go short one unit of stock, and then just wait. The reasoning is analogous.

This exercise shows an important point: we cannot have two bank accounts accruing at different rates, or we have immediately arbitrage.

Also, note that in this case the Radon Nykodym derivative dQ/dP would not be well defined.

Exercise + Solutions: Put Call Parity Violation and Arbitrage I

Consider the Black and Scholes basic economy given by a bank account and a stock, whose prices are given respectively by

$$dB_t = rB_t dt, \quad B_0 = 1, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t, \; S_0 = 1$$

where r, μ, σ are positive constants and W is a brownian motion under the physical measure P.

Consider two options that are at the money forward (namely having strike $K = S_0 e^{rT}$), respectively a call option and a put option with maturity T and payout

$$(S_T - K)^+ = max(S_T - K, 0) \text{ (Call)}, \quad (K - S_T)^+ = max(K - S_T, 0) \text{ (Put)}.$$

Exercise + Solutions: Put Call Parity Violation and Arbitrage II

In this case, put call parity tells us that the initial value of the call option, at time 0, must be equal to the initial value of the put, given that the forward contract value is zero (K is the at-the-money forward strike that sets the forward price to zero).

Show that if this condition is violated, and for example

$$\text{CallPrice}_0 = \text{PutPrice}_0 + X$$

for a positive amount $X > 0$, one has arbitrage.

Exercise + Solutions: Put Call Parity Violation and Arbitrage III

SOLUTION.
Since the price of the call is larger than the one of the put when they should actually be the same, we can try by buying a put and short-selling a call, and buying an at the money forward contract to balance put minus call at maturity, buying some bank account with the difference between the call and the put at time 0.

We enter into one position in a put option at time 0, and short sell one call option at time 0, both options with strike K and maturity T. We also enter into a forward contract at the same maturity with strike K. This means that we accept to receive $S_T - S_0e^{rT}$ at maturity (meaning that if this quantity is positive we receive it, if it is negative we pay its absolute value to our counterparty in the trade). Then we also buy an amount X of bank account.

Exercise + Solutions: Put Call Parity Violation and Arbitrage IV

The cost of starting this strategy is:

- We pay PutPrice_0 to enter into the put option
- we receive $\text{CallPrice}_0 = \text{PutPrice}_0 + X$ by short-selling the Call option
- We pay X to buy a quantity X of bank account B_0 at time 0.
- We pay nothing to enter into the forward contract since its initial cost is $S_0 - Ke^{-rT} = 0$.

Exercise + Solutions: Put Call Parity Violation and Arbitrage V

These four operations have a total cost of

$$\text{PutPrice}_0 - \text{CallPrice}_0 + X + 0 = 0$$

So it costs nothing to set up the strategy.
Following the initial setup, we just wait. This clearly preserve the self financing condition since we do not inject external funds or extracts funds from the strategy.

At maturity, we have the following cash flows:

- We receive $(K - S_T)^+$ from the Put option.
- We pay $(S_T - K)^+$ for the Call option we have been short-selling
- We receive $S_T - K$ from the forward contract

Exercise + Solutions: Put Call Parity Violation and Arbitrage VI

- We have Xe^{rT} in the bank account.

The total value of this strategy at T is hence

$$\text{PutPayout}_T - \text{CallPayout}_T + \text{FwdContractPayout}_T + XB_T =$$
$$(K - S_T)^+ - (S_T - K)^+ + S_T - K + Xe^{rT} =$$
$$= K - S_T + S_T - K + Xe^{rT} = Xe^{rT} > 0$$

So we have a self-financing trading strategy whose initial cost is zero and that produces a positive final cash flow Xe^{rT} in all scenarios. Hence this is an arbitrage opportunity and the market is arbitrageable.

Exercise + Solution: FRA I

Consider a Forward Rate Agreement where one party agrees to a fixed rate of 2% in exchange for the 180-day Libor rate prevailing in the market in 90 days time. This is typically a 3×9 FRA contract since its expiration is in 3 months (90 days) and the payment occurs in a further 6 months (so the maturity is 9 months from inception).

Assume further that after 90 days the 6-month Libor rate is 3%. Compute the payment that our original party has to make.

Exercise + Solution: FRA II

SOLUTIONS: It seems that the original party is paying the constant rate, so from this point of view it is a payer FRA. The amount paid is: $\tau(K - L(90, 270))$, where the fraction of the year is $\tau = 180/360$ (half a year). Then:

$$\frac{180}{360}(0.02 - 0.03) = -0.005.$$

This amount is negative, so the party is actually receiving this amount, which is equivalent to locking the fixed interest rate and getting compensated when the market rate is above that.
The amount is small, but usually there would be a large notional amount multiplying it.

In practice the settlement may happen already at expiration (earlier than the maturity date), when all quantities are known. The calculation would have an extra discount factor involving the Libor rate and the time difference.

Exercise + Solutions: FRA replication I

Replicating FRA with par zero bonds and Libor

Show that the FRA can be replicated with a portfolio consisting of zero coupon bonds with maturities T and S. In particular, prove that the price of the FRA at any time $t < T$ can be obtained as the time t value of the replicating portfolio.

Exercise + Solutions: FRA replication II

Replicating FRA with par zero bonds and Libor

SOLUTIONS:

Construct a portfolio:

_ buy zero coupon bond with maturity T, paying a face value $1 + K\tau$ at S. Its price at time 0 is $(1 + K\tau)P(0, S)$, and at any time t is $(1 + K\tau)P(t, S)$.

_ sell a zero coupon bond with maturity T (and get $P(0, T)$ at time 0, which is worth $P(t, T)$ at time t), and then at time T the proceeds of the bond are invested in a bank account earning Libor rate $L(T, S)$, resulting in $(1 + \tau L(T, S))$ at time S.

Exercise + Solutions: FRA replication III

Replicating FRA with par zero bonds and Libor

The value of this portfolio at time S is $(1 + K\tau) - (1 + \tau L(T, S)) = \tau K - \tau L(T, S)$, equal to the payoff of the FRA.

By no-arbitrage and the law of one price the value of the portfolio at time t, which is $(1 + K\tau)P(t, S) - P(t, T)$, should match the FRA price.

But this is exactly what was calculated in class as $FRA(t, T, S, K)$.

Exercise + Solutions: Zero coupon bonds and rates I

Assume that the zero coupon bond term structure is given by

$$P(0,T) = \exp(-kT^2), \;\; 0 < k \quad (*)$$

a) Write an expression for the LIBOR spot rate $L(0,T)$ as a function of T.

b) Write an expression for the LIBOR forward rate $F(0,T,T+1)$ as a function of T.

c) Study $F(0,T,T+1)$ as a function of k. Is it increasing, decreasing, concave, convex, how does it behave at 0 and infinity?

d) Compute the swap rate $S(0,0,2) = S_{0,2}(0)$ for a swap with annual tenor in both legs, and compute the limits for k going to zero and infinity of this swap rate.

Exercise + Solutions: Zero coupon bonds and rates II

e) The continuously compounded spot rate at time t for maturity T is defined as

$$R(t, T) = -\frac{1}{T - t} \ln P(t, T).$$

Find an explicit expression for $R(0, T)$ when the bond follows (*) above.

f) By making use of point e), find the short rate r_0 at time 0.

Exercise + Solutions: Zero coupon bonds and rates III

SOLUTIONS:

a)

$$L(0,T) = \frac{1}{T}(1/P(0,T) - 1) = \frac{1}{T}(1/e^{-kT^2} - 1) = \frac{1}{T}(e^{kT^2} - 1)$$

b)

$$F(0,T,T+1) = \frac{1}{T+1-T}(P(0,T)/P(0,T+1) - 1) =$$
$$= (e^{-kT^2}/e^{-k(T+1)^2} - 1) = (e^{k(2T+1)} - 1)$$

Exercise + Solutions: Zero coupon bonds and rates IV

c) The map

$$k \mapsto \frac{1}{T}(e^{k(2T+1)} - 1)$$

is increasing, convex, at infinity goes to infinity, at 0 goes to 0.

d)

$$S(0,0,2) = \frac{1 - P(0,2)}{P(0,1) + P(0,2)} = \frac{1 - e^{-4k}}{e^{-k} + e^{-4k}} = \frac{e^{4k} - 1}{e^{3k} + 1}$$

This tends to 0 for k going to 0, and to infinity for k going to infinity.

e)

$$R(0,T) = kT$$

f)

$$r_0 = \lim_{T\downarrow 0} R(0,T) = \lim_{T\downarrow 0} kT = 0$$

Exercise with Solutions: Spot, fwd and swap rates I

Consider the following curve of zero coupon bonds for the maturities $T_0 = 1y, T_1 = 2y, \ldots, T_9 = 10y$:

$$P(0, T_0) = 0.961538462 \quad P(0, T_1) = 0.924556213$$

$$P(0, T_2) = 0.888996359 \quad P(0, T_3) = 0.854804191$$

$$P(0, T_4) = 0.821927107$$

a) Compute the forward swap rates $S_{1,4}(0)$ and $S_{2,4}(0)$.
b) Compute the forward libor rates $F_1(0), F_2(0), ...$ and verify that the swap rate $S_{2,4}(0)$ is a weighted average of the forward LIBOR rates (in particular, compute the weights).

Exercise with Solutions: Spot, fwd and swap rates II

SOLUTIONS:

a) We have an annual tenor structure where $T_i - T_{i-1} = 1$ year. This is important in defining all rates.

$$S_{1,4}(0) = \frac{P(0,T_1) - P(0,T_4)}{1\ P(0,T_2) + 1\ P(0,T_3) + 1\ P(0,T_4)} \approx 0.0399999999$$

Similarly we obtain

$$S_{2,4}(0) = \frac{P(0,T_2) - P(0,T_4)}{1\ P(0,T_3) + 1\ P(0,T_4)} \approx 0.0400000000$$

Substitution of the given values for the P's gives the result.

b) We know that

$$S_{2,4}(0) = w_3(0)F_3(0) + w_4(0)F_4(0),$$

Exercise with Solutions: Spot, fwd and swap rates III

where

$$w_3(0) = P(0, T_3)/(P(0, T_3) + P(0, T_4)) \approx 0.5098039215,$$

$$w_4(0) = P(0, T_4)/(P(0, T_3) + P(0, T_4)) \approx 0.4901960785.$$

Also,

$$F_3(0) = \left(\frac{P(0, T_2)}{P(0, T_3)} - 1 \right) \approx 0.0400000005,$$

$$F_4(0) = \left(\frac{P(0, T_3)}{P(0, T_4)} - 1 \right) \approx 0.0399999997.$$

Substitution of the given values for the P's gives the result.

Exercise with Solution: Transformations of the Vasicek model I

Consider the SDE for the Vasicek model

$$dx_t = k(\theta - x_t)dt + \sigma dW_t,\ x_0$$

where W is a brownian motion under the risk neutral measure.
a) Assume $r_t = x_t + 0.01$. Compute a formula for the risk neutral probability that the short rate is negative at a given time T.
b) Assume $\theta = 0$ and set $r_t = (x_t)^3$. Find the stochastic differential equation for this short rate.

Exercise with Solution: Transformations of the Vasicek model II

SOLUTIONS: a) We know that in the Vasicek model the short rate at time t conditional on info at time s is normally distributed with mean and variance given by

$$E[x(t)|x(s)] = x(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\mathrm{VAR}[x(t)|x(s)] = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right].$$

Then the short rate $r_t = x_t + 0.01$ at time T conditional on info at time 0 is normally distributed

$$r_T \sim \mathrm{Normal}(0.01 + m_T, V_T^2)$$

Exercise with Solution: Transformations of the Vasicek model III

where

$$m_T = x(0)e^{-kT} + \theta\left(1 - e^{-kT}\right), \quad V_T^2 = \frac{\sigma^2}{2k}\left[1 - e^{-2kT}\right]$$

The probability that a normally distributed random variable is negative:

$$Q\{r_T < 0\} = Q\{\text{Normal}(0.01 + m_T, V_T^2) < 0\} =$$

$$= Q\{0.01 + m_T + V_T\text{Normal}(0,1) < 0\} =$$

$$= Q\{\text{Normal}(0,1) < -(0.01 + m_T)/V_T\} = \Phi(-(0.01 + m_T)/V_T)$$

b). Apply Ito's formula with $f(x) = x^3$ to get

$$d(x_t)^3 = 3x_t^2 dx_t + \frac{1}{2}3 \cdot 2x_t \cdot dx_t dx_t =$$

Exercise with Solution: Transformations of the Vasicek model IV

$$= 3x_t^2 k(\theta - x_t)dt + 3x_t^2 \sigma dW_t + 3x_t \sigma^2 dt$$

$$d(x_t)^3 = 3[x_t^2 k(\theta - x_t) + x_t \sigma^2]dt + 3x_t^2 \sigma dW_t$$

Set $x_t^3 = r_t$, so that $x_t = r_t^{1/3}$. We have

$$dr_t = 3[r_t^{2/3} k\theta - kr_t + r_t^{1/3}\sigma^2]dt + 3r_t^{2/3}\sigma dW_t$$

Setting $\theta = 0$ gives the SDE:

$$dr_t = 3[-kr_t + r_t^{1/3}\sigma^2]dt + 3r_t^{2/3}\sigma dW_t$$

Exercise with Solution: Two-factor Vasicek I

Consider two factors, each following an SDE for the Vasicek model

$$dx_t = k(\theta - x_t)dt + \sigma dW_t,\ x_0 > 0, \quad dy_t = k(\theta - y_t)dt + v dW_t,\ \ y_0 = x_0$$

where W is a Brownian motion under the risk neutral measure, k and θ are non-negative real constants, and σ and v are positive real constants.

a) Assume $r_t = x_t - y_t + \alpha(m - \exp(-\alpha t))$, for a positive real constant $\alpha > 0$ and a positive real constant $m > 0$. Derive a stochastic differential equation for r_t.

b) Write the probability distribution of the short rate r given in point a) at time t conditional on the short rate at time s, $0 \leq s < t$.

Exercise with Solution: Two-factor Vasicek II

c) Compute a formula for the risk neutral probability that the short rate r_t is negative at time t.

d) Study the probability in point c) at the two time extremes $t \downarrow 0$ and $t \uparrow +\infty$ and comment on your findings.

e) Now set $r_t = (x_t)^3 - (y_t)^3$. Compute the probability that the short rate is negative at time t. Pay particular attention to the initial condition for r_t based on the above assumptions on x and y.

Exercise with Solution: Two-factor Vasicek III

Solutions.
a) We have

$$dr_t = dx_t - dy_t + \alpha^2 \exp(-\alpha t)dt = -k(x_t - y_t)dt + \alpha^2 \exp(-\alpha t)dt + (\sigma - v)dW_t$$

hence

$$dr_t = -kr_t dt + k\alpha(m - \exp(-\alpha t))dt + \alpha^2 \exp(-\alpha t)dt + (\sigma - v)dW_t.$$

For calculations however it is best to work with the process

$$z_t = x_t - y_t$$

that follows the SDE

$$dz_t = -kz_t dt + (\sigma - v)dW_t, \quad z_0 = 0$$

Exercise with Solution: Two-factor Vasicek IV

which is a Vasicek process starting at 0 with long term mean 0, and then use

$$r_t = z_t + \alpha(m - \exp(-\alpha t)).$$

b) Since our model z is a special case of Vasicek, we can use the results of Vasicek and then add the shift.
We know that in the Vasicek model the short rate at time t conditional on time s is normally distributed with mean and variance given by

$$E[z(t)|z_s] = z(s)e^{-k(t-s)}$$

$$\mathrm{VAR}[z(t)|z_s] = \frac{(\sigma - v)^2}{2k}\left[1 - e^{-2k(t-s)}\right] =: V_{t|s}^2.$$

Exercise with Solution: Two-factor Vasicek V

Adding the deterministic shift only changes the mean of the normal distribution:

$$m_{t|s} := E[r(t)|r_s] = z(s)e^{-k(t-s)} + \alpha(m - \exp(-\alpha t)) =$$

$$= [r_s - \alpha(m - \exp(-\alpha s))]e^{-k(t-s)} + \alpha(m - \exp(-\alpha t))$$

while the variance is the same as z's. Thus $r_t|r_s \sim N(m_{t|s}, V^2_{t|s})$.

c) Set $s = 0$ and use the above distribution to find that

$$Q(r_t < 0) = Q(\mathcal{N}(m_{t|0}, V^2_{t|0}) < 0) = Q(m_{t|0} + V_{t|0}\mathcal{N}(0,1) < 0) =$$

$$= Q(\mathcal{N}(0,1) < -m_{t|0}/V_{t|0}) = \Phi(-m_{t|0}/V_{t|0}) = \Phi\left(\frac{-\alpha(m - \exp(-\alpha t))}{\sqrt{\frac{(\sigma - v)^2}{2k}\left[1 - e^{-2kt}\right]}}\right)$$

Exercise with Solution: Two-factor Vasicek VI

d) For $t \downarrow 0$ we get easily (fill in the calculations)

$$\Phi(\alpha(1-m)/0^+) = 1 \text{ if } m < 1, \quad 0 \quad \textit{if} \quad m > 1,$$

while for $m = 1$ we have an indeterminate $0/0$ limit. Applying l'Hôpital's rule we see immediately that the limit is zero in this case (verify this limit).
For $t \uparrow \infty$ we get easily (check)

$$\Phi(-\alpha m\sqrt{2k}/|\sigma - v|)$$

Comment: at time 0 it is natural to expect probability one for negative rates if the initial condition is negative and deterministic. This happens if $m > 1$. Analogous comments apply to the other two cases.

Exercise with Solution: Two-factor Vasicek VII

e) We use the decomposition

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

so that the probability that r is negative if the same as the probability that $x - y$ is negative, namely the probability that our z above is negative. Since z is normally distributed with zero mean ($z_0 = 0$), we have immediately

$$Q(z_t < 0) = \Phi(0/V_{t|0}) = 1/2.$$

Exercise: G2++ model and HJM I

In the G2++ model, the instantaneous short-rate process is given by

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0,$$

where the processes $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ satisfy

$$dx(t) = -ax(t)dt + \sigma dW_1(t), \;\; x(0) = 0,$$

$$dy(t) = -by(t)dt + \eta dW_2(t), \;\; y(0) = 0,$$

and (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ:

$$dW_1(t)dW_2(t) = \rho dt.$$

Exercise: G2++ model and HJM II

Here r_0, a, b, σ, η are positive constants, and $-1 \leq \rho \leq 1$. The function φ is deterministic and well defined in the time interval $[0, T^*]$, with T^* a given time horizon, typically 10, 30 or 50 (years).

The bond price formula for G2++ model is given by

$$\begin{aligned} P(t,T) &= \exp\left\{ -\int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a} x(t) \right. \\ &\qquad \left. - \frac{1-e^{-b(T-t)}}{b} y(t) + \frac{1}{2} V(t,T) \right\} \end{aligned}$$

Exercise: G2++ model and HJM III

where

$$V(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]$$

$$+\frac{\eta^2}{b^2}\left[T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right]$$

$$+2\rho\frac{\sigma\eta}{ab}\Bigg[T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b}$$

$$-\frac{e^{-(a+b)(T-t)} - 1}{a+b}\Bigg].$$

Exercise: G2++ model and HJM IV

a) Write an expression for the instantaneous forward rate $f(t, T)$ in this model.

b) Write the SDEs for the dynamics of $f(t, T)$ and, in particular, the expression for the instantaneous volatility $\sigma(t, T)$ of $f(t, T)$.

Exercise: G2++ model and HJM V

SOLUTION.

a) With reference to previous calculations done in the course, where we had found that

$$P(t,T) = \exp\left\{ -\int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t) + \frac{1}{2}V(t,T) \right\},$$

with

Exercise: G2++ model and HJM VI

$$V(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]$$

$$+\frac{\eta^2}{b^2}\left[T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right]$$

$$+2\rho\frac{\sigma\eta}{ab}\left[T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b}\right.$$

$$\left.-\frac{e^{-(a+b)(T-t)} - 1}{a+b}\right],$$

Exercise: G2++ model and HJM VII

we now write

$$
\begin{aligned}
f(t,T) &= -\frac{\partial \ln P(t,T)}{\partial T} \\
&= \frac{\partial}{\partial_T}\left\{\int_t^T \varphi(u)du + \frac{1-e^{-a(T-t)}}{a}x(t) + \frac{1-e^{-b(T-t)}}{b}y(t)\right. \\
&\qquad \left. -\frac{1}{2}V(t,T)\right\} \\
&= \varphi(T) + e^{-a(T-t)}x(t) + e^{-b(T-t)}y(t) - \frac{1}{2}\frac{\partial V}{\partial T},
\end{aligned}
$$

Exercise: G2++ model and HJM VIII

where

$$\begin{aligned}\frac{\partial V}{\partial T} &= \frac{\sigma^2}{a^2}\left(1-2e^{-a(T-t)}+e^{-2a(T-t)}\right)+\frac{\eta^2}{b^2}\left(1-2e^{-b(T-t)}+e^{-2b(T-t)}\right)\\ &\quad +2\rho\frac{\sigma\eta}{ab}\left(1-e^{-a(T-t)}-e^{-b(T-t)}+e^{-(a+b)(T-t)}\right)\\ &= \frac{\sigma^2}{a^2}\left(1-e^{-a(T-t)}\right)^2+\frac{\eta^2}{b^2}\left(1-e^{-b(T-t)}\right)^2\\ &\quad +2\rho\frac{\sigma\eta}{ab}\left(1-e^{-a(T-t)}\right)\left(1-e^{-b(T-t)}\right).\end{aligned}$$

So we have

$$\begin{aligned}f(t,T) &= \varphi(T)+e^{-a(T-t)}x(t)+e^{-b(T-t)}y(t)-\frac{\sigma^2}{2a^2}\left(1-e^{-a(T-t)}\right)^2\\ (*)\qquad &\quad -\frac{\eta^2}{2b^2}\left(1-e^{-b(T-t)}\right)^2-\rho\frac{\sigma\eta}{ab}\left(1-e^{-a(T-t)}\right)\left(1-e^{-b(T-t)}\right).\end{aligned}$$

Exercise: G2++ model and HJM IX

Since $dx(t) = -ax(t)dt + \sigma dW_1(t)$, if we apply Itô's formula to $e^{at}x(t)$, we have

$$\begin{aligned} d(e^{at}x(t)) &= ae^{at}x(t)dt + e^{at}dx(t) \\ &= ae^{at}x(t)dt - ae^{at}x(t)dt + \sigma e^{at}dW_1(t) = \sigma e^{at}dW_1(t). \end{aligned}$$

Integrate from time 0 to t for both side, we get

$$e^{at}x(t) = \sigma \int_0^t e^{as}dW_1(s),$$

$$x(t) = \sigma \int_0^t e^{-a(t-s)}dW_1(s).$$

Exercise: G2++ model and HJM X

Similarly, we can get

$$y(t) = \eta \int_0^t e^{-b(t-s)} dW_2(s).$$

Substitute $x(t)$ and $y(t)$ into the formula $(*)$ we have just derived for $f(t, T)$, and we have

$$\begin{aligned} f(t, T) = \varphi(T) &- \frac{\sigma^2}{2a^2}\left(1 - e^{-a(T-t)}\right)^2 - \frac{\eta^2}{2b^2}\left(1 - e^{-b(T-t)}\right)^2 \\ &- \rho\frac{\sigma\eta}{ab}\left(1 - e^{-a(T-t)}\right)\left(1 - e^{-b(T-t)}\right) \\ &+ \sigma\int_0^t e^{-a(T-s)} dW_1(s) + \eta\int_0^t e^{-b(T-s)} dW_2(s). \end{aligned}$$

Exercise: G2++ model and HJM XI

b) Apply Itô's formula starting from (*) a couple of pages back:

$$df(t,T) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t + \frac{\partial f}{\partial Y}dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}dX_t dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial Y_t^2}dY_t dY_t + \frac{\partial^2 f}{\partial X \partial Y}dX_t dY_t.$$

Now we can easily calculate different terms from (*) as follows

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{a}e^{-a(T-t)}\left(1-e^{-a(T-t)}\right) + \frac{\eta^2}{b}e^{-b(T-t)}\left(1-e^{-b(T-t)}\right) + \rho\frac{\sigma\eta}{b}e^{-a(T-t)}\left(1-e^{-b(T-t)}\right) + \rho\frac{\sigma\eta}{b}e^{-b(T-t)}\left(1-e^{-a(T-t)}\right)) + ae^{-a(T-t)}x(t) + be^{-b(T-t)}y(t),$$

$$\frac{\partial f}{\partial X} = e^{-a(T-t)}, \qquad \frac{\partial f}{\partial Y} = e^{-b(T-t)},$$

Exercise: G2++ model and HJM XII

and

$$\frac{\partial^2 f}{\partial X_t^2} = \frac{\partial^2 f}{\partial Y_t^2} = \frac{\partial^2 f}{\partial X \partial Y} = 0.$$

Thus,

$$\begin{aligned}
df(t,T) &= \frac{\partial f}{\partial t} dt - a e^{-a(T-t)} x_t dt + \sigma e^{-a(T-t)} dW_{1,t} \\
&\quad - b e^{-b(T-t)} y_t dt + \eta e^{-b(T-t)} dW_{2,t} \\
&= \left[\frac{\sigma^2}{a} e^{-a(T-t)} \left(1 - e^{-a(T-t)}\right) + \frac{\eta^2}{b} e^{-b(T-t)} \left(1 - e^{-b(T-t)}\right) \right. \\
&\quad + \rho \frac{\sigma\eta}{b} e^{-a(T-t)} \left(1 - e^{-b(T-t)}\right) \\
&\quad \left. + \rho \frac{\sigma\eta}{b} e^{-b(T-t)} \left(1 - e^{-a(T-t)}\right) \right] dt \\
&\quad + \sigma e^{-a(T-t)} dW_{1,t} + \eta e^{-b(T-t)} dW_{2,t}.
\end{aligned}$$

Exercise: G2++ model and HJM XIII

We denote as $\mu_f(t,T)$ the drift of this model, and $\bar{\sigma}_f(t,T)$ as the volatility of the model, and the dynamics of $f(t,T)$ can be written as

$$df = \mu_f(t,T)dt + \bar{\sigma}_f(t,T)\begin{pmatrix} dW_{1,t} \\ dW_{2,t} \end{pmatrix},$$

where the volatility can be written explicitly as a row vector

$$\bar{\sigma}_f(t,T) = \left(\sigma e^{-a(T-t)}, \eta e^{-b(T-t)}\right).$$

This volatility vector however does not represent the whole covariance of the system, since correlation is also present in the brownian motions, as we have

$$d\langle W_1, W_2\rangle_t = dW_{1,t}\, dW_{2,t} = \rho dt.$$

Exercise: G2++ model and HJM XIV

To have a vector volatility representing the entire instantaneous covariance structure of the system of interest rates, we now incorporate the correlation into the volatility vector. This is achieved through a Cholesky decomposition technique.
In dimension 2, this is as follows. We construct another two dimensional Brownian Motion $\left(B_{1,t}, B_{2,t}\right)$ (not to be confused with the bank account B) such that $B_{1,t}$ and $B_{2,t}$ *are independent*. Assume

$$
\begin{aligned}
W_{1,t} &= B_{1,t} \\
W_{2,t} &= \rho B_{1,t} + \sqrt{1-\rho^2} B_{2,t}.
\end{aligned}
$$

It is immediate to check that the single processes W_1 and W_2 defined this way are standard Brownian motions in dimension one, and also that their quadratic covariation is ρ. Hence we have constructed a

Exercise: G2++ model and HJM XV

representation of our two correlated Brownian motion in terms of two independent ones.
Finally, we have

$$df = \mu_f(t,T)dt + \left(\sigma e^{-a(T-t)}, \eta e^{-b(T-t)}\right) \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} dB_{1,t} \\ dB_{2,t} \end{pmatrix}.$$

and by carring out the matrix product in front of dB, we obtain the vector volatility of the model with respect to $\left(B_{1,t}, B_{2,t}\right)$:

$$\begin{aligned} \sigma_f(t,T) &= \left(\sigma e^{-a(T-t)}, \eta e^{-b(T-t)}\right) \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \\ &= \left(\sigma e^{-a(T-t)} + \rho\eta e^{-b(T-t)}, \sqrt{1-\rho^2}\eta e^{-b(T-t)}\right). \end{aligned}$$

This vector volatility embeds the whole instantaneous covariation of the system.

Exercise with Solutions: LIBOR model calibration I

Assume the following tenor structure: $T_0 = 1y$, $T_1 = 2y$, $T_2 = 3y$, $\ldots$, $T_5 = 6y$. Consider the associated forward LIBOR rates $F_i(t) = F(t; T_{i-1}, T_i)$, $i = 1, \ldots, 6$, whose instantaneous volatility is denoted by $\sigma_i(t)$. Consider the Caplet volatilities $v_{T_{i-1}}^{\text{Caplet}} =: v_{i-1}$ for the caplet resetting at T_{i-1} with maturity T_i.

a) If the caplet volatilities are:

$$\begin{aligned} v_0 &= 0.1;\ v_1 = 0.12;\ v_2 = 0.15; \\ v_3 &= 0.14;\ v_4 = 0.13;\ v_5 = 0.12 \end{aligned}$$

compute the LIBOR model vols $\sigma_i(t)$ consistent with these data in case we assume $\sigma_i(t) = \psi_{i-k} \quad \text{for} \quad t \in [T_{k-1}, T_k]$, where:

Exercise with Solutions: LIBOR model calibration II

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	ψ_1	Dead	Dead	...	Dead
$F_2(t)$	ψ_2	ψ_1	Dead	...	Dead
$\vdots$	...	...	...	...	...
$F_M(t)$	ψ_M	ψ_{M-1}	ψ_{M-2}	...	ψ_1

b) Repeat the calculation but with $v_5 = 0.08$ and identify a qualitative characteristic of the output.

c) Given the same input as in part a), compute the LIBOR model volatilities under the assumption $\sigma_i(t) = \Phi_i$.

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	Φ_1	Dead	Dead	...	Dead
$F_2(t)$	Φ_2	Φ_2	Dead	...	Dead
$\vdots$	...	...	...	...	...
$F_M(t)$	Φ_M	Φ_M	Φ_M	...	Φ_M

Exercise with Solutions: LIBOR model calibration III

d) Again, repeat the same calculation as in c) but with $v_5 = 0.08$. Do you find the same problems as in point b) ?

e) Find the exponential Full rank instantaneous correlation structure

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)e^{-\beta|j-i|},$$

with $\beta = 0.2$ implying $\rho_{1,2} = 0.9$.

Exercise with Solutions: LIBOR model calibration IV

SOLUTION:

a) We know that the $T_0 - T_1$ caplet volatility in the LIBOR model is v_0, where

$$v_0^2 = \frac{1}{T_0}\int_0^{T_0} \sigma_1(t)^2 dt = \frac{1}{1}\int_0^{1y} \psi_1^2 dt = \psi_1^2,$$

so that $\psi_1 = v_0 = 0.1$.

Similarly, the $T_1 - T_2$ caplet volatility is v_1, where

$$v_1^2 = \frac{1}{T_1}(\int_0^{T_1} \sigma_2(t)^2 dt) = \frac{1}{2y}(\int_0^{2y} \sigma_2(t)^2 dt) =$$

$$= \frac{1}{2}(\int_0^{1y} \sigma_2(t)^2 dt + \int_{1y}^{2y} \sigma_2(t)^2 dt) = \frac{1}{2}(\int_0^{1y} \psi_2^2 dt + \int_{1y}^{2y} \psi_1^2 dt)$$

$$= \frac{1}{2}(\psi_1^2 + \psi_2^2),$$

Exercise with Solutions: LIBOR model calibration V

from which

$$\psi_2 = \sqrt{2v_1^2 - \psi_1^2} = 0.1371$$

Then, analogously,

$$v_2^2 = \frac{1}{3}(\psi_3^2 + \psi_2^2 + \psi_1^2)$$

and

$$\psi_3 = \sqrt{3v_2 - \psi_2^2 - \psi_1^2} = 0.1967,$$

and similarly

$$\psi_4 = 0.1044, \quad \psi_5 = 0.0781, \quad \psi_6 = 0.0436.$$

b). With $v_5 = 0.08$ we follow the same procedure, but when we reach

$$\psi_6 = \sqrt{6v_5^2 - \psi_5^2 - \psi_4^2 - \psi_3^2 - \psi_2^2 - \psi_1^2} = \sqrt{-0.0461} =$$

Exercise with Solutions: LIBOR model calibration VI

$$= 0.2147i$$

the square root of a negative number, i.e. an imaginary number, unacceptable as a volatility. Then in this case the chosen parameterization of volatilities cannot be calibrated to caplet volatilities.
c) As before, we know that the $T_0 - T_1$ caplet volatility in the LIBOR model is v_0, where

$$v_0^2 = \frac{1}{T_0}\int_0^{T_0} \sigma_1(t)^2 dt = \frac{1}{1}\int_0^{1y} \Phi_1^2 dt = \Phi_1^2,$$

so that $\Phi_1 = v_0 = 0.1$.
Similarly, the $T_1 - T_2$ caplet volatility is v_1, where

$$v_1^2 = \frac{1}{T_1}\left(\int_0^{T_1} \sigma_2(t)^2 dt\right) = \frac{1}{2y}\left(\int_0^{2y} \sigma_2(t)^2 dt\right) =$$

Exercise with Solutions: LIBOR model calibration VII

$$= \frac{1}{2}\left(\int_0^{1y} \sigma_2(t)^2 dt + \int_{1y}^{2y} \sigma_2(t)^2 dt\right) = \frac{1}{2}\left(\int_0^{1y} \Phi_2^2 dt + \int_{1y}^{2y} \Phi_2^2 dt\right)$$

$$= \frac{1}{2}(2\Phi_2^2) = \Phi_2^2,$$

from which

$$\Phi_2 = v_1 = 0.12$$

Then, analogously,

$$v_2^2 = \frac{1}{3}(\Phi_3^2 + \Phi_3^2 + \Phi_3^2)$$

and

$$\Phi_3 = v_2 = 0.15,$$

and similarly

$$\Phi_4 = v_3 = 0.14, \quad \Phi_5 = v_4 = 0.13, \quad \Phi_6 = v_5 = 0.12$$

Exercise with Solutions: LIBOR model calibration VIII

d) This time $v_5 = 0.08$ gives no problem, since $\Phi_6 = v_5 = 0.08$ is fine. The Φ parameterization never breaks down if the input caplet volatilities make sense, whereas the ψ parameterization can give problems for steep caplet volatility graphs, as seen in case b) above.
e) If $\beta = 0.2$, then

$$\rho_{1,2} = \rho_\infty + (1 - \rho_\infty)e^{-\beta|2-1|} = \rho_\infty + (1 - \rho_\infty)e^{-0.2}.$$

Since we know that $\rho_{1,2} = 0.9$, we solve

$$0.9 = \rho_\infty + (1 - \rho_\infty)e^{-0.2}$$

in ρ_∞. We obtain $\rho_\infty = 0.4483$.

Exercise with Solution: LIBOR fourth payoff I

Consider a contract paying in $T_4 =$four years the fourth power of the LIBOR rate that resets in $T_3 =$three years plus a positive strike $K > 0$, if the total of this sum is positive, and zero otherwise.

a) Compute the pricing formula of this product, namely

$$V = E\left[\frac{B(0)}{B(T_4)}(L(T_3, T_4)^4 + K)^+\right] = E\left[\frac{B(0)}{B(T_4)}(F_4(T_3)^4 + K)^+\right]$$

in a LIBOR market model setting

$$dF_4(t) = \sigma_4 F_4(t) dZ_4, \quad Q^4$$

as a function of the constant volatility σ_4, of the initial forward $F_4(0)$, of the bond $P(0, T_4)$ and of the strike K.

b) Is this product sensitive to volatility of interest rates? Motivate.

Exercise with Solution: LIBOR fourth payoff II

c) If you answered ”Yes” to the previous question, compute the vega sensitivity of the product price to volatility, namely

$$\mathcal{V} = \frac{\partial V}{\partial \sigma_4}$$

Exercise with Solution: LIBOR fourth payoff III

SOLUTION:

a)

We change the numeraire from B to the T_4-forward measure numeraire $P(\cdot, T_4)$:

$$E^B\left[\frac{B(0)}{B(T_4)}(F_4(T_3)^4 + K)^+\right] = E^4\left[\frac{P(0,T_4)}{P(T_4,T_4)}(F_4(T_3)^4 + K)^+\right]$$

$$= P(0,T_4)E^4[(F_4(T_3)^4 + K)^+].$$

Now, since the rate F is positive and K is positive too, we have that the sum $F_4(T_3)^4 + K$ is positive in all scenarios,

$$F_4(T_3)^4 + K \geq 0.$$

It follows that the positive part in the option term is not necessary, namely

Exercise with Solution: LIBOR fourth payoff IV

$$(F_4(T_3)^4 + K)^+ = \max(F_4(T_3)^4 + K, 0) = F_4(T_3)^4 + K > 0$$

in all scenarios. Hence our pricing formula reduces to

$$V = P(0, T_4)E^4[(F_4(T_3)^4 + K)^+] = P(0, T_4)E^4[F_4(T_3)^4 + K] =$$

$$= P(0, T_4)(E^4[F_4(T_3)^4] + K).$$

All we have to compute then is

$$E^4[(F_4(T_3)^4].$$

Since we know that the dynamics of F_4 under numeraire $P(., T_4)$ is

$$dF_4 = \sigma_4 F_4 dZ,$$

by Ito's formula

Exercise with Solution: LIBOR fourth payoff V

$$d(F_4)^4 = 4(F_4)^3 dF_4 + \frac{1}{2} 4 \cdot 3 \cdot (F_4)^2 dF_4 dF_4$$

$$d(F_4)^4 = 4(F_4)^3 \sigma_4 F_4 dZ + 6 F_4^2 \sigma_4^2 F_4^2 dZdZ$$

$$d(F_4)^4 = 4(F_4)^4 \sigma_4 dZ + 6 F_4^4 \sigma_4^2 dt$$

Set $Y_t = (F_4(t))^4$. From the last equation we have

$$dY = (6\sigma_4^2) Y dt + (4\sigma_4) Y dZ.$$

This is a "Black Scholes" geometric brownian motion of type

$$dY = aYdt + bYdZ$$

$(a = 6\sigma_4^2,\ b = (4\sigma_4))$ whose expected value is

$$E[Y(T)] = Y_0 e^{aT}$$

Exercise with Solution: LIBOR fourth payoff VI

Therefore the price is

$$P(0, T_4)(E^4[F_4(T_3)^4] + K) = P(0, T_4)(E[Y(T_3)] + K) =$$

$$= P(0, T_4)(Y_0 e^{aT_3} + K)$$

$$= P(0, T_4)(F_4(0)^4 e^{6\sigma_4^2 T_3} + K)$$

Exercise with Solution: LIBOR fourth payoff VII

b) We see clearly that the price

$$V = P(0, T_4)(F_4(0)^4 e^{6\boxed{\sigma_4^2}T_3} + K)$$

depends on the volatility σ_4, so the answer is "yes".

c)

$$\mathcal{V} = \frac{\partial V}{\partial \sigma_4} = \frac{\partial \left[P(0, T_4)(F_4(0)^4 e^{6\sigma_4^2 T_3} + K) \right]}{\partial \sigma_4}$$

$$= P(0, T_4) \frac{\partial \left[F_4(0)^4 e^{6\sigma_4^2 T_3} \right]}{\partial \sigma_4} = P(0, T_4) F_4(0)^4 \; e^{6\sigma_4^2 T_3} \; 12 \; \sigma_4 T_3.$$

CDS, constant hazard rates and defaultable bonds I

a) Explain how a Credit Default Swap (CDS) is structured. Illustrate the role of the protection buyer “A” and of the protection seller “B”, the role of the reference entity “C”, and introduce the default leg and the premium leg. Provide a formula for the discounted cash flows seen by the protection seller in terms of the default time τ of the reference entity, the recovery rate *Rec* for the default of the reference entity, and the default-free short interest rate r.

b) Assume the following simplifications on the valuation of a CDS:
1) the premium leg pays the premium rate R continuosly rather than quarterly, and
2) the default of the reference entity is modelled with constant deterministic intensity γ.

CDS, constant hazard rates and defaultable bonds II

Show that if the CDS rate R is such that the initial value of the CDS at time 0 is zero, then one has

$$\gamma = \frac{R}{LGD}$$

where $LGD = 1 - Rec$ is the loss given default fraction, and Rec is the recovery rate.

c) Consider now a defaultable zero-coupon bond issued by the entity "C" with maturity T and a CDS offering protection against the default of the same entity "C" in the time window $[0, T_b]$, with $T_b \geq T$. Assume you buy the bond at time 0 from "C" and that you buy protection against default of "C" from a bank "A" via the above CDS, again at time 0. Compute the total discounted payout of the combined bond and CDS position to you at time 0. Explain whether the notional of the bond is still at risk in the combined position.

CDS, constant hazard rates and defaultable bonds III

Solutions.

a) For this question and notation see the lecture notes. Here we only repeat the formally we may write the (Running) CDS discounted payoff to “B” (protection seller) at time $t < T_a$ as

$$\begin{aligned}\Pi\mathrm{CDS}_{a,b}(t) := D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} \\ + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}} \\ -\mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\,\mathrm{LGD}\end{aligned}$$

(cash flows seen from the protection buyer would have opposite signs) where $T_{\beta(\tau)}$ is the first of the T_i's following τ and α_i is the year fraction between T_{i-1} and T_i. The stochastic discount factor at time t for maturity T is $D(t,T) = \exp(-\int_t^T r_u du)$.

CDS, constant hazard rates and defaultable bonds IV

These are random discounted cash flows, not yet the CDS price. To get the price, we need to take a risk neutral expectation.
Note that we are ignoring default risk of both protection buyer and seller "A" and "B", and we will do so for the remaining part of this problem.

b) Instead of paying $(T_i - T_{i-1})R$ at T_i as the standard CDS, given that there has been no default before T_i, we replace this premium leg by assuming that it pays "$dt\ R$" in $[t, t + dt)$ if there has been no default before $t + dt$.
This amounts to replace the original pricing formula of a CDS (receiver case, spot CDS with $T_a = 0 =$ today)

CDS, constant hazard rates and defaultable bonds V

$$\mathrm{CDS}_0 = R\left[\int_0^{T_b} P(0,t)(t - T_{\beta(t)-1})d_t Q(\tau \le t) + \right.$$

$$\left. + \sum_{i=1}^{b} P(0,T_i)\alpha_i Q(\tau \ge T_i)\right] - \mathrm{LGD}\left[\int_0^{T_b} P(0,t)\, d_t\, Q(\tau \le t)\right]$$

with (accrual term vanishes because payments continuous now)

$$R\int_0^{T_b} P(0,t)Q(\tau \ge t)dt - \mathrm{LGD}\int_0^{T_b} P(0,t)\, d_t Q(\tau \le t)$$

If the intensity is a constant γ we have

$$Q(\tau > t) = e^{-\gamma t}, \quad d_t Q(\tau \le t) = d_t(1 - Q(\tau > t)) =$$

CDS, constant hazard rates and defaultable bonds VI

$$= -d_t Q(\tau > t) = -(-\gamma)e^{-\gamma t}dt = \gamma Q(\tau > t)dt,$$

and the receiver CDS price we have seen earlier becomes

$$\text{CDS}_0 = -\text{LGD}\left[\int_0^{T_b} P(0,t)\gamma Q(\tau \geq t)dt\right] + R\left[\int_0^{T_b} P(0,t)Q(\tau \geq t)dt\right]$$

Take the constant γ out of the integral and set the initial price at time 0 to zero.

$$\text{LGD}\ \gamma\left[\int_0^{T_b} P(0,t)Q(\tau \geq t)dt\right] = R\left[\int_0^{T_b} P(0,t)Q(\tau \geq t)dt\right]$$

Now simplify by dividing both sides by the positive squared brackets term to obtain

$$\text{LGD}\ \gamma = R \;\;\Rightarrow\;\; \gamma = R/LGD.$$

CDS, constant hazard rates and defaultable bonds VII

c) The bond offers the discounted cash flows

$$X = Rec1_{\{\tau \le T\}} D(0,\tau) + 1_{\{\tau > T\}} D(0,T),$$

namely pays the recovery at default τ of the issuer if this default is before the bond maturity T, and pays the full notional 1 at maturity T if default is after maturity. Both flows are discounted back from the time when they happen to 0, hence the two discount terms.
The CDS for a protection buyer was written above (minus sign) and is (we have $T_a = 0$ and $T_b > T$)

$$Y = -D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{\tau < T\}} - \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau > T_i\}} +$$

$$+\mathbf{1}_{\{\tau \le T\}} D(t,\tau)\ \text{(1-Rec)}$$

CDS, constant hazard rates and defaultable bonds VIII

Call Z the discounted cash flows in the premium leg of the CDS:

$$Z = D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{\tau<T\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}}$$

so that

$$Y = -Z + \mathbf{1}_{\{\tau\leq T_b\}}D(t,\tau)\ \text{(1-Rec)}$$

Adding up the discounted cash flows of the two positions we have

$$X+Y = Rec1_{\{\tau\leq T\}}D(0,\tau)+1_{\{\tau>T\}}D(0,T)-Z+\mathbf{1}_{\{\tau\leq T_b\}}D(t,\tau)\ \text{(1-Rec)}$$

Now write $\mathbf{1}_{\{\tau\leq T_b\}} = \mathbf{1}_{\{\tau\leq T\}} + \mathbf{1}_{\{\tau\in(T,T_b]\}}$ and substitute in the last equation to get

CDS, constant hazard rates and defaultable bonds IX

$$X+Y = Rec1_{\{\tau\leq T\}}D(0,\tau)+1_{\{\tau>T\}}D(0,T)-Z+\mathbf{1}_{\{\tau\leq T\}}D(t,\tau)\ \text{(1-Rec)}+$$

$$+\mathbf{1}_{\{\tau\in(T,T_b]\}}D(t,\tau)\ \text{(1-Rec)}$$

Call M the last term, $M = \mathbf{1}_{\{\tau\in(T,T_b]\}}D(t,\tau)$ (1-Rec). The first two Rec terms simplify and we end up with

$$X+Y = 1_{\{\tau>T\}}D(0,T) - Z + \mathbf{1}_{\{\tau\leq T\}}D(t,\tau) + M.$$

If not for the difference on the discount maturities and the last term, we could simplify further. We then write

$$X+Y = 1_{\{\tau>T\}}D(0,\min(\tau,T)) + \mathbf{1}_{\{\tau\leq T\}}D(t,\min(\tau,T)) - Z + M$$

CDS, constant hazard rates and defaultable bonds X

We can formally simplify by noticing that

$$\mathbf{1}_{\{\tau > T\}} + \mathbf{1}_{\{\tau \leq T\}} = 1$$

so that

$$\boxed{X + Y = D(0, \min(\tau, T)) \times 1 - Z + M}.$$

Hence the combined BOND and CDS position is equivalent to receiving the bond notional at the first time between τ and T, whoever comes first, as shown by the term

$$D(0, \min(\tau, T)) \times 1,$$

and to pay the protection leg of the CDS, as in the term

$$-Z,$$

CDS, constant hazard rates and defaultable bonds XI

obtaining also further protection for the amount $1 - Rec$ in case "C" defaults after the Bond maturity T and before the CDS maturity T_b, as from the term

$$M = \mathbf{1}_{\{\tau \in (T, T_b]\}} D(t, \tau) \text{ (1-Rec)}.$$

Basically we have a bond whose maturity is no longer T but the first time between default of the issuer of C and T, but this bond is no longer defaultable, as it will pay us for sure, whether in τ or T. This "for sure" has the cost of paying the protection leg Z of the CDS. We have eliminated the risk of default in the Bond thanks to the protection bought via the CDS, and this will cost us the premium leg in the CDS. Note finally that we are paying protection longer than necessary, up to the time T_b rather than T, so that Z will include the cost of the further protection in $(T, T_b]$, i.e. the cost of the protection cash flow M. If we could manage to find a CDS with the maturity exactly equal to that of

CDS, constant hazard rates and defaultable bonds XII

the Bond, i.e. $T_b = T$, then we would have no M term and the cost of making the bond default free would be just the CDS premium leg Z up to T rather than T_b (and hence cheaper).

Deterministic intensities / hazard rates I

Assume that, following calibration to CDS, we have a piecewise constant hazard rare $\gamma(t)$ that takes the following values.

$$\gamma(t) = 0.02 \;\; \textit{for} \;\; 0 <= t < 1y$$

$$\gamma(t) = 0.04 \;\; \textit{for} \;\; 1y <= t < 2y$$

$$\gamma(t) = 0.02 \;\; \textit{for} \;\; 2y <= t < 3y$$

Compute:

a) The probability of defaulting in one year, two years and three years.

b) The probability of surviving in two years.

c) assuming that the recovery is 0.5, interest rates are zero and the CDS sells protection from today to three years, compute the price of the default or protection leg of the CDS.

Deterministic intensities / hazard rates II

Solutions.

a) Probability of default in T year is

$$\mathbb{Q}(\tau \leq T) = 1 - \exp(-\int_0^T \gamma(t)dt) =?$$

Compute

$$\int_0^1 \gamma(t)dt = \int_0^1 0.02dt = 0.02 \cdot 1 = 0.02$$

since $\gamma(t)$ is constant and equal to 0.02 in the integration interval. Compute

$$\int_0^2 \gamma(t)dt = \int_0^1 \gamma(t)dt + \int_1^2 \gamma(t)dt = \int_0^1 0.02dt + \int_1^2 0.04dt =$$

Deterministic intensities / hazard rates III

$$= 0.02 \cdot 1 + 0.04 \cdot 1 = 0.06$$

Analogously,

$$\int_0^3 \gamma(t)dt = 0.02 \cdot 1 + 0.04 \cdot 1 + 0.02 \cdot 1 = 0.08$$

We can now compute the probabilities of defaulting in 1, 2 and 3 years respectively as

$$1 - \exp(-0.02), 1 - \exp(-0.06), 1 - \exp(-0.08)$$

b) The probability of surviving in two years is

$$\mathbb{Q}(\tau > T) = \exp(-\int_0^T \gamma(t)dt)$$

Deterministic intensities / hazard rates IV

for $T = 2y$ and hence $\exp(-0.06)$.

c) The default leg price is $L_{GD}E[D(0,\tau)1_{\{\tau\leq 3y\}}]$. However, since interest rates are zero, $r = 0$, all discounts are equal to one, $D(s,u) = \exp(-r(u-s)) = \exp(0) = 1$. Hence we have

$$L_{GD}E[D(0,\tau)1_{\{\tau\leq 3y\}}] = L_{GD}E[1\cdot 1_{\{\tau\leq 3y\}}] = L_{GD}E[1_{\{\tau\leq 3y\}}] = L_{GD}\mathbb{Q}(\tau \leq 3y) =$$

$$= L_{GD}(1-\exp(-0.08)) = (1-R_{EC})(1-\exp(-0.08)) = 0.5(1-\exp(-0.08))$$

Exercise with Solution: CVA for Bonds I

Consider a default-free zero coupon bond with final maturity T in a market with constant-in-time and random interest rates r that are uniformly distributed in $[0, R]$.

Denote by $P(t, T)$ the price of the bond at time t for maturity T.

Assume that the bond portfolio is held at time t by a party "A" and that the payment of the notional 1 at maturity T is expected from a counterparty "C".

Assume further that the probability of default of "C" is associated to an intensity model with random intensity λ that is constant in time and uniformly distributed in the interval $[0, L]$.

Exercise with Solution: CVA for Bonds II

a) Compute the Credit Valuation Adjustment (CVA) for the price of the bond as seen from "A".

b) Compute the CVA sensitivity

$$\frac{\partial CVA}{\partial R}$$

to the interest rates range R.

c) Compute the CVA limit when the intensity range contracts to 0, namely $L \downarrow 0$, and comment your finding.

d) Compute the CVA limit when the interest rates range contracts to 0, namely $R \downarrow 0$, and comment your finding.

Exercise with Solution: CVA for Bonds III

Solution.
We compute

$$CVA = (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)(\mathrm{NPV}(\tau))^+] =$$

where $NPV(\tau)$ is the residual NPV of the bond at the default time τ of the counterparty. The residual NPV in our case is the residual value of the bond at time τ for maturity T, which is $P(\tau, T)$.

$$CVA = (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)(P(\tau,T))^+] =$$

Since a zero coupon bond price is always positive, the positive part $(...)^+$ can be removed without affecting our equation:

$$= (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)P(\tau,T)] =$$

Remembering the general definition of zero coupon bond price, namely

Exercise with Solution: CVA for Bonds IV

$P(t,T) = E_t[D(t,T)]$, we have

$$= (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)E_\tau[D(\tau,T)]] =$$

Since $D(0,\tau)$ is measurable for E_τ, we can bring it inside the expected value:

$$= (1 - Rec)E_0[1_{\{\tau<T\}}E_\tau[D(0,\tau)D(\tau,T)]] =$$

$$= (1 - Rec)E_0[1_{\{\tau<T\}}E_\tau[D(0,T)]] =$$

where we have used $D(0,s)D(s,t) = D(0,t)$. Then we notice that also $1_{\{\tau<T\}}$ is measurable for E_τ, and hence can be brought inside the expected value

$$= (1 - Rec)E_0[E_\tau[1_{\{\tau<T\}}D(0,T)]] =$$

Exercise with Solution: CVA for Bonds V

Now we use the tower property of conditional expectation: $E_0[E_\tau[X]] = E_0[X]$, so that

$$= (1 - Rec)E_0[1_{\{\tau<T\}}D(0,T)] = (1 - Rec)\mathbf{Q}(\tau < T)E_0[D(0,T)] =$$

where we have used independence of τ and $D(0,T)$, which follows from the independence of r and λ. We obtain

$$= (1 - Rec)\mathbf{Q}(\tau < T)P(0,T)$$

Hence for CVA we just need $\mathbf{Q}(\tau < T)$ and $P(0,T)$. We start from

$$\mathbf{Q}\{\tau \geq T\} = E[\exp(-\int_0^T \lambda_t dt)] = E[\exp(-\lambda T)] =$$

Exercise with Solution: CVA for Bonds VI

since in our case λ is constant in time. Then

$$= \int_0^L \exp(-xT) p_\lambda(x) dx = \int_0^L \exp(-xT) \frac{1}{L} dx = \frac{1}{LT}(1 - \exp(-LT))$$

where we have used the fact that λ is uniformly distributed (with density $1/L$) in $[0, L]$. Then $\mathbf{Q}\{\tau < T\} = 1 - \mathbf{Q}\{\tau \geq T\}$ above.
The bond price is similarly computed:

$$P(0, T) = E_0[\exp(-\int_0^T r_t dt)] = E_0[\exp(-rT)] =$$

where we have used the fact that interest rates are constant. We get

$$= \int_0^R \exp(-xT) \frac{1}{R} dx = \frac{1}{RT}(1 - \exp(-RT))$$

Exercise with Solution: CVA for Bonds VII

where the calculation is completely analogous to the one for λ. By substituting the above expressions we finally obtain the CVA formula

$$CVA = (1 - REC)\left(1 - \frac{1}{LT}(1 - \exp(-LT))\right)\frac{1}{RT}(1 - \exp(-RT))$$

b)
Set

$$A(L) := (1 - REC)\left(1 - \frac{1}{LT}(1 - \exp(-LT))\right).$$

$$B(R) := \frac{1}{RT}(1 - \exp(-RT)).$$

Exercise with Solution: CVA for Bonds VIII

$A(L)$ is the part of CVA that does not depend on R. CVA is given by

$$CVA = A(L)\; B(R) = A(L)\; \frac{1}{RT}(1 - \exp(-RT)).$$

Then by the differentiation rule for a quotient of two functions

$$\frac{\partial CVA}{\partial R} = A(L)\; \frac{1}{R^2 T}(RT \exp(-RT) + \exp(-RT) - 1)$$

c)

When $L \downarrow 0$ we can compute the limit

$$\lim_{L \downarrow 0} A(L) B(R) = B(R) \lim_{L \downarrow 0} A(L) =$$

$$= B(R)(1 - REC) \lim_{L \downarrow 0} \left(1 - \frac{1}{LT}(1 - \exp(-LT))\right)$$

Exercise with Solution: CVA for Bonds IX

$$= B(R)(1 - REC)\left(1 - \lim_{L\downarrow 0} \frac{1}{LT}(1 - \exp(-LT))\right) =$$

using well known limits of the exponential function, or expand the exponential function in Taylor series around 0, or L'Hopital

$$= B(R)(1 - REC)(1 - 1) = 0.$$

This result is natural: If the uniform distribution of λ collapses towards 0, given that $\lambda\, dt$ is the local probability of default around $[t, t + dt)$ for all t, it follows that when $L \downarrow 0$ one has zero probability of defaulting around $[t, t + dt)$ for all t's. Hence default cannot happen with zero intensity and CVA is zero because we are pricing counterparty default risk in a situation where default cannot happen.

Exercise with Solution: CVA for Bonds X

d) When $R \downarrow 0$ we can compute the limit

$$\lim_{R\downarrow 0} A(L)B(R) = A(L) \lim_{R\downarrow 0} B(R) =$$

$$= A(L) \lim_{R\downarrow 0} \frac{1}{RT}(1 - \exp(-RT)) =$$

using well known limits of the exponential function, or expand the exponential function in Taylor series around 0, or L'Hopital

$$= A(L)1 = A(L)$$

This is also natural. If interest rates collapse to zero, with R going to zero, then the zero coupon bond will be worth 1 whatever the maturity. Hence CVA reduces to the probability of default times (1-REC), given that the residual value of the bond will be 1 regardless of when default happens.

Exercise with Solution: CVA for Call option I

Consider a call option C with final maturity T on a default-free equity stock S in a market with constant and deterministic interest rate r. Assume the stock S follows a geometric brownian motion with volatility σ under the risk neutral measure and with initial value S_0 at time 0.

Assume that the call portfolio is held at time t by a party "A" and, if the option is exercised, the underlying S will be delivered at maturity T by a counterparty "C" at the agreed strike price K.

Assume further that the probability of default of "C" is associated to an intensity model with random intensity λ that is constant in time and uniformly distributed in the interval $[0, L]$. Assume λ is independent of the stock price S. The recovery rate associated to default of "C" is assumed to be a deterministic constant REC.

Compute the Credit Valuation Adjustment (CVA) for the price of the call as seen from "A".

Exercise with Solution: CVA for Call option II

Solution.
We compute

$$CVA = (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)(\mathrm{NPV}(\tau))^+] =$$

where $NPV(\tau)$ is the residual NPV of the call at the default time τ of the counterparty. The residual NPV in our case is the expected value at time τ of the discounted payoff of the call, which is $\mathrm{NPV}(\tau) = E_\tau[D(\tau, T)(S_T - K)^+]$.

$$CVA = (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)(E_\tau[D(\tau, T)(S_T - K)^+])^+] =$$

Exercise with Solution: CVA for Call option III

Since the discount factor and the payoff of a call are always positive, the outer positive part $(...)^+$ can be removed without affecting our equation:

$$
\begin{aligned}
&= (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)E_\tau[D(\tau,T)(S_T - K)^+]] \\
&= (1 - Rec)E_0[1_{\{\tau<T\}}E_\tau[D(0,\tau)D(\tau,T)(S_T - K)^+]] \\
&= (1 - Rec)E_0[1_{\{\tau<T\}}E_\tau[D(0,T)(S_T - K)^+]] =
\end{aligned}
$$

where we also used the fact that $D(0,\tau)$ is measurable for E_τ and the property $D(0,s)D(s,t) = D(0,t)$. Now, noticing that $1_{\{\tau<T\}}$ is also measurable for E_τ we can then apply the tower property of conditional expectation $E_0[E_\tau[X]] = E_0[X]$

$$
\begin{aligned}
&= (1 - Rec)E_0[E_\tau[1_{\{\tau<T\}}D(0,T)(S_T - K)^+]] = \\
&= (1 - Rec)E_0[1_{\{\tau<T\}}D(0,T)(S_T - K)^+] =
\end{aligned}
$$

Exercise with Solution: CVA for Call option IV

We assume independence of λ and S,

$$
\begin{aligned}
&= (1 - Rec)E_0[1_{\{\tau<T\}}]E_0[D(0,T)(S_T - K)^+] = \\
&= (1 - Rec)\mathbf{Q}(\tau < T)\,\mathrm{CallPrice}_0.
\end{aligned}
$$

Hence, to compute CVA we just need to compute $\mathbf{Q}(\tau < T)$, and the call price at $t = 0$. We already know from the previous exercise that

$$\mathbf{Q}(\tau < T) = 1 - \frac{1}{LT}(1 - \exp(-LT))$$

Assuming the Black-Scholes model, we know that the price of a call is

$$\mathrm{CallPrice}_0 = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

Exercise with Solution: CVA for Call option V

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{(T)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T)}$$

By substituting the above expressions we finally obtain the CVA formula

$$CVA = (1 - REC)\left(1 - \frac{1}{LT}(1 - \exp(-LT))\right)(S_0 N(d_1) - Ke^{-rT}N(d_2))$$

Exercise with Solution: CVA for Bonds portfolio I

Consider two default-free zero coupon bond withs final maturities T_1 and $T_2 = T$, respectively, $T_1 < T_2$, in a market with interest rates that are independent of credit and default risk.

Denote by $P(t, T)$ the price of a default-free bond at time t for maturity T.

Assume that the bonds are held at time t by a bank "B" and that the payment of the notionals 1 at maturities T_1 and T_2 is expected from a counterparty "C".

Try to answer all questions below in terms of default probabilities, recovery rates, and zero coupon bond prices $P(0, T_i)$. If at some point you need information that is not in this dataset, please specify at that point what further information you would need.

Exercise with Solution: CVA for Bonds portfolio II

a) Compute the total Credit Valuation Adjustment (CVA) seen from the Bank for the two bonds position, assuming that they are not in a netting agreement with C.

b) Compute the total Credit Valuation Adjustment (CVA) seen from the Bank for the two bonds position, assuming that they are in a netting agreement with C.

c) Comment on the differences between a) and b).

d) Repeat points a), b) and c) but in case now we have the following two bond positions: Short one bond with maturity T_1, long one bond with maturity T_2.

e) Repeat points a), b) and c) but in case now we have the following two bond positions: Long one bond with maturity T_1, short one bond with maturity T_2.

Exercise with Solution: CVA for Bonds portfolio III

Solution.

a). If there is no netting agreement, the losses on the two positions, due to default risk, are to be priced separately, and then they are added up.

We need then to compute CVA for the first bond and CVA for the second bond, and add them up.

We have seen in a previous exercise that the price of CVA at time 0 for a zero coupon default-free bond with maturity T is

$$CVA = (1 - Rec)\mathbf{Q}(\tau < T)P(0, T).$$

This formula is derived exactly in the same way as before. It is enough to assume that interest rates and default are independent for this formula to hold (rederive the formula and check this). Hence it holds also in this case. The total CVA for the two bonds is then

Exercise with Solution: CVA for Bonds portfolio IV

$$CVA_{TOT} = CVA_1 + CVA_2,$$

$CVA_1 = (1 - Rec)\mathbf{Q}(\tau < T_1)P(0, T_1), \; CVA_2 = (1 - Rec)\mathbf{Q}(\tau < T_2)P(0, T_2)$

By substituting the default probabilities formulas derived in the earlier exercise for the uniform intensity case, we conclude.

b) If there is a netting agreement, the residual values on the two positions are to be netted at default, before checking positivity.
It is best to write the discounted cash flows of the netted portfolio at a time $t < T_2$: we obtain, recalling that $T = T_2$,

$$\Pi(t, T) = 1_{t \leq T_1} D(t, T_1) \cdot 1 + D(t, T_2) \cdot 1.$$

In other terms, we need to take into account that after T_1, the first bond expires, and the notional is paid back. Hence it makes a difference

Exercise with Solution: CVA for Bonds portfolio V

whether we are looking at a case before T_1 or after T_1. Before T_1 we have two bond positions, after T_1 we only have one, namely the T_2 maturity bond.
We compute the CVA of the two bond positions under netting as follows:

$$CVA = (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)(E_\tau[\Pi(\tau,T)])^+] = ...$$

Now we pause a second to compute

$$E_\tau[\Pi(\tau,T)] = E_\tau[1_{\tau\leq T_1}D(\tau,T_1)\cdot 1 + D(\tau,T_2)\cdot 1] =$$

$$= 1_{\tau\leq T_1}E_\tau[D(\tau,T_1)\cdot 1] + E_\tau[D(\tau,T_2)\cdot 1] = 1_{\tau\leq T_1}P(\tau,T_1) + P(\tau,T_2) \geq 0,$$

where we have used the fact that $1_{\tau\leq T_1}$ is E_τ-measurable, and the definition of zero coupon bond price. We see that the residual NPV is

Exercise with Solution: CVA for Bonds portfolio VI

always non-negative, given that it is the sum of two positive zero coupon bonds. It follows that

$$(E_\tau[\Pi(\tau, T)])^+ = (1_{\tau \leq T_1} P(\tau, T_1) + P(\tau, T_2))^+ = 1_{\tau \leq T_1} P(\tau, T_1) + P(\tau, T_2)$$

so that, going back to our CVA calculation, writing LGD = 1-Rec,

$$... = (1 - Rec) E_0[1_{\{\tau < T\}} D(0, \tau)(1_{\tau \leq T_1} P(\tau, T_1) + P(\tau, T_2))] =$$

$$= LGD\ E_0[1_{\{\tau < T\}} D(0, \tau) 1_{\tau \leq T_1} P(\tau, T_1)] + LGD\ E_0[1_{\{\tau < T\}} D(0, \tau) P(\tau, T_2))] =$$

$$= LGD\ E_0[1_{\{\tau < T_1\}} D(0, \tau) P(\tau, T_1)] + LGD\ E_0[1_{\{\tau < T\}} D(0, \tau) P(\tau, T_2))] = ...$$

where we have used the fact that

$$1_{\{\tau < T_1\}} 1_{\{\tau < T_2\}} = 1_{\{\tau < T_1\}}.$$

Exercise with Solution: CVA for Bonds portfolio VII

The second expected value is the CVA on a single T_2 bond position, and is the same as the CVA_2 we found in point a) above.
We can pause for a second to make the first expected value explicit, as we have already done earlier:

$$E_0[1_{\{\tau<T_1\}}D(0,\tau)(E_\tau(D(\tau,T_1)))] = E_0[E_\tau(1_{\{\tau<T_1\}}D(0,\tau)D(\tau,T_1))] =$$

$$= E_0[E_\tau(1_{\{\tau<T_1\}}D(0,T_1))] = E_0[1_{\{\tau<T_1\}}D(0,T_1)] = \mathbb{Q}(\tau<T_1)P(0,T_1)$$

where we have used the tower property of expectations and the independence between default τ and interest rates discounts $D(0,T_1)$.
Putting all the pieces together:

$$... = LGD\,\mathbb{Q}(\tau<T_1)P(0,T_1) + LGD\,E_0[1_{\{\tau<T\}}D(0,\tau)P(\tau,T_2))]$$

So we conclude

$$CVA_{NETTING} = CVA_1 + CVA_2$$

Exercise with Solution: CVA for Bonds portfolio VIII

exactly as in the previous case a).

c) There is no difference between a) and b). The reason is that, being the NPV of the two assets positive or zero in all scenarios, and never negative, the two asset values never offset each other. There is no scenario where one asset is positive and the other one is negative, giving some benefit to netting. Hence netting is irrelevant here.

d)

We restart from a), and call this d.a)

d.a) If there is no netting agreement, the losses on the two positions, due to default risk, are to be priced separately, and then they are added up.

We need then to compute CVA for the first (short) bond and CVA for the second (long) bond, and add them up.

Exercise with Solution: CVA for Bonds portfolio IX

For the second (long) bond, we still have

$$CVA = (1 - Rec)\mathbf{Q}(\tau < T_2)P(0, T_2) = CVA_2.$$

This formula is derived exactly in the same way as before. For the first (short) bond, the price of the position $E_\tau[\Pi(\tau, T)]$ at default time τ is

$$\begin{array}{rll} -P(\tau, T_1) & \text{if} & \tau \leq T_1, \\ 0 & \text{if} & \tau > T_1 \ \text{(bond expired before default).} \end{array}$$

Then the residual NPV is always negative or zero for the first bond, so that when we take its positive part,

$$(E_\tau[\Pi(\tau, T)])^+ = (-P(\tau, T_1)1_{\{\tau \leq T_1\}})^+ = 0,$$

Exercise with Solution: CVA for Bonds portfolio X

so that the CVA for the first bond position is zero, since the positive part of the residual NPV is zero in all scenarios. So in this case

$$CVA_{TOT} = CVA_2.$$

d.b)
If there is a netting agreement, the residual values on the two positions are to be netted at default, before checking positivity.
It is best to write the discounted cash flows of the netted portfolio at a time $t < T_2$: we obtain, recalling that $T = T_2$,

$$\Pi(t, T) = -1_{t \leq T_1} D(t, T_1) \cdot 1 + D(t, T_2) \cdot 1.$$

Exercise with Solution: CVA for Bonds portfolio XI

The first bond, with maturity T_1, has a minus because we are in a short position in that bond. The second bond has a plus because we are in a long position.
In other terms, we need to take into account that after T_1, the first bond expires. Hence it makes a difference whether we are looking at a case before T_1 or after T_1. Before T_1 we have two bond positions, after T_1 we only have one, namely the T_2 maturity bond.
We compute the CVA of the two bond positions under netting as follows:

$$CVA = (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)(E_\tau[\Pi(\tau, T)])^+] = \dots \rightarrow$$

Now we pause a second to compute

$$E_\tau[\Pi(\tau, T)] = E_\tau[-1_{\tau\leq T_1}D(\tau, T_1)\cdot 1 + D(\tau, T_2)\cdot 1] =$$

Exercise with Solution: CVA for Bonds portfolio XII

$$= -1_{\tau \le T_1} E_\tau[D(\tau, T_1) \cdot 1] + E_\tau[D(\tau, T_2) \cdot 1] = -1_{\tau \le T_1} P(\tau, T_1) + P(\tau, T_2),$$

where we have used the fact that $1_{\tau \le T_1}$ is E_τ-measurable, and the definition of zero coupon bond price. It follows that

$$(E_\tau[\Pi(\tau, T)])^+ = (-1_{\tau \le T_1} P(\tau, T_1) + P(\tau, T_2))^+ =$$

$$= (-1_{\tau \le T_1} P(\tau, T_1) + (1_{\tau \le T_1} + 1_{\tau > T_1}) P(\tau, T_2))^+ =$$

$$= (-1_{\tau \le T_1} (P(\tau, T_1) - P(\tau, T_2)) + 1_{\tau > T_1} P(\tau, T_2))^+$$

This last expression can take two different values depending on whether $\tau \le T_1$ or $\tau > T_1$. We see that

$$\begin{aligned} (E_\tau[\Pi(\tau, T)])^+ &= (-P(\tau, T_1) + P(\tau, T_2))^+ = 0 \ \text{ if } \ \tau \le T_1 \\ (E_\tau[\Pi(\tau, T)])^+ &= (P(\tau, T_2))^+ = P(\tau, T_2) \ \text{ if } \ \tau > T_1. \end{aligned}$$

Exercise with Solution: CVA for Bonds portfolio XIII

The first row "0" is due to the fact that a zero coupon bond with a shorter maturity is always worth more than a zero coupon bond with a longer maturity, as we get the money back earlier. Hence $P(\tau, T_1) > P(\tau, T_2)$, which also implies $P(\tau, T_2) - P(\tau, T_1) < 0$ and hence $(P(\tau, T_2) - P(\tau, T_1))^+ = 0$.
We can write this in a single equation as

$$(E_\tau[\Pi(\tau, T)])^+ = 1_{\{\tau > T_1\}} P(\tau, T_2).$$

Going back to our CVA calculation, writing LGD = 1-Rec,

$$\rightarrow \ldots = LGD\ E_0[1_{\{\tau < T\}} D(0, \tau) 1_{\{\tau > T_1\}} P(\tau, T_2)] =$$

$$= LGD\ E_0[1_{\{T_1 \leq \tau < T_2\}} D(0, \tau) P(\tau, T_2) = \ldots$$

Exercise with Solution: CVA for Bonds portfolio XIV

where we have used $1_{\{\tau<T\}}1_{\{\tau>T_1\}} = 1_{\{\tau<T_2\}}1_{\{\tau>T_1\}} = 1_{\{T_1<\tau<T_2\}}$.
We continue as usual

$$... = LGD\ E_0[1_{\{T_1<\tau<T_2\}}D(0,\tau)E_\tau[D(\tau, T_2)]] =$$

$$= LGD\ E_0[E_\tau[1_{\{T_1<\tau<T_2\}}D(0,\tau)D(\tau, T_2)]] =$$

$$= LGD\ E_0[E_\tau[1_{\{T_1<\tau<T_2\}}D(0, T_2)]] = LGD\ E_0[1_{\{T_1<\tau<T_2\}}D(0, T_2)] =$$

$$= LGD\ \ E_0[1_{\{T_1<\tau<T_2\}}]E_0[D(0, T_2)] = LGD\ \mathbb{Q}\{T_1 < \tau < T_2\}P(0, T_2) =$$

$$= LGD(\mathbb{Q}\{\tau < T_2\} - \mathbb{Q}\{\tau < T_1\})P(0, T_2) =$$

$$= CVA_2 - LGD\ \mathbb{Q}\{\tau < T_1\}P(0, T_2).$$

We conclude

$$CVA_{NETTING} = CVA_2 - LGD\ \mathbb{Q}\{\tau < T_1\}P(0, T_2).$$

Exercise with Solution: CVA for Bonds portfolio XV

d.c) There is now a relevant difference between d.a) and d.b). The reason is that now the two assets offset each other in several scenarios. When default happens before T_1, the short T_1 bond has a negative value that dominates, in absolute value, the positive bond T_2. Hence the total NPV in this case is negative and there is no contribution to the CVA payoff. When instead default happens after T_1 and before T_2, the first bond is gone and we only have the second (positive) bond T_2, which is not offset by any other cash flow. Therefore netting benefits us only in scenarios where default is before T_1. This is confirmed if we compute the difference between CVA_{TOT} and $\text{CVA}_{NETTING}$, giving us the price benefit of netting:

$$CVA_{TOT} - CVA_{NETTING} = CVA_2 - (CVA_2 - LGD\,\mathbb{Q}\{\tau < T_1\}P(0,T_2)) =$$

$$= LGD\,\mathbb{Q}\{\tau < T_1\}P(0,T_2).$$

This is the netting benefit.

Exercise with Solution: CVA for Bonds portfolio XVI

e)
We restart from a), and call this e.a)
e.a) If there is no netting agreement, the losses on the two positions, due to default risk, are to be priced separately, and then they are added up.
We need then to compute CVA for the first (long) bond and CVA for the second (short) bond, and add them up.
For the first (short) bond, we still have

$$CVA = (1 - Rec)\mathbf{Q}(\tau < T_1)P(0, T_1) = CVA_1.$$

This formula is derived exactly in the same way as before. For the second (short) bond, similarly to what we have seen in d.a), we see

Exercise with Solution: CVA for Bonds portfolio XVII

that the residual value at default is always negative, so that this yields a zero CVA. So in this case

$$CVA_{TOT} = CVA_1.$$

e.b)
If there is a netting agreement, the residual values on the two positions are to be netted at default, before checking positivity.
It is best to write the discounted cash flows of the netted portfolio at a time $t < T_2$: we obtain, recalling that $T = T_2$,

$$\Pi(t, T) = 1_{t \leq T_1} D(t, T_1) \cdot 1 - D(t, T_2) \cdot 1.$$

Exercise with Solution: CVA for Bonds portfolio XVIII

The second bond, with maturity T_2, has a minus because we are in a short position in that bond. The first bond has a plus because we are in a long position.

In other terms, we need to take into account that after T_1, the first bond expires. Hence it makes a difference whether we are looking at a case before T_1 or after T_1. Before T_1 we have two bond positions, after T_1 we only have one, namely the T_2 maturity bond.

We compute the CVA of the two bond positions under netting as follows:

$$CVA = (1 - Rec)E_0[1_{\{\tau<T\}}D(0,\tau)(E_\tau[\Pi(\tau,T)])^+] = ... \rightarrow$$

Now we pause a second to compute

$$E_\tau[\Pi(\tau,T)] = E_\tau[1_{\tau \leq T_1}D(\tau,T_1)\cdot 1 - D(\tau,T_2)\cdot 1] =$$

Exercise with Solution: CVA for Bonds portfolio XIX

$$= 1_{\tau \le T_1} E_\tau[D(\tau, T_1) \cdot 1] - E_\tau[D(\tau, T_2) \cdot 1] = 1_{\tau \le T_1} P(\tau, T_1) - P(\tau, T_2),$$

where we have used the fact that $1_{\tau \le T_1}$ is E_τ-measurable, and the definition of zero coupon bond price. It follows that

$$(E_\tau[\Pi(\tau, T)])^+ = (1_{\tau \le T_1} P(\tau, T_1) - P(\tau, T_2))^+ =$$

$$= (1_{\tau \le T_1} P(\tau, T_1) - (1_{\tau \le T_1} + 1_{\tau > T_1}) P(\tau, T_2))^+ =$$

$$= (1_{\tau \le T_1} (P(\tau, T_1) - P(\tau, T_2)) - 1_{\tau > T_1} P(\tau, T_2))^+$$

This last expression can take two different values depending on whether $\tau \le T_1$ or $\tau > T_1$. We see that

$$\begin{aligned} (E_\tau[\Pi(\tau, T)])^+ &= (P(\tau, T_1) - P(\tau, T_2))^+ \text{ if } \tau \le T_1 \\ (E_\tau[\Pi(\tau, T)])^+ &= (-P(\tau, T_2))^+ = 0 \text{ if } \tau > T_1. \end{aligned}$$

Exercise with Solution: CVA for Bonds portfolio XX

This is the same as

$$
\begin{aligned}
(E_\tau[\Pi(\tau,T)])^+ &= P(\tau,T_1) - P(\tau,T_2) \ \text{ if } \ \tau \le T_1 \\
(E_\tau[\Pi(\tau,T)])^+ &= (-P(\tau,T_2))^+ = 0 \ \text{ if } \ \tau > T_1
\end{aligned}
$$

because the positive part in the first row is not acting. Indeed, the difference $P(\tau, T_1) - P(\tau, T_2)$ is already positive in all scenarios, as it is the difference of a more valuable bond with a less valuable one, as we explained earlier in d.b).
We can write this in a single equation as

$$
(E_\tau[\Pi(\tau,T)])^+ = 1_{\{\tau < T_1\}}(P(\tau,T_1) - P(\tau,T_2)).
$$

Going back to our CVA calculation, writing LGD = 1-Rec,

$$
\rightarrow \ldots = LGD\ E_0[1_{\{\tau<T\}} D(0,\tau) 1_{\{\tau<T_1\}}(P(\tau,T_1) - P(\tau,T_2))] =
$$

Exercise with Solution: CVA for Bonds portfolio XXI

$$= LGD\ E_0[1_{\{\tau<T_1\}}D(0,\tau)(P(\tau,T_1)-P(\tau,T_2))] =$$

$$= LGD\ E_0[1_{\{\tau<T_1\}}D(0,\tau)E_\tau[D(\tau,T_1)-D(\tau,T_2)]] =$$

$$= LGD\ E_0[1_{\{\tau<T_1\}}E_\tau[D(0,\tau)D(\tau,T_1)-D(0,\tau)D(\tau,T_2)]] =$$

$$= LGD\ E_0[1_{\{\tau<T_1\}}E_\tau[D(0,T_1)-D(0,T_2)]] =$$

$$= LGD\ E_0[E_\tau[1_{\{\tau<T_1\}}(D(0,T_1)-D(0,T_2))]] =$$

$$= LGD\ E_0[1_{\{\tau<T_1\}}(D(0,T_1)-D(0,T_2))] =$$

$$= LGD\ E_0[1_{\{\tau<T_1\}}]\ E_0[D(0,T_1)-D(0,T_2)] =$$

$$= LGD\ \mathbb{Q}\{\tau<T_1\}[P(0,T_1)-P(0,T_2)].$$

We conclude $CVA_{NETTING} = LGD\ \mathbb{Q}\{\tau<T_1\}[P(0,T_1)-P(0,T_2)] =$

$$= CVA_1 - LGD\ \mathbb{Q}\{\tau<T_1\}P(0,T_2).$$

Exercise with Solution: CVA for Bonds portfolio XXII

e.c) There is now a relevant difference between e.a) and e.b). The reason is that now the two assets offset each other in several scenarios. When default happens before T_1, the short T_1 bond has a positive value that dominates, in absolute value, the negative bond T_2. Hence the total NPV in this case is positive but less than if we had only the first bond. When instead default happens after T_1 and before T_2, the first bond is gone and we only have the second (negative) bond T_2, which being negative gives no contribution to the CVA payout. Therefore netting benefits us only in scenarios where default is before T_1. This is confirmed if we compute the difference between CVA_{TOT} and $\text{CVA}_{NETTING}$, giving us the price benefit of netting:

$$CVA_{TOT} - CVA_{NETTING} = CVA_1 - (CVA_1 - LGD\,\mathbb{Q}\{\tau < T_1\}P(0,T_2)) =$$

$$= LGD\,\mathbb{Q}\{\tau < T_1\}P(0,T_2).$$

Risk Measures I

Consider the dynamics of an equity asset price S in the Black and Scholes model, under both probability measures P (the Physical or Historical measure) and Q (the risk neutral measure).

a) Define Value at Risk (VaR) for a time horizon T with confidence level α for a general portfolio.

b) Compute VaR for horizon T and confidence level α for a portfolio with N units of equity, where the equity price follows the Black Scholes process above.

c) Explain at least one drawback of VaR as a risk measure

d) Is the equity dynamics you used for VaR the same you would have used to price an equity call option in Black Scholes?

e) Define Expected Shortfall, and explain in which sense Expected Shortfall (ES) is a better risk measure than VaR.

Risk Measures II

f) For the VaR computed in point b), compute the second order sensitivity of VaR to the risk free interest rate r and explain your result.

SOLUTIONS.

a)

VaR is related to the potential loss on our portfolio over the time horizon T. Define this loss L_T as the difference between the value of the portfolio today (time 0) and in the future T.

$$L_T = \text{Portfolio}_0 - \text{Portfolio}_T.$$

VaR with horizon T and confidence level α is defined as that number $q = q_{T,\alpha}$ such that

$$P[L_T < q] = \alpha$$

Risk Measures III

so that our loss at time T is smaller than q with P-probability α. We are assuming the loss to have a continous distribution for simplicity.
In other terms, it is that level of loss over a time T that we will not exceed with probability α. It is the α P-percentile of the loss distribution over T.

b)
In Black Scholes the equity process follows the dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ, σ are positive constants and W is a brownian motion under the physical measure P.
We know that S_T can be written as

Risk Measures IV

$$S_T = S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma W_T \right\}, \tag{1}$$

and recalling the distribution of W_T,

$$S_T = S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sqrt{T}\sigma \mathcal{N}(0,1) \right\} \tag{2}$$

so that in our case $L_T = N(S_0 - S_T)$, namely

$$L_T = N S_0 \left(1 - \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sqrt{T}\sigma \mathcal{N}(0,1) \right\} \right)$$

Hence

Risk Measures V

$$\alpha = P[L_T < q] = P\left[\left(1 - \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sqrt{T}\sigma\mathcal{N}(0,1)\right\}\right) < \frac{q}{NS_0}\right]$$

$$= P\left[\left(\mu - \frac{1}{2}\sigma^2\right)T + \sqrt{T}\sigma\mathcal{N}(0,1) > \ln\left(1 - \frac{q}{NS_0}\right)\right]$$

$$= P\left[\mathcal{N}(0,1) > \frac{\ln\left(1 - \frac{q}{NS_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sqrt{T}\sigma}\right] =$$

$$= 1 - \Phi\left(\frac{\ln\left(1 - \frac{q}{NS_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sqrt{T}\sigma}\right)$$

Risk Measures VI

$$= \Phi\left(-\frac{\ln\left(1-\frac{q}{NS_0}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)T}{\sqrt{T}\sigma}\right)$$

So we have obtained

$$\alpha = \Phi\left(-\frac{\ln\left(1-\frac{q}{NS_0}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)T}{\sqrt{T}\sigma}\right)$$

or

$$\Phi^{-1}(\alpha) = -\frac{\ln\left(1-\frac{q}{NS_0}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)T}{\sqrt{T}\sigma}$$

and therefore

$$\exp\left(-\sqrt{T}\sigma\Phi^{-1}(\alpha)+\left(\mu-\frac{1}{2}\sigma^2\right)T\right) = \left(1-\frac{q}{NS_0}\right)$$

Risk Measures VII

$$q = NS_0 \left[1 - \exp\left(-\sqrt{T}\sigma\Phi^{-1}(\alpha) + \left(\mu - \frac{1}{2}\sigma^2 \right) T \right) \right]$$

c) VaR is not subadditive, hence it does not recognize the benefit of diversification. Also, VaR ignores the structure of the loss distribution after the percentile. So if 99% VaR is 10 billions, we can have the remaining 1% loss concentrated

(i) either on 10.1 billions,

(ii) or on 10 trillions,

Risk Measures VIII

as two stylized cases, without VaR being able to tell us anything on whether we are in case (i) or (ii).

d) No the dynamics is not the same, to price an option we need to use the risk neutral dynamics, where the drift parameter μ of S is replaced by the risk free rate r of the bank account.

e) Expected shortfall is defined as the expectation of the loss given that the loss is larger or equal than VaR. Formally, with our previous notation from point a),

$$\mathrm{ES}_{T,\alpha} = E[L_T | L_T \geq q_{T,\alpha}].$$

Again for simplicity we assume the loss distribution to be continuous. In terms of comparison with VaR, ES is subadditive, ie it recognizes the benefit of diversification, whereas VaR is not. In other terms, if our

Risk Measures IX

portfolio P is the sum of two sub-portfolios P_1 and P_2, namely $P = P_1 + P_2$, then VaR is NOT sub-additive, in that we may have in some cases

$$VaR(P1 + P2) > VaR(P_1) + VaR(P_2) \text{ for some } P_1,\ P_2$$

whereas with expected shortfall this never happens, as we always have

$$ES(P1 + P2) \leq ES(P_1) + ES(P_2) \text{ for all } P_1,\ P_2.$$

This means that with ES the risk of a more diversified portfolio P, obtained adding two different portfolios $P1$ and $P2$, is always smaller or equal than the sum of the two portfolios P_1 and P_2 risks.

Risk Measures X

f) We see immediately that the VaR formula in *b*) does not depend on r, hence the first derivative is

$$\frac{\partial \text{VaR}}{\partial r} = 0.$$

Clearly, the second derivative is zero as well:

$$\frac{\partial^2 \text{VaR}}{\partial r^2} = \frac{\partial 0}{\partial r} = 0.$$

The reason why this happens is that risk measures are computed under the physical/historical measure P statistics. The statistics of the asset S do not depend on r under the measure P.
Moreover, usually one may expect the presence of r in discount terms giving the value of the portfolio at time 0 and T in the loss expression. However, discounting does not play a role here because we have a

Risk Measures XI

direct (linear) position on the asset rather than an option contract. Hence the value of the portfolio at times 0 and T is the asset itself, S_0 or S_T, and there is no presence of r in the value of the portfolio at 0 or T. Hence VaR does not depend on r and all first or higher order sensitivities wrt to r are zero.

The CIR model for stochastic intensity I

Assume we are given a stochastic intensity process λ_t for a default time τ, with

$$\mathbb{Q}(\tau \in [t, t+dt)) = \lambda_t \, dt,$$

where we set $\lambda_t = y_t$,

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW(t)$$

is a CIR process, where y_0, κ, μ, ν are positive constants. W is a brownian motion under the risk neutral measure.
a) Increasing κ increases or decreases randomness in the intensity? And ν?

b) The mean of the intensity at future times is affected by k? And by ν?

c) What happens to mean of the intensity when time grows to infinity?

The CIR model for stochastic intensity II

d) Is it true that, because of mean reversion, the variance of the intensity goes to zero (no randomness left) when time grows to infinity?

e) Can you compute a rough approximation of the percentage volatility in the intensity?

f) Suppose that $y_0 = 400bps = 0.04$, $\kappa = 0.3$, $\nu = 0.001$ and $\mu = 400bps$. Can you guess the behaviour of the future random trajectories of the stochastic intensity after time 0?

g) Can you guess the spread of a CDS with 10y maturity with the above stochastic intensity when the recovery is 0.35?

h) If we are modeling the default intensity λ_t for a default time τ, what are the pros' and cons' of the following three models:

The CIR model for stochastic intensity III

- (i) $\lambda_t = y_t$, the CIR model above;
- (ii) a geometric brownian motion for λ_t, like in the Dothan model, ie

$$d\lambda_t = a\lambda_t dt + b\lambda_t dW(t), \quad b \geq 0.$$

  with a and b real numbers;
- (iii) a Vasicek model for λ, ie

$$d\lambda_t = \kappa(\mu - \lambda_t)dt + \nu dW(t).$$

The CIR model for stochastic intensity IV

SOLUTION.

a) We can refer to the formulas for the mean and variance of y_T in a CIR model as seen from time 0, at a given T. The formula for the variance is known to be (see for Example Brigo and Mercurio (2006))

$$\mathrm{VAR}(y_T) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2$$

whereas the mean is

$$E[y_T] = y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

We can see that for k becoming large the variance becomes small, since the exponentials decrease in k and the division by k gives a small value for large k. In the limit

$$\lim_{\kappa \to +\infty} \mathrm{VAR}(y_T) = 0$$

so that for very large κ there is no randomness left.

The CIR model for stochastic intensity V

We can instead see that VAR(y_T) is proportional to ν^2, so that if ν increases randomness increases, as is obvious from $\nu\sqrt{y_t}$ being the instantaneous volatility in the process y.

b) As the mean is

$$E[y_T] = y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

we clearly see that this is impacted by κ (indeed, "speed of mean reversion") and by μ clearly ("long term mean") but not by the instantaneous volatility parameter ν.
c) As T goes to infinity, we get for the mean

$$\lim_{T \to +\infty} y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T}) = \mu$$

so that the mean tends to μ (this is why μ is called "long term mean").

The CIR model for stochastic intensity VI

d) In the limit where time goes to infinity we get, for the variance

$$\lim_{T \to +\infty} [y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2] = \mu \frac{\nu^2}{2\kappa}$$

So this does not go to zero. Indeed, mean reversion here implies that as time goes to infinite the mean tends to μ and the variance to the constant value $\mu \frac{\nu^2}{2\kappa}$, but not to zero.

The CIR model for stochastic intensity VII

e) Rough approximations of the percentage volatilities in the intensity would be as follows. The instantaneous variance in dy_t, conditional on the information up to t, is (remember that $VAR(dW(t)) = dt$)

$$VAR(dy_t) = \nu^2 y_t dt$$

The percentage variance is

$$VAR\left(\frac{dy_t}{y_t}\right) = \frac{\nu^2 y_t}{y_t^2} dt = \frac{\nu^2}{y_t} dt$$

and is state dependent, as it depends on y_t. We may replace y_t with either its initial value y_0 or with the long term mean μ, both known. The two rough percentage volatilities estimates will then be, for $dt = 1$,

$$\sqrt{\frac{\nu^2}{y_0}} = \frac{\nu}{\sqrt{y_0}}, \quad \sqrt{\frac{\nu^2}{\mu}} = \frac{\nu}{\sqrt{\mu}}$$

The CIR model for stochastic intensity VIII

These however do not take into account the important impact of κ in the overall volatility of finite (as opposed to instantaneous) credit spreads and are therefore relatively useless.

The CIR model for stochastic intensity IX

f) First we check if the positivity condition is met.

$$2\kappa\mu = 2 \cdot 0.3 \cdot 0.04 = 0.024; \quad \nu^2 = 0.001^2 = 0.000001$$

hence $2\kappa\mu > \nu^2$ and trajectories are positive. Then we observe that the variance is very small: Take $T = 5y$,

$$\mathrm{VAR}(y_T) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \theta \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2 \approx 0.0000006.$$

Take the standard deviation, given by the square root of the variance:

$$\mathrm{STDEV}(y_T) \approx \sqrt{0.0000006} = 0.00077.$$

which is much smaller of the level 0.04 at which the intensity refers both in terms of initial value and long term mean. Therefore there is almost no randomness in the system as the variance is very small compared to the initial point and the long term mean.

The CIR model for stochastic intensity X

Hence there is almost no randomness, and since the initial condition y_0 is the same as the long term mean $\mu_0 = 0.04$, the intensity will behave as if it had the value 0.04 all the time. All future trajectories will be very close to the constant value 0.04.

g) In a constant intensity model the CDS spread can be approximated by

$$y = \frac{R_{CDS}}{1 - REC} \Rightarrow R_{CDS} = y(1 - REC) = 0.04(1 - 0.35) = 260bps$$

h) (i) CIR has the advantage of having closed form solutions for bond prices,

$$E\left[\exp\left(-\int_0^T \lambda_s ds\right)\right]$$

The CIR model for stochastic intensity XI

which in the intensity context are survival probabilities $\mathbb{Q}(\tau > T)$. Hence it is possible to compute survival (and default) probabilities, and hence CDS prices and defaultable bond prices, in closed form under deterministic default-free rates r_t. Calibration to CDS curves is thus easier.
CIR has also the good advantage of being positive. Since $\lambda\, dt$'s are local default probabilities, λ's need to be positive, as probabilities are positive. CIR has this property.

(ii) Dothan has the good feature of being positive as CIR but the calculation of the survival probabilities is not as tractable as in CIR. The formula for

$$E\left[\exp\left(-\int_0^T \lambda_s ds\right)\right]$$

The CIR model for stochastic intensity XII

is much more cumbersome and practically not very useful, and requires special functions contrary to CIR. So survival probabilities are not very tractable here and CDS curve calibration may be done through numerical methods.

(iii) Vasicek has the pro of having an easy expression for the survival probability

$$E\left[\exp\left(-\int_0^T \lambda_s ds\right)\right],$$

even simpler than CIR. However, since the model implies a Gaussian distribution for λ at all future times, λ can be negative with a positive probability, which is not good for an intensity model, as we explained above.

CVA for an Asset or Nothing Option I

Consider an asset or Nothing option (ANO) and the related cash-or-nothing option (CNO) with final maturity T and strike K on a default-free equity stock with price S in a market with constant and deterministic interest rates r. Let $Y^{AN} = S_T 1_{\{S_T > K\}}$, $Y^{CN} = K 1_{\{S_T > K\}}$ be the final payoff of the ANO and CNO options at maturity T, respectively. Assume that the stock price S follows a geometric brownian motion under the risk neutral measure, with volatility σ and with deterministic initial value s_0 at time 0, namely

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s_0$$

where r, σ are positive constants and W is a brownian motion under the risk neutral measure Q.

CVA for an Asset or Nothing Option II

Assume further that the ANO is held by a party “A” and, if the option is exercised by “A” at maturity, the asset S will be delivered at maturity T by the counterparty “C” to “A”.

Assume further that the probability of default of “C” is associated to an intensity model with random intensity λ that is constant in time and uniformly distributed in the interval $[0, L]$. Assume λ is independent of the stock price S. The recovery rate associated to default of ”C” is assumed to be a deterministic constant R.

a) Derive a formula for the price of the ANO seen by “A” in a world without default risk.

b) Compute the Credit Valuation Adjustment (CVA), as seen from “A” at time 0, for the price of the ANO. Make the formula as explicit as possible.

CVA for an Asset or Nothing Option III

c) Compute the CVA sensitivity to the underlying equity price, namely

$$\frac{\partial \mathrm{CVA}}{\partial s_0},$$

and comment on the sign of the resulting sensitivity.

CVA for an Asset or Nothing Option IV

Solution.
a) We compute both option prices via Risk Neutral Valuation,

$$V_{BS}(0) = \mathbb{E}^Q\left(e^{-rT}Y\right).$$

We notice immediately that

$$Y^{AN} - Y^{CN} = (S_T - K)1_{\{S_T > K\}} = (S_T - K)1_{\{S_T - K > 0\}} = (S_T - K)^+,$$

namely a call option payoff. Hence it is enough to derive the price of one of the two options and the remaining one can be derived using the fact that $\mathbb{E}^Q\left(e^{-rT}Y^{CALL}\right) =$

$$= \mathbb{E}^Q\left(e^{-rT}(Y^{AN} - Y^{CN})\right) = \mathbb{E}^Q\left(e^{-rT}Y^{AN}\right) - \mathbb{E}^Q\left(e^{-rT}Y^{CN}\right)$$

CVA for an Asset or Nothing Option V

or

$$V_{BS}^{CALL}(0) = V_{BS}^{AN}(0) - V_{BS}^{CN}(0).$$

It is easier to compute the cash or nothing option price. Compute

$$V_{BS}^{CN}(0) = \mathbb{E}^Q\left[e^{-rT}K1_{\{S_T>K\}}\right] = e^{-rT}K\mathbb{E}^Q\left[1_{\{S_T>K\}}\right] = e^{-rT}K\mathbb{Q}(S_T > K)$$

since from basic probability we know that $E^Q[1_A] = \mathbb{Q}(A)$. We are now left with computing $\mathbb{Q}(S_T > K)$. We recall the SDE for S under the risk neutral measure $\mathbb{Q}$:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0.$$

Ito's formula for the natural logarithm $\ln S_t$ gives easily (exercise, write this in detail)

$$d\ln(S_t) = (r - \sigma^2/2)dt + \sigma dW_t$$

CVA for an Asset or Nothing Option VI

from which, writing in integral form and recalling that $W_0 = 0$

$$\ln S_T - \ln S_0 = (r - \sigma^2/2)T + \sigma W_T \sim (r - \sigma^2/2)T + \sigma\sqrt{T}\mathcal{N}(0,1)$$

Now we write

$$\mathbb{Q}(S_T > K) = \mathbb{Q}(\ln S_T > \ln K) = ...$$

because logarithm is an increasing function; by substituing our expression for $\ln S_T$

$$... = \mathbb{Q}(\ln S_0 + (r - \sigma^2/2)T + \sigma\sqrt{T}\mathcal{N}(0,1) > \ln K) =$$

$$= \mathbb{Q}(\sigma\sqrt{T}\mathcal{N}(0,1) > -\ln(S_0/K) - (r - \sigma^2/2)T) =$$

$$= \mathbb{Q}\left(-\mathcal{N}(0,1) < \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) =$$

CVA for an Asset or Nothing Option VII

$$= \Phi\left(\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) = \Phi(d_2(0))$$

where Φ is the cdf of the standard normal and where we used the fact that the opposite of a standard normal is still a standard normal. Going back to our formula

$$V_{BS}^{CN}(0) = e^{-rT}K\mathbb{Q}(S_T > K) = e^{-rT}K\Phi(d_2(0)).$$

Recall the BS formula for a Call option we wrote earlier, written at time 0

$$V_{BS}^{CALL}(0) = S_0\Phi(d_1(0)) - e^{-rT}K\Phi(d_2(0)).$$

Hence

$$V_{BS}^{CALL}(0) = S_0\Phi(d_1(0)) - V_{BS}^{CN}(0).$$

CVA for an Asset or Nothing Option VIII

Since we have seen a few lines above than

$$V_{BS}^{CALL}(0) = V_{BS}^{AN}(0) - V_{BS}^{CN}(0)$$

it follows immediately by difference that

$$V_{BS}^{AN}(0) = S_0\Phi(d_1(0)).$$

b) With a procedure completely analogous to what we did for the Call option CVA formula exercise (but write all the steps anyway: CVA definition, use positivity of payout to remove $(\cdot)^+$, use tower property and use independence of τ from S), we arrive at

$$CVA = (1 - Rec)E_0[1_{\{\tau<T\}}]E_0[D(0,T)Y^{AN}] =$$

$$= (1 - Rec)\mathbf{Q}(\tau < T)V_{BS}^{AN}(0)$$

CVA for an Asset or Nothing Option IX

Again using the above exercises and point a) (but write all calculations anyway), we can write

$$CVA = (1 - REC)\left(1 - \frac{1}{LT}(1 - \exp(-LT))\right) S_0 \Phi(d_1(0))$$

c)
Call $A = (1 - REC)\left(1 - \frac{1}{LT}(1 - \exp(-LT))\right)$, so that

$$CVA = A\, S_0 \Phi(d_1(0)).$$

Then

$$\frac{\partial \mathrm{CVA}}{\partial S_0} = A \frac{\partial (S_0 \Phi(d_1(0)))}{\partial S_0} =$$

CVA for an Asset or Nothing Option X

$$= A[\Phi(d_1(0)) + S_0 \partial_{S_0} \Phi(d_1(0))]$$

Now compute

$$\partial_{S_0} \Phi(d_1(0)) = \Phi'(d_1(0))) \frac{\partial d_1(0)}{\partial S_0} =$$

$$= p(d_1(0)) \partial_{S_0} \left(\frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \right) =$$

$$= p(d_1(0)) \left(\frac{1}{S_0 \sigma \sqrt{T}} \right)$$

where p is the derivative of the standard normal cdf, ie the standard normal density

$$p(x) = \exp(-x^2/2)/(\sqrt{2\pi}).$$

CVA for an Asset or Nothing Option XI

We obtain the sensitivity

$$\frac{\partial \text{CVA}}{\partial S_0} = A\left[\Phi(d_1(0)) + S_0 p(d_1(0))\left(\frac{1}{S_0\sigma\sqrt{T}}\right)\right]$$

One may also notice that $\Phi(d_1)$ is the call option delta and $\partial_{S_0}\Phi(d_1)$ is the call Gamma.

The sensitivity is positive since each single term and factor in the formula is positive and we are adding positive quantities. Positivity of the sensitivity means that when the underlying stock increases our CVA will increase as well. This is intuitive, since when the spot increases the asset or nothing option gets more in the money, as $S > K$ becomes more likely, and also S increases in the $S\,1_{\{S>K\}}$ product, and the option embedded in CVA also gets in the money in more scenarios and larger, increasing the overall value of CVA at time 0.