Hopf Fibration

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A quick adventure

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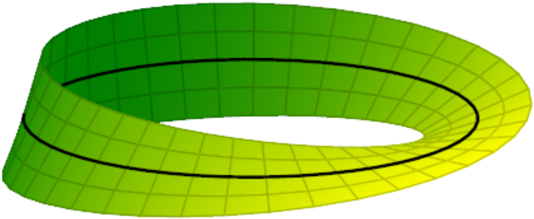
Background: Fiber Bundles



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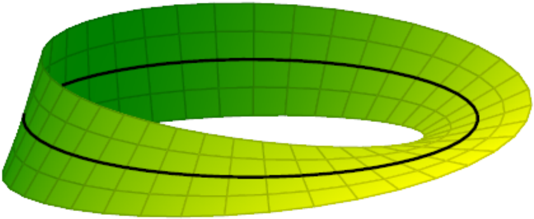
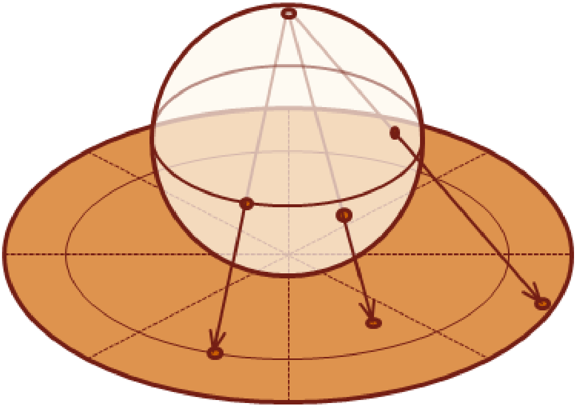


Definition

X is a fiber bundle over B if there exists $\pi\colon X \to B$ s.t. each $x \in B$ has a neighborhood U, so that $\pi^{-1}(U)$ is homeomorphic to $F \times B$, where F is the fiber space, or just fiber.

Stereographic Projection



Hopf Fibration

Construction

$$\zeta \colon S^3 \to S^2$$
$$(z_1, z_2) \mapsto \pi^{-1}\left(\frac{z_1}{z_2}\right)$$

where $z_1, z_2 \in \mathbb{C}$ and π is the stereographic projection map.

Remarks

- "Great circles" are of the form $\lambda(z_1, z_2)$ where $\lambda \in \mathbb{C}$ and $|\lambda| = 1$
- and are each mapped by ζ to single points in S^2.

Hopf Fiber Bundle

More Remarks

- $S^2 \times S^1 \neq S^3$
- however locally, it is!

Fibers

- S^3 can be viewed as a fiber bundle, where the circles are the fibers!
- This is because ζ is actually a submersion onto S^2, making it diffeomrphically, locally the same as the canonical projection map.

How can we see it?

Projections

$$S^2 \xrightarrow{\zeta^{-1}} S^3 \xrightarrow{\pi} \mathbb{R}^3 \xrightarrow{\phi} B$$

where B is the unit ball in $\mathbb{R}^3$.

ζ^{-1} in $\mathbb{R}^4$

$$x_1 = \cos\left(\frac{\xi_1 + \xi_2}{2}\right)\sin\eta$$

$$x_2 = \sin\left(\frac{\xi_1 + \xi_2}{2}\right)\sin\eta$$

$$x_3 = \cos\left(\frac{\xi_1 - \xi_2}{2}\right)\cos\eta$$

$$x_4 = \sin\left(\frac{\xi_1 - \xi_2}{2}\right)\cos\eta$$

Rendering in OpenGL

Caveat no. 1

- "Speed" of circle paramterizations was NOT uniform.
- To fix this, we can reparameterize them w.r.t. arc length

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- To fix this, we can reparameterize them w.r.t. arc length
- or do it the less math way.

Caveat no. 2

Point at infinity.

Improvements and next project

Improvements

- Recoloring
- Easier parameter manipulation
- Generalize to more fibrations?

Next project ideas

- Live in a manifold (start with hyperbolic?)
- Shape analysis (vision processing with manifolds)