Notes on Diversification and the Capital Asset Pricing Model

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March 13, 2016

1 Mean-Variance Optimization

What do we want to do when we invest? The answer for most is that we want to make the greatest gains at the lowest possible risk to ourselves. Thus, in this section, I will lay out the math for doing this, and thus *optimizing the mean with respect to variance.*

1.1 The Benefits of Diversification

Where do the benefits of diversification come from? For those familiar with statistics, you will recognize the equation:

$$Var(aX + bY) = a^2Var(x) + b^2Var(Y) + abCov(X, Y) \tag{1}$$

So what does this tell us? It means that spreading our money across different securities is the best plan of action. Here's an example: Using the standard deviation of returns on 50 different stocks, we see

Min	1st Qu	Median	3rd Qu.	Max
.04640	.07543	.09380	0.11630	.23270

but as it turns out, an equal weighted portfolio of this leads to a standard deviation of returns of .05667.

But we can do better than just reducing the expected standard deviation.

1.2 Minimum Variance Portfolio

To find the portfolio with the minimum variance, we must first conceptualize a way to represent the portfolio. What is a portfolio? It is a collection of holdings. Let us then define each holding as being indexed from $1, 2, ..., k$. And to think about how much we invest in each security, we can create weights, or percentages of the portfolio in each security, with the condition that $\omega_1 + \omega_2 + ... + \omega_k = 1$. The return of the portfolio is now really easy to compute, as it is simply

$$\vec{\omega} * \mathbf{r} \tag{2}$$

where $\vec{\omega}$ is the vector of weights and $\mathbf{r}$ is the vector of returns.

Let $\mathbf{V}$ be the variance-covariance matrix of all of the assets in our universe. That is to say that the i, j position in the matrix tracks the covariance of the ith and jth securities. If i and j are the same, it will simply be the variance of that security.

Thus to find the variance of our portfolio with generic weights, we can use the formula

$$\sigma^2 = \vec{\omega}^T * \mathbf{V} * \vec{\omega} \tag{3}$$

With the problem of notation out of our way, we should now turn to the problem of actually minimizing variance. To find the lowest possible variance of a portfolio, we can use the equation:

$$\frac{\mathbf{V}^{-1} * \vec{1}}{\vec{1}^T * \mathbf{V}^{-1} * \vec{1}} \tag{4}$$

This particular equation can be justified by linear algebra and multivariate calculus.

1.3 Tangency Portfolio

1.3.1 The Sharpe Ratio

The Sharpe Ratio is a measure of performance of portfolios and securities, and is given by:

$$SR_p = \frac{R_p - R_{rf}}{\sigma_p} \tag{5}$$

Where R_p is the return of the portfolio, R_{rf} is the return of an appropriate risk-free asset, and σ_p is the standard deviation of the portfolio. It's generally seen as a good measure of optimization. So why not optimize that?

1.3.2 The Tangency Portfolio

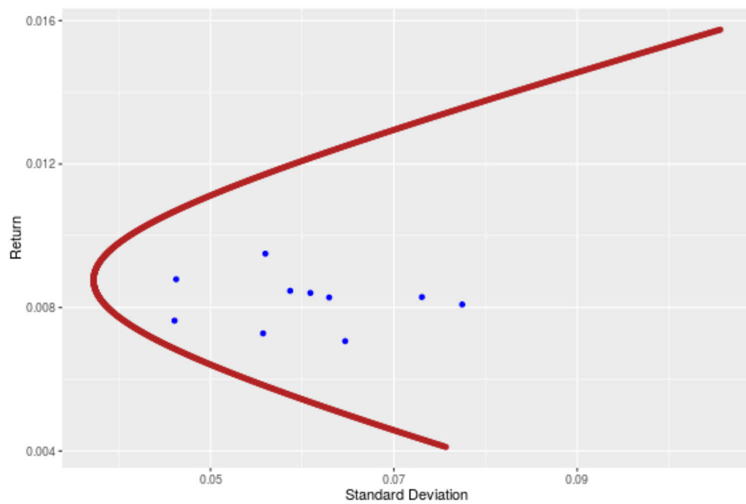
Using a similar notation to above, we can show that the optimal weights are

$$\frac{\mathbf{V}^{-1} * (\overrightarrow{R_p - R_{rf}})}{\vec{1}^T * \mathbf{V}^{-1} * (\overrightarrow{R_p - R_{rf}})} \tag{6}$$

This comes from a similar form of optimization as to finding the minimum variance portfolio.

1.4 The Efficient Frontier

Here's the coolest part of the whole process. We can create a "frontier" of portfolios with the best and worst possible returns for any given level of risk (standard deviation). It will form a hyperbola. We do this by finding all linear combinations of any two points on the frontier. As it happens, I just walked you through finding two! Here's what it looks like for the given dataset.



The blue dots represents the historical returns and standard deviations of the ten industries tracked in the dataset.

2 Capital Asset Pricing Model

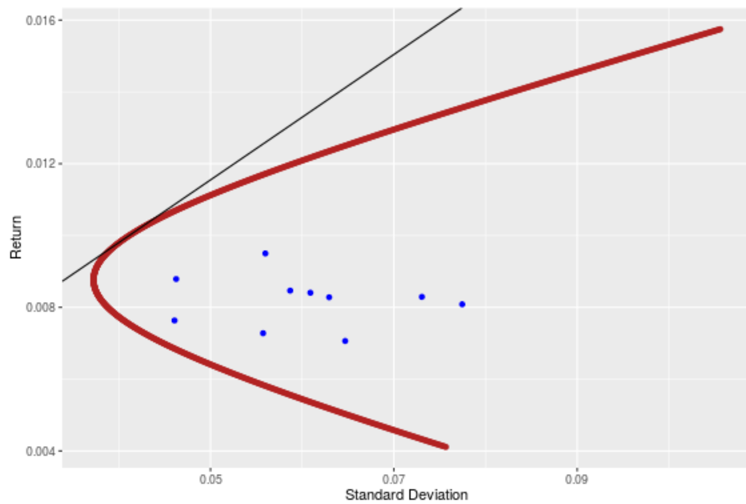
We can actually do better just mean-variance optimization- we can get a mathematical model from it! This model is called the Capital Asset Pricing Model (CAPM), and was introduced to financial theory in the 1960s. The equation is

$$E(R_s) = E(R_{rf}) + \beta E(R_m - R_{rf}) \tag{7}$$

Where R_s, R_m are the returns of the security and the market respectively, and $\beta = \frac{Cov(S,M}{\sigma_s^2}$. The derivation of β comes from standard linear algebraic and econometric theory.

The line given by this equation should be the highest expected return, and it takes into account the risk. Thus, just as we can draw a hyperbola in mean-variance space, we can also draw a line representing the "market" securities line in mean-variance space. In fact, every security on this line (with the exception of the risk free asset) will have the same Sharpe Ratio. But how can we actually draw this line? We need at least two points to determine the line itself, **and**

we have both a risk free asset and an asset with a maximized Sharpe Ratio!

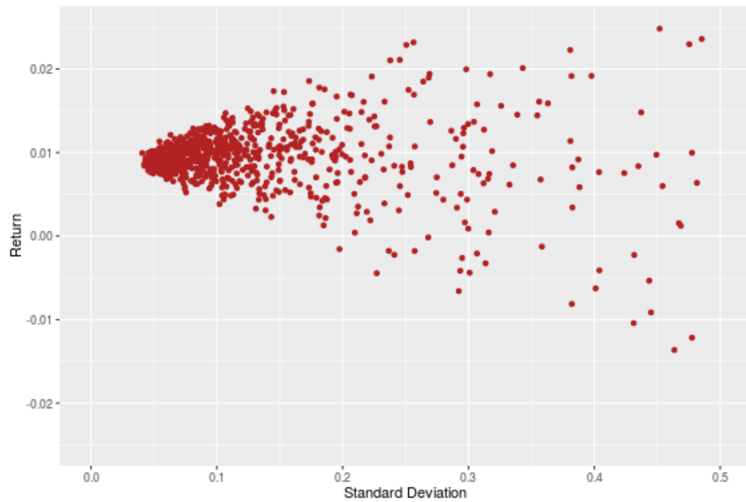


In fact, if this theory is true, there are some phenomenal consequences, namely that holding the market portfolio will have the same Sharpe Ratio as optimally balancing your portfolio. This is why CAPM has been so historically connected to the efficient markets hypothesis, which implies that people can't consistently beat the market. There are several benefits to the CAPM, namely that it is easy to determine the "beta" or implicit risk of a stock under the assumptions of CAPM and that it gives us a conceptual framework to think about the market.

2.1 Problems with the CAPM and Markowitz Theory

There are two big problems with the CAPM and the underlying Markowitz Portfolio Theory: the portfolios perform poorly out of sample, and the expected return of a stock seems to depend on more than just the expected market return.

What do I mean when I say that it performs poorly out of sample? Mean returns are actually a random variable. The mean of a time series varies ever so slightly from day to day, month to month, getting ever closer to the "true" mean, but not truly reaching it. This is also a (smaller) problem with estimating variances and covariances. So when we estimate our tangent weights, we can actually create some gross miscalculations in weights.



The above picture is a Monte Carlo simulation of tangent weights and subsequent returns and standard deviations. Using 5 years worth of simulated historical data, I calculated the tangent weights implied by this, and then found the returns and standard deviation when applied to the entire time period, 1926-2015. I then repeated this process 1000 times. The above chart leaves off the 85 riskiest portfolios.

As I mentioned, there are other problems with the CAPM- for instance, the "small cap" premium that is often observed means that smaller companies have higher expected returns than otherwise expected.

3 Where To Go From Here?

There should be three things that I want you to take away from this presentation and set of notes.

Diversification As it turns out, diversification is extremely important. If you don't hedge, expect to lose!

Modeling The process of modeling is essential to approaching the market and making investing decisions. When we have the right knowledge, tool set, and paradigm, we can hope to do well and perhaps even "beat the market".

Skepticism Always be cautious of any strategy. If it works too well, you're probably doing it wrong. The CAPM and modern portfolio theory worked really well in theory, but as I showed you, we were able to poke holes in them until the market makers abandoned them by the 1980s.