# ECE-3111: System Analysis Project

## I. Project Description

In this project, we will model the quadcopter system from the first principles of physics. The main aim of the project is to control the position and attitude (aircraft orientation relative to the vehicle's center of gravity) of a quadcopter by manipulating the angular velocities of its rotors. You will have to design a PID and LQR controller that drives the quadcopter to a specified height. The quadcopter should able to fly within $\pm 0.01524$ meters of the specified height.

The project is divided into three parts:

1. Mathematical modeling of the quadcopter platform. We will start with the nonlinear equations of motion (EoM). We will then linearize the EoM to form a linearized system. We will derive transfer functions and state-space models.
2. We will study the open-loop response of the quadcopter plant in Matlab.
3. We will design a Proportional-Integral-Derivative (PID) and linear quadratic regulator (LQR) controllers to stabilize the quadcopter system.

## II. Quadcopter Dynamics

A quadcopter is an aerial vehicle that is able to hover. It has four identical rotors arranged at the corners of a square body, and its propellers or blades have a fixed angle of attack. Figure 1 shows the diagram of the quadcopter that you will have to control. Notice that the rotors are paired, and each pair rotates in a different direction. Motors 1 and 3 rotate clockwise when looked from above, whereas motors 2 and 4 have a counter-clockwise rotation. When all the motors rotate at the same angular velocity, the torques $\tau_1$, $\tau_2$, $\tau_3$ and $\tau_4$ (these are the counter torques applied to the aircraft as a consequence of the motors rotation) will cancel each other out and the quadcopter will not spin about its $z^b$-axis ( $\psi = 0$). The quadcopter will hover when the angular velocities are such that the total thrust $(f_1 + f_2 + f_3 + f_4)$ generated by the rotors is equal to the force of gravity.

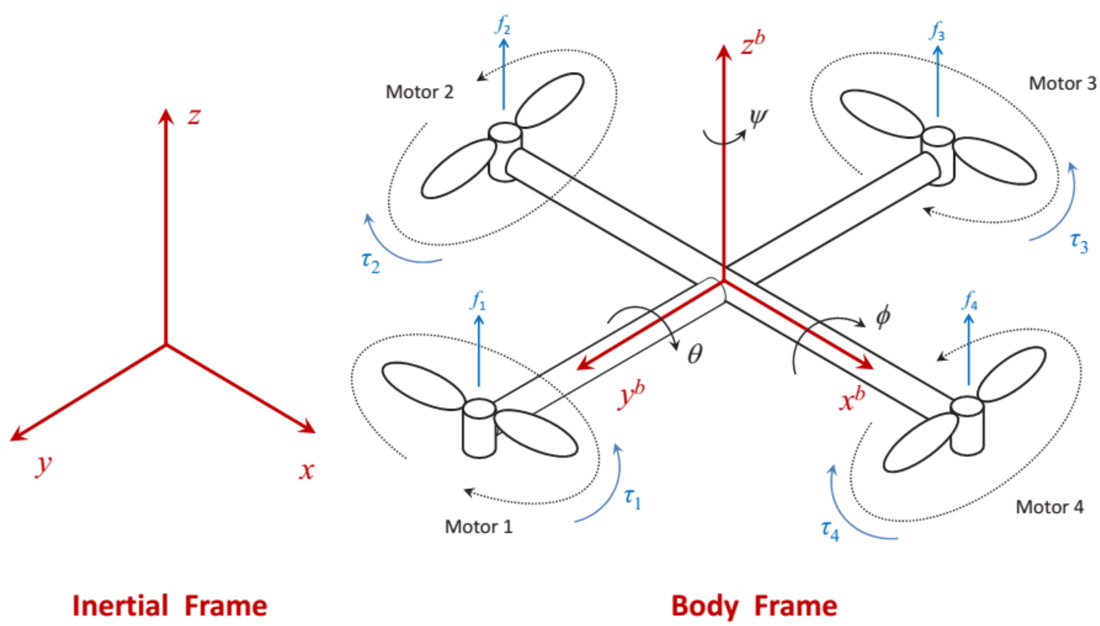


Figure 1. Quadcopter configuration. The roll, pitch and yaw angles are denoted by $\phi$,$\theta$ and $\psi$ respectively. Motor 1 and 3 rotate clockwise and motor 2 and 4 rotate counter-clockwise as indicated by the arrows with black dotted lines. $f_1$, $f_2$, $f_3$and $f_4$are the thrusts generated by the rotors about their centers of rotation. $\tau_1$, $\tau_2$, $\tau_3$ and $\tau_4$ are the torques applied to the aircraft (counter torques) are a consequence of the spinning of the rotors.

In order describe the movement of the quadrotor and its attitude, two frames of reference are used, namely, the inertial frame and the body frame (see Figure 1). The inertial frame is defined by the ground, with gravity pointing in the negative z direction. The body frame is defined by the orientation of the quadcopter, with the rotor axes pointing in the positive $z^b$ direction and the arms pointing in the $x^b$ and $y^b$ directions.

The attitude of the quadcopter is determined by three angles, namely, roll-$\phi$, pitch-$\theta$ and yaw-$\psi$. The way of changing these angles by playing with the angular velocities of the rotors. Changes in the roll and pitch angles are accompanied by translational motion. It should be clear that a quadcopter is an underactuated vehicle, since it only has $4$ actuators (rotors) for controlling $6$ degrees of freedom (three translational, $x$, $y$, $z$ and three rotational, $\phi$, $\theta$, and $\psi$). Let's denote the angular velocities of the motors by $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$, and the angular velocity to which the quadcopter hovers by $\omega_{hover}$. The vertical takeoff and landing motions (change in the $z$-axis) are obtained by equally augmenting or diminishing the angular speed of all motors with respect to hover.

## III. Quadcopter model

The translational motion of the quadcopter in the inertial frame is described by the following set of equations:

$$m\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}=\begin{bmatrix}0\\0\\-mg\end{bmatrix}+RT^b+F_D \tag{1}$$

where $x$, $y$ and $z$ are the coordinates of the position of the quadcopter in the inertial frame, m is the mass of the system, g is the acceleration due to gravity, $F_D$ is the drag force due to air friction, $T^b \in \mathbb{R}^3$ is the thrust vector in the body frame, and $R \in \mathbb{R}^{3\times 3}$ is the rotation matrix that relates the body frame with the inertial frame and is defined as follows:

$$R=\begin{bmatrix}\cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi-\cos\phi\sin\psi & \sin\psi\sin\phi+\cos\psi\cos\phi\sin\theta\\ \cos\theta\sin\psi & \cos\psi\cos\phi+\sin\psi\sin\theta\sin\phi & \cos\phi\sin\psi\sin\theta-\cos\psi\sin\phi\\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi\end{bmatrix}$$

The drag $F_D$ due to air friction is modeled as a force proportional to the linear velocity in each direction

$$F_D=-k_d\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}.$$

Here, $k_d$ is the air friction coefficient. The thrust $f_i$ generated by the $i^{th}$ rotor is given by the following expression

$$f_i=k\omega_i^2, \qquad \text{for } i=1,...,4,$$

where $k$ is the propeller/rotor lift coefficient and $\omega_i$ is the angular velocity of the $i^{th}$ motor. The total thrust $T^b$ generated by the $4$ rotors (in the body frame) is given by

$$T^b=\sum_{i=1}^{4} f_i=k\begin{bmatrix}0\\0\\ \sum_{i=1}^{4}\omega_i^2\end{bmatrix}.$$

In this study it is assumed that the dynamics of the motors is much faster than the one of the quadcopter, and therefore is not taken into account. Since the angular velocity of each rotor is typically proportional to the applied voltage, we have that

$$\omega_i^2=c_m v_i^2, \qquad \text{for } i=1,...,4,$$

where $c_m$ is a constant and $v$ is the voltage applied to the rotor.

While it is convenient to have the linear equations of motion in the inertial frame, the rotational equations of motion are useful to us in the body frame, so that we can express rotations about the center of the quadcopter instead of about the inertial center. To this end we can use the Euler's equations for rigid body dynamics, which are defined as follows:

$$\mathbf{I}\dot{\omega}+\omega\times(\mathbf{I}\omega)=\tau \tag{2}$$

where "$\times$" denotes cross product, $\mathbf{I}\in\mathbb{R}^{3\times 3}$ is the inertia matrix, $\omega=[\omega_x\omega_y\omega_z]^T$ is the angular velocity vector, and $\tau=[\tau_\phi\tau_\theta\tau_\psi]^T$ is the vector of external torques.

We can model the quadcopter as two thin uniform rods crossed at the origin with a point mass (motor) at each end. This results in a diagonal inertia matrix of the following form

$$\mathbf{I}=\begin{bmatrix}I_{xx} & 0 & 0\\ 0 & I_{yy} & 0\\ 0 & 0 & I_{zz}\end{bmatrix},$$

where $I_{xx}$, $I_{yy}$ and $I_{zz}$ are the moments of inertia of the quadcopter about the $x^b$, $y^b$ and $z^b$ axes respectively. After computing the cross product, equation (2) reduces to

$$\begin{bmatrix}I_{xx} & 0 & 0\\ 0 & I_{yy} & 0\\ 0 & 0 & I_{zz}\end{bmatrix}\begin{bmatrix}\dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\omega}_z\end{bmatrix}=\begin{bmatrix}\tau_\phi\\ \tau_\theta\\ \tau_\psi\end{bmatrix}-\begin{bmatrix}(I_{yy}-I_{zz})\,\omega_y\omega_z\\ (I_{zz}-I_{xx})\,\omega_z\omega_x\\ (I_{xx}-I_{yy})\,\omega_x\omega_y\end{bmatrix} \tag{3}$$

The roll $\tau_\phi$ and pitch $\tau_\theta$ torques are derived from standard mechanics as follows

$$\tau_\phi = L(f_1 - f_3) = Lk\left(\omega_1^2 - \omega_3^2\right) = Lkc_m\left(v_1^2 - v_3^2\right),$$
$$\tau_\theta = L(f_2 - f_4) = Lk\left(\omega_2^2 - \omega_4^2\right) = Lkc_m\left(v_2^2 - v_4^2\right),$$

where $L$ is the distance between the rotor and the quadcopter center (radius). As it was discussed earlier, all the rotors apply torques to the aircraft about its $z^b$-axis while they rotate. In order to have an angular acceleration about the $z^b$-axis, the total torque generated by the rotors has to overcome the drag forces. The total torque about the $z^b$-axis, that is the yaw torque is given by the following equation:

$$\tau_\psi = b\left(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2\right) = bc_m\left(v_1^2 - v_2^2 + v_3^2 - v_4^2\right)$$

where $b$ is the propellers drag coefficient.

The roll, pitch and yaw rates are related to the components of the angular velocity vector by means of the following expression:

$$\begin{bmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{bmatrix} = \begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\theta\\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \tag{4}$$

where Q is a projection matrix.

Finally, from equations (1), (3), and (4) we can write down the the entire model of the quadcopter in a state space form as follows:

$$\dot x = v_x \tag{5}$$
$$\dot y = v_y \tag{6}$$
$$\dot z = v_z \tag{7}$$
$$\dot v_x = -\frac{k_d}{m}v_x + \frac{kc_m}{m}\left(\sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta\right)\left(v_1^2 + v_2^2 + v_3^2 + v_4^2\right) \tag{8}$$
$$\dot v_y = -\frac{k_d}{m}v_y + \frac{kc_m}{m}\left(\cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi\right)\left(v_1^2 + v_2^2 + v_3^2 + v_4^2\right) \tag{9}$$
$$\dot v_z = -\frac{k_d}{m}v_z - g + \frac{kc_m}{m}\left(\cos\theta\cos\phi\right)\left(v_1^2 + v_2^2 + v_3^2 + v_4^2\right) \tag{10}$$
$$\dot\phi = \omega_x + \omega_y\left(\sin\phi\tan\theta\right) + \omega_z\left(\cos\phi\tan\theta\right) \tag{11}$$
$$\dot\theta = \omega_y\left(\cos\phi\right) + \omega_z\left(-\sin\theta\right) \tag{12}$$
$$\dot\psi = \omega_y\left(\sin\phi/\cos\theta\right) + \omega_z\left(\cos\phi/\cos\theta\right) \tag{13}$$
$$\dot\omega_x = \frac{Lkc_m}{I_{xx}}\left(v_1^2 - v_3^2\right) - \left(\frac{I_{yy} - I_{zz}}{I_{xx}}\right)\omega_y\omega_z \tag{14}$$
$$\dot\omega_y = \frac{Lkc_m}{I_{yy}}\left(v_2^2 - v_4^2\right) - \left(\frac{I_{zz} - I_{xx}}{I_{yy}}\right)\omega_x\omega_z \tag{15}$$
$$\dot\omega_z = \frac{Lkc_m}{I_{zz}}\left(v_1^2 - v_2^2 + v_3^2 - v_4^2\right) - \left(\frac{I_{xx} - I_{yy}}{I_{zz}}\right)\omega_x\omega_y \tag{16}$$

Notice that in this model we have not considered the ground effect. This means that in principle we can have negative values for the $z$ coordinate of the quadcopter. If we denote the state vector of the system by $\mathbf{x} \in \mathbb{R}^{12} = [x\ y\ z\ v_x\ v_y\ v_z\ \phi\ \theta\ \psi\ \omega_x\ \omega_y\ \omega_z]^T$ , and we define the input and output vectors as $\mathbf{u} \in \mathbb{R}^4 = [v_1^2\ v_2^2\ v_3^2\ v_4^2]^T$ and $\mathbf{y} \in \mathbb{R}^6 = [x\ y\ z\ \phi\ \theta\ \psi]^T$ respectively, we can write down the model of the quacopter in a more compact way

$$\dot{\mathbf{x}} = \xi\left(\mathbf{x}, \mathbf{u}\right) \tag{17}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{18}$$

where $\mathbf{C} \in \mathbb{R}^{6\times 12}$ is the output matrix and is a nonlinear vector function defined by (5)-(16). Notice that the the input vector is in terms of the **squared voltages** of the rotors, and therefore **your control system should compute** $v_1^2, v_2^2, v_3^2$, and $v_4^2$ instead of $v_1, v_2, v_3$, and $v_4$. The maximum voltage that can be applied to the motors is 10 volt. To make sure that this constraint is not violated, the circuitry within the quadcopter incorporates a clipping mechanism. Observe that negative voltages would change the spinning direction of the motors and therefore would invalid the model that has been derived. Based on the previous considerations, it is clear that the input constraints of the system are given by

$$0\ \text{volt}^2 \le u_1, u_2, u_3, u_4 \le 100\ \text{volt}^2$$

where $u_1 = v_1^2$, $u_2 = v_2^2$, $u_3 = v_3^2$ and $u_4 = v_4^2$. In the present setup, the attitude $(\phi, \theta, \psi)$ of the quadcopter is computed from the readings of a gyroscope and an accelerometer installed in the aircraft. The position $(x, y, z)$ of the vehicle is determined by a set of cameras installed within the room.

| Parameter | Symbol | Value | Unit |
|---|---|---|---|
| Mass of the quadcopter | $m$ | 0.5 | kg |
| Radius of the quadcopter | $L$ | 0.25 | m |
| Propeller lift coefficient | $k$ | $3 \cdot 10^{-6}$ | $\mathrm{N\,s^2}$ |
| Propeller drag coefficient | $b$ | $1 \cdot 10^{-7}$ | $\mathrm{N\,m\,s^2}$ |
| Acceleration of gravity | $g$ | 9.81 | $\mathrm{m/s^2}$ |
| Air friction coefficient | $k_d$ | 0.25 | kg/s |
| Quadcopter inertia about the $x^b$-axis | $I_{xx}$ | $5 \cdot 10^{-3}$ | $\mathrm{kg\ m^2}$ |
| Quadcopter inertia about the $y^b$-axis | $I_{yy}$ | $5 \cdot 10^{-3}$ | $\mathrm{kg\ m^2}$ |
| Quadcopter inertia about the $z^b$-axis | $I_{zz}$ | $1 \cdot 10^{-2}$ | $\mathrm{kg\ m^2}$ |
| Motor constant | $c_m$ | $1 \cdot 10^{4}$ | $\mathrm{v^{-2}s^{-2}}$ |

Figure 2. Parameters of the quadcopter.

## IV. Assignment

Make a report that answers following assignment questions:

*Problem #1 (Linearization)*

In order to design a controller using linear system analysis tools, it is necessary to have a linear approximation of the nonlinear model around an operating point. In this exercise such operating point, which is also an equilibrium point, is the one of a stationary flight (the quadcopter is in hover).

- Determine the linearization/equilibrium point of the vehicle for $\phi = \theta = \psi = 0$ and $x = y = 0$, $z = 1m$. Provide symbolic expression and numerical values of matrices.

*Problem #2 (System Modeling - State Space & Transfer Function)*

1) Compute the state space model with $\mathbf{x} = [x\ y\ z\ v_x\ v_y\ v_z\ \phi\ \theta\ \psi\ \omega_x\ \omega_y\ \omega_z]^T$ as the state, $\mathbf{u} = [u_1, u_2, u_3, u_4]^T$ as the control input and $\mathbf{y} = [x\ y\ z\ \phi\ \theta\ \psi]^T$ as the measurement. Write down your result in the following form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$
$$\mathbf{y} = C\mathbf{x}$$

Your job is the figure out $A$, $B$, $C$ matrices, $D$ will be zero in this case. Write down what are the dimensions of $A$, $B$, and $C$?

2) Compute the transfer functions from the inputs $u_1$, $u_2$, $u_3$, and $u_4$ to the outputs $x$, $y$, $z$, $\phi$, $\psi$, $\theta$. You should use Matlab symbolic toolbox for this purpose. You can use the formula to compute $ij$th transfer function $G_{ij}(s) = C_i(sI - A)^{-1}B_j + D$. Pick the row of $C$ corresponding to $i$th output $\mathbf{y}$ and column corresponding to $j$th input $\mathbf{u}$. For example, for $x$ as the output choose the first row of $C$ matrix, for $y$ as the output choose the second row of $C$ matrix, for first control input $u_1$ choose first column of $B$, for second control input $u_2$ choose second column of $B$, and so on...


Figure 3. Quadcopter control structure

*Problem #3 (Control Conversion)*

Typical control structure of a Quadcopter looks like the one in Fig. 3. Given the measurements $\mathbf{y} = [x\ y\ z\ \phi\ \theta\ \psi]^T$ (in certain cases you can measure angular rates, i.e., $[\dot{\phi},\ \dot{\theta},\ \dot{\psi}]^T$) the control variables or inputs are $\phi, \theta, \psi$ and $z$. This controller is called as attitude controller. The attitude controller computes the commanded roll, pitch, yaw commands based on the current roll, pitch, yaw and the desired (or reference) roll, pitch, yaw and altitude (height). The model that you designed in Problems 1 and 2 uses lower level control inputs of motor speeds. The roll, pitch, yaw and height of the quadcopter is decided by how much motor speeds you give to four motors. So, there is a mapping between the motor speeds and the roll, pitch, yaw, altitude control commands. The mapping is as follows:

$$\begin{bmatrix} u_z \\ u_\psi \\ u_\theta \\ u_\phi \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0 & 0 \\ 1 & 0 & -0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \\ \omega_4^2 \end{bmatrix} \tag{19}$$

Use the transformation in (19) to convert the transfer function from $z$, $\phi$, $\theta$, $\psi$ to $u_1$,$u_2$, $u_3$ and $u_4$ **to** $z$, $\phi$, $\theta$, $\psi$ to $u_z$,$u_\psi$, $u_\theta$ and $u_\psi$. This is just an algebraic conversion. Use the formula $\omega_i^2 = c_m v_i^2 = c_m u_i$ for $i = 1, 2, 3, 4$.

*Problem #4 (PID Control Design and Analysis)*

Design a P and PID controller for the transfer functions. Do it using the following steps:

1. Draw a block diagram of a closed loop system with a plant, controller and sensor transfer function blocks. Mark all the signals. Your block diagram should include a disturbance input.

2. Draw a root locus using maltab 'rlocus' command and determine if:

a) P controller can stabilize the system?

b) PI controller can stabilize the system?

c) PID controller can stabilize the system?

3. Draw Bode plot and determine the gain and phase margin of the open loop transfer function.

4. Design a PID controller so that the vehicles hovers at 5 feet height starting at ground level. Use Matlab PID tool to tune the PID gains.

5. Plot the following quantities:

a. $\phi$, $\psi$, $\theta$ vs time;

b. $x$, $y$, $z$,

c. $\mathbf{u}$ vs time.

*Problem #5 (State space Control Design and Analysis) (Honors section only)*

Design a state space controller such that following criterion is satisfied: Settling time of less than 10 seconds.

## V. Remarks about your report

The report should be written preferably in English, and it should address every single point in Section 4. Your report should describe the technical aspects of the designed controllers, and clearly it should state which design specifications were met and which were not (and why they could not be achieved).

Not only the technical correctness of your report is evaluated but also the quality and presentation. So, make sure that everything is clear and well explained. For example, make sure you label every axis of every plot, that your figures include legends if necessary, that in the text you discuss or address what is presented in every figure of the manuscript, that the text is free of typos and well redacted, etc. In addition, your report must be self-contained. Do not refer the reader to your Matlab files. Keep in mind that we only ask you for your Matlab files for verification purpose only.