Tutorial session 3 - Integer Linear Programming

Many optimization problems can be formulated as ILP problems. This session concerns a permutation problem which can be efficienty solved with an ILP solver.

Problem. The restaurant OPTIPLAT is moving. It's time to fill the boxes. Let N be the number of items to pack. The items are numbered from 1 to N. Let v_n be the volume of item n.

The restaurant has boxes, all with a volume capacity equal to C. The goal is to pack the items into the various boxes so that :

— the total volume of the items in each box does not exceed the capacity C (Constraint $\mathcal{C}_1$),

— all items are packed (Constraint $\mathcal{C}_2$),

— the number of used boxes is minimized (Objective $\mathcal{O}_1$).

This is a problem known as « Bin packing problem ».

Instructions. The main objective of this session is to consider problems with various constraints, to bring them to ILP problems and to solve them. The developed scripts have to be tested with the file `VolumeItems.txt` which gives the volumes v_n for N=50 items.

The Matlab command `intlinprog` allows to solve ILP and MILP problems. The provided Matlab script `BinPacking.mlx` illustrates the use of a convenient type of representation for the desired minimizer : the `struct` data type.

Necessary files (data, scripts) may be downloaded from the Edunao webpage.

Q1. Propose a greedy heuristic to quickly obtain an upper bound on the optimal number of boxes. Implement this heuristic and indicate the number of boxes obtained.

Q2. We want to obtain a global minimizer of this bin packing problem by modeling it as an ILP problem. Let B be the upper bound on the number of boxes obtained in Q1. Let the following binary variables :

— $\{x_{n,b}\}_{1\leq n\leq N,\, 1\leq b\leq B}$: variables indicating that item n is placed in box b,

— $\{y_b\}_{1\leq b\leq B}$: variables indicating that box b is used.

Write the constraints $\mathcal{C}_1$ and $\mathcal{C}_2$ and the objective $\mathcal{O}_1$ using linear equalities or inequalities with respect to x and y (Hint for constraint $\mathcal{C}_1$: inequalities are of the form $\forall b$, $\sum_{1\leq n\leq N} \alpha_n x_{n,b} \leq \beta y_b$ where constants α_n and β are to be determined).

Q3. Formulate the problem as an ILP problem. Solve it numerically. Indicate the minimal number of boxes obtained.

Q4. Consider the Constraint $\mathcal{C}_3$: The items numbered from 1 to 5 are within the same group (group $\mathcal{G}_1$). They must be packed in at most 2 boxes.

Why is it possible to consider without loss of optimality that these items are packed in boxes 1 and 2 ? Add Constraint to the previous ILP problem and solve the new problem.

Indicate the minimal number of boxes as well as how items from group $\mathcal{G}_1$ are distributed in boxes 1 and 2.

Q5. Constraint $\mathcal{C}_4$ is similar to Constraint $\mathcal{C}_3$: The items numbered from 6 to 10 are within group $\mathcal{G}_2$ and must be packed in at most 3 boxes. We allow boxes to contain objects from the two groups.

Constraints $\mathcal{C}_3$ and $\mathcal{C}_4$ are to be satisfied. Why is it no longer possible to set the boxes where items from both groups will be packed ? We suggest to formulate Constraint $\mathcal{C}_4$ with linear inequalities using the linear expression $\sum_{1\leq b\leq B} b\, x_{n,b}$. What does this expression represent ? How to use it to formulate Constraint $\mathcal{C}_4$?

Adding these inequalities, solve the ILP problem and indicate the distribution of items from both groups.

Q6. Propose a strategy, still based on ILP modeling, to test the uniqueness of the minimizer, modulo permutations of box numbers. Test the proposed strategy on the data in the file `VolumeItems50.txt`.[1]

Q7. We now wish to :

— (Objective $\mathcal{O}_1$) Minimize the number of boxes,

— (Objective $\mathcal{O}_2$) Among all minimizers, choose the one with the maximal unused volume for the least filled box.

Propose a model using MILP taking into account the constraints $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$. Write the MILP problem with all of its equations (objective function, linear equalities and inequalities). No simulation is required.

1. The proposed strategy should not be specific to the data in the given file.