The University of Sydney

AERO2705: Space Engineering 1
AERO3760: Space Engineering 2

The Space Engineering Companion

An Introduction to the Mathematical Tools for Undergraduate Space Engineering Students

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Preface

To the reader, I write this document in the hopes that you might benefit from the extensive time I spent learning and researching various topics throughout my time in space engineering.

This is intended as a mathematical reference for space engineering students, with less of a focus on spacecraft design. Instead I hope that this serves as a useful reference that you can return to when you are unsure what the relevant equations are, or how they work.

Contents

Chapter 1: Coordinate Systems

Coordinate systems are one of those things that go by un-noticed. You learn about them in relatively random stages of your education and sometimes you don't even realise what it is you're learning. However, understanding the different kinds of coordinate systems can be extremely useful, so it is worth knowing about them when the time comes.

1.1 What is a Coordinate System?

Though they are rarely discussed in enough detail, coordinate systems are at the heart of all of the mathematics that you've likely ever learned.

A coordinate system can be as simple as the number line, where going in one direction is positive and the other is negative. More confusing coordinates can come up, though they are not very common in engineering problems, such as parabolic or hyperbolic coordinates.

1.2 A Quick Note on Metrics

A metric is a way of measuring how distance in a coordinate system is related to changes in each of the coordinates. Really to discuss this properly requires venturing into scary-sounding topics like differential and Riemannian geometry, where you would prove the existence of a Riemannian metric tensor for a given coordinate system.

In our case we call the metric $\mathrm{d}s^2$ the square of the very very small (infinitesimal) distance travelled. In cartesian coordinates we can just use the Pythagorean theorem to find the metric, and most other metrics are derived from the conversion to cartesian coordinates. Fundamentally the metric describes the square of the total distance travelled due to small changes in each of the coordinates.

Why would we need this? Because sometimes we want to describe the kinetic energy in not-so-nice coordinates, which means we need to know the speed of something, in this case we just use the metric, where

$$v^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{\mathrm{d}s^2}{\mathrm{d}t^2} \tag{1.1}$$

will end up giving us v^2 as a function of the velocities in each coordinate.

1.3 Linear Coordinate Systems

Linear coordinates are the typical xy coordinates you have used since the beginning. They are called linear because changing each coordinate at a constant rate results in straight line motion. More technically, they obey the rules of linear algebra.

Because they obey the rules of linear algebra, positions in linear coordinate systems can be represented as vectors. You can transform between different linear coordinate systems using matrix multiplication (which you might recall from linear algebra is actually why matrix multiplication is a linear transform).

In general, we can write any coordinate in a linear system $(q_1, q_2, \ldots, q_n)$ as a vector which is a linear

combination of the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$ i.e.

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + \cdots + q_n\mathbf{e}_n. \tag{1.2}$$

The most common form of this is $\mathbb{R}^3$ where a point is (x, y, z) and the basis vectors are $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ or $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ (whichever notation you prefer, throughout this I will usually choose the latter as it's more obvious which coordinate matches up with which basis vector).

We can write any transform between two linear systems (of the same dimension n) as a matrix multiplication or vector addition. For example, if you want to shift the origin of one linear system to another, you simply set the position in frame B to be the position in frame A plus the position of the origin of frame B in frame A i.e.

$$(\mathbf{r})_B = (\mathbf{r})_A + \mathbf{r}_{A \text{ rel. to } B}. \tag{1.3}$$

The circle brackets $(\cdot)_A$ denote that the vector is in the frame of A.

If the axes of B are rotated or scaled relative to A then you multiply this expression by a matrix $\mathbf{M}$

$$(\mathbf{r})_B = \mathbf{M}\left((\mathbf{r})_A + \mathbf{r}_{A \text{ rel. to } B}\right) \tag{1.4}$$

where you can find $\mathbf{M}$ by finding the basis vectors of B in the reference frame of A

$$\mathbf{M} = \begin{bmatrix} (\mathbf{e}_{B1})_A^\mathsf{T} \\ (\mathbf{e}_{B2})_A^\mathsf{T} \\ \vdots \\ (\mathbf{e}_{Bn})_A^\mathsf{T} \end{bmatrix}. \tag{1.5}$$

Here each basis vector is transposed so that it is the row of $\mathbf{M}$.

These transforms are discussed in much more detail in chapter 9.

1.3.1 Orthonormal Bases

Definition 1.1: Orthonormal Basis

A linear coordinate system is orthonormal if its basis vectors are all orthogonal to each other and have a magnitude of 1.

The matrix that transforms between such systems is called a rotation matrix (see chapter 2) and is a member of the special orthogonal group $\mathrm{SO}(n)$, which more generally defines orthogonal matrices.

This is by far the most common form of coordinate system as they possess special properties and identities such as these which make them easier than any other coordinate system to work with.

1.3.2 The Cartesian Coordinate System

Cartesian coordinates, formally referred to as $\mathbb{R}^2$, is the standard x-y plane that you are likely familiar with. It is the most basic form of a linear orthonormal coordinate system, as a change in just x or just y moves you in a straight line and the basis vectors are orthogonal (or perpendicular) to each other.

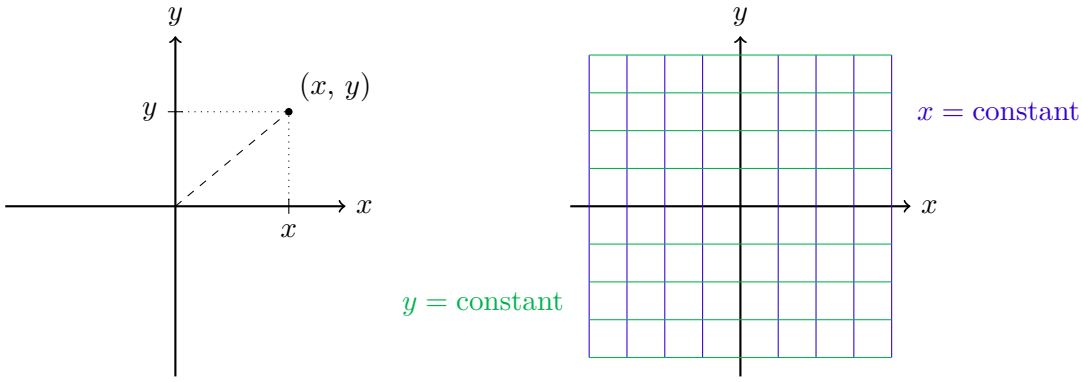

Figure 1.1: Cartesian coordinates in the Cartesian plane.

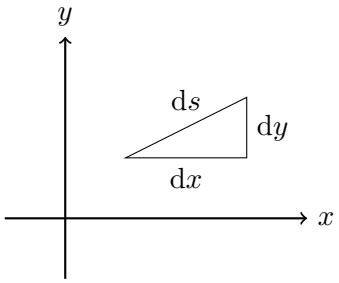

Figure 1.2: Calculation of the metric for cartesian coordinates using the Pythagorean theorem.

The metric in cartesian coordinates can be found using the Pythagorean theorem, as per figure 1.2, which gives

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \,. \tag{1.6}$$

1.3.3 Cartesian-Like Coordinates in 3 Dimensions

In 3D we commonly expand the cartesian coordinates and add a third coordinate, z to get (x, y, z). This is the standard 3D coordinate system that you are likely familiar with. Instead of lines of constant x and y we now have planes of constant x, y, and z.

The metric is found using the Pythagorean theorem in 3D, which gives

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{1.7}$$

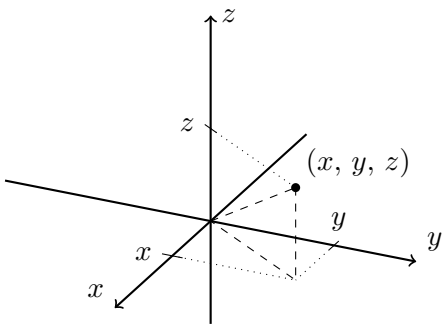

Figure 1.3: Cartesian coordinates in 3D space.

1.4 Non-Linear Coordinate Systems

Non-linear coordinate systems are the more general form of coordinates. Linear coordinate systems are a special subset of non-linear coordinates. Linear systems are so named because increasing the coordinates at a constant results in linear motion. Therefore a coordinate system that is not linear simply doesn't obey this.

1.4.1 Non-Linear Point Transforms

A point transform is simply a transform that describes a change in coordinate, it moves points from one coordinate system to another.

Take again as our example polar coordinates $(r,\, \theta)$, and the relevant transform from polar coordinates to cartesian coordinates $(x,\, y)$.

$$x = r\cos\theta, \quad y = r\sin\theta.$$

This transform maps points in polar coordinates to points in cartesian coordinates. Such a transform is difficult to invert and in many non-linear systems has certain cases where it is impossible to invert the coordinate transform.

1.4.2 Polar Coordinates

Polar coordinates describe a point in terms of a distance from the origin r and an angle θ from the x-axis. One typically chooses polar coordinates when the problem has circular symmetry, or if something is undergoing periodic motion around the origin.

The point transform from polar coordinates $(r,\, \theta)$ to cartesian coordinates $(x,\, y)$ is

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned} \tag{1.8}$$

Infinitesimal changes in x and y can be written as

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}r\cos\theta - r\,\mathrm{d}\theta\sin\theta, \\ \mathrm{d}y &= \mathrm{d}r\sin\theta + r\,\mathrm{d}\theta\cos\theta. \end{aligned} \tag{1.9}$$

We then find the metric for polar coordinates, derived from equation (1.6), with

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\,. \tag{1.10}$$

We can also define orthonormal basis vectors in cartesian coordinates that point in the direction of increasing r and θ. These vectors are

$$\hat{\mathbf{r}} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad \hat{\boldsymbol{\theta}} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}. \tag{1.11}$$

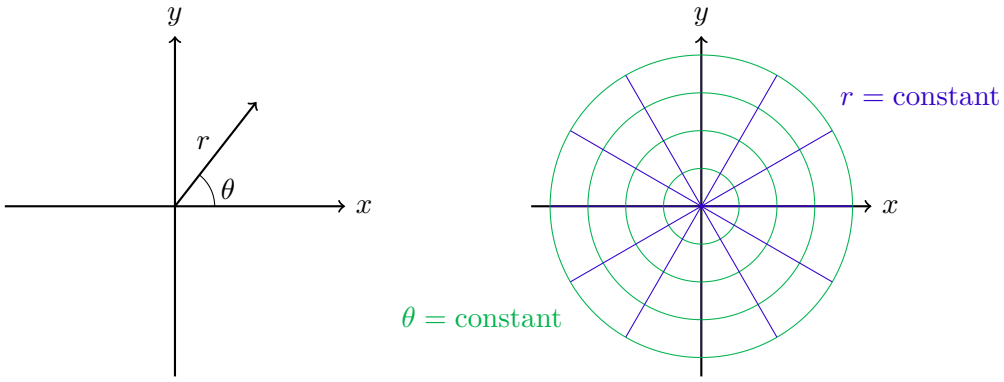

Figure 1.4: Polar coordinates in the Cartesian plane.

1.4.3 Cylindrical Polar Coordinates

Cylindrical polar coordinates describe a point in terms of a distance from the z-axis r, an angle θ from the x-axis around the z-axis, and a height z. The reason for the name is that curves of constant r form cylinders. One typically chooses cylindrical coordinates when the problem has cylindrical symmetry, or if something is undergoing periodic motion around the z-axis.

The point transform from cylindrical polar coordinates $(r,\, \theta,\, z)$ to cartesian coordinates $(x,\, y,\, z)$ is

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned} \tag{1.12}$$

The associated metric is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2\,. \tag{1.13}$$

We can also define orthonormal basis vectors in cartesian coordinates that point in the direction of increasing r, θ and z. These vectors are

$$\hat{\mathbf{r}} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\theta}} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.14}$$

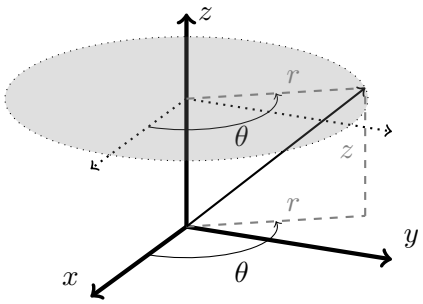

Figure 1.5: Cylindrical coordinates shown relative to Cartesian coordinates.

1.4.4 Spherical Polar Coordinates

Spherical polar coordinates are named as such because curves of constant radius r are spheres. There are various different conventions that all do the same thing here, the most important thing to remember is that any coordinate system using two angles and a distance in 3D is some form of spherical coordinate system.

1.4.4.1 Right Ascension and Declination

The coordinate system pictured in figure 1.6 uses right ascension α and declination δ to describe a point in 3D space. The point transform from right ascension-declination spherical coordinates $(r,\, \alpha,\, \delta)$ to cartesian coordinates $(x,\, y,\, z)$ is

$$\begin{aligned} x &= r\cos\delta\cos\alpha, \\ y &= r\cos\delta\sin\alpha, \\ z &= r\sin\delta. \end{aligned} \tag{1.15}$$

The associated metric is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\delta^2 + r^2\cos^2\delta\,\mathrm{d}\alpha^2\,. \tag{1.16}$$

We can also define orthonormal basis vectors in cartesian coordinates that point in the direction of increasing r, α and δ. These vectors are

$$\hat{\mathbf{r}} = \begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix}, \qquad \hat{\boldsymbol{\alpha}} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\delta}} = \begin{bmatrix} -\sin\delta\cos\alpha \\ -\sin\delta\sin\alpha \\ \cos\delta \end{bmatrix}. \tag{1.17}$$

It is also often a useful fact that

$$\frac{\partial\mathbf{r}}{\partial r} = \hat{\mathbf{r}}, \qquad \frac{\partial\mathbf{r}}{\partial\alpha} = r\cos\delta\hat{\boldsymbol{\alpha}}, \qquad \frac{\partial\mathbf{r}}{\partial\delta} = r\hat{\boldsymbol{\delta}}. \tag{1.18}$$

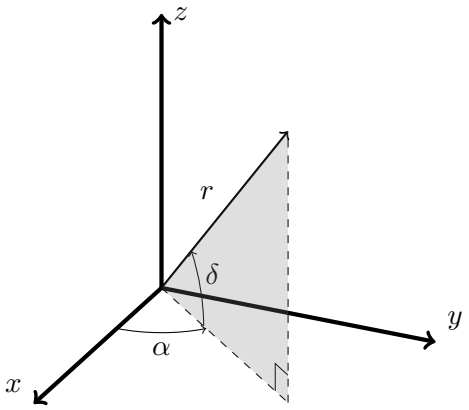

Figure 1.6: Spherical coordinates using right ascension α and declination δ.

1.4.4.2 Azimuth and Inclination

The coordinate system pictured in figure 1.7 uses azimuth ϕ and inclination θ to describe a point in 3D space. The point transform from azimuth-inclination spherical coordinates $(r,\, \phi,\, \theta)$ to cartesian coordinates $(x,\, y,\, z)$ is

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \tag{1.19}$$

The associated metric is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\,. \tag{1.20}$$

We can also define orthonormal basis vectors in cartesian coordinates that point in the direction of increasing r, ϕ and θ. These vectors are

$$\hat{\mathbf{r}} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \qquad \hat{\boldsymbol{\phi}} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\theta}} = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix}. \tag{1.21}$$

It is also often a useful fact that

$$\frac{\partial \mathbf{r}}{\partial r} = \hat{\mathbf{r}}, \qquad \frac{\partial \mathbf{r}}{\partial \phi} = r\sin\theta\hat{\boldsymbol{\phi}}, \qquad \frac{\partial \mathbf{r}}{\partial \theta} = r\hat{\boldsymbol{\theta}}. \tag{1.22}$$

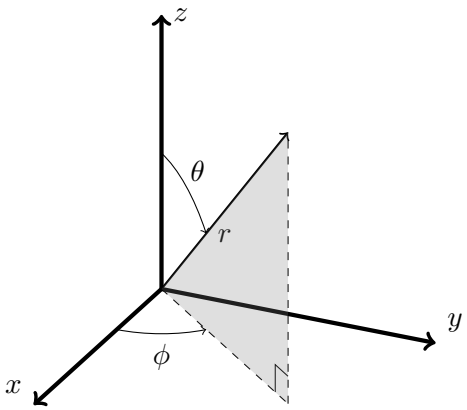

Figure 1.7: Spherical coordinates using right ascension α and declination δ.

1.4.5 Parabolic Coordinates

Parabolic coordinates are so named because curves of constant u and v are parabolas. As can be seen in figure 1.8, changing either u or v moves you along various parabolic curves.

They are useful for solving problems using separation of variables, they tend to be quite unintuitive to work with for most people, though they can be very powerful for solving certain types of problems.

The point transform from parabolic coordinates $(u,\, v)$ to cartesian coordinates $(x,\, y)$ is

$$\begin{aligned} x &= uv, \\ y &= \frac{1}{2}(u^2 - v^2). \end{aligned} \tag{1.23}$$

Infinitesimal changes in x and y can be written as

$$\begin{aligned} \mathrm{d}x &= v\,\mathrm{d}u + u\,\mathrm{d}v\,, \\ \mathrm{d}y &= u\,\mathrm{d}u - v\,\mathrm{d}v\,. \end{aligned} \tag{1.24}$$

We then find the metric for parabolic coordinates, derived from equation (1.6), with

$$\mathrm{d}s^2 = (u^2 + v^2)(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{1.25}$$

We can also define orthonormal basis vectors in cartesian coordinates that point in the direction of increasing u and v. These vectors are

$$\hat{\mathbf{u}} = \begin{bmatrix} \dfrac{v}{\sqrt{u^2+v^2}} \\ \dfrac{u}{\sqrt{u^2+v^2}} \end{bmatrix}, \qquad \hat{\mathbf{v}} = \begin{bmatrix} \dfrac{u}{\sqrt{u^2+v^2}} \\ \dfrac{-v}{\sqrt{u^2+v^2}} \end{bmatrix}. \tag{1.26}$$

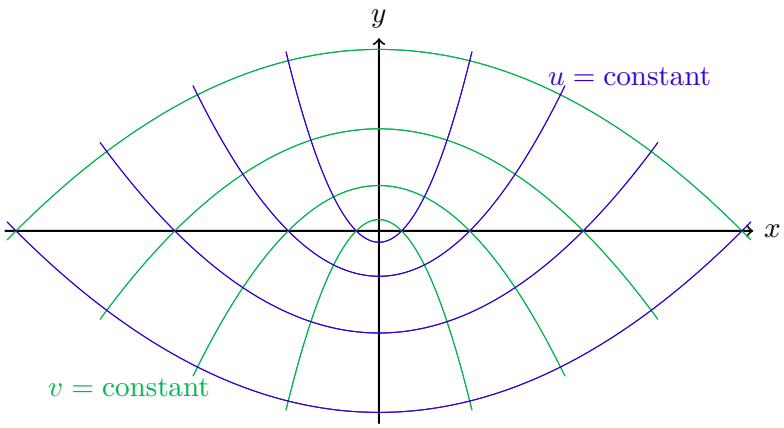

Figure 1.8: Parabolic coordinates in the Cartesian plane.

Chapter 2: Rotation Matrices and other Linear Algebra

Rotation matrices are a core concept in linear algebra and describe perfect rotations about a given axis by an angle. They specifically have the property that they are orthogonal matrices, meaning that

$$\mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{R}\mathbf{R}^\mathsf{T} = \mathbf{I} \tag{2.1}$$

which also implies that

$$\mathbf{R}^{-1} = \mathbf{R}^\mathsf{T} \tag{2.2}$$

and

$$\det|\mathbf{R}| = 1. \tag{2.3}$$

We generally denote a rotation matrix in n dimensions as being part of the special orthogonal group $\mathrm{SO}(n)$ and for 3D or $\mathbb{R}^3$ we call this $\mathrm{SO}(3)$. We would therefore say that any 3×3 rotation matrix $\mathbf{R} \in \mathrm{SO}(3)$ if it satisfies the properties in equations equation (2.1), equation (2.2), and equation (2.3).

2.1 The Base Rotation Matrices

The base rotation matrices are the matrices which rotate coordinates about the three principle axes. We rotate about the x, y, and z axes respectively with

$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \tag{2.4a}$$

$$\mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{2.4b}$$

$$\mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.4c}$$

2.2 Cross Product Matrices

A cross product matrix is a member of the Lie algebra group $\mathfrak{so}(3)$ and is a 3×3 matrix which represents the cross product operation. We can define the cross product matrix $[\mathbf{a}]_\times$ for a vector $\mathbf{a} \in \mathbb{R}^3$ as

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2.5}$$

whereby we can compute the cross product

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}. \tag{2.6}$$

This matrix shares a property with its Lie group $\mathfrak{so}(3)$ that it is skew-symmetric, meaning that

$$[\mathbf{a}]_\times^\mathsf{T} = -[\mathbf{a}]_\times. \tag{2.7}$$

This is particularly useful for the cross product due to its anti-commutative property, so we can write a cross product in a few ways

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = [\mathbf{a}]_\times \mathbf{b} = -[\mathbf{b}]_\times \mathbf{a} = [\mathbf{b}]_\times^\mathsf{T} \mathbf{a}. \tag{2.8}$$

In particular we can swap which vector we take the cross product of by transposing the cross product matrix. For example let's say we have $\mathbf{b} \times \mathbf{a} = \mathbf{c}$, but we would really love to write $\mathbf{c}$ in terms of the cross-product matrix of $\mathbf{a}$. This isn't an issue because

$$\mathbf{c} = \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = -[\mathbf{a}]_\times \mathbf{b} = [\mathbf{a}]_\times^\mathsf{T} \mathbf{b}. \tag{2.9}$$

2.2.1 The Commutator and the Cross Product Matrix

The commutator of two square matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}. \tag{2.10}$$

If the commutator is zero then the two matrices commute, as in $\mathbf{AB} = \mathbf{BA}$.

More generally the commutator is a specific form of the Lie bracket or adjoint representation of the Lie algebra.

In this case it turns out that the commutator of two cross product matrices is the cross product matrix of the cross product of the two vectors

$$\Big[[\mathbf{a}]_\times, [\mathbf{b}]_\times\Big] = [\mathbf{a} \times \mathbf{b}]_\times. \tag{2.11}$$

2.3 The Exponential Map and Rodrigues' Rotation Formula

We shall not touch on the complexities of the exponential map here, however in general it maps the Lie algebra $\mathfrak{so}(3)$ to the Lie group SO(3).

Let's consider that we want to rotate some coordinate about some arbitrary axis $\hat{\mathbf{n}}$ by some angle θ. It is important that $\hat{\mathbf{n}}$ is a unit vector, otherwise we must normalize it.

It turns out that to rotate a vector $\mathbf{v}$ about $\hat{\mathbf{n}}$ by θ we can use Rodrigues' rotation formula

$$\mathbf{R}_{\hat{\mathbf{n}}}(\theta) = \exp\Big(\theta[\hat{\mathbf{n}}]_\times\Big) \tag{2.12}$$

where

$$\mathbf{v}_{\text{rot}} = \mathbf{R}_{\hat{\mathbf{n}}}(\theta)\mathbf{v}. \tag{2.13}$$

Using Taylor series expansion this can also be written as

$$\mathbf{R}_{\hat{\mathbf{n}}}(\theta) = \mathbf{I} + (\sin\theta)[\hat{\mathbf{n}}]_\times + (1 - \cos\theta)[\hat{\mathbf{n}}]_\times^2. \tag{2.14}$$

While the equation in equation (2.14) is useful computationally as it avoids taking the matrix exponential, equation (2.12) is more useful for considering the rate of change of the rotation matrix with respect to θ.

2.3.1 Time Derivatives of Rodrigues' Rotation Formula

Taking the derivative with respect to θ of equation (2.12) we get

$$\frac{\partial \mathbf{R}_{\hat{\mathbf{n}}}(\theta)}{\partial \theta} = [\hat{\mathbf{n}}]_\times \mathbf{R}_{\hat{\mathbf{n}}}(\theta). \tag{2.15}$$

To compute the time derivative for a time-varying angle $\theta(t)$ we can use the chain rule to get

$$\dot{\mathbf{R}}_{\hat{\mathbf{n}}}(\theta(t)) = \frac{\mathrm{d}\theta}{\mathrm{d}t}[\hat{\mathbf{n}}]_\times \mathbf{R}_{\hat{\mathbf{n}}}(\theta(t)). \tag{2.16}$$

If the axis about which the rotation is occurring is changing then the expansion is not as simple.

2.4 Time Derivatives of the Base Rotation Matrices

We can treat the base rotation matrices as Rodrigues' rotation formulae about the principle axes. We get new equivalent formulas for the three rotation matrices of

$$\begin{aligned} \mathbf{R}_1(\theta) &= \mathbf{R}_{\hat{\mathbf{x}}}(\theta) \\ \mathbf{R}_2(\theta) &= \mathbf{R}_{\hat{\mathbf{y}}}(\theta) \\ \mathbf{R}_3(\theta) &= \mathbf{R}_{\hat{\mathbf{z}}}(\theta). \end{aligned} \tag{2.17}$$

Therefore using equation (2.16) we can compute the time derivatives of the base rotation matrices as

$$\begin{aligned} \dot{\mathbf{R}}_1(\theta) &= \dot{\mathbf{R}}_{\hat{\mathbf{x}}}(\theta) = \dot{\theta}[\hat{\mathbf{x}}]_\times \mathbf{R}_{\hat{\mathbf{x}}}(\theta) = \dot{\theta}[\hat{\mathbf{x}}]_\times \mathbf{R}_1(\theta) \\ \dot{\mathbf{R}}_2(\theta) &= \dot{\mathbf{R}}_{\hat{\mathbf{y}}}(\theta) = \dot{\theta}[\hat{\mathbf{y}}]_\times \mathbf{R}_{\hat{\mathbf{y}}}(\theta) = \dot{\theta}[\hat{\mathbf{y}}]_\times \mathbf{R}_2(\theta) \\ \dot{\mathbf{R}}_3(\theta) &= \dot{\mathbf{R}}_{\hat{\mathbf{z}}}(\theta) = \dot{\theta}[\hat{\mathbf{z}}]_\times \mathbf{R}_{\hat{\mathbf{z}}}(\theta) = \dot{\theta}[\hat{\mathbf{z}}]_\times \mathbf{R}_3(\theta). \end{aligned} \tag{2.18}$$

2.5 Time Derivative of the Matrix Inverse

A slightly more complex problem arises when taking the derivative of the inverse of a matrix. This is especially the case for rotation matrices. One might intuit that because

$$\frac{\mathrm{d}\mathbf{U}^\mathsf{T}(t)}{\mathrm{d}t} = \left(\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t}\right)^\mathsf{T} \tag{2.19}$$

that we would have

$$\frac{\mathrm{d}\mathbf{R}^{-1}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{R}^\mathsf{T}(t)}{\mathrm{d}t} = -\left(\frac{\mathrm{d}\mathbf{R}(t)}{\mathrm{d}t}\right)^\mathsf{T} \tag{2.20}$$

and while this will work this is not the general case, as it is only due to a special property of rotation matrices. Since the rate of change of a rotation matrix is not a rotation matrix we can apply this rule to rotation matrices but not their derivatives.

So for a rotation matrix $\mathbf{R}$ we have

$$\frac{\mathrm{d}\mathbf{R}^{-1}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{R}^\mathsf{T}}{\mathrm{d}t} = \dot{\mathbf{R}}^\mathsf{T} \tag{2.21}$$

For any non-rotation matrix we instead we need to consider the proper derivative of the matrix inverse.

Consider the expression which defines the inverse of a matrix

$$\mathbf{U}^{-1}\mathbf{U} = \mathbf{I}. \tag{2.22}$$

Taking the derivative of both sides with respect to time we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{U}^{-1}\mathbf{U}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{I} \\ \frac{\mathrm{d}\mathbf{U}^{-1}}{\mathrm{d}t}\mathbf{U} + \mathbf{U}^{-1}\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} &= \mathbf{0} \\ \frac{\mathrm{d}\mathbf{U}^{-1}}{\mathrm{d}t}\mathbf{U} &= -\mathbf{U}^{-1}\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \end{aligned} \tag{2.23}$$

which then gives the final result

$$\frac{\mathrm{d}\mathbf{U}^{-1}}{\mathrm{d}t} = -\mathbf{U}^{-1}\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\mathbf{U}^{-1}. \tag{2.24}$$

This relationship is very useful in general, but it must also hold for rotation matrices. Therefore we have that

$$\frac{\mathrm{d}\mathbf{R}^{-1}}{\mathrm{d}t} = -\mathbf{R}^{\mathsf{T}}\dot{\mathbf{R}}\mathbf{R}^{\mathsf{T}} = \dot{\mathbf{R}}^{\mathsf{T}}. \tag{2.25}$$

2.6 Change of Basis Transforms

A rotation matrix rotates coordinates from one orthonormal basis to another – this is their defining characteristic. However, it is possible to formulate a more general form of change of basis.

Let's say that we have two bases A and B, each respectively with basis vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\dots \mathbf{a}_n$ and $\mathbf{b}_1$, $\mathbf{b}_2$, $\dots \mathbf{b}_n$.

Let us denote a vector $\mathbf{v}$ viewed in basis A as $(\mathbf{v})_A$ and in basis B as $(\mathbf{v})_B$.

When viewed in basis A these basis vectors are just

$$(\mathbf{a}_1)_A = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (\mathbf{a}_2)_A = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots \quad (\mathbf{a}_n)_A = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and the same is true for the basis vectors of B when viewed in basis B.

As it turns out, the matrix that transforms a coordinate from frame A to frame B is given by

$$\mathbf{T}_{B\leftarrow A} = \begin{bmatrix} (\mathbf{a}_1)_B & (\mathbf{a}_2)_B & \cdots & (\mathbf{a}_n)_B \end{bmatrix} = \begin{bmatrix} a_{1\,B1} & a_{2\,B1} & \cdots & a_{n\,B1} \\ a_{1\,B2} & a_{2\,B2} & \cdots & a_{n\,B2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1\,Bn} & a_{2\,Bn} & \cdots & a_{n\,Bn} \end{bmatrix} \tag{2.26}$$

where its columns are the basis vectors of A expressed in the coordinates of B.

We can perhaps see this most easily by example. A vector $(\mathbf{v})_A$ in frame A would be represented as

$$(\mathbf{v})_A = v_{a1}(\mathbf{a}_1)_A + v_{a2}(\mathbf{a}_2)_A + \cdots + v_{an}(\mathbf{a}_n)_A = \begin{bmatrix} v_{a1} \\ v_{a2} \\ \vdots \\ v_{an} \end{bmatrix}.$$

If we just brute force compute the matrix multiplication we can see that

$$\begin{aligned}
(\mathbf{v})_B &= \mathbf{T}_{B\leftarrow A}(\mathbf{v})_A \\
&= \mathbf{T}_{B\leftarrow A} \begin{bmatrix} v_{a1} \\ v_{a2} \\ \vdots \\ v_{an} \end{bmatrix} \\
&= \begin{bmatrix} v_{a1}\, a_{1\,B1} + v_{a2}\, a_{2\,B1} + \cdots + v_{an}\, a_{n\,B1} \\ v_{a1}\, a_{1\,B2} + v_{a2}\, a_{2\,B2} + \cdots + v_{an}\, a_{n\,B2} \\ \vdots \\ v_{a1}\, a_{1\,Bn} + v_{a2}\, a_{2\,Bn} + \cdots + v_{an}\, a_{n\,Bn} \end{bmatrix} \\
&= v_{a1}(\mathbf{a}_1)_B + v_{a2}(\mathbf{a}_2)_B + \cdots + v_{an}(\mathbf{a}_n)_B
\end{aligned}$$

so multiplying the transform matrix by the vector in frame A gives us the exact same result as if we had figured out how to express the basis vectors of A in the coordinates of B and then scaled each of those basis vectors by the components of the vector in frame A.

The change of basis transform $\mathbf{T}_{A\leftarrow B}$ from frame B to frame A is a **rotation matrix** if and only if the two bases are orthonormal.

2.6.1 Change of Basis of a Matrix

While perhaps not as commonly discussed, matrices also depend on the frame of reference in which they are defined.

Suppose we have some square matrix $(\mathbf{M})_A \in \mathbb{R}^n$ in frame A such that

$$(\mathbf{y})_A = (\mathbf{M})_A(\mathbf{x})_A.$$

If we want to express this same result in frame B we need to change each of the vectors to frame B using the change of basis transform, i.e.

$$\begin{aligned}
(\mathbf{y})_B &= \mathbf{T}_{B\leftarrow A}(\mathbf{y})_A \\
&= \mathbf{T}_{B\leftarrow A}(\mathbf{M})_A(\mathbf{x})_A \\
&= \mathbf{T}_{B\leftarrow A}(\mathbf{M})_A\mathbf{T}_{A\leftarrow B}(\mathbf{x})_B
\end{aligned}$$

so we can see by inspection that this matrix in frame B is

$$(\mathbf{M})_B = \mathbf{T}_{B\leftarrow A}(\mathbf{M})_A\mathbf{T}_{A\leftarrow B} \tag{2.27}$$

where $\mathbf{T}_{A\leftarrow B} = \mathbf{T}^{-1}_{B\leftarrow A}$.

2.6.2 Change of Basis of a Cross Product Matrix

A less intuitive result is that the change of basis of a cross product matrix can be expressed in the same was as for any other matrix so long as the basis transform is a rotation matrix i.e.

$$\left[(\mathbf{a})_B\right]_\times = \left[\mathbf{R}_{B\leftarrow A}(\mathbf{a})_A\right]_\times = \mathbf{R}_{B\leftarrow A}\left[(\mathbf{a})_A\right]_\times \mathbf{R}_{A\leftarrow B}. \tag{2.28}$$

We can use the identity that

$$\mathbf{R}_{B\leftarrow A}\Big((\mathbf{a})_A \times (\mathbf{b})_A\Big) = \mathbf{R}_{B\leftarrow A}(\mathbf{a})_A \times \mathbf{R}_{B\leftarrow A}(\mathbf{b})_A \tag{2.29}$$

for any rotation matrix to prove this.

We begin simply by using this identity to show that

$$\mathbf{R}_{B\leftarrow A}\Big((\mathbf{a})_A \times (\mathbf{b})_A\Big) = (\mathbf{a})_B \times (\mathbf{b})_B = \left[(\mathbf{a})_B\right]_\times (\mathbf{b})_B. \tag{2.30}$$

Writing the left hand side instead in terms of the cross product matrix we have

$$\begin{aligned}\mathbf{R}_{B\leftarrow A}\left[(\mathbf{a})_A\right]_\times (\mathbf{b})_A &= \left[(\mathbf{a})_B\right]_\times (\mathbf{b})_B \\ \mathbf{R}_{B\leftarrow A}\left[(\mathbf{a})_A\right]_\times \mathbf{R}_{A\leftarrow B}(\mathbf{b})_B &= \left[(\mathbf{a})_B\right]_\times (\mathbf{b})_B\end{aligned}$$

and by equating coefficients we see that indeed the expected result holds. It is worth noting that equation (2.29) does not generally hold for any transform matrix and therefore this formula is specific for rotation matrices.

Chapter 3: Vector and Matrix Calculus

3.1 Matrix Calculus

Matrix calculus is the general name given to calculus using vector and matrix notation. There are two notations that are convenient in different situations, one called numerator notation and one called denominator notation. In numerator notation the shape of the numerator takes precedent over that of the denominator, the converse is true for denominator notation. Identities and other rules that can be derived from first principles will not be presented here, but can be found at [1].

3.1.1 Scalar Derivatives

Let there be a scalar u which is some function of either a vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{3.1}$$

or a matrix

$$\mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}. \tag{3.2}$$

When transposing a vector or matrix derivative of a scalar the following is important in either notation.

$$\begin{aligned} \left(\frac{\mathrm{d}u}{\mathrm{d}\mathbf{x}}\right)^T &= \frac{\mathrm{d}u}{\mathrm{d}\mathbf{x}^T} \\ \left(\frac{\mathrm{d}u}{\mathrm{d}\mathbf{X}}\right)^T &= \frac{\mathrm{d}u}{\mathrm{d}\mathbf{X}^T} \end{aligned} \tag{3.3}$$

This is equivalent to swapping notation.

3.1.1.1 Numerator Notation

In numerator notation we can take a derivative with respect to either this vector or matrix where each entry corresponds to a derivative with respect to the transpose of the denominator.

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\mathbf{x}} &= \begin{bmatrix} \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u}{\partial x_n} \end{bmatrix}, \\ \frac{\mathrm{d}u}{\mathrm{d}\mathbf{X}} &= \begin{bmatrix} \frac{\partial u}{\partial x_{11}} & \cdots & \frac{\partial}{\partial x_{n1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial u}{\partial x_{1m}} & \cdots & \frac{\partial u}{\partial x_{nm}} \end{bmatrix}. \end{aligned} \tag{3.4}$$

3.1.1.2 Denominator Notation

In denominator notation we can take a derivative with respect to either this vector or matrix where each entry corresponds to a derivative with respect to the denominator.

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\mathbf{x}} &= \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix}, \\ \frac{\mathrm{d}u}{\mathrm{d}\mathbf{X}} &= \begin{bmatrix} \frac{\partial u}{\partial x_{11}} & \cdots & \frac{\partial}{\partial x_{1m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial u}{\partial x_{n1}} & \cdots & \frac{\partial u}{\partial x_{nm}} \end{bmatrix}. \end{aligned} \tag{3.5}$$

3.1.2 Vector Derivatives

Let there be a vector $\mathbf{u}$ that is some function another vector $\mathbf{x}$[1].

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} u_1(\mathbf{x}) \\ \vdots \\ u_k(\mathbf{x}) \end{bmatrix} \tag{3.6}$$

When transposing a vector derivative of a vector the following is important in either notation.

$$\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{x}}\right)^T = \frac{\mathrm{d}\mathbf{u}^T}{\mathrm{d}\mathbf{x}^T}. \tag{3.7}$$

This is equivalent to swapping notations.

3.1.2.1 Numerator Notation

In numerator notation we can take a derivative of each entry in the numerator with respect to the denominator, where each row corresponds to the derivative of the corresponding row of the numerator, for each entry in the denominator. This can also be thought of as a derivative with respect to $\mathbf{x}$ of every entry of $\mathbf{u}$ in numerator notation i.e.

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{x}} = \begin{bmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}\mathbf{x}} \\ \vdots \\ \frac{\mathrm{d}u_k}{\mathrm{d}\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_k}{\partial x_1} & \cdots & \frac{\partial u_k}{\partial x_n} \end{bmatrix}. \tag{3.8}$$

3.1.2.1.1 *The Chain Rule*

The chain rule in numerator notation is very similar to that of the regular chain rule, however one must be very careful of the order of operations due to the shape of the matrices.

Say that there is a function $\mathbf{u}(\mathbf{v})$ and $\mathbf{v}(\mathbf{x})$, then the chain rule in numerator notation is

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{x}} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{v}} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}}. \tag{3.9}$$

You can justify this as taking $\mathbf{u} \in \mathbb{R}^1$ (i.e. it is a scalar) you will end up with a row vector multiplied by a matrix, which reduces to a row vector of the same shape as the left hand side of the equation.

[1] We cannot take the derivative of a vector with respect to a matrix as the resulting shape would be undefined.

3.1.2.2 Denominator Notation

In denominator notation we take a derivative of each entry in the numerator with respect to the denominator, where each row corresponds to the derivative of each entry of the numerator, with respect to the corresponding row of the denominator, this can also be thought of as a row vector where each entry is the derivative of the corresponding entry of $\mathbf{u}$ with respect to $\mathbf{x}$ in denominator notation.

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{x}} = \begin{bmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}\mathbf{x}} & \cdots & \frac{\mathrm{d}u_k}{\mathrm{d}\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_k}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_1}{\partial x_n} & \cdots & \frac{\partial u_k}{\partial x_n} \end{bmatrix}. \tag{3.10}$$

3.1.2.2.1 *The Chain Rule*

The chain rule in denominator notation is very similar to that of numerator notation, however the order is swapped.

Say that there is a function $\mathbf{u}(\mathbf{v})$ and $\mathbf{v}(\mathbf{x})$, then the chain rule in numerator notation is

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{x}} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{v}}. \tag{3.11}$$

You can justify this as taking $\mathbf{u} \in \mathbb{R}^1$ (i.e. it is a scalar) you will end up with a matrix multiplied by a column vector, which reduces to a column vector of the same shape as the left hand side of the equation.

Chapter 4: Calculus of Variations, Lagrangian and Hamiltonian Mechanics

Calculus of variations is very similar to regular calculus, except instead of finding a point that minimises a function, you instead find a function that minimises something called a functional. It was discovered by mathematicians Euler and Lagrange, then later turned into the easier forms that we use it today by Hamilton and Jacobi. So it may not surprise you that calculus of variations is the basis of both Lagrangian and Hamiltonian mechanics as well.

While calculus of variations is not as useful in many day-to-day problems as regular calculus, it can often be a very useful tool for those that know how to use it.

Within this chapter we shall explore a relatively high-level overview of the calculus of variations, Lagrangian mechanics and also to a lesser extent Hamiltonian mechanics. The goal of this chapter is to explore the fundamental principles, including derivations where it is relevant, but only to enough detail to appreciate when the calculus of variations is useful and how it can be used.

4.1 What is Calculus of Variations?

To understand calculus of variations it is easiest to consider the following classical problem.

We would like to find the shortest distance between two points in 3D. In cartesian coordinates a small path length $\mathrm{d}s$ is described by something called a metric, which in cartesian coordinates ends up being just like the Pythagorean theorem. A very small path length is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} \tag{4.1}$$

where $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ are each small steps in each of the coordinates. We would like to minimise the total path length

$$S = \int_{s_0}^{s_1} \mathrm{d}s = \int_{s_0}^{s_1} \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} \tag{4.2}$$

however solving that equation is not possible by any standard methods. Notice that we can't solve this problem using regular calculus since S is just some number. The bounds of the integral are fixed numbers, so taking any derivative just gives us zero.

We can rewrite the problem by dividing equation (4.1) by $\mathrm{d}t$ to get the velocity terms

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{4.3}$$

and therefore minimuse the total path length is equivalent to minimising

$$S = \int_{t_0}^{t_1} v\,\mathrm{d}t = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t = \int_{t_0}^{t_1} v(t,\, x,\, y,\, z,\, \dot{x},\, \dot{y},\, \dot{z})\,\mathrm{d}t\,. \tag{4.4}$$

We call S a *functional* as it is a scalar that is dependent on a function F. The calculus of variations is about finding that function f that minimises the functional $S[f]$. We write S similar to function

notation as $S[f]$ where S is a functional with respect to f. This remains true even if S or f are vector quantities, though the computations simply become more complex.

4.2 The Euler-Lagrange Equations

The Euler-Lagrange equation describes the function that is an *extremal* (either minimal or maximal) solution to the problem

$$S[y] = \int_{x_1}^{x_2} F(x,\, y,\, y')\,\mathrm{d}x \tag{4.5}$$

where $y = y(x)$ and the problem is constrained by boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. The extremal function $y(x)$ is the function that minimises or maximises the functional $S[F]$. In most practical cases the function y that maximuses the functional S is infinite, so the solution ends up being a function that minimises (at least locally) the functional.

Recall how in regular calculus a regular derivative is defined by considering the slope of a line between two points on a curve, then taking the limit as those points get infinitesimally close.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \tag{4.6}$$

We can follow a very similar principle for calculus of variations. We begin by considering a function $y_*(x)$ that we assume is an extremal solution to the problem. We then consider the case where we add another function to that i.e.

$$y(x) = y_*(x) + \varepsilon\eta(x) \tag{4.7}$$

where $\eta(x)$ is some other function and ε is some number or coefficient. The only constraint we put on this is that the new function $y(x)$ must satisfy the same boundary conditions as the original function $y_*(x)$. We can therefore derive that

$$\begin{aligned} y_1 = y(x_1) = y_*(x_1) + \varepsilon\eta(x_1) = y_1 + \varepsilon\eta(x_1) \\ y_2 = y(x_2) = y_*(x_2) + \varepsilon\eta(x_2) = y_2 + \varepsilon\eta(x_2) \end{aligned} \tag{4.8}$$

which tells us that $\eta(x)$ can be any function so long as

$$\eta(x_1) = \eta(x_2) = 0. \tag{4.9}$$

We can substitute this function back into our functional S to get

$$S[y] = S[y_* + \varepsilon\eta] = \int_{x_1}^{x_2} F(x,\, y(x),\, y'(x))\,\mathrm{d}x = \int_{x_1}^{x_2} F(x,\, y_*(x) + \varepsilon\eta(x),\, y_*'(x) + \varepsilon\eta'(x))\,\mathrm{d}x \tag{4.10}$$

which probably seems a bit strange to do. The insight we need here is similar to regular calculus.

Just like in regular calculus, when we are at a local minimum or maximum, any small step $\mathrm{d}x$ either side of the extremal point won't change the value of the function, at least in the limit as that step goes to zero. The exact same can also be said for the slope, except in that case we know that at an extremal point the slope is known, it's just zero.

Applying this concept to the functional S, we know that the function y_* is an extremal solution if a small change in the function $\varepsilon\eta$ doesn't change the value of the functional S. In other words, the rate of change of the functional with respect to ε, as we send ε to zero (making the "step size" infinitesimally small), must be zero. We write this as

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} S[y_* + \varepsilon\eta]\bigg|_{\varepsilon=0} = 0 \tag{4.11}$$

where the vertical line means "evaluated at $\varepsilon = 0$". We take the derivative with respect to ε since this should work for any function $\eta(x)$, and the ε ensures that the change due to that function is small.

Applying the multivariable chain rule and the Leibniz differentiation under the integral rule to equation (4.11) gives

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\varepsilon} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}\varepsilon} \right) \mathrm{d}x \tag{4.12}$$

since

$$y(x) = y_*(x) + \varepsilon\eta(x)$$
$$y'(x) = y'_*(x) + \varepsilon\eta'(x)$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}\varepsilon} = \eta(x)$$
$$\frac{\mathrm{d}y'}{\mathrm{d}\varepsilon} = \eta'(x).$$

Since there is multiplication inside the integral we should really integrate by parts. It is also equivalent to also consider the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\eta(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\eta(x) + \frac{\partial F}{\partial y'}\eta'(x). \tag{4.13}$$

Notice that the term on the right is inside our integral so we can rearrange equation (4.12) to get

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\eta(x) \right) \mathrm{d}x + \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\eta(x)\right) \mathrm{d}x\,. \tag{4.14}$$

The integral on the right is the integral of a derivative which simplifies nicely to

$$\int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\eta(x)\right) \mathrm{d}x = \frac{\partial F}{\partial y'}\eta(x)\Big|_{x_1}^{x_2}$$

and since $\eta(x_1) = \eta(x_2) = 0$ this term is zero. Equation (4.14) becomes

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) \right) \eta(x)\,\mathrm{d}x\,. \tag{4.15}$$

Recall that we defined in equation (4.11) that we require that this integral is zero, when $\varepsilon = 0$ and as a result, that $y(x) = y_*(x)$. Since $\eta(x)$ is free to be any function, this integral being zero requires that the term inside the brackets.

The Euler-Lagrange Equation (ELE) is now defined as

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{4.16}$$

where for a function S which is the integral of F with respect to x, the extremal function $y(x)$ is the solution to equation (4.18).

This idea generalises to a functional of many variables

$$S[q_1, \ldots, q_n] = \int_{x_1}^{x_2} L\left(t, q_1(t), \ldots, q_n(t), \frac{\mathrm{d}q_1}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}q_n}{\mathrm{d}t}\right) \mathrm{d}t \tag{4.17}$$

where we normally use t as the independent variable instead of x because x typically becomes one of the variables.

Definition 4.1: Euler-Lagrange Equations

Each of these variables q_i, called *generalised coordinates*, are functions of the independent variable t. Each coordinate has an associated ELE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{4.18}$$

where

$$\dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t}. \tag{4.19}$$

We can compact this notation using matrix calculus to get

$$S[\mathbf{q}] = \int_{t_1}^{t_2} L(t, \mathbf{q}, \dot{\mathbf{q}})\,\mathrm{d}t \tag{4.20}$$

where $\mathbf{q}$ is a vector of generalised coordinates and $\dot{\mathbf{q}}$ is the vector of the time derivatives of those coordinates.

Result 4.1: Euler-Lagrange Equations – Vector Form

The Euler-Lagrange equations can then be written as

$$\frac{\partial L}{\partial \mathbf{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \mathbf{0}. \tag{4.21}$$

with the partial derivatives being taken in denominator notation. (See § 3.1)

This gives all of the Euler-Lagrange equations in one compact form, though equation (4.18) is often easier to compute by hand.

4.2.1 First Integrals and Conserved Quantities

The name "first integral" might seem a bit strange, especially when we discuss in a moment how they have nothing to do with performing an integral. However, the way to perhaps think about them, is that a first integral is a constant of motion which can be treated as constant under an integral sign.

Each first integral corresponds to a conserved quantity in the system. Finding first integrals often results in equations being easier to solve, as knowing a first integral almost immediately solves at least one ELE.

Consider some function $L = L(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$. If the function does not depend on some coordinate q_i then the term

$$\frac{\partial L}{\partial q_i} = 0 \tag{4.22}$$

and so the ELE for that coordinate becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{4.23}$$

In other words,

$$\frac{\partial L}{\partial \dot{q}_i} = C_i = \text{constant} \tag{4.24}$$

is a constant of motion (it can change based on the initial conditions but will not change over time). If the function L does not depend on time, then we can find another first integral by considering the total derivative of L

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\right) \tag{4.25}$$

and if the function L is not strictly a function of time (as in there is no t in the formula, it is ok if $q_i = q_i(t)$ is a function of time) then

$$\frac{\partial L}{\partial t} = 0. \tag{4.26}$$

We can use the ELE to simplify the expression to

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\right) = \sum_{i=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \tag{4.27}$$

which rearranges to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = 0 \tag{4.28}$$

or perhaps more simply,

$$\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\dot{\mathbf{q}} - L = E = \text{constant}. \tag{4.29}$$

When L is the Lagrangian this quantity is the total energy of the system.

Each first integral corresponds to a conserved quantity in the system. Similar to how n variables require n equations to solve, a system is solvable if there are n conserved quantities.

4.3 Lagrangian Mechanics

Before jumping right into Lagrangian mechanics, we should first discuss the quantity called action and how it leads to the Lagrangian.

4.3.1 What is Action?

Action, S, is a quantity very similar to work, it describes something akin to the "cost" for an object to move between two places. Originally action was represented as

$$S = \int_{\mathbf{s}_1}^{\mathbf{s}_2} m\mathbf{v}\cdot\mathrm{d}\mathbf{s} = \int_{\mathbf{s}_1}^{\mathbf{s}_2}\mathbf{p}\cdot\mathrm{d}\mathbf{s} \tag{4.30}$$

where the boundary conditions are the initial and final positions of the object. The quantity $\mathbf{p}$ is the momentum of the object. Hamilton, responsible for Hamiltonian mechanics, later changed the form of this equation into the form we use today. This changed the boundary conditions to instead be the initial and final positions as well as the times of each of them. The action is therefore re-written as

$$S = \int_{t_1}^{t_2}(T - V)\,\mathrm{d}t = \int_{t_1}^{t_2} L\left(t,\,\mathbf{q},\,\dot{\mathbf{q}}\right)\mathrm{d}t \tag{4.31}$$

where T is the kinetic energy, V is the potential energy and $L = T - V$ is the Lagrangian. The Lagrangian is a function of time t, the generalised coordinates $\mathbf{q}$ and their time derivatives $\dot{\mathbf{q}}$.

4.3.2 Finding Lagrangians and Using Lagrangian Mechanics

If in your problem you can write down the kinetic energy $T(t,\, \mathbf{q},\, \dot{\mathbf{q}})$ and the potential energy $V(t,\, \mathbf{q},\, \dot{\mathbf{q}})$ then you can write down the Lagrangian. Take the simple example of a ball moving under gravity near Earth's surface, in 2D. The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \tag{4.32}$$

and the potential energy is

$$V = mgh = mgy. \tag{4.33}$$

The Lagrangian is

$$L(t,\, x,\, y,\, \dot{x},\, \dot{y}) = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy. \tag{4.34}$$

Writing out the Euler-Lagrange equations for each coordinate we have

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \qquad\qquad \frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

$$0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}\right) = 0 \qquad\qquad (-mg) - \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{y}\right) = 0$$

$$\ddot{x} = 0 \qquad\qquad \ddot{y} = -g.$$

This is also exactly the result we would expect from Newtonian mechanics. These equations still need to be solved, given the initial position and time. So why bother? Well the true usefulness is being able to use non-linear coordinates such as polar coordinates.

Consider a ball of mass m rolling around a frictionless cone with height $z = h(r)$ where r is the distance from the centre of the cone and k is a constant. The kinetic energy in cylindrical polar coordinates can be written using the metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2 \tag{4.35}$$

which can be derived from the cartesian metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{4.36}$$

by using the point transformation equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{4.37}$$

The kinetic energy is then

$$T = \frac{1}{2}m\frac{\mathrm{d}s^2}{\mathrm{d}t^2} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \tag{4.38}$$

and this principle of using the metric to find the kinetic energy is very useful for other more tricky coordinates too. The potential energy is

$$V = mgz. \tag{4.39}$$

By fixing the ball to be rolling on the cone defined by $z = h(r)$ we have applied a *constraint*. Constraints are typically very annoying in Newtonian mechanics, because you need to analyse the forces and calculate normal forces and such so that the ball is truly constrained to the surface. In Lagrangian mechanics the answer just sort of falls out.

The Lagrangian is

$$L(t,\, r,\, \theta,\, z,\, \dot{r},\, \dot{\theta},\, \dot{z}) = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz. \tag{4.40}$$

Applying our constraint gives $z = h(r)$ and $\dot{z} = h'(r)\dot{r}$ so the Lagrangian becomes

$$L(t,\, r,\, \theta,\, \dot{r},\, \dot{\theta}) = \frac{1}{2}m\left(\dot{r}^2(1 + h'(r)^2) + r^2\dot{\theta}^2\right) - mgh(r). \tag{4.41}$$

Solving the Euler-Lagrange equations for r and θ gives the equations of motion for the ball rolling on the cone. Analysis of such a problem is straightforward, though perhaps algebraically annoying (to simplify such working one might note that because θ is not present in the Lagrangian that there is an associated first integral).

The power of using Lagrangian mechanics here is that we have been able to write out the equations of motion in terms of polar coordinates rather than cartesian coordinates, something very hard to do using Newtonian mechanics.

4.4 Hamiltonian Mechanics

Hamiltonian mechanics is a re-stating of Lagrangian mechanics, but instead of using the Lagrangian, L, we use the Hamiltonian, H. There are a few approaches to Hamiltonian mechanics, in almost all classical cases it is easier to find a Lagrangian and then convert it to a Hamiltonian, though in many quantum cases there is no such Lagrangian and instead one must directly find the Hamiltonian.

Hamiltonian mechanics is useful for two primary reasons. The first is that it provides a set of $2n$ first order differential equations, instead of n second order differential equations. The second is a result of this fact, whereby state-space motion of the system can be interpreted geometrically, completely ignoring any real physics. Such methods often prove useful for systems that are exceptionally difficult to solve.

4.4.1 The Legendre Transform

The Legendre transform is able to transform us from the Lagrangian to the Hamiltonian. The Legendre transform is actually a very interesting mathematical transform as it is its own inverse, so it can also transform us back from the Hamiltonian to the Lagrangian. It is used in other areas of physics such as thermodynamics to convert between Gibbs and Helmholtz free energies.

For the Legendre transform to exist we require that the function is convex on the interval of interest, so that the derivatives of the function are invertible.

Let us consider our function to be $L = L(t,\, q_1,\, \ldots,\, q_n,\, \dot{q}_1,\, \ldots,\, \dot{q}_n)$. The total time derivative of the function is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\right) \tag{4.42}$$

using the ELE we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{4.43}$$

simplifying the expression to

$$0 = \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right). \tag{4.44}$$

It's perhaps useful to recall from the discussion in § 4.2.1 that we call the quantity inside the derivative the energy E and its rate of change is therefore

$$\dot{E} = -\frac{\partial L}{\partial t}. \tag{4.45}$$

We now define

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{4.46}$$

to be the conjugate momenta to the generalised coordinates q_i. The quantity E is the Legendre transform of L with respect to its generalised velocities $\dot{q}_i$ and is defined as

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \tag{4.47}$$

where H is the Hamiltonian. This quantity is also the energy from earlier, so the Hamiltonian does indeed represent the total energy of the system. The Hamiltonian is a function of time t, the generalised coordinates q_i and the conjugate momenta p_i. To perform the Legendre transform you must be able to write $\dot{q}_i = \dot{q}_i(t, q_1, \ldots, q_n, p_1, \ldots, p_n)$ in terms of the generalised coordinates and the conjugate momenta. This means replacing $\dot{q}_i$ in the sum and the Lagrangian.

It is now important to write out the differential for this quantity as well,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\mathrm{d}p_i}{\mathrm{d}t}\dot{q}_i + \sum_{i=1}^{n} p_i \frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t} - \frac{\mathrm{d}L}{\mathrm{d}t} \tag{4.48}$$

which simplifies using equation (4.42) and equation (4.46) to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \dot{E} - \sum_{i=1}^{n} \dot{p}_i \frac{\mathrm{d}q_i}{\mathrm{d}t} + \sum_{i=1}^{n} \dot{q}_i \frac{\mathrm{d}p_i}{\mathrm{d}t}. \tag{4.49}$$

4.4.2 Hamilton's Equations

Consider the Hamiltonian to be $H = H(t, q_1, \ldots, q_n, p_1, \ldots, p_n)$. From the Legendre transform we know that if L does not explicitly depend on t then neither can H, so

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = \dot{E} \tag{4.50}$$

This is also why the Hamiltonian can often just be written as $H = T + V$ since it is the total energy of the system and is conserved so long as it is independent of time.

We can write the total time derivative of $H = H(t, q_1, \ldots, q_n, p_1, \ldots, p_n)$ as

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \tag{4.51}$$

Definition 4.2: Hamilton's Equations

Comparing this to equation (4.49) we can equate terms to get Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{4.52}$$

We can then define the vector of generalised coordinates $\mathbf{q}$ and their associated conjugate momenta $\mathbf{p}$ such that the Hamiltonian is $H = H(t, \mathbf{q}, \mathbf{p})$.

Result 4.2: Hamilton's Equations – Vector Form

We can write Hamilton's equations in vector form as

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}. \tag{4.53}$$

In denominator notation.

4.5 Generalised Forces & the Augmented Euler-Lagrange Equations

4.5.1 d'Alembert's Principle

Consider a system of n particles with positions $\mathbf{r}_i$ and velocities $\mathbf{v}_i$ which are constrained to move in some way. Figure 4.1 shows a particle constrained to move on a surface, which is an example of a *holonomic* constraint.

Definition 4.3: Holonomic Constraints

A holonomic constraint is one that constrains only positions $\mathbf{q}$, e.g. $\mathbf{C}(\mathbf{q}, t) = 0$.
Non-holonomic constraints are constraints that constrain something other than just position.

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Let's denote a small change in position that obeys the constraints as $\delta\hat{\mathbf{r}}_i$, with examples pictured in figure 4.1. Such a small change in position is called a *virtual displacement*. Importantly, we can consider this change in position as not physically possible, for example, even if the particle is on some known trajectory, $\delta\hat{\mathbf{r}}_i$ may point along another trajectory, so long as that trajectory also obeys the constraints.

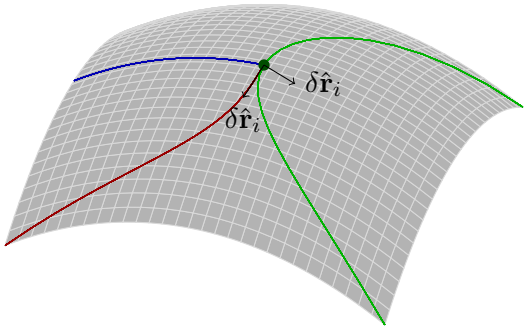


Figure 4.1: A particle constrained to move on an arbitrarily shaped surface.

A particle will be constrained to obey constraints by constraint forces $\hat{\mathbf{F}}_i$. There are also external forces acting on the particle, which we denote as $\mathbf{F}_i$. An example of a constraint force is the normal force from a table stopping a book from falling through it, while an external force, one that always acts the same way no matter the constraint, would be gravity.

We can generalise this concept to the generalised coordinates, with

$$\dot{\mathbf{p}} = \mathbf{Q} + \hat{\mathbf{Q}} = \mathbf{Q}_{\text{net}} \tag{4.54}$$

where $\mathbf{p}$ are the conjugate momenta, $\mathbf{Q}$ are the external generalised forces and $\hat{\mathbf{Q}}$ are the constraint generalised forces. This is a general form of Newton's second law

$$\mathbf{F}_{\text{net}} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} + \hat{\mathbf{F}} \tag{4.55}$$

where $\mathbf{p}$ are the regular momenta.

Let's consider a system with dynamics given by the Hamiltonian $H(\mathbf{q}, \mathbf{p})$ where $\mathbf{q}$ is the vector of generalised coordinates and $\mathbf{p}$ is the vector of conjugate momenta. For the unconstrained system the dynamics are given by Hamilton's equations

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}.$$

For the unconstrained system the generalised forces are just the external generalised forces, so

$$\mathbf{Q} = -\frac{\partial H}{\partial \mathbf{q}}. \tag{4.56}$$

The transpose on the denominator is to keep the results of the derivative as a column vector.

Regular *holonomic* constraints are written in the form

$$\mathbf{C}(\mathbf{q}) = \mathbf{0} \tag{4.57}$$

such that

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}^{\mathsf{T}}}\dot{\mathbf{q}} = \mathbf{0}. \tag{4.58}$$

Since our constraint is always zero we can formulate an augmented Hamiltonian which incorporates the constraints. Using a Lagrange multiplier method we obtain

$$\mathcal{H}(\mathbf{q}, \mathbf{p}; \boldsymbol{\lambda}) = H(\mathbf{q}, \mathbf{p}) + \mathbf{C}^{\mathsf{T}}(\mathbf{q})\boldsymbol{\lambda} \tag{4.59}$$

where $\mathcal{H}(\mathbf{q}, \mathbf{p}; \boldsymbol{\lambda}) = H(\mathbf{q}, \mathbf{p}) \; \forall \; \boldsymbol{\lambda}$.

The new dynamics of our constrained system are given by Hamilton's equations

$$\dot{\mathbf{q}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}$$

This gives a new equation of motion, now with the generalised constraint forces

$$\dot{\mathbf{p}} = \underbrace{-\frac{\partial H}{\partial \mathbf{q}}}_{\mathbf{Q}} \underbrace{-\frac{\partial \mathbf{C}^{\mathsf{T}}}{\partial \mathbf{q}}\boldsymbol{\lambda}}_{\hat{\mathbf{Q}}} \tag{4.60}$$

which we can write in a more clear way as

$$\hat{\mathbf{Q}} = -\frac{\partial \mathbf{C}^{T}}{\partial \mathbf{q}}\boldsymbol{\lambda} = -\left(\frac{\partial \mathbf{C}}{\partial \mathbf{q}^{\mathsf{T}}}\right)^{\mathsf{T}}\boldsymbol{\lambda}. \tag{4.61}$$

Note that the power transmitted to the system by these forces is

$$\dot{\mathbf{q}} \cdot \hat{\mathbf{Q}} = \dot{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{Q}} = -\left(\frac{\partial \mathbf{C}}{\partial \mathbf{q}^{\mathsf{T}}}\dot{\mathbf{q}}\right)^{\mathsf{T}}\boldsymbol{\lambda} = \mathbf{0}^{\mathsf{T}}\boldsymbol{\lambda} = 0$$

which we know from equation (4.58).

We have therefore proved what is known as d'Alembert's Principle.

Definition 4.4: d'Alembert's Principle

For holonomic constraints, the constraint forces do no work on the system.

4.5.2 What are Generalised Forces?

We have discussed generalised forces in the context of Hamiltonian mechanics, but what are they?

Work done by a force is defined as

$$W = \int \mathbf{F} \cdot \mathrm{d}\mathbf{s}. \tag{4.62}$$

Many forces are conservative forces and can be represented as a potential energy, V, so that $\mathbf{F} = -\boldsymbol{\nabla} V$. However, forces like friction are not conservative and therefore have no associated potential, meaning they cannot be included in a Lagrangian or Hamiltonian.

We define a generalised force as

$$Q_i = \frac{\partial W}{\partial q_i} = \mathbf{F} \cdot \frac{\partial \mathbf{s}}{\partial q_i} \tag{4.63}$$

such that the work by a generalised force on a given generalised coordinate is

$$W_i = \int Q_i \, \mathrm{d}q_i \, . \tag{4.64}$$

In vector form (using denominator notation) we can write this as

$$\mathbf{Q}_{\text{ext}} = \frac{\partial W_{\text{ext}}}{\partial \mathbf{q}}. \tag{4.65}$$

Another way you can think about this is how we define potential energy. For a conservative force the associated potential is

$$V = -W = -\int \mathbf{F} \cdot \mathrm{d}\mathbf{s}. \tag{4.66}$$

So we can include external forces into our Lagrangian or Hamiltonian with

$$\mathcal{L} = T - (V - W_{\text{ext}}) = L + W_{\text{ext}}, \tag{4.67}$$

or

$$\mathcal{H} = T + (V - W_{\text{ext}}) = H - W_{\text{ext}}. \tag{4.68}$$

We represent the Lagrangian and Hamiltonian of the base system as L and H respectively, while the augmented Lagrangian and Hamiltonian are represented as $\mathcal{L}$ and $\mathcal{H}$ respectively.

4.5.3 The Augmented Euler-Lagrange Equations

The augmented Euler-Lagrange equations are a generalisation of the Euler-Lagrange equations to include generalised forces.

Definition 4.5: Augmented Euler-Lagrange Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_{i,\text{ext}} \tag{4.69}$$

where $Q_{i,\text{ext}}$ is the external generalised force associated with the generalised coordinate q_i.

In vector form we can write this as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q}_{\text{ext}} \tag{4.70}$$

where W_{ext} is the work done by external forces and $\mathbf{Q}_{\text{ext}}$ is the vector of generalised external forces. *All derivatives here are in denominator notation.*

4.5.4 The Augmented Hamilton's Equations

The augmented Hamilton's equations are a generalisation of Hamilton's equations to include generalised forces. They can be obtained by performing a Legendre transform on the Lagrangian from equation (4.70).

Definition 4.6: Augmented Hamilton's Equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_{i,\text{ext}} \tag{4.71}$$

where $Q_{i,\text{ext}}$ is the external generalised force associated with the generalised coordinate q_i.

In vector form we can write this as

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{Q}_{\text{ext}} \tag{4.72}$$

where $\mathbf{Q}_{\text{ext}}$ is the vector of generalised external forces. *All derivatives here are in denominator notation.*

Chapter 5: Useful Mathematics

5.1 Newton's Method

5.1.1 Root Finding

Newton's method is a numerical method for finding the roots of a function. It is based on the idea of linear approximation, where we approximate the function with a tangent line at a given point and then find where that tangent line intersects the x-axis.

The formula for Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{5.1}$$

where x_n is the current approximation, $f(x_n)$ is the value of the function at that point, and $f'(x_n)$ is the derivative of the function at that point. One can find a root that is close to the initial guess x_0 by iterating this formula until the change in x is sufficiently small.

5.1.2 Finding Minima

We can also re-formulate Newton's method for finding a minimum of a function $f(\mathbf{x})$ where $\mathbf{x}$ is a vector of variables. This is typically most useful when the function is convex (or in simpler terms only has one set of connected points that contain the minimum).

This is like a weighted gradient descent method, with the weighting given by the Hessian

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{H}^{-1}\boldsymbol{\nabla} f(\mathbf{x}_n) \tag{5.2}$$

where

$$\boldsymbol{\nabla} f(\mathbf{x}) = \frac{\partial f}{\partial \mathbf{x}^\mathsf{T}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \tag{5.3}$$

and

$$\mathbf{H} = \frac{\partial^2 f}{\partial \mathbf{x} \partial \mathbf{x}^\mathsf{T}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \tag{5.4}$$

5.2 Matrix Mathematics

I'm going to assume you can multiply matrices together. However, determinants and inverses are more complex.

5.2.1 Minor Matrices

Let's consider a 3×3 matrix of the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{5.5}$$

Let's say we wanted to find the minor matrix from entry a_{12}. We do this by deleting the row and column that a_{12} lies in, and keep the remaining 2×2 matrix. e.g.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}. \tag{5.6}$$

5.2.2 Determinants

We can only take the determinant of a square matrix. The determinant of a 1×1 matrix is itself i.e.

$$\det(a) = a. \tag{5.7}$$

The determinant of a 2×2 matrix is more complicated, with

$$\det(\mathbf{A}) = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21} \tag{5.8}$$

being the definition that lays the foundation for the rest of the formulas we shall use. The determinant of any larger matrix is found by choosing any row or column, moving along that row or column, you multiply that number by the determinant of the matrix with the row and column that the number is in deleted. You then alternate multiplying it by -1 and sum the answers. For a matrix larger than 3×3 you continue this process until all determinants remaining are 2×2.

That might be a little confusing. Let's take the determinant of the matrix

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}.$$

The first column has the most zeros so it is a convenient choice. We then take the determinant by moving down the first column, multiplying each entry by the determinant of the remaining 2×2 matrix when we delete that column and row. We alternate signs with the following rule

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}. \tag{5.9}$$

Taking the determinant by alternating signs down the first column,

$$\begin{aligned}\det(\mathbf{M}) &= 0 \cdot \det\left(\begin{bmatrix}-1 & 1\\ 1 & -3\end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix}1 & 2\\ 1 & -3\end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix}1 & 2\\ 1 & -3\end{bmatrix}\right)\\ &= -2(-3-2)\\ &= 10\end{aligned}$$

We could, for example also move along the second row. This involves more calculations but can be done. Also note that as per equation (5.9) we start at a negative

$$\begin{aligned}\det(\mathbf{M}) &= -2 \cdot \det\left(\begin{bmatrix}1 & 2\\ 1 & -3\end{bmatrix}\right) + (-1) \cdot \det\left(\begin{bmatrix}0 & 2\\ 0 & -3\end{bmatrix}\right) - 1 \cdot \det\left(\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right)\\ &= -2(-3-2) + (0-0) - (0-0)\\ &= 10\end{aligned}$$

These smaller matrices are called minor matrices as covered in § 5.2.1.

Taking the determinant of a 4×4 matrix and larger follows the same process. You keep reducing the matrix to minor matrices until you reach 2×2 matrices.

5.2.3 Cofactor and Adjugate Matrices

Every entry of a cofactor matrix $\mathbf{C}$ is the determinant of the minor matrix around that entry. If we have the matrix

$$\mathbf{A} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

then the cofactor matrix of this is the minor matrix associated with each entry, with alternating negative signs as per equation (5.9).

$$\mathbf{C} = \begin{bmatrix}\det\left(\begin{bmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix}\right) & -\det\left(\begin{bmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{bmatrix}\right) & \det\left(\begin{bmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix}\right)\\ -\det\left(\begin{bmatrix}a_{12} & a_{13}\\ a_{32} & a_{33}\end{bmatrix}\right) & \det\left(\begin{bmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{bmatrix}\right) & -\det\left(\begin{bmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{bmatrix}\right)\\ \det\left(\begin{bmatrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{bmatrix}\right) & -\det\left(\begin{bmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{bmatrix}\right) & \det\left(\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right)\end{bmatrix}. \tag{5.10}$$

This expands as one would expect to higher and lower dimensioned matrices.

The adjugate matrix of $\mathbf{A}$ is the transpose of $\mathbf{C}$ i.e.

$$\text{Adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}}. \tag{5.11}$$

5.2.4 Inverse of a Matrix

The inverse of a matrix $\mathbf{A}$ is generally given by the adjugate of $\mathbf{A}$ divided by the determinant of $\mathbf{A}$ i.e.

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\text{Adj}(\mathbf{A}). \tag{5.12}$$

One can find the inverse by first finding all the minor matrices associated with each matrix entry, filling in the cofactor matrix, taking its transpose to find the adjugate matrix and dividing the adjugate matrix by the determinant.

The inverse of a 2×2 matrix is

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{5.13}$$

and for a 3×3 matrix is

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} - a_{13}a_{22} \\ -a_{21}a_{33} + a_{23}a_{31} & a_{11}a_{33} - a_{13}a_{31} & -a_{11}a_{23} + a_{13}a_{21} \\ a_{21}a_{32} - a_{22}a_{31} & -a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

$$\det(\mathbf{A}) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{5.14}$$

5.2.5 Matrix Exponential

The Taylor Series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{5.15}$$

and as such we define that the exponential of a matrix is

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots \tag{5.16}$$

If a matrix is diagonalisable for eigenvalues and eigenvectors

$$\mathbf{A}\mathbf{a}_i = \lambda_i \mathbf{a}_i \tag{5.17}$$

with

$$\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}^{-1} \tag{5.18}$$

then the matrix exponential is

$$e^{\mathbf{A}} = \mathbf{T}e^{\mathbf{D}}\mathbf{T}^{-1} = \mathbf{T} \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix} \mathbf{T}^{-1}. \tag{5.19}$$

5.2.6 Symmetric Matrices

A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric if $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$. This means that the matrix is equal to its transpose. For example, the matrix.

A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is skew-symmetric if $\mathbf{A} = -\mathbf{A}^{\mathsf{T}}$. This means that the matrix is equal to the negative of its transpose.

5.2.7 Positive, Negative and Semi Definite Matrices

A symmetric square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is positive definite if, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} > 0, \implies \mathbf{A} \succ 0, \tag{5.20}$$

negative definite if, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} < 0, \implies \mathbf{A} \prec 0, \tag{5.21}$$

positive semi-definite if, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} \geq 0, \implies \mathbf{A} \succeq 0, \tag{5.22}$$

and negative semi-definite if, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} \leq 0, \implies \mathbf{A} \preceq 0. \tag{5.23}$$

5.3 Trigonometry

arctan2 is a common function in programming languages as it avoids the ambiguity of the arctan function in some of the quadrants. Its definition is

$$\operatorname{arctan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{5.24}$$

5.4 Set Notation and Set Functions

5.4.1 Ordered Sets

A small but not insignificant piece of set theory is whether a set is ordered or unordered. In general an ordered set is one where the elements of the set can be put in sequence. A set can be infinite e.g. the real numbers $\mathbb{R}$ or finite e.g. the integers $\mathbb{Z}$.

5.4.2 Subsets

Let's say we have two sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Very clearly all of the elements of A are in B, so we say that A is a subset of B, or $A \subset B$.

5.4.3 Minimum and Maximum Values

The minimum and maximum values of a set are the smallest and largest values in the set respectively. For example, the minimum value of the set $\{1, 2, 3\}$ is 1 and the maximum value is 3. Or alternatively $\min A = 1$ and $\max A = 3$.

Let's instead consider the case where we want a set to include all real numbers between 0 and 1, but to not include 0 and 1. We would write this as $D = (0, 1)$. What is the maximum or minimum of such a set? That doesn't quite make sense.

To better answer this question we need to define new operations, called the infimum and supremum. The infimum is the largest possible value that all values in a set are greater than, so in this case the infimum of D is $\inf D = 0$. The supremum is the smallest possible value that all values in a set are less than, so in this case the supremum of D is $\sup D = 1$.

5.5 Determining Line of Sight Around an Ellipsoid

To determine the line of sight between two satellites involves checking whether the line passing between them intersects with an object. In this case the only object that can meaningfully obscure the two objects from seeing each other is Earth. We can generally solve whether the line between the two objects intersects an arbitrary ellipsoid, then substitute in Earth's ellipsoid parameters to check if the two objects can see each other.

We first consider an ellipsoid with radii along each of the axes given by

$$\mathbf{d} = \begin{bmatrix} R_x & R_y & R_z \end{bmatrix}. \tag{5.25}$$

This vector does not denote a position but is simply convenient to denote the ellipsoid parameters. The surface of such an ellipsoid, with centre at $\mathbf{r}_c = \begin{bmatrix} r_{cx} & r_{cy} & r_{cz} \end{bmatrix}^T$, is parametrised by

$$\left(\frac{x - r_{cx}}{R_x}\right)^2 + \left(\frac{y - r_{cy}}{R_y}\right)^2 + \left(\frac{z - r_{cz}}{R_z}\right)^2 = 1 \tag{5.26}$$

which can be reduced using vector notation to

$$\left\|\mathbf{D}^{-1}\left(\mathbf{r} - \mathbf{r}_c\right)\right\|^2 = 1, \qquad \mathbf{D} = \operatorname{diag}(\mathbf{d}) = \begin{bmatrix} R_x & & \\ & R_y & \\ & & R_z \end{bmatrix}. \tag{5.27}$$

The line between two points in space is given by a start point $\mathbf{r}_1$ and an end point $\mathbf{r}_2$. Letting $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ the line between the two objects is parametrised by

$$\mathbf{r} = \mathbf{r}_1 + \lambda\Delta\mathbf{r}, \qquad \lambda \in [0,\, 1]. \tag{5.28}$$

Substituting equation (5.28) into equation (5.27) yields

$$\left\|\mathbf{D}^{-1}\left(\mathbf{r}_1 - \mathbf{r}_c + \lambda\Delta\mathbf{r}\right)\right\|^2 = 1 \tag{5.29}$$

which expanded gives a quadratic equation of the form

$$\left\|\mathbf{D}^{-1}\Delta\mathbf{r}\right\|^2 \lambda^2 + 2\left(\left[\mathbf{D}^{-1}\left(\mathbf{r}_1 - \mathbf{r}_c\right)\right] \cdot \left[\mathbf{D}^{-1}\Delta\mathbf{r}\right]\right)\lambda + \left(\left\|\mathbf{D}^{-1}\left(\mathbf{r}_1 - \mathbf{r}_c\right)\right\|^2 - 1\right) = 0. \tag{5.30}$$

Solving for the discriminant of the quadratic equation yields the number of real solutions for λ, if the discriminant is negative then there are no real solutions and there is a direct line of sight between the two objects. There is also a line of sight between the two objects if there are real solutions for λ but at least one not in the range [0, 1].

Chapter 6: Epochs and Time Systems

An epoch is a specific moment in time. However, due to Einstein and General Relativity, that stuff gets spooky pretty quickly. As it currently stands almost all measurements are taken around Earth, so all the time systems we use are based on the time relative to Earth.

Satellites use atomic clocks to keep time, however they need to adjust due to the different rate at which time passes for something in distant orbit. Despite all of this complexity, the time systems we use are designed to keep this complexity away from us.

6.1 Julian Days (JD)

Julian days are so named because they are the length of a day in the Julian calendar, which is exactly 86 400 s. The J2000 epoch is exactly JD 2 451 545.

The number of Julian centuries since the J2000 epoch is given by

$$T_J = \frac{JD - 2\,451\,545}{36\,525}. \tag{6.1}$$

6.2 International Atomic Time (TAI)

International Atomic Time (abbreviated to TAI from its French name temps atomique international) and is a weighted average of time measured by atomic clocks around the world. It is a time system without leap-seconds and is the basis for UTC.

6.3 Coordinated Universal Time (UTC)

Yes the acronym is in the wrong order, yes it's the official acronym.

UTC is the current most widely used time system. It tracks the time of the Greenwhich meridian relative to the Sun, so the time for any other part of the Earth can be calculated using its longitude.

UTC is exactly 37 s behind TAI due to historical leap seconds. So

$$\text{UTC} = \text{TAI} - 37\,\text{s} \tag{6.2}$$

In UTC the J2000 epoch is 12:00 January 1, 2000[1] or 2000:01:01:12:00.000 in YYYY:MM:DD:hh:mm:ss format. Numbers after the decimal point are fractions of a second.

Using this we can then convert from UTC to Julian Days as per [2]. The Julian Day of the start of the current UTC day is

$$J_0 = 367y - \text{int}\left(\frac{7}{4}\left[y + \text{int}\left(\frac{m+9}{12}\right)\right]\right) + \text{int}\left(\frac{275m}{9}\right) + d + 1\,721\,013.5 \tag{6.3}$$

where

$$\text{int}(x) = \text{integer part of } x = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x < 0 \end{cases} \tag{6.4}$$

[1] To add the confusion, there is now also an adjusted J2000 epoch that is 00:00 January 1, 2000.

for instance $\text{int}(1.7) = 1$ and $\text{int}(-3.5) = -3$. We then get the fractional Julian day of the UTC epoch with

$$J = J_0 + \frac{H_{UTC}}{24} \tag{6.5}$$

where H_{UTC} is the current fractional hours passed in the day. For instance for 12:30 April 1, 2025, we have $y = 2025$, $m = 4$, $d = 1$, and $H_{UTC} = 12.5$.

It is worth noting that this only works for

$$1901 \leq y \leq 2099, \qquad 1 \leq m \leq 12, \qquad 1 \leq d \leq 31.$$

6.4 Terrestrial Time (TT)

Terrestrial Time is now also based off TAI, but with a different offset to that of UTC. It is used for the prediction or recording of the positions of celestial bodies as measured by an observer on Earth [3]. Conversion from TAI to TT is given by

$$\text{TT} = \text{TAI} + 32.184\,\text{s}. \tag{6.6}$$

Most of the time TT times are not used in astronomical measurements such as TLEs, however it does crop up sometimes.

6.5 Barycentric Coordinate/Dynamical Time (TDB/TCB)

These are time systems intended to be used as a common measurement for all objects in the Solar System. They account for effects such as time dilation due to gravitational effects and is therefore the true standard time that we currently have for astronomical observations. Due to this complexity I have omitted the conversions as they aren't particularly relevant.

Chapter 7: Orbits

Orbits are curved paths around bodies, typically due to gravity, but more generally some central force. While most orbits are drawn as closed shapes, they need not be, in fact all orbits in reality are not closed, slowly drifting over time.

Orbits of a massless object around a body at the origin, with gravity given by Newton's law of universal gravitation, are called Keplerian orbits. Orbits that do not obey a special case like this, rather obviously called non-Keplerian orbits, typically require numerical methods to solve.

7.1 Keplerian Orbits

Consider an object orbiting due to the centrally acting force

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \tag{7.1}$$

where G is the gravitational constant, M is the mass of the body at the origin, m is the mass of the orbiting object, and $\mathbf{r}$ is the position of the orbiting object relative to the body at the origin. The force is negative because it is attractive, and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$.

Before we dive into the maths, let's make some useful observations that really are quite cool.

7.1.1 Conservation of Angular Momentum

The angular momentum of the orbiting object is given by

$$\mathbf{H} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} \tag{7.2}$$

where $\mathbf{v}$ is the velocity of the orbiting object. The rate of change of angular momentum is the torque on the object relative to the origin, or

$$\begin{aligned}\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times m\mathbf{v}) \\ &= \cancelto{\mathbf{0}}{\mathbf{v} \times m\mathbf{v}} + \mathbf{r} \times m\mathbf{a} \\ &= \mathbf{r} \times m\mathbf{a} \\ &= \mathbf{r} \times \mathbf{F}\end{aligned}$$

or

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \left(-\frac{GMm}{r^2}\hat{\mathbf{r}}\right) \tag{7.3}$$

because $\mathbf{r}$ and $\hat{\mathbf{r}}$ are parallel their cross product is always zero, so we have that

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{0}. \tag{7.4}$$

Conservation of angular momentum directly gives Kepler's second law. Consider the diagram in figure 7.1.

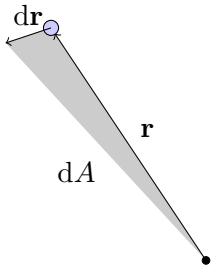


Figure 7.1: The area traced out by the orbiting object in a small time dt.

The magnitude of the cross product of two vectors is the area of the parallelogram formed by them, and half that is the area of the triangle formed by the two vectors. Therefore the area traced out over a small time dt is given by

$$\mathrm{d}A = \frac{1}{2}\left\|\mathbf{r} \times \mathrm{d}\mathbf{r}\right\| \tag{7.5}$$

and the rate of area traced over time is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\left\|\mathbf{r} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right\| = \frac{1}{2}\left\|\mathbf{r} \times \mathbf{v}\right\| = \frac{1}{2}\frac{H}{m} \tag{7.6}$$

where H is the magnitude of the angular momentum, which is conserved. We divide by the mass but because this is constant the law holds. We actually have a special name for this quantity, which we call the specific angular momentum, which is given by

$$\mathbf{h} = \frac{1}{m}\mathbf{H} = \mathbf{r} \times \mathbf{v} \tag{7.7}$$

and the area traced over time is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}h. \tag{7.8}$$

7.1.2 Reduction to 2D

It is a useful fact that all orbits given by central potentials can be reduced to two dimensions (even when they are in more than three dimensions). Though this is harder to prove in n-dimensions, it is quite easy in 3D.

From a purely mathematical standpoint, because three points form a plane, it is also true that two vectors that form a triangle also form a plane. The cross product of two vectors forms a vector that is orthogonal to both vectors, and therefore also orthogonal to their plane.

Since the angular momentum is conserved/constant in all directions, its direction is also conserved. This means that the plane of the orbit always has the same normal. Equivalently, the plane of the orbit is always the same plane, with its normal given by the direction of the angular momentum vector. Since the motion always reduces to a being on a plane, the problem can always be reduced to two dimensions.

7.1.3 Kinematics of Orbits

Let's now consider the reduced 2D problem of a Keplerian orbit. We can write the position of the orbiting object in polar coordinates as (r, θ), where r is the distance from the origin and θ is the angle

from the x axis. Speed has the generic formula

$$v^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \tag{7.9}$$

which can be converted to polar coordinates using the polar coordinate metric in equation (1.10) as

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{7.10}$$

It is at this point far easier to compute the dynamics using Lagrangian mechanics. The potential energy of the system is

$$V(r) = -\frac{GMm}{r} \tag{7.11}$$

so the Lagrangian is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}. \tag{7.12}$$

Applying the Euler-Lagrange equations gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{r}}\right) - \frac{\mathrm{d}L}{\mathrm{d}r} = 0 \qquad & \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}}\right) - \frac{\mathrm{d}L}{\mathrm{d}\theta} = 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\dot{\theta}^2 + \frac{GMm}{r^2} = 0 \qquad & \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) = 0 \\
m\ddot{r} = mr\dot{\theta}^2 - \frac{GMm}{r^2} = 0 \qquad & \qquad mr^2\dot{\theta} = \text{constant} = J.
\end{aligned}$$

Notice that because θ does not appear in the Lagrangian we get a conserved quantity J. What is this quantity? In polar coordinates the position and velocity can be written as

$$\mathbf{r} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{bmatrix}. \tag{7.13}$$

The angular momentum very nicely simplifies, with

$$\begin{aligned}
\mathbf{H} &= \mathbf{r} \times m\mathbf{v} \\
&= \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \times m \begin{bmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{bmatrix} \\
&= mr^2\dot{\theta}\hat{\mathbf{z}}
\end{aligned}$$

and so the magnitude of the angular momentum is our conserved quantity J. In fact this now helps us remove the angular velocity, using

$$\dot{\theta} = \frac{J}{mr^2} = \frac{H}{mr^2}.$$

Substituting this into the radial equation gives

$$m\ddot{r} = \frac{H^2}{mr^3} - \frac{GMm}{r^2}. \tag{7.14}$$

Let's now make the substitution $r = 1/u$, so that

$$\begin{aligned}\ddot{r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) \\ &= \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) \\ &= \frac{H}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{H}{mr^2}\right) \\ &= \frac{H}{m}u^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{H}{m}u^2\right) \\ &= -\frac{H^2}{m^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}.\end{aligned}$$

which simplifies equation (7.14) to

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = \frac{GMm^2}{H^2} - u. \tag{7.15}$$

We now simplify this by knowing that the specific angular momentum is

$$h = \frac{\|\mathbf{H}\|}{m} = \frac{H}{m} \tag{7.16}$$

and by defining the standard gravitational parameter

$$\mu = GM. \tag{7.17}$$

This gives the final form

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\mu}{h^2}. \tag{7.18}$$

This is an inhomogeneous second order linear ordinary differential equation, with a constant inhomogeneous term. The solution to this is given by

$$u(\theta) = A\cos(\theta - \theta_0) + \frac{\mu}{h^2} \tag{7.19}$$

for some constants A and θ_0. If we define that the periapsis, the point of closest approach, is at $\theta = 0$, then $\theta = 0$ should maximise $u = 1/r$, which means that $\theta_0 = 0$.

We now have a formula for $r(\theta)$, given by

$$r(\theta) = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}. \tag{7.20}$$

where $e = \frac{h^2}{\mu}A$ is the eccentricity of the orbit. The eccentricity is a measure of how elliptical the orbit is, with $e = 0$ being a circle, $0 < e < 1$ being an ellipse, $e = 1$ being a parabola, and $e > 1$ being a hyperbola. We call θ the true anomaly, which is the angle from the periapsis of the orbit by definition.

This combined with

$$h = \|\mathbf{r} \times \mathbf{v}\| = \text{constant} \tag{7.21}$$

and

$$\dot{\theta} = \frac{h}{r^2} \tag{7.22}$$

gives the time dependent equations of motion of the orbit. Therefore, to describe the position of an orbiting object in its orbital plane, we need three orbital elements: r, θ, and h.

7.2 Nomenclature & Orbital Parameters

Orbits in reality are weirdly shaped curves with no particular shape. However, orbits obeying Newton's law of gravitation involving two bodies do have well defined shapes – the conic sections. The conic sections are the circle, ellipse, parabola and hyperbola, and are so named as they can be obtained by slicing a cone at different angles.

7.2.1 Elliptical Orbits

These are the most common kind of orbit as they are the closed periodic orbits. Most of the naming from these orbits extends to parabolic and hyperbolic orbits. A diagram of an elliptical orbit is shown in figure 7.2, with the nomenclature given in table 7.1.

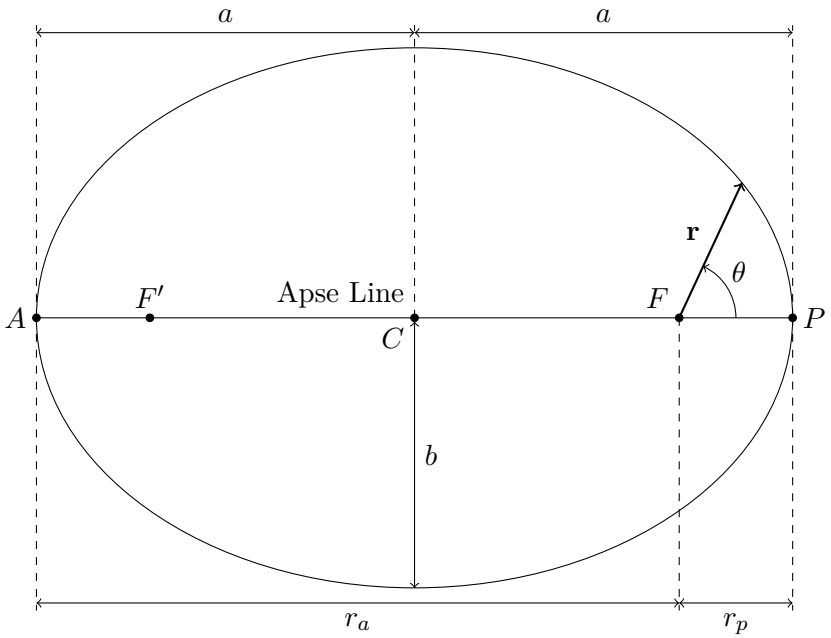


Figure 7.2: Diagram of an elliptical orbit.

Table 7.1: Nomenclature for orbits, as per figure 7.2.

Name	Variable	Explanation
Occupied Focus	F	The focus of the ellipse where object being orbited is located. In two-body orbits both bodies orbit this is the centre of mass.
Unoccupied Focus	F'	The other focus of the orbit which is unoccupied.
Periapsis	P	The closest point in the orbit to F.
Apoapsis	A	The furthest point in the orbit from F.
Perigee	P	Periapsis but specifically when an orbit is around Earth.

Continued on next page

Table 7.1: Nomenclature for orbits, as per figure 7.2. (Continued)

Name	Variable	Explanation
Apogee	A	Apoapsis but specifically when an orbit is around Earth.
Perihelion	P	Periapsis but specifically when an orbit is around the Sun.
Aphelion	A	Apoapsis but specifically when an orbit is around the Sun.
Apse Line	–	The line connecting the periapsis and apoapsis.
Centre	C	The centre of the orbit, also the midpoint of the apse line.
Semi-major axis	a	The distance from the centre C to the widest point on the orbit.
Semi-minor axis	b	The distance from the centre to the orbit, orthogonal to the apse line.
Periapsis Distance	r_p	The distance from the occupied focus F to the periapsis.
Apoapsis Distance	r_a	The distance from the occupied focus F to the apoapsis.
Position	$\mathbf{r}$	The position vector of the orbiting body, relative to the occupied focus F.
True Anomaly	θ	The angle between the periapsis and the position vector $\mathbf{r}$. $\theta = 0$ when the body is at periapsis.
Eccentricity	e	Describes the shape of the orbit. Orbits are circular when $e = 0$, elliptical for $0 < e < 1$, parabolic for $e = 1$ and hyperbolic for $e > 1$.

7.2.1.1 Orbital Parameters

It is common to parametrise orbits in terms of the typical conic section parameters, as well as write positions with respect to the occupied focus F in polar coordinates (r, θ).

Using equation (7.20) we can write this position in the orbital plane, in a frame called the *perifocal frame*, as

$$\mathbf{r} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \frac{h^2}{\mu} \begin{bmatrix} \frac{\cos\theta}{1+e\cos\theta} \\ \frac{\sin\theta}{1+e\cos\theta} \end{bmatrix}. \tag{7.23}$$

The periapsis is when $r(\theta)$ is at a minimum, which we defined as when $\theta = 0$, so

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} \tag{7.24}$$

and the apoapsis is when $r(\theta)$ is at a maximum, i.e when $\theta = \pi$, so

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e}. \tag{7.25}$$

We know geometrically from figure 7.2 that the semi-major axis is given by

$$a = \frac{r_p + r_a}{2} = \frac{h^2}{\mu}\frac{1}{1-e^2} \tag{7.26}$$

allowing us to define the periapsis and apoapsis distances as

$$r_p = a(1-e), \qquad r_a = a(1+e). \tag{7.27}$$

7.2.1.2 Energies

As energy is conserved in an orbit, we can compute the energy at any point that is convenient, let's take the periapsis. The periapsis is when the distance r is at a minimum, so $\dot{r} = 0$ at the periapsis.

The kinetic energy per mass is

$$\mathcal{T} = \frac{1}{2}v^2 = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{7.28}$$

and the potential energy per mass is

$$\mathcal{V} = -\frac{\mu}{r} \tag{7.29}$$

and with some algebra it can be shown that the total energy per mass is

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1-e^2\right) = -\frac{\mu}{2a}. \tag{7.30}$$

This gives a new formula for the eccentricity in terms of inherent properties of the orbit, rather than other parameters for its shape,

$$e = \sqrt{1 + \frac{2\mathcal{E}h^2}{\mu^2}}. \tag{7.31}$$

7.2.1.3 Eccentricity Vector

A curious quantity is the vector

$$\mathbf{e} = \frac{1}{\mu}\mathbf{v}\times\mathbf{h} - \hat{\mathbf{r}} \tag{7.32}$$

which I claim has the properties that its magnitude is the eccentricity of the orbit, and its direction is the direction of the periapsis.

Let's start with a few vector identities.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \tag{7.33}$$

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0 \tag{7.34}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{7.35}$$

The squared magnitude of equation (7.32) is

$$\begin{aligned}
\|\mathbf{e}\|^2 &= \mathbf{e}\cdot\mathbf{e} \\
&= \left(\frac{1}{\mu}\mathbf{v}\times\mathbf{h} - \hat{\mathbf{r}}\right)\cdot\left(\frac{1}{\mu}\mathbf{v}\times\mathbf{h} - \hat{\mathbf{r}}\right) \\
&= \frac{1}{\mu^2}(\mathbf{v}\times\mathbf{h})\cdot(\mathbf{v}\times\mathbf{h}) - 2\frac{1}{\mu}\hat{\mathbf{r}}\cdot(\mathbf{v}\times\mathbf{h}) + 1 \\
&= \frac{1}{\mu^2}\left(\|\mathbf{v}\|^2\|\mathbf{h}\|^2 - \cancelto{0}{(\mathbf{v}\cdot\mathbf{h})^2}\right) - 2\frac{1}{\mu r}\mathbf{h}\cdot(\mathbf{r}\times\mathbf{v}) + 1 && \text{(by equations (7.33) to (7.35))} \\
&= \frac{2}{\mu^2}h^2\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right) + 1 \\
&= 1 + \frac{2}{\mu^2}h^2\mathcal{E} \\
&= e^2 && \text{(by equation (7.31))}
\end{aligned}$$

So therefore $\|\mathbf{e}\| = e$.

It can also be shown that $\mathbf{e}$ lies in the orbital plane, with

$$\begin{aligned}
\mathbf{e}\cdot\mathbf{h} &= \frac{1}{\mu}(\mathbf{v}\times\mathbf{h})\cdot\mathbf{h} - \hat{\mathbf{r}}\cdot\mathbf{h} \\
&= 0 - \frac{1}{r}\mathbf{r}\cdot(\mathbf{r}\times\mathbf{v}) \\
&= 0
\end{aligned}$$

meaning that $\mathbf{e}$ is orthogonal to $\mathbf{h}$, so must lie in the orbital plane.

The angle made between $\mathbf{e}$ and the position $\mathbf{r}$, ϕ, is found with

$$\begin{aligned}
\mathbf{e}\cdot\mathbf{r} &= er\cos\phi \\
\left(\frac{1}{\mu}\mathbf{v}\times\mathbf{h} - \hat{\mathbf{r}}\right)\cdot\mathbf{r} &= er\cos\phi \\
\frac{1}{\mu}\mathbf{h}\cdot(\mathbf{r}\times\mathbf{v}) - r &= er\cos\phi && \text{(by equation (7.33))} \\
\frac{h^2}{\mu} - r &= er\cos\phi \\
\frac{h^2}{\mu}\left(1 - \frac{1}{1+e\cos\theta}\right) &= er\cos\phi && \text{(by equation (7.20))} \\
\frac{h^2}{\mu}\left(\frac{1}{1+e\cos\theta}\right)e\cos\theta &= er\cos\phi \\
er\cos\theta &= er\cos\phi
\end{aligned}$$

and therefore the angle ϕ between $\mathbf{e}$ and $\mathbf{r}$ is the same as the true anomaly θ, meaning that $\mathbf{e}$ must always point to the periapsis of the orbit.

7.2.2 Circular Orbits

Circular orbits are just a special case of an elliptical orbit, where the two foci merge to become the centre. The periapsis and apoapsis distances become the same and the semi-major and semi-minor axes are equal as well. Mathematically we can state this as just

$$e = 0 \implies r_p = r_a = a = b. \tag{7.36}$$

This also means that there is no periapsis to define $\theta = 0$. Normally the periapsis is arbitrarily chosen to be a point on the orbit, perhaps simply the starting point of the orbit to be studied.

7.2.3 Hyperbolic Orbits

Hyperbolic orbits inherit many of their parameters from the elliptical orbit parameters listed in table 7.1, though they appear very differently, as can be seen in figure 7.3. An orbit is hyperbolic if the eccentricity is greater than 1.

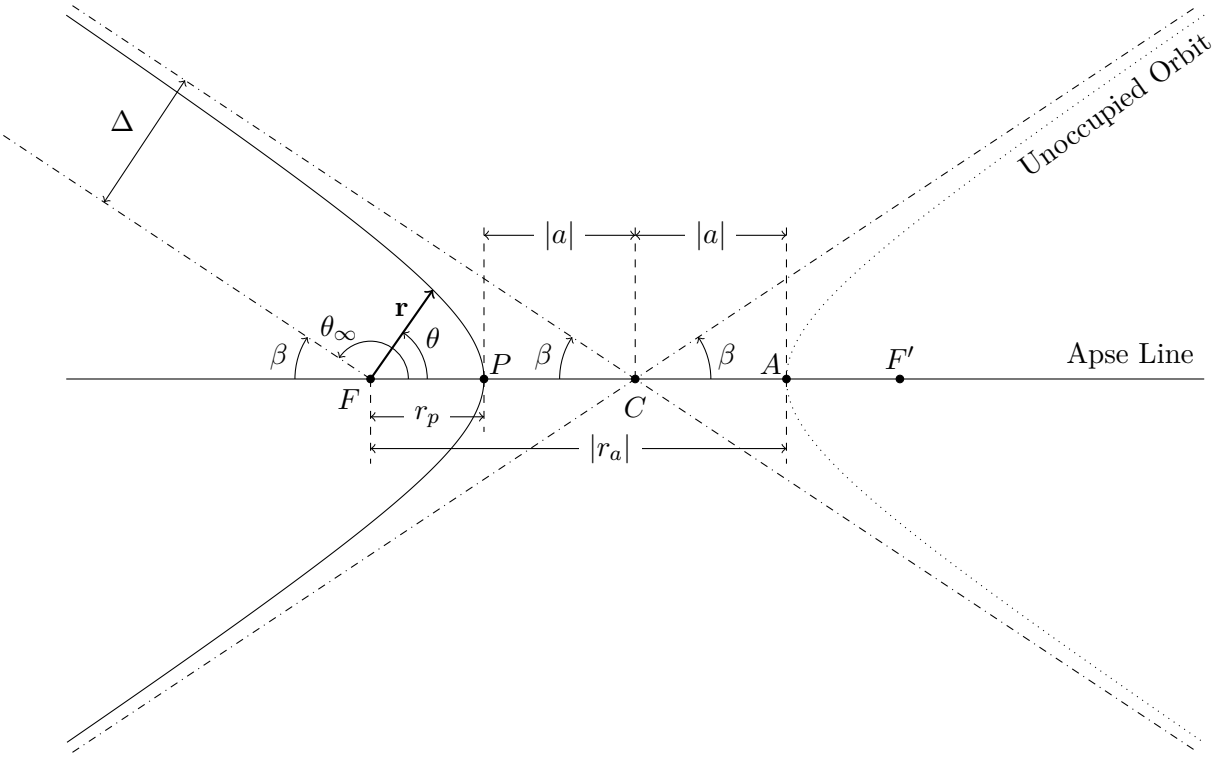


Figure 7.3: Diagram of a hyperbolic orbit.

7.2.3.1 Orbital Parameters

All of the equations from § 7.2.1.1 hold for hyperbolic orbits too, except that many of the quantities are now negative. Notice that the semi-major axis and apoapsis distances are now negative, with

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} < 0, \qquad r_a = \frac{h^2}{\mu}\frac{1}{1-e} < 0.$$

There is a maximum angle that can be reached as the orbiting body reaches an infinite distance away, called θ_∞. We can find this by finding where the denominator of equation (7.20) goes to 0. This yields the equation

$$\cos\theta_\infty = -\frac{1}{e}. \tag{7.37}$$

Because the frame is always defined so that the body moves counter-clockwise, we can take the positive solution, giving

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right). \tag{7.38}$$

Though it is worth noting that the negative solution simply represents where the object would have come from if it started infinitely far away.

The angle of the asymptotes β is given by

$$\theta_\infty + \beta = 180^\circ \tag{7.39}$$

which yields

$$\beta = \cos^{-1}\left(\frac{1}{e}\right). \tag{7.40}$$

7.2.3.2 Energies

The energy per mass of a hyperbolic orbit is given by

$$\mathcal{E} = -\frac{\mu}{2a}, \tag{7.41}$$

which, because $a < 0$, is positive. This represents the fact that the object is moving faster than the escape velocity.

7.2.4 Parabolic Orbits

Parabolic orbits are orbits where $e = 1$ and are best thought of as an edge case of hyperbolic orbits. The object will tend to reach infinity, however the true anomaly at which this happens is

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right) = 180^\circ \tag{7.42}$$

and the total energy per mass is

$$\mathcal{E} = -\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2) = 0. \tag{7.43}$$

This means that the object has exactly the minimum energy to escape the gravitational pull of the body it is orbiting but as it moves away, it will slow down and eventually come to a stop at infinity.

Some of the other orbital quantities have strange values such as

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} = \infty, \qquad r_a = \frac{h^2}{\mu}\frac{1}{1-e} = \infty, \qquad r_p = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{h^2}{2\mu}.$$

Other than theoretically though, you won't really encounter parabolic orbits, because most escape orbits have a speed at least slightly greater than the escape velocity.

7.2.5 Mean and Eccentric Anomalies

While typically quite usesless concepts, the mean anomaly M is given as a key orbital parameter in Two-Line-Element sets (TLEs) and therefore requires some discussion.

We can define the mean motion as the average angular velocity of the orbiting body, given by

$$n = \frac{2\pi}{T} \tag{7.44}$$

where T is the orbital period.

Substituting equation (7.20) into equation (7.22) gives

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h^3}{\mu^2}(1 + e\cos\theta)^2 \tag{7.45}$$

which can then be rearranged to give

$$\frac{h^3}{\mu^2}\Delta t = \int_{\theta_i}^{\theta} \frac{\mathrm{d}\theta}{(1 + e\cos\theta)^2}. \tag{7.46}$$

where $\Delta t = t - t_i$ is the time interval between the two true anomalies.

Setting $\theta_i = 0$ yields the time since periapsis.

The mean anomaly is defined so that its rate of change is the mean motion, so

$$M = n\Delta t \tag{7.47}$$

where Δt is the time since periapsis.

Solutions to equation (7.46) tell us that

$$n = \begin{cases} \dfrac{\mu^2}{h^3} & \text{if } e = 1 \\ \dfrac{\mu^2}{h^3}|1 - e^2|^{3/2} & \text{otherwise} \end{cases} \tag{7.48}$$

Similarly, as per [2], we can find solutions to equation (7.46) of the form

$$M = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \qquad (e < 1)$$

$$M = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2} \qquad (e = 1)$$

$$M = \frac{e\sqrt{e^2-1}\sin\theta}{1+e\cos\theta} - \ln\left(\frac{\sqrt{e+1} + \sqrt{e-1}\tan\frac{\theta}{2}}{\sqrt{e+1} - \sqrt{e-1}\tan\frac{\theta}{2}}\right) \qquad (e > 1)$$

There are then associated equations for the eccentric anomaly E and the true anomaly θ that can be solved more easily.

$$E - e\sin E = M \qquad (e < 1)$$

$$e\sinh E - E = M \qquad (e > 1)$$

and

$$\theta = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) \qquad (e < 1)$$

$$\theta = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tanh\frac{E}{2}\right) \qquad (e > 1)$$

One can solve these equations using root-finding methods such as Newton's method by solving for where the equations are zero for a known mean anomaly M.

$$E - e\sin E - M = 0 \qquad (e < 1)$$

$$e\sinh E - E - M = 0 \qquad (e > 1)$$

Once the eccentric anomaly E is found, the true anomaly θ can be easily computed using the equations derived earlier.

In the case of $e = 1$, one need only solve the mean anomaly equation numerically.

7.2.6 Universal Anomaly

Another way to describe an orbit over time is the universal anomaly ψ such that

$$\dot{\psi} = \frac{1}{\sqrt{\mu}\, r}. \tag{7.49}$$

If we can solve for the universal anomaly over time then we can also compute the position of the object with time. Most notably, the universal anomaly can be solved for an arbitrary time after the initial conditions in the same way, and doesn't require integrating from the initial conditions, which can be computationally useful for calculating over long time spans.

7.2.6.1 Stumpff Functions

Stumpff functions [4] are a set of functions discovered by Karl Stumpff in 1957, which are used to solve the universal anomaly. We define them using a series

$$C_k(x) = \frac{1}{k!} + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n+k)!} \quad \text{for } k = 0,\, 1,\, 2,\, \ldots \tag{7.50}$$

but is often shortened to

$$C_k(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n+k)!} \quad \text{for } k = 0,\, 1,\, 2,\, \ldots \tag{7.51}$$

Equation (7.51) is equivalent to equation (7.50) except for $x = 0$, which when performing calculations in a computer is quite important.

The m^{th} derivative of the Stumpff functions is given by the series

$$C_k^{(m)}(x) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{(-1)^n x^{n-m}}{(2n+k)!} \quad \text{for } k = 0,\, 1,\, 2,\, \ldots \tag{7.52}$$

Computing the series reveals that they also have very nice recursive relationships,

$$C_0(x) = \cos\sqrt{x} \tag{7.53}$$

$$C_1(x) = \begin{cases} \dfrac{\sin\sqrt{x}}{\sqrt{x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \tag{7.54}$$

$$x\, C_{k+2}(x) = \frac{1}{k!} - C_k(x) \quad \text{for } k = 0,\, 1,\, 2,\, \ldots. \tag{7.55}$$

This holds for positive and negative x, but is often simplified for $x < 0$ with

$$\cos(ix) = \cosh(x), \quad \sin(ix) = i\sinh(x)$$

where $i = \sqrt{-1}$ is the imaginary unit.

Here we will require $C_2(x)$ and $C_3(x)$, which can be written

$$C_2(x) = \begin{cases} \dfrac{1-\cos\sqrt{x}}{x} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases} \qquad C_3(x) = \begin{cases} \dfrac{\sqrt{x}-\sin\sqrt{x}}{(\sqrt{x})^3} & \text{if } x \neq 0 \\ \frac{1}{6} & \text{if } x = 0 \end{cases} \tag{7.56}$$

or, simplifying for $x < 0$,

$$C_2(x) = \begin{cases} \dfrac{1-\cosh\sqrt{-x}}{x} & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ \dfrac{1-\cos\sqrt{x}}{x} & \text{if } x > 0 \end{cases} \qquad C_3(x) = \begin{cases} \dfrac{\sqrt{-x}-\sinh\sqrt{-x}}{-(\sqrt{-x})^3} & \text{if } x < 0 \\ \frac{1}{6} & \text{if } x = 0 \\ \dfrac{\sqrt{x}-\sin\sqrt{x}}{(\sqrt{x})^3} & \text{if } x > 0 \end{cases} \tag{7.57}$$

The derivatives of these functions, through arduous algebra, can be found to be

$$C_2'(x) = \begin{cases} \dfrac{1}{2x}\Big(1 - 2C_2(x) - xC_3(x)\Big) & \text{if } x \neq 0 \\ -\dfrac{1}{24} & \text{if } x = 0 \end{cases} \tag{7.58a}$$

$$C_3'(x) = \begin{cases} \dfrac{1}{2x}\Big(C_2(x) - 3C_3(x)\Big) & \text{if } x \neq 0 \\ -\dfrac{1}{120} & \text{if } x = 0 \end{cases} \tag{7.58b}$$

7.2.6.2 Universal Anomaly Equation

These Stumpff functions can be shown using rather extensive algebra (see [5]) to form solutions to the two-body problem. This then yields an explicit solution to equation (7.46) of the form

$$\sqrt{\mu}\Delta t = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\psi^2 C_2\left(\frac{\psi^2}{a}\right) + \left(1 - \frac{r_0}{a}\right)\psi^3 C_3\left(\frac{\psi^2}{a}\right) + r_0\psi \tag{7.59}$$

where ψ is the universal anomaly, $\mathbf{r}_0$ is the initial position vector at $\Delta t = 0$, $r_0 = \|\mathbf{r}_0\|$, and $\mathbf{v}_0$ is the initial velocity vector. a is the semi-major axis which can be calculated using equation (7.26).

One can solve for ψ numerically using Newton's method, or any other root-finding method, by solving for where

$$F(\psi) = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\psi^2 C_2\left(\frac{\psi^2}{a}\right) + \left(1 - \frac{r_0}{a}\right)\psi^3 C_3\left(\frac{\psi^2}{a}\right) + r_0\psi - \sqrt{\mu}\Delta t = 0$$

for any given initial conditions and Δt.

Its derivative can be calculated with equations (7.58a) and (7.58b) and then this can be used with Newton's method to numerically solve for ψ.

The universal anomaly ψ can then be used to find the position and velocity using the Lagrange coefficient equations

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0, \tag{7.60a}$$
$$\mathbf{v} = \dot{f}\mathbf{v}_0 + \dot{g}\mathbf{r}_0, \tag{7.60b}$$

where f and g are the Lagrange coefficients. The Lagrange coefficients are given by

$$f = 1 - \frac{\psi^2}{r_0} C_2\left(\frac{\psi^2}{a}\right), \qquad g = \Delta t - \frac{\psi^3}{\sqrt{\mu}} C_3\left(\frac{\psi^2}{a}\right) \tag{7.61a}$$
$$\dot{f} = \frac{\psi\sqrt{\mu}}{r r_0}\left(\frac{\psi^2}{a} C_3\left(\frac{\psi^2}{a}\right) - 1\right), \qquad \dot{g} = 1 - \frac{\psi^2}{r} C_2\left(\frac{\psi^2}{a}\right), \tag{7.61b}$$

where r is the magnitude of $\mathbf{r}$ after time Δt. These equations require knowledge of r, however only for calculating $\mathbf{v}$, so as long as $\mathbf{r}$ is calculated first, this is not an issue.

7.2.7 Key Equations

There is what feels like an infinite number of equations for Keplerian orbits, so here is a summary of the most common ones with explanations of where they come from and what they mean.

7.2.7.1 Conic Sections Properties

We know that Keplerian orbits are defined by parametric equations for the conic sections, so many of the equations that we obtain through geometry carry over. While many textbooks will swap signs for the conic sections here we shall treat them as the same for all conic sections, though for $e \geq 1$ some of these formulas loose their meaning. One notable is the semi-minor axis b is imaginary for parabolas and hyperbolas.

$$r_p = a(1-e), \qquad r_a = a(1+e)$$
$$e = \frac{r_a - r_p}{r_a + r_p}, \qquad \frac{r_a}{r_p} = \frac{1+e}{1-e}$$
$$a = \frac{r_a + r_p}{2}$$
$$r = \frac{a(1-e^2)}{1 + e\cos\theta}$$
$$b = a\sqrt{1-e^2}$$

7.2.7.2 Orbital Dynamics Equations

The equations relevant to the motion of the body can be stated quite succinctly. Equations specific to each conic section can be found in their relative sections.

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\hat{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

$$\mathbf{H} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

$$\dot{\mathbf{H}} = \mathbf{0}, \qquad \dot{\mathbf{h}} = \mathbf{0}$$

$$\mathbf{e} = \frac{1}{\mu}\mathbf{v} \times \mathbf{h} - \hat{\mathbf{r}}$$

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$$

$$h = \sqrt{2\mu\frac{r_p r_a}{r_p + r_a}}$$

$$\dot{\theta} = \frac{h}{r^2} = \frac{h^3}{\mu^2}(1 + e\cos\theta)^2$$

$$a = \frac{h^2}{\mu}\frac{1}{1 - e^2}$$

$$\mathcal{E} = \frac{E}{m} = \mathcal{T} + \mathcal{V} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} = \frac{\mu^2(e^2 - 1)}{2h^2}$$

$$e = \sqrt{1 + \frac{2\mathcal{E}h^2}{\mu^2}}$$

7.3 The Perifocal Frame

In the derivation of the orbital parameters we defined that $\theta = 0$ when the body is at periapsis. More than that, we defined that the orbit always occurs in a plane with normal vector $\mathbf{h}$.

We therefore define that the x, $\hat{\mathbf{p}}$, axis points from the occupied focus F to the periapsis, the z axis, $\hat{\mathbf{w}}$, is given by $\hat{\mathbf{h}}$ (the unit vector of the specific angular momentum) and the y axis, $\hat{\mathbf{q}}$, is given by $\hat{\mathbf{q}} = \hat{\mathbf{w}} \times \hat{\mathbf{p}}$. This forms an orthonormal basis pqw with an origin at F.

7.4 The Two-Body Problem

The two-body problem is to solve for the shape of the orbits of two bodies, orbiting each other due to their gravitational attraction, pictured in figure 7.4.

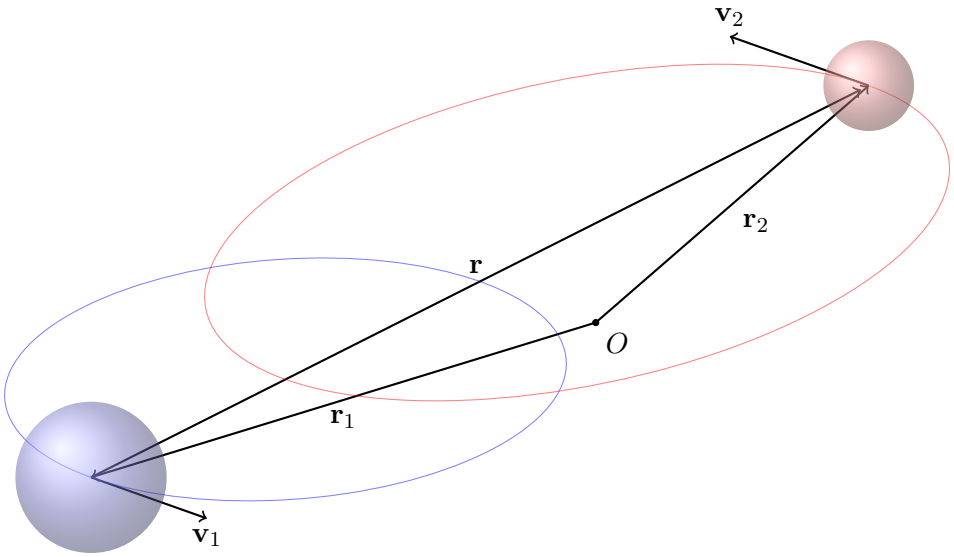


Figure 7.4: Two bodies orbiting around their centre of mass. For the sake of generality the origin is shown offset.

Let's begin by defining the problem mathematically. The position of the first body is $\mathbf{r}_1$, its' velocity is $\mathbf{v}_1$, its mass is m_1 and similarly for the second body. The vector pointing from body 1 to body 2 is

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \tag{7.62}$$

so that the force on body 2 due to body 1 is

$$m_2\mathbf{a}_2 = \mathbf{F}_2 = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{7.63}$$

and

$$m_1\mathbf{a}_1 = \mathbf{F}_1 = \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{7.64}$$

where G is the gravitational constant, $r = \|\mathbf{r}\|$ is the distance between the two bodies, and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$.

The centre of mass is given by

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{7.65}$$

and as it turns out, the centre of mass has no acceleration, with

$$\begin{aligned}
\ddot{\mathbf{R}} &= \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2} \\
&= \frac{\mathbf{F}_1 + \mathbf{F}_2}{m_1 + m_2} \\
&= \frac{\mathbf{F}_1 - \mathbf{F}_1}{m_1 + m_2} \\
&= \mathbf{0}.
\end{aligned}$$

This means that the centre of mass moves with constant velocity, and can be the origin of an inertial frame of reference.

We also have an equation of motion for the relative position of the two bodies, with

$$\begin{aligned}\ddot{\mathbf{r}} &= \mathbf{a}_2 - \mathbf{a}_1 \\ &= \frac{\mathbf{F}_2}{m_2} - \frac{\mathbf{F}_1}{m_1} \\ &= \frac{\mathbf{F}_2}{m_2} + \frac{\mathbf{F}_2}{m_1} \\ &= \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_2 \\ &= \frac{m_1 + m_2}{m_1 m_2}\mathbf{F}_2\end{aligned}$$

Note that $\mathbf{F}_2$ only depends on $\mathbf{r}$, making this a vector ordinary differential equation.

If we denote the total mass of the system

$$M = m_1 + m_2 \tag{7.66}$$

and a *reduced mass* of the system

$$m = \frac{m_1 m_2}{m_1 + m_2} \tag{7.67}$$

then we can write the differential equation for $\mathbf{r}$ as

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}) \tag{7.68}$$

where

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}_2 = -\frac{Gm_1 m_2}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \tag{7.69}$$

This simplifies further to

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\hat{\mathbf{r}} \tag{7.70}$$

where

$$\mu = GM = G(m_1 + m_2) \tag{7.71}$$

is the *gravitational parameter* of the system.

7.4.1 Centre of Mass Frame

Since the centre of mass does not accelerate, it is equivalent to treat it as the stationary origin of an inertial frame of reference. The equations of motion of the two bodies can more easily be solved in this frame.

This simplification allows us to make two simplifications. The centre of mass being the origin yields

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \mathbf{0} \implies \mathbf{r}_1 = -\frac{m_2}{m_1}\mathbf{r}_2 \tag{7.72}$$

and similarly for velocity

$$\mathbf{v}_1 = -\frac{m_2}{m_1}\mathbf{v}_2. \tag{7.73}$$

The reduced mass can also be used to define the angular momentum of the system and the kinetic energy in this frame.

The angular momentum of the system is given by

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = \mathbf{r}_1 \times m_1\mathbf{v}_1 + \mathbf{r}_2 \times m_2\mathbf{v}_2 \tag{7.74}$$

but we can rearrange this to be

$$\begin{aligned}
\mathbf{H} &= \mathbf{H}_1 + \mathbf{H}_2 \\
&= \frac{1}{M}\left(m_1\mathbf{H}_1 + m_2\mathbf{H}_2 + m_2\mathbf{H}_1 + m_1\mathbf{H}_2\right) \\
&= \frac{1}{M}\left(m_1\mathbf{r}_1 \times m_1\mathbf{v}_1 + m_2\mathbf{r}_2 \times m_2\mathbf{v}_2 + m_2\mathbf{H}_1 + m_1\mathbf{H}_2\right)
\end{aligned}$$

Using equation (7.72)

$$\begin{aligned}
&= \frac{1}{M}\left(-m_2\mathbf{r}_2 \times m_1\mathbf{v}_1 - m_1\mathbf{r}_1 \times m_2\mathbf{v}_2 + m_2\mathbf{H}_1 + m_1\mathbf{H}_2\right) \\
&= \frac{1}{M}\left(-m_2\mathbf{r}_2 \times m_1\mathbf{v}_1 - m_1\mathbf{r}_1 \times m_2\mathbf{v}_2 + m_2\mathbf{r}_1 \times m_1\mathbf{v}_1 + m_1\mathbf{r}_2 \times m_2\mathbf{v}_2\right) \\
&= \frac{m_1m_2}{M}\left(-\mathbf{r}_2 \times \mathbf{v}_1 - \mathbf{r}_1 \times \mathbf{v}_2 + \mathbf{r}_1 \times \mathbf{v}_1 + \mathbf{r}_2 \times \mathbf{v}_2\right) \\
&= \left(\frac{m_1m_2}{M}\right)\left(\mathbf{r}_2 - \mathbf{r}_1\right) \times \left(\mathbf{v}_2 - \mathbf{v}_1\right)
\end{aligned}$$

finally giving

$$\mathbf{H} = \mathbf{r} \times m\dot{\mathbf{r}} \tag{7.75}$$

where m is the reduced mass given in equation (7.67).

Similarly we can rearrange the kinetic energy of the system to be

$$\begin{aligned}
T &= T_1 + T_2 \\
&= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\
&= \frac{1}{2}m_1\left(\mathbf{v}_1 \cdot \mathbf{v}_1\right) + \frac{1}{2}m_2\left(\mathbf{v}_2 \cdot \mathbf{v}_2\right) \\
&= \frac{1}{2M}\left((m_1 + m_2)m_1\left(\mathbf{v}_1 \cdot \mathbf{v}_1\right) + (m_1 + m_2)m_2\left(\mathbf{v}_2 \cdot \mathbf{v}_2\right)\right) \\
&= \frac{1}{2M}\left(m_2m_1\mathbf{v}_1 \cdot \mathbf{v}_1 + m_1m_2\mathbf{v}_2 \cdot \mathbf{v}_2 + m_1\mathbf{v}_1 \cdot m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \cdot m_2\mathbf{v}_2\right)
\end{aligned}$$

Using equation (7.73)

$$\begin{aligned}
&= \frac{1}{2M}\left(m_2m_1\mathbf{v}_1 \cdot \mathbf{v}_1 + m_1m_2\mathbf{v}_2 \cdot \mathbf{v}_2 - m_2\mathbf{v}_2 \cdot m_1\mathbf{v}_1 - m_1\mathbf{v}_1 \cdot m_2\mathbf{v}_2\right) \\
&= \frac{1}{2M}\left(m_2m_1\mathbf{v}_1 \cdot \mathbf{v}_1 + m_1m_2\mathbf{v}_2 \cdot \mathbf{v}_2 - m_2\mathbf{v}_2 \cdot m_1\mathbf{v}_1 - m_1\mathbf{v}_1 \cdot m_2\mathbf{v}_2\right) \\
&= \frac{1}{2}\frac{m_1m_2}{M}\left(\mathbf{v}_2 - \mathbf{v}_1\right) \cdot \left(\mathbf{v}_2 - \mathbf{v}_1\right)
\end{aligned}$$

finally giving

$$T = \frac{1}{2} m \|\dot{\mathbf{r}}\|^2 \tag{7.76}$$

where m is the reduced mass given in equation (7.67).

The potential energy of the system is the gravitational potential energy stored in the field, given by

$$V = -\frac{Gm_1m_2}{r} = -\frac{GMm}{r} = -\frac{\mu m}{r} \tag{7.77}$$

notably there is no need to "double count" for each body, as the energy is stored in the field, not in the bodies themselves, and only exists when there is attraction between the two bodies.

7.4.2 Conservation of Angular Momentum and Reduction to 2D

Using equation (7.75), we can see that the angular momentum of the system is conserved quite easily.

Using the fact that a vector cross product with itself is the zero vector, the rate of change of the angular momentum is

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times m\dot{\mathbf{r}}) \\
&= \cancelto{\mathbf{0}}{\dot{\mathbf{r}} \times m\dot{\mathbf{r}}} + \mathbf{r} \times m\ddot{\mathbf{r}} \\
&= \mathbf{r} \times \mathbf{F}(\mathbf{r}) \\
&= \left(-\frac{GMm}{r^3}\right) \cancelto{\mathbf{0}}{\mathbf{r} \times \mathbf{r}} \\
&= \mathbf{0}.
\end{aligned}$$

More importantly now, we can study the rates of change of angular momentum of each object. The angular momentums of the two objects can be related to each other using equations (7.72) and (7.73), with

$$\begin{aligned}
\mathbf{H}_1 &= \mathbf{r}_1 \times m_1\mathbf{v}_1 \\
&= \left(-\frac{m_2}{m_1}\mathbf{r}_2\right) \times m_1 \left(-\frac{m_2}{m_1}\mathbf{v}_2\right) \\
&= \frac{m_2}{m_1}\mathbf{r}_2 \times m_2\mathbf{v}_2 \\
&= \frac{m_2}{m_1}\mathbf{r}_2\mathbf{H}_2
\end{aligned}$$

And since

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

it can be found that

$$\mathbf{H} = \left(1 + \frac{m_1}{m_2}\right)\mathbf{H}_1 = \left(1 + \frac{m_2}{m_1}\right)\mathbf{H}_2 \tag{7.78}$$

which, since $\dot{\mathbf{H}} = \mathbf{0}$,

$$\dot{\mathbf{H}} = \mathbf{0} = \left(1 + \frac{m_1}{m_2}\right)\dot{\mathbf{H}}_1 = \left(1 + \frac{m_2}{m_1}\right)\dot{\mathbf{H}}_2$$

means that the angular momentum of each body is conserved as well.

The angular momentum describes the normal of the orbital plane. Since the angular momentums of the two bodies are related by a scalar factor, their angular momentums share the same direction and their orbital planes are the same. This reduces the dimensionality of the problem to two dimensions.

7.4.3 Solving the Equations of Motion

The pieces put together in §7.4.1 and §7.4.2 may at first seem random, however they allow us to produce the solution quite quickly.

It is now convenient (and common practice) to define that the relative position is written in polar coordinates. We also assume, without loss of generality, that the orbits occur in the xy-plane, which can be done due to the reduction of dimensionality found in §7.4.2.

Letting

$$\mathbf{r} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

we can write the velocity as

$$\dot{\mathbf{r}} = \begin{bmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{bmatrix}$$

and notably we have that

$$\|\dot{\mathbf{r}}\|^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{7.79}$$

and the angular momentum is

$$\mathbf{H} = \mathbf{r} \times m\dot{\mathbf{r}} = mr^2\dot{\theta}\hat{\mathbf{z}} \tag{7.80}$$

where m is the reduced mass given in equation (7.67) and $\hat{\mathbf{z}}$ is the unit vector in the z-direction.

Using Lagrangian mechanics makes solving for the equations of motion much easier. The Lagrangian is given by

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu m}{r} \tag{7.81}$$

By reducing the equations of motion to centre of mass frame we have reduced the number of variables to solve for from 6 (both objects' positions in 3D) to 2 (r and θ).

Applying the Euler-Lagrange equations gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{r}}\right) - \frac{\mathrm{d}L}{\mathrm{d}r} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}}\right) - \frac{\mathrm{d}L}{\mathrm{d}\theta} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\mu m}{r^2} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) = 0$$

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\mu m}{r^2} = 0 \qquad mr^2\dot{\theta} = \text{constant} = H.$$

The constant of motion H allows us to remove $\dot{\theta}$,

$$\dot{\theta} = \frac{H}{mr^2}.$$

Substituting this into the radial equation gives

$$m\ddot{r} = \frac{H^2}{mr^3} - \frac{\mu m}{r^2}. \tag{7.82}$$

Let's now make the substitution $r = 1/u$, so that

$$\begin{aligned}\ddot{r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) \\ &= \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) \\ &= \frac{H}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{H}{mr^2}\right) \\ &= \frac{H}{m}u^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{H}{m}u^2\right) \\ &= -\frac{H^2}{m^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}.\end{aligned}$$

which simplifies equation (7.14) to

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = \frac{\mu m^2}{H^2} - u. \tag{7.83}$$

We now simplify this by defining the specific angular momentum as

$$\mathbf{h} = \frac{1}{m}\mathbf{H} \tag{7.84}$$

such that

$$h = \frac{H}{m}. \tag{7.85}$$

This gives the final form

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\mu}{h^2}. \tag{7.86}$$

This is an inhomogeneous second order linear ordinary differential equation, with a constant inhomogeneous term. The solution to this is given by

$$u(\theta) = A\cos(\theta - \theta_0) + \frac{\mu}{h^2} \tag{7.87}$$

for some constants A and θ_0. If we define that the periapsis, the point of closest approach, is at $\theta = 0$, then $\theta = 0$ should maximise $u = 1/r$, which means that $\theta_0 = 0$.

We now have a formula for $r(\theta)$, given by

$$r(\theta) = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}. \tag{7.88}$$

where $e = \frac{h^2}{\mu}A$ is the eccentricity of the orbit. The eccentricity is a measure of how elliptical the orbit is, with $e = 0$ being a circle, $0 < e < 1$ being an ellipse, $e = 1$ being a parabola, and $e > 1$ being a hyperbola. We call θ the true anomaly, which is the angle from the periapsis of the orbit by definition.

This combined with

$$h = \|\mathbf{r} \times \mathbf{v}\| = \text{constant} \tag{7.89}$$

and

$$\dot{\theta} = \frac{h}{r^2} \tag{7.90}$$

gives the time dependent equations of motion of the orbits.

To describe the positions of both bodies in their orbital plane one calculates $\mathbf{r}$ and then, setting the origin as the centre of mass, the positions can be given by

$$\mathbf{r} = \left(1 + \frac{m_2}{m_1}\right)\mathbf{r}_2 = -\left(1 + \frac{m_1}{m_2}\right)\mathbf{r}_1. \tag{7.91}$$

By the definition of this coordinate system this means that when both bodies are at their periapsis body 2 lies on the positive x-axis.

Therefore both objects orbit about their centre of mass, either in closed circles or ellipses, or escape in parabolic or hyperbolic trajectories.

7.4.3.1 Reduction to Keplerian Orbits

The equations of motion derived above can be reduced to the equations for Keplerian orbits quite quickly.

In Keplerian orbits the first object is very large i.e. $m_1 \gg m_2$. This means that the total mass is

$$M = m_1 + m_2 \approx m_1$$

and the reduced mass is

$$m = \frac{m_1 m_2}{m_1 + m_2} \approx \frac{m_1 m_2}{m_1} = m_2.$$

The gravitational parameter $\mu = GM$ becomes the specific gravitational parameter, specific to the central body.

Since the first body is very large, the centre of mass is approximately the position of the first body, so the second body orbits the first body, which is stationary at the origin.

Chapter 8: Orbital Transfers

Orbital transfers are perhaps the most complex topics in space from a technical perspective. The easiest kind of transfer to understand are impulsive manoeuvres, and even those can be quite complex. Continuous thrust transfers require a mathematics and control theory background to fully grasp the underlying concepts.

The efficiency of a transfer is measured by how much fuel it uses. The amount of fuel used directly corresponds to the change in velocity, $\Delta\mathbf{v}$, of the object. The total amount of fuel used is the sum of the $\Delta\mathbf{v}$ for each manoeuvre, so

$$\Delta v_{\text{total}} = \sum_{i=1}^{n} \|\Delta\mathbf{v}_i\|, \tag{8.1}$$

for impulsive burns, or

$$\Delta v_{\text{total}} = \int_{t_0}^{t_f} \|\mathbf{a}_{\text{thrust}}\| \, \mathrm{d}t \tag{8.2}$$

for continuous burns.

The efficiency of a transfer is often measured in terms of the total Δv required to complete the transfer.

8.1 Impulsive Manoeuvres

An impulsive manoeuvre is one where a change in velocity, a $\Delta\mathbf{v}$, is applied instantaneously. Obviously practically this isn't possible, but on the time scales that most space missions operate on, this is often a good-enough approximation for mission plans.

8.1.1 Hohmann Transfers

A Hohmann transfer moves an object from an initial orbit A to a final orbit B, so long as both orbits are co-planar and have the same apse line. In this case a Hohmann transfer is the most efficient way to move between two orbits.

There are two kinds of Hohmann transfers, one from the initial periapsis, one from the apoapsis. Typically orbits are aligned, with both periapses on the same side of the orbit, as is illustrated in figure 8.1. There are other cases where the periapses are on opposite sides.

Let us assume we know

- The initial orbit periapsis and apoapsis distances $r_{p,i}$ and $r_{a,i}$,
- The final orbit periapsis and apoapsis distances $r_{p,f}$ and $r_{a,f}$,
- Whether the periapses are on the same side or not,
- Whether the transfer should start from the periapsis or apoapsis of the initial orbit.

A Hohmann transfer always covers a 180° ellipsoidal arc, so if the transfer is from periapsis to periapsis or apoapsis to apoapsis that means that the periapsis and apoapsis of the final orbit are on the opposite side of the initial orbit.

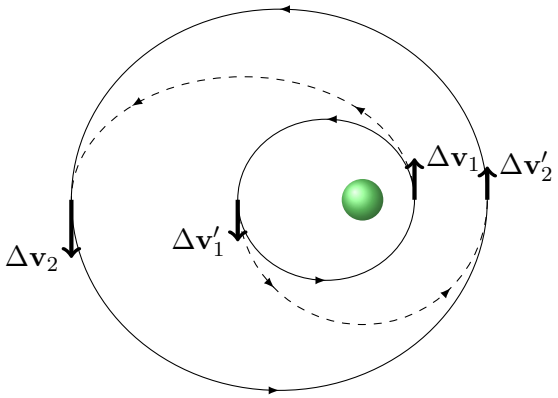
Let's denote the periapsis and apoapsis distances of the transfer orbit as $r_{p,t}$ and $r_{a,t}$. We can represent what these distances of the transfer orbit are from where we want to start the transfer and what

orientation the final orbit has. These are given in table 8.1.

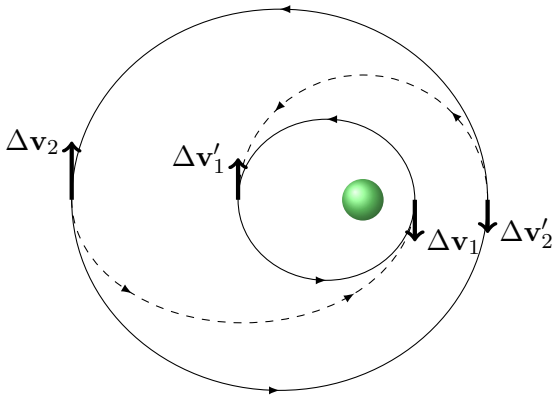
Table 8.1: Hohmann Transfer Orbit Parameters

		To	
		Periapsis	Apoapsis
From	Periapsis	$r_{p,t} = r_{p,i}$ $r_{a,t} = r_{p,f}$	$r_{p,t} = r_{p,i}$ $r_{a,t} = r_{a,f}$
	Apoapsis	$r_{p,t} = r_{a,i}$ $r_{p,t} = r_{a,f}$	$r_{p,t} = r_{a,i}$ $r_{a,t} = r_{a,f}$

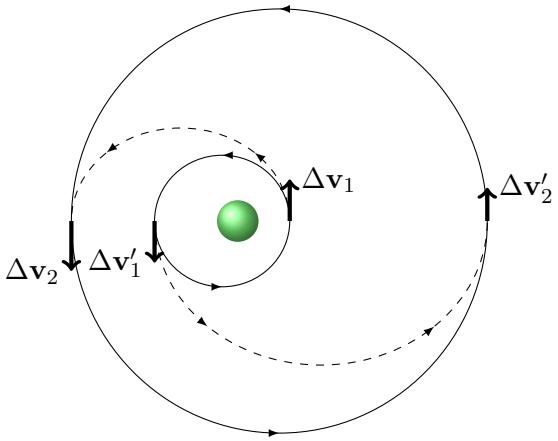
One can select the appropriate parameters by checking the shape of the orbits. A representation of the different combinations of transfers is shown in figure 8.1.



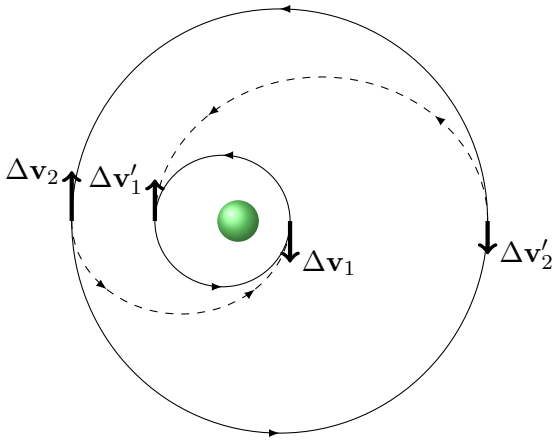
(a) Transfers to a larger orbit where the two periapses are near each other.



(b) Transfers to a smaller orbit where the two periapses are near each other.



(c) Transfers to a larger orbit where the two periapses are opposite each other.



(d) Transfers to a smaller orbit where the two periapses are opposite each other.

Figure 8.1: Hohmann transfer examples.

The Δv for each manoeuvre is easy to calculate since at the periapsis and apoapsis velocity is orthogonal to the position, so one only needs to increase or decrease the current magnitude of the velocity, not change its' direction.

As a result we can simplify the specific angular momentum to

$$\begin{aligned}\|\mathbf{h}\| &= \|\mathbf{r} \times \mathbf{v}\| \\ &= \|\mathbf{r}\| \, \|\mathbf{v}\| \sin\theta \\ &= \|\mathbf{r}\| \, \|\mathbf{v}\|\end{aligned}$$

because they are orthogonal at peri/apoapsis so $\theta = 90°$. This simplifies the specific angular momentum to just

$$h = rv.$$

We know that the specific angular momentum for any of the orbits can be written as

$$h = \sqrt{2\mu \left(\frac{r_p r_a}{r_p + r_a} \right)},$$

allowing the Hohmann transfer Δv to be calculated as

$$\Delta v = \frac{\Delta h}{r}. \tag{8.3}$$

8.1.2 Plane Change Manoeuvres

A plane change manoeuvre changes the orientation of the plane of the orbit. The most important fact to remember:

> The point at which you perform the plane change manoeuvre must be a point that intersects the current orbital plane and the desired orbital plane.

In the simplest case this is an inclination change, where the points where a plane change can be performed are the ascending and descending nodes.

The two orbital planes are defined by their normal vectors, which are their specific angular momentum direction vectors $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$. The vector that plane 1 needs to be rotated about to match plane 2 is orthogonal to both $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$, which is conveniently given by the cross product

$$\boldsymbol{\epsilon} = \hat{\mathbf{h}}_1 \times \hat{\mathbf{h}}_2. \tag{8.4}$$

The angle to be rotated by, ϕ, is also in this formula

$$\phi = \sin^{-1} \|\boldsymbol{\epsilon}\| \tag{8.5}$$

This also means that there is a rotation matrix that rotates coordinates from plane 1 to plane 2, which is given by the Rodrigues' rotation formula (equation (2.14)). Finding the unit vector of the rotation axis, $\hat{\boldsymbol{\epsilon}}$, gives the rotation matrix by Rodrigues' formula

$$\mathbf{R}_{1\to 2} = \mathbf{I} + \sin\phi \left[\hat{\boldsymbol{\epsilon}}\right]_\times + (1 - \cos\phi) \left[\hat{\boldsymbol{\epsilon}}\right]_\times^2$$

where $\left[\hat{\boldsymbol{\epsilon}}\right]_\times$ is the cross product matrix of $\hat{\boldsymbol{\epsilon}}$.

A desired plane change manoeuvre is only possible if the line formed by the rotation axis is contained within both orbital planes. Mathematically this means that attempting to map the eigenvector $\hat{\boldsymbol{\epsilon}}$ back into either perifocal frames results in a zero w component i.e.

$$\begin{aligned}\left(\mathbf{R}^{\mathsf{T}}_{pqw1\leftarrow \mathrm{XCI}} \hat{\boldsymbol{\epsilon}}\right) \cdot \hat{\mathbf{w}}_1 = 0\,, \\ \left(\mathbf{R}^{\mathsf{T}}_{pqw2\leftarrow \mathrm{XCI}} \hat{\boldsymbol{\epsilon}}\right) \cdot \hat{\mathbf{w}}_2 = 0\,.\end{aligned} \tag{8.6}$$

If the condition in equation (8.6) is true, then there are two valid locations in orbit 1, to plane change to an equivalent orbit in plane 2. The true anomalies of these locations θ_+ and θ_- can be found by first transforming $\pm\hat{\boldsymbol{\epsilon}}$ to the perifocal frame of 1 and then finding their true anomalies.

$$\pm(\hat{\boldsymbol{\xi}}_1)_{pqw1} = \pm\mathbf{R}^{\mathsf{T}}_{pqw1\leftarrow\mathrm{XCI}}\hat{\boldsymbol{\epsilon}} \tag{8.7}$$

$$\theta_{1,\pm} = \arctan2\left(\pm\xi_{1,q},\, \pm\xi_{1,p}\right) \tag{8.8}$$

Similarly, the true anomaly of the plane change location in orbit 2 is found by transforming $\pm\hat{\boldsymbol{\epsilon}}$ to the perifocal frame of 2 and then finding their true anomalies. Transforming is done with

$$\pm(\hat{\boldsymbol{\xi}}_2)_{pqw2} = \pm\mathbf{R}^{\mathsf{T}}_{pqw2\leftarrow\mathrm{XCI}}\hat{\boldsymbol{\epsilon}} \tag{8.9}$$

and then the true anomalies are found as per equation (8.8).

From the true anomaly and the other known orbital parameters, the locations and velocities at these points in the perifocal frame and ECI frame can be found, where

$$\begin{aligned} r_{1/2,\pm} &= \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_{1/2,\pm}}\,, \\ \left(\mathbf{r}_{1/2,\pm}\right)_{pqw} &= \begin{bmatrix} r_{1/2,\pm}\cos\theta_{1/2,\pm} \\ r_{1/2,\pm}\sin\theta_{1/2,\pm} \\ 0 \end{bmatrix}\,, \\ (\mathbf{r}_\pm)_{\mathrm{XCI}} &= \mathbf{R}_{\mathrm{XCI}\leftarrow pqw1}\left(\mathbf{r}_{1,\pm}\right)_{pqw1} = \mathbf{R}_{\mathrm{XCI}\leftarrow pqw2}\left(\mathbf{r}_{2,\pm}\right)_{pqw2}\,. \end{aligned} \tag{8.10}$$

The $\Delta\mathbf{v}$ associated with the plane change manoeuvre, such that the orbit shape is the same, is given by

$$\Delta\mathbf{v} = \mathbf{v}_{\text{after}} - \mathbf{v}_{\text{before}} = \left(\mathbf{R}_{1\to2} - \mathbf{I}\right)\mathbf{v}_{\text{before}}. \tag{8.11}$$

8.1.3 An Algorithm for Hohmann-like Transfers Between Two Points

Disclaimer: This algorithm reduces to a Hohmann transfer if the two points are in the plane as the starting orbit and along the apse line of the starting orbit. However, I don't have a proof that this is the most efficient transfer method.

Let us take two points in space in an inertial frame, $\mathbf{r}_1$ and $\mathbf{r}_2$, orbiting an object at the origin. There exists some elliptical path between the two, such that the object starts at $\mathbf{r}_1$, with initial velocity $\mathbf{v}_0$, and ends at $\mathbf{r}_2$.

The plane in which the orbit occurs is the plane containing the origin, $\mathbf{r}_1$ and $\mathbf{r}_2$. As such, its normal is

$$\hat{\mathbf{N}} = \frac{\mathbf{r}_1\times\mathbf{r}_2}{\|\mathbf{r}_1\times\mathbf{r}_2\|}\,. \tag{8.12}$$

Ideally we preserve the motion direction of the orbit in the plane, with the velocity in the plane given by

$$\mathbf{v}_{0\,\text{plane}} = \mathbf{v}_0 - \left(\mathbf{v}_0\cdot\hat{\mathbf{N}}\right)\hat{\mathbf{N}}\,. \tag{8.13}$$

The specific angular momentum of the system in the plane of desired motion is therefore

$$\mathbf{h}_0 = \mathbf{r}_1\times\mathbf{v}_{0\,\text{plane}}\,. \tag{8.14}$$

In the case where $\hat{\mathbf{N}} = \mathbf{0}$ (i.e. the transfer should be a perfect Hohmann transfer) we select the normal to the plane as simply

$$\hat{\mathbf{N}} = \hat{\mathbf{h}}_0. \tag{8.15}$$

In the alternate case where $\mathbf{h}_0 = \mathbf{0}$, it can be reassigned the plane normal direction

$$\hat{\mathbf{h}}_0 = \hat{\mathbf{N}}. \tag{8.16}$$

If $\hat{\mathbf{N}} = \hat{\mathbf{h}} = \mathbf{0}$ then the problem has infinitely many solutions and a different trajectory planning method that is more constrained is required. To be more specific, if neither vector is defined then the initial orbit is a straight line into/away from the origin and the target point is on that line, so you can choose any direction for the orbit.

We assign the apogee of the orbit to be the location with the greatest distance from the origin.

$$\mathbf{r}_a = \begin{cases} \mathbf{r}_1 & \text{if} \quad r_1 \geq r_2 \\ \mathbf{r}_2 & \text{if} \quad r_1 < r_2 \end{cases} \tag{8.17}$$

The other vector becomes $\mathbf{r}_o$.

$$\mathbf{r}_o = \begin{cases} \mathbf{r}_2 & \text{if} \quad r_1 \geq r_2 \\ \mathbf{r}_1 & \text{if} \quad r_1 < r_2 \end{cases} \tag{8.18}$$

The eccentricity direction vector of the system is in the opposite direction to the apogee vector i.e.

$$\hat{\mathbf{e}} = -\hat{\mathbf{r}}_a \,. \tag{8.19}$$

Using the specific angular momentum and the eccentricity vector, we can now find the perifocal to XCI matrix, where vectors form the columns of the matrix.

$$\mathbf{R}_{\text{XCI}\leftarrow pqw} = \begin{bmatrix} \hat{\mathbf{e}} & \hat{\mathbf{h}}_0 \times \hat{\mathbf{e}} & \hat{\mathbf{h}}_0 \end{bmatrix} \tag{8.20}$$

The angle traversed going from $\mathbf{r}_1$ to $\mathbf{r}_2$ can be computed with

$$\begin{aligned} \cos\theta_{12} &= \hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 \,, \\ \sin\theta_{12} &= (\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2) \cdot \hat{\mathbf{h}}_0 \,, \\ \theta_{12} &= \operatorname{arctan2}\left(\sin\theta_{12},\, \cos\theta_{12}\right) \,, \end{aligned} \tag{8.21}$$

which now yields the eccentricity for the new orbit

$$e = \frac{r_a - r_o}{r_a - r_o \cos\theta_{12}} \,. \tag{8.22}$$

The start and end true anomalies are conditionally given by

$$\begin{aligned} \theta_1 &= \begin{cases} \pi & \text{if} \quad r_1 \geq r_2 \\ \pi - \theta_{12} & \text{if} \quad r_1 < r_2 \end{cases} \\ \theta_2 &= \begin{cases} \theta_{12} - \pi & \text{if} \quad r_1 \geq r_2 \\ \pi & \text{if} \quad r_1 < r_2 \end{cases} \end{aligned} \tag{8.23}$$

with the remaining orbital elements required to propagate the orbit given by

$$\begin{aligned} a &= r_a(1+e) \,, \\ h &= \sqrt{a\mu(1-e^2)} \,, \\ n &= \sqrt{\frac{\mu}{a^3}} \,. \end{aligned} \tag{8.24}$$

Chapter 9: Frames of Reference & General Coordinate Transforms

9.1 What is a Frame of Reference?

A frame of reference, or reference frame, is a coordinate system that varies with time.
Not only does it have an origin and axes directions, it also can move and rotate. A reference frame that isn't rotating and is moving at a constant velocity (or just stationary) is called an *inertial reference frame*. An inertial reference frame is one in which Newton's laws of motion are valid, what is also particularly interesting about these things is that you can't actually tell when you're inside an inertial reference frame if you're moving or if everything is just moving around you.

The moment you start accelerating or rotating almost everything falls apart.

9.2 How do we convert between frames of reference?

Often frames of reference are attached to some object, and we define the axes and origin relative to that object's orientation and position respectively.

As a result of this it is quite common that we would like to convert measurements of position, velocity etc. taken in one reference frame and convert them into another reference frame. An example of this is to convert a measurement of the position of a planet's position, velocity, orientation and angular velocity, as measured in the observer's reference frame on Earth, to the ecliptic (solar system) reference frame.

To do this conversion between frames we perform coordinate transforms.

9.2.1 Coordinate Transforms

Let's consider a generic case, let frame A be our initial frame and frame B be the frame we would like to convert our measurements into.

As an example let measurements in frame A be in spherical polar coordinates $(r(t),\, \theta(t),\, \phi(t))$ – as per § 1.4.4. We would like to convert to cartesian coordinates $(x,\, y,\, z)$ in frame B, which is moving at a constant velocity away from A and rotating at a constant angular velocity relative to A. In this example frame B is moving at a constant velocity relative to A, but it is also rotating, which is a form of acceleration, so we must convert positions and velocities from A into positions, velocities and accelerations in B. In general it is easier to state that we require a function that takes position, velocity and acceleration measurements in A and outputs the same in frame B, i.e.

$$(x,\, y,\, z,\, \dot{x},\, \dot{y},\, \dot{z},\, \ddot{x},\, \ddot{y},\, \ddot{z}) = f(r,\, \theta,\, \phi,\, \dot{r},\, \dot{\theta},\, \dot{\phi},\, \ddot{r},\, \ddot{\theta},\, \ddot{\phi}). \tag{9.1}$$

A transform that just converts coordinates, e.g.

$$(x,\, y,\, z) = f(r,\, \theta,\, \phi). \tag{9.2}$$

is called a point transform. If two frames are not rotating relative to each other, just at a constant different orientation, then it is sometimes possible to compute a coordinate transform of velocities and accelerations with just a point transform.

As a concrete example consider the transformation from spherical coordinates to cartesian coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

which is a point transform of the form in equation (9.2).

9.3 Linear Coordinate Transforms Between Frames of Reference

Here we will specifically discuss linear coordinate systems that have orthonormal basis vectors. The details on these can be found in § 1.3.1, though the main reason for this is that this is by far the most common form of coordinate system.

To shift coordinate frames we need to apply a shift to the origin and a rotation. These operations are instantaneous and therefore can either be constant with time or change with time. Let there be one frame of reference A described by an origin O and basis XYZ. Sometimes it is more convenient to simply denote this reference frame $OXYZ$. Inside that reference frame A let there be another reference frame B or $oxyz$. To describe this frame B from within A we need to know the position of o within A, $(\mathbf{r}_B)_A$. We also need to know the basis vectors of B written in terms of the basis vectors of A, which we would write as

$$\hat{\mathbf{x}} = x_1\hat{\mathbf{X}} + x_2\hat{\mathbf{Y}} + x_3\hat{\mathbf{Z}}, = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)_A \tag{9.3}$$

and similar for the other basis vectors $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. We obviously require that these vectors are unit length and dimensionless and in most cases orthonormal, meaning each pair of the vectors is also orthogonal.

We wrap a column vector in parentheses and subscript it with the frame of reference to denote that it is a vector in that frame. This is because if we were to write these vectors in a different frame of reference the vectors would be the same vectors but their components would be different in the new frame of reference.

We can see an example of this graphically in § 9.3.

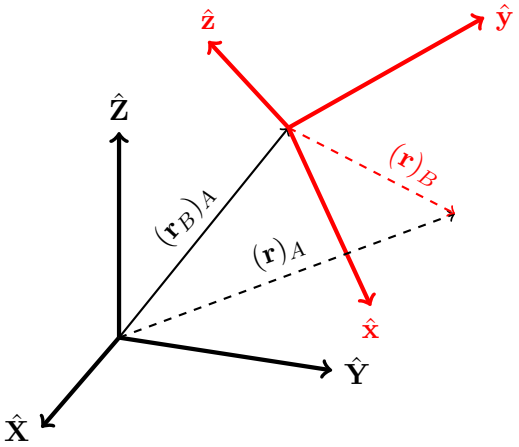


Figure 9.1: A general diagram of a reference frame B (red) within another reference frame A (black). Both reference frames measure coordinates of the same point differently.

9.3.1 Coordinate Transform Rotation Matrix

We can take equation (9.3) and write a rotation matrix in block form as

$$\mathbf{R}_{A\leftarrow B} = \begin{bmatrix} (\hat{\mathbf{x}})_A & (\hat{\mathbf{y}})_A & (\hat{\mathbf{z}})_A \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \tag{9.4}$$

where the columns of $\mathbf{R}_{B\leftarrow A}$ are the basis vectors of B written in terms of the basis vectors of A. While it may not be obvious it is true that this matrix takes any vector written in terms of the basis vectors of B and writes it in terms of the basis vectors of A. Take as an example the multiplication $\mathbf{R}_{A\leftarrow B}(\hat{\mathbf{x}})_B$.

$$\mathbf{R}_{A\leftarrow B}(\hat{\mathbf{x}})_B = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (\hat{\mathbf{x}})_A \tag{9.5}$$

The more general proof of this is complex and can be found in § 2.6. This matrix is called the direction cosine matrix.

From chapter 2 we know that the inverse of a rotation matrix is its transpose. Therefore the inverse of $\mathbf{R}_{A\leftarrow B}$ is

$$\mathbf{R}_{B\leftarrow A} = \mathbf{R}_{A\leftarrow B}^{\mathsf{T}} = \begin{bmatrix} (\hat{\mathbf{x}})_A^{\mathsf{T}} \\ (\hat{\mathbf{y}})_A^{\mathsf{T}} \\ (\hat{\mathbf{z}})_A^{\mathsf{T}} \end{bmatrix} \tag{9.6}$$

and this matrix does the exact same thing but in reverse.

It is worth noting here that it is assumed that these matrices are rotation matrices. In the case where one of the bases is not orthonormal (which is very rare) then the mathematics above holds except the matrices are not rotation matrices.

9.3.1.1 Time Derivatives of the Rotation Matrix Transform

Imagine the scenario where frame B is rotating relative to A. This rotation can be described by an angular velocity vector $\boldsymbol{\Omega}_B$ such that the rate of change of the basis vectors is given by the cross product

$$\frac{\mathrm{d}(\hat{\mathbf{x}})_A}{\mathrm{d}t} = (\boldsymbol{\Omega}_B)_A \times (\hat{\mathbf{x}})_A \tag{9.7}$$

and similarly for the other basis vectors. What this means is that each basis vector rotates like a rigid body does, where the rotation of frame B is described by $\boldsymbol{\Omega}_B$, which is typically measured in frame A.

The rate of change of the rotation matrix $\mathbf{R}_{A\leftarrow B}$ is then given by

$$\dot{\mathbf{R}}_{A\leftarrow B} = \begin{bmatrix} \dfrac{\mathrm{d}(\hat{\mathbf{x}})_A}{\mathrm{d}t} & \dfrac{\mathrm{d}(\hat{\mathbf{y}})_A}{\mathrm{d}t} & \dfrac{\mathrm{d}(\hat{\mathbf{z}})_A}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\Omega}_B)_A \times (\hat{\mathbf{x}})_A & (\boldsymbol{\Omega}_B)_A \times (\hat{\mathbf{y}})_A & (\boldsymbol{\Omega}_B)_A \times (\hat{\mathbf{z}})_A \end{bmatrix}. \tag{9.8}$$

As discussed in § 2.2, a cross product can be written as

$$\mathbf{a} \times \mathbf{b} = \left[\mathbf{a}\right]_{\times} \mathbf{b} \tag{9.9}$$

where $[\mathbf{a}]_\times$ is the cross product matrix of $\mathbf{a}$, given by equation (2.5). Therefore equation (9.8) simplifies to

$$\dot{\mathbf{R}}_{A\leftarrow B} = \begin{bmatrix} [(\boldsymbol{\Omega}_B)_A]_\times(\hat{\mathbf{x}})_A & [(\boldsymbol{\Omega}_B)_A]_\times(\hat{\mathbf{y}})_A & [(\boldsymbol{\Omega}_B)_A]_\times(\hat{\mathbf{z}})_A \end{bmatrix} = [(\boldsymbol{\Omega}_B)_A]_\times \mathbf{R}_{A\leftarrow B}. \tag{9.10}$$

The second time derivative is then given by the product rule

$$\ddot{\mathbf{R}}_{A\leftarrow B} = \frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{R}}_{A\leftarrow B} = [(\dot{\boldsymbol{\Omega}}_B)_A]_\times \mathbf{R}_{A\leftarrow B} + [(\boldsymbol{\Omega}_B)_A]_\times^2 \mathbf{R}_{A\leftarrow B} \tag{9.11}$$

where $(\dot{\boldsymbol{\Omega}}_B)_A$ is the angular acceleration of frame B, as measured in frame A.

9.3.2 Shifting the Origin

To transform between frames we need to not only rotate our coordinates as per § 9.3.1 but also shift the origin.

As per § 9.3 we have a vector $(\mathbf{r}_B)_A$ which describes the position of the origin of frame B within frame A. To see how positions would be measured from the origin of B we simply subtract this vector

$$(\mathbf{r}_{\text{rel}})_A = (\mathbf{r})_A - (\mathbf{r}_B)_A. \tag{9.12}$$

9.3.3 A Full Position Coordinate Transform

Now we have all the pieces to describe a coordinate transform. Let there be some position measured in reference frame A called $(\mathbf{r})_A$. We want to know what this position is in reference frame B. First shifting the origin and then rotating the coordinates into the basis of B we find

$$(\mathbf{r})_B = \mathbf{R}_{B\leftarrow A}\Big((\mathbf{r})_A - (\mathbf{r}_B)_A\Big) = \mathbf{R}_{B\leftarrow A}(\mathbf{r})_A - \mathbf{R}_{B\leftarrow A}(\mathbf{r}_B)_A. \tag{9.13}$$

Usually when you perform a coordinate transform you normally measure the orientation of frame B relative to A, and measure the position, velocity and angular velocity of frame B from within A. The goal is then to transform a measurement $(\mathbf{r})_B$ from frame B, into the correct location in A (and with all its' associated derivatives). This means you know $(\mathbf{r})_B$, $(\mathbf{r}_B)_A$ and $\mathbf{R}_{A\leftarrow B}$ and their associated derivatives. Therefore we find the more useful form of equation (9.13) is

$$(\mathbf{r})_A = \mathbf{R}_{A\leftarrow B}(\mathbf{r})_B + (\mathbf{r}_B)_A. \tag{9.14}$$

9.3.4 Velocity Transforms

Now let's consider the case where the position and orientation of frame B within frame A is not constant. Mathematically we would write that we now have $(\mathbf{r})_A = (\mathbf{r})_A(t)$, $(\mathbf{r}_B)_A = (\mathbf{r}_B)_A(t)$ and $\mathbf{R}_{B\leftarrow A} = \mathbf{R}_{B\leftarrow A}(t)$.

Computing the velocity of some coordinate in frame A, as per equation (9.14), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r})_A = (\mathbf{v})_A = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{R}_{A\leftarrow B}(\mathbf{r})_B + (\mathbf{r}_B)_A\right)$$

and

$$(\mathbf{v})_A = \dot{\mathbf{R}}_{A\leftarrow B}(\mathbf{r})_B + \mathbf{R}_{A\leftarrow B}(\mathbf{v})_B + (\mathbf{v}_B)_A \tag{9.15}$$

Here we have simply used the chain rule to expand the derivative and denoted $\dot{\mathbf{r}} = \mathbf{v}$.

Using equation (9.10) this can be simplified to

$$\begin{aligned}(\mathbf{v})_A &= \left[(\boldsymbol{\Omega}_B)_A\right]_\times \mathbf{R}_{A\leftarrow B}(\mathbf{r})_B + \mathbf{R}_{A\leftarrow B}(\mathbf{v})_B + (\mathbf{v}_B)_A \\ &= (\boldsymbol{\Omega}_B)_A \times \Big(\mathbf{R}_{A\leftarrow B}(\mathbf{r})_B\Big) + \mathbf{R}_{A\leftarrow B}(\mathbf{v})_B + (\mathbf{v}_B)_A\end{aligned} \tag{9.16}$$

In the case where the angular velocity of frame B is measured in frame B instead of frame A, it can be transformed into frame A using the rotation matrix

$$(\boldsymbol{\Omega}_B)_A = \mathbf{R}_{A\leftarrow B}(\boldsymbol{\Omega}_B)_B \tag{9.17}$$

and given that

$$(\mathbf{Ra}) \times (\mathbf{Rb}) = \mathbf{R}(\mathbf{a} \times \mathbf{b}) \tag{9.18}$$

for any rotation matrix $\mathbf{R}$, we can change equation (9.16) to

$$\begin{aligned}(\mathbf{v})_A &= \mathbf{R}_{A\leftarrow B}\Big(\left[(\boldsymbol{\Omega}_B)_B\right]_\times (\mathbf{r})_B + (\mathbf{v})_B\Big) + (\mathbf{v}_B)_A \\ &= \mathbf{R}_{A\leftarrow B}\Big((\boldsymbol{\Omega}_B)_B \times (\mathbf{r})_B + (\mathbf{v})_B\Big) + (\mathbf{v}_B)_A\end{aligned} \tag{9.19}$$

Therefore, if we know the angular velocity of frame B relative to frame A, in either frame A or B, then we can compute the velocity of a point in frame A from the position and velocity of that point in frame B.

It follows for higher derivatives that we continue to apply the chain rule to get the rates of change such as acceleration and so on.

9.3.5 Acceleration Transforms

First taking the derivative of equation (9.15) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v})_A = (\mathbf{a})_A = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{\mathbf{R}}_{A\leftarrow B}(\mathbf{r})_B + \mathbf{R}_{A\leftarrow B}(\mathbf{v})_B + (\mathbf{v}_B)_A\Big)$$

which yields

$$(\mathbf{a})_A = \ddot{\mathbf{R}}_{A\leftarrow B}(\mathbf{r})_B + 2\dot{\mathbf{R}}_{A\leftarrow B}(\mathbf{v})_B + \mathbf{R}_{A\leftarrow B}(\mathbf{a})_B + (\mathbf{a}_B)_A. \tag{9.20}$$

For the acceleration transform we again simply take the derivative of the velocity transform from equations (9.16) and (9.19) and find

$$\begin{aligned}(\mathbf{a})_A = {} & \left[(\dot{\boldsymbol{\Omega}}_B)_A\right]_\times \mathbf{R}_{A\leftarrow B}(\mathbf{r})_B + \left[(\boldsymbol{\Omega}_B)_A\right]_\times^2 \mathbf{R}_{A\leftarrow B}(\mathbf{r})_B \\ & + 2\left[(\boldsymbol{\Omega}_B)_A\right]_\times \mathbf{R}_{A\leftarrow B}(\mathbf{v})_B + \mathbf{R}_{A\leftarrow B}(\mathbf{a})_B + (\mathbf{a}_B)_A \\ = {} & (\dot{\boldsymbol{\Omega}}_B)_A \times \Big(\mathbf{R}_{A\leftarrow B}(\mathbf{r})_B\Big) + (\boldsymbol{\Omega}_B)_A \times \Big((\boldsymbol{\Omega}_B)_A \times \Big(\mathbf{R}_{A\leftarrow B}(\mathbf{r})_B\Big)\Big) \\ & + 2(\boldsymbol{\Omega}_B)_A \times \Big(\mathbf{R}_{A\leftarrow B}(\mathbf{v})_B\Big) + \mathbf{R}_{A\leftarrow B}(\mathbf{a})_B + (\mathbf{a}_B)_A\end{aligned} \tag{9.21}$$

If you measure the angular velocity and angular acceleration of frame B from within frame B, then the transform from equation (9.17) yields a very nice result:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\Omega}_B)_A &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{R}_{A\leftarrow B}(\boldsymbol{\Omega}_B)_B\right)\\ &= (\boldsymbol{\Omega}_B)_A \times \left(\mathbf{R}_{A\leftarrow B}(\boldsymbol{\Omega}_B)_B\right) + \mathbf{R}_{A\leftarrow B}(\dot{\boldsymbol{\Omega}}_B)_B\\ &= \mathbf{R}_{A\leftarrow B}\Big(\cancelto{\mathbf{0}}{(\boldsymbol{\Omega}_B)_B \times (\boldsymbol{\Omega}_B)_B}\Big) + \mathbf{R}_{A\leftarrow B}(\dot{\boldsymbol{\Omega}}_B)_B\end{aligned}$$

giving

$$(\dot{\boldsymbol{\Omega}}_B)_A = \mathbf{R}_{A\leftarrow B}(\dot{\boldsymbol{\Omega}}_B)_B. \tag{9.22}$$

This is hopefully an obvious result, that the current orientation of the frames doesn't affect the magnitude of measured relative angular velocities or accelerations, just their orientations.

This then gives new forms of equation (9.21) as

$$\begin{aligned}(\mathbf{a})_A &= \mathbf{R}_{A\leftarrow B}\Big(\left[(\dot{\boldsymbol{\Omega}}_B)_B\right]_\times(\mathbf{r})_B + \left[(\boldsymbol{\Omega}_B)_B\right]_\times^2(\mathbf{r})_B + 2\left[(\boldsymbol{\Omega}_B)_B\right]_\times(\mathbf{v})_B + (\mathbf{a})_B\Big) + (\mathbf{a}_B)_A\\ &= \mathbf{R}_{A\leftarrow B}\Big((\dot{\boldsymbol{\Omega}}_B)_B \times (\mathbf{r})_B + (\boldsymbol{\Omega}_B)_B \times \left((\boldsymbol{\Omega}_B)_B \times (\mathbf{r})_B\right)\\ &\qquad + 2(\boldsymbol{\Omega}_B)_B \times (\mathbf{v})_B + (\mathbf{a})_B\Big) + (\mathbf{a}_B)_A\end{aligned} \tag{9.23}$$

9.3.6 A Bit More on Shifts in the Origin of a Coordinate System

While so far in this section we have discussed rotation operations on positions and velocities, sometimes we also need to shift the coordinate system.

There are two cases for this.

1. In one instance we want to shift from frame A to B and we know the position of the origin of B in reference frame A.
2. The other case is when we know the position of the origin of frame B but in the rotated frame B' (where we have rotated the axes of A but not yet shifted the origin).

This second case might seem a bit contrived however take the example where we want to shift from the ecliptic frame to the ECI coordinate frame, where we know the rotation easily as per equation (10.2) but the origin is only known in the ecliptic frame (since it doesn't really make sense to have the Earth's position in the Earth-centred frame).

Let's denote the origin or centre of B in A as $(\mathbf{r}_B)_A$ and its velocity $(\mathbf{v}_B)_A$. In the first case we have positions and velocities in B

$$\begin{aligned}\mathbf{r}_B &= \mathbf{R}_{B\leftarrow A}\left(\mathbf{r}_A - (\mathbf{r}_B)_A\right),\\ \mathbf{v}_B &= \dot{\mathbf{R}}_{B\leftarrow A}\left(\mathbf{r}_A - (\mathbf{r}_B)_A\right) + \mathbf{R}_{B\leftarrow A}\left(\mathbf{v}_A - (\mathbf{v}_B)_A\right).\end{aligned} \tag{9.24}$$

In the second case we have

$$\begin{aligned}\mathbf{r}_B &= (\mathbf{r}_A)_{B'} - (\mathbf{r}_B)_{B'} = \mathbf{R}_{B\leftarrow A}\mathbf{r}_A - (\mathbf{r}_B)_{B'}\\ \mathbf{v}_B &= \dot{\mathbf{R}}_{B\leftarrow A}\mathbf{r}_A + \mathbf{R}_{B\leftarrow A}\mathbf{v}_A - (\mathbf{v}_B)_{B'}.\end{aligned} \tag{9.25}$$

By comparing equations (9.24) and (9.25) we can see that in the second case it is simply as though we have "pre-rotated" our shift vector with

$$\begin{aligned}(\mathbf{r}_B)_{B'} &= \mathbf{R}_{B\leftarrow A}(\mathbf{r}_B)_A.\\ (\mathbf{v}_B)_{B'} &= \dot{\mathbf{R}}_{B\leftarrow A}(\mathbf{r}_B)_A + \mathbf{R}_{B\leftarrow A}(\mathbf{v}_B)_A.\end{aligned} \tag{9.26}$$

9.4 Dealing with Einstein

Einstein tells us that instead of just spatial coordinates (x, y, z) we should also consider time as a coordinate. We normally convert this to a 'distance' by multiplying by the speed of light c to give us coordinates (ct, x, y, z), which we can then treat as a 4-vector. This is called a *spacetime coordinate*.

While it is not my intention to go into the details of relativity, it is important to understand its implications.

9.4.1 Special Relativity Transforms

Special relativity deals with transforms between inertial frames of reference, i.e. frames that are not accelerating or rotating relative to each other. The most famous of these is the Lorentz transform, which describes how to convert between two inertial frames moving at a constant velocity relative to each other.

Standard notation for Special Relativity is to consider two frames S and S' with the same axes directions and with a difference in position only along their x axes. In this case S' is moving at a constant velocity $\mathbf{u}$ relative to S. These frames can be seen in figure 9.2.

Each frame also has its own time coordinate, t and t', which are the time measured in each frame. You can think of each frame as having 3 axes and a clock which ticks at a constat rate within that frame. We define that $t = t' = 0$ when the two frames are at the exact same position.

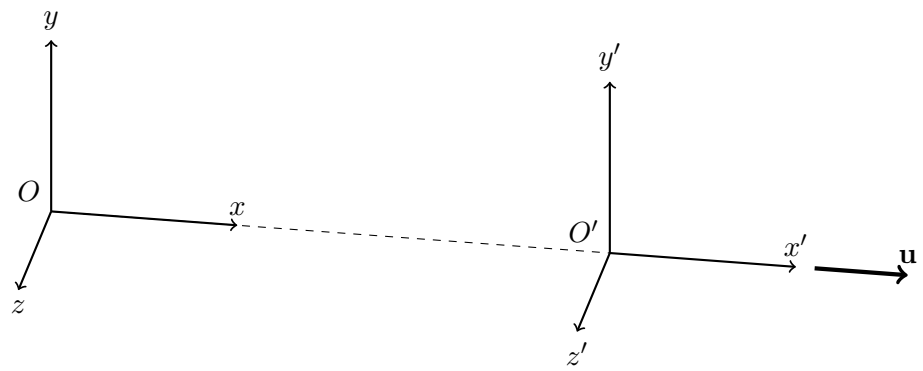


Figure 9.2: A diagram of two frames, S has axes x, y, z and origin O and S' has axes x', y', z' and origin O'. The axes of the frames are parallel and S' is moving at a constant velocity $\mathbf{u}$ relative to frame S in the x direction.

A natural question that you might ask is *but what if the origins don't align along the direction of motion?* The answer is that we don't need our origin to be the location of the observer, we can define the origin in a different location so that the coordinate system setup in figure 9.2 still holds.

The Lorentz transforms are given by

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{u}{c}x\right) & x' &= \gamma\left(x - \frac{u}{c}ct\right) \\ y' &= y & z' &= z \end{aligned} \tag{9.27}$$

where c is the speed of light and γ is the Lorentz factor given by

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{9.28}$$

9.4.2 Is Gravity is an Acceleration?

If you've been around physics communities long enough then you've probably heard that "gravity isn't an acceleration it's then warping of spacetime." This statement, to the best of our knowledge, is true. But why then isn't gravity an acceleration?

Veritasium has a great video on this topic, which you can view here[1]. To summarise, general relativity tells us that the motion of an object through space is given by the acceleration through space and the geodesic through spacetime (what we call gravity)

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = a^r \underbrace{-\Gamma^r_{tt}\left(u^t\right)^2}_{\text{gravity}} \tag{9.29}$$

and when those terms cancel you remain stationary. For instance on Earth when you stand still on the ground you feel a force through your feet because the ground is accelerating you upwards so that the acceleration exactly cancels the induced motion due to space-time curvature.

Therefore, technically, gravity is not an acceleration. But objects moving under gravity *look* like they are accelerating, to an external observer. So it's completely fine to numerically model gravity as an acceleration or a force, however this distinction is important, because we can consider an object orbiting due to gravity as being a local inertial reference frame.

Despite this, the special relativity coordinate transforms do not hold for frames under the influence of gravity, as the curvature of spacetime means changes lengths and times in ways not accounted for by special relativity.

[1] https://youtu.be/XRr1kaXKBsU?si=ZhnQ1gl-BTlLSV4n

Chapter 10: Useful Frames of Reference

10.1 Orbits

10.1.1 Perifocal Frame

The perifocal frame has its origin at the centre of the object causing the orbit (e.g. Earth), the x axis pointing towards and passing through the perigee, the z axis points out of the orbital plane, and the y axis points such that the xyz axes form a right-handed coordinate system. We normally label these axes pqw so as to avoid confusion with the Body-Centred Inertial frame axes.

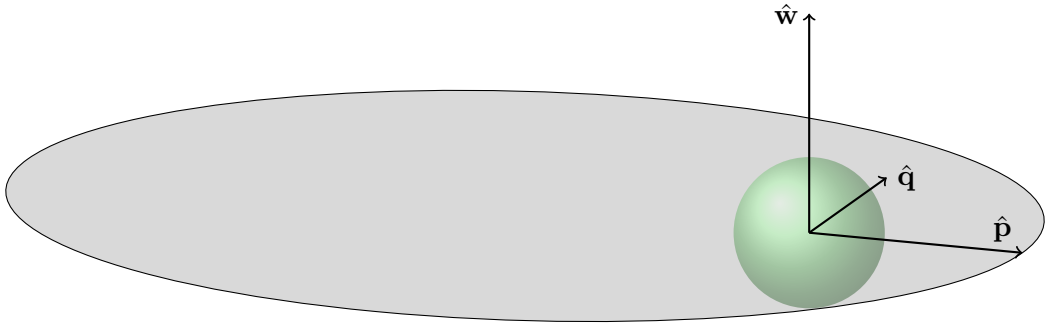


Figure 10.1: The Perifocal Frame. $\hat{\mathbf{p}}$ points towards the perigee, $\hat{\mathbf{q}}$ is 90° from $\hat{\mathbf{p}}$ in the direction of motion, and $\hat{\mathbf{w}}$ points out of the orbital plane. $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ lie in the orbital plane.

10.1.2 Body-Centred-Inertial

The Body-Centred-Inertial frame e.g. Earth-Centred-Inertial or ECI is a commonly used reference frame for orbits. In these frames of reference Keplerian orbits are still conic sections and elliptical orbits are still closed loops.

These coordinate frames are defined relative to the Earth's Body-Centred-Inertial frame at the J2000 epoch, which can be found in § 10.6.

Orbits within these frames are defined by three parameters

ω – The argument of periapsis

Ω – The right ascension of the ascending node

i – The inclination

The inclination is the angle of the orbit from the equatorial plane of the planet. The right ascension of the ascending node is the angle from the x-axis, in the xy-plane, of the point where the orbiting body ascends through the xy-plane so that its z coordinate goes from negative to positive. The argument of periapsis is the angle of the periapsis from the ascending node.

The can all be seen in figure 10.2. To rotate from perifocal to body-centred-inertial, one rotates by

$$\mathbf{R}_{\text{XCI}\leftarrow\text{Perifocal}} = \mathbf{R}_3(\Omega)\mathbf{R}_1(i)\mathbf{R}_3(\omega) \tag{10.1}$$

where X refers to an arbitrary body.

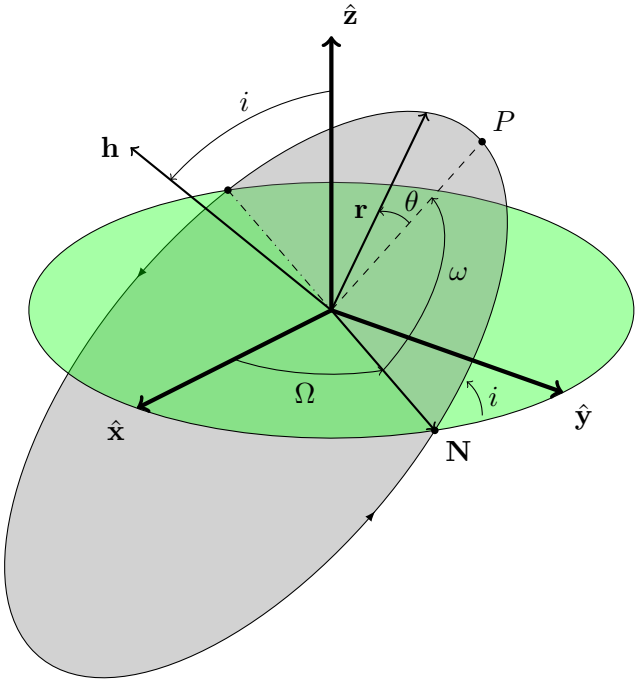


Figure 10.2: An orbit shown in a body-centred-inertial frame (XCI) with axes xyz. The equatorial plane is shown in green. **N** points to the ascending node, the intersection point of the orbit and the equatorial plane where it crosses from $-z$ to $+z$. The descending node is the point on the opposite side. Something important to note, for readability the ascending and descending nodes are shown to be at the same distance from the origin, however they could be at completely different distances.

10.2 The EME2000 or ECI J2000 Reference Frame

The Earth Mean Equator J2000 (EME2000) reference frame or Earth-Centred Inertial (ECI) reference frame (with axes determined from the J2000 Earth orientation) is a reference frame that is fixed to the Earth's equator with axes pointed towards the vernal equinox at the J2000 epoch.

We treat this frame of reference as inertial, however because it follows Earth in its' orbit there are some caveats to this, with the primary being that while Earth's frame is accelerating with respect to the sun, as per the discussion in § 9.4.2, the laws of physics will behave normally since the only reason its' accelerating is gravity.

Typically the axes of this coordinate system are denoted with capital letters i.e. $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$, $\hat{\mathbf{Z}}$.

10.3 The ICRS Reference System

The International Celestial Reference System (ICRS) is an inertial reference frame that is fixed to the stars. From the reference system then defined coordinate systems which have been updated over time, generally referred to as ICRF1, ICRF2 etc. This frame of reference is the best choice for tracking objects in space as it is the easiest to measure when observing distant stars. The ICRS is defined by the International Astronomical Union (IAU) and is based on the positions of 212 extragalactic sources.

The ICRF3 (which shall hereafter simply be referred to as ICRF) is defined so that its' axes almost exactly line up with the EME2000 reference frame, but its centre is at the centre of mass of the solar

system (the barycentre). However, because the precision with which stars' positions are known, it is possible to be off from the EME2000 reference frame by a few milliarcseconds, simply because the ICRF frame directions are known to greater precision than EME2000.

While typically the ICRS is just a set of axes with no origin, it is common to define the origin of the ICRS as the barycentre of the solar system. As a result the Sun moves in a small orbit in the ICRS too.

10.4 The Ecliptic Frame

The ecliptic frame is a frame of reference that is fixed to the time-averaged plane of the Earth's orbit around the sun.

It is the most common reference frame when plotting the solar system. We shall denote its axes $\hat{\boldsymbol{\mathcal{X}}}$, $\hat{\boldsymbol{\mathcal{Y}}}$, $\hat{\boldsymbol{\mathcal{Z}}}$.

One rotates to the ecliptic frame from the EME2000/ECI frame by rotating about the $\hat{\mathbf{X}}$ axis of the EME2000 frame by an angle of $\varepsilon = -23.439\,291°$. This angle is the inclination of Earth's spin axis from its orbital plane, it is called the obliquity of the ecliptic.

$$\mathbf{R}_{\text{ECL}\leftarrow\text{ECI}} = \mathbf{R}_1(\varepsilon) \tag{10.2}$$

The Ecliptic frame is also generally centred on the Sun, rather than the barycentre (centre of mass) of the solar system.

The true value of ϵ is the time varying angle based on the orientation of Earth's pole. For the time-varying conversion from the time-varying ECI frame to the Ecliptic frame one must evaluate the angle between the spin axis of Earth and the Ecliptic plane at the time of interest. It is easiest to do this via a conversion to the time-fixed EME2000 frame and then from EME2000 to the Ecliptic frame using a Rodrigues rotation.

10.5 The J2000 Mean Ecliptic Frame

The J2000 Mean Ecliptic frame is a frame of reference that is fixed to the time-averaged plane of the Earth's orbit around the sun at the J2000 epoch. This reference frame is the rotated version of the EME2000 reference frame by the obliquity of the ecliptic at the J2000 epoch. In other words we rotate the axes of the EME2000 reference frame by $\varepsilon_0 = -23.439\,291°$ with

$$\mathbf{R}_{\text{J2000ECL}\leftarrow\text{ICRF}} = \mathbf{R}_1(\varepsilon). \tag{10.3}$$

10.6 Body Fixed Inertial and Rotating Reference Frames

Body fixed reference frames rotate with the body they are fixed to. For instance, the International Terrestrial Reference Frame (ITRF) or Earth-Centred-Earth-Fixed (ECEF) reference frame is a body fixed reference frame that is fixed to the Earth. This frame of reference is useful for tracking objects on the a body's surface. An example of a situation where a transform must be performed is visually shown in figure 10.3.

As is described in figure 10.4, the body fixed reference frame is defined by the instantaneous direction of

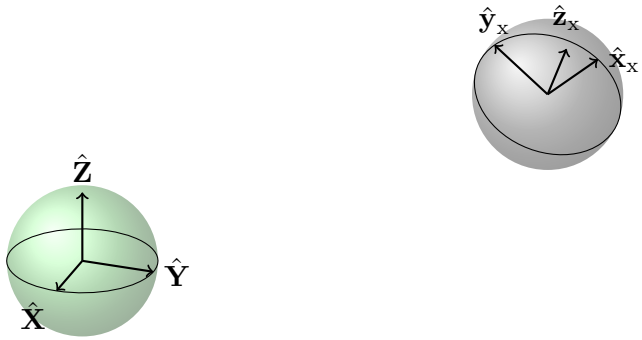


Figure 10.3: EME2000 Earth shown on the left, with the body fixed frame of reference shown attached to the body on the right. The transform described by figure 10.4 performs the transform from the EME2000 frame shown into the body fixed frame. Bodies are treated as rotating about their z axis.

the spin axis of the object. The intersection of the unit circles on the equatorial plane and the equatorial plane of the object is Q and the position of the prime meridian Eastward from that intersection is W. The right ascension and declination of the spin axis are α and δ respectively. We rotate from the EME2000 reference frame to the body fixed reference frame by rotating first the prime-meridian by an angle $90^\circ + W$, then rotating such that the polar direction is correct. Note that we choose the standard spherical coordinate rotation transform for this except using the angle from the $\hat{\mathbf{Z}}$ axis $\theta = 90^\circ - \delta$.

Definition 10.1: Body Centred to ICRF Rotation

$$\mathbf{R}_{\text{ICRF}\leftarrow\text{XCI/XCXF}} = \mathbf{R}_3(\alpha)\mathbf{R}_2(90^\circ - \delta)\mathbf{R}_3(90^\circ + W) \tag{10.4}$$

Definition 10.2: ICRF Rotation to Body Centred

$$\mathbf{R}_{\text{XCI/XCXF}\leftarrow\text{ICRF}} = \mathbf{R}_3^\mathsf{T}(90^\circ + W)\mathbf{R}_2^\mathsf{T}(90^\circ - \delta)\mathbf{R}_3^\mathsf{T}(\alpha) \tag{10.5}$$

One can derive this rotation by considering the unit vectors defined by figure 10.4. As per equation (9.4) finding the basis vectors directly leads to the ICRF to body fixed (XCXF) rotation matrix. Using spherical coordinates we find that the spin axis or $\hat{\mathbf{z}}_x$ is

$$\hat{\mathbf{z}}_x = \begin{bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{bmatrix} = \begin{bmatrix} \cos\alpha\sin(90^\circ - \delta) \\ \sin\alpha\sin(90^\circ - \delta) \\ \cos(90^\circ - \delta) \end{bmatrix} = \begin{bmatrix} \cos\alpha\sin\theta \\ \sin\alpha\sin\theta \\ \cos\theta \end{bmatrix}. \tag{10.6}$$

$$\hat{\mathbf{x}}_x = \begin{bmatrix} -\cos\alpha\sin\delta\sin W - \sin\alpha\cos W \\ -\sin\alpha\sin\delta\sin W + \cos\alpha\cos W \\ \cos\delta\sin W \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos(90^\circ - \delta)\cos(90^\circ + W) - \sin\alpha\sin(90^\circ + W) \\ \sin\alpha\cos(90^\circ - \delta)\cos(90^\circ + W) + \cos\alpha\sin(90^\circ + W) \\ -\sin(90^\circ - \delta)\cos(90^\circ + W) \end{bmatrix} \tag{10.7}$$

and

$$\hat{\mathbf{y}}_x = \hat{\mathbf{z}}_x \times \hat{\mathbf{x}}_x = \begin{bmatrix} -\cos(90^\circ + W)\sin\alpha - \sin(90^\circ + W)\cos\alpha\cos(90^\circ - \delta) \\ \cos\alpha\cos(90^\circ + W) - \sin(90^\circ + W)\sin\alpha\cos(90^\circ - \delta) \\ \sin(90^\circ + W)\sin(90^\circ - \delta) \end{bmatrix} \tag{10.8}$$

These are the basis vectors of the body fixed frame of reference, written in ICRF coordinates, i.e. $(\hat{\mathbf{x}}_{\mathrm{x}})_{\mathrm{ICRF}}$, $(\hat{\mathbf{y}}_{\mathrm{x}})_{\mathrm{ICRF}}$, $(\hat{\mathbf{z}}_{\mathrm{x}})_{\mathrm{ICRF}}$. From equation (9.4) we therefore know that

$$\mathbf{R}_{\mathrm{ICRF}\leftarrow\mathrm{XCI/XCXF}} = \begin{bmatrix}(\hat{\mathbf{x}}_{\mathrm{x}})_{\mathrm{ICRF}} & (\hat{\mathbf{y}}_{\mathrm{x}})_{\mathrm{ICRF}} & (\hat{\mathbf{z}}_{\mathrm{x}})_{\mathrm{ICRF}}\end{bmatrix} \tag{10.9}$$

which just so happens to be the same as equation (10.4).

Choosing the body fixed inertial frame requires choosing $(\alpha, \delta, W) = (\alpha_0, \delta_0, W_0)$ from the J2000 (or any other) epoch. This defines a body fixed inertial reference frame. Choosing instead the time varying values for (α, δ, W) produces a transform that rotates the EME2000 reference frame to/from the rotating body fixed reference frame. Values for both the instantaneous J2000 and time varying values for these angles for planets and moons are available in [6].

It is worth noting that GMAT's official documentation states that GMAT assumes that α and δ are constant and it also sets $W = 0$ for all inertial planet-fixed frames. It does however use the time-varying value of W for the rotating frames of reference.

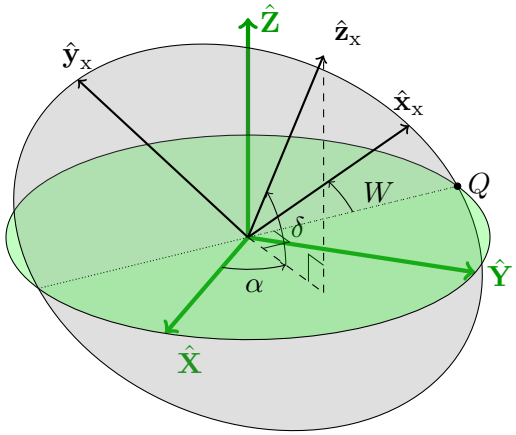


Figure 10.4: Definition of the body fixed reference frames, relative to the EME2000 reference frame. The EME2000 reference frame is shown and the Earth's equatorial plane in green. The equatorial plane of object x is in grey. The instantaneous direction of the spin axis of the object x is $\hat{\mathbf{z}}_{\mathrm{x}}$ and is described on the unit sphere using a right ascension α and declination δ in the EME2000 reference frame. The intersection of the unit circles on each of these planes is Q and the position of the prime meridian Eastward from that intersection is W.

10.6.1 EME2000/ICRF to & from Body Centred Body Fixed Transformation

To compute positions and velocities in rotating reference frames (X-Centred-X-Fixed) let's first consider the rotated position vector.

Result 10.1: ICRF to Body-Centred-Body-Fixed Frame Position Transformation

$$(\mathbf{r})_{\mathrm{XCXF}} = \mathbf{R}_{\mathrm{XCXF}\leftarrow\mathrm{ICRF}}\Big((\mathbf{r})_{\mathrm{ICRF}} - (\mathbf{r}_{\mathrm{X}})_{\mathrm{ICRF}}\Big) \tag{10.10}$$

where $(\mathbf{r})_{\mathrm{ICRF}}$ is the position vector in the ICRF to be transformed, and $(\mathbf{r}_{\mathrm{X}})_{\mathrm{ICRF}}$ is the position vector of the centre of mass of the target frame.

It is worth noting that the angles shown in figure 10.4 change with time, so this rotation matrix changes with time as well.

The velocity in the rotating frame can now be found.

Result 10.2: ICRF to Body-Centred-Body-Fixed Frame Velocity Transformation

$$(\mathbf{v})_{\text{XCXF}} = \dot{\mathbf{R}}_{\text{XCXF}\leftarrow\text{ICRF}}\Big((\mathbf{r})_{\text{ICRF}} - (\mathbf{r}_{\text{X}})_{\text{ICRF}}\Big) + \mathbf{R}_{\text{XCXF}\leftarrow\text{ICRF}}\Big((\mathbf{v})_{\text{ICRF}} - (\mathbf{v}_{\text{X}})_{\text{ICRF}}\Big) \tag{10.11}$$

where $(\mathbf{v})_{\text{ICRF}}$ is the velocity vector in the ICRF to be transformed, and $(\mathbf{v}_{\text{X}})_{\text{ICRF}}$ is the velocity vector of the centre of mass of the target frame.

The rotation matrix from XCXF to ICRF is

$$\mathbf{R}_{\text{ICRF}\leftarrow\text{XCXF}} = \mathbf{R}_3(\alpha)\mathbf{R}_2(90^\circ - \delta)\mathbf{R}_3(90^\circ + W) \tag{10.12}$$

giving the equations for the required rotation matrices of

$$\begin{aligned}\mathbf{R}_{\text{XCXF}\leftarrow\text{ICRF}} &= \mathbf{R}^{\mathsf{T}}_{\text{ICRF}\leftarrow\text{XCXF}} = \mathbf{R}_3^{\mathsf{T}}(90^\circ + W(t))\mathbf{R}_2^{\mathsf{T}}(90^\circ - \delta(t))\mathbf{R}_3^{\mathsf{T}}(\alpha(t)),\\ \dot{\mathbf{R}}_{\text{XCXF}\leftarrow\text{ICRF}} &= \dot{\mathbf{R}}^{\mathsf{T}}_{\text{ICRF}\leftarrow\text{XCXF}}\end{aligned} \tag{10.13}$$

(recall that $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$).

The rate of change of the rotation matrix from the body fixed reference frame to the EME2000 coordinate frame is found by using equation (2.18) and the chain rule, arriving at

$$\begin{aligned}\dot{\mathbf{R}}_{\text{ICRF}\leftarrow\text{XCXF}} = {}& \dot{\mathbf{R}}_3(\alpha)\mathbf{R}_2(90^\circ - \delta)\mathbf{R}_3(90^\circ + W)\\ &+ \mathbf{R}_3(\alpha)\dot{\mathbf{R}}_2(90^\circ - \delta)\mathbf{R}_3(90^\circ + W)\\ &+ \mathbf{R}_3(\alpha)\mathbf{R}_2(90^\circ - \delta)\dot{\mathbf{R}}_3(90^\circ + W).\end{aligned} \tag{10.14}$$

Result 10.3: Derivative of Body-Centred-Body-Fixed to ICRF Rotation

$$\begin{aligned}\dot{\mathbf{R}}_{\text{ICRF}\leftarrow\text{XCXF}} = {}& \dot{\alpha}\big[\hat{\mathbf{Z}}\big]_\times\mathbf{R}_3(\alpha)\mathbf{R}_2(90^\circ - \delta)\mathbf{R}_3(90^\circ + W)\\ &+ \mathbf{R}_3(\alpha)(-\dot{\delta})\big[\hat{\mathbf{Y}}\big]_\times\mathbf{R}_2(90^\circ - \delta)\mathbf{R}_3(90^\circ + W)\\ &+ \mathbf{R}_3(\alpha)\mathbf{R}_2(90^\circ - \delta)\dot{W}\big[\hat{\mathbf{Z}}\big]_\times\mathbf{R}_3(90^\circ + W).\end{aligned} \tag{10.15}$$

The get the inverse transform we assume that the position of X is still known in ICRF (which it normally is).

Result 10.4: Body-Centred-Body-Fixed to ICRF State Transformation

The inverse transform is found by rearranging equations (10.10) and (10.11) into the form

$$(\mathbf{r})_{\text{ICRF}} = \mathbf{R}_{\text{ICRF}\leftarrow\text{XCXF}}(\mathbf{r})_{\text{XCXF}} + (\mathbf{r}_{\text{X}})_{\text{ICRF}} \tag{10.16}$$

and

$$(\mathbf{v})_{\text{ICRF}} = \dot{\mathbf{R}}_{\text{ICRF}\leftarrow\text{XCXF}}(\mathbf{r})_{\text{XCXF}} + \mathbf{R}_{\text{ICRF}\leftarrow\text{XCXF}}(\mathbf{v})_{\text{XCXF}} + (\mathbf{v}_{\text{X}})_{\text{ICRF}}. \tag{10.17}$$

The inverse transform's associated matrices are

$$\begin{aligned}\mathbf{R}_{\text{ICRF}\leftarrow\text{XCXF}} &= \mathbf{R}^{\mathsf{T}}_{\text{XCXF}\leftarrow\text{ICRF}}\\ \dot{\mathbf{R}}_{\text{ICRF}\leftarrow\text{XCXF}} &= \dot{\mathbf{R}}^{\mathsf{T}}_{\text{XCXF}\leftarrow\text{ICRF}} = -\mathbf{R}^{\mathsf{T}}_{\text{XCXF}\leftarrow\text{ICRF}}\dot{\mathbf{R}}_{\text{XCXF}\leftarrow\text{ICRF}}\mathbf{R}^{\mathsf{T}}_{\text{XCXF}\leftarrow\text{ICRF}}\end{aligned} \tag{10.18}$$

which are what we expect from the generalised transforms in § 9.3.

This in fact holds in general for a position and velocity transform between two frames with a relative rotation and rate of change of rotation as per equation (2.25).

10.6.2 Body Centred Inertial Frames

To convert to the Body-Centred-Inertial (XCI) frames, we use the same transforms as equations (10.10), (10.11), (10.16) and (10.17) except we fix the angles (α, δ, W) to the J2000 epoch values (α_0, δ_0, W_0), which can be found in [6].

Definition 10.3: Body Centred Inertial to ICRF Rotation

$$\mathbf{R}_{\text{ICRF}\leftarrow\text{XCI}} = \mathbf{R}_3(\alpha_0)\mathbf{R}_2(90^\circ - \delta_0)\mathbf{R}_3(90^\circ + W_0) \tag{10.19}$$

Definition 10.4: ICRF Rotation to Body Centred Inertial

$$\mathbf{R}_{\text{XCI}\leftarrow\text{ICRF}} = \mathbf{R}_3^\mathsf{T}(90^\circ + W_0)\mathbf{R}_2^\mathsf{T}(90^\circ - \delta_0)\mathbf{R}_3^\mathsf{T}(\alpha_0) \tag{10.20}$$

Result 10.5: ICRF to Body-Centred-Inertial Frame State Transform

$$(\mathbf{r})_{\text{XCI}} = \mathbf{R}_{\text{XCI}\leftarrow\text{ICRF}}\Big((\mathbf{r})_{\text{ICRF}} - (\mathbf{r}_\text{X})_{\text{ICRF}}\Big) \tag{10.21}$$

$$(\mathbf{v})_{\text{XCI}} = \mathbf{R}_{\text{XCI}\leftarrow\text{ICRF}}\Big((\mathbf{v})_{\text{ICRF}} - (\mathbf{v}_\text{X})_{\text{ICRF}}\Big) \tag{10.22}$$

Result 10.6: Body-Centred-Inertial to ICRF Frame State Transform

$$(\mathbf{r})_{\text{ICRF}} = \mathbf{R}_{\text{ICRF}\leftarrow\text{XCI}}(\mathbf{r})_{\text{XCI}} + (\mathbf{r}_\text{X})_{\text{ICRF}} \tag{10.23}$$

$$(\mathbf{v})_{\text{ICRF}} = \mathbf{R}_{\text{ICRF}\leftarrow\text{XCI}}(\mathbf{v})_{\text{XCI}} + (\mathbf{v}_\text{X})_{\text{ICRF}} \tag{10.24}$$

As a result of the frame being inertial there is no rate of change of the rotation matrix, so any occurrences of $\dot{\mathbf{R}}$ are set to zero.

10.7 Topocentric Coordinate Systems (LLH)

The topocentric coordinates are the coordinates used to describe the position of an object relative to the surface of a planet. These are typically the latitude ϕ, longitude λ and height h above that point on the surface. Often this is abbreviated to LLH.

Since almost all planets are not perfect spheres, these angles are still spherical coordinates, however the distance of the surface of the planet from its centre changes. As a result calculating the height is more complex than just subtracting the radius of the planet from the distance away from its centre.

In the case of an oblate spheroid, where a planet is wider at the equator, there exists an analytic solution called Ferrari's Solution [7]. If the position measured in the XCI frame to be $(X,\, Y,\, Z)$, a is the planet's equatorial radius and b is the polar radius, then the following quantities can be calculated

$$e^2 = \frac{a^2 - b^2}{a^2} \qquad \varepsilon^2 = \frac{a^2 - b^2}{b^2}$$

$$p = \sqrt{X^2 + Y^2} \qquad F = 54 b^2 Z^2$$

$$G = p^2 + (1 - e^2)Z^2 - e^2(a^2 - b^2) \qquad c = \frac{e^4 F p^2}{G^3}$$

$$s = \sqrt[3]{1 + c + \sqrt{c^2 + 2c}} \qquad k = s + 1 + \frac{1}{s}$$

$$P = \frac{F}{3k^2 G^2} \qquad Q = \sqrt{1 + 2e^4 P}$$

$$r_0 = \frac{-Pe^2 p}{1+Q} + \sqrt{\frac{1}{2}a^2\left(1 + \frac{1}{Q}\right) - \frac{P(1-e^2)Z^2}{Q(1+Q)} - \frac{1}{2}Pp^2} \qquad U = \sqrt{(p - e^2 r_0)^2 + Z^2}$$

$$V = \sqrt{(p - e^2 r_0)^2 + (1 - e^2)Z^2} \qquad z_0 = \frac{b^2 Z}{aV}$$

With all the above variables defined, the latitude, longitude and height can be calculated with

$$\phi = \arctan\left(\frac{Z + \varepsilon^2 z_0}{p}\right), \tag{10.25a}$$

$$\lambda = \operatorname{arctan2}(Y,\, X), \tag{10.25b}$$

$$h = U\left(1 - \frac{z_0}{Z}\right). \tag{10.25c}$$

10.8 Local Tangent Plane Coordinate Systems

Local tangent plane coordinate systems are used to describe the orientation of an object, with the xy-plane of the coordinate system being tangent to the surface of a sphere. Planets are not perfect spheres, which means these tangent planes are not quite tangent to the planet's actual surface. However, these coordinates very useful for describing the orientation of something in orbit around a planet.

An important fact to note is that we can use the right-ascension declination convention for spherical coordinates (as per § 1.4.4.1) to represent positions using the latitude and longituide. Specifically, the longitude ϕ is the right ascension and the latitude λ is the declination.

10.8.1 East North Up (ENU)

In ENU coordinates we would like our x-axis to point east which is the direction of increasing longitude ϕ, our y-axis to point north which is the direction of increasing latitude λ, and our z-axis to point away from the surface of the planet. Therefore we can use the basis vectors from equation (1.17) and equation (9.6) to find that

$$\mathbf{R}_{\text{ENU}\leftarrow\text{XCXF}} = \begin{bmatrix} \hat{\boldsymbol{\phi}}^{\mathsf{T}} \\ \hat{\boldsymbol{\lambda}}^{\mathsf{T}} \\ \hat{\mathbf{r}}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\sin\lambda\cos\phi & -\sin\lambda\sin\phi & \cos\lambda \\ \cos\lambda\cos\phi & \cos\lambda\sin\phi & \sin\lambda \end{bmatrix}. \tag{10.26}$$

It is worth noting that, by default, this conversion only works when converting to/from Body-Centred-Body-Fixed (X-Centred-X-Fixed) frames, because the location in spherical coordinates of the orbiting craft is the latitude and longitude. When converting to/from Body-Centred-Body-Fixed frames, the latitude λ and longitude ϕ in these formulas must, respectively, be replaced with the declination δ and right-ascension α of the position.

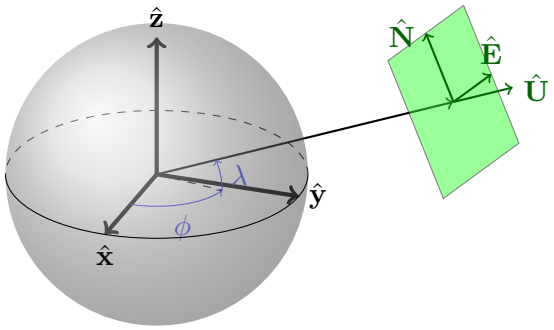


Figure 10.5: East-North-Up (ENU) local tangent plane coordinate system. The plane is a projection of the tangent plane at the equivalent point (latitude and longitude) on the planet's surface. The x-axis points east, the y-axis points north, and the z-axis away from the surface, which makes the coordinate system right-handed.

10.8.2 North East Down (NED)

In NED coordinates we would like our x-axis to point north which is the direction of increasing latitude λ, our y-axis to point east which is the direction of increasing longitude ϕ, and our z-axis to point towards the surface of the planet. Therefore we can use the basis vectors from equation (1.17) and equation (9.6) to find that

$$\mathbf{R}_{\text{NED}\leftarrow\text{XCXF}} = \begin{bmatrix} \hat{\boldsymbol{\lambda}}^{\mathsf{T}} \\ \hat{\boldsymbol{\phi}}^{\mathsf{T}} \\ -\hat{\mathbf{r}}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} -\sin\lambda\cos\phi & -\sin\lambda\sin\phi & \cos\lambda \\ -\sin\phi & \cos\phi & 0 \\ -\cos\lambda\cos\phi & -\cos\lambda\sin\phi & -\sin\lambda \end{bmatrix}. \tag{10.27}$$

It is worth noting that, by default, this conversion only works when converting to/from Body-Centred-Body-Fixed (X-Centred-X-Fixed) frames, because the location in spherical coordinates of the orbiting craft is the latitude and longitude. When converting to/from Body-Centred-Body-Fixed frames, the latitude λ and longitude ϕ in these formulas must, respectively, be replaced with the declination δ and right-ascension α of the position.

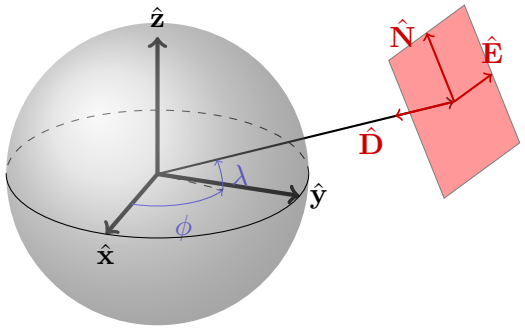


Figure 10.6: North-East-Down (NED) local tangent plane coordinate system. The plane is a projection of the tangent plane at the equivalent point (latitude and longitude) on the planet's surface. The x-axis points north, the y-axis points east, and the z-axis points down towards the surface, which makes the coordinate system right-handed.

Chapter 11: Useful Coordinate Transforms

Here is a summary of the most common and useful coordinate transforms that appear in space engineering. For brevity I have excluded the actual coordinate transforms and just included the rotation matrices and their time derivatives where it's relevant. The formulas for the transforms can be found in chapter 9.

11.1 Perifocal to Body Centred Inertial

As per figure 10.2, the transformation from perifocal pqw to body centred inertial XCI is given by the rotation matrix

$$\mathbf{R}_{\text{XCI}\leftarrow pqw} = \mathbf{R}_3(\Omega)\mathbf{R}_1(i)\mathbf{R}_3(\omega) \tag{11.1}$$

where the principle rotation matrices are defined by equation (2.4).

The reverse transform is simply

$$\mathbf{R}_{pqw\leftarrow\text{XCI}} = \mathbf{R}^{\mathsf{T}}_{\text{XCI}\leftarrow pqw}. \tag{11.2}$$

11.2 State Vector to Orbital Parameters

It is quite common that the state vector containing position and velocity,

$$\mathbf{x} = \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} \tag{11.3}$$

is measured as a result of either a simulation or orbit determination. However, it is very common to present orbital elements instead of the state vector.

These orbital parameters are

- θ – True Anomaly
- h – Specific Angular Momentum
- e – Eccentricity
- ω – Argument of Periapsis
- Ω – Right Ascension of Ascending Node
- i – Inclination

Let's assume that the state vector is measured in some inertial frame, e.g. ECI, which we shall denote as XCI, around some body X.

The easiest parameter to define is the specific angular momentum which is just

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}, \qquad h = \|\mathbf{h}\|. \tag{11.4}$$

The eccentricity vector is

$$\mathbf{e} = \frac{1}{\mu}\mathbf{v} \times \mathbf{h} - \hat{\mathbf{r}}, \qquad e = \|\mathbf{e}\|, \tag{11.5}$$

with the magnitude of the vector giving the eccentricity and the direction of the vector pointing towards the periapsis. By the right hand rule we therefore have all axes of our perifocal frame defined, with

$$\hat{\mathbf{p}} = \hat{\mathbf{e}}, \qquad \hat{\mathbf{q}} = \hat{\mathbf{h}} \times \hat{\mathbf{e}}, \qquad \hat{\mathbf{w}} = \hat{\mathbf{h}}.$$

From equation (9.4) we now have enough information to construct the rotation matrix to the perifocal frame.

$$\mathbf{R}_{\text{XCI}\leftarrow pqw} = \begin{bmatrix} \hat{\mathbf{e}} & \hat{\mathbf{h}} \times \hat{\mathbf{e}} & \hat{\mathbf{h}} \end{bmatrix}, \qquad \mathbf{R}_{pqw\leftarrow\text{XCI}} = \mathbf{R}^{\mathsf{T}}_{\text{XCI}\leftarrow pqw}$$

Where the vectors make up the columns of the rotation matrix.

The inclination is always between $0 \leq i < \pi$, so we can simply find the angle by finding the angle between $\hat{\mathbf{h}}$ and $\hat{\mathbf{z}}$.

$$i = \cos^{-1}\left(\hat{\mathbf{h}} \cdot \hat{\mathbf{z}}\right) \tag{11.6}$$

The cross product of $\hat{\mathbf{z}}$ and $\mathbf{h}$

$$\mathbf{N} = \hat{\mathbf{z}} \times \mathbf{h}$$

produces what is called the node-line vector. $\hat{\mathbf{z}}$ is orthogonal to the xy-plane, and $\mathbf{h}$ is perpendicular to the orbital plane, with the orbit revolving counter-clockwise around it. Therefore, $\mathbf{N}$ points in the direction of the ascending node of the orbit. The angle that $\mathbf{N}$ makes with the x-axis is the right ascension of the ascending node, and can be found in all cases except where there is no ascending node.

$$\Omega = \begin{cases} 0 & \mathbf{N} = \mathbf{0} \\ \arctan2\left(N_y,\, N_x\right) & \text{otherwise} \end{cases} \tag{11.7}$$

The rest of the calculations are easier in the perifocal frame. The argument of periapsis is the angle between the node-line vector and the eccentricity vector. In the perifocal frame $\hat{\mathbf{e}}$ is just the x-axis, and therefore $\hat{\mathbf{N}}$ is located $-\omega$ radians from the x-axis. One can intuit this from figure 10.2. If $\mathbf{N} = \mathbf{0}$ then the orbit lies in the equatorial plane and the eccentricity vector's angle from the x-axis is the argument of periapsis.

Calculating the $\hat{\mathbf{N}}$ in the perifocal frame is done with

$$\left(\hat{\mathbf{N}}\right)_{pqw} = \mathbf{R}_{pqw\leftarrow\text{XCI}}\,\hat{\mathbf{N}} = \begin{bmatrix} N_p \\ N_q \\ 0 \end{bmatrix}$$

and the argument of periapsis is then

$$\omega = \begin{cases} \arctan2\left(e_y,\, e_x\right) & \mathbf{N} = \mathbf{0} \\ \arctan2\left(N_q,\, N_p\right) & \text{otherwise} \end{cases} \tag{11.8}$$

note that e_x and e_y are the components of the eccentricity vector in the XCI frame.

Finally, the true anomaly is the angle that the position vector makes from the x-axis in the perifocal frame. The position in the perifocal frame is

$$(\mathbf{r})_{pqw} = \mathbf{R}_{pqw \leftarrow \text{XCI}}\, \mathbf{r} = \begin{bmatrix} r_p \\ r_q \\ 0 \end{bmatrix}$$

and the true anomaly, the final element, is

$$\theta = \arctan2\left(r_q,\, r_p\right). \tag{11.9}$$

11.3 EME2000/ICRF to Body Centred Inertial/Body Fixed

While definitely one of the most complex transformations, this is perhaps the most important transformation for calculating interplanetary trajectories, or even just trajectories outside of Earth. Here we consider the coordinate frame as aligned with Earth's at the J2000 epoch. The Z axis as Earth's spin axis, the X axis as the direction of vernal equinox, and the Y axis as the cross product of the other two axes.

The International Celestial Reference System (ICRS) and it's associated coordinate frame the International Celestial Reference Frame (ICRF) were misaligned from the EME2000. By looking at figure 10.4 one can see that the two angles α and δ represent the orientation of the equatorial plane of a planet relative to the EME2000 frame. The angle W represents the orientation of the planet that varies over time due to its' spin.

Values for of these angles for every major body in the solar system, and formulas for their values at times since J2000, can be found in [6].

11.3.1 EME2000/ICRF to Body Centred Body Fixed

To rotate from the body-centred-body-fixed coordinates of the planet to the EME2000/ICRF coordinates, we simply need to know α, δ and W at the epoch of interest, then multiply by the matrix

$$\mathbf{R}_{\text{ICRF} \leftarrow \text{XCXF}} = \mathbf{R}_3(\alpha(t))\mathbf{R}_2(90^\circ - \delta(t))\mathbf{R}_3(90^\circ + W(t)). \tag{11.10}$$

To rotate from the EME2000/ICRF coordinates to the body-centred-body-fixed coordinates of the planet, we simply take the transpose of the above matrix,

$$\mathbf{R}_{\text{XCXF} \leftarrow \text{ICRF}} = \mathbf{R}^{\mathsf{T}}_{\text{ICRF} \leftarrow \text{XCXF}} \tag{11.11}$$

This is a rotating frame of reference and so the velocity transform requires the rate of change of the rotation matrix as well. Simple application of the product rule and equation (2.18) gives

$$\begin{aligned} \dot{\mathbf{R}}_{\text{ICRF} \leftarrow \text{XCXF}} = {} & \dot{\alpha}\left[\hat{\mathbf{Z}}\right]_\times \mathbf{R}_3(\alpha)\mathbf{R}_2(90^\circ - \delta)\mathbf{R}_3(90^\circ + W) \\ & + \mathbf{R}_3(\alpha)(-\dot{\delta})\left[\hat{\mathbf{Y}}\right]_\times \mathbf{R}_2(90^\circ - \delta)\mathbf{R}_3(90^\circ + W) \\ & + \mathbf{R}_3(\alpha)\mathbf{R}_2(90^\circ - \delta)\dot{W}\left[\hat{\mathbf{Z}}\right]_\times \mathbf{R}_3(90^\circ + W). \end{aligned} \tag{11.12}$$

and again the transpose gives the inverse

$$\dot{\mathbf{R}}_{\text{XCXF} \leftarrow \text{ICRF}} = \dot{\mathbf{R}}^{\mathsf{T}}_{\text{ICRF} \leftarrow \text{XCXF}} \tag{11.13}$$

The position and velocity transforms are taken from equations (9.14) and (9.15). Depending on where the vector to shift the origin is measured, one may need the formulations of these transforms from § 9.3.6.

The full set of transforms is detailed in § 10.6.1.

11.3.2 EME2000/ICRF to Body Centred Inertial

To shift from the body-centred-inertial coordinates of the planet to the EME2000/ICRF coordinates, we simply replace all time-varying angles with their values at the J2000 epoch,

$$\mathbf{R}_{\text{ICRF}\leftarrow\text{XCI}} = \mathbf{R}_3(\alpha_{\text{J2000}})\mathbf{R}_2(90^\circ - \delta_{\text{J2000}})\mathbf{R}_3(90^\circ + W_{\text{J2000}}). \tag{11.14}$$

The inverse rotation is again given by the transpose

$$\mathbf{R}_{\text{XCI}\leftarrow\text{ICRF}} = \mathbf{R}^{\mathsf{T}}_{\text{ICRF}\leftarrow\text{XCI}} \tag{11.15}$$

This matrix does not change, so the transforms from equations (9.14) and (9.15) are relatively easy to compute. Depending on where the vector to shift the origin is measured, one may need the formulations of these transforms from § 9.3.6.

The full set of transforms is detailed in § 10.6.1.

11.4 Body Centred Inertial to Body Centred Body Fixed

While the most appropriate way to transform from body centred inertial (XCI) to body centred body fixed (XCXF) is to transfrom into the ICRF axes and then to the other frame, this often isn't performed. The more common transform is to set $\Delta W(t) = W(t) - W_{\text{J2000}}$ and then use the rotation matrix

$$\mathbf{R}_{\text{XCXF}\leftarrow\text{XCI}} = \mathbf{R}_3(\Delta W(t)) \tag{11.16}$$

The transpose gives the inverse transform

$$\mathbf{R}_{\text{XCI}\leftarrow\text{XCXF}} = \mathbf{R}^{\mathsf{T}}_{\text{XCXF}\leftarrow\text{XCI}}. \tag{11.17}$$

In the case of Earth this quantity ΔW is often referred to as *Greenwhich Mean Sidereal Time* (GMST). Calculation of this angle for all large bodies in the solar system is given in [6].

The time derivative of this rotation matrix is simply

$$\dot{\mathbf{R}}_{\text{XCXF}\leftarrow\text{XCI}} = \dot{W}(t)\left[\hat{\mathbf{z}}\right]_\times \mathbf{R}_3(\Delta W(t)) \tag{11.18}$$

where $\left[\hat{\mathbf{z}}\right]_\times$ is the cross product matrix of the unit vector $\hat{\mathbf{z}}$.

The transpose gives the inverse for this too

$$\dot{\mathbf{R}}_{\text{XCXF}\leftarrow\text{XCI}} = \dot{\mathbf{R}}^{\mathsf{T}}_{\text{XCXF}\leftarrow\text{XCI}}. \tag{11.19}$$

Since the origins are shared between these frames, one can quite easily use equations (9.14) and (9.15) to compute the transformed positions and velocities, with the shift term at the end being zero due to the origins being the same point.

11.5 Cartesian State to/from Spherical Coordinate State

11.5.1 Right Ascension and Declination (Ra-Dec)

Representing a position with distance r, right-ascension α, and declination δ yields a simple conversion to Cartesian coordinates as per § 1.4.4.1.

Definition 11.1: Ra-Dec Spherical Coordinates to Cartesian Coordinates

$$\mathbf{r} = \begin{bmatrix} r\cos\delta\cos\alpha \\ r\cos\delta\sin\alpha \\ r\sin\delta \end{bmatrix} \tag{11.20}$$

One can find through relatively simple rearranging the inverse transformation.

Definition 11.2: Cartesian Coordinates to Ra-Dec Spherical Coordinates

$$r = \|\mathbf{r}\|, \qquad \alpha = \arctan2\left(r_y,\, r_x\right), \qquad \delta = \sin^{-1}\left(\frac{r_z}{r}\right). \tag{11.21}$$

It is worth noting that this inverse transform breaks down at $r = 0$ and when $r_x = r_y = 0$. In fact, at these point spherical coordinates don't even really make sense.

To convert from Cartesian velocities to spherical velocities, we can use the basis vectors in equation (1.17), repeated here for convenience.

$$\hat{\mathbf{r}} = \begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix}, \qquad \hat{\boldsymbol{\alpha}} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\delta}} = \begin{bmatrix} -\sin\delta\cos\alpha \\ -\sin\delta\sin\alpha \\ \cos\delta \end{bmatrix}.$$

Using equation (1.18) we then find the velocity transform.

Definition 11.3: Ra-Dec Spherical Rates to Cartesian Velocity

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\alpha}\cos\delta\,\hat{\boldsymbol{\alpha}} + r\dot{\delta}\,\hat{\boldsymbol{\delta}}. \tag{11.22}$$

When performing the reverse transform from velocity $\mathbf{v}$ to the spherical rates, you already have the angles by equation (11.21), which then allows the basis vectors to be used to find the rates of each spherical coordinate.

Definition 11.4: Cartesian Velocity to Ra-Dec Spherical Rates

$$\dot{r} = \mathbf{v}\cdot\hat{\mathbf{r}}, \qquad \dot{\alpha} = \frac{\mathbf{v}\cdot\hat{\boldsymbol{\alpha}}}{r\cos\delta}, \qquad \dot{\delta} = \frac{\mathbf{v}\cdot\hat{\boldsymbol{\delta}}}{r}. \tag{11.23}$$

This inverse transform is undefined at $r = 0$ as well as $r_x = r_y = 0$ (or equivalently $\delta = \pm\frac{\pi}{2}$).

11.5.2 Azimuth and Inclination (Az-Inc)

Representing a position with distance r, azimuth ϕ, and inclination θ yields a simple conversion to Cartesian coordinates as per § 1.4.4.1.

Definition 11.5: Az-Inc Spherical Coordinates to Cartesian Coordinates

$$\mathbf{r} = \begin{bmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{bmatrix} \tag{11.24}$$

One can find through relatively simple rearranging the inverse transformation.

Definition 11.6: Cartesian Coordinates to Az-Inc Spherical Coordinates

$$r = \|\mathbf{r}\|, \qquad \phi = \arctan2\left(r_y, r_x\right), \qquad \theta = \cos^{-1}\left(\frac{r_z}{r}\right). \tag{11.25}$$

It is worth noting that this inverse transform breaks down at $r = 0$ and when $r_x = r_y = 0$. In fact, at these point spherical coordinates don't even really make sense.

To convert from Cartesian coordinates to right-ascension and declination, we can use the basis vectors in equation (1.21), repeated here for convenience.

$$\hat{\mathbf{r}} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \qquad \hat{\boldsymbol{\phi}} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\theta}} = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix}.$$

Using equation (1.22) we then find the velocity transform.

Definition 11.7: Az-Inc Spherical Rates to Cartesian Velocity

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\sin\theta\,\hat{\boldsymbol{\phi}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}. \tag{11.26}$$

When performing the reverse transform from velocity $\mathbf{v}$ to the spherical rates, you already have the angles by equation (11.21), which then allows the basis vectors to be used to find the rates of each spherical coordinate.

Definition 11.8: Cartesian Velocity to Az-Inc Spherical Rates

$$\dot{r} = \mathbf{v}\cdot\hat{\mathbf{r}}, \qquad \dot{\phi} = \frac{\mathbf{v}\cdot\hat{\boldsymbol{\phi}}}{r\sin\theta}, \qquad \dot{\theta} = \frac{\mathbf{v}\cdot\hat{\boldsymbol{\theta}}}{r}. \tag{11.27}$$

This inverse transform is undefined at $r = 0$ as well as $r_x = r_y = 0$ (or equivalently $\theta = 0$ and $\theta = \pi$).

Add diagram for order of transforms

Chapter 12: Spherical Harmonics Gravity Models

12.1 What are Spherical Harmonics?

Spherical harmonics are a concept whereby we model some function U in spherical coordinates using a sum of a set of functions $Y_{lm}(r,\,\theta,\,\varphi)$ multiplied by some coefficients C_{lm}. The functions $Y_{lm}(\theta,\,\varphi)$ are called the spherical harmonics and the best way to think of them is the same way we think of basis vectors in $\mathbb{R}^2$. In $\mathbb{R}^2$ we require two basis vectors which are not co-linear, or better yet are completely orthogonal, to be able to then describe any point in $\mathbb{R}^2$. In the same way if we want to describe **any** function in spherical coordinates we need to add together sufficiently many coefficients multiplied by *orthogonal functions* to then describe that function.

A real valued spherical harmonic function $U : \mathbb{R}^n \to \mathbb{R}$ must satisfy Laplace's equation

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2} = 0. \quad (12.1)$$

These are often referred to as the solid harmonics and in general functions that satisfy Laplace's equation are called harmonic functions of the form

$$U = R(r)\Theta(\theta)\Phi(\varphi). \quad (12.2)$$

12.2 Spherical Harmonics as a model of Gravity

As you would intuitively know, some parts of Earth's surface are denser than others, for instance solid ground is denser than water and therefore the density is greater close to the surface where there is land. This results in the gravitational strength varying with both latitude ϕ and longitude λ. Typically gravity is modelled as being from a spherically symmetric object with

$$\mathbf{g}(\mathbf{r}) = -\frac{\mu}{r^2}\hat{\mathbf{r}} \quad (12.3)$$

or in spherical coordinates as

$$\mathbf{g}(r,\,\theta,\,\phi) = -\frac{\mu}{r^2}\hat{\mathbf{r}} = -\frac{\mu}{r^2}\begin{bmatrix}\cos\lambda\cos\phi\\ \sin\lambda\cos\phi\\ \sin\phi\end{bmatrix}. \quad (12.4)$$

Since gravity is a conservative force it also has an associated potential energy V or potential energy per unit mass U (hereafter referred to as simply its potential function) such that

$$\mathbf{g} = -\boldsymbol{\nabla}U, \qquad \mathbf{F}_g = -\boldsymbol{\nabla}V. \quad (12.5)$$

For a spherically symmetric mass the potential function is simply

$$U = -\frac{\mu}{r}. \quad (12.6)$$

To add the effects of a body's non-spherical shape we can add a spherical harmonics expansion to the potential function that is dependent on the latitude and longitude of the point. Harmonic functions of the form in equation (12.2) which are solutions to equation (12.1) take a more specific form given by

$$\begin{aligned} g_{nm}(\theta,\,\varphi) &= \frac{1}{r^{n+1}} P_{nm}(\sin\theta)\cos(m\varphi), \\ h_{nm}(\theta,\,\varphi) &= \frac{1}{r^{n+1}} P_{nm}(\sin\theta)\sin(m\varphi), \end{aligned} \tag{12.7}$$

where P_{nm} are the associated Legendre polynomials of degree n and order m. The associated Legendre polynomials [8] are defined as

$$P_{nm}(x) = (1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m} \tag{12.8}$$

using the Legendre polynomials $P_n(x)$ which are defined using Rodrigues' formula [9] as

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n. \tag{12.9}$$

It is worth noting that we have omitted the Condon-Shortley phase factor $(-1)^m$ from the definition of the associated Legendre polynomials, as this is common practice in spherical harmonic expansions. You could include the factor however it all ends up being accounted for by the unknown coefficients anyway.

First let's swap our spherical coordinates to latitude and longitude. We go from $(r,\,\theta,\,\varphi)$ (range, inclination and azimuth) to $(r,\,\phi,\,\lambda)$. We can now write the potential function as a sum of the spherical harmonics with

$$U = -\mu\left(\sum_{n=0}^{\infty} \frac{R^n}{r^{n+1}} \sum_{m=0}^{n} P_{nm}(\sin\phi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right) \tag{12.10}$$

using R as some nominal radius to cancel out the dimensions to be correct. There are a few things to note here. The first is that whenever $m = 0$ the S term goes away since $\sin(m\lambda) = 0$ and the C term reduces to a constant since $\cos(m\lambda) = 1$. Therefore some documents write the sum by removing the $m = 0$ term and instead writing the sum as

$$U = -\mu\left(\sum_{n=0}^{\infty} \frac{R^n}{r^{n+1}} J_n P_{n0}(\sin\phi) + \sum_{n=1}^{\infty} \frac{R^n}{r^{n+1}} \sum_{m=1}^{n} P_{nm}(\sin\phi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right) \tag{12.11}$$

where the J terms are the zonal terms, dividing the body into vertical zones and the nm terms are the tesseral terms, dividing those zones into smaller sections or tessellations.

The next simplification that is made is that since terms for $n = m = 0$ are constant and as such do not change the potential function (this is otherwise known as Gauge symmetry). We can also observe that terms for $n = 1$ result in some function multiplied by $1/r$, which we know must therefore reduce to just $1/r$ or the standard gravitational potential function. Therefore we can arrive at two possible forms of the potential function. We either have

$$U = -\frac{\mu}{r}\left(1 + \sum_{n=2}^{\infty} \frac{R^n}{r^n} \sum_{m=0}^{n} P_{nm}(\sin\phi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right) \tag{12.12}$$

where S_{n0} is necessarily defined as 0 for all n. Or

$$U = -\frac{\mu}{r}\left(1 + \sum_{n=2}^{\infty} \frac{R^n}{r^n} J_n P_{n0}(\sin\phi) + \sum_{n=2}^{\infty} \frac{R^n}{r^n} \sum_{m=1}^{n} P_{nm}(\sin\phi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right) \tag{12.13}$$

where the transformation to get from equation (12.13) back to equation (12.12) is simply a renaming of the J_n terms to C_{n0} and allowing the summation over m to begin at $m = 0$.

Equation (12.12) is the more common choice for gravity models using more than just the first J_2 coefficient as it reduces the number of coefficients for each nm pair to 2.

12.2.1 Recursive Expansion

As it turns out there is a more computationally convenient way to write the spherical harmonic expansion in equation (12.12). We first define the reduced associated Legendre functions as

$$A_{nm} = \frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m}. \tag{12.14}$$

Note in equation (12.8) we can reduce the term out the front using $x = \sin\phi$ to

$$P_{nm}(\sin\phi) = (1 - \sin^2\phi)^{\frac{m}{2}} A_{nm}(\sin\phi) = \cos^m\phi A_{nm}(\sin\phi) \tag{12.15}$$

ignoring the negative solution because $-\pi/2 < \phi < \pi/2$. We can then write the potential function as

$$U = -\frac{\mu}{r}\left(1 + \sum_{n=2}^{\infty}\frac{R^n}{r^n}\sum_{m=0}^{n} A_{nm}(\sin\phi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\cos^m\phi\right). \tag{12.16}$$

As per [10] we can write this using the recursive functions $r_m(s,\, t)$ and $i_m(s,\, t)$. We first define

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{12.17a}$$
$$s = \frac{x}{r} = \cos\lambda\cos\phi, \tag{12.17b}$$
$$t = \frac{y}{r} = \sin\lambda\cos\phi, \tag{12.17c}$$
$$u = \frac{z}{r} = \sin\phi, \tag{12.17d}$$
$$1 = s^2 + t^2 + u^2. \tag{12.17e}$$

Setting $r_0 = 1$ and $i_0 = 0$ we have recursive definitions

$$r_m(s,\, t) = s\, r_{m-1} - t\, i_{m-1}, \tag{12.18a}$$
$$i_m(s,\, t) = s\, i_{m-1} + t\, r_{m-1}, \tag{12.18b}$$

which give the final equation for the recursive expansion of the potential function

$$U = -\frac{\mu}{r}\left(1 + \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n\sum_{m=0}^{n} A_{nm}(u)\left[C_{nm}r_m(s,\, t) + S_{nm}i_m(s,\, t)\right]\right). \tag{12.19}$$

12.3 Earth and Mars Gravity Model Coefficients

There have been many iterations of the Earth Gravity Model including EGM84 and EMG96 as well as the Joint Gravity Model JGM-1 through JGM-3. The most recent model is the EGM2008 model which is a spherical harmonics model of the Earth's gravity field to degree and order 2159, though this model is quite computationally excessive for space applications.

While the JGM models quote the above coefficients C_{nm} and S_{nm}, the EGM models normalise the coefficients due to the fact that often coefficients past $n = 100$ are very small, near floating point limit. The EGM models as well as the Goddard Mars Models (GMM1, GMM2) normalise with the factor

$$N_{nm} = \sqrt{(2 - \delta_{0m})(2n + 1)\frac{(n - m)!}{(n + m)!}} \tag{12.20}$$

where δ_{0m} is the Kronecker delta function

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{12.21}$$

such that $\delta_{0m} = 1$ if $m = 0$ and $\delta_{0m} = 0$ if $m \neq 0$.

The lists of coefficients from these models quotes the normalised coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$ where to de-normalise the coefficients we multiply by the normalisation factor

$$C_{nm} = N_{nm}\bar{C}_{nm}, \tag{12.22}$$

$$S_{nm} = N_{nm}\bar{S}_{nm}. \tag{12.23}$$

12.4 The J_2 Effect

The J_2 effect is so named because it comes from the J_2 coefficient in the spherical harmonics expansion from equation (12.11). This term physically represents the effect on the gravitational field caused by the planet's widening at the equator, which happens due to its rotation. In most cases this has the largest effect on orbits and as a result most of the other spherical harmonics terms are ignored.

Typically the gravitational acceleration is given by

$$\mathbf{g} = -\boldsymbol{\nabla} U = -\boldsymbol{\nabla}\left(-\frac{\mu}{r}\right) = -\frac{\mu}{r^2}\hat{\mathbf{r}} \tag{12.24}$$

however including the J_2 term and taking the gradient of equation (12.11) yields a slightly different gravitational acceleration

$$\mathbf{g}_{J_2} = -\frac{\mu}{r^2}\hat{\mathbf{r}} + \underbrace{\frac{3}{2}J_2\left(\frac{R}{r}\right)^2\frac{\mu}{r^2}\begin{bmatrix}\frac{x}{r}\left(5\left(\frac{z}{r}\right)^2 - 1\right)\\ \frac{y}{r}\left(5\left(\frac{z}{r}\right)^2 - 1\right)\\ \frac{z}{r}\left(5\left(\frac{z}{r}\right)^2 - 3\right)\end{bmatrix}}_{J_2 \text{ term}} \tag{12.25}$$

Lagrange's variation of parameters can be used to obtain time-averaged rates of change for the orbital parameters. In the case of the J_2 perturbation two of the planar parameters oscillate and increase or decrease slowly over time [2, ch. 4.7], with average rates of change given by

$$\left\langle\frac{\mathrm{d}\omega}{\mathrm{d}t}\right\rangle = -\left(\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1 - e^2)^2a^{\frac{7}{2}}}\right)\left(\frac{5}{2}\sin^2 i - 2\right), \qquad \left\langle\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right\rangle = -\left(\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1 - e^2)^2a^{\frac{7}{2}}}\right)\cos i. \tag{12.26}$$

where $\langle x\rangle$ means 'take the average of x'.

The time-averaged motion of each angle after a time Δt as

$$\langle\omega\rangle = \left\langle\frac{\mathrm{d}\omega}{\mathrm{d}t}\right\rangle\Delta t, \qquad \langle\Omega\rangle = \left\langle\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right\rangle\Delta t. \tag{12.27}$$

All other orbital parameters are unaffected by the J_2 effect. As a result the overall result of the J_2 effect is simply that the perigee either advances for $0 \leq i < 1.107$ and $2.035 < i \leq \pi$ or regresses for $1.107 < i < 2.035$. The right ascension of the ascending node precesses clockwise with time for $0 \leq i < \frac{\pi}{2}$ and counterclockwise with time for $\frac{\pi}{2} < i \leq \pi$. It is worth noting that this is a time-averaged effect, so in reality these parameters will oscillate over time, around this trend.

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