ADVANCED MECHANICS

Base Units

Mass (m) – Kilograms (kg)
Displacement $(\vec{s})$ – metres (m)
Time (t) – Seconds (s)
Speed (v) – Metres per second $(ms^{-1}$ **or** $m/s)$
Velocity $(\vec{v})$ – Metres per second $(ms^{-1}$ **or** $m/s)$
Acceleration $(\vec{a})$ – $(ms^{-2}$ **or** $m/s^2)$
Force $(\vec{F})$ – Newtons (N)
Energy (E) – Joules (J)
Work (W) – Newton Meters (Nm) **or** Joules (J)
Angular Displacement (θ) – Radians (rad)
Angular Velocity (ω) – Radians per second $(rad\ s^{-1})$
Angular Acceleration (α) – Radians per second per second $(rad\ s^{-2})$

Constants

Gravitational Constant: $G = 6.67 \times 10^{-11}\ (N\ m^2\ kg^{-2})$
Mass of the Earth: $m_E = 6.0 \times 10^{24}\ (kg)$
Radius of the Earth: $r_E = 6.371 \times 10^6\ (m)$

Equations

$\vec{F}_{net} = m\vec{a}$

Newtons is a measure of Force; One Newton is equivalent to one: kilogram metres-per-second-per-second (kg ms^{-2}). The net means sum of all forces which accounts for the fact that there can be forces on an object but if they cancel there is no acceleration.

$$F_c = \frac{mv^2}{r}$$

$$a_c = \frac{v^2}{r}$$

Centripetal force or acceleration on an object derived from circular motion. Centripetal force is the description of the sum of forces on an object in circular motion. The actual force is often generated by tension in a string or a gravitational field.

$$\theta = \frac{l}{r}$$

$$\omega = \frac{\Delta\theta}{t} = \frac{v_{\perp}}{r}$$

$$\alpha = \frac{\Delta\omega}{t} = \frac{a_{\perp}}{r}$$

*Angular equivalents of linear factors. Equations for angular displacement, angular velocity and angular acceleration for an object spinning in a circle. Measured in **radians**, **radians** per second and **radians** per second per second.*

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r}_{\perp}\vec{F} = rF\sin\theta$$

Torque on a rotating object is the Force on that object multiplied by the radius from the axis of rotation multiplied by sine of the angle between them.

$$F_G = G\frac{m_1 m_2}{r^2}$$

$$a_G = G\frac{m_1}{r^2}$$

Gravitational Force between two objects of mass at some radius. G is the gravitational constant.

The acceleration due to gravity is dependent on the mass of the object generating the field.

$$v = \frac{2\pi r}{T}$$

Kepler's second law of planetary motion. It states the average speed of a planet in motion is proportional to its radius divided by the time for one orbit around the sun.

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

Kepler's third law of planetary motion. The relationship between radius and period of orbit with a constant (where M is the mass of the object it orbits).

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Escape velocity of an object with mass at an initial distance, r, away. It is the minimum velocity an object must have to escape the gravity of that object.

$$U_G = -\frac{Gm_1m_2}{r}$$

The potential energy due to gravity of an object in another objects gravitational field. Note that it is not necessarily the total potential energy, but often is.

Projectile Motion Equations

$$\vec{s} = \frac{\vec{a}}{2}t^2 + \vec{v}_i t + \vec{s}_i$$

The displacement of an object in a given direction is given by this equation, where the displacement, acceleration and initial velocity are all vector components in the same direction. i.e. If the displacement is the displacement in the x direction, then the acceleration is the acceleration in the x-direction. It is possible for this to be zero.

The extra term $\vec{s}_i$ on the end is the initial displacement from the target. E.g. if a ball falls from a table, its initial y-displacement from the floor is the height (if positive direction is up).

Note that it is a quadratic in terms of t.

$$\vec{v} = \vec{a}t + \vec{v}_i$$

The velocity of an object is given by the acceleration in that direction and the initial velocity in that direction.

$$v^2 = {v_i}^2 + 2as$$

A rearrangement of the above formulae. Useful when time is not given.

Extension Equations

$$I = \int_0^m r^2 \, dm$$

The Moment of Inertia of a rotating object. Although it is a strange integral it is solvable by defining dm *in terms of some area or length mass density and some area or length* dA *or* dl*.*

$$\Sigma\vec{\tau} = \frac{d\vec{L}}{dt} = I\vec{\alpha}$$

The sum of torques is the rate of change of angular momentum with respect to time.

Torque is also moment of inertia multiplied by angular acceleration. Note the similarity for force where moment of inertia takes the place of mass.

$$\oiint \vec{g} \cdot d\vec{A} = -4\pi GM$$

$$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho$$

Gauss' Law for gravity where $\vec{g}$ *is the gravitational field, M the mass of the object creating the field and* ρ *the density* $\left(\rho = \frac{dM}{dV}\right)$*.*

$$\vec{F}_G = -G\frac{m_1 m_2}{r^2}\hat{r}$$

The direction of a gravitational force is given by the unit vector (a vector of length 1 with some direction) $\hat{r}$*, which points from the mass exerting the force to the other mass.*

$$U_G = \int_\infty^r G\frac{m_1 m_2}{r^2} \, dr = -\frac{Gm_1 m_2}{r}$$

The potential energy of a particle in a gravitational field. The integral is from infinity to r as the potential energy is equivalent to the work done by the field and is therefore, the Force done by the field dragging that mass from infinity to its current point.

Derivative Forms

$$\frac{d\vec{s}}{dt} = \vec{v}$$

The first derivative of Displacement is velocity.

$$\frac{d^2\vec{s}}{dt^2} = \frac{d\vec{v}}{dt} = \vec{a}$$

The second derivative of Displacement, or the first derivative of Velocity, is Acceleration.

$$\frac{d^3\vec{s}}{dt^3} = \frac{d^2\vec{v}}{dt^2} = \frac{d\vec{a}}{dt} = \vec{j}$$

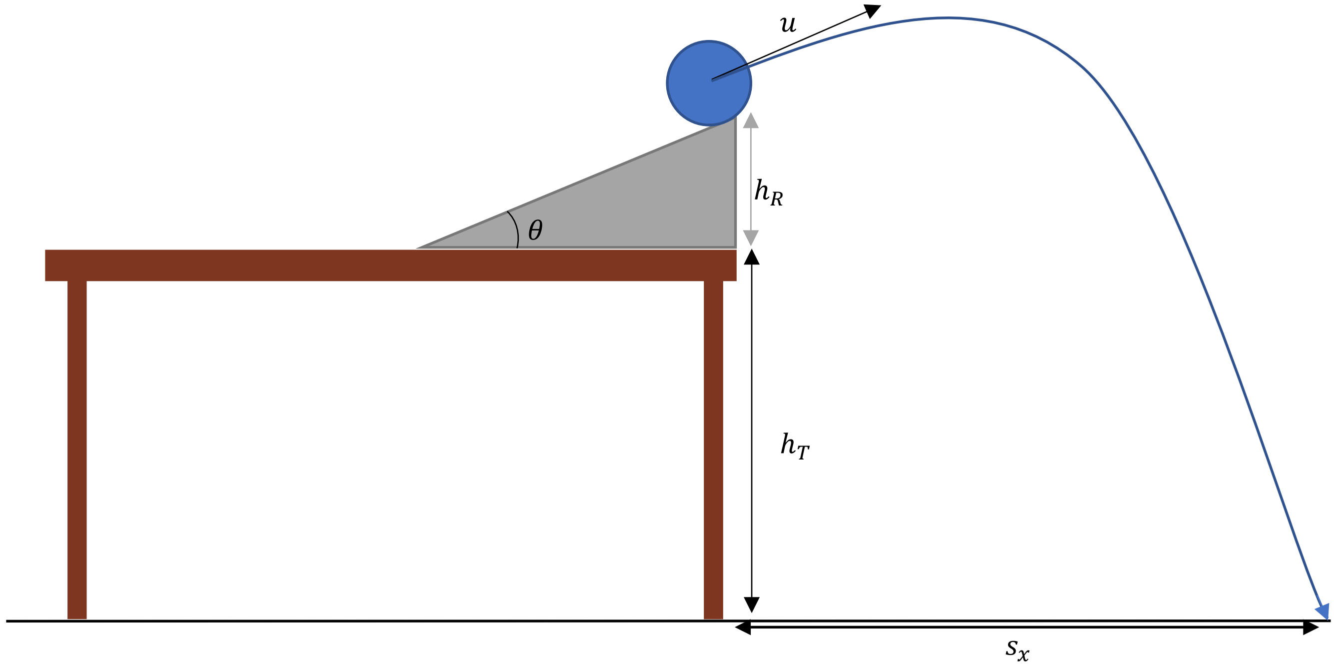
The derivative of Acceleration is Jerk. (Well beyond the course as all Senior Physics assumes Jerk is 0.)

$$\vec{\omega} = \frac{\overrightarrow{d\theta}}{dt}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{\overrightarrow{d^2\theta}}{dt^2}$$

The derivative forms of angular velocity and acceleration. Note that θ can never be defined as a vector or pseudo-vector, however $\overrightarrow{d\theta}$ can. This means that although angular displacement cannot be added as a vector, angular velocity and acceleration can as they are functions of a vector, meaning they too, are vectors.

Using the Projectile Motion Equations



The below is intended to work through the problem algebraically with explanations such that you can try to solve it yourself. Numerical solutions are provided however working through the problem without numbers first is encouraged first.

Question:

A ball is rolled up a slope off a table with initial velocity v at angle θ from the vertical.

a) How long does it take the ball to hit the ground?
b) How far is it from the edge of the table when it hits the ground?
c) What is the ball's velocity when it hits the ground?
d) What is the ball's displacement from the edge of the table when it is at the highest point of its arc? (Extension Question)

Solve the above problems if:

$u = 15ms^{-1}$
$\theta = 30°$
$h_T = 1.2m$
$h_R = 0.24m$

Next 3 pages for algebraic and numerical answers .

Algebraic Answers

a) To solve this first question, we need to construct an equation that solves for when the height of the ball is the height of the floor, which we are going to define as when $s_y = 0$. We are also going to define up as positive.

On the formula sheet, the equation for displacement is $s = ut + \frac{1}{2}at^2$. Although this formula is good, it has an issue. When $t = 0$, displacement is also zero ($s = 0$) but at $t = 0$ the height of the ball is $h_T + h_R$ which means our equation is slightly wrong. We can fix this by using the equation detailed above in the notes: $y = \frac{a_y}{2}t^2 + u_y t + y_i$ where $y_i = h_T + h_R$.

Now we have a displacement equation in the y-direction, we need to find the initial velocity and acceleration in the y-direction. Using trig rules, $u_y = u \sin\theta$. Acceleration is just gravity which can be defined as $g = -9.8$ ($-$ because up is $+$).

Now to find t, we need to use the quadratic formula:

$$t = \frac{-u_y \pm \sqrt{\left(u_y\right)^2 - 4\left(\frac{g}{2}\right)(y_i)}}{2\left(\frac{g}{2}\right)}$$

b) To find the horizontal distance from the table (s_x in the diagram) we take the time taken to hit the ground and multiply it by the initial horizontal velocity. This can be derived from the displacement formula $x = \frac{a_x}{2}t^2 + u_x t + x_i$ where $a_x = 0,\ \ x_i = 0$.

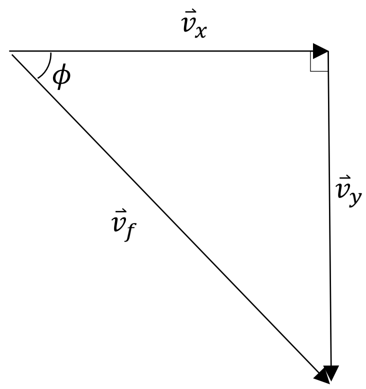
$$\therefore s_x = u_x t$$

u_x is given by the horizontal component of the initial velocity, i.e. $u_x = u\cos\theta$

c) To find the final velocity, we have to add the two velocity vectors at the time of the collision with the ground. We already found t in a) and we know the horizontal velocity is constant throughout the flight, therefore the only new fact we need is the final vertical velocity.
$v_y = gt + u_y$ gives the final velocity after some time and we use this to find the downward velocity at the time of collision.
Now for vector addition:

$$|\vec{v}_f| = \sqrt{{\vec{v}_x}^2 + {\vec{v}_y}^2}$$

$$\phi = \tan^{-1}\left(\frac{\vec{v}_y}{\vec{v}_x}\right)$$



d) (Extension) To solve this question, we have to find where the vertical velocity is zero as this is when the ball has risen to its peak. Because we want the height above
To find this we set $v_y = 0 = gt + u_y$ and solve for time at the peak:

$$t_p = -\frac{u_y}{g}$$

Now we plug this time into our displacement equations:

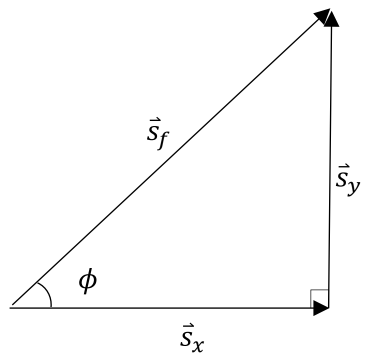
$$\vec{s}_x = \vec{v}_x t$$

$$s_y = \frac{\vec{g}}{2}t^2 + \vec{v}_{i_y}t + (h_T + h_R)$$

However, there is again a problem with the height, as we want the displacement from the edge of the table. This means at $t = 0$ we only want our height to be h_R

$$\therefore s_y = \frac{\vec{g}}{2}t^2 + u_y t + h_R$$

Now we do vector addition:



$$|\vec{s}_f| = \sqrt{{s_x}^2 + {s_y}^2}$$

$$\phi = \tan^{-1}\left(\frac{\vec{s}_y}{\vec{s}_x}\right)$$

Numerical Answers

a) $t = 1.7s$

b) $\vec{s}_x = 22m$

c) $\vec{v}_f = 16ms^{-1}$ $[To\ the\ right, 62°\ below\ horizontal]$

d) $\vec{s} = 22m$ $[Right, 51°\ above\ horizontal]$

Course Notes

Converting Degrees to Radians to Degrees

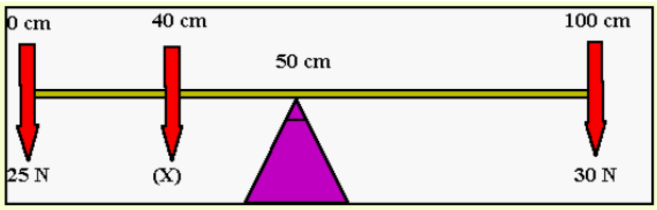
To convert degrees to radians, multiply by $\frac{\pi}{180}$

To convert radians to degrees, multiply by $\frac{180}{\pi}$

RPM to Radians per Second

To convert RPM to Radians per second, first convert it to Revs per second by dividing it by 60 (i.e. 60rpm will be 1 rotation per second). Then since one revolution is 2π radians, multiply the Revs per second by 2π.

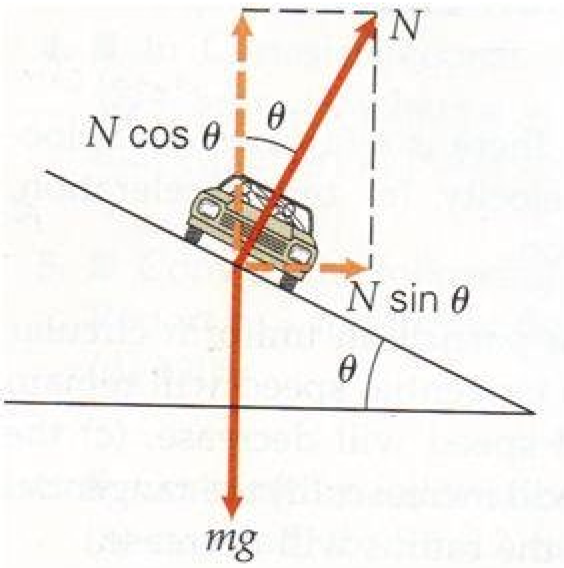
Torque Equilibrium Questions



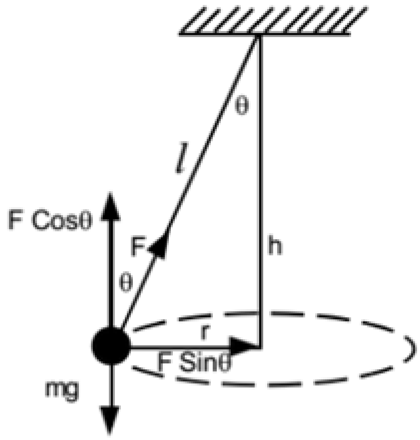
The torque questions like what is pictured above rely on the idea that $\sum\tau = 0$. This means the torques on the left of the pivot are equal to the torques on the right.
These questions can also be tricky in that the centre is defined as $50cm$, which means the force at $0cm$ is actually at a radius of $50cm$ etc.

Banked Curves

In a typical banked curve question the centripetal force is considered to be the inward component of the normal force and the vertical component of the normal force is the opposite of gravity.



Tension as Centripetal Force



In this example of circular motion, the vertical component of tension is the force opposing gravity and the horizontal is the centripetal force.

The interesting implication of this is that when you swing an object on a string in real life, it can never be perfectly flat, there will always be a slight angle between the string and the vertical.

Static Friction around a Corner

Going around a flat corner, the static friction acts as the centripetal force. In physics we don't consider other effects which are why you typically accelerate around a bend when driving.

Summary: $\mu_s N \geq \frac{mv^2}{r}$

Change in Potential Energy in a Gravitational Field

By definition, any change is the final state minus the initial state:

$\Delta U = U_f - U_i$

Given $U_G = -\frac{Gm_1m_2}{r}$, $\Delta U = -Gm_1m_2\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$

This ΔU will also return a positive or negative value, if it is positive, work has been done against the field. If it is negative, then the field has done work.

The final useful definition is that $-\Delta U = \Delta E_K$

Harder Potential Energy Concepts

Applying this formula is harder when the concept of '*altitude*' is introduced. When the word *altitude* is in the question it is implying that $r = r_E + a$, where a is the altitude.

Some other problems also introduced where energy is lost to something like air resistance as heat. In this instance the heat energy lost is subtracted from the kinetic energy gained due to the loss of potential energy i.e. $\Delta E_k = -\Delta U - Q$ (The increase in kinetic energy is the decrease in potential energy minus the heat energy)

Orbital Motion in a Gravitational Field

In orbital motion, the gravitational force is the centripetal force ($F_G = F_c$)

i.e. $G\frac{m_1 m_2}{r^2} = \frac{mv^2}{r}$

Geostationary Satellites

The definition of a geostationary satellite is that it orbits with a period of 24hrs (86400s). Every other fact about geostationary satellites is consistent with circular motion in a gravitational field.

Formula Derivations

Kepler's Third Law

$F_G = F_c$

$$G\frac{m_1 m_2}{r^2} = \frac{m_2 v^2}{r}$$

$$\frac{Gm_1}{r} = v^2$$

$$v = \frac{2\pi r}{T}$$

$$\frac{Gm_1}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$\frac{Gm_1}{r^3} = \frac{4\pi^2}{T^2}$$

$$\frac{Gm_1}{4\pi^2} = \frac{r^3}{T^2}$$ *Q.E.D.*

Escape Velocity

$E_K = -U_G$

$$\frac{1}{2}mv^2 = G\frac{Mm}{r}$$

$$v^2 = \frac{2GM}{r}$$

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$ *Q.E.D.*

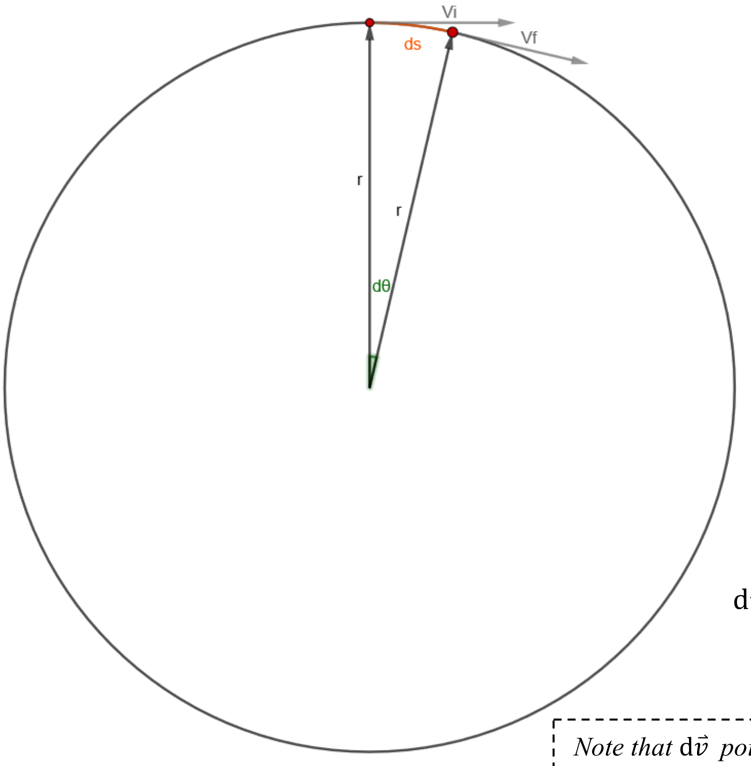
Velocity of a Satellite

$F_c = F_G$

$$\frac{mv^2}{r} = G\frac{Mm}{r^2} \quad \Rightarrow \quad v^2 = \frac{GM}{r} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{r}}$$ *Q.E.D.*

Extension Notes

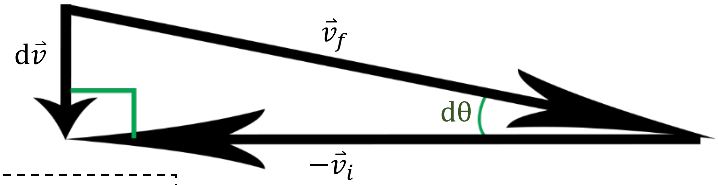
Derivation of Centripetal Acceleration



Let $d\theta$ be the angle traced by the object in some infinitesimally small time interval dt, and ds is the distance travelled.

Also: $|\vec{v}_i| = |\vec{v}_f| = v$

$$\mathrm{d}\vec{v} = \vec{v}_f - \vec{v}_i$$



Note that $\mathrm{d}\vec{v}$ points inward (i.e. $d\vec{v} = (-\hat{r})dv$).

Even though the two vectors are equal, because $d\theta$ is infinitesimally small, it is still a right-angled triangle.

$$\sin d\theta = d\theta = \frac{\mathrm{d}\vec{v}}{\vec{v}_f}$$

Sine approximation holds true for infinitesimally small $d\theta$.

$$\sin d\theta = d\theta = \frac{ds}{r}$$

Derived from the circle diagram where the distance travelled is perpendicular to $\vec{r}$.

$$\therefore \frac{\mathrm{d}\vec{v}}{\vec{v}_f} = \frac{ds}{r}$$

$$\mathrm{d}\vec{v} = ds\frac{\vec{v}_f}{r}$$

$$d\vec{v} = ds\frac{v}{r}$$

$$\frac{d\vec{v}}{dt} = \frac{ds}{dt}\frac{v}{r} = v(-\hat{r})\frac{v}{r}$$

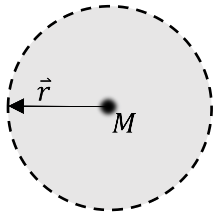
$\mathrm{d}\vec{v} = (-\hat{r})dv$

$$\vec{a_c} = -\frac{v^2}{r}\hat{r}$$

Q.E.D.

Derivation of Newton’s Law of Gravity with Gauss’ Law

We begin by considering a point mass with mass M, surrounded by an invisible sphere of radius r.



Since the point mass is identical when observing it from any point on the sphere it is spherically symmetric.

Now we consider Gauss’ law, where A is the area of the invisible sphere:

$$\oiint \vec{g} \cdot d\vec{A} = -4\pi G M$$

Because the object is spherically symmetric, and we are considering the area of a sphere, the field is constant on all points on the sphere. Therefore, we can remove it from the integral as a constant (though we will have to remove the vector lines to undo the dot product):

$$g \oiint dA = -4\pi G M$$

$$g\,A = -4\pi G M$$

$$g\,4\pi r^2 = -4\pi G M$$

$$g = -G\frac{M}{r^2}$$

Derivation of the Moment of Inertia of a disk rotating about its centre

Let the disk have some mass m, mass per area λ, and radius R.

The origin will be the centre of rotation which is the centre of the disk

$$I = \int_0^m r^2\,dm$$

$$= \int_0^R R^2\,\lambda\,2\pi R\,dR$$

$$= \lambda\,2\pi \int_0^R R^3\,dR$$

$$= \lambda\frac{2\pi}{4}R^4$$

$$= \frac{1}{2}\lambda(\pi R^2)R^2 = \frac{1}{2}\lambda A\,R^2$$

$$I = \frac{1}{2}mR^2$$

$$m = \lambda A = \lambda\,\pi R^2$$

$$\frac{dm}{dR} = \lambda\,2\pi R$$

$$dm = \lambda\,2\pi R\,dR$$

$$\lambda A = m$$

Radians as a Unit

Most people refer to Radians as rad and as a unit for rotating objects. This is not incorrect, but it is important to know what it means to do this. Many people say a number n has no units and is just a number because it says how many things there are. Similarly, rad denotes how many radians there are. As a result, radians are not technically a unit.

This is why when angular velocity (ω) is multiplied by a radius (r) it has units $m\ s^{-1}$ not $m\ rad\ s^{-1}$

Interestingly this is also why, despite having units Nm, torque is not energy, but its path integral is. Since torque is $\tau = rF$ it must be multiplied by the angle to be energy (since work is the energy along the path and $r\theta$ is the path length). Notice how despite multiplying by an angle, it retains the units $Nm\ or\ J$.