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Pre-Module I: Linear Algebra

There is lots of vector maths associated with what is typically Linear Algebra that is used quite often in physics. The relevant maths is covered here, though the topic of Linear Algebra is far more expansive than what is covered here.
Since the most common dimension set dealt with in physics is 3D this shall be what is explicitly covered here. Though it is worth noting that, for the most part, the definitions carry to further dimensions in the way that you would expect.

Coordinate & Vector Notation

$\mathbb{R}^n$ Notation

This may seem odd at first but this notation is actually pretty simple. Real numbers are numbers with a decimal place (potentially all 0s) and the set of all real numbers is $\mathbb{R}$. $\mathbb{R}^n$ just means the set of all vectors with n real number components.
There is also the set of all complex numbers $\mathbb{C}$ and $\mathbb{C}^n$ would mean vectors with n complex components.

Technically $\mathbb{R}^n$ is actually something called a *vector space* and has a predefined set of rules, but that's a bit much formal maths for this.

3D Point Notation

Points in a 3D system are typically referred two as existing in $\mathbb{R}^3$ (they have an x, y and z coordinate and each of them is a decimal number / not complex). A point is denoted by comma-separated numbers enclosed in round brackets e.g. (1, 2, 3).

3D Vector Notation

A vector is the same, except a vector "points" to a point. Vectors are denoted most often using 3 numbers written vertically inside square brackets, though sometimes they may be written inside round brackets, or horizontally like a point except in square brackets. To maintain consistency, only square brackets will be used here to refer to vectors.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad [x,\, y,\, z]\,, \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

A vector can also be represented algebraically using an arrow, a harpoon, a squiggle underneath, or even in bold.

$$\vec{v}, \qquad \overrightharpoon{v}, \qquad \underset{\sim}{v}, \qquad \mathbf{v}$$

This is not to say that the vector cannot be written in component form, this is just a simplified way of representing it. They are often written this way to reduce the amount of working needed.

$$\vec{\mathbf{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

The Zero Vector

The zero vector is denoted by $\vec{\mathbf{0}}$ and represents the vector of all zeros.

$$\vec{\mathbf{0}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{I.1}$$

Vectors Between Points

Let's say you have two points A and B and you want to draw a vector that points from point A to point B. To denote this you simply write $\overrightarrow{AB}$.
Let's also say that the points have vectors pointing to them $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

$$\vec{\mathbf{a}} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

$$\vec{\mathbf{b}} = \begin{bmatrix} B_x \\ B_y \end{bmatrix}$$

To calculate what this vector is, we do $final - initial$ and, since this vector starts at A and ends at B, we get Eq. I.2
(See §I: Addition & Subtraction for more information on vector subtraction)

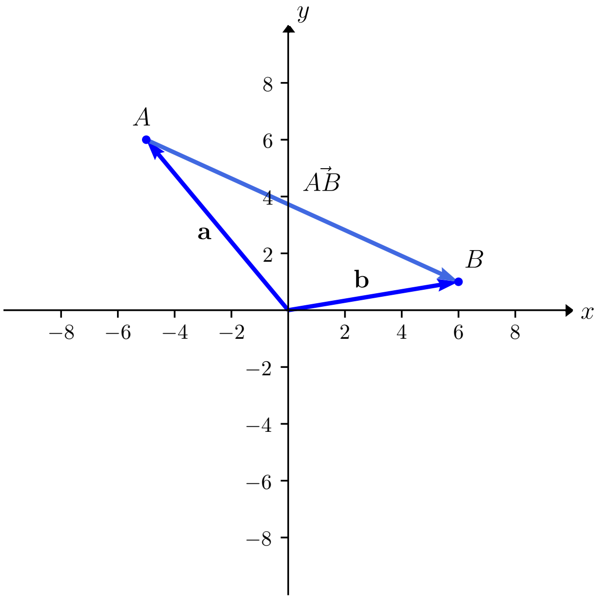
$$\overrightarrow{AB} = final - initial = \vec{\mathbf{a}} - \vec{\mathbf{b}} \tag{I.2}$$

The cool thing about vectors is that they point at something, but they do not have a fixed starting point, in some sense they are free to float around the coordinates so long as they still have the same components.

This can be seen in Fig. I.1

It might now also be more obvious that when we talk about a vector $\vec{\mathbf{a}}$ we are also essentially talking about a vector pointing to the point A from the origin. So the final point is V and the initial is 0

$$\vec{\mathbf{a}} = final - initial = \vec{\mathbf{a}} - \vec{\mathbf{0}} \tag{I.3}$$



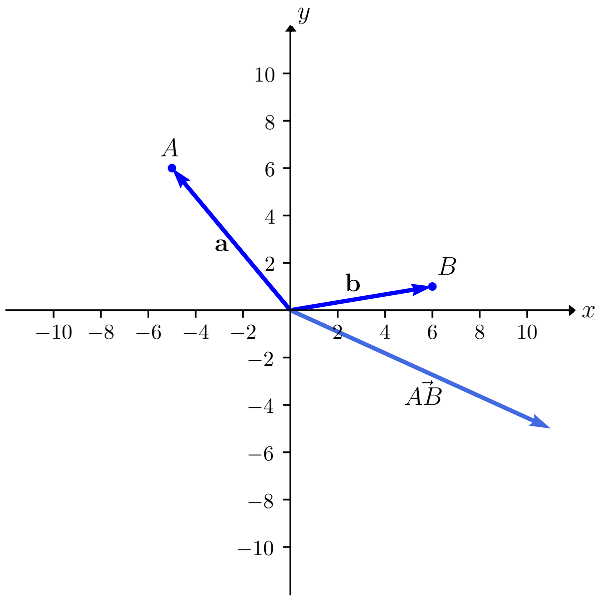


Figure I.1: Diagrams showing the vector between points A and B.

Vector Operations

Scalar Multiplication

Vectors can be multiplied by a real scalar (a number) such that Eq. I.4 is satisfied.

$$\gamma\vec{\mathbf{v}} = \gamma \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} := \begin{bmatrix} \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{bmatrix}, \quad \gamma \in \mathbb{R} \tag{I.4}$$

(Dividing is also fine since you could write $\gamma = \frac{1}{2}$)

Properties of Scalar Multiplication

Scalar multiplication is distributive with addition.

$$(\gamma + \mu)\vec{\mathbf{v}} = \gamma\vec{\mathbf{v}} + \mu\vec{\mathbf{v}} \tag{I.5}$$

The order that you multiply also doesn't matter.

$$\gamma(\mu\vec{\mathbf{v}}) = (\gamma\mu)\vec{\mathbf{v}} \tag{I.6}$$

Addition & Subtraction

Vectors can be added and subtracted e.g. $\vec{\mathbf{v}}+\vec{\mathbf{u}}$ or $\vec{\mathbf{v}}-\vec{\mathbf{u}}$ and this is represented component-wise as

$$\vec{\mathbf{v}} + \vec{\mathbf{u}} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} := \begin{bmatrix} v_x + u_x \\ v_y + u_y \\ v_z + u_z \end{bmatrix} \tag{I.7}$$

Subtraction can also be thought of as adding the negative of another vector i.e. $\vec{\mathbf{v}}-\vec{\mathbf{u}} = \vec{\mathbf{v}}+(-\vec{\mathbf{u}})$

Properties of Addition

Addition is what is called commutative (order doesn't matter). Subtraction is not though.

$$\vec{\mathbf{v}} + \vec{\mathbf{u}} = \vec{\mathbf{u}} + \vec{\mathbf{v}} \tag{I.8}$$

$$\vec{\mathbf{v}} - \vec{\mathbf{u}} = -\vec{\mathbf{u}} + \vec{\mathbf{v}} \tag{I.9}$$

Addition of vectors is what we call associative, just like normal addition (i.e. the order of bracket operations doesn't matter).

$$\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} \tag{I.10}$$

Subtraction is as well, as long as you group the negative sign with the vector.

$$\vec{\mathbf{a}} + (\vec{\mathbf{b}} - \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) - \vec{\mathbf{c}} = -\vec{\mathbf{c}} + (\vec{\mathbf{a}} + \vec{\mathbf{b}}) \tag{I.11}$$

Scalar multiplication is distributive across addition (it works like normal expanding).

$$\gamma(\vec{\mathbf{v}} + \vec{\mathbf{u}}) = \gamma\vec{\mathbf{v}} + \gamma\vec{\mathbf{u}} \tag{I.12}$$

Adding Vectors Visually

It is also possible to perform component-wise addition and subtraction with vectors visually. To add two vectors simply place the tail of one vector on the tip of the other vector. For instance, with $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ you could move $\vec{\mathbf{b}}$ so that its tail is on the tip of $\vec{\mathbf{a}}$. The important step is that the first vector must have its tail on the origin.

You can perform subtraction this way but you must also remember that the order matters with subtraction. The way to get around this is to add the negative version of the vectors (and since order doesn't matter with addition you can then do this in any order).

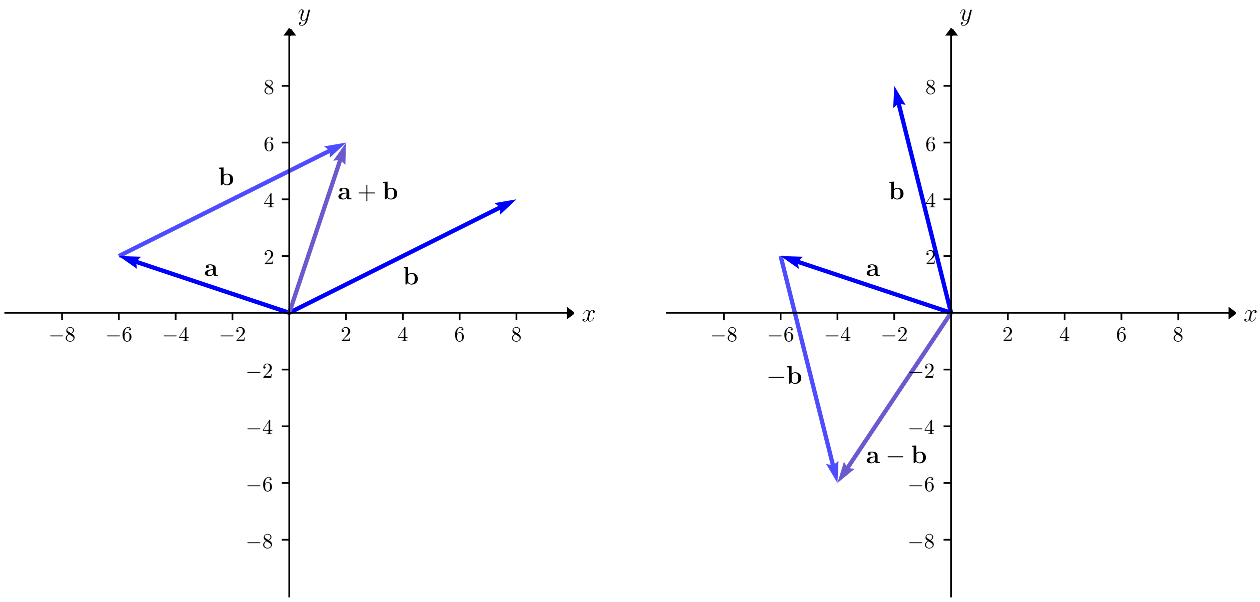


Figure I.2: Diagrams showing visual vector addition and subtraction of vectors **a** and **b**.

Modulus

Vectors are said to have “magnitude and direction” since they point somewhere and have a length. The modulus of a vector returns this size. For 2D and 3D it is derived from pythagoras' theorem which gives the length of the hypotenuse, though for higher dimensions it is assumed that the formula continues to work. It is denoted by either one or two sets of absolute value signs.

$$\|\vec{\mathbf{v}}\|\,, \qquad |\vec{\mathbf{v}}|$$

Sometimes, if there is a vector such as radial position $\vec{\mathbf{r}}$, the modulus will just be denoted without the arrow i.e. r (though this is really a definition of $\|\vec{\mathbf{r}}\| = r$).

The modulus function is defined in Eq. I.13

$$\|\vec{\mathbf{v}}\| := \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{I.13}$$

For vectors in $\mathbb{R}^n$, this is defined by Eq. I.14

$$\|\vec{\mathbf{v}}\| := \sqrt{\sum_{i=1}^{n} v_i^2} \tag{I.14}$$

Properties of Modulus

It is important to be careful when using modulus, as modulus does not carry through calculations as one might initially expect. It should be obvious when considering the formula that these properties are true, but it is still good to be careful.

$$\|\gamma\vec{\mathbf{v}}\| = |\gamma|\,\|\vec{\mathbf{v}}\| \tag{I.15}$$

$$\|\vec{\mathbf{v}} + \vec{\mathbf{u}}\| = \|\vec{\mathbf{v}}\| + \|\vec{\mathbf{u}}\| \,:\, \vec{\mathbf{u}} = \gamma\vec{\mathbf{v}} \tag{I.16}$$

Eq. I.16 highlights the fact that modulus is only additive if two vectors are scalar multiples of each other.

Unit Vectors

Unit vectors are vectors with a *unit* length (1). In that sense they have only a direction. In physics this is even more true since a unit vector also has no dimension, because it is defined as being divided by its own length (Eq. I.17). A unit vector is denoted with a “hat” above the vector symbol when written algebraically i.e. $\hat{\mathbf{v}}$.

$$\hat{\mathbf{v}} := \begin{cases} \vec{\mathbf{0}} & \|\vec{\mathbf{v}}\| = 0 \\ \dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} & \|\vec{\mathbf{v}}\| \neq 0 \end{cases} \tag{I.17}$$

If the vector is not the zero vector, then the length is always 1.

$$\|\hat{\mathbf{v}}\| = 1 \tag{I.18}$$

The Dot Product

The dot product (sometimes called the scalar product) of two vectors is written as $\vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$.
It produces a scalar result and the definition is given in Eq. I.19.

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \cdot \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} := v_x u_x + v_y u_y + v_z u_z \tag{I.19}$$

In $\mathbb{R}^n$ this is defined by Eq. I.20

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} := \sum_{i=1}^{n} v_i u_i \tag{I.20}$$

The dot product is also related to trigonometry by the identity in Eq. I.21

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = \|\vec{\mathbf{v}}\| \, \|\vec{\mathbf{u}}\| \cos\theta \tag{I.21}$$

Where θ is the angle between the two vectors.
In this equation it should be clear that the dot product also measures how parallel two vectors are, as when the vectors are completely parallel then $\cos\theta = 1$ and if they are perpendicular then $\cos\theta = 0$.

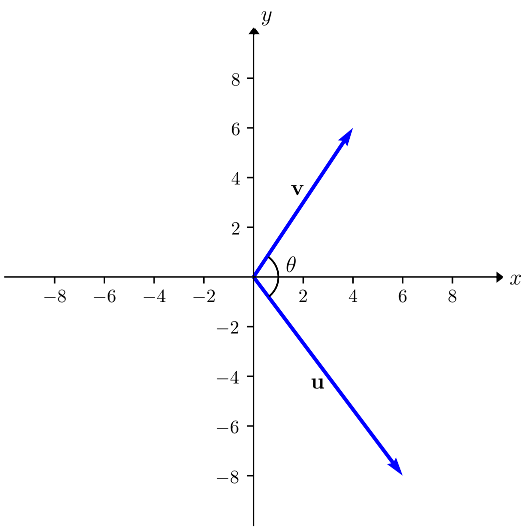


Figure I.3: A diagram showing two vectors **v** and **u**, and the angle between them.

Properties of the Dot Product

The dot product still obeys most of the standard rules of multiplication.

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \tag{I.22}$$

$$\gamma(\vec{\mathbf{v}} \cdot \vec{\mathbf{u}}) = (\gamma\vec{\mathbf{v}}) \cdot \vec{\mathbf{u}} = \vec{\mathbf{v}} \cdot (\gamma\vec{\mathbf{u}}) \tag{I.23}$$

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} \tag{I.24}$$

It also gives the vector modulus.

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2 \tag{I.25}$$

The Cross Product

The cross product is more complicated and the formula can be a little more confusing so, if you aren't too sure what's going on, don't worry yet.
The cross product takes two vectors and considers how *orthogonal* (perpendicular) they are. If they are fully orthogonal then the cross product is at its maximum size, if they are parallel then it is zero.

The cross product is written as $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ and is defined in Eq. I.26

$$\vec{\mathbf{v}} \times \vec{\mathbf{u}} \coloneqq \begin{bmatrix} v_y u_z - v_z u_y \\ v_z u_x - v_x u_z \\ v_x u_y - v_y u_x \end{bmatrix} \tag{I.26}$$

The cross product does not exist in any general $\mathbb{R}^n$, just in $\mathbb{R}^3$ and, for some reason, $\mathbb{R}^7$.

The magnitude of the cross product reflects the statements about orthogonality from earlier.

$$\|\vec{\mathbf{v}} \times \vec{\mathbf{u}}\| = \|\vec{\mathbf{v}}\| \, \|\vec{\mathbf{u}}\| \, |\sin\theta| \tag{I.27}$$

The direction of the final vector is given by the right hand rule (Fig. I.4). The rule is generally applicable in $\mathbb{R}^3$ with any cross product of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$. You point your pointer finger in the direction of $\vec{\mathbf{a}}$ and then your middle finger in the direction of $\vec{\mathbf{b}}$ and your thumb points in the direction of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$.

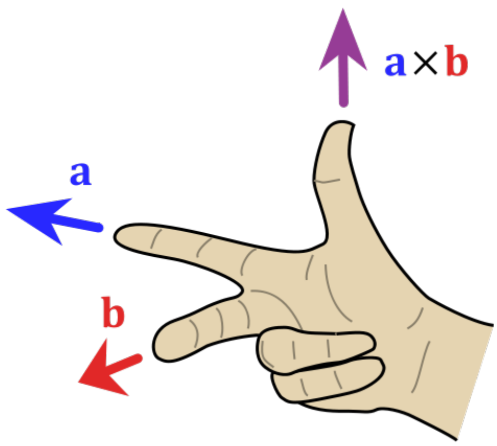


Figure I.4: A diagram showing the right hand rule.
Source: Acdx, CC BY-SA 3.0

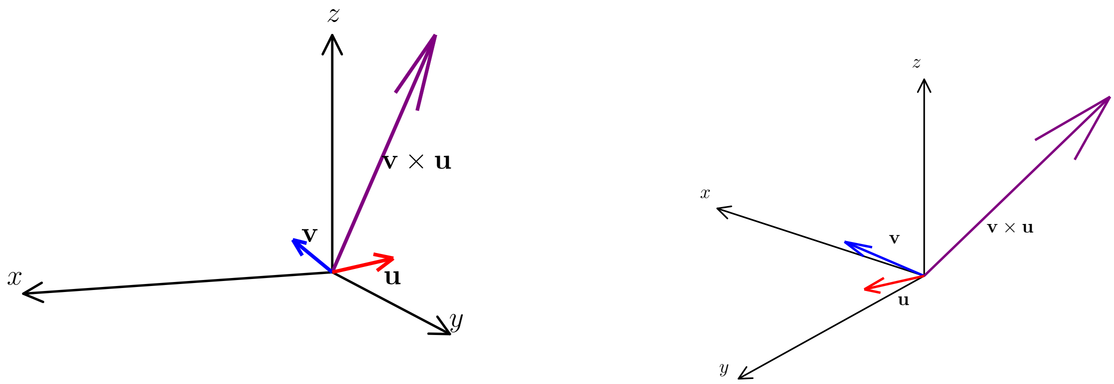


Figure I.5: Diagrams showing the cross product of two vectors $\mathbf{v}$ and $\mathbf{u}$.

Properties of the Cross Product

The cross product has some strange identities associated with it, primarily because it is not commutative or associative.

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -(\vec{\mathbf{b}} \times \vec{\mathbf{a}}) \tag{I.28}$$

$$\gamma(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = (\gamma\vec{\mathbf{a}}) \times \vec{\mathbf{b}} = \vec{\mathbf{a}} \times (\gamma\vec{\mathbf{b}}) \tag{I.29}$$

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}} \tag{I.30}$$

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = \vec{\mathbf{b}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = 0 \tag{I.31}$$

You might recall that the area of a triangle with side lengths a, b, c and opposite angles A, B, C has an area $A = \frac{1}{2}ab\sin C$ (where C is the angle between the sides a and b).
We can instead look at Fig. I.3 or Fig. I.6 and think of these sides as $\mathbf{v}$ and $\mathbf{u}$, with the angle between them as θ.
By Eq. I.27 we then know that the magnitude of the cross product of these two vectors gives double the area of this triangle. Therefore, the magnitude of the cross product gives *double the area of the triangle* inscribed by the vectors or *the area of the parallelogram* inscribed by the vectors (Fig. I.6).

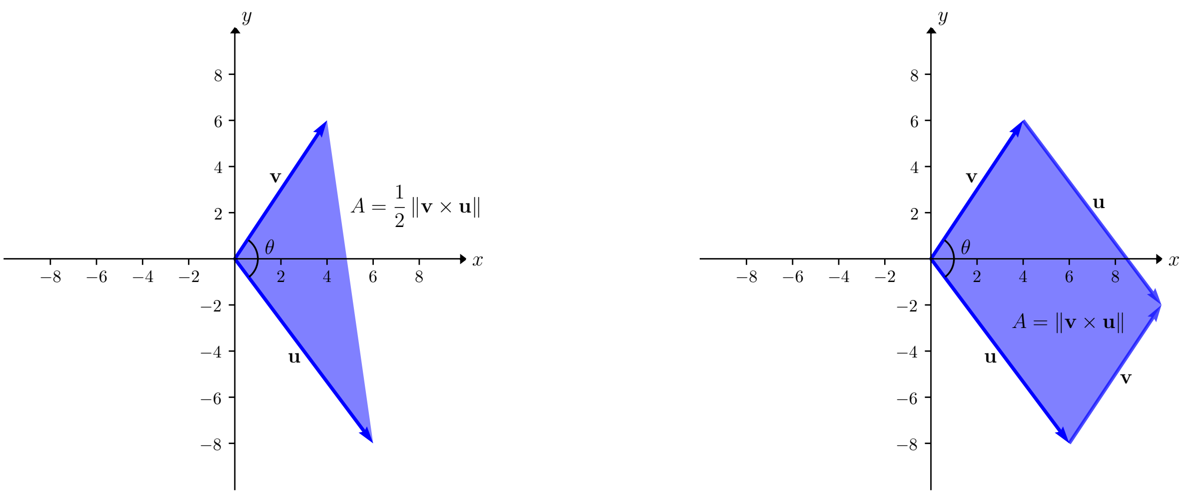


Figure I.6: Diagrams showing areas of the triangle and parallelogram inscribed by vectors $\mathbf{v}$ and $\mathbf{u}$ and the relationship to the cross product of two.

Some Useful Vector Identities

Here in this section we will simply sate some vector identities that can be rather tedious to prove.

$$\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = \vec{\mathbf{B}} \cdot (\vec{\mathbf{C}} \times \vec{\mathbf{A}}) = \vec{\mathbf{C}} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \tag{I.32}$$

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = (\vec{\mathbf{A}} \cdot \vec{\mathbf{C}})\vec{\mathbf{B}} - (\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})\vec{\mathbf{C}} \tag{I.33}$$

$$(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot (\vec{\mathbf{C}} \times \vec{\mathbf{D}}) = (\vec{\mathbf{A}} \cdot \vec{\mathbf{C}})(\vec{\mathbf{B}} \cdot \vec{\mathbf{D}}) - (\vec{\mathbf{A}} \cdot \vec{\mathbf{D}})(\vec{\mathbf{B}} \cdot \vec{\mathbf{C}}) \tag{I.34}$$

Matrices

Matrix Notations & Definitions

A matrix is similar to a vector, but instead of being a column of numbers, it is a rectangle of numbers.

Convention is to use uppercase letters to denote a matrix. To denote a matrix A that is m items tall and n items across we write $A_{m\times n}$. The element in the i^{th} row and j^{th} column is written with the lowercase letter i.e. a_{ij}.

We would write such a matrix as in Eq. I.35

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{I.35}$$

for example

$$A_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Sometimes to make it explicit that a variable is a matrix it will be written with square brackets around it i.e. $[A]$.

Definitions

A square matrix is a matrix of the form $A_{n\times n}$.

The n identity matrix I_n is a square matrix with width and height n that has 1s along its diagonal and 0s in all other entries.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{I.36}$$

Multiplying a Matrix by a Scalar

This works basically as one would expect. If you have a matrix $A_{m\times n}$ and multiply it by some scalar γ then every element is multiplied by γ.

For example

$$3A = 3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

Matrix Addition

To add two matrices together they must be of the same dimension (i.e. both must be $m \times n$ where m and n are the same for both matrices).

Matrix Multiplication

Multiplying two matrices together is far more complex than one would expect. It is very different to addition in that here the operation is not done component-wise.

We begin by setting some rules:

- To multiply two matrices together A and B, B must have the same number of rows as A has columns i.e. the matrices must be of the form $A_{m\times n}B_{n\times k}$
- The multiplication is not commutative (i.e. $AB \neq BA$), in fact, the multiplication of A and B in reverse order from above (i.e.$B_{n\times k}A_{m\times n}$) is not allowed unless $k = m$.
- Multiplication of the form $A_{m\times n}B_{n\times k}$ gives a third matrix $C_{m\times k}$.

Take the multiplication of the two matrices A and B produces a matrix C of the form $A_{m\times n}B_{n\times k} = C_{m\times k}$.

To find the ij^{th} term of C you, for lack of a better explanation, take the dot product of the i^{th} row of A with the j^{th} column of B.

For example:

$$A_{3\times 2}B_{2\times 3} = C_{3\times 3}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) & (a_{11}b_{13} + a_{12}b_{23}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) & (a_{21}b_{13} + a_{22}b_{23}) \\ (a_{31}b_{11} + a_{32}b_{21}) & (a_{31}b_{12} + a_{32}b_{22}) & (a_{31}b_{13} + a_{32}b_{23}) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} ((1)(2) + (2)(8)) & ((1)(4) + (2)(10)) & ((1)(6) + (2)(12)) \\ ((3)(2) + (4)(8)) & ((3)(4) + (4)(10)) & ((3)(6) + (4)(12)) \\ ((5)(2) + (6)(8)) & ((5)(4) + (6)(10)) & ((5)(6) + (6)(12)) \end{bmatrix} = \begin{bmatrix} 18 & 24 & 30 \\ 38 & 52 & 66 \\ 58 & 80 & 102 \end{bmatrix}$$

Left multiplication of an $m\times n$ matrix by I_m (Eq. I.36) or right multiplication by I_n is equivalent to multiplying by 1.

$$I_m A_{m\times n} = A_{m\times n} I_n = A_{m\times n} \tag{I.37}$$

Matrix Transposes

The transpose of a matrix A is denoted by A^T.

To get the transpose of A the n^{th} row of A becomes the n^{th} column of A^T (or the n^{th} column of A becomes the n^{th} row of A^T, either works).

The easiest way to remember this is '*Rows to columns, columns to rows.*'

For example:

$$\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right)^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Determinants

The determinant of a square matrix A is denoted either by $\det(A)$ or $|A|$.

Take an $n \times n$ matrix A. To use the formula for the determinant one must first make an arbitrary choice of either $1 \leq i \leq n$ or $1 \leq j \leq n$.

Some Required Notation

We denote a matrix A_{ij} as the matrix A with the i^{th} row and j^{th} column removed.

For example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

The formulas

If an arbitrary i was chosen then the formula is iterated through all j:

$$|A| \coloneqq \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij}\, |A_{ij}| \tag{I.38}$$

For an arbitrary j then the formula is iterated through all i:

$$|A| \coloneqq \sum_{i=1}^{n} (-1)^{i+j}\, a_{ij}\, |A_{ij}| \tag{I.39}$$

This formula is a rare example of a recursive formula, in that to define the determinant for an $n \times n$ matrix you need to know how to find the determinant of a $(n-1) \times (n-1)$ matrix.
It is worth noting that we define the determinant of a 1×1 matrix as just the number (i.e. the determinant of the matrix $[x]$ is x.)
Therefore this formula can then be used to define the determinant of a 2×2 matrix using this definition, then the 2×2 for a 3×3 etc.

Matrix Inversion

While you cannot divide by a matrix, in some cases you can multiply by another matrix which cancels out a matrix. This is called matrix inversion and the inverse of a matrix is denoted by a -1 power i.e. A^{-1}.
For a matrix to be invertible it **must** be a square matrix. A square matrix does not always have an inverse, but if it does then the following is true.

Let A be an $n \times n$ matrix that has an inverse:

$$A^{-1}A = AA^{-1} = I_n \tag{I.40}$$

There are many conditions that can be used to determine whether a matrix is invertible, but perhaps the simplest is the requirement that the determinant be non 0 (i.e. $|A| \neq 0$).

Provided this condition is met, the inverse is given by Eq. I.41.

$$A^{-1} = \frac{1}{|A|}\left(C_{ij}\right)^T \tag{I.41}$$

Here C_{ij} is the matrix where every ij^{th} element c_{ij} of the matrix is given by

$$c_{ij} = (-1)^{i+j}|A_{ij}| \tag{I.42}$$

Where A_{ij} is A with the i^{th} row and j^{th} column removed.

If a matrix is invertible then there is the identity that $|A^{-1}| = \frac{1}{|A|}$

Pre-Module II: Calculus

There are two areas of calculus that can make it difficult. One is the conceptual understanding and the other is the manual aspect of doing the maths.
There are plenty of both online and school textbook resources on calculus that will be infinitely more helpful for understanding how to take the derivative and how to integrate, so we won't try to cover those here.
Instead, we'll cover the key ideas of calculus.

More advanced resources such as those form university courses can be very useful but often use notation that is difficult to understand.
Hopefully this will arm you with enough knowledge on a variety of topics so that you will be able to understand the maths related to physics and, if you want, research them further.

Derivatives

A regular derivative is denoted in two different ways. The most common when dealing with functions in maths is to just add a dash above the function to say that it is the derivative.

$$f(x) \to f'(x) \to f''(x)$$

If you are giving more than the second derivative it is common to just write a number in place of the dashes.

$$f''(x) \to f^{(3)}(x)$$

The most common and most intuitive explanation of taking the derivative is the *slope of the graph* understanding.

This understanding is based on the formula in Eq. II.1 and basically says that the slope of the line between two points on a graph becomes the slope of the graph at the first point as you compress the second point ever closer to the first point (Fig. II.1).

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right) = \lim_{\Delta x \to 0} \left(\frac{\Delta f}{\Delta x} \right) \tag{II.1}$$

This is also where the second form of notation comes from. We signify the Δ getting infinitesimally small by writing it as a d.

$$f(x) \to \frac{d\,f(x)}{dx} \to \frac{d^2 f(x)}{dx^2}$$

The x on the bottom is actually helpful since it tells us what we are measuring the slope with respect to. In this case it's obvious but when we get to functions of time it is important to specify.

Having the d^2 on top but the x^2 on the bottom is also important. Every time we differentiate we take the difference of the function and then divide by the horizontal axis, therefore every time we differentiate we divide by x again. The power on the top shows that this is the n^{th} derivative and the power on the bottom tells us how many times we've had to divide by that variable (Eq. II.2).

$$\frac{d}{dx}(f') = \frac{d}{dx}\left(\frac{d}{dx}f\right) = \frac{d^2 f}{dx^2} \tag{II.2}$$

This may seem like an odd distinction but it's extremely important for units and also explains why the powers are where they are.

For instance, we say that x velocity is given by Eq. II.3

$$v_x = \frac{dx}{dt} \tag{II.3}$$

The units of velocity are $m\,s^{-1}$ because it is distance (x) divided by time (t).

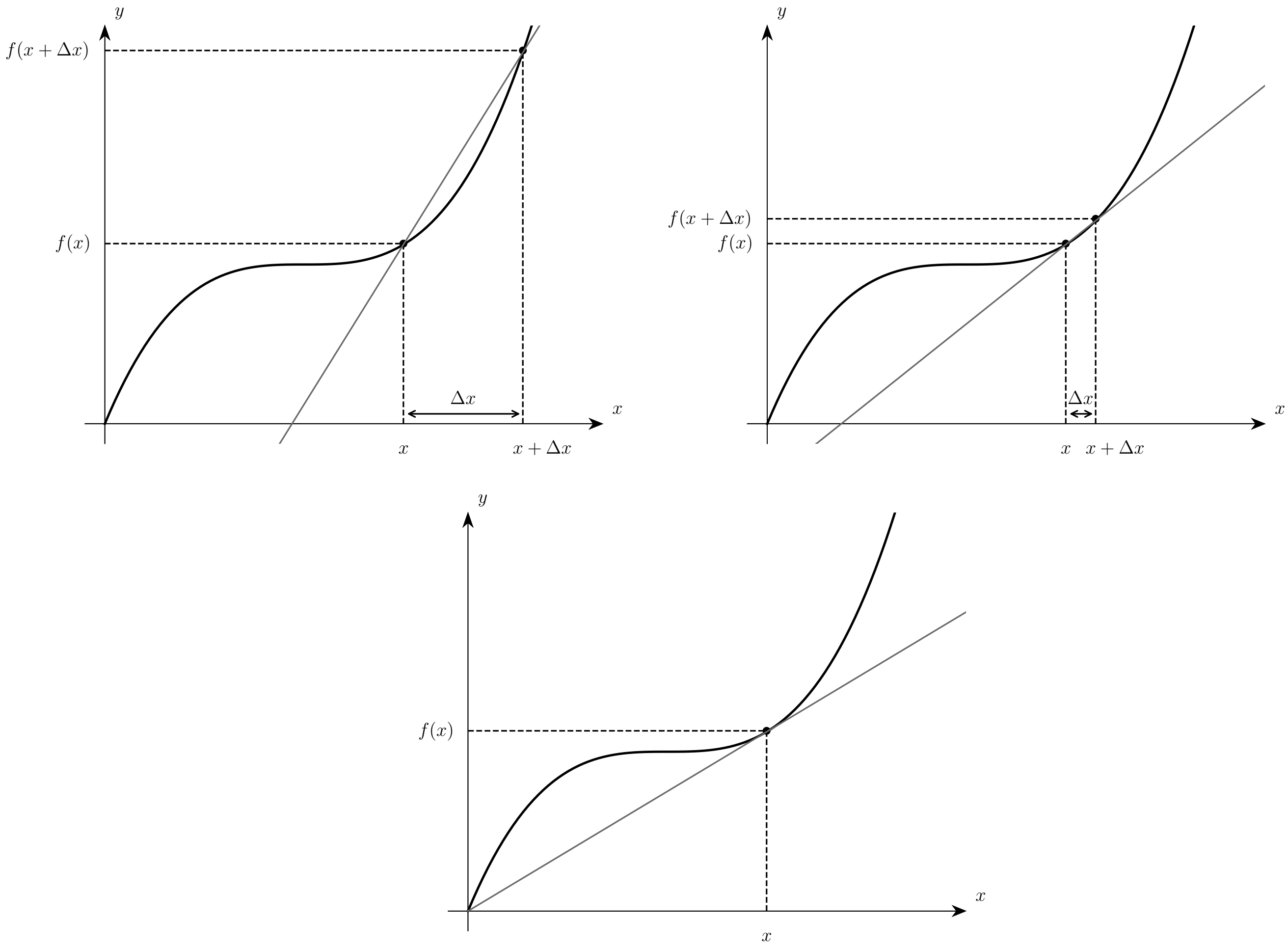


Figure II.1: Diagram showing the method of taking the derivative. The distance Δx is shortened until it is infinitesimally small such that the slope of the secant approaches ever closer to the slope of the graph.

Δx can be Negative

One important factor that is not reflected in Fig. II.1 is that Δx must be able to be negative or positive and have Eq. II.1 still work (§II: Undefined or Ambiguous Derivatives).

If there is a case where Δx being negative gives a different result than it being positive, e.g. $|x|$, about $x = 0$ (Fig. II.2) then the derivative is indeterminate at that given x.
This is due to the definition of a limit having to be equal to its left and right limits (§IV: Limits), so if the limit as $\Delta x \to 0$ of the slope is different depending on whether Δx is positive or negative then the derivative cannot be defined.

Undefined or Ambiguous Derivatives

There are some functions where, even though they are continuous, they are not smooth. In this case smooth means that the gradient of the graph is different depending on what side you approach a given point from.

For example, take $f(x) = |x|$ (Fig. II.2).

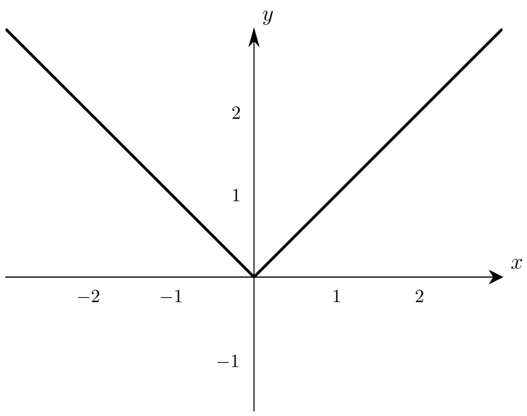


Figure II.2: A graph of $y = |x|$.

As we approach $x \to 0^-$ (from the left side) the gradient appears to be -1. but as we approach $x \to 0^+$ (from the right hand side) the gradient appears to be 1. This means that the derivative is undefined at $x = 0$.

Partial Derivatives (∂)

Partial derivatives are a bit strange when you first see them but believe me they're actually really simple.
When you do a normal derivative you do it on a function with respect to one variable. For instance you might be finding $f'(x)$.
But what if you had the function $f(x, t)$? What are you differentiating with respect to?

Your answer to this might be *"Well I'll just write it as $\frac{df}{dx}$, easy!"*
And, to be fair, you wouldn't be far off. But then when you differentiate do you just ignore t? Why should you, it also changes?

This is all solved by partial differentiation.

A partial derivative is written with curly d's.

$$\frac{\partial f}{\partial x} \tag{II.4}$$

You might say *"But why can't I just write it with a normal d?"* – a reasonable question.
The reason is that when you use a d you are claiming this is the complete derivative, but with the ∂ you are just saying *this is one differential, not the whole thing.*

Ok, time for an example.

Let's say your function is given by Eq. II.5

$$f(x,t) = ax^2 + bt^3 \tag{II.5}$$

To partially differentiate it with respect to x or t you just treat the other variable as a constant (like b is a constant in $y = mx + b$).

$$\frac{\partial f}{\partial x} = 2ax \tag{II.6}$$

$$\frac{\partial f}{\partial t} = 3bt^2 \tag{II.7}$$

Loss of Information

There are many functions in maths where, when you perform them, you lose information. Taking the derivative is one of these functions.
Once you lose information, it is impossible to get it back.

As an example, let's say that you know $f'(x) = 2x$. We know that differentiating $f(x) = x^2$ can get us this function, but so does $f(x) = x^2 + 5$, or $f(x) = x^2 + C$.

Differentiating loses information about constants.
Partial Differentiation loses information about constants and the other variables.

Integration

There are two main and very different interpretations of what integration is.

The first is that it is the anti-derivative, i.e. it undoes taking the derivative.
The second is the area under a curve and it is this idea that is more important to have a strong conceptual grasp on.

The method of finding the area is called a Riemann Sum or the Trapezoidal Rule. It involves finding the area of all the trapezoids of width Δx between $x = a$ and $x = b$ of some function, then squishing that Δx down to be infinitesimally small (Fig. II.3, Eq. II.8).

$$I = \lim_{\Delta x \to 0} \left(\sum_{x=a}^{b} \frac{f(x) + f(x + \Delta x)}{2} \Delta x \right) \tag{II.8}$$

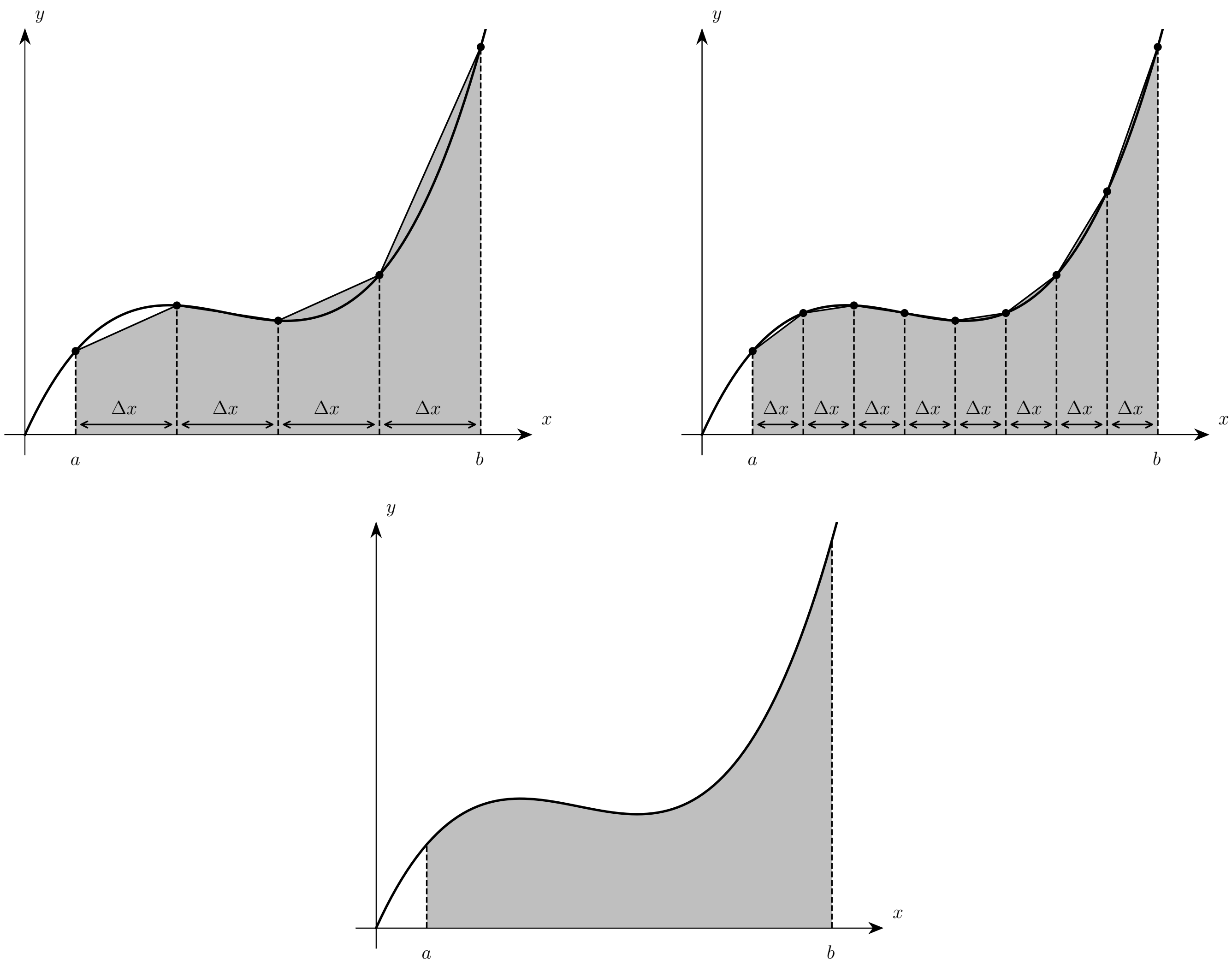


Figure II.3: Diagram showing the method of performing a Riemann Sum. The distance Δx is shortened until it is infinitesimally small such that the total area of the trapezoids becomes the area under the curve.

As we take the limit, $f(x+\Delta x) \to f(x)$, so it becomes Eq. II.9

$$I = \lim_{\Delta x \to 0} \left(\sum_{x=a}^{b} f\left(x\right) \Delta x \right) \tag{II.9}$$

$$I = \lim_{\Delta x \to 0} \left(\sum_{k=1}^{n} f\left(a + (k-1)\Delta x\right) k \Delta x \right), \qquad n = \frac{b-a}{\Delta x} \tag{II.10}$$

(Eq. II.10 is the more formal definition where we iterate by an integer up to n, with the length between a and b divided into n trapezoids of width Δx and then the areas are added.)

Then, since Sigma ($\sum$) means sum, we make the limit with the sum symbol an s-like symbol ($\int$) and make the Δ into a d (Eq. II.11)

$$I = \int_a^b f(x)\ dx \tag{II.11}$$

We say that the function describing the area under the curve $y = f(x)$ between $x = a$ and some movable point x is $F(x)$ (Eq. II.12).

$$F(x) = \int_a^x f(t)\ dt \tag{II.12}$$

It is not inherently obvious from this that integrating is the inverse of differentiation. The proofs are a bit tedious but it can be proven as generally true (Eq. II.13).

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\ dt = f(x) \tag{II.13}$$

Negative or Signed Area

When we spoke about the integral between two regions being the area under the curve, we neglected to consider what happens when the curve is below the x axis.

In this case the area below the axis is treated as 'negative'. The area isn't really negative but it must be treated this way for more practical uses of integration. For instance in physics we say that an object's position vector is given by the integral of its velocity vector with time (Eq. II.14)

$$\Delta x = \int_{t_1}^{t_2} v_x \; dt \tag{II.14}$$

In Fig. II.4 we see that there is an area above the horizontal (t) axis and an area below the horizontal axis. The total area is all of the positive area (the stuff above the axis) minus all of the negative area (the stuff below the axis).

In this case we can see that the areas are the same so the total integral is 0.
This is representative of the fact that the object will have the same position at each of these times.

The function used here was $v_x = \sin t$ with $t_1 = \frac{\pi}{2},\ t_2 = \frac{3\pi}{2}$.
Integrating manually gives the same result (as we would hope):

$$\begin{aligned}\Delta x &= \int_{t_1}^{t_2} v_x \; dt \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin t \; dt \\ &= [-\cos t]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -0 + 0 = 0\end{aligned}$$

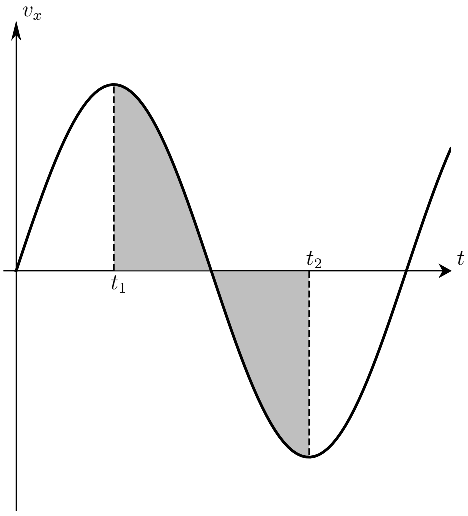


Figure II.4: Diagram of an integral of velocity with respect to time that has areas both above and below the horizontal axis.

Area vs Anti-Derivative

To make the distinction as to whether something is trying to find the area under a curve or trying to be an anti-derivative you have to look at the bounds.

If one of the bounds is moveable (i.e. x) or there are no bounds at all, then it is probably performing the function of an anti-derivative.
If it has fixed bounds (i.e. a and b) then it is finding the area under the curve between the bounds.

Integration as a Sum

You can also think of an integral as a sum. This is actually the case, it's why the integral sign is basically just an 'S' shape, since it's meant to be the sum of the infinitely many infinitesimal rectangular areas under the graph.

But sometimes it can also be a useful way to understand an integral which has more of a practical meaning, such as those found in physics.
For instance, in physics we say that a single point mass with mass m rotating at a radius r from the spin axis has a moment of inertia $I = mr^2$. Don't worry about what this is relating to for this case, but it is covered in more depth in §5: Advanced Mechanics.
But most things aren't a point mass, they have size or are at least made of multiple particles (which aren't really point masses but can be treated as point masses).
To find the total moment of inertia of the object we need to add up all of the moments of inertia of the object (Eq. II.15).

$$I_{total} = \sum mr^2 \tag{II.15}$$

But, when dealing with a continuous evenly distributed mass, we can't "find" all of the individual masses, and as we break up the mass into smaller and smaller pieces the mass term gets smaller and smaller but we get more and more masses, almost like it becomes the addition of infinitely many infinitesimal terms.

So, because the mass term becomes super small as we break up the mass it becomes the total sum of the small masses dm multiplied by their distance squared from the rotation axis r^2 (Eq. II.16)

$$I_{total} = \int_0^M r^2 \; dm \tag{II.16}$$

The bounds of 0 and M represent that we are finding the sum of all of the contributions from each of the small dm's over the whole object with mass M.

Total length of a Curve Using Integration

On the topic of using an integral as a sum, we can use an integral as a way of finding the total length of a function curve between two bounds.

Let's consider some function $f(x)$ that has a derivative that exists $f'(x)$ (there are some functions that don't).
In Fig. II.5 we can see a function such as this. Between two arbitrary points on that curve we have drawn the 'rise' (dy) and the 'run' (dx). With pythagoras' theorem we can see that the small piece of curve dl is given by Eq. II.17.
Though the diagram is exaggerated, as dx takes the limit to 0 (gets really small) the graph approaches being linear over that range and so the use of pythagoras's theorem is justified.

$$dl = \sqrt{dx^2 + dy^2} \tag{II.17}$$

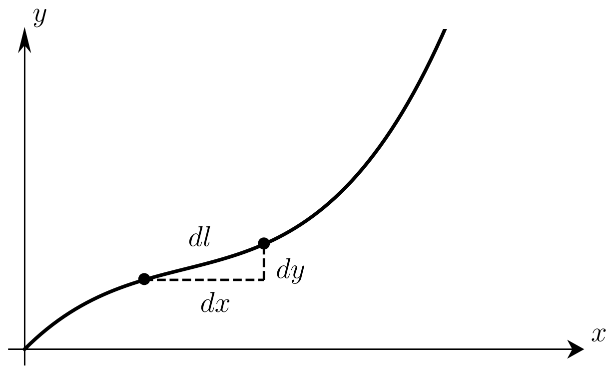


Figure II.5: Diagram showing the relationship between the small lengths dx & dy and the length of the curve over that step dl.

Rearranging Eq. II.17 gives us Eq. II.18

$$dl = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{II.18}$$

Then, taking the sum of all of the small lengths dl gives us Eq. II.19

$$\int_0^L dl = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx \tag{II.19}$$

Regaining Lost Information

In §II: Loss of Information it was mentioned how, when differentiating, you lose information about constants.
As a result, when we integrate without bounds (anti-differentiate) we have to remember to add $+C$ to the end of our final answer, since this represents the constant that we lost, and still do not know, when differentiating.

In that sense we can say that differentiation is the anti-integral but integration is an imperfect anti-differentiator.
We can say that Eq. II.20 is true, but we cannot say that Eq. II.21 is true.

$$f(x) = \frac{d}{dx}\int f(x)\ dx \tag{II.20}$$

$$f(x) \stackrel{?}{=} \int \left(\frac{d}{dx}f(x)\right) dx \tag{II.21}$$

The Meaning of an Integral in the Context of Physics

In physics it is very common to hear the statement *"velocity is the rate of change of position"*.

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{s}}}{dt}$$

The other way of thinking about this is that the velocity is the small change in position that occurs over a small period of time, divided by that time.

We can rearrange the formula to say that a small change in position over a small period of time is given by the velocity at that time, multiplied by the small change in time.

$$d\vec{\mathbf{s}} = \vec{\mathbf{v}}\, dt$$

This can be thought of as the (signed) area of the trapezoid under the curve $\vec{\mathbf{v}}(t)$ with width dt at some time t.
To find the total change of $\vec{\mathbf{s}}$ between the times t_1 and t_2 we just need to add up all of these small contributions from $\vec{\mathbf{v}}$, in other words add up all of the areas of the trapezoids.
An integral is shaped like an 'S' since it is a type of sum, just a sum of infinitely many infinitesimal pieces. So, to add up these infinitely many infinitesimal contributions from $\vec{\mathbf{v}}$ we need to integrate $\vec{\mathbf{v}}$ with respect to t.

Pre-Module III: Vector Calculus

There are really two forms of vector calculus, one is performing regular calculus operations on the vector algebra covered in §I: Linear Algebra, the other is operations on vector fields.

For the sake of simplicity, only vectors in $\mathbb{R}^3$ will be covered here.

Differentiation of Vectors

Since a large part of physics involves differentiating with respect to time, that is what will be covered here. However, basic rules of calculus apply if you were to replace t with something else.

$$\frac{d\vec{\mathbf{v}}}{dt} = \begin{bmatrix} \frac{d}{dt}v_x \\ \frac{d}{dt}v_y \\ \frac{d}{dt}v_z \end{bmatrix} \tag{III.1}$$

$$\frac{d}{dt}(\vec{\mathbf{v}} + \vec{\mathbf{u}}) = \frac{d\vec{\mathbf{v}}}{dt} + \frac{d\vec{\mathbf{u}}}{dt} \tag{III.2}$$

$$\frac{d}{dt}(\gamma\vec{\mathbf{v}}) = \frac{d\gamma}{dt}\vec{\mathbf{v}} + \gamma\frac{d\vec{\mathbf{v}}}{dt} \tag{III.3}$$

Special Differentiation Identities

Differentiating Vector Products

Each of the vector products follow the product rule. What is important to remember is that the order of the cross products matters and is kept the same as the original (Eq. III.5).

$$\frac{d}{dt}(\vec{\mathbf{v}} \cdot \vec{\mathbf{u}}) = \frac{d\vec{\mathbf{v}}}{dt} \cdot \vec{\mathbf{u}} + \vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{u}}}{dt} \tag{III.4}$$

$$\frac{d}{dt}(\vec{\mathbf{v}} \times \vec{\mathbf{u}}) = \frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{u}} + \vec{\mathbf{v}} \times \frac{d\vec{\mathbf{u}}}{dt} \tag{III.5}$$

Differentiating the Modulus

$$\frac{d\,\|\vec{\mathbf{v}}\|}{dt} = \frac{d}{dt}\sqrt{\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}}$$

$$= \frac{1}{2}\frac{1}{\sqrt{\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}}}\ \frac{d}{dt}(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}})$$

$$= \frac{1}{2}\frac{1}{\|\vec{\mathbf{v}}\|}\left(\frac{d\vec{\mathbf{v}}}{dt}\cdot\vec{\mathbf{v}} + \vec{\mathbf{v}}\cdot\frac{d\vec{\mathbf{v}}}{dt}\right)$$

$$= \frac{\frac{d\vec{\mathbf{v}}}{dt}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}$$

$$\frac{d\,\|\vec{\mathbf{v}}\|}{dt} = \frac{d\vec{\mathbf{v}}}{dt}\cdot\hat{\mathbf{v}} \tag{III.6}$$

Differentiating the Unit Vector

$$\frac{d\hat{\mathbf{v}}}{dt} = \frac{d}{dt}\left(\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}\right)$$

$$= \frac{\frac{d\vec{\mathbf{v}}}{dt}\,\|\vec{\mathbf{v}}\| - \vec{\mathbf{v}}\frac{d\,\|\vec{\mathbf{v}}\|}{dt}}{\|\vec{\mathbf{v}}\|^2}$$

$$= \frac{\frac{d\vec{\mathbf{v}}}{dt}\,\|\vec{\mathbf{v}}\| - \vec{\mathbf{v}}\left(\frac{d\vec{\mathbf{v}}}{dt}\cdot\hat{\mathbf{v}}\right)}{\|\vec{\mathbf{v}}\|^2} \qquad \text{(From Eq. III.6)}$$

$$\frac{d\hat{\mathbf{v}}}{dt} = \frac{\frac{d\vec{\mathbf{v}}}{dt} - \hat{\mathbf{v}}\left(\frac{d\vec{\mathbf{v}}}{dt}\cdot\hat{\mathbf{v}}\right)}{\|\vec{\mathbf{v}}\|} \tag{III.7}$$

$$\frac{d\hat{\mathbf{v}}}{dt} = \frac{\hat{\mathbf{v}}\times\left(\frac{d\vec{\mathbf{v}}}{dt}\times\hat{\mathbf{v}}\right)}{\|\vec{\mathbf{v}}\|} \tag{III.8}$$

Differentiation of Fields

Thinking back to §??:?? you should remember that there are two types of fields: scalar fields and vector fields.

Sometimes you will encounter a field which changes with respect to something like time. In this instance you can partially differentiate it to find its rate of change with time(i.e. $\frac{\partial}{\partial t}$).

In most cases however, you will be dealing with spatial derivatives (i.e. $\frac{\partial}{\partial x}$).

Differentiating Scalar Fields

Let's say that you have a scalar electric potential V field being generated by a proton with charge q, position $\vec{\mathbf{s}}$ and, for the sake of argument, let's give that proton a velocity $\vec{\mathbf{v}}$.

$$\vec{\mathbf{s}} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$$

$$\vec{\mathbf{s}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad t = 0$$

$$\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \vec{\mathbf{s}}$$

$$\frac{d\vec{\mathbf{r}}}{dt} = -\frac{d\vec{\mathbf{s}}}{dt} = -\vec{\mathbf{v}} \tag{III.9}$$

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{||\vec{\mathbf{r}}||} \tag{III.10}$$

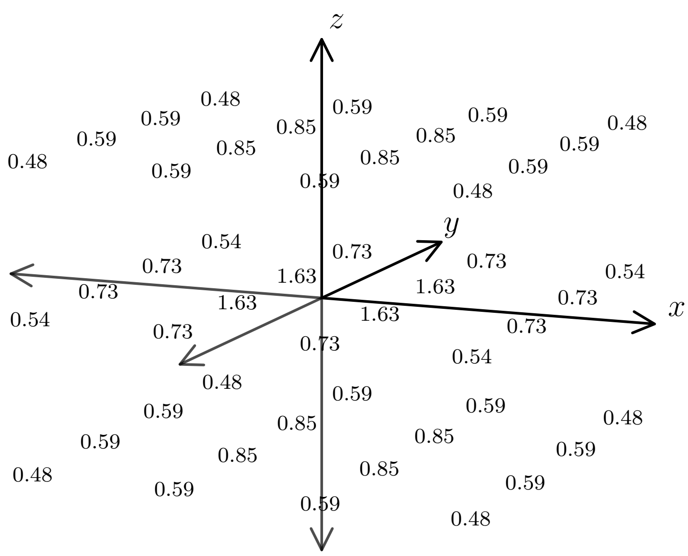


Figure III.1: A visual representation of the scalar field V (the numbers) described in Eq. III.10 where the position of the proton is the origin (i.e. $t = 0$).

Differentiating Scalar Fields with Respect to Time

It is reasonably simple to differentiate this field with respect to time using regular differentiation and some vector identities from §III: Differentiation of Vectors.

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\left(\frac{1}{\|\vec{\mathbf{r}}\|}\right) \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{-1}{\|\vec{\mathbf{r}}\|^2}\right)\frac{\partial\,\|\vec{\mathbf{r}}\|}{\partial t} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{\|\vec{\mathbf{r}}\|^2}\,\vec{\mathbf{v}}\cdot\hat{\mathbf{r}} && \text{(Eq. III.6 \& Eq. III.9)}
\end{aligned}$$

Since the unit vector of $\vec{\mathbf{r}}$ (position) is included in this equation it should be clear that this is also a scalar field. It is generally true that differentiating a scalar field with respect to time also produces a scalar field, since you are essentially differentiating all of the values for every position with respect to time.

The same general process of differentiating applies to all scalar fields, just differentiate using the chain rule until you're done.
(Admittedly this is a relatively simple example since velocity is constant.)

The $\vec{\boldsymbol{\nabla}}$ Operator

It is more than reasonable to differentiate a field with respect to one of the spacial coordinates (e.g. $\frac{\partial}{\partial x}$) or even the radial coordinate (i.e. $\frac{\partial}{\partial r}$: $r = \|\vec{\mathbf{r}}\|$).
However, here we will talk about the **gradient** operator. The operator itself comes in multiple different forms, but only this one is relevant to scalar fields.

The operator takes a scalar field and creates a new vector field, with the vector representing the gradient of the field strength across space.

$$\vec{\boldsymbol{\nabla}} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \tag{III.11}$$

The **gradient** operator can also be written in terms of other coordinate systems e.g. spherical coordinates (r, θ, ϕ). For more coordinate systems see https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates.

$$\vec{\boldsymbol{\nabla}} = \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{III.12}$$

Just as an example, to use this operator, you fill in all of the partial operators with the field.

$$\vec{\boldsymbol{\nabla}} V = \begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{bmatrix} = \frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} \tag{III.13}$$

It is actually more practical in this case to apply the spherical coordinate version here. If we apply the form in Eq. III.12 to our V field from Eq. III.10, we get the following result.

$$\begin{aligned} \vec{\boldsymbol{\nabla}} V &= \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}} \\ &= \left(\frac{q}{4\pi\varepsilon_0}\right)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\right)\hat{\boldsymbol{\phi}}\right] \end{aligned}$$

The equation does not depend on θ or ϕ, so those derivatives go to zero.

$$\vec{\boldsymbol{\nabla}} V = \left(\frac{q}{4\pi\varepsilon_0}\right)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} + \frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right)}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\phi}\left(\frac{1}{r}\right)}\hat{\boldsymbol{\phi}}\right]$$

$$\vec{\boldsymbol{\nabla}} V = \left(\frac{q}{4\pi\varepsilon_0}\right)\left[\frac{-1}{r^2}\right]\hat{\mathbf{r}}$$

$$\vec{\boldsymbol{\nabla}} V = \frac{-1}{4\pi\varepsilon_0}\,\frac{q}{\|\vec{\mathbf{r}}\|^2}\hat{\mathbf{r}} \tag{III.14}$$

And, just like that, we have a new vector field as a function of $\vec{\mathbf{r}}$.
(It also isn't a coincidence that this looks a lot like the formula for the electric field around a charged particle)

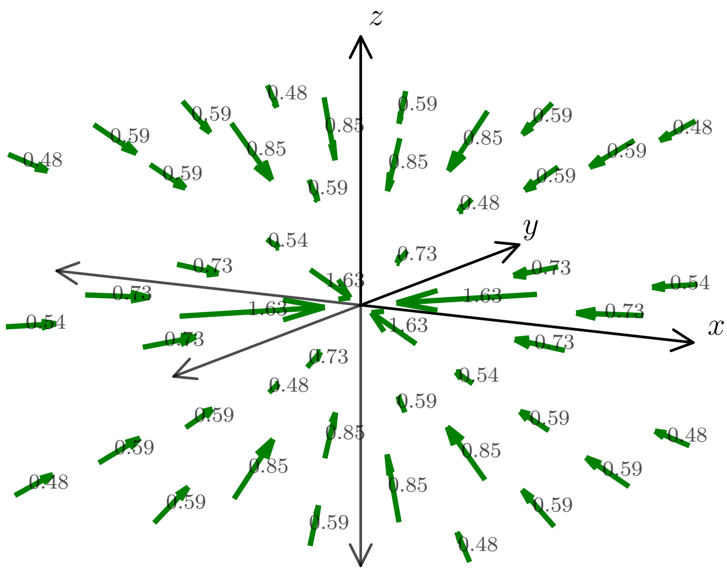


Figure III.2: A visual representation of the vector field $\vec{\boldsymbol{\nabla}} V$ (green) described in Eq. III.14 where the position of the proton is the origin. The original scalar field is shown as well.

Differentiating Vector Fields

You'll notice that when we used the gradient operator on the scalar field, we differentiated it with respect to some variable and multiplied by the unit vector of that variable (Eq. III.11). However, when we differentiate a vector we still get a vector, so we can't just differentiate a vector field like this.

Instead, there are special operators closely linked to the dot and cross products that we use to differentiate vector fields.

The $\vec{\nabla}\cdot$ Operator

The **divergence** operator tells you how much a vector field changes its strength along the direction that it points.

The $\cdot$ in the operator should remind you of the dot product and that's because these are conceptually very similar. While the dot product asks *how much of one vector is in the direction of the other?*, the divergence is asking *how much does the field change when I travel in the direction it is pointing?*

If we use the divergence operator on some field $\vec{\mathbb{F}}$, we get Eq. III.16.

Just like the dot product produces a scalar result, the divergence of a field is also a scalar result.

$$\vec{\nabla}\cdot\vec{\mathbb{F}} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \mathbb{F}_x \\ \mathbb{F}_y \\ \mathbb{F}_z \end{bmatrix} \tag{III.15}$$

$$\vec{\nabla}\cdot\vec{\mathbb{F}} = \frac{\partial \mathbb{F}_x}{\partial x} + \frac{\partial \mathbb{F}_y}{\partial y} + \frac{\partial \mathbb{F}_z}{\partial z} \tag{III.16}$$

Notice how, in Eq. III.15, the result is the same as taking the dot product with the $\vec{\nabla}$ operator and the field vector? This is the literal application of the dot product analogy.

The divergence operation also has versions in other coordinate systems. For more information see `https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates`.

Below is an example of a divergent field given by Eq. III.17

$$\vec{\mathbb{F}} = \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^{\frac{3}{2}}} \tag{III.17}$$

First we'll look at this field (Eq. III.17) in $\mathbb{R}^2$:

$$\vec{\mathbb{F}} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}^{\frac{3}{2}}} \\ \dfrac{y}{\sqrt{x^2+y^2}^{\frac{3}{2}}} \end{bmatrix}, \qquad \vec{\mathbb{F}} \in \mathbb{R}^2 \tag{III.18}$$

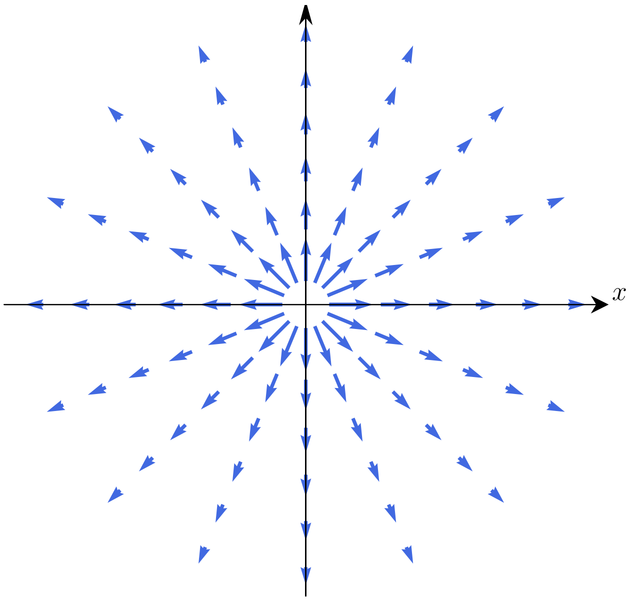


Figure III.3: A visual representation of the field $\vec{\mathbb{F}}$ described in Eq. III.18

The divergence of this field is therefore given by Eq. III.19

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbb{F}} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)^{\frac{3}{4}}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2)^{\frac{3}{4}}}\right) \tag{III.19}$$

The result is below. For the sake of space working isn't included but one can verify the result themselves (or check WolframAlpha because that is the strategic way).

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbb{F}} = \frac{2y^2 - x^2}{2(x^2+y^2)^{\frac{7}{4}}} + \frac{2x^2 - y^2}{2(x^2+y^2)^{\frac{7}{4}}}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbb{F}} = \frac{x^2 + y^2}{2(x^2+y^2)^{\frac{7}{4}}}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbb{F}} = \frac{1}{2(x^2+y^2)^{\frac{3}{4}}} \tag{III.20}$$

The scalar field produced by the divergence operation can be represented in multiple ways but it is most easily represented as a field of numbers (instead of a colour-based representation or a 3-axis representation).

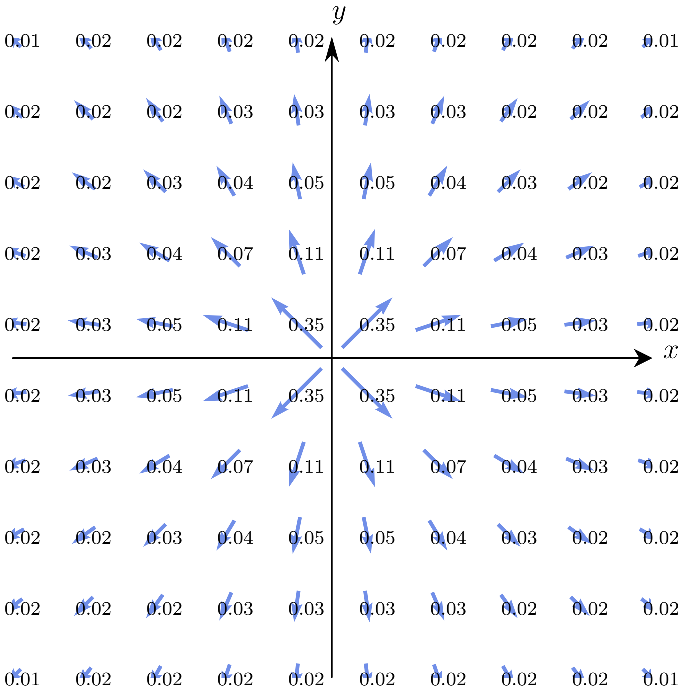


Figure III.4: A visual representation of the scalar field $\vec{\nabla} \cdot \vec{\mathbb{F}}$ (numbers) described in Eq. III.20
The original field is represented on the $x - y$ axis in blue.

Now we'll look at Eq. III.17 in $\mathbb{R}^3$:

$$\vec{\mathbb{F}} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}^{\frac{3}{2}}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}^{\frac{3}{2}}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}^{\frac{3}{2}}} \end{bmatrix}, \qquad \vec{\mathbb{F}} \in \mathbb{R}^3 \tag{III.21}$$

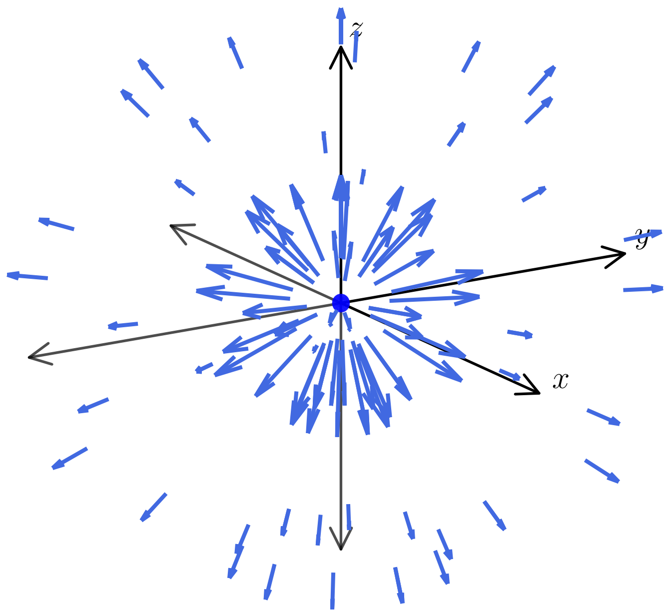


Figure III.5: A visual representation of the field $\vec{\mathbb{F}}$ described in Eq. III.21

The divergence of this field is given by Eq. III.22

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbb{F}} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{4}}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{\frac{3}{4}}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{\frac{3}{4}}}\right) \tag{III.22}$$

As with before, we'll skip straight past the differentiation, though the working is reasonably straightforward if you want to have a go.

$$\vec{\nabla} \cdot \vec{\mathbb{F}} = \frac{2(y^2+z^2)-x^2}{2(x^2+y^2+z^2)^{\frac{7}{4}}} + \frac{2(x^2+z^2)-y^2}{2(x^2+y^2+z^2)^{\frac{7}{4}}} + \frac{2(x^2+y^2)-z^2}{2(x^2+y^2+z^2)^{\frac{7}{4}}}$$

$$\vec{\nabla} \cdot \vec{\mathbb{F}} = \frac{3x^2+3y^2+3z^2}{2(x^2+y^2+z^2)^{\frac{7}{4}}}$$

$$\vec{\nabla} \cdot \vec{\mathbb{F}} = \frac{3}{2(x^2+y^2+z^2)^{\frac{3}{4}}} \tag{III.23}$$

Notice how the results from Eq. III.20 and Eq. III.23 are actually different (one has a 1 on the numerator, one has a 3). What this means is that it is important to specify what dimension you are dealing with.

We can also represent the scalar field produced by Eq. III.23, though one would not regularly do so (Fig. III.6).

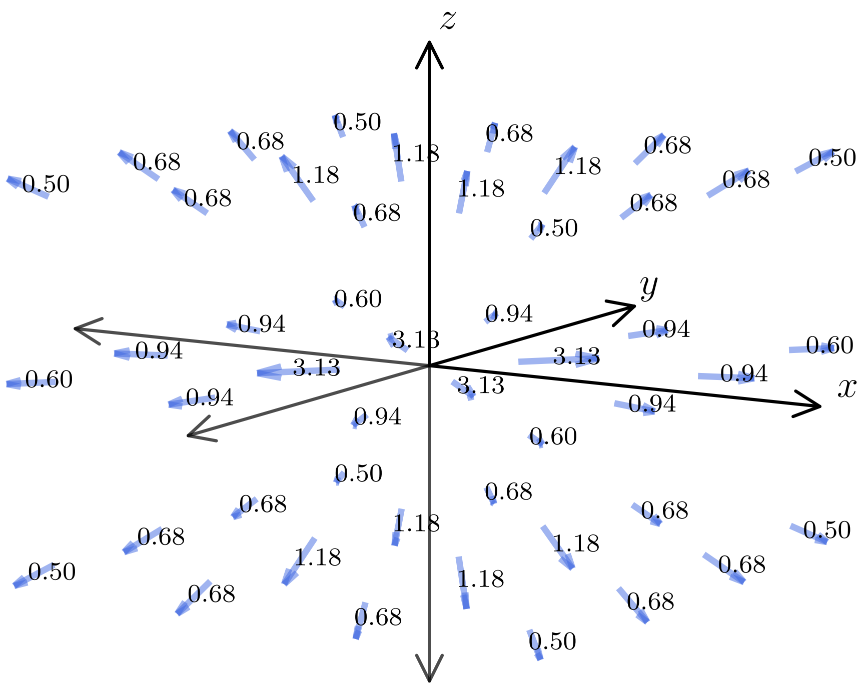


Figure III.6: A visual representation of the scalar field $\vec{\nabla} \cdot \vec{\mathbb{F}}$ described in Eq. III.23
The original field $\vec{\mathbb{F}}$ is represented in blue.

The $\vec{\nabla}\times$ Operator

The **curl** operator is slightly more difficult to understand than the divergence operator.

One way one can understand the curl operator is that it asks *how much does the field change when I move orthogonal to the direction of the field vector at a given point?*
This is a somewhat useful way of thinking about it and can help to interpret some of the results that you will see about the curl operator. However, there are some possible alternatives.

Let's start with a more fundamental understanding of the cross product. When you take the cross product of $\vec{\mathbf{v}}\times\vec{\mathbf{u}}$ you are saying *if I start at $\vec{\mathbf{v}}$, what vector is there that describes how much and in what direction I need to rotate such that I find $\vec{\mathbf{u}}$?* The cross product then generates a vector which points normal to the plane of rotation and with a magnitude proportional to the sine of the angle that you need to rotate through.

In a similar way, the curl operator ($\vec{\nabla}\times\vec{\mathbb{F}}$) asks *can I generate a vector which describes the direction and magnitude of rotation an object placed in a force field represented by $\vec{\mathbb{F}}$ would experience?*

The mathematical representation of $\vec{\nabla}\times\vec{\mathbb{F}}$ can be seen in Eq. III.25.

$$\vec{\nabla}\times\vec{\mathbb{F}} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} \mathbb{F}_x \\ \mathbb{F}_y \\ \mathbb{F}_z \end{bmatrix} \tag{III.24}$$

$$\vec{\nabla}\times\vec{\mathbb{F}} = \begin{bmatrix} \frac{\partial\mathbb{F}_z}{\partial y} - \frac{\partial\mathbb{F}_y}{\partial z} \\ \frac{\partial\mathbb{F}_x}{\partial z} - \frac{\partial\mathbb{F}_z}{\partial x} \\ \frac{\partial\mathbb{F}_y}{\partial x} - \frac{\partial\mathbb{F}_x}{\partial y} \end{bmatrix} = \left(\frac{\partial\mathbb{F}_z}{\partial y} - \frac{\partial\mathbb{F}_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial\mathbb{F}_x}{\partial z} - \frac{\partial\mathbb{F}_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial\mathbb{F}_y}{\partial x} - \frac{\partial\mathbb{F}_x}{\partial y}\right)\hat{\mathbf{z}} \tag{III.25}$$

Sometimes, purely for ease of notation, the partial derivatives are shortened using subscripts.

$$\vec{\nabla}\times\vec{\mathbb{F}} = \begin{bmatrix} \partial_y\mathbb{F}_z - \partial_z\mathbb{F}_y \\ \partial_z\mathbb{F}_x - \partial_x\mathbb{F}_z \\ \partial_x\mathbb{F}_y - \partial_y\mathbb{F}_x \end{bmatrix}$$

It is again worth noting, just as with the divergence operator, that there is a close similarity between taking the cross product of the $\vec{\nabla}$ vector and the field $\vec{\mathbb{F}}$, and the curl operation.

The curl operation also has versions in other coordinate systems. For more information see `https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates`.

Below is an example of a curling field. First we'll look at the field in a single 2D plane, then analyse it when we let it take up 3D space as well.

In 2D the field equation is given by Eq. III.26 and can be seen in Fig. III.7

$$\vec{\mathbb{F}} = \begin{bmatrix} \dfrac{-y}{(x^2+y^2)^{\frac{3}{4}}} \\ \dfrac{x}{(x^2+y^2)^{\frac{3}{4}}} \end{bmatrix}, \qquad \vec{\mathbb{F}} \in \mathbb{R}^2 \tag{III.26}$$

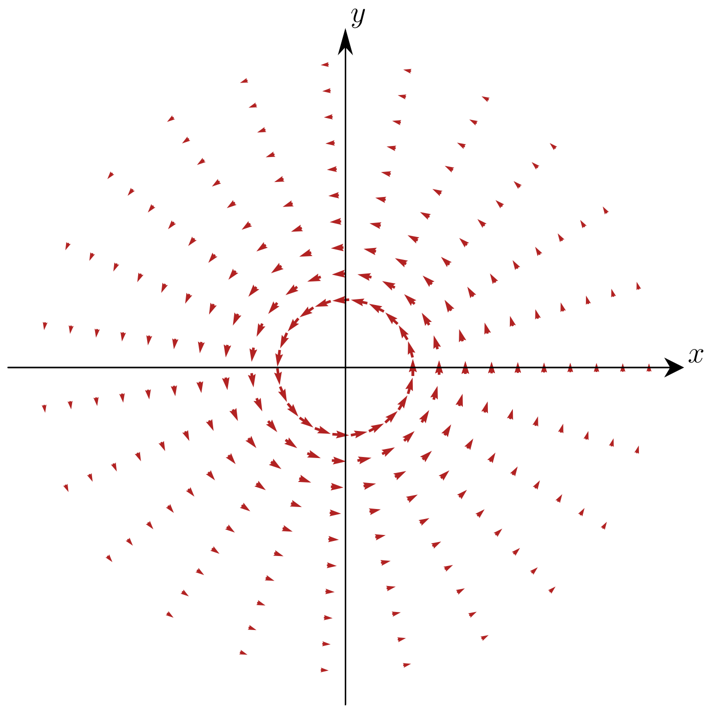


Figure III.7: A visual representation of the field $\vec{\mathbb{F}}$ described in Eq. III.26

We notice that as we move a small distance $\delta\vec{\mathbf{v}}$ along each of the vectors, the vectors remain the same length but rotate by some small angle $\delta\theta$ anticlockwise. This is where the 'curl' part of the name comes from.

If we move orthogonal to the vectors (radially outwards) then we see the vectors reduce in strength. This is what was alluded to at the beginning of the section.

We can calculate the curl of the field directly using the formula (Eq. III.27). However, we have to consider the world in $\mathbb{R}^3$, even though we defined the field in $\mathbb{R}^2$.

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{F}} = \begin{bmatrix} 0 \\ 0 \\ \partial_x \left(\dfrac{x}{(x^2+y^2)^{\frac{3}{4}}} \right) - \partial_y \left(\dfrac{-y}{(x^2+y^2)^{\frac{3}{4}}} \right) \end{bmatrix} \tag{III.27}$$

As with before, we'll skip past the differentiation, though the working isn't too bad if you'd like to have a go.

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{F}} = \begin{bmatrix} 0 \\ 0 \\ \left(\frac{2y^2 - x^2}{2(x^2+y^2)^{\frac{7}{4}}}\right) - \left(\frac{y^2 - 2x^2}{2(x^2+y^2)^{\frac{7}{4}}}\right) \end{bmatrix}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{F}} = \begin{bmatrix} 0 \\ 0 \\ \frac{(x^2+y^2)}{(x^2+y^2)^{\frac{7}{4}}} \end{bmatrix}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{F}} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{(x^2+y^2)^{\frac{3}{4}}} \end{bmatrix} \tag{III.28}$$

This is a new vector field which can be seen in Fig. III.8

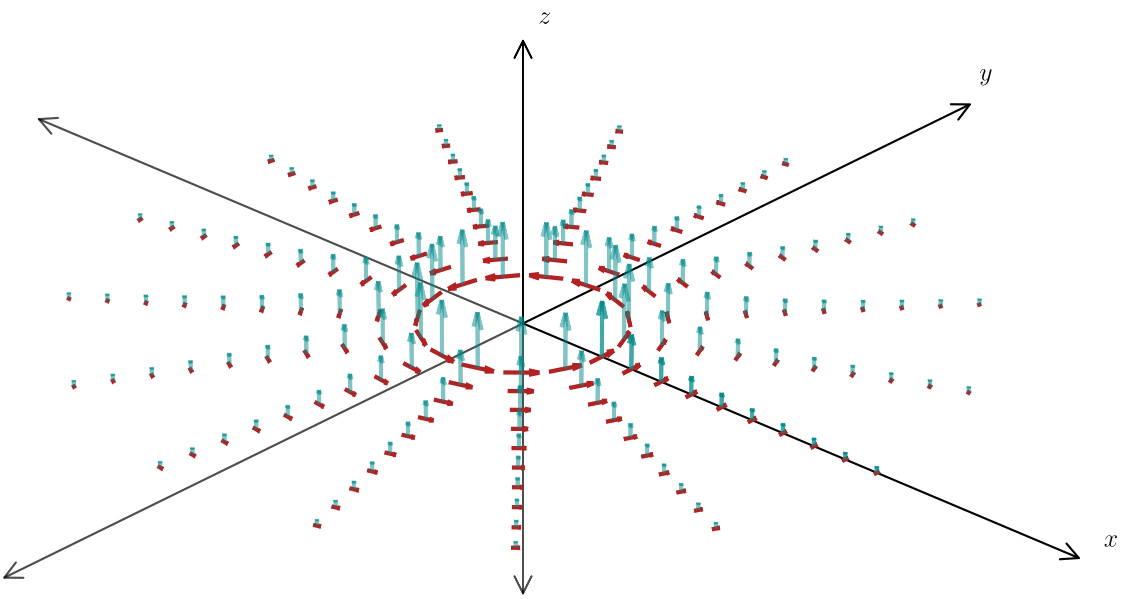


Figure III.8: A visual representation of the vector field $\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{F}}$ (blue) described in Eq. III.28
The original field is represented on the $x - y$ axis in red.

Now we'll look at the same field but in $\mathbb{R}^3$. Instead of being like the divergence example where we expand the field to depend on z, we're just going to let it exist in the z dimension as well. (This has physical significance for electromagnetism).

Fig. III.9 shows a visual representation of the field.

$$\vec{\mathbb{F}} = \begin{bmatrix} \frac{-y}{(x^2+y^2)^{\frac{3}{4}}} \\ \frac{x}{(x^2+y^2)^{\frac{3}{4}}} \\ 0 \end{bmatrix}, \qquad \vec{\mathbb{F}} \in \mathbb{R}^3 \tag{III.29}$$

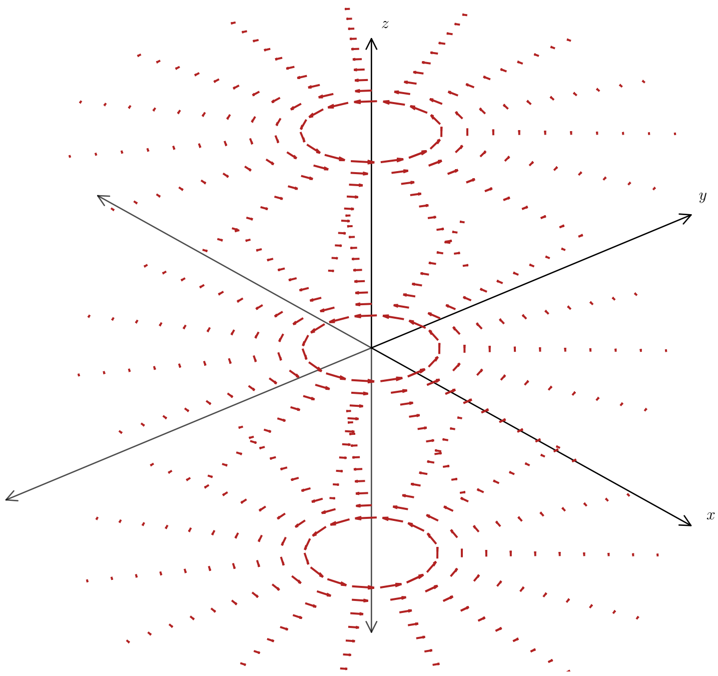


Figure III.9: A visual representation of the field $\vec{\mathbb{F}}$ described in Eq. III.29

It turns out that the curl of this field is exactly the same, except that the new curl field exists at all of the z coordinates as well, not just on the $x - y$ plane.

This can be seen in Fig. III.10

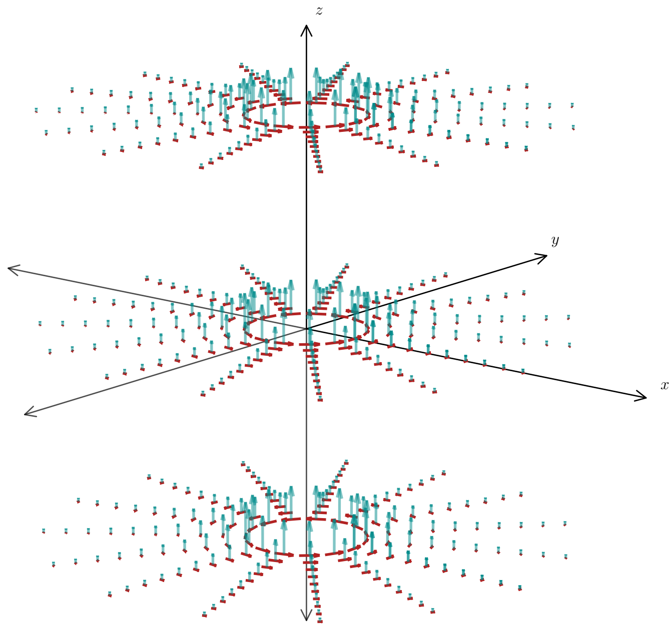


Figure III.10: A visual representation of the vector field (in $\mathbb{R}^3$) $\vec{\nabla} \times \vec{\mathbb{F}}$ (blue) described in
Eq. III.28. The original field is represented in red.

There are other types of curling fields as well that are less intuitive.

For example the field given by Eq. III.30 also has a non-zero curl, since as we move orthogonal to the field direction the strength changes.

$$\vec{\mathbb{F}} = \begin{bmatrix} y \\ 0 \end{bmatrix} \tag{III.30}$$

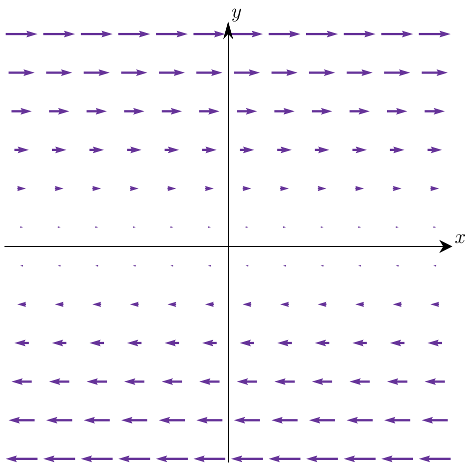


Figure III.11: A visual representation of the field $\vec{\mathbb{F}}$ described in Eq. III.30
This field doesn't appear to be curling, though it still has a non-zero curl field.

The curl field of this field is given by the Eq. III.31

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{F}} = \begin{bmatrix} 0 \\ 0 \\ \partial_x 0 - \partial_y y \end{bmatrix}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{F}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{III.31}$$

One might interpret this as representing the field of rotational axes that an object would have at each point if it had forces exerted on it as given by the field.

Think of a log placed so that it is parallel to the y axis. Because the force at the top is always stronger in the positive x direction it will always tend to rotate clockwise. This angular velocity would have a unit vector of $-\hat{\mathbf{z}}$, the same as the curl field.

The curl field is visually represented in Fig. III.12. Notice that even when the field is 0 along the x axis the curl is still constant.

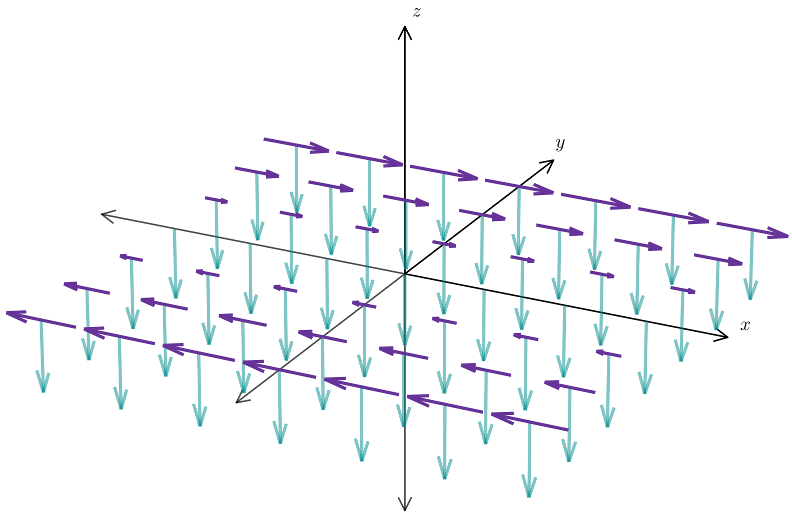


Figure III.12: A visual representation of the vector field $\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{F}}$ (blue) described in Eq. III.31. The original field is represented in purple.

Pre-Module IV: Notation

Mathematical Notation

Subscripts

In maths subscripts are (most of the time) used to give extra information about the number it is below. For example, adding a 0 below an I in a circuits problem i.e. I_0 often represents the fact that it is the current I at time $t = 0$.

Limits

Limits are a bit weird in maths because they can be thought of in their purest sense:
"what happens to this number a as another number b gets ever closer to some value?".
But they can also be thought of in a very pure and definitive way using what is called the epsilon-delta definition.

The epsilon-delta definition of limits is super tedious and really not helpful for understanding limits, so it won't be discussed here.
Instead we'll cover the difference between a pure limit, a limit from the right and a limit from the left.

A Standard Limit

Let's say that there is some function $f(x)$ where, as x gets ever closer to some number a, f gets ever closer to L. We would say that as x approaches a that $f(x)$ approaches L.

$$\lim_{x \to a} f(x) = L \tag{IV.1}$$

However, the crucial rule with this is that this must be true no matter what side you approach a from. If it is true from one side but not the other then the limit does not exist.

As an example let's say that our function is given by Eq. IV.2.

$$f(x) = \frac{\sin(x)}{x} \tag{IV.2}$$

This function is continuous and smooth for all values of x except for $x = 0$ where there is a divide by 0 error.
But, as x approaches 0, $f(x)$ approaches 1 (Fig. IV.1), and this is true from either side of $x = 0$.

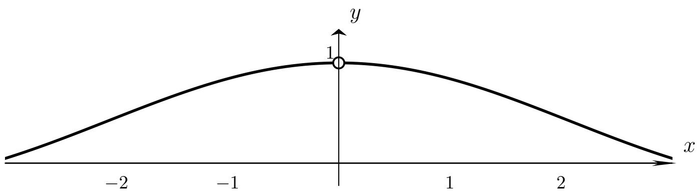


Figure IV.1: A graph of $y = \dfrac{\sin x}{x}$.

As an example of when a limit doesn't exist let's look at the function in Eq. IV.3

$$f(x) = \begin{cases} -x^2 + 2 & \text{if } x < 0 \\ x^2 - 2 & \text{if } x \geq 0 \end{cases} \tag{IV.3}$$

If we tried to find $\lim\limits_{x \to 0} f(x)$ we would not be able to since it is different depending on what side we come from.
This is more obvious in Fig. IV.2.

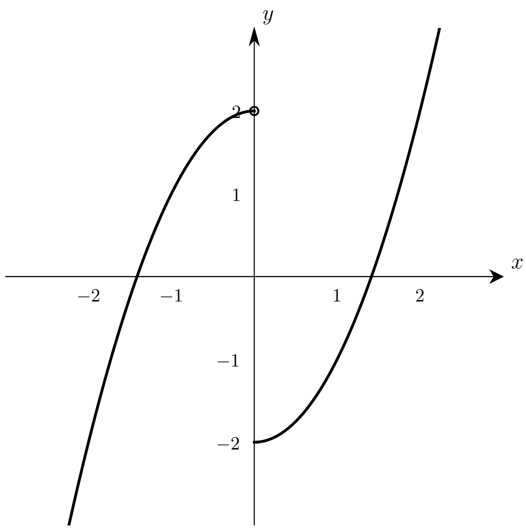


Figure IV.2: A graph of Eq. IV.3.

Limits from the Left and Right

Taking a limit from the left or the right is exactly what it sounds like. Instead of asking what does the function do as we approach this point, we are specifically asking what does it do as we approach it from a given side.

Taking the limit from the left is denoted with a superscript '−' above the number (Eq. IV.4) and a limit from the right is denoted with a superscript '+' above the number (Eq. IV.5)

$$\lim_{x \to a^-} f(x) \tag{IV.4}$$

$$\lim_{x \to a^+} f(x) \tag{IV.5}$$

In the case of the function in Eq. IV.3 and Fig. IV.2 we get the following limits.

$$\lim_{x \to 0^-} f(x) = 2$$

$$\lim_{x \to 0^+} f(x) = -2$$

Physics Notation

Change (Δ)

When you see something like ΔE_k in physics this means "change in" E_k. The change is always defined as the final value of that number minus the initial value.

$$\Delta v = v_{final} - v_{initial} \tag{IV.6}$$

Module 1: Kinematics

Kinematics is the study of motion. It studies things like how different accelerations result in different motion. It does not concern itself with how those accelerations arise, that is §2: Dynamics.

Base Units

Mass	(m)	Kilogram (kg)
Distance	(s)	Metres (m)
Displacement	($\vec{\mathbf{s}}$)	Metres (m)
Time	(t)	Seconds (s)
Speed	(v)	Metres per Second ($\mathrm{m\,s^{-1}}$ **or** m/s)
Velocity	($\vec{\mathbf{v}}$)	Metres per Second ($\mathrm{m\,s^{-1}}$ **or** m/s)
Acceleration	($\vec{\mathbf{a}}$)	Metres per Second, per Second ($\mathrm{m\,s^{-2}}$ **or** m/s/s **or** $\mathrm{m/s^2}$)

Constants

Gravitational acceleration at Earth's surface (g) – $9.8\,m\,s^{-2}$

Equations

$$v = \|\vec{\mathbf{v}}\| \tag{1.1}$$

Speed is the magnitude of velocity.

$$\vec{\mathbf{v}}_{avg} = \frac{\Delta\vec{\mathbf{s}}}{\Delta t} = \frac{\vec{\mathbf{s}}_{final} - \vec{\mathbf{s}}_{initial}}{t_{final} - t_{initial}} \tag{1.2}$$

$$\vec{\mathbf{a}}_{avg} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{\vec{\mathbf{v}}_{final} - \vec{\mathbf{v}}_{initial}}{t_{final} - t_{initial}} \tag{1.3}$$

The average velocity vector is the change in the displacement vector divided by the time taken to change.
The average acceleration vector is the change in the velocity vector divided by the time taken to change.

$$\vec{\mathbf{s}} = \vec{\mathbf{u}}t + \frac{1}{2}\vec{\mathbf{a}}t^2 \tag{1.4}$$

$$\vec{\mathbf{v}} = \vec{\mathbf{u}} + \vec{\mathbf{a}}t \tag{1.5}$$

$$v^2 = u^2 + 2\,\vec{\mathbf{a}} \cdot \vec{\mathbf{s}} \tag{1.6}$$

Standard equations of motion for an object with constant acceleration.

Eq. 1.6 uses the dot product of two vectors, for more information see §I: The Dot Product.

$$\vec{\mathbf{v}}_{A\,rel.\,B} = \vec{\mathbf{v}}_A - \vec{\mathbf{v}}_B \tag{1.7}$$

The velocity of object A as seen in the reference frame of object B.
(The velocity of A relative to B)

Non-Syllabus Equations

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{s}}}{dt} \tag{1.8}$$

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{s}}}{dt^2} \tag{1.9}$$

Derivative defintions of Eq. 1.2 and Eq. 1.3 where Δt is taken as limiting towards 0.

$$\vec{\mathbf{j}} = \frac{d\vec{\mathbf{a}}}{dt} = \frac{d^2\vec{\mathbf{v}}}{dt} = \frac{d^3\vec{\mathbf{s}}}{dt^3} \tag{1.10}$$

Definition of jerk. This is set to 0 for all of high school physics.

Module Notes

Significant Figures & Precision

Significant figures are weird when you first encounter them. The concept of an insignificant figure revolves around recursive 0s.

Take the number 1.234. We could write this as $\cdots 0001.234000 \cdots$, but we don't write the 0s because they don't change what the number is, they are insignificant. The first significant figure is the first digit after the beginning 0s. So in this case 1 is the first significant figure. Then we just say that the next digit is the 2^{nd} significant figure etc.

The number of significant figures would normally be considered the number of digits until you reach the trailing 0s, so 1.234 has 4.

However, in science significant figures are a bit different.

Let's say that you have a ruler which measures to the nearest mm and you measure the length of two rods. One you measure as $50.5\,cm$ and one you measure as $1\,m$.

Now, you could just say that the second rod is $1\,m$ and leave it at that, but in reality you know that it's exactly a metre to the nearest millimetre. So, instead you should say that it's $1.00m$, since you know that the second decimal place (nearest mm) is definitely 0.

Measurement Uncertainty

There is always uncertainty (i.e. potential error) associated with any measurement. When you measure with a millimetre ruler, was the table exactly a metre long, or was it a little bit longer (say 0.5 mm)?

This is the crux of the issue of uncertainty and there are general rules for finding the uncertainty of a given device.

- Physical Instruments: ± Half the smallest measurement increment.
- Digital Instruments: ± The smallest increment (or, if it flickers back and forth, the amount it flickers by).

When you report a measurement, you can only report to a precision down to the first digit of your uncertainty. For example, if you measure something to be $2.32\,m \pm 0.1\,m$, then you have to report that as $2.3\,m \pm 0.1\,m$

Scientific Notation

It can be quite common for big numbers to crop up quite quickly in the sciences, and it can make it quite difficult to compare results.
As a result numbers that are very small or very large tend to be compressed using scientific notation.

Typically the rule is that you move the decimal place behind the first significant figure and then multiply by 10 to the power of something so that, if you were to actually multiply, the decimal place would return to the correct spot.

For example, take the number 1234000000000. This number is very large but we can move the decimal point behind the 1 and then multiply by 10^{12} so that the number's value doesn't change. We then also get rid of the trailing *insignificant* 0s. (If a 0 is significant as discussed in §1:Significant Figures & Precision then it should stay).

$$1234000000000 = 1.234 \times 10^{12}$$

In the same way we can shorten very small numbers such as 0.0000001234 (remembering that $10^{-n} = \frac{1}{10^n}$).

$$0.0000001234 = 1.234 \times 10^{-7}$$

There is also another way in which this is written, which you can see when you write $2mm$ (2 millimetres). The 'milli' or m in front is equivalent to writing $\times 10^{-3}$, so you could equally say $2 \times 10^{-3}m$.

There are lots of different prefixes like this and they each represent a different power of 10 (Table 1.1).

Table 1.1: Scientific Notation Prefixes

Prefix	**Symbol**	**Value**
Peta	P	$\times 10^{15}$
Tera	T	$\times 10^{12}$
Giga	G	$\times 10^{9}$
Mega	M	$\times 10^{6}$
kilo	k	$\times 10^{3}$
hecto	h	$\times 10^{2}$
deca	da	$\times 10^{1}$
–	–	–
deci	d	$\times 10^{-1}$
centi	c	$\times 10^{-2}$
milli	m	$\times 10^{-3}$
micro	μ	$\times 10^{-6}$
nano	n	$\times 10^{-9}$
pico	p	$\times 10^{-12}$
femto	f	$\times 10^{-15}$

Frames of Reference

When we talk about reference frames we are asking *"what does someone with the motion of that thing see?"*

Think of the situation where you're on a train going past a station. As you move past you see the people on the platform moving backwards. But you know that they would see you and the train going forwards. How can we describe this mathematically?

You'll notice that when we change into the reference frame of the people on the platform, we have to make it so that their velocity is 0. So, if they have some velocity as a function of time in our frame of reference, we just need to subtract that velocity function from all objects, which will make their velocity 0 and tell us what they see.

This is how we get Eq. 1.7: $\vec{\mathbf{v}}_{A\,rel.\,B} = \vec{\mathbf{v}}_A - \vec{\mathbf{v}}_B$

The General Non-Relativistic Case

Ignoring the case where motion is occurring near the speed of light (i.e. relativity becomes a problem), there is a somewhat simple generalised process for shifting into another reference frame.

Consider the case where the observer sits in an inertial (non-accelerating) reference frame A and observes an object passing by. Let's call the reference frame of the object B. To "shift into" reference frame B from A (i.e. find out what the object in frame B sees something in reference frame A doing) we must subtract all of the equations of motion of B, as seen in A, from the motion otherwise observed in B.

Mathematically we can prove this by using some basic vector principles and calculus.
Let's now consider a third (inertial) reference frame C in which we can assign the position of the observer in A as $\vec{\mathbf{s}}_A$ and the position of the object in B as $\vec{\mathbf{s}}_B$. The vector starting at the position of A and ending at the position of B (i.e. the position of B relative to A) is $\vec{\mathbf{s}}_{B\,rel.\,A} = \vec{\mathbf{s}}_B - \vec{\mathbf{s}}_A$
We then continue taking the derivative with respect to time (noting that all positions, velocities, accelerations etc. are as they are observed in frame C.).

$$\frac{d}{dt}(\vec{\mathbf{s}}_{B\,rel.\,A}) = \frac{d}{dt}(\vec{\mathbf{s}}_B - \vec{\mathbf{s}}_A)$$

$$\vec{\mathbf{v}}_{B\,rel.\,A} = \vec{\mathbf{v}}_B - \vec{\mathbf{v}}_A$$

$$\frac{d}{dt}(\vec{\mathbf{v}}_{B\,rel.\,A}) = \frac{d}{dt}(\vec{\mathbf{v}}_B - \vec{\mathbf{v}}_A)$$

$$\vec{\mathbf{a}}_{B\,rel.\,A} = \vec{\mathbf{a}}_B - \vec{\mathbf{a}}_A$$

$$\vdots$$

This process allows us to shift into the reference frame of A and determine how an observer in A would observe the object B. This is important since we are not shifting from B to A but shifting from C to A and determining how A sees B moving.

Derivations of Formulas

To derive Eq. 1.4, Eq. 1.5 & Eq. 1.6 we need to start with the fact that $\vec{\mathbf{j}} = \vec{\mathbf{0}}$ (Jerk is defined in Eq. 1.10).

$$\vec{\mathbf{j}} = \frac{d\vec{\mathbf{a}}}{dt} = \vec{\mathbf{0}}$$

So, we can just let acceleration be a constant $\vec{\mathbf{a}}$ (it could also be $\vec{\mathbf{0}}$). This allows us to integrate it easily.

$$\frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{a}}$$

$$d\vec{\mathbf{v}} = \vec{\mathbf{a}}\, dt$$

$$\int_{\vec{\mathbf{u}}}^{\vec{\mathbf{v}}} d\vec{\mathbf{v}} = \int_0^t \vec{\mathbf{a}}\, dt$$

$$\vec{\mathbf{v}} - \vec{\mathbf{u}} = \vec{\mathbf{a}}t$$

$$\vec{\mathbf{v}} = \vec{\mathbf{u}} + \vec{\mathbf{a}}t \tag{1.5}$$

$$\frac{d\vec{\mathbf{s}}}{dt} = \vec{\mathbf{u}} + \vec{\mathbf{a}}t$$

$$d\vec{\mathbf{s}} = (\vec{\mathbf{u}} + \vec{\mathbf{a}}t)\, dt$$

$$\int_{\vec{\mathbf{s}}_0}^{\vec{\mathbf{s}}} d\vec{\mathbf{s}} = \int_0^t (\vec{\mathbf{u}} + \vec{\mathbf{a}}t)\, dt$$

$$\vec{\mathbf{s}} - \vec{\mathbf{s}}_0 = \vec{\mathbf{u}}t + \frac{1}{2}\vec{\mathbf{a}}t^2 \qquad (\textit{Let } \vec{\mathbf{s}}_0 = \vec{\mathbf{0}})$$

$$\vec{\mathbf{s}} = \vec{\mathbf{u}}t + \frac{1}{2}\vec{\mathbf{a}}t^2 \tag{1.4}$$

The step where we let $\vec{\mathbf{s}}_0 = \vec{\mathbf{0}}$ is important because it means we can go back and let it equal something if we want to more easily deal with a situation where our displacement at $t = 0$ is non-zero.

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = \frac{dv_x}{dx}v_x = \frac{dv_x}{dx}\frac{d}{dv_x}\left(\frac{1}{2}{v_x}^2\right) = \frac{d}{dx}\left(\frac{1}{2}{v_x}^2\right)$$

This is generally applicable to all of the dimensions, not just x.

$$a_x \ dx = d\left(\frac{1}{2}{v_x}^2\right)$$

$$\int_{x_0}^{x} a_x \ dx = \frac{1}{2}\int_{{u_x}^2}^{{v_x}^2} d\left({v_x}^2\right)$$

$$a_x\left(x - x_0\right) = \frac{1}{2}\left({v_x}^2 - {u_x}^2\right)$$

$$2a_x x - 2a_x x_0 = {v_x}^2 - {u_x}^2$$

Now we add together the equations for x, y and z.

$$\left(2a_x x - 2a_x x_0\right) + \left(2a_y y - 2a_y y_0\right) + \left(2a_z z - 2a_z z_0\right) = \left({v_x}^2 - {u_x}^2\right) + \left({v_y}^2 - {u_y}^2\right) + \left({v_z}^2 - {u_z}^2\right)$$

$$2\left(a_x x + a_y y + a_z z\right) - 2\left(a_x x_0 + a_y y_0 + a_z z_0\right) = \left({v_x}^2 + {v_y}^2 + {v_z}^2\right) - \left({u_x}^2 + {u_y}^2 + {u_z}^2\right)$$

$$2\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{s}}\right) - 2\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{s}}_0\right) = \left\|\vec{\mathbf{v}}\right\|^2 - \left\|\vec{\mathbf{u}}\right\|^2$$

As with before, we'll let $\vec{\mathbf{s}}_0 = \vec{\mathbf{0}}$.

$$2\vec{\mathbf{a}} \cdot \vec{\mathbf{s}} = v^2 - u^2$$

$$v^2 = u^2 + 2\vec{\mathbf{a}} \cdot \vec{\mathbf{s}} \tag{1.6}$$

Module 2: Dynamics

Dynamics is the study of forces and how they give rise to motion.

Base Units

Mass	(m)	Kilogram (kg)
Distance	(s)	Metres (m)
Displacement	$(\vec{\mathbf{s}})$	Metres (m)
Time	(t)	Seconds (s)
Speed	(v)	Metres per Second ($\mathrm{m\,s^{-1}}$ **or** m/s)
Velocity	$(\vec{\mathbf{v}})$	Metres per Second ($\mathrm{m\,s^{-1}}$ **or** m/s)
Acceleration	$(\vec{\mathbf{a}})$	Metres per Second, per Second ($\mathrm{m\,s^{-2}}$ **or** m/s/s **or** $\mathrm{m/s^2}$)
Force	$(\vec{\mathbf{F}})$	Newtons (N) **or** ($\mathrm{kg\,m\,s^{-2}}$)
Energy	(E)	Joules (J) **or** ($\mathrm{kg\,m^2\,s^{-2}}$)
Work	(W)	Joules (J) **or** (N m) **or** ($\mathrm{kg\,m^2\,s^{-2}}$)
Power	(P)	Joules per second ($\mathrm{J\,s^{-1}}$) **or** ($\mathrm{kg\,m^2\,s^{-3}}$)

Constants

Gravitational acceleration at Earth's surface (g) – 9.8 $m\,s^{-2}$

Equations

$$\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}} \tag{2.1}$$

The equation describing the relationship between the net force on an object and the acceleration on an object with constant mass.
The net force is found by adding all of the force vectors on an object (important because you cannot just add the magnitudes).

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \tag{2.2}$$

The momentum of an object.

$$I = \vec{\mathbf{F}}\Delta t = \Delta\vec{\mathbf{p}} \tag{2.3}$$

Impulse is the change in momentum due to a force over some (usually small) time.

$$f_k = \mu_k N \tag{2.4}$$

$$f_s \leq \mu_s N \tag{2.5}$$

The equations describing the strength of the friction force between two objects.
f_k describes the kinetic friction which is present when the objects are sliding across each other.
f_s describes the upper limit on the static friction force which occurs when two objects are not moving relative to each other but have other forces trying to slide them across each other.

$$E_k = \frac{1}{2}mv^2 \tag{2.6}$$

The kinetic energy of an object with mass m and speed v.

$$W = \Delta E_k \tag{2.7}$$

The work done on an object describes it's change in kinetic energy.

$$W_A = \vec{\mathbf{F}}_A \cdot \vec{\mathbf{s}} \tag{2.8}$$

The work done by some force A (e.g. W_f would be the work by friction).
This description is where the we get Work is a force over a distance from.

$$U_g = mgh \tag{2.9}$$

The gravitational potential energy for an object a distance h off the ground. (This equation only applies close to Earth's surface).

$$W_g = -\Delta U_g \tag{2.10}$$

The work done by gravity is the negative change

$$P = \frac{\Delta E_k}{\Delta t} = \frac{W}{\Delta t} \tag{2.11}$$

$$P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} \tag{2.12}$$

Power is the rate of change of kinetic energy.

Non-Syllabus Equations

$$\vec{\mathbf{r}}_{com} = \frac{\sum\limits_{n=1}^{N} \vec{\mathbf{r}}_n m_n}{\sum\limits_{n=1}^{N} m_n} \tag{2.13}$$

The formula for the centre of mass of an object is the average of each of the positions of the small masses which make it up, weighted according to their mass.

$$\vec{\mathbf{r}}_{com} = \frac{\int_0^M \vec{\mathbf{r}}\ dm}{M} \tag{2.14}$$

The formula for the centre of mass of an object. This is just Eq. 2.13 for continuous mass distributions.

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}_{com}}{dt} \tag{2.15}$$

The linear velocity of an object is the time derivative of its centre of mass' position.

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}_{com}}{dt^2} \tag{2.16}$$

The linear acceleration of an object is the time derivative of its linear velocity.

$$\vec{\mathbf{F}}_{net} = \frac{d\vec{\mathbf{p}}}{dt} \tag{2.17}$$

The actual definition of force is the derivative of momentum. This equation comes in handy when dealing with situations with non-constant mass (such as rockets).

$$I = \int_{t_1}^{t_2} \vec{\mathbf{F}}\ dt = \Delta\vec{\mathbf{p}} \tag{2.18}$$

The more formal definition of impulse as the change in momentum. By integrating the rate of change of momentum (force) you find the true change in momentum whether the force is constant or not.

$$W = \int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} \vec{\mathbf{F}} \cdot \, d\vec{\mathbf{s}} \tag{2.19}$$

This is the path integral definition of Work. This allows us to take the sum of all of the force components along small straight paths $d\vec{\mathbf{s}}$.

$$P = \frac{dW}{dt} = \frac{dE_K}{dt} \tag{2.20}$$

The definition of power as the rate of change of work. This is the dynamics definition, though there are many other forms of power in areas such as electromagnetism where this definition of power does not apply.

Newton's Laws

1st Law – Inertia

"An object in motion will retain its state of motion unless acted on by an external force"

This law is just verbalising Eq. 2.17. It is really saying that where there is no force there will be no change in motion.

2nd Law – Force

$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ – Only applies for constant mass.

3rd Law – Force Pairs

The typical way of stating this is that *"Every action (force) has an equal and opposite reaction (force)."*

This is not super obvious until you consider where all forces come from. The four fundamental forces are gravity, electromagnetic, the strong nuclear force & the weak nuclear force. Most of the forces you deal with in life will be gravitation and electrostatic. For example, the tension in a rope is just the total effect of the electrostatic attraction of atoms.

It just so happens that when two charges push or pull on each other, the strength of the force on each charge is the same and the direction of one force is opposite to the other (so that they either attract or repel).

The same is true of gravitation, with the force that the Earth attracts you with being equal to the gravitational force you put on the Earth (the accelerations are just very different since the masses are very different).

Therefore all forces must have an equal and opposite reaction since they are just the total effect of more fundamental forces for which this is true.

The real world application of this can be seen when you sit on a table. The table holds you up with the electrostatic repulsion of your atoms and its atoms. Therefore your atoms must be repelling the table downwards with the same force, meaning there is both a normal force on you from the table and a normal force on the table from you.

Contact Forces

When two surfaces are in contact, their atoms begin to attract each other. As you push them together harder they repel again. These are called the Van der Waals forces and describe the forces on atoms when they are not bonded together.

Friction Forces

Friction is the horizontal resistance of motion between two surfaces.

If the objects are in motion relative to each other then the attraction resists the motion and is called "kinetic friction" or sometimes "sliding friction".

If the surfaces are not moving relative to each other then it is called "static friction" and the force acts such that the part of the objects in contact remain in contact (which is important since the force of friction may be lower than the force pushing the objects if the objects can also rotate).

One aspect of friction is the attractive end of the Van der Waals forces (the forces between non-bonded atoms). Friction is also caused by surface deformations and two surfaces grinding / getting caught on each other. Fig. 2.1

Friction is strictly the component of the contact force which is parallel to the contact plane.

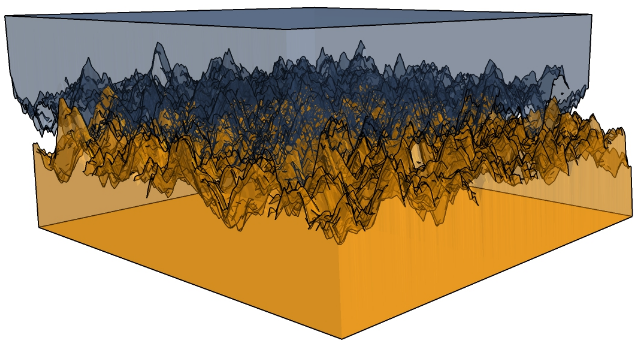


Figure 2.1: Diagram showing the surface deformations that contribute to the friction forces between objects.
Source: CaoHao, CC BY-SA 4.0

Normal Forces

The normal force from a surface is perpendicular to the plane of the contact (normal means orthogonal to the surface) and acts such that one object does not push through the other object.

The normal force of object 1 on object 2 is always equal and opposite to the normal force of object 2 on object 1.

Force as the Time Derivative of Momentum

The definition of force as the derivative of momentum is important because in most situations it gives us the usual equation $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ but it also gives the correct result for situations with changing mass by giving Eq. 2.21

$$\vec{\mathbf{F}} = \frac{dm}{dt}\vec{\mathbf{v}} + m\vec{\mathbf{a}} \tag{2.21}$$

Derivation that Net Force is Related to the Acceleration of the Centre of Mass of an Object

It is a very common assertion that the equation $\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}}$ is stating that the net force on an object is directly related to the acceleration of the centre of mass of the object.
This is not a statement that should seem inherently true, it is often just said enough that you start to believe it.

We begin by establishing that the formula for the centre of mass of an object is given by Eq. 2.13 (also below).
This equation defines the centre of mass for a set of N particles each with a constant mass.

$$\vec{\mathbf{r}}_{com} = \frac{\sum_{n=1}^{N} \vec{\mathbf{r}}_n m_n}{\sum_{n=1}^{N} m_n}$$

For simplicity's sake, let's set the total mass to be M.

$$\vec{\mathbf{r}}_{com} = \frac{\sum_{n=1}^{N} \vec{\mathbf{r}}_n m_n}{M}$$

Taking the derivative of this equation gives the velocity of the centre of mass.

$$\frac{d}{dt}\vec{\mathbf{r}}_{com} = \frac{d}{dt}\left(\frac{\sum_{n=1}^{N} \vec{\mathbf{r}}_n m_n}{M}\right)$$

$$\vec{\mathbf{v}}_{com} = \frac{\sum_{n=1}^{N} \vec{\mathbf{v}}_n m_n}{M}$$

Taking the derivative again gives us the acceleration on the centre of mass.

$$\frac{d}{dt}\vec{\mathbf{v}}_{com} = \frac{d}{dt}\left(\frac{\sum_{n=1}^{N} \vec{\mathbf{v}}_n m_n}{M}\right)$$

$$\vec{\mathbf{a}}_{com} = \frac{\sum_{n=1}^{N} \vec{\mathbf{a}}_n m_n}{M}$$

The sum in the numerator is just the sum of all of the net forces on each particle.

$$M\vec{\mathbf{a}}_{com} = \sum_{n=1}^{N} \vec{\mathbf{F}}_n$$

And the sum of all of the net forces is just the net force on the system as a whole. Therefore the net force on the system of particles is equal to its total mass multiplied by the acceleration on the centre of mass of the system.
One particularly nice result of this is that it doesn't matter whether these particles are stuck together or not, since we haven't enforced that this condition must be true while performing the derivation.

$$M\vec{\mathbf{a}}_{com} = \vec{\mathbf{F}}_{net}$$

Energy

Energy is a widely used concept in physics. Because in almost all cases energy is said to be conserved (§2: The Law of Conservation of Energy) it is often easier to use energy to describe what will happen rather than forces.

For example, a way of describing motion in terms of energy could be *"an object falls to Earth because potential energy tends to decrease and energy must be conserved so it must lose height and gain speed."*

The Law of Conservation of Energy

It is often outright asserted that energy is conserved in any physics problem. That **energy cannot be created or destroyed, only transformed into other types of energy**.
Mathematically this means that if you add up all of the different types of energies an object has, they should always add to the same number.
For the most part this statement is true and it's generally pretty safe to assume that it is.

However, mathematician Emmy Noether was able to discover what is now known as Noether's Theorem which states that energy is always conserved so long as the space-time has 0 curvature. She made this discovery at the same time that Einstein was completing his work on General Relativity, the theory which describes how mass and energy curve space-time.
As a result of Noether's Theorem and Einstein's General Relativity, it cannot be said that energy is conserved on a universal scale. However, just like a small piece of a curve can be approximated as linear, a small part of space-time can be approximated as flat, so for most physics occurring on Earth, conservation of energy holds true.

Kinetic Energy

Kinetic Energy has one of the most confusing types of energy since its formula is seemingly arbitrary (Eq. 2.6).

$$E_k = \frac{1}{2}mv^2 \tag{2.6}$$

Work

The most basic definition of work is that it is the amount of force multiplied by the distance that it acts in or, as it is regularly quoted, "force over a distance". For an object travelling in a straight path $\vec{\mathbf{s}}$, with a constant force on it $\vec{\mathbf{F}}$, the work done is the parallel component of the force multiplied by the distance (i.e. the dot product of the vectors) $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}}$.

But how is this useful? Well, what we have to do is consider the case where the force isn't constant across the path, or where the path is not straight. In either of these cases we have to add up the infinitesimally small bits of work done along small pieces of the path $d\vec{\mathbf{s}}$, adjusting for the changing force or path direction along the way.
The way we do this is by adding using integration, formally this type of integral is called a *path integral*. From this we get the formal definition of work (Eq. 2.19)

$$W = \int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} \vec{\mathbf{F}} \cdot \, d\vec{\mathbf{s}} \tag{2.19}$$

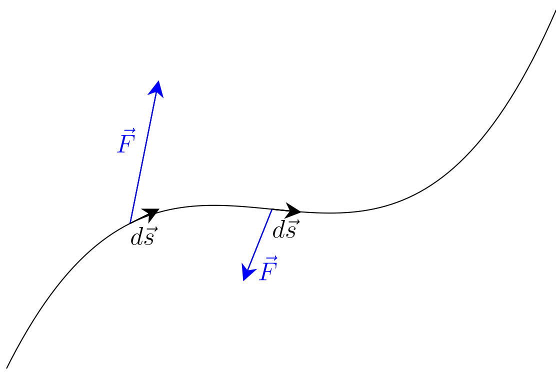


Figure 2.2: A path followed by an object, where a small part of the path $d\vec{\mathbf{s}}$, with the same direction as the direction of motion at that point, and a changing force $\vec{\mathbf{F}}$.

In this case we treat the integral as a sum of infinitely many infinitesimal amounts of work done along the path (§II:Integration as a Sum).

Work is the change in Kinetic Energy

It is often asserted that the total work done on an object results in a change in Kinetic Energy of the same amount. Using the traditional (non-integral) definition of work, this can be a bit tricky to argue. However, if we assume that the mass of the object is constant (i.e. $\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}}$) then it is a rather simple proof.

First let's begin by reminding ourselves of the quantities we're dealing with:

$$d\vec{\mathbf{s}} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = m\frac{d\vec{\mathbf{v}}}{dt} = m\ \frac{d}{dt} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{s}}}{dt} = \frac{d}{dt} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}} = a_x b_x + a_y b_y + a_z b_z$$

We can now rearrange our equation for work using these identities.

$$W_{total} = \int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} \vec{\mathbf{F}}_{net} \cdot d\vec{\mathbf{s}}$$

$$= \int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} m\vec{\mathbf{a}} \cdot d\vec{\mathbf{s}}$$

$$= m \int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} \frac{d\vec{\mathbf{v}}}{dt} \cdot d\vec{\mathbf{s}}$$

$$= m \int_{\vec{\mathbf{v}}_a}^{\vec{\mathbf{v}}_b} d\vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{s}}}{dt}$$

$$= m \int_{\vec{\mathbf{v}}_a}^{\vec{\mathbf{v}}_b} \vec{\mathbf{v}} \cdot d\vec{\mathbf{v}}$$

$$= m \left(\int_{v_{x\,a}}^{v_{x\,b}} v_x\ dv_x + \int_{v_{y\,a}}^{v_{y\,b}} v_y\ dv_y + \int_{v_{z\,a}}^{v_{z\,b}} v_z\ dv_z \right)$$

$$= m\left(\frac{1}{2}({v_{xb}}^2 - {v_{xa}}^2) + \frac{1}{2}({v_{y_b}}^2 - {v_{y_a}}^2) + \frac{1}{2}({v_{zb}}^2 - {v_{za}}^2)\right)$$

$$= \frac{1}{2}m\left(({v_{xb}}^2 + {v_{y_b}}^2 + {v_{zb}}^2) - ({v_{xa}}^2 + {v_{y_a}}^2 + {v_{za}}^2)\right)$$

$$= \frac{1}{2}m\left(\|\vec{\mathbf{v}}_b\|^2 - \|\vec{\mathbf{v}}_a\|^2\right)$$

$$= \frac{1}{2}m\,\|\vec{\mathbf{v}}_b\|^2 - \frac{1}{2}m\,\|\vec{\mathbf{v}}_a\|^2$$

$$= E_{Kb} - E_{Ka}$$

$$= \Delta E_K$$

Potential Energy

Potential Energy is often given the symbol U and, conceptually, is the amount of work a force could to do on an object. Potential energy was invented to make conservation of energy as a law work.
Consider the case where you lift an object above your head and hold it there. While you lifted it you did work, but the object has no kinetic energy, so where did the energy go? The answer to this was potential energy.

Mathematically we write that the sum of the work done by a force and its potential energy are held constant at some value E.

$$W + U = E$$

Not all forces have an associated potential energy, typically only *conservative* forces have an associated potential.
A conservative force is one where the work done by that force does not depend on the path travelled, only on the final and initial positions.
An example of a force which is not conservative is kinetic friction, since the longer your path the more distance you travel and, since it always opposes motion, it does more work. The extra lost energy here goes into other forms of energy such as heat and sound.
An example of a conservative force is gravity, since it always points in a fixed direction - down (even the spherically symmetric version is conservative but that's a little more complex for right now).

Power

Power is the rate at which energy changes or transfers per second. Take for instance a crane lifting a load at a constant speed. Since every second it does a certain amount of work, the power being transmitted to the load is some constant value.

There are a few mechanisms for energy transfer, for instance heat energy (Q) can move between objects. Work (W) transfers kinetic energy (E_K) to an object.
Since here we are considering the case of mechanical power, we will treat power as the rate at which work transfers energy to/from a system.

Take a small amount of work dW done by a force $\vec{\mathbf{F}}$ over a small path element $d\vec{\mathbf{s}}$

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} \tag{2.22}$$

Power is the derivative of work with respect to time $\left(P = \frac{dW}{dt}\right)$, therefore giving the result in Eq. 2.12.

$$P = \frac{dW}{dt} = \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{s}}}{dt}$$

$$P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} \tag{2.12}$$

Module 3: Waves & Thermodynamics

Waves

Waves crop up in many areas of physics, not just mechanics. The most common forms of waves you will deal with are sound waves since they are the easiest mathematically and also the type that you are most used to.
There are also waves in strings, solids, water or even the electromagnetic field (light).

A wave is characterised by an oscillation of some value which travels through space over some period of time. For instance a sound wave is an oscillation of pressure which propagates (moves) through space. What is important is that although there is a high pressure zone which moves through the air, the air particles do not move with the pressure wave, they simply oscillate around their original position.

Rules for all Waves

All waves follow a key set of rules which will be set out here. Since we're talking about all waves not just mechanical waves, we're going to let the wave be represented by a wave function $\Psi(x, t)$, where Ψ could represent the pressure value at a given x (displacement) and t (time) of a pressure wave, the height of a wave in a string, or the displacement between turns in a spring.

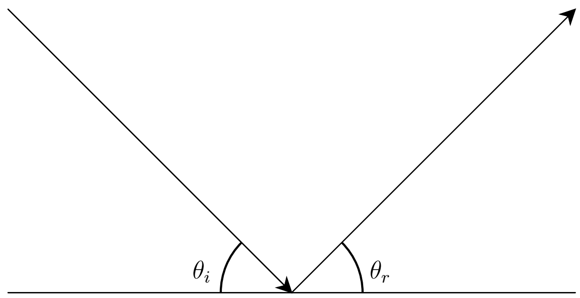
Wave functions are often represented mathematically as some form of sine wave dependant on time and position, for example $\Psi(x, t) = A\sin(\omega t - kx)$

Addition of Waves (Interference)

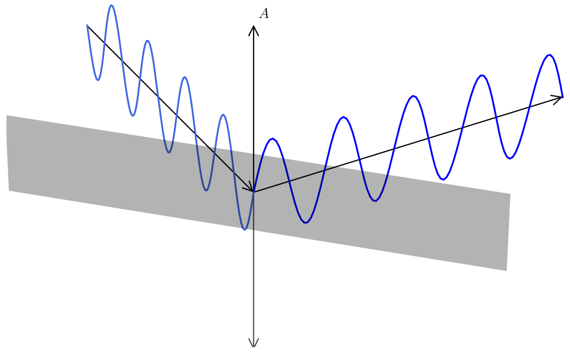
If you have two waves with respective wave functions $\Psi_1(x, t)$ and $\Psi_2(x, t)$ then the final wave that you see is just $\Psi_1 + \Psi_2$ (where each value at a given x and t is added, just like if you added the functions $f(x) = x^2$ and $g(x) = x$ giving $f + g = x^2 + x$).

Reflection of Waves

Let's say that you have a sound wave travelling towards a wall (we'll get to that in §**??**:**??**).
The sound wave will reflect off the wall just like you might expect a ball hitting a wall to reflect (though the wave isn't affected by gravity), i.e. its angle of incidence θ_i is equal to its angle of reflection θ_r.
This is because the velocity of the wave into the wall is reversed.



(a) Digram showing the reflection behaviour of a ray of a wave in 2D.



(b) Diagram showing the reflection of a wave off of a surface, with the vertical axis representing the amplitude A of oscillation of the wave.

Figure 3.1: Diagram showing the reflection of a single ray of a wave.

Intensity of a Wave

Each type of wave has what's called intensity, which is related to the amount of energy stored in the wave. A ray of light can be said to have a certain amount of intensity as well as sound waves.

Thermodynamics

Module 4: Electricity & Magnetism

Here electricity and magnetism will be treated separately, though as may become apparent, the two are actually strongly linked.

Electrostatics

Here we will deal with the rules which govern static (stationary) charged particles.

Charge

We are all used to talking about positively or negatively charged things, but how do we quantify that?
Like most things we have to give it a unit, in this case named after a very influential physicist in this area: Coulomb.

Because it was historically hard to measure charge directly, this unit is based off the Amp (also named after a physicist Ampere) such that $1\,C = 1\,A \cdot 1\,s$

The Electric Field

The electric field ($\vec{\mathbf{E}}$) is a vector field representing the force that would be exerted on a $1\,C$ charged particle.

Module 5: Advanced Mechanics

Mechanics is the study of both Kinematics and Dynamics. Advanced Mechanics covers and extends upon the linear Kinematics and Dynamics covered in §1: Kinematics and §2: Dynamics as well as covering their rotational equivalents. Just like how Kinematics is the study of motion and Dynamics is the study of how that motion arises, rotational Kinematics is the study of rotational motion and rotational Dynamics is the study of how rotation arises.

Base Units

Mass	(m)	Kilogram (kg)
Distance	(s)	Metres (m)
Displacement	$(\vec{\mathbf{s}})$	Metres (m)
Time	(t)	Seconds (s)
Speed	(v)	Metres per Second ($\mathrm{m\,s^{-1}}$ **or** m/s)
Velocity	$(\vec{\mathbf{v}})$	Metres per Second ($\mathrm{m\,s^{-1}}$ **or** m/s)
Acceleration	$(\vec{\mathbf{a}})$	Metres per Second, per Second ($\mathrm{m\,s^{-2}}$ **or** m/s/s **or** $\mathrm{m/s^2}$)
Force	$(\vec{\mathbf{F}})$	Newtons (N) **or** ($\mathrm{kg\,m\,s^{-2}}$)
Angular Position	(θ)	Radians (rad)
Angular Speed	(ω)	Radians per Second ($\mathrm{rad\,s^{-1}}$ **or** rad/s)
Angular Velocity	$(\vec{\boldsymbol{\omega}})$	Radians per Second ($\mathrm{rad\,s^{-1}}$ **or** rad/s)
Angular Acceleration	$(\vec{\boldsymbol{\alpha}})$	Radians per Second, per Second ($\mathrm{rad\,s^{-2}}$ **or** $\mathrm{rad/s^2}$)
Torque	$(\vec{\boldsymbol{\tau}})$	Newton Metres (N m)
Energy	(E)	Joules (J) **or** ($\mathrm{kg\,m^2\,s^{-2}}$)
Work	(W)	Joules (J) **or** (N m) **or** ($\mathrm{kg\,m^2\,s^{-2}}$)
Power	(P)	Joules per second ($\mathrm{J\,s^{-1}}$) **or** ($\mathrm{kg\,m^2\,s^{-3}}$)

Constants

Gravitational acceleration at Earth's surface	(g)	$9.8\ \mathrm{m\,s^{-2}}$
The Gravitational Constant	(G)	$6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Mass of the Earth	(M_E)	6.0×10^{24} kg
Radius of the Earth	(r_E)	6.371×10^6 m

Equations

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \tag{5.1}$$

The linear momentum of an object with mass m and velocity $\vec{\mathbf{v}}$.

$$\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}} \tag{5.2}$$

The net (total) force on an object with constant mass is equal to its mass mutlplied by its acceleration due to that force.

$$a_c = \frac{v^2}{r} \tag{5.3}$$

$$\vec{\mathbf{a}}_c = -\frac{v^2}{r}\hat{\mathbf{r}} \tag{5.4}$$

The magnitude and vector form of the acceleration on an object undergoing circular motion. This is known as centripetal acceleration since it always points inwards towards to axis of rotation.

$$l = \theta r \tag{5.5}$$

The arc length travelled over an angle θ at a radius r.

$$v_\perp = \omega r \tag{5.6}$$

The tangential speed of a particle a distance r from the rotation axis that is part of an object rotating with angular speed ω. In this case r represents the perpendicular radial length from the spin axis.

$$\omega = 2\pi f \tag{5.7}$$

The angular speed is also given by 2π times the rotational frequency (e.g. 1 rotation per second = 2π $rad\,s^{-1}$).

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \tag{5.8}$$

The torque induced by a force $\vec{\mathbf{F}}$ acting at a position $\vec{\mathbf{r}}$ relative to the centre of rotation. $\times$ represents the vector cross product.

$$\vec{\mathbf{g}} = -G\frac{m}{r^2}\hat{\mathbf{r}} \tag{5.9}$$

The gravitational acceleration that will be induced on a mass if placed at a radial position $\vec{\mathbf{r}}$ from the centre of mass of the object with mass m.
This is also known as the gravitational field of the mass m.

$$\vec{\mathbf{F}}_g = m\vec{\mathbf{g}} \tag{5.10}$$

The general expression for the force induced on an object with mass m by a gravitational field g.

$$\vec{\mathbf{F}}_g = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} \tag{5.11}$$

The gravitational force induced on an object with mass m_1 when its centre of mass is placed at a position such that the vector beginning at the centre of mass of the object with mass m_2 and ending at m_1 is $\vec{\mathbf{r}}$.

$$U_g = -G\frac{m_1 m_2}{r} \tag{5.12}$$

The gravitational potential energy of a mass m_1 when its centre of mass is a distance r from the centre of mass of m_2.

Non-Syllabus Equations

$$\vec{\mathbf{r}}_{com} = \frac{\sum_{n=1}^{N} \vec{\mathbf{r}}_n m_n}{\sum_{n=1}^{N} m_n} \tag{5.13}$$

The formula for the centre of mass of an object is the average of each of the positions of the small masses which make it up, weighted according to their mass.

$$\vec{\mathbf{r}}_{com} = \frac{\int_0^M \vec{\mathbf{r}}\, dm}{M} \tag{5.14}$$

The formula for the centre of mass of an object. This is just Eq. 5.13 for continuous mass distributions.

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}_{com}}{dt} \tag{5.15}$$

The linear velocity of an object is the time derivative of its centre of mass' position.

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}_{com}}{dt^2} \tag{5.16}$$

The linear acceleration of an object is the time derivative of its linear velocity.

$$\vec{\mathbf{F}}_{net} = \frac{d\vec{\mathbf{p}}}{dt} \tag{5.17}$$

The net force on an object gives the rate of change of its momentum. For a case where the mass is constant, this gives Eq. 5.2

$$\vec{\boldsymbol{\omega}} = \frac{d\vec{\theta}}{dt} \tag{5.18}$$

The angular velocity is the time derivative of the angular position. Notably absolute angular displacement cannot be treated as a vector but infinitesimally small angular displacements can, so angular velocity is what is known as a pseudovector.

$$\vec{\boldsymbol{\alpha}} = \frac{d\vec{\boldsymbol{\omega}}}{dt} \tag{5.19}$$

The definition of angular acceleration as the rate of change of angular velocity.

$$\vec{\mathbf{v}}_{rot} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} \tag{5.20}$$

The velocity due to rotation of a point on an object with angular velocity ω is given by the cross product of the angular velocity vector and the radius vector from the centre of rotation to the point $\vec{\mathbf{r}}$.

$$\vec{\mathbf{a}}_{\perp} = \vec{\boldsymbol{\alpha}} \times \vec{\mathbf{r}} \tag{5.21}$$

The tangential acceleration on a point on an object which is undergoing angular acceleration $\vec{\boldsymbol{\alpha}}$ and has a position vector relative to the centre of rotation $\vec{\mathbf{r}}$. The $\times$ represents the vector cross product.

$$I = mr^2 \tag{5.22}$$

The moment of inertia of a single point mass with mass m and radius from the axis of rotation r.

$$I = \sum_{n=1}^{N} m_n {r_n}^2 \tag{5.23}$$

The total moment of inertia of a set of N point masses all rotating together about a given rotational axis.

$$I_z = \int_m (r_x^2 + r_y^2)\ dm \tag{5.24}$$

The total Moment of Inertia of an object with a continuous mass distribution about the z axis.

$$I_{xy} = \int_m r_x r_y\ dm \tag{5.25}$$

The Product of Inertia in the xy plane (generalises to all other permutations of axes). Unlike the Moment of Inertia this can take on a negative value. It is also worth noting that the equation is symmetric, i.e. $I_{xy} = I_{yx}$.

$$[I] = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix} \tag{5.26}$$

The Moment of Inertia matrix or tensor.

$$I' = I_{com} + Md^2 \tag{5.27}$$

$$[I]' = [I] + M[(D,\, D)] - 2M[(D,\, C)] \tag{5.28}$$

The Parallel Axis Theorem for both Moment of Inertia (special case) and the Inertia Tensor (generalisation). Allows for the recalculation of moment of inertia when moving the point of rotation a distance d to a point D. C is the location of the centre of mass.

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} \tag{5.29}$$

The angular momentum of a single point mass with mass m and position r. This equation is particularly strange as r does not necessarily have to point from an axis of rotation, though it often does.

$$\vec{\mathbf{L}} = [I]\,\vec{\boldsymbol{\omega}} \tag{5.30}$$

The angular momentum of a mass with moment of inertia tensor $[I]$ and angular velocity $\vec{\boldsymbol{\omega}}$.

$$\vec{\boldsymbol{\tau}}_{net} = \frac{d\vec{\mathbf{L}}}{dt} \tag{5.31}$$

The derivative form of torque. This equation describes the net torque on an object but can also be used with Eq. 5.29 to define the torque from a single force.

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \tag{5.32}$$

The contribution to the total torque by a force $\vec{\mathbf{F}}$.

$$\vec{\boldsymbol{\tau}}_{net} = [I]\,\vec{\boldsymbol{\alpha}} \tag{5.33}$$

The equation for net torque on an object with constant moment of inertia I. This equation has clear parallels to Newton's Second Law (Eq. 5.2)

Circular Motion of a Single Point Mass

A key concept within mechanics is that of circular motion. It is important to understand that circular motion is a very common phenomenon in physics, though it is still technically a special case.
The ideas discussed here **assume** that the object is undergoing circular motion, which requires that you know beforehand that, for some reason, this object must undergo circular motion.
For instance, an object on the end of a string will almost always undergo circular motion since the string cannot extend, therefore forcing the object to move in a circular path.

One cool thing is that it is also possible to show that an object is undergoing circular motion if certain conditions are met, but you have to be careful about when and where you assume circular motion.

Derivation of Centripetal Acceleration

When an object moves in a circle, the net acceleration on that object must be pointed towards the central axis of rotation and have a magnitude of $a_c = \frac{v^2}{r}$.
This acceleration might arise because of tension or the electric field but the acceleration must be this strength.
If the acceleration is greater than this then the object will tend towards the centre and if it is less then it will tend to fly outwards. Notably this would be non-circular motion, though it may still result in orbital motion in a different shape. An example would be the planets which orbit in ellipses since the acceleration is not exactly the correct magnitude.

To derive that the acceleration is pointed inwards and has magnitude $\frac{v^2}{r}$, we merely start with the assumption that an object moves in a circular path of fixed radius r.

There are multiple ways to derive that the acceleration must be of the form in Eq. 5.4, however some ways are more intuitive than others.
The less intuitive proofs are the most general and provide the most accurate derivations (such as §5: Vector Derivation) but are also the most mathematically involved and tend to purely serve the purpose of giving a good result, rather than helping with understanding.

Graphical Derivation

This derivation is perhaps the most intuitive derivation hence its inclusion, however it must use the assumption that the speed is constant. Centripetal acceleration does not require this assumption in general but it shall be made here.

Over some small time interval δt an object moves from a position $\vec{\mathbf{r}}_1$ to position $\vec{\mathbf{r}}_2$. At each of these positions the object has velocities $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. At every point the velocity vector is perpendicular to its associated position vector.
Over this time a small angle $\delta\theta$ is traced out. (Fig. 5.1)

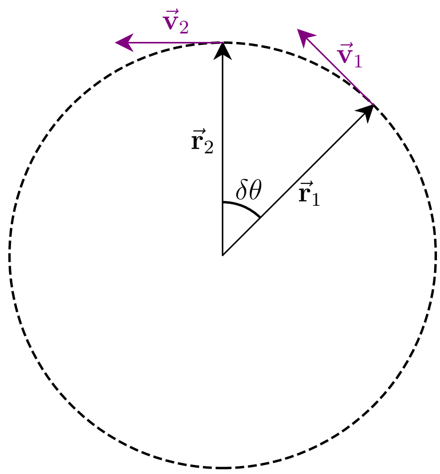


Figure 5.1: Diagram showing the different positions and velocities of an object undergoing circular motion over some time span δt.

Now we can find the change in position and velocity over this time using trigonometry. To do this we shall assume that $\|\vec{\mathbf{v}}_1\| = \|\vec{\mathbf{v}}_2\|$ (i.e. the speed is constant).
Also, since the velocity vector is always perpendicular to the radius, it rotates by the same angle. Therefore, since the angle between the two radius vectors is $\delta\theta$, the angle between the two velocity vectors must also be $\delta\theta$.

One idea which gets a bit confusing for this is the assertion that, as we compress δt towards 0, the change in position gets super small and becomes perpendicular to both $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$.
This allows us to apply trigonometric ratios to the triangles created.

$$\delta\theta + 2\phi = 180^\circ$$

$$\lim_{\delta t \to 0} \delta\theta = 0, \quad \lim_{\delta t \to 0} \phi = 90^\circ$$

Figure 5.2: Diagrams showing the relationship between the initial & final positions and velocities to the change in each of these quantities.

Taking the sine of $\delta\theta$ on each of the diagrams in Fig. 5.2 gives Eq. 5.34 and Eq. 5.35.

$$\sin\delta\theta = \frac{\|\delta\vec{\mathbf{r}}\|}{\|\vec{\mathbf{r}}_1\|} = \frac{\delta r}{r} \tag{5.34}$$

$$\sin\delta\theta = \frac{\|\delta\vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}_1\|} = \frac{\delta v}{v} \tag{5.35}$$

Now we have to set our variables correctly. So far we've dealt with small changes but now we want to take the limit as δt tends towards 0.
This re-establishes our variables as infinitesimal changes.

$$\lim_{\delta t\to 0} \delta t = dt \qquad \lim_{\delta t\to 0} \delta\theta = d\theta \qquad \lim_{\delta t\to 0} \delta r = dr \qquad \lim_{\delta t\to 0} \delta v = dv$$

Also, the sine approximation becomes valid.

$$\lim_{x\to 0}(\sin x) = x \implies \sin d\theta = d\theta$$

This gives a new set of equations and the final result of Eq. 5.36.

$$d\theta = \frac{dr}{r}, \quad d\theta = \frac{dv}{v}$$

$$\frac{dv}{v} = \frac{dr}{r}$$

$$dv = dr\frac{v}{r}$$

$$\frac{dv}{dt} = \frac{dr}{dt}\frac{v}{r} \qquad \frac{d\theta}{dt} = \frac{dr}{dt}\frac{1}{r} \qquad \frac{d\theta}{dt} = \frac{dv}{dt}\frac{1}{v}$$

$$a = v\frac{v}{r} \qquad \omega = \frac{v}{r} \qquad \omega = \frac{a}{v}$$

$$a_c = \frac{v^2}{r} \tag{5.36}$$

We know that the acceleration must be pointed inwards because $\delta\vec{\mathbf{v}}$ is perpendicular to $\vec{\mathbf{v}}$ (as is $\vec{\mathbf{r}}$), so the acceleration (which is in the same direction as $\delta\vec{\mathbf{v}}$) must be inwards along the radius. From Fig. 5.2 we can see that $\delta\vec{\mathbf{v}}$ is 90° anticlockwise from $\vec{\mathbf{v}}_1$, putting it in the opposite direction as $\vec{\mathbf{r}}_1$ (i.e. inwards).

Vector Derivation

Let's start with the assumption that an object has a position vector $\vec{\mathbf{r}}$ where $\|\vec{\mathbf{r}}\| = r$, and r is constant (Eq. 5.37).
Let the velocity of the object be $\vec{\mathbf{v}}$.

$$\frac{dr}{dt} = 0 \tag{5.37}$$

$$\frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{v}}, \quad \|\vec{\mathbf{v}}\| = v \tag{5.38}$$

Applying Eq. III.6 to Eq. 5.37 we get Eq. 5.39.

$$\vec{\mathbf{v}} \cdot \hat{\mathbf{r}} = 0 \tag{5.39}$$

Taking the derivative of both sides then rearranging we get the following.

$$\frac{d}{dt}\left(\vec{\mathbf{v}} \cdot \hat{\mathbf{r}}\right) = 0$$

$$\vec{\mathbf{a}} \cdot \hat{\mathbf{r}} + \vec{\mathbf{v}} \cdot \frac{d\hat{\mathbf{r}}}{dt} = 0 \qquad \text{(from Eq. III.4)}$$

$$\vec{\mathbf{a}} \cdot \hat{\mathbf{r}} + \vec{\mathbf{v}} \cdot \left(\frac{\vec{\mathbf{v}}r - \hat{\mathbf{r}}\left(\vec{\mathbf{v}} \cdot \hat{\mathbf{r}}\right)}{r^2}\right) = 0 \qquad \text{(from Eq. III.7)}$$

$$\vec{\mathbf{a}} \cdot \hat{\mathbf{r}} + \frac{\|\vec{\mathbf{v}}\|^2 r - \left(\vec{\mathbf{v}} \cdot \hat{\mathbf{r}}\right)^2}{r^2} = 0$$

$$\vec{\mathbf{a}} \cdot \hat{\mathbf{r}} + \frac{v^2 r - \cancelto{0}{\left(\vec{\mathbf{v}} \cdot \hat{\mathbf{r}}\right)^2}}{r^2} = 0 \qquad \text{(from Eq. 5.39)}$$

$$\vec{\mathbf{a}} \cdot \hat{\mathbf{r}} = -\frac{v^2}{r} \tag{5.40}$$

Eq. 5.40 is telling us that, for an object moving in a circle, the total acceleration which is parallel to the radius must have a magnitude of $\frac{v^2}{r}$.
Since the dot product is negative it tells us that the acceleration must always be pointed opposite to the radius vector.

This is an important result because it's specifically telling us there could be other accelerations which are perpendicular to the radius which act to change the velocity, and that is completely fine.
Multiplying both sides by $\hat{\mathbf{r}}$ can allow us to re-assign the acceleration a direction and gives us the final result in Eq. 5.41 where $\vec{\mathbf{a}}_r$ represents all acceleration components parallel to $\vec{\mathbf{r}}$.

$$\vec{\mathbf{a}}_r = -\frac{v^2}{r}\hat{\mathbf{r}} \tag{5.41}$$

An In-depth Exploration of Newton's Laws

Newton's 1st Law

"An object will retain its state of motion unless acted upon by an unequal force."

This statement can't really be examined properly without the invocation of his 2nd law $\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}_{net}}{m}$.

What this statement is really saying is that, because $\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt}$, if there is no acceleration then the velocity will not change with time (this is really a definitional thing rather than a 'Law'). And, because accelerations result from forces, there must be a non-zero net force (unequal forces) on the object.

Newton's 2nd Law

$$\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}} \tag{5.2}$$

The derivation for this formula for a single object or particle is based on the derivative definition of force (Eq. 5.17)

$$\vec{\mathbf{F}}_{net} = \frac{d\vec{\mathbf{p}}}{dt} \tag{5.17}$$

Using $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ (Eq. 5.1) and by assuming the mass to be constant we arrive at Newton's result.

However, this definition actually applies to any set of particles (whether part of a rigid solid or not), so long as the velocity used in the momentum definition is the velocity of the centre of mass $\vec{\mathbf{v}}_{com}$ and the mass is to total mass of the particles M (each with constant mass).

Recall the formula for the centre of mass of a system of n particles.

$$\vec{\mathbf{r}}_{com} = \frac{\sum\limits_{n=1}^{N} \vec{\mathbf{r}}_n m_n}{\sum\limits_{n=1}^{N} m_n} \tag{5.13}$$

$$M = \sum_{n=1}^{N} m_n$$

Taking the time derivative of Eq. 5.13 we get the formula for the velocity of the centre of mass.

$$\vec{\mathbf{v}}_{com} = \frac{\sum\limits_{n=1}^{N} \vec{\mathbf{v}}_n m_n}{M}$$

Here is the first important result, the total momentum of the system (the sum of all momentums from each of the small constituent particles) is equal to the total mass multiplied by the velocity of the centre of mass.

$$\vec{\mathbf{p}}_{total} = \sum_{n=1}^{N} \vec{\mathbf{v}}_n m_n = M\vec{\mathbf{v}}_{com} \tag{5.42}$$

So, it turns out that using the centre of mass and total mass is equivalent to just mass and velocity for a system of masses.

The total force on the system is just the sum of all forces on each particle which is just the time derivative of the total momentum.

$$\begin{aligned}
\vec{\mathbf{F}}_{net} &= \frac{d\vec{\mathbf{p}}_{total}}{dt} \\
&= \frac{d}{dt}\left(\sum_{n=1}^{N} \vec{\mathbf{v}}_n m_n\right) \\
&= \sum_{n=1}^{N} \vec{\mathbf{a}}_n m_n \\
&= M\frac{d\vec{\mathbf{v}}_{com}}{dt} \\
&= M\vec{\mathbf{a}}_{com}
\end{aligned}$$

So here we see that Newton's law is generalisable to any system of constant mass particles, so long as the position of the object is given by its centre of mass and its mass given by the total mass.
Perhaps a surprising part of this result is that the system need not be made of bound particles (i.e. the equation applies to systems of particles like gases).

Newton's 3rd Law

"For every action (force) there is an equal and opposite reaction."

This is quite a strange law because it is really purely by luck that Newton was correct here.

So, as it turns out, the forces between two particles due to the Electric and Magnetic forces are symmetric (i.e. two particles attract/repel each other with the same force regardless of charge). This is important because these are the most dominant forces in terms of macroscopic interactions.

$$\vec{\mathbf{F}}_E = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}\hat{\mathbf{r}} \qquad\qquad \vec{\mathbf{F}}_B = \frac{\mu_0}{4\pi}\frac{q_1q_2(\vec{\mathbf{v_2}} \times (\vec{\mathbf{v_1}} \times \vec{\mathbf{r}}_{12}))}{r^3}$$

The non-relativistic formulas for the forces between two charged particles.

The same is true for gravity (yes, you apply the same force on the Earth due to your gravity as it applies to you, just the acceleration on the Earth is much less because it is so big).

$$\vec{\mathbf{F}}_G = -G\frac{m_1m_2}{r^2}\hat{\mathbf{r}}$$

Again, for both of the nuclear forces it is also true regardless of the differing properties of the two particles (though this one is a little harder to show).

As a result, whenever there is a force on one object, there must be an equal and opposite force on the other object.
But, since the forces Newton was observing were primarily gravitational and electrostatic, it was quite possible that when we discovered the nuclear forces that they would have been non-symmetric, rendering Newton's statement false. As such it is really purely blind luck that Newton's 3rd Law is still considered true.

Rotation of a Rigid Body

While rotation of a single point mass is a key concept in physics, most real world examples require that we acknowledge the fact that an object is really made of many point masses at different locations.

Defining & Deriving Some Key Equations

Angular Velocity

Angular velocity ($\vec{\boldsymbol{\omega}}$) is a vector[1] which points along the axis of rotation of an object such that if you were to point your right thumb in the direction of the vector, your other fingers curl in the direction of rotation.

While not often explicitly said, angular velocity is always considering motion "about a point" $P = (P_x,\, P_y,\, P_z)$ – meaning it is different if considering motion about two different points.

For a point mass with displacement from a point P $\vec{\mathbf{r}}$ and angular velocity about P of $\vec{\boldsymbol{\omega}}$ we define its velocity due to rotation to be given by Eq. 5.20.

$$\vec{\mathbf{v}}_{rot} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} \tag{5.20}$$

Angular Momentum

Angular momentum ($\vec{\mathbf{L}}$) can seem like an arbitrary concept at first but it has some very useful properties. We begin by defining angular momentum as being about a point $P = (P_x,\, P_y,\, P_z)$. The angular momentum for a point mass about point P is given by

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} \tag{5.29}$$

Where $\vec{\mathbf{p}}$ is the momentum of the point mass and $\vec{\mathbf{r}}$ is the displacement vector from P to the location of the mass.

For an infinitesimally small point mass this equation can be written as in Eq. 5.43.

$$\vec{\mathbf{L}} = \int_m \vec{\mathbf{r}} \times d\vec{\mathbf{p}} = \int_m \vec{\mathbf{r}} \times \vec{\mathbf{v}}\ dm \tag{5.43}$$

Moment of Inertia & The Inertia Tensor

Moment of Inertia is a somewhat contrived concept in that it is more mathematical than it is intuitive. As such, seeing where it comes from is perhaps easier than trying to justify its existence conceptually (though there is a conceptual way of understanding it which will be covered in a later section).

Using Eq. 5.20 and the identity $\vec{\mathbf{A}} \times (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = (\vec{\mathbf{A}} \cdot \vec{\mathbf{C}})\vec{\mathbf{B}} - (\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})\vec{\mathbf{C}}$, Eq. 5.43 can be re-written

[1] Technically it is a pseudovector for a multitude of complex reasons that don't really matter. One of which is because it does not describe one motion but the collective motion of a set of particles.

$$\vec{\mathbf{L}} = \int_m \vec{\mathbf{r}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})\ dm = \int_m \left(\|\vec{\mathbf{r}}\|^2 \vec{\boldsymbol{\omega}} - (\vec{\mathbf{r}} \cdot \vec{\boldsymbol{\omega}})\,\vec{\mathbf{r}}\right)\ dm \tag{5.44}$$

Further expanding Eq. 5.44 into its separate components can then allow for the derivation of the inertia tensor.
It is worth making it explicit that here is where we impose the rigid body restriction, which forces $\vec{\boldsymbol{\omega}}$ as constant across the entire mass, allowing it to be factored out of any integrals as a constant (with respect to the location / mass).

$$\vec{\mathbf{L}} = \int_m \left(\|\vec{\mathbf{r}}\|^2 \vec{\boldsymbol{\omega}} - (\vec{\mathbf{r}} \cdot \vec{\boldsymbol{\omega}})\,\vec{\mathbf{r}}\right)\ dm$$

$$= \begin{bmatrix} \int_m \left(\omega_x\left(r_x^2 + r_y^2 + r_z^2\right) - \left(r_x^2\omega_x + r_xr_y\omega_y + r_xr_z\omega_z\right)\right)\ dm \\ \int_m \left(\omega_y\left(r_x^2 + r_y^2 + r_z^2\right) - \left(r_yr_x\omega_x + r_y^2\omega_y + r_yr_z\omega_z\right)\right)\ dm \\ \int_m \left(\omega_z\left(r_x^2 + r_y^2 + r_z^2\right) - \left(r_zr_x\omega_x + r_zr_y\omega_y + r_z^2\omega_z\right)\right)\ dm \end{bmatrix}$$

$$= \begin{bmatrix} \int_m \left(\omega_x\left(r_y^2 + r_z^2\right) - (r_xr_y\omega_y + r_xr_z\omega_z)\right)\ dm \\ \int_m \left(\omega_y\left(r_x^2 + r_z^2\right) - (r_yr_x\omega_x + r_yr_z\omega_z)\right)\ dm \\ \int_m \left(\omega_z\left(r_x^2 + r_y^2\right) - (r_zr_x\omega_x + r_zr_y\omega_y)\right)\ dm \end{bmatrix}$$

$$= \begin{bmatrix} \omega_x \int_m (r_y^2 + r_z^2)\ dm - \omega_y \int_m r_xr_y\ dm - \omega_z \int_m r_xr_z\ dm \\ -\omega_x \int_m r_yr_x\ dm + \omega_y \int_m (r_x^2 + r_z^2)\ dm - \omega_z \int_m r_yr_z\ dm \\ -\omega_x \int_m r_zr_x\ dm - \omega_y \int_m r_zr_y\ dm + \omega_z \int_m (r_x^2 + r_y^2)\ dm \end{bmatrix}$$

$$= \begin{bmatrix} \left(\int_m (r_y^2 + r_z^2)\ dm\right) & \left(-\int_m r_xr_y\ dm\right) & \left(-\int_m r_xr_z\ dm\right) \\ \left(-\int_m r_yr_x\ dm\right) & \left(\int_m (r_x^2 + r_z^2)\ dm\right) & \left(-\int_m r_yr_z\ dm\right) \\ \left(-\int_m r_zr_x\ dm\right) & \left(-\int_m r_zr_y\ dm\right) & \left(\int_m (r_x^2 + r_y^2)\ dm\right) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

The above uses matrix multiplication to factor out the angular velocity vector, see §I:Matrix Multiplication for further clarification on the process of matrix multiplication.

We then simply define the matrix (Eq. 5.45) as the inertia tensor or inertia matrix.

$$[I] \coloneqq \begin{bmatrix} \left(\int_m (r_y^2 + r_z^2)\ dm\right) & \left(-\int_m r_x r_y\ dm\right) & \left(-\int_m r_x r_z\ dm\right) \\ \left(-\int_m r_y r_x\ dm\right) & \left(\int_m (r_x^2 + r_z^2)\ dm\right) & \left(-\int_m r_y r_z\ dm\right) \\ \left(-\int_m r_z r_x\ dm\right) & \left(-\int_m r_z r_y\ dm\right) & \left(\int_m (r_x^2 + r_y^2)\ dm\right) \end{bmatrix} \tag{5.45}$$

By using the below definitions for the moment of inertia and product of inertia the matrix can be re-written using simpler notation.

Moments of Inertia

$$I_x = \int_m (r_y^2 + r_z^2)\ dm \qquad I_y = \int_m (r_x^2 + r_z^2)\ dm \qquad I_z = \int_m (r_x^2 + r_y^2)\ dm$$

Products of Inertia

$$I_{xy} = \int_m r_x r_y\ dm \qquad I_{xz} = \int_m r_x r_z\ dm$$

$$I_{yx} = \int_m r_y r_x\ dm \qquad I_{yz} = \int_m r_y r_z\ dm$$

$$I_{zx} = \int_m r_z r_x\ dm \qquad I_{zy} = \int_m r_z r_y\ dm$$

$$[I] = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix} \tag{5.26}$$

This finally allows us to re-write the definition of angular momentum for any rigid body (Eq. 5.30).

$$\vec{\mathbf{L}} = [I]\,\vec{\boldsymbol{\omega}} \tag{5.30}$$

In many cases for an object rotating about its centre of mass (and even some other cases) the products of inertia are 0 (Eq. 5.46), simplifying the matrix to a diagonal matrix, making it much simpler to invert (Eq. 5.47).

$$[I] = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{5.46}$$

$$[I]^{-1} = \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix} \tag{5.47}$$

Torque

The formal definition of torque is as the time derivative of angular momentum (Eq. 5.31).

$$\vec{\boldsymbol{\tau}}_{net} = \frac{d\vec{\mathbf{L}}}{dt} \tag{5.31}$$

For a rigid body with a constant inertia matrix, the torque is given by Eq. 5.33.

$$\vec{\boldsymbol{\tau}}_{net} = \frac{d}{dt}\left([I]\,\vec{\boldsymbol{\omega}}\right)$$

$$\vec{\boldsymbol{\tau}}_{net} = [I]\,\frac{d}{dt}\left(\vec{\boldsymbol{\omega}}\right)$$

$$\vec{\boldsymbol{\tau}}_{net} = [I]\,\vec{\boldsymbol{\alpha}} \tag{5.33}$$

In linear motion we have $\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}}$, where m describes how hard the object is to push. Torque is analogous to force, in that $[I]$ describes how hard the object is to rotate in each direction and $\vec{\boldsymbol{\alpha}}$ describes the acceleration of rotational motion.

There is one other definition which is required for torque which is based on the original definition for angular momentum of a particle.

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

$$\frac{d}{dt}\vec{\mathbf{L}} = \frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{p}})$$

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{v}} \times \vec{\mathbf{p}} + \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

Here we use $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$ to get the final relationship (Eq. 5.32).

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \tag{5.32}$$

This equation describes the contribution to the net torque by a force $\vec{\mathbf{F}}$. The net torque is then just the vector sum of all of these contributions.

$$\vec{\boldsymbol{\tau}}_{net} = \sum \vec{\mathbf{r}} \times \vec{\mathbf{F}} \tag{5.48}$$

Module 6: Electromagnetism

Electromagnetism is the area of physics that defines the behaviour of the electric ($\vec{\mathbf{E}}$) and magnetic ($\vec{\mathbf{B}}$) fields and their eventual combination into the electromagnetic field.

Base Units

Charge	(q)	Coulombs (C)
Electric Field	$(\vec{\mathbf{E}})$	Newtons per Coulomb ($\mathrm{N\,C^{-1}}$)
Electric Potential Energy	(U_E)	Joules (J)
Electric Potential	(V)	Volts (V) **or** Joules per Coulomb ($\mathrm{J\,C^{-1}}$)
Electric Flux	(Φ_E)	Volt Metres (V m)
Magnetic Field	$(\vec{\mathbf{B}})$	Tesla (T)
Magnetic Flux	(Φ_B)	Weber (Wb) **or** Tesla Square Metres ($\mathrm{T\,m^2}$)

Constants

Permittivity of Free Space	(ε_0)	$8.754 \times 10^{-12}\ \mathrm{A^2\,s^4\,kg^{-1}\,m^{-3}}$
Permeability of Free Space	(μ_0)	$4\pi \times 10^{-7}\ \mathrm{N\,A^{-2}}$
Mass of an Electron	(m_e)	9.109×10^{-31} kg
Mass of a Proton	(m_p)	1.673×10^{-27} kg
Mass of a Neutron	(m_n)	1.675×10^{-27} kg
Charge of an Electron	(q_e)	-1.602×10^{-19} C
Charge of a Proton	(q_p)	$+1.602 \times 10^{-19}$ C

Equations

Electrostatics

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} \tag{6.1}$$

The force on a charge q in an electric field $\vec{\mathbf{E}}$.

$$E = \frac{V}{d} \tag{6.2}$$

The uniform electric field strength between two large parallel charged plates, seperated by a distance d, with an applied voltage difference of V across them.

$$W = qV = q\vec{\mathbf{E}} \cdot \vec{\mathbf{d}} \tag{6.3}$$

The work done on a charge q moving across a voltage difference V. The work is also given by the dot product of the electric field $\vec{\mathbf{E}}$ (assumed uniform) and the displacement vector $\vec{\mathbf{d}}$.

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \tag{6.4}$$

The electric field vector at a point such that the vector from the charge Q to that point is given by $\vec{\mathbf{r}}$.

$$\vec{\mathbf{F}} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{\mathbf{r}} \tag{6.5}$$

The force on a charge q from the electric field of charge Q. $\vec{\mathbf{r}}$ is defined as beginning at Q and ending at q.

$$\vec{\mathbf{p}} = q\vec{\mathbf{d}} \tag{6.6}$$

The dipole moment of a pair of opposite charges (with charges q and $-q$). $\vec{\mathbf{d}}$ points from the negative charge to the positive charge.

$$\vec{\tau}_{\mathbf{p}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}} \tag{6.7}$$

The torque on an electric dipole with dipole moment $\vec{\mathbf{p}}$ in an external uniform electric field $\vec{\mathbf{E}}$.

$$\sigma = \frac{dq}{dA} \tag{6.8}$$

$$\rho = \frac{dq}{dV} \tag{6.9}$$

The standard surface charge density (σ) and volume charge density (ρ) forms.

$$\Phi_E = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \tag{6.10}$$

The electric flux Φ_E through a surface is the sum of the parllel components of the field vector at a given point $\vec{\mathbf{E}}$, multiplied by the area of the surface at that point $d\vec{\mathbf{A}}$.

$$\oiint_{surface} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{enc}}{\varepsilon_0} \tag{6.11}$$

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\mathbf{E}} = \frac{\rho}{\varepsilon} \tag{6.12}$$

Gauss' Law for the electric field.

$$\oiint_{surface} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \tag{6.13}$$

$$\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\mathbf{B}} = 0 \tag{6.14}$$

Gauss' Law for the magnetic field

Gauss' Law

Integral Form

$$\oiint_{surface} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{enc}}{\varepsilon_0} \tag{6.11}$$

$$\oiint_{surface} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \tag{6.13}$$

The first thing to get right here is the integral part of the equations and what they are actually saying and that requires us to talk about flux.