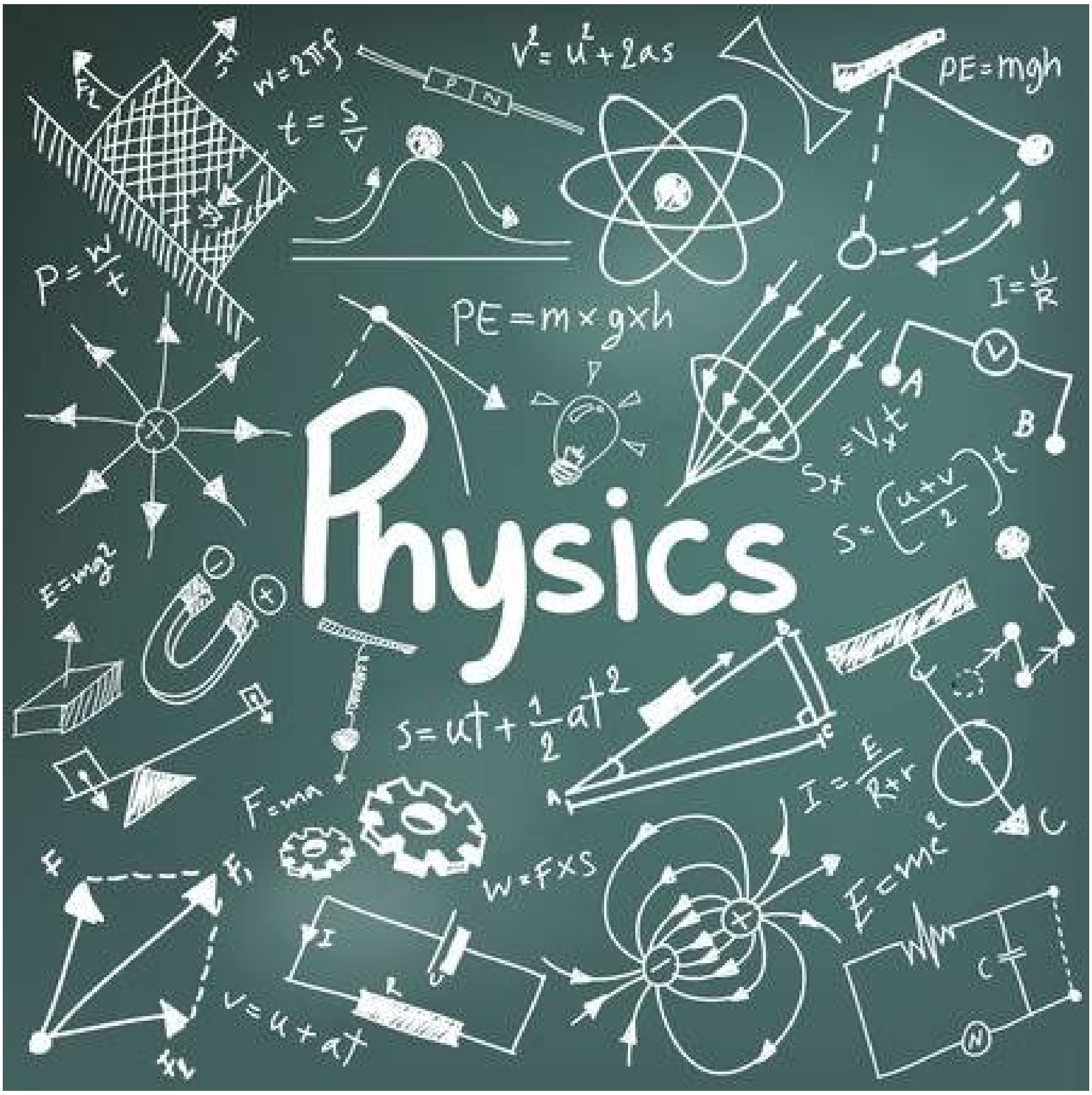
Year 11 & 12 Physics Notes (Modules 1-8)

By Alex Gray



Contents

MODULE 1: KINEMATICS

Base Units

Mass (m) – Kilograms (kg)
Distance (s) – metres (m)
Displacement ($\vec{s}$) – metres (m)
Time (t) – Seconds (s)
Speed (v) – Metres per second (ms^{-1} **or** m/s)
Velocity ($\vec{v}$) – Metres per second (ms^{-1} **or** m/s)
Acceleration ($\vec{a}$) – (ms^{-2} **or** m/s^2)

Equations

$s = |\vec{s}|$
$v = |\vec{v}|$

Distance is equal to the magnitude of Displacement (i.e. without a direction)

Speed is equal to the magnitude of the Velocity (i.e. without a direction)

$$\vec{v} = \frac{\Delta\vec{s}}{t} = \frac{\vec{s}_f - \vec{s}_i}{t}$$

$$\vec{a} = \frac{\Delta\vec{v}}{t} = \frac{\vec{v}_f - \vec{v}_i}{t}$$

<u>Average</u> Velocity is the change in displacement over time

<u>Average</u> Acceleration is the change in velocity over time

$$\vec{s} = \vec{v}_i t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}_f = \vec{v}_i + \vec{a}t$$

$$v_f^2 = v_i^2 + 2\vec{a}s$$

Standard Kinematics equations.

NOTE: For interest sake, you can derive the first equation with respect to time to find the second equation. The equations themselves are integrals (reverse derivatives) assuming $\frac{d\vec{a}}{dt} = 0$ *(i.e. acceleration is constant, or jerk is 0).*

Derivative Forms

(only for those interested)

$$\frac{d\vec{s}}{dt} = \vec{v}$$

The first derivative of Displacement is velocity.

$$\frac{d^2\vec{s}}{dt^2} = \frac{d\vec{v}}{dt} = \vec{a}$$

The second derivative of Displacement, or the first derivative of Velocity, is Acceleration.

$$\frac{d^3\vec{s}}{dt^3} = \frac{d^2\vec{v}}{dt^2} = \frac{d\vec{a}}{dt} = \vec{j}$$

The derivative of Acceleration is Jerk. (All Senior Physics assumes Jerk is 0.)

Course Notes

Significant Figures

Sig. Figs are weird. However, the concept revolves around recursion and can be explained as such: *Any zero which can be written an infinite number of times in any direction while keeping the number the same, is not significant i.e. 1 = 1.0000000000000000… (The zeros can continue forever and are not significant). Therefore, any digit which is non-zero can be considered significant.*

When rounding to n significant figures, if there are more significant figures than required, merely round n^{th} digit up if the next digit is 5 or higher and down if it is less.
If there are less significant figures then all you have to do is add zeros to show that there is more precision in the answer. This is used when writing measurements, allowing the precision of the measurement to be given.

For example:
Your ruler measures to the nearest cm, but you found a stick to be exactly 100cm long. You could write that as '=1m' but since your ruler measures to the nearest cm, you know with more precision how close it is to 1m. Therefore, you write it as =1.00m long.

Measurement Error

Any measurement device has a recording error of ±1/2 of the units it measures in e.g. *a ruler that measures to the nearest cm has a possible error of ±5mm.*

Measurement error also relates to how you answer questions. Numbers are always given to the correct number of significant figures based off the measurement error. As such, you base your answer off the least accurate measurement.

When giving answers, ensure to give answers to the minimum number of significant figures present in the question.
e.g. ***Find the force when accelerating a 5.00kg block at 10.00ms^{-2} (ignoring friction and air resistance):***
The number of sig. figs in the mass of the block is 3 and the number of sig. figs in the acceleration is 4. Given the Force will equal 50N, we write it to the smallest number of sig. figs: 3. So our answer is 50.0N

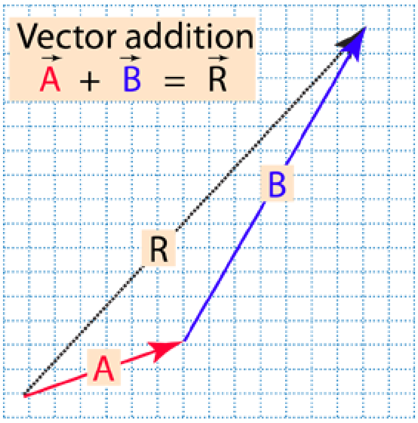
Vectors

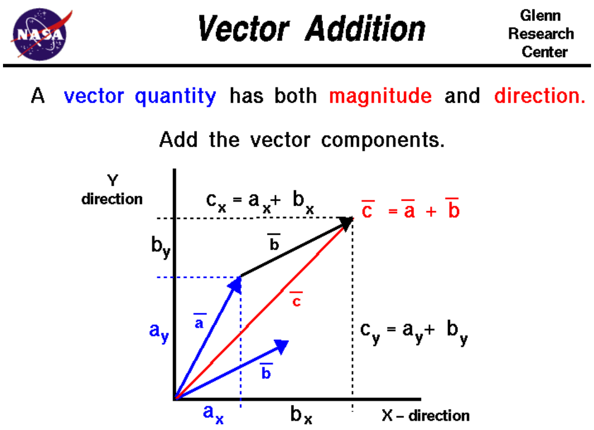
A vector quantity, denoted by an arrow ($\overrightarrow{e.g.}$), possesses a magnitude and a direction. Unlike **Scalar** quantities, **Vector** quantities can be negative or positive, based on their direction. Vectors are a construct of the coordinate plane which we humans invented. Because this plane is relative, we can orient it however we wish. As such, it isn't important which direction is positive, as long as vectors facing the opposite direction are negative with respect to each other.
This can be seen in a velocity vector i.e. *5m/s North* is also *-5m/s South*. Equally, a *5m/s South* vector is *-5m/s North.*

Vectors can also be in multiple directions at once, for instance a velocity vector defined as *10 ms*$^{-1}$ *(N 30˚ E)* is in both the North direction and the East direction. The magnitude which is in each direction is determined by the angle of the vector. In this instance we can use Sine, Cosine and Tangent ratios to solve for the magnitude in each direction. You can also apply this if you are given the separate components of a vector but not the whole vector, using the sum of the vectors to find the total vector. (Refer to Tutorial 5 in Book 1 for questions if unsure)

Vectors can also be added (*and multiplied but that's not relevant yet*). When adding vectors, fundamentally you are adding their x-components and adding their y-components, therefore the first method to vector addition is just that. As such this is called the component method and uses the standard trigonometric ratios to find each component.

The second method is a bit less indirect but works just the same, the Cosine rule [$a^2 = b^2 + c^2 - 2bc \cdot cos(A)$]. It has arguably less computation to it but requires that you know what you're doing to a greater degree.
When adding vectors, they are drawn as such:





The component method is used as above, where the resulting vector, $\vec{c}$, is the conjoining side for the triangle constructed from the sum of $\vec{a}$ and $\vec{b}$.

The Cosine Rule method requires you use the rule $a^2 = b^2 + c^2 - 2bc\cos(A)$ to find the magnitude of the resulting vector. However, finding the angle of the resulting vector can prove more difficult and will require the Sine Rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \textbf{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

For more information on vector addition with both methods see:
Component → https://www.youtube.com/watch?v=6Kw2nIwWYL0
Cosine Rule → https://www.youtube.com/watch?v=ZElOxG7_m3c

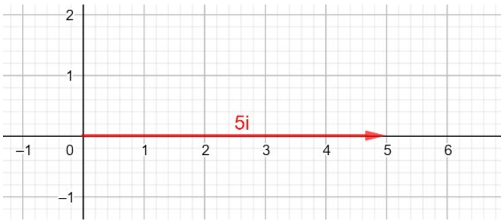
Mathematical Notation for Vectors

In mathematics, vectors are denoted using the same vector arrow but can also be represented as a matrix. This is known as component form. For a 2D vector this would look like:

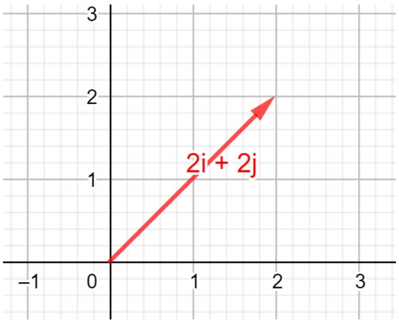
$$\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Vectors are also denoted using i, j, k notation (linear form). i is a vector of length 1 which points along the positive x direction; j is a vector of length 1 pointing along the positive y direction; and k is a vector of length 1 which points along the positive z direction.

So, the vector $5i$ would look like this:



Similarly, the vector $2i + 2j$ would look like:



We could have also written $5i$ as $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $2i + 2j$ as $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Adding vectors can be best represented using component form:

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$$

We can also use linear form to do this, though it is less intuitive:

$$(ai + bj) + (ci + dj) = (a + c)i + (b + d)j$$

This is why we break vectors into x and y components, it makes adding them easier.

Relative Vectors

Vectors can also be relative to each other; in fact you will find any two vectors are always relative. Relative vectors are used to change reference frame

The typical example for this is two trains travelling towards each other at equal and opposite velocities. To an observer outside the trains, both trains appear to be travelling at the same speed. But to someone on one of the trains, the other train appears to be travelling at double the speed.

Relative velocity is often spoken of in terms of *Velocity of A relative to B*.
What this means is: If you are object B, how fast does object A appear to be travelling.

$$\vec{v}_{A\,rel.B} = \vec{v}_A - \vec{v}_B$$

This can also be used for vector addition, where Velocity of A, relative to B can be drawn as
$\vec{v}_A + (-\vec{v}_B)$
By doing this we are shifting into the reference frame of train A.

We can apply this to our trains from before. If both our trains are travelling at $10ms^{-1}$ but train A is going North, while train B is going South, then the Velocity of A relative to B is:

$$(10ms^{-1}N) - (10ms^{-1}S) = (10ms^{-1}N) + (10ms^{-1}N) = 20ms^{-1}North$$

The reasoning behind this is that we take the velocity of train A as positive and since train B is going in the opposite direction, we change its direction to North but negative (since it is going South).

Another way of rationalising relative vectors is thinking about the simplest scenario, a moving train passing a stationary person. To the train, the relative velocity of the person is in the opposite direction to the train's motion. As such, the relative velocity of the person is the negative velocity of the train, as when the train moves past the person while going North, the person appears to be going at the same speed but South.

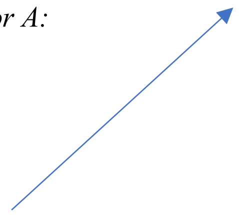
Now if the person was running South, then they would still appear to be going South, but they would appear to be travelling South faster. This is because they are now running in that direction, so we add their running velocity to the negative velocity of the train to get the final relative velocity.

A Note on Negative Vectors

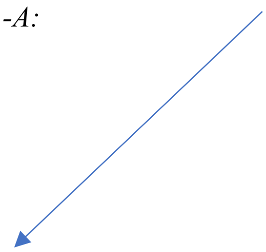
When performing vector addition, a negative vector is the reverse of the vector, such that all components have their direction reversed. So, a vector (N 30˚ E) would become (S 30˚ W).

If you are using the traditional vector notation with *x* and *y* components, multiply both components by -1 before adding.

Vector A:



Vector -A:



MODULE 2: DYNAMICS

Base Units

Force ($\vec{F}$) – Newtons (N) **or** $(kg\ m\ s^{-2})$
Energy (E) – Joules (J) **or** $(kg\ m^2\ s^{-2})$
Work (W) – Newton-Metres (Nm) **or** Joules (J)
Power (P) – Watts (W **or** Js^{-1})
Gravity (g) – Acceleration (ms^{-2})

Equations

$\vec{F}_{net} = m\vec{a}$

Newtons is a measure of Force, One Newton is equivalent to one kilogram metres-per-second^2 (kg ms^{-2}). The net comes from the fact there can be a force pair on an object where it is immobile, therefore the net force is 0N.

$\vec{F}_f = \mu_f F_N$

$\vec{F}_f \leq \mu_s N$ **and** $\vec{F}_f = \mu_k N$

Force of friction is equal to the coefficient of friction between two surfaces, μ, multiplied by the force pushing the two together, N (normal force).

$\vec{F}_{\parallel} = F\cos\theta$ **and** $\vec{F}_{\perp} = F\sin\theta$

$\vec{F}_x = F\cos\theta$ **and** $\vec{F}_y = F\sin\theta$

Component breakdown of the gravity force on an object which is on an inclined slope .

$W = \vec{F}\cdot\vec{s} = \vec{F}_{\parallel}\vec{s} = Fs\cos\theta$

Work done by a Force is equal to the dot product of the Force and distance covered. This is the same as the parallel component of the Force to the distance, multiplied by the distance.

$W = \Delta U$

Work done is also the change in potential energy. If the potential energy goes down, work has been done by the field. If potential energy has increased, work has been input into the system to increase the potential.

$E_K = \frac{1}{2}mv^2$

The ordered kinetic energy of a system.

$U_G = mgh$

Gravitational Potential Energy – The potential of a gravitational body to do work near Earth's surface.

Proof:

$m\vec{g} \cdot \Delta\vec{h} = \vec{F} \cdot \vec{s} = W$

$W_g = -mg\Delta h$

Work done in a uniform gravitational system is equal to the negative change in potential energy. An increase in potential energy requires work be done, while if gravity did the work there is a corresponding reduction in potential.

Note:

$\frac{W}{t} = \frac{\vec{F} \cdot \vec{s}}{t} = Fv\cos\theta$

$P = \frac{\Delta U}{t} = \frac{W}{t} = Fv$

Power is work done per second.

$\vec{p} = m\vec{v}$

Momentum: mass multiplied by velocity.

Note:

$m\Delta\vec{v} = m\vec{a}\Delta t = \vec{F}_{net}\Delta t$

$I = \Delta\vec{p} = \Delta m\Delta\vec{v}$

Impulse describes change in momentum

Derivative Forms

(only for those interested)

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}m\vec{v} = m \cdot \frac{d}{dt}\vec{v} = m\vec{a} = \vec{F}$$

The first derivative of Momentum with respect to time is Force.

$$\frac{dE_k}{dt} = \frac{dE_k}{dv} \cdot \frac{dv}{dt} = \frac{d}{dv}\left(\frac{1}{2}mv^2\right) \cdot \frac{d\vec{v}}{dt} = m\vec{v} \cdot \vec{a} = \vec{F} \cdot \vec{v} = P$$

The first derivative of Kinetic Energy with respect to time is Force times Velocity, or Power (using chain rule).

Course Notes

Newton's Laws

1st Law - *Inertia*

"***An object in motion will retain its state of motion unless acted on by an external force***"

This law has a logical derivative which is not often stated but is assumed knowledge in Physics:
An object that has a uniform state of motion or is not moving has 0 net Force acting on it,
i.e. a box hanging from the ceiling has an equal amount of force acting on it from gravity as there is from the string holding it, therefore the force from the string vertically is equal to the force from gravity.

2nd Law - *Force*

$\vec{F} = m\vec{a}$

3rd Law – *Law of Pairs*

Any Force will exist in an equal pair. Often these forces are referred to as "reaction forces" however this leads to the misconception that the mere existence of a Force creates another. In actuality, the fact that a force exists demands that another force also exists which is equal and opposite, however it must already exist. For example, a proton attracting an electron. The reaction force is the electron attracting the proton; however, this force is created by the electron due to the electron's charge, not because the proton is pulling on the electron.

Friction Forces

Friction is one of the simpler forces that occur in Physics, however, there are a couple of important details to note about friction. The first is that static friction is a **maximum** value. The force of static friction is always equal the force pushing it, otherwise the object would not be "static".
Kinetic friction is different. Kinetic friction has a constant value, no matter what velocity the objects are moving at (except $\vec{v} = 0$). The force of kinetic friction is always in the opposite direction to the motion of the object.

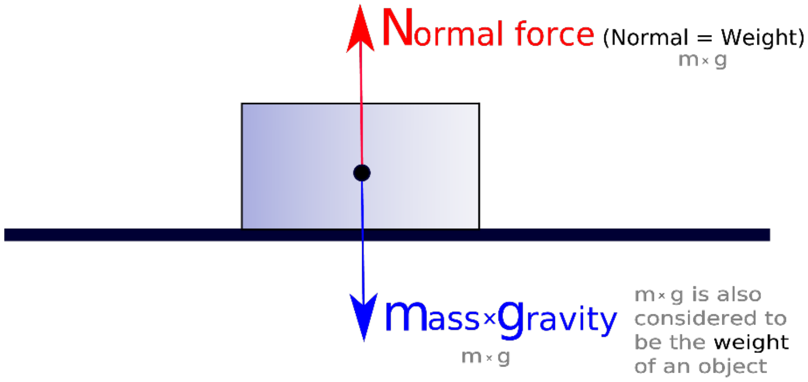
Tension Forces

Tension forces can be one of the most confusing part of this topic if you don't understand it. To summarise, tension force occurs due to bonds between molecules which pulls any two molecules back together. Because there are many thousands of molecules all pulling each other together, the sum of the forces results in two forces, each pulling with equal strength towards the centre of mass of the string but in opposite directions (creating our force pair). Objects like chains can also exert a tension force. *[See Tutorial 1 for practice]*

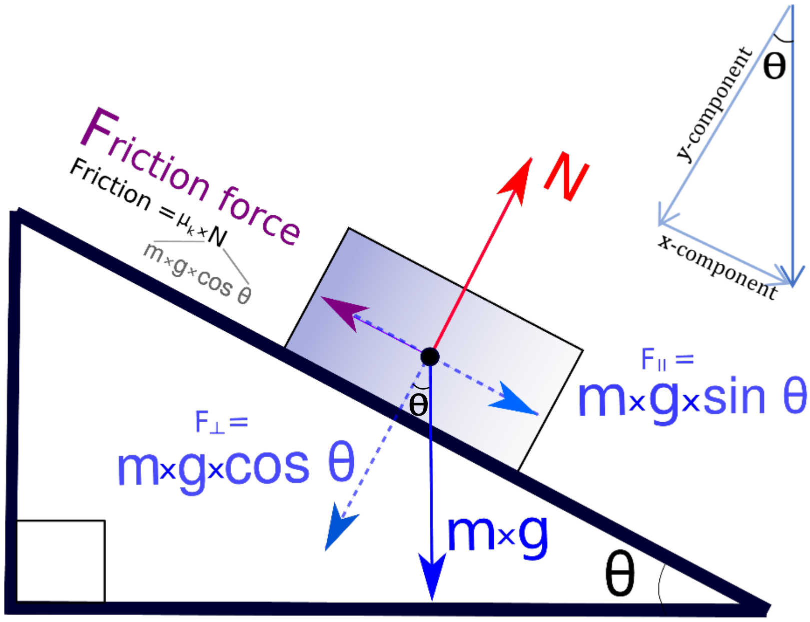
The tension in a string is the sum of the forces pulling in the direction that would stretch the string.

Vector Diagrams

Force diagrams, especially solving for components, can get tricky. They require that you are meticulous and aware of all possibilities.



As seen in the first example, the net force is 0N and, as such, the object is at rest. The force due to gravity is the mass (m) x acceleration (g)



In this example, the vectors are more difficult to figure out. The first thing which we know is that the angle of elevation is equal to the angle the gravity vector makes with its' y-component. The two components can then be solved by using $mg \sin\theta$ and $mg \cos\theta$.

The y-component always points perpendicular to the slope and the x-component always points parallel down the slope*.

The y-component is almost always equal and opposite to the normal force.

If the object is in equilibrium (motionless) then the x-component is equal to the Force of static friction.
If the object is moving, then the x-component is equal to or greater than the Force of kinetic friction.

*This is what is expected for a component breakdown however which directions you choose to break up a vector into is arbitrary and not important. As long as it is broken up correctly (the sum of the components is the original) then it is valid. The trick is knowing which components to break it into for the given scenario.

Conservation of Energy

The law of Conservation of Energy states that the total energy in a closed system remains un-changed.

$$\sum E_{before} = \sum E_{after}$$

Although this law really requires that all energies (including chemical potential) be accounted for, it can be applied to kinetic energy in certain scenarios.

$$\sum \frac{1}{2} m v^2{}_{before} = \sum \frac{1}{2} m v^2{}_{after}$$

Conservation of Momentum

The law of Conservation of Momentum is closely related to the law of Conservation of Energy. The law states that momentum in a closed system is always conserved.

$$\sum m\vec{v}_{before} = \sum m\vec{v}_{after}$$

This law has led to the development of the terms Elastic and In-Elastic collisions. An elastic collision is one where the ordered velocity of objects before and after is perfectly conserved, or equally, it is a collision where kinetic energy is conserved.

An Inelastic collision is a collision where kinetic energy is converted to heat energy or some other energy. Think about clapping, clap for long enough and your hands get red and warm up, the kinetic energy of the clap has converted to heat, this collision is therefore, inelastic.

It is important to note that momentum is still conserved in this situation, the momentum is merely no longer the ordered kinetic motion of the hand, rather the momentum has transferred to the molecules in the hand and has moved them randomly, increasing heat.

Collisions

Elastic

An Elastic collision is one where both ordered kinetic energy and momentum are conserved (i.e. two objects collide and rebound back).

$$\sum \frac{1}{2} m v^2{}_{before} = \sum \frac{1}{2} m v^2{}_{after}$$

$$\sum m\vec{v}_{before} = \sum m\vec{v}_{after}$$

Inelastic

In an inelastic collision, only momentum is conserved (i.e. two objects hit and both stop).

$$\sum m\vec{v}_{before} = \sum m\vec{v}_{after}$$

MODULE 3: WAVES & THERMODYNAMICS

Base Units

Frequency (f) – Hertz (Hz **or** s^{-1})
Period (T) – Seconds (s)
Wavelength (λ) – Metres (m)
Amplitude – Metres (m)
Intensity (I) – Energy of light in an area – *Luminosity or Candela* (cd)

Refractive Index of substance-x – (n_x)

Speed of Light (c) – [$3.00 \times 10^8\ ms^{-1}$]

Temperature (T) – Kelvin (K) **or** Celsius ($^\circ C$)
Thermal Energy (Q) – Joules (J)
Specific Heat (c) – Joules *per* kg *per* Kelvin ($J \cdot kg^{-1} \cdot K^{-1}$)
Thermal Conductivity (k) – Joules *per* metre *per* second *per* Kelvin ($J \cdot m^{-1} \cdot s^{-1} \cdot K^{-1}$)
Latent Heat (L) – Joules *per* kg ($J \cdot kg^{-1}$)

Equations

Waves:

$f = \frac{1}{T} = T^{-1}$ $\qquad$ $T = \frac{1}{f} = f^{-1}$

Frequency is the inverse of Period.

$$v = f\lambda$$

Universal wave equation, describing the relationship between speed, frequency and wavelength.

$$f_{beat} = |f_2 - f_1|$$

Beats of interfering waves

$$f' = f\left(\frac{\vec{v}_{wave} - \vec{v}_{observer}}{\vec{v}_{wave} - \vec{v}_{source}}\right)$$

Doppler Effect – Note that the equation is different to the official formula sheet.

$$n_x = \frac{c}{v_x}$$

The refractive index, n, is the ratio of the universal constant, c, to the speed of light in the given substance, v_x.

$$n_i \sin\theta_i = n_r \sin\theta_r$$

Snell's Law. Rewritten using different notation but same formula.
i *signifies incidence,* r *signifies refraction.*

$$\theta_c = \sin^{-1}\frac{n_r}{n_i}$$

Critical angle of refraction. The critical angle is defined as the angle at which a light ray will refract to 90˚ from the normal.
Any angle > θ_c *will reflect.*

Mathematical Proof from Snell's Law:
$\theta_i = \theta_c$ when $\theta_r = 90^\circ$.
$\therefore\ \sin\theta_r = 1$
$\therefore n_i \sin\theta_c = n_r \times 1$ $\therefore \sin\theta_c = \frac{n_r}{n_i}$

$$I_1 {r_1}^2 = I_2 {r_2}^2$$

The intensity of light in a given area will degrade over a distance as a function of the surface area of a sphere. This equation only applies to a single source at a time.

Thermodynamics:

$$Q = mc\Delta T$$

Internal heat energy will increase dependant on the specific heat, c*, of a substance and its mass.*

$$Q = mL$$

Latent Heat Formula.
Heat required for a phase change / to overcome chemical potential of phase.

$$\frac{Q}{t} = \frac{kA\Delta T}{d}$$

The rate of heat transfer (in Watts) of an object where:
A → Cross-section Surface area
k → Thermal conductivity of substance
d → Thickness of object parallel to direction of heat transfer

Note:

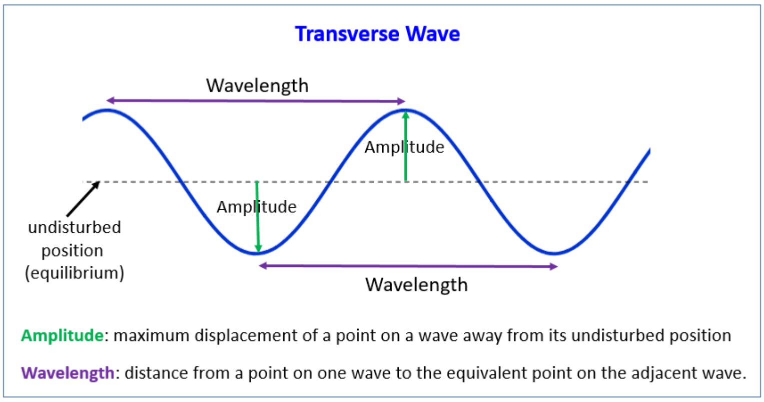
Where t is not given, take as 1

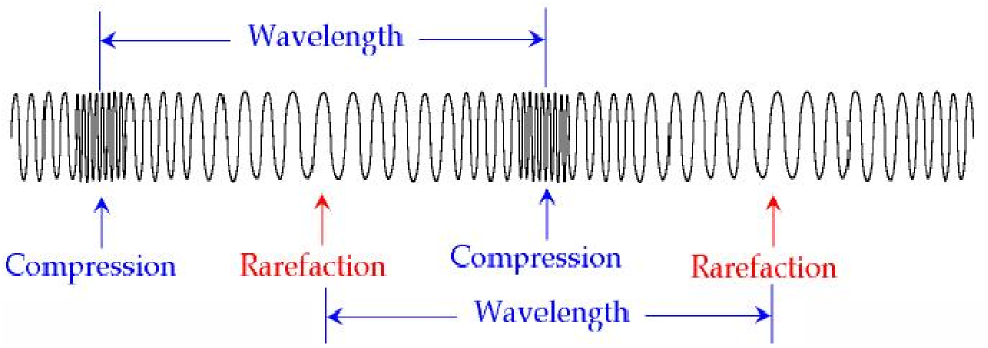
Glossary

amplitude	The measure of the displacement of the wave from its rest position. The greater the amplitude of a wave, the greater its energy.
beats	Two sound waves of different frequency add by constructive and destructive interference producing an alternatively soft and loud sound.
coherent	Two waves are said to be coherent when they have a constant phase difference between them.
compression	The region in a longitudinal wave where the particles are closest together.
crest	The crest is the highest point of a wave.
diffraction	Diffraction is when a wave remains in the same medium but bends around an obstacle or passes through an aperture.
electromagnetic wave	Waves that can travel through a vacuum. They do not need a medium. Light is an example of an electromagnetic wave.
equilibrium position	The position the medium would have if there were no wave. It is represented on a graph by a line through the centre of the wave.
frequency f	The number of crests of a wave that move past a given point in a second. The SI unit is the Hertz. The frequency is the inverse of the period.
interference	When two waves add together to form a resultant wave of lower or greater or the same amplitude.
longitudinal wave	A wave where the disturbance moves parallel to the direction of the wave.
mechanical wave	A wave that travels through a medium. Mechanical waves cannot travel through a vacuum.
medium	The matter that a wave travels through.
opaque	Does not allow light to pass through.
period T	The time between wave crests. The SI unit is the second. It is the inverse of the frequency.
progressive	A progressive or travelling wave moves away from its source.
propagation	The process by which a wave is transmitted through a medium.
rarefaction	The region in a longitudinal wave where the particles are furthest apart.
reflection	Occurs when a wave bounces off a boundary, changing direction but remaining in the same medium.
refraction	The change in direction and wavelength when a wave moves from one medium to another.
refractive index	A number that describes how light travels through a specific medium. Different mediums have different refractive indexes. The refractive index of a vacuum is defined to be 1.
resonance	The tendency for a system to oscillate with greater amplitude at some frequencies than at others.
ripple tank	A shallow glass tank of water used to demonstrate the properties of waves.
signal generator	A machine that can produce different patterns of voltage at a range of frequencies and amplitudes.
standing wave	A wave that remains in a constant position. Also called a stationary wave.
transducer	Converts a signal from one type of energy into a signal of another type; for example, a sound transducer changes sound energy into electrical energy.
transparent	A material that allows light through, for example, glass.
transverse wave	A wave where the disturbance moves perpendicular to the direction of the wave.
trough	The trough is the lowest part of the wave.
wave equation	The relationship between velocity, frequency and wavelength. $v = f\lambda$
wave velocity	The speed at which the wave travels through a medium. The SI unit is ms^{-1}.
wavelength λ	The distance between successive crests or troughs of a wave. The SI unit is the metre.

Course Notes

Properties of Waves





Frequency and Period

Frequency is defined as the number of oscillations of a wave in one second.
Period is defined as the time taken for one oscillation to occur.

Energy of a Wave

The energy of a wave is determined by many factors and while there are equations for the energy of waves, they are not covered in this course. However, for any given wave, the energy is proportional to its **amplitude**.
A practical example of this is the fact that louder music has a larger amplitude.

A sound wave with double the amplitude has double the energy. However, the energy of a wave will change depending on many other factors including wavelength and frequency, so the Amplitude is only relevant to a wave where all other factors are constant.

Transverse vs Longitudinal

Transverse Waves

A transverse wave is defined as a wave where the axis of propagation is perpendicular to the axis of oscillation (i.e. a light wave).

Longitudinal Waves

A longitudinal wave is defined as a wave where the axis of propagation is parallel to the axis of oscillation (i.e. a sound wave).
Any compression wave is a longitudinal wave.

Graphing Compression Waves

Graphing a wave such as a sound wave may seem unintuitive, however there are two different ways to graph a compression wave.
1: Displacement
2: Pressure

The displacement method graphs the average distance between particles at a given point on the wave.
The pressure method graphs the air pressure of the wave at a given point.

These two methods are the inverse of each other (as when displacement is large pressure is low etc.) however the graph will still show the same wave.

The Ray Model

Light is often modelled using a ray model. This model is a gross simplification of how light behaves, however in the context it is used, it is accurate enough for correct calculations to be made.

The reason the ray model works is that parallel light rays interfere with each other in such a way that only a ray is visible. What this means is the ray model is applicable to any situation where photons may be travelling in parallel, however it is not applicable to situations where a single wave is present.

It is also applicable because light travels across perfectly straight lines along space-time. (Unless space itself is curved)

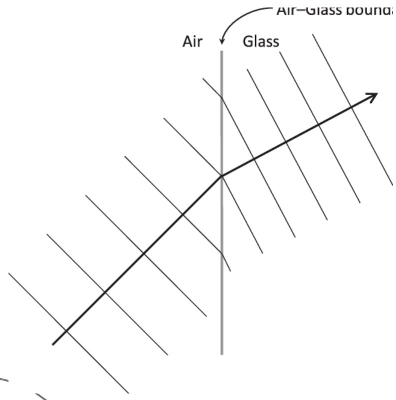
Refraction

All waves possess the property of refraction. When a wave moves from one substance to another, its' speed changes. Going from a slower medium to a faster medium will bend the wave away from the normal.
Going from a faster medium to a slower medium will bend it towards the medium.

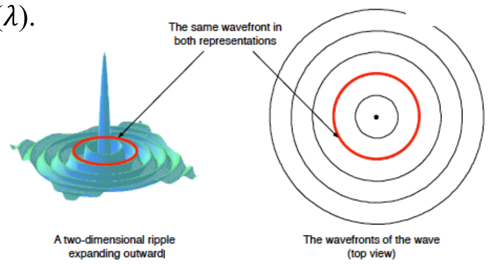
This refraction can be imagined using the *wavefronts*.

Wavefronts

Any wave can be drawn using wavefronts. The wavefront image is often used to explain refraction using a "marching soldiers" metaphor.
Although this metaphor is inaccurate, if it helps with understanding then use it.
Explanation of how refraction of light really works



A wavefront is a line perpendicular to the motion of the wave and denotes the point of highest amplitude. Therefore, the distance between fronts is the wavelength (λ).



Refractive Index

The refractive index, as defined earlier, denotes the ratio of the universal constant to the velocity of light in a given substance. The larger the refractive index, the slower light travels through that substance.
It is impossible to have a refractive index smaller than 1.

Refraction and Refractive Index

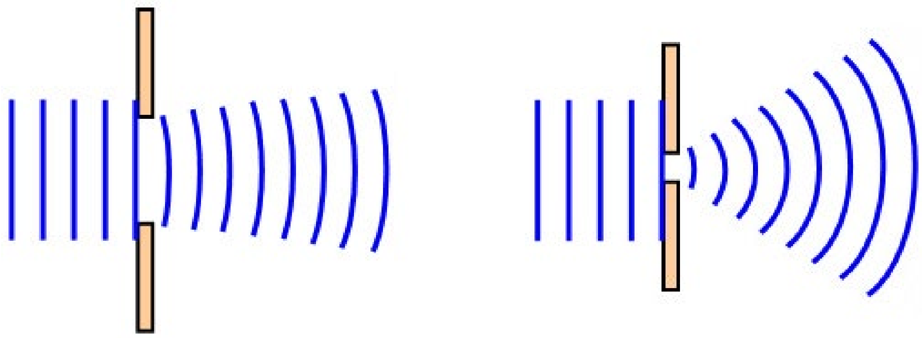
In relation to Refraction, given that light bends towards the normal when going into a slower substance, we can also say it bends towards the normal when going into a substance with a larger refractive index.

The inverse is also true when going into a faster substance. The ray will bend away from the normal when travelling into a faster medium as it will bend away from the normal when going into a substance with a smaller refractive index.

Diffraction

Another property of waves is diffracting around obstacles.

When the gap between two obstacles is much greater than the wavelength, there is little to no diffraction.
As the gap approaches the size of wavelength and gets smaller, more diffraction will occur.



Huygens’ Principle

This principle states that all points on a wave are a source of a smaller wave.

It is often stated that only wavefronts are sources and that they are sources of wavelets, however this is disingenuous to Huygens.

If you imagine a water wave, the reason it waves is because of how water interacts with itself under gravity. As a water molecule moves away from another water molecule, it attracts that molecule to it. However, there is a small delay since the force the first molecule exerts gets stronger with the distance from the other molecule.
So, after a small delay, the other molecule starts moving with its neighbour. And then that molecule does the same thing to its next neighbour. The delay of the interaction is the speed of sound in that material.
The same logic applies to pushing molecules closer together except the force is repulsive and gets stronger the closer they get.

So, if a particle in a wave is moving, it is pulling and pushing on its’ neighbouring particles.

If all other particles immediately stopped moving, that particle would become a source of a new wave, no matter what stage of its ‘waving’ it was in.

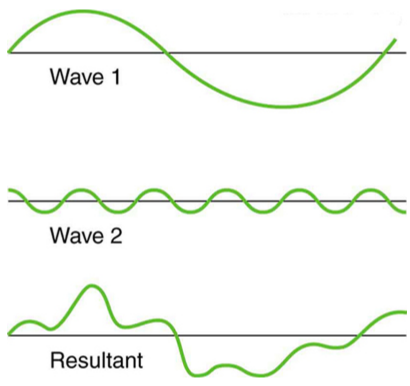
This is the reason waves diffract. The walls act as the ‘suddenly stop’ function.

Huygens’ principle can be written mathematically and must be true for any wave however it would merely be restating the above in a much less useful way.

Superposition

Waves possess the ability to superimpose themselves on other waves of the same type.
Light waves and sound waves will interfere with themselves, but a light wave cannot interfere with a sound wave and visa-versa.

Superposition is responsible for many phenomena, including beats and resonance.
A simple way to add the waves is as though they are sine functions on a graph.



Superposition, as with the above, results in beats, where the amplitude of the wave oscillates at a different frequency to the frequency of the wave.

The Doppler Effect

The Doppler Effect is a phenomenon with any wave that occurs when the relative velocities of the observer and the source are different. When the source moves, the frequency of the wave changes. When the source moves in the same direction as the wave, the wavelength shortens.
When the source moves in the opposite direction of the wave, the wavelength lengthens.

The Doppler Effect is slightly different when the observer moves. When the observer moves the apparent frequency of the wave changes, but the actual frequency does not.

When the observer moves *towards* the wave (in the opposite direction of the wave), the observer observes more oscillations per second, increasing the apparent frequency.
When the observer moves *away* from the wave (in the same direction as the wave), it is as though the observer is running away from the wave, as such they observe less oscillations per second than the actual frequency, lowering the apparent frequency.

The Doppler Effect Equation – A Strange Mistake

On the 2019+ HSC Physics formula sheet, you will note that the doppler effect equation is given as:

$$f' = f\left(\frac{\vec{v}_{wave} + \vec{v}_{observer}}{\vec{v}_{wave} - \vec{v}_{source}}\right)$$

This equation is wrong, do not use it.

Now, I say it is wrong, however, this is not necessarily the case. Let's state the assumptions of the above formula:
The direction of the wave is the positive direction.

The observer is by default, travelling towards the wave. Therefore, the formula takes the observer's direction as positive when it is travelling in the pre-defined negative direction.

The source is by default, travelling with the wave, and its' direction is positive when it is going in the same direction as the wave.

But this is stupidly confusing and so I have concluded that it is functionally worse than useless.

What we notice about this formula is you can re-write it to be much simpler.

$$f' = f\left|\frac{\vec{v}_{wave} - \vec{v}_{observer}}{\vec{v}_{wave} - \vec{v}_{source}}\right|$$

The - sign on the top now allows us to use this single assumption:
The direction of the wave is the positive direction for all vectors.

So, there we go, the newly adjusted formula works for the doppler effect and allows the use of normal vector addition.

Standing Waves

Standing waves occur as a result of superposition within a system. A standing wave is a wave where the direction of propagation of the wave is 0, i.e. it is not moving. However, the direction of propagation of energy is still present. In a standing wave, the wave does not propagate because there is an equal amount of energy propagating in either direction, meaning that at any point on the wave, there is an equal amount of energy travelling in both directions, causing the appearance of a "standing" wave.

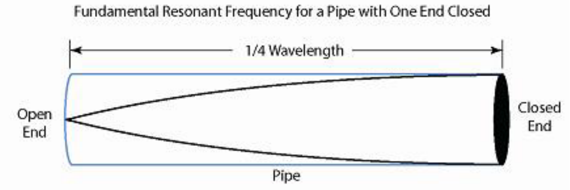
Resonance

There is a very complex explanation of how resonance works, regarding air pressure and frequency of oscillation at a given end of a tube. Because of this, diagrams vary in consistency across the internet. However, here is a close guide to what you need to know.

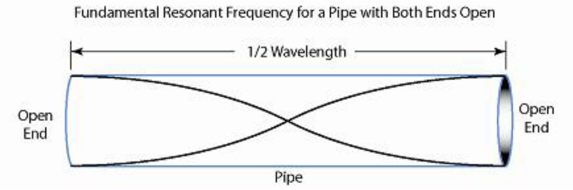
Here is the simple explanation:

Resonance occurs when a standing wave is able to be formed with a *node* or *anti-node* at each end. For a tube with air, the fundamental resonant frequencies are as such:

One Open – One Closed: $\frac{1}{4}\lambda$

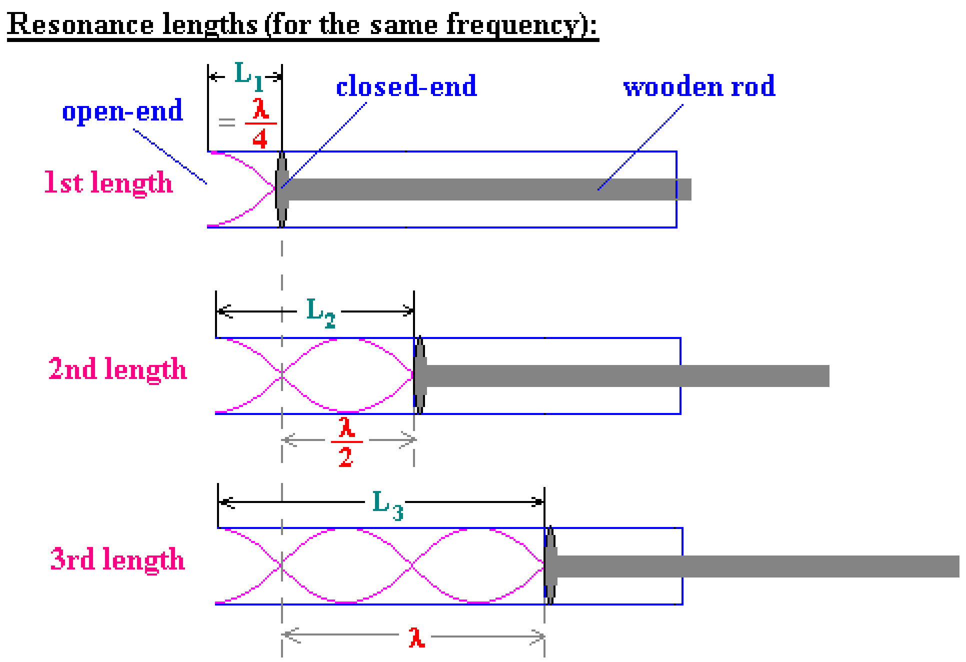


Both Open or Both Closed: $\frac{1}{2}\lambda$



Following these resonant frequencies, the next consecutive frequencies will be the base frequency $+\frac{1}{2}\lambda$. This is the case no matter what combination of end types.

One rule for drawing resonance is that at an open end, you draw an anti-node and at a closed end you draw a node.



Entropy
Entropy describes the relative disorder of particles. Entropy is a measure of how disordered a given substance is at a given temperature and, as such, is measured in $J \cdot K^{-1}$.

For Year 11 Thermodynamics, this unit is less relevant.

Entropy is the Second Law of Thermodynamics and states that the measure of average entropy ($\bar{S}$) will never decrease, only remain the same or increase.
The law also describes the tendency for a thermodynamical system to approach thermal equilibrium over time. In essence, heat will spread out over time until the distribution of Entropy is even.

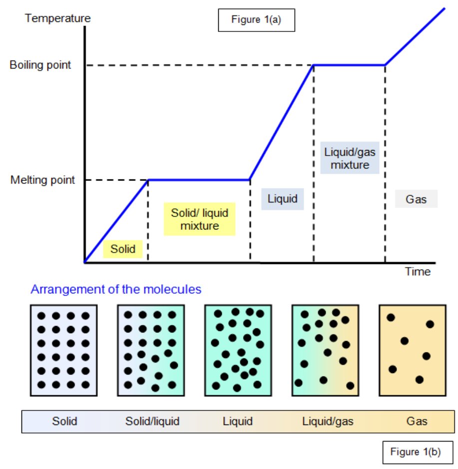
The final important note of the law is that heat moves from hot to cold, or more specifically, Work must be done to move heat energy from something cold to something warm.

Specific Heat
Specific heat is a property of all matter, with a *specific* value for each substance at each phase (s, l, g). Specific heat describes the heat energy (in Joules) required to heat one kg of that substance by one degree Kelvin.

Specific heat exists because of the different potential energies between molecules. The stronger a chemical bond, the more energy must be put in to increase the velocities of each particle. Therefore, substances with stronger bonds require more heat energy to increase their internal kinetic energy by the same amount.

Latent Heat
Latent heat describes the phase change between solid and liquid and, liquid and gas.

There is no simple explanation for latent heat that works, so here is the simplest one:
Latent heat is the amount of heat energy required to increase the kinetic energies of the molecules past the escape velocities of the inter-molecular bonds of each phase.
This latent heat is only measured once a state of fusion or vaporisation has been reached, as such, latent heat is the amount of heat energy which is input which only goes towards breaking chemical bonds, not raising the temperature.



Thermal Conductivity
Thermal conductivity is another inherent property of matter which denotes how quickly heat can be transferred through the substance per area.

Objects like metals or liquids which have free flowing particles (metals have electrons) are able to more effectively transfer heat than a rigid material like salt.

“The Speed of Light”: The Universal Constant - c

In the above Units section, is listed “Speed of Light (c) – [$3.00 \times 10^8\ ms^{-1}$]”. This number is a special number because it is a Universal Constant. If you are uninterested, just remember this as the speed of light.

This ‘Universal Constant’ exists as such because it is the maximum speed at which anything can move in our universe. It is even the maximum relative velocity and is the speed at which light can be observed to move, no matter what speed you are travelling at.

Another property of The Speed of Light is that anything without mass in the universe will move at this speed. Light is the most common thing to move at this speed, however things like gravity waves and electric fields propagate at the speed of light.

The Speed of Light was assigned the letter “c” because it is “the constant”. No matter what, it is the only thing in the universe which cannot be exceeded. It is also known as the “speed of causality”, signifying that information itself is limited by this speed.

MODULE 4: ELECTROMAGNETISM

Base Units

Charge (q) – Coulombs (C)
Electric Field ($\vec{E}$) – Newtons *per* Coulomb (NC^{-1})
Voltage (V) – Volts (V **or** $J\ C^{-1}$)
Current (I) – Amperes (A **or** Cs^{-1})
Resistance (R) – Ohms (Ω)
Conductance (G) – Mhos (S)
Magnetic Flux Density ($\vec{B}$) – Teslas (T) **or** Newtons *per* metre *per* Ampere ($N\ m^{-1}\ A^{-1}$)

Constants

Charge of Electron (q_e) $= -1.602 \times 10^{-19}\ C$
Charge of Proton (q_p) $= 1.602 \times 10^{-19}\ C$

Mass of Electron (m_e) $= 9.109 \times 10^{-31}\ kg$
Mass of Proton (m_p) $= 1.673 \times 10^{-27}\ kg$

Electric Permittivity Constant (ε_0) $= 8.854 \times 10^{-12}\ \ A^2 s^4 kg^{-1} m^{-3}$
Magnetic Permeability Constant (μ_0) $= 4\pi \times 10^{-7}\ \ NA^{-2}$

Equations

$$\vec{F} = q\vec{E}$$

Force on a charged particle due to an Electric field.

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

The strength of an Electric field caused by particle q*, at a given radius,* r.

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$$

Mathematical Explanation:

$$F = q_2 E = q_2 \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1}{r^2} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$$

Describes the Force on particle q_1 *due to field emitted by* q_2.
Also describes the force on particle q_2 *due to field from* q_1.

The equation describes the full pair of forces (Newton's 3rd Law).

$W_E = q\vec{E}\Delta d$

The work done in an Electric field, moving a particle a distance.
It describes the change in potential energy in the field.

$U_E = q\vec{E}d$

The Electric Potential Energy of a charged particle in a uniform Electric field at distance.
It describes the potential of the system to do Work.

This equation is only true when the particle is able to be moved by the field. If the particle is restricted from moving, like by a plate, then the potential is zero.

$U_E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r}$

The Electric Potential Energy of a charged particle in an Electric field caused by another particle at distance.
It describes the potential of the system to do Work.

$V = Ed$

The Electric Potential of any charged particle in a field. Voltage is the Electric Potential Energy per Charge.

$I = \frac{q}{t}$

Current is measured in charges per second (Coulombs per second)

$V = IR$

Ohm's Law

$P = VI$

Power output (Watts) of a circuit.

$I_{series} = I_n$ **or** $I_{series} = I_1 = I_2 = \cdots = I_n$

$V_{series} = \sum V_n$ **or** $V_{in} = V_1 + V_2 + \cdots + V_n$ **or** $V_{in} - V_1 - V_2 - \cdots - V_n = 0$

$R_{series} = \sum R_n$ **or** $R_{series} = R_1 + R_2 + \cdots + R_n$

Current, Voltage and Resistance in Series Circuits

Note: 'Σ' means *sum of all* from *1* to *n*.

$I_{parallel} = \sum I_n$ **or** $I_{in} = I_1 + I_2 + \cdots + I_n$ **or** $I_{in} - I_1 - I_2 - \cdots - I_n = 0$

$V_{parallel} = V_n$ **or** $V_{parallel} = V_1 = V_2 = \cdots = V_n$

${R_{parallel}}^{-1} = \sum {R_n}^{-1}$ **or** $\frac{1}{R_p} = \Sigma \frac{1}{R_n}$ **or** $\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$

Current, Voltage and Resistance in Parallel Circuits

$$G = \frac{1}{R}$$

Conductance of a resistor. (I have no idea why it's G)

$$B = \frac{\mu_0 I}{2\pi r}$$

Strength of a Magnetic Field in a wire with a flow of electrons.

$$B = \frac{\mu_0 N I}{L} = \mu_0 I \cdot \frac{N}{L}$$

Strength of a Magnetic Field in a uniform Solenoid.
$\frac{N}{L}$ *is the number of loops per metre.*

Course Notes

Fields

Fields are a strange topic as they are not traditionally observable and can interfere with each other in strange ways. Fields can be thought of as being emitted by all things which possess the given property the field requires. Gravity fields are "emitted" by all objects with mass, Electric fields by everything with a charge. The other thing fields appear to be is a constant. Just as The Universal Constant is the maximum speed anything can move at through space; Fields appear to exist as a Universal Constant. There is no reason for fields to exist, they merely do.

There is also a constant behaviour of Fields, they apply a Force per unit of fundamental that thing possesses. Gravitational Fields apply a force per unit mass and Electric Fields apply a force per unit charge.

The reason they can only be thought of as being emitted is because they are always present. The electromagnetic field is present throughout all of space and a charged particle merely excites that field. It is similar with gravity however space itself is the gravitational field.

Force Pairs of Fields

As shown by Coulomb's Law: $\vec{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$

and Newton's Law of Universal Gravitation: $\vec{F} = G\frac{m_1 m_2}{r^2}$,

Fields exert a force on an object and the force exerted by object a on b is equal to the force exerted on b by a. Therefore, when using Coulomb's Law, both Forces in the pair are being found, such that one Force, $\vec{F}_1$, is equal to $-\vec{F}_2$.

This allows momentum to be conserved as the net force in the universe is 0.

Electric Potential Energy

Electric Potential Energy is a difficult concept to grasp as it differs depending on what particles are present and their charges. For a particle which is attracted by the field being emitted, the Potential Energy of that particle is $E_{EPE} = q\vec{E}d$, where *d* is the distance from the field emitter.

For a particle which is repelled by the field it gets harder. In this case the particle must be contained by something like a capacitor, otherwise the particle's Potential Energy is infinite (*d* is infinite). If the particle is contained, take *d* as the distance from the point at which it can no longer move (i.e. the wall of the capacitor).

Electric Potential and Electric Potential Difference – Voltage

Electric Potential is Voltage and describes the Potential Energy per Coulomb of a charge in a given uniform field.

Voltage is also known as Electric Potential Difference. A better explanation is that Voltage drop is the difference in Electric Potential of two points. As such, there can be a voltage between two points in an Electric Field emitted by a proton, as the difference in the Electric Field and distance for the two points will change the Voltage.

Voltage is not Electric Potential Energy, however since it is Potential Energy per Charge, multiplying Voltage by the Charge will get the Potential Energy of a Particle or the Work done on a Particle by a system.

In a circuit, Voltage at a point is a measure of the Kinetic Energy per charge, and a Voltage "drop" measures the decrease in kinetic energy per Coulomb of each electron.

Current

Current is a measure of Coulombs per second. In reality, this is electrons passing per second however, since they have a charge in Coulombs, it is measured in Coulombs per second.
Current does not necessarily denote velocity (that is Voltage), rather it denotes how many electrons are passing in parallel. Given even a small wire can be thousands of atoms thick, thousands of electrons can be passed through the wire next to each other at once. This is how current can increase / remain constant while Voltage decreases.

Electrical Kinetic Energy and Power

Electrical Energy is akin to the amount of kinetic energy the electrons in a circuit possess. Increasing the Energy output by a circuit is done by increasing potential energy of the electrons ($U = qV$) in the circuit
In doing so this increases the current which is the number of electrons passing per second.
As such, an increase in Voltage will increase the kinetic energy of the electrons.

This is how we get the law $P = VI$.
It is the kinetic energy per charge lost through the circuit, multiplied by the number of charges per second passing through the circuit. This is what gives us our energy per second.

Resistance

Resistance is a property of every form of matter, the higher the resistance of a substance, the more insulating it is. Resistance, measured in Ohms (Ω), denotes how much energy is taken away from the electrons as they pass through that object.

As stated before, Electrical Energy is the energy of all the electrons in a circuit. As such, a resistor must slow down the electrons and convert the lost energy into something like heat, sound or motion. Any device which uses electrical energy is an appliance of electrical resistance to perform a task.

A good analogy for Resistance is like trying to shoot a bullet completely straight through something like a dense forest. Theoretically it's possible but more likely is that the bullet will deflect slightly off a tree and lose energy. This is the same but instead of a bullet, use an electron, and instead of trees, the atoms of the resistor.

Although this is just an analogy and doesn't properly represent the true mechanics of resistance, it may help with understanding.

The Slightly more complicated explanation

Electrons in a circuit all travel at the same speed. When a voltage is applied to a circuit it propagates down the circuit at the speed of light (i.e. there is a slight time delay at the end of the circuit before it 'knows' a voltage has been applied).
This delay is enough to make the electrons push on each other and, in doing so, allows the conductor to conduct the field around the circuit. It is almost like pushing beads through a curved hose. As you push on each bead it pushes on the next bead and the force is redirected around the curve.

The resistance is how much the material stops the field from being conduced through the material.

What this means is that, in reality, the electrons aren't being slowed down by the material because they were never going too fast. Instead, the field spends some of the energy it would have spent pushing the electrons trying to push the wires of the circuit and this is what gives rise to resistance.

A Strange Property of Resistance

A somewhat strange behaviour of resistors is Ohm's Law ($V = IR$) which states that an increase in current across a resistor will increase the Voltage drop across that resistor. If we define Voltage in a circuit as a measure of Kinetic Energy per charge, then as we increase the number of electrons flowing into the resistor, we also increase the amount of energy lost per electron as it passes through.

Electron Spin and Magnetism

Electrons possess two fundamental properties which allow them to create magnetic fields.
1. Charge
2. Quantum Spin

Electrons are magnetic, it is a fundamental property of Electrons due to their spin. However, Electrons are also quantum and, as such, are random.
Electrons *basically* have a North and South pole, however, their rotation is random. This concept is known as spin and it describes the fact that when an Electron encounters a magnetic field it has a 50% chance of being spin-up and 50% chance of being spin-down. Once the electron encounters the field, it becomes one of these spins and is attracted to magnets as such, where spin-up places the North pole of the electron towards the magnet, and spin-down the South.

Traditionally Electrons exist in electron shells and inside those, in pairs. These pairs, unless forced by another magnet, always exist with one spin-up and one spin-down electron, resulting in the cancellation of the magnetic fields. An object becomes magnetised when the outer orbitals which contain a single electron all possess an electron which has a spin direction parallel to all or most of the other atoms in the object. The more outer single electrons in their own orbital, the more magnetic a substance can be.

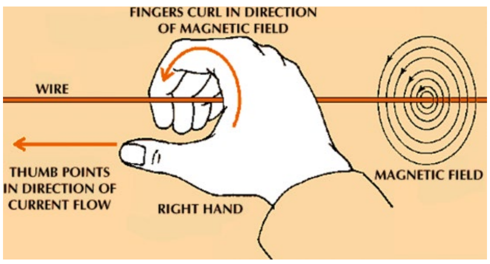
Electrons also create magnetic fields by moving as this is the traditional way in which a magnetic field is created.

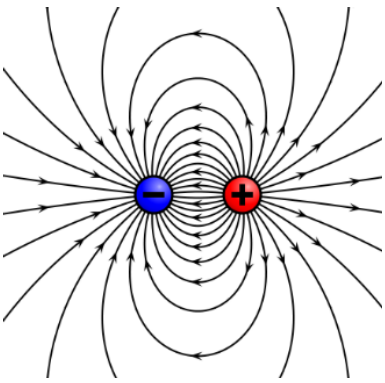
Magnetic Fields

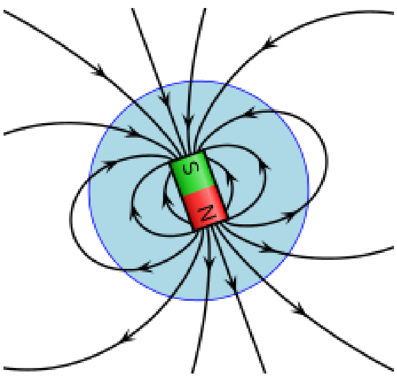
Fundamentally, Magnetic Fields are caused by Electrons and their spin arrangement or by moving charges. Magnetic fields interact with objects and each other in the same way Gravitational and Electric Fields do, where the stronger the magnetism of an object, the greater the force exerted on it by a magnetic field.

Unlike Electric Fields, where a singular point charge can emit a spherical field, magnets always exist with two poles, almost like having an electric field due to an electron and a proton being stuck to each other. As such, a magnetic field will always have some form of interference and behave like a typical bar magnet diagram.

Magnetic Fields can also be caused by current flowing through a wire. The direction of the magnetic field's rotational flow is a cross (×) multiplication and as such, can be denoted using the right-hand rule.







Comparison of Magnetic Fields and Electrostatic Fields

Magnetism in a Wire with Current

Although the direct explanation of this phenomena is very difficult to understand mathematically, see Veritasium's video on the topic for a visual explanation: https://www.youtube.com/watch?v=1TKSfAkWWN0

In short, the video above details how special relativity and length contraction explains electromagnets. The contraction of distances between charges along the vector of motion increases the density of negative charge along the wire and creates the magnetic field.

Magnetism in a Ferromagnet

As with the above, magnetism in a permanent magnet is a little difficult to explain. What we know is that permanent magnets exist because of all the outer valence electrons being able to exist in their own orbits and then align their magnetic moments (spin) in the same direction, generating a net magnetic field.

The difficulty is that although it's possible to calculate the magnetic moment of an electron with extreme accuracy, it is somewhat unclear how it produces that field. One possible explanation is that since electrons spin, the rotation of its electric charge causes relativistic effects which cause it to create a magnetic field.

Magnetic Field Lines

Magnetic field lines are a little different from Electric field lines as they show the direction a small magnet would be aligned at that spot in the field.

The force on a moving charge in a magnetic field is also perpendicular to this field line.

MODULE 5: ADVANCED MECHANICS

Base Units

Mass (m) – Kilograms (kg)
Displacement $(\vec{s})$ – metres (m)
Time (t) – Seconds (s)
Speed (v) – Metres per second $(ms^{-1}$ **or** $m/s)$
Velocity $(\vec{v})$ – Metres per second $(ms^{-1}$ **or** $m/s)$
Acceleration $(\vec{a})$ – $(ms^{-2}$ **or** $m/s^2)$
Force $(\vec{F})$ – Newtons (N)
Energy (E) – Joules (J)
Work (W) – Newton Meters (Nm) **or** Joules (J)
Angular Displacement (θ) – Radians (rad)
Angular Velocity (ω) – Radians per second $(rad\ s^{-1})$
Angular Acceleration (α) – Radians per second per second $(rad\ s^{-2})$

Constants

Gravitational Constant: $G = 6.67 \times 10^{-11}\ (N\ m^2\ kg^{-2})$
Mass of the Earth: $m_E = 6.0 \times 10^{24}\ (kg)$
Radius of the Earth: $r_E = 6.371 \times 10^6\ (m)$

Equations

$\vec{F}_{net} = m\vec{a}$

Newtons is a measure of Force; One Newton is equivalent to one: kilogram metres-per-second-per-second (kg ms^{-2}). The net means sum of all forces which accounts for the fact that there can be forces on an object but if they cancel there is no acceleration.

$$F_c = \frac{mv^2}{r}$$

$$a_c = \frac{v^2}{r}$$

Centripetal force or acceleration on an object derived from circular motion. Centripetal force is the description of the sum of forces on an object in circular motion. The actual force is often generated by tension in a string or a gravitational field.

$$\theta = \frac{l}{r}$$

$$\omega = \frac{\Delta\theta}{t} = \frac{v_{\perp}}{r}$$

$$\alpha = \frac{\Delta\omega}{t} = \frac{a_{\perp}}{r}$$

Angular equivalents of linear factors. Equations for angular displacement, angular velocity and angular acceleration for an object spinning in a circle. Measured in ***radians****,* ***radians*** *per second and* ***radians*** *per second per second.*

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r}_{\perp}\vec{F} = rF\sin\theta$$

Torque on a rotating object is the Force on that object multiplied by the radius from the axis of rotation multiplied by sine of the angle between them.

$$F_G = G\frac{m_1 m_2}{r^2}$$

$$a_G = G\frac{m_1}{r^2}$$

Gravitational Force between two objects of mass at some radius. G is the gravitational constant.

The acceleration due to gravity is dependent on the mass of the object generating the field.

$$v = \frac{2\pi r}{T}$$

Kepler's second law of planetary motion. It states the average speed of a planet in motion is proportional to its radius divided by the time for one orbit around the sun.

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

Kepler's third law of planetary motion. The relationship between radius and period of orbit with a constant (where M is the mass of the object it orbits).

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Escape velocity of an object with mass at an initial distance, r, away. It is the minimum velocity an object must have to escape the gravity of that object.

$$U_G = -\frac{Gm_1m_2}{r}$$

The potential energy due to gravity of an object in another objects gravitational field. Note that it is not necessarily the total potential energy, but often is.

Projectile Motion Equations

$$\vec{s} = \frac{\vec{a}}{2}t^2 + \vec{v}_i t + \vec{s}_i$$

The displacement of an object in a given direction is given by this equation, where the displacement, acceleration and initial velocity are all vector components in the same direction. i.e. If the displacement is the displacement in the x direction, then the acceleration is the acceleration in the x-direction. It is possible for this to be zero.

The extra term $\vec{s}_i$ on the end is the initial displacement from the target. E.g. if a ball falls from a table, its initial y-displacement from the floor is the height (if positive direction is up).

Note that it is a quadratic in terms of t.

$$\vec{v} = \vec{a}t + \vec{v}_i$$

The velocity of an object is given by the acceleration in that direction and the initial velocity in that direction.

$$v^2 = {v_i}^2 + 2as$$

A rearrangement of the above formulae. Useful when time is not given.

Extension Equations

$$I = \int_0^m r^2 \, dm$$

The Moment of Inertia of a rotating object. Although it is a strange integral it is solvable by defining dm *in terms of some area or length mass density and some area or length* dA *or* dl*.*

$$\Sigma\vec{\tau} = \frac{d\vec{L}}{dt} = I\vec{\alpha}$$

The sum of torques is the rate of change of angular momentum with respect to time.

Torque is also moment of inertia multiplied by angular acceleration. Note the similarity for force where moment of inertia takes the place of mass.

$$\oiint \vec{g} \cdot d\vec{A} = -4\pi GM$$

$$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho$$

Gauss' Law for gravity where $\vec{g}$ *is the gravitational field, M the mass of the object creating the field and* ρ *the density* $\left(\rho = \frac{dM}{dV}\right)$*.*

$$\vec{F}_G = -G\frac{m_1 m_2}{r^2}\hat{r}$$

The direction of a gravitational force is given by the unit vector (a vector of length 1 with some direction) $\hat{r}$*, which points from the mass exerting the force to the other mass.*

$$U_G = \int_\infty^r G\frac{m_1 m_2}{r^2} \, dr = -\frac{Gm_1 m_2}{r}$$

The potential energy of a particle in a gravitational field. The integral is from infinity to r as the potential energy is equivalent to the work done by the field and is therefore, the Force done by the field dragging that mass from infinity to its current point.

Derivative Forms

$$\frac{d\vec{s}}{dt} = \vec{v}$$

The first derivative of Displacement is velocity.

$$\frac{d^2\vec{s}}{dt^2} = \frac{d\vec{v}}{dt} = \vec{a}$$

The second derivative of Displacement, or the first derivative of Velocity, is Acceleration.

$$\frac{d^3\vec{s}}{dt^3} = \frac{d^2\vec{v}}{dt^2} = \frac{d\vec{a}}{dt} = \vec{j}$$

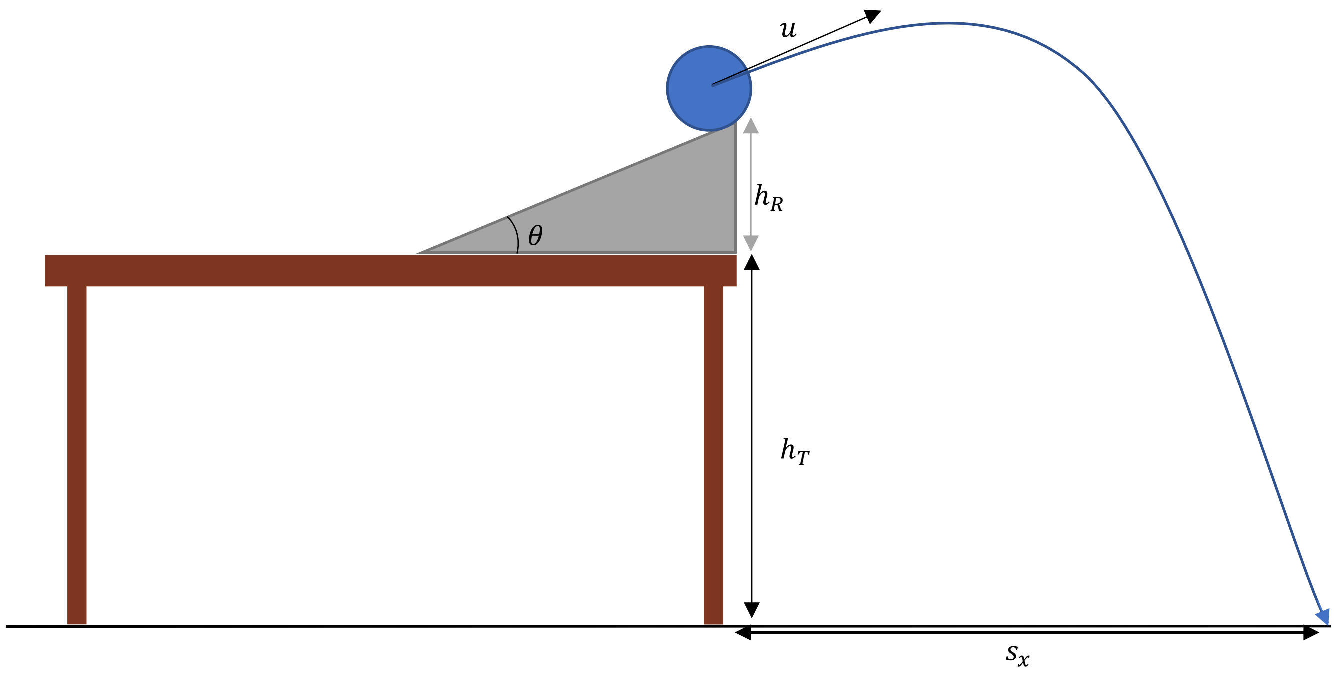
The derivative of Acceleration is Jerk. (Well beyond the course as all Senior Physics assumes Jerk is 0.)

$$\vec{\omega} = \frac{\overrightarrow{d\theta}}{dt}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{\overrightarrow{d^2\theta}}{dt^2}$$

The derivative forms of angular velocity and acceleration. Note that θ can never be defined as a vector or pseudo-vector, however $\overrightarrow{d\theta}$ can. This means that although angular displacement cannot be added as a vector, angular velocity and acceleration can as they are functions of a vector, meaning they too, are vectors.

Using the Projectile Motion Equations



The below is intended to work through the problem algebraically with explanations such that you can try to solve it yourself. Numerical solutions are provided however working through the problem without numbers first is encouraged first.

Question:

A ball is rolled up a slope off a table with initial velocity v at angle θ from the vertical.

a) How long does it take the ball to hit the ground?
b) How far is it from the edge of the table when it hits the ground?
c) What is the ball's velocity when it hits the ground?
d) What is the ball's displacement from the edge of the table when it is at the highest point of its arc? (Extension Question)

Solve the above problems if:

$u = 15ms^{-1}$
$\theta = 30°$
$h_T = 1.2m$
$h_R = 0.24m$

Next 3 pages for algebraic and numerical answers .

Algebraic Answers

a) To solve this first question, we need to construct an equation that solves for when the height of the ball is the height of the floor, which we are going to define as when $s_y = 0$. We are also going to define up as positive.

On the formula sheet, the equation for displacement is $s = ut + \frac{1}{2}at^2$. Although this formula is good, it has an issue. When $t = 0$, displacement is also zero ($s = 0$) but at $t = 0$ the height of the ball is $h_T + h_R$ which means our equation is slightly wrong. We can fix this by using the equation detailed above in the notes: $y = \frac{a_y}{2}t^2 + u_y t + y_i$ where $y_i = h_T + h_R$.

Now we have a displacement equation in the y-direction, we need to find the initial velocity and acceleration in the y-direction. Using trig rules, $u_y = u\sin\theta$. Acceleration is just gravity which can be defined as $g = -9.8$ ($-$ because up is $+$).

Now to find t, we need to use the quadratic formula:

$$t = \frac{-u_y \pm \sqrt{\left(u_y\right)^2 - 4\left(\frac{g}{2}\right)(y_i)}}{2\left(\frac{g}{2}\right)}$$

b) To find the horizontal distance from the table (s_x in the diagram) we take the time taken to hit the ground and multiply it by the initial horizontal velocity. This can be derived from the displacement formula $x = \frac{a_x}{2}t^2 + u_x t + x_i$ where $a_x = 0,\ \ x_i = 0$.

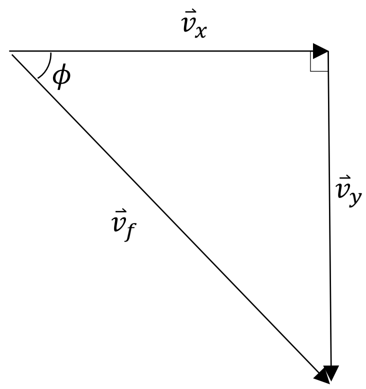
$$\therefore s_x = u_x t$$

u_x is given by the horizontal component of the initial velocity, i.e. $u_x = u\cos\theta$

c) To find the final velocity, we have to add the two velocity vectors at the time of the collision with the ground. We already found t in a) and we know the horizontal velocity is constant throughout the flight, therefore the only new fact we need is the final vertical velocity.
$v_y = gt + u_y$ gives the final velocity after some time and we use this to find the downward velocity at the time of collision.
Now for vector addition:

$$|\vec{v}_f| = \sqrt{\vec{v}_x^{\,2} + \vec{v}_y^{\,2}}$$

$$\phi = \tan^{-1}\left(\frac{\vec{v}_y}{\vec{v}_x}\right)$$



d) (Extension) To solve this question, we have to find where the vertical velocity is zero as this is when the ball has risen to its peak. Because we want the height above
To find this we set $v_y = 0 = gt + u_y$ and solve for time at the peak:

$$t_p = -\frac{u_y}{g}$$

Now we plug this time into our displacement equations:

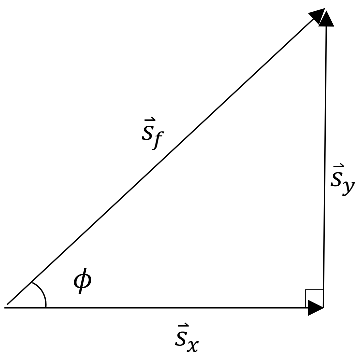
$$\vec{s}_x = \vec{v}_x t$$

$$s_y = \frac{\vec{g}}{2}t^2 + \vec{v}_{i_y}t + (h_T + h_R)$$

However, there is again a problem with the height, as we want the displacement from the edge of the table. This means at $t = 0$ we only want our height to be h_R

$$\therefore s_y = \frac{\vec{g}}{2}t^2 + u_y t + h_R$$

Now we do vector addition:



$$|\vec{s}_f| = \sqrt{{s_x}^2 + {s_y}^2}$$

$$\phi = \tan^{-1}\left(\frac{\vec{s}_y}{\vec{s}_x}\right)$$

Numerical Answers

a) $t = 1.7s$
b) $\vec{s}_x = 22m$
c) $\vec{v}_f = 16ms^{-1}\ [To\ the\ right, 62°\ below\ horizontal]$
d) $\vec{s} = 22m\ [Right, 51°\ above\ horizontal]$

Course Notes

Converting Degrees to Radians to Degrees

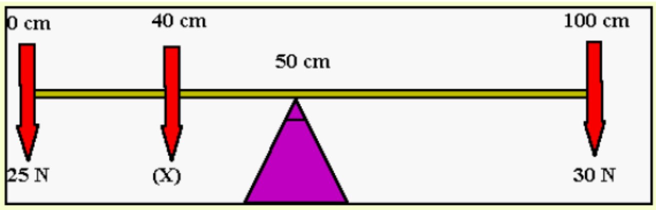
To convert degrees to radians, multiply by $\frac{\pi}{180}$

To convert radians to degrees, multiply by $\frac{180}{\pi}$

RPM to Radians per Second

To convert RPM to Radians per second, first convert it to Revs per second by dividing it by 60 (i.e. 60rpm will be 1 rotation per second). Then since one revolution is 2π radians, multiply the Revs per second by 2π.

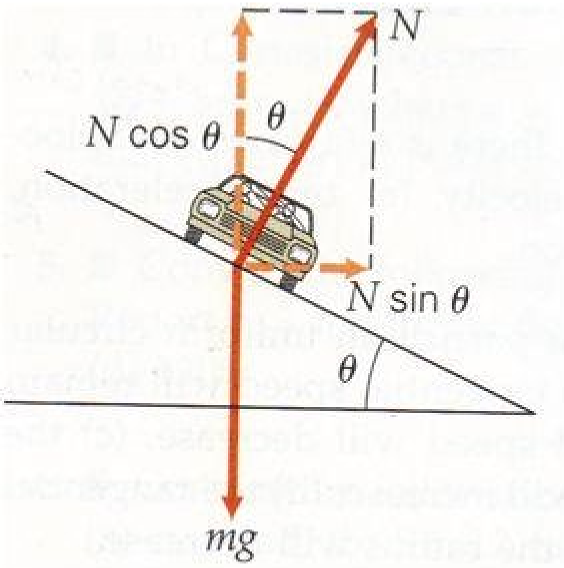
Torque Equilibrium Questions



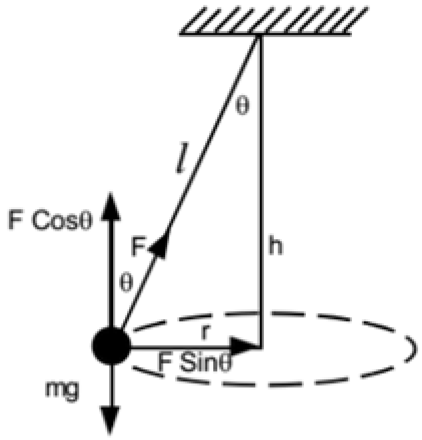
The torque questions like what is pictured above rely on the idea that $\sum \tau = 0$. This means the torques on the left of the pivot are equal to the torques on the right.
These questions can also be tricky in that the centre is defined as $50cm$, which means the force at $0cm$ is actually at a radius of $50cm$ etc.

Banked Curves

In a typical banked curve question the centripetal force is considered to be the inward component of the normal force and the vertical component of the normal force is the opposite of gravity.



Tension as Centripetal Force



In this example of circular motion, the vertical component of tension is the force opposing gravity and the horizontal is the centripetal force.

The interesting implication of this is that when you swing an object on a string in real life, it can never be perfectly flat, there will always be a slight angle between the string and the vertical.

Static Friction around a Corner

Going around a flat corner, the static friction acts as the centripetal force. In physics we don't consider other effects which are why you typically accelerate around a bend when driving.

Summary: $\mu_s N \geq \frac{mv^2}{r}$

Change in Potential Energy in a Gravitational Field

By definition, any change is the final state minus the initial state:

$\Delta U = U_f - U_i$

Given $U_G = -\frac{Gm_1m_2}{r}$, $\quad \Delta U = -Gm_1m_2\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$

This ΔU will also return a positive or negative value, if it is positive, work has been done against the field. If it is negative, then the field has done work.

The final useful definition is that $-\Delta U = \Delta E_K$

Harder Potential Energy Concepts

Applying this formula is harder when the concept of '*altitude*' is introduced. When the word *altitude* is in the question it is implying that $r = r_E + a$, where a is the altitude.

Some other problems also introduced where energy is lost to something like air resistance as heat. In this instance the heat energy lost is subtracted from the kinetic energy gained due to the loss of potential energy i.e. $\Delta E_k = -\Delta U - Q$ (The increase in kinetic energy is the decrease in potential energy minus the heat energy)

Orbital Motion in a Gravitational Field

In orbital motion, the gravitational force is the centripetal force ($F_G = F_c$)

i.e. $G\frac{m_1 m_2}{r^2} = \frac{mv^2}{r}$

Geostationary Satellites

The definition of a geostationary satellite is that it orbits with a period of 24hrs (86400s). Every other fact about geostationary satellites is consistent with circular motion in a gravitational field.

Kepler's Laws

1st Law

Planets orbit in ellipses with the Sun at one of the foci.

2nd Law

The imaginary line connecting the planet and the sun sweeps out the same area per unit time no matter how fast it is going or where it is.

3rd Law

$$\frac{GM}{4\pi^2} = \frac{r^3}{T^2}$$

Where r is the orbital radius, T is the time for one orbit and M is the mass of the object causing the orbit.

Formula Derivations

Kepler's Third Law

$F_G = F_c$

$G\frac{m_1 m_2}{r^2} = \frac{m_2 v^2}{r}$

$\frac{Gm_1}{r} = v^2$

$v = \frac{2\pi r}{T}$

$\frac{Gm_1}{r} = \frac{4\pi^2 r^2}{T^2}$

$\frac{Gm_1}{r^3} = \frac{4\pi^2}{T^2}$

$\frac{Gm_1}{4\pi^2} = \frac{r^3}{T^2}$ *Q.E.D.*

Escape Velocity

$E_K = -U_G$

$\frac{1}{2}mv^2 = G\frac{Mm}{r}$

$v^2 = \frac{2GM}{r}$

$v_{esc} = \sqrt{\frac{2GM}{r}}$ *Q.E.D.*

Velocity of a Satellite

$F_c = F_G$

$\frac{mv^2}{r} = G\frac{Mm}{r^2} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$ *Q.E.D.*

Total Energy of an orbiting Object

$$E_{net} = E_k + U_G$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\frac{GM}{r} = v^2$$

$$E_{net} = \frac{1}{2}m\left(\frac{GM}{r}\right) - \frac{GMm}{r}$$

$$= \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$

$$= -\frac{GMm}{2r}$$

Extension Notes

Derivation of Standard Motion Equations

We shall begin by assuming there to be a constant acceleration vector such that

$$\frac{d\vec{a}}{dt} = 0$$

$$\frac{d\vec{v}}{dt} = \vec{a}$$

$$d\vec{v} = \vec{a}\,dt$$

$$\int_{\vec{u}}^{\vec{v}} d\vec{v} = \int_0^t \vec{a}\,dt$$

$$\vec{v} - \vec{u} = \vec{a}t$$

$$\vec{v} = \vec{a}t + \vec{u}$$

$$\vec{v} = \frac{d\vec{s}}{dt} = \vec{a}t + \vec{u}$$

$$d\vec{s} = (\vec{a}t + \vec{u})\,dt$$

$$\int_{\vec{s}_0}^{\vec{s}} d\vec{s} = \int_0^t (\vec{a}t + \vec{u})\,dt$$

$$\vec{s} - \vec{s}_0 = \frac{1}{2}\vec{a}t^2 + \vec{u}t$$

$$\vec{s} = \frac{1}{2}\vec{a}t^2 + \vec{u}t + \vec{s}_0$$

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v = \frac{dv}{ds}\frac{d}{dv}\left(\frac{1}{2}v^2\right) = \frac{d}{ds}\left(\frac{1}{2}v^2\right)$$

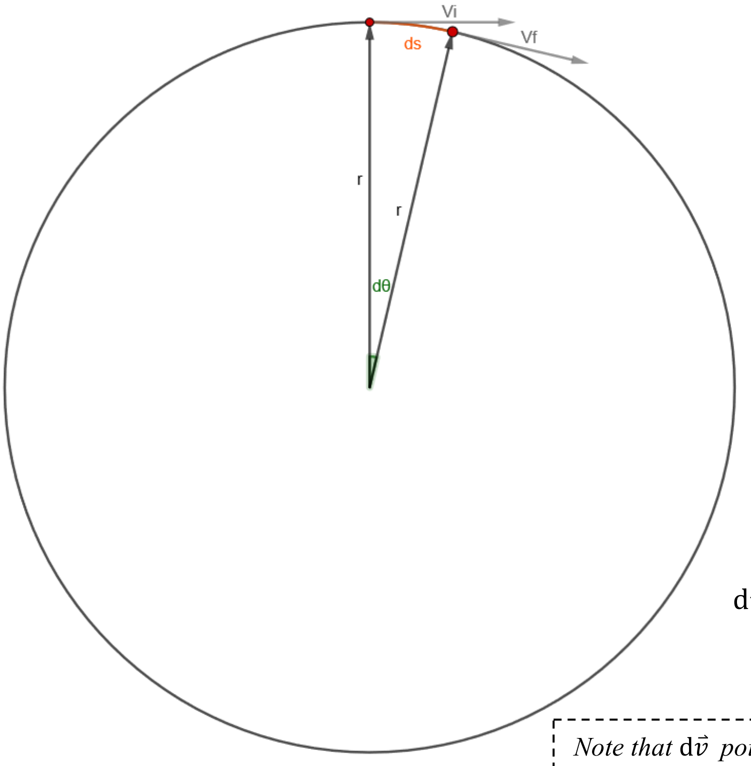
$$a\,ds = \frac{1}{2}d(v^2)$$

$$\int_0^s 2a\,ds = \int_{u^2}^{v^2} d(v^2)$$

$$2as = v^2 - u^2$$

$$v^2 = u^2 + 2as$$

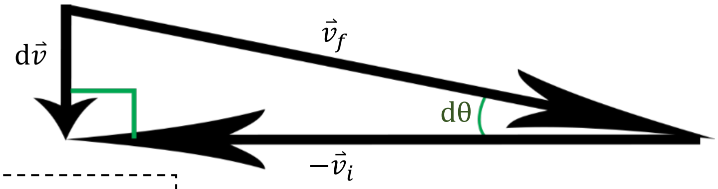
Visual Derivation of Centripetal Acceleration



Let $d\theta$ be the angle traced by the object in some infinitesimally small time interval dt, and ds is the distance travelled.

Also: $|\vec{v}_i| = |\vec{v}_f| = v$

$$\mathrm{d}\vec{v} = \vec{v}_f - \vec{v}_i$$



$$\sin d\theta = d\theta = \frac{\mathrm{d}\vec{v}}{\vec{v}_f}$$

Note that $\mathrm{d}\vec{v}$ points inward (i.e. $d\vec{v} = (-\hat{r})dv$).

Even though the two vectors are equal, because $d\theta$ is infinitesimally small, it is still a right-angled triangle.

Sine approximation holds true for infinitesimally small $d\theta$.

$$\sin d\theta = d\theta = \frac{ds}{r}$$

Derived from the circle diagram where the distance travelled is perpendicular to $\vec{r}$.

$$\therefore \frac{\mathrm{d}\vec{v}}{\vec{v}_f} = \frac{ds}{r}$$

$$\mathrm{d}\vec{v} = ds\frac{\vec{v}_f}{r}$$

$$d\vec{v} = ds\frac{v}{r}$$

$$\frac{d\vec{v}}{dt} = \frac{ds}{dt}\frac{v}{r} = v(-\hat{r})\frac{v}{r}$$

$\boxed{\mathrm{d}\vec{v} = (-\hat{r})dv}$

$$\overrightarrow{a_c} = -\frac{v^2}{r}\hat{r}$$

Q.E.D.

Vector Derivation of Centripetal Acceleration

We shall begin by stating that since velocity ($\vec{v}$) is defined with a size and a direction, it can be defined by $\vec{v} = |\vec{v}|\hat{v}$

In this case $\hat{v}$ is the unit vector for velocity meaning it points in the direction of velocity but has a length of 1.

$$\frac{d\vec{v}}{dt} = \vec{a}$$

$$\frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right) = \frac{d}{dt}(\vec{v}\cdot\vec{v})^{\frac{1}{2}} = \frac{1}{2}\frac{1}{(\vec{v}\cdot\vec{v})^{\frac{1}{2}}}\,2\vec{v}\cdot\vec{a} = \frac{\vec{v}\cdot\vec{a}}{|\vec{v}|} = \hat{v}\cdot\vec{a}$$

$$\frac{d\hat{v}}{dt} = \frac{d}{dt}\left(\frac{\vec{v}}{|\vec{v}|}\right) = \frac{\frac{d\vec{v}}{dt}|\vec{v}| - \vec{v}\frac{d|\vec{v}|}{dt}}{|\vec{v}|^2} = \frac{\vec{a}|\vec{v}| - \vec{v}(\hat{v}\cdot\vec{a})}{|\vec{v}|^2} = \frac{\vec{a} - \hat{v}(\hat{v}\cdot\vec{a})}{|\vec{v}|} = \frac{\vec{a}_\perp}{|\vec{v}|}$$

In this case $\vec{a}_\perp$ is the acceleration perpendicular to the velocity.

Since $\hat{v}$ is a unit vector, it can be represented like a number on the unit circle with the x component given by cos of the angle and y by the sin. (We shall omit three dimensions for the sake of simplicity)

$$\hat{v} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$\frac{d\hat{v}}{dt} = \frac{d}{dt}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\frac{d\theta}{dt} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\omega(t)$$

$$\therefore \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\omega(t) = \frac{\vec{a}_\perp}{|\vec{v}|}$$

$$\vec{a}_\perp = |\vec{v}|\,\omega(t)\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$|\vec{a}_\perp| = \left||\vec{v}|\,\omega(t)\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\right| = |\vec{v}|\,\omega(t) = v\,\omega$$

*Note that $\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$ is $\hat{v}$ rotated 90° anticlockwise.

Even at changing velocities, the relationship always holds that $\omega = \frac{v}{r}$ where v is the speed of a point at radius r from the centre of rotation.

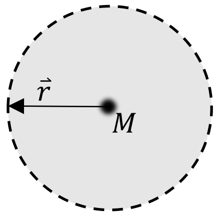
$$\therefore |\vec{a}_\perp| = v\frac{v}{r} = \frac{v^2}{r}$$

$$a_\perp = \frac{v^2}{r}$$

Therefore, any perpendicular acceleration will induce rotational motion, though a changing acceleration will result in changing values for radius and speed.

Derivation of Newton's Law of Gravity with Gauss' Law

We begin by considering a point mass with mass M, surrounded by an invisible sphere of radius r.



Since the point mass is identical when observing it from any point on the sphere it is spherically symmetric.

Now we consider Gauss' law, where A is the area of the invisible sphere:

$$\oiint \vec{g} \cdot d\vec{A} = -4\pi GM$$

Because the object is spherically symmetric, and we are considering the area of a sphere, the field is constant on all points on the sphere. Therefore, we can remove it from the integral as a constant (though we will have to remove the vector lines to undo the dot product):

$$g \oiint dA = -4\pi GM$$

$$g\, A = -4\pi GM$$

$$g\, 4\pi r^2 = -4\pi GM$$

$$g = -G\frac{M}{r^2}$$

Derivation of the Moment of Inertia of a disk rotating about its centre

Let the disk have some mass m, mass per area μ, and radius R.

The origin will be the centre of rotation which is the centre of the disk

$$I = \int_0^m r^2\, dm \qquad\qquad m = \mu A = \mu\, \pi R^2$$

$$= \int_0^R R^2\, \mu\, 2\pi R\, dR \qquad\qquad \frac{dm}{dR} = \mu\, 2\pi R$$

$$= \mu\, 2\pi \int_0^R R^3\, dR \qquad\qquad dm = \mu\, 2\pi R\, dR$$

$$= \mu \frac{2\pi}{4} R^4$$

$$= \frac{1}{2}\mu(\pi R^2)R^2 = \frac{1}{2}\mu A\, R^2 \qquad\qquad \mu A = m$$

$$I = \frac{1}{2} m R^2$$

Radians as a Unit

Most people refer to Radians as rad and as a unit for rotating objects. This is not incorrect, but it is important to know what it means to do this. Many people say a number n has no units and is just a number because it says how many things there are. Similarly, rad denotes how many radians there are. As a result, radians are not technically a unit.

This is why when angular velocity (ω) is multiplied by a radius (r) it has units $m\ s^{-1}$ not $m\ rad\ s^{-1}$

Interestingly this is also why, despite having units Nm, torque is not energy, but its path integral is. Since torque is $\tau = rF$ it must be multiplied by the angle to be energy (since work is the energy along the path and $r\theta$ is the path length). Notice how despite multiplying by an angle, it retains the units $Nm\ or\ J$.

MODULE 6: ELECTROMAGNETISM

Base Units

Charge (q) – Coulombs (C) or Amp Seconds (As)
Electric Field ($\vec{E}$) – Newtons per Coulomb (NC^{-1}) **or** Volts per Meter (Vm^{-1})
Magnetic Field ($\vec{B}$) – Tesla (T)
Magnetic Flux (ϕ_B) – Weber (Wb) **or** Tesla Square Metres (Tm^2)
Electric Potential (U_E) – Joules (J) **or** Volt Coulombs (VC)
Voltage (V) – Volts (V) **or** Joules per Coulomb (JC^{-1})
Current ($\vec{I}$) – Amperes (A) **or** Coulombs per Second (Cs^{-1})

Constants

Permittivity of Free Space $\varepsilon_0 = 8.854 \times 10^{-12}\ (A^2s^4kg^{-1}m^{-3})$
Permeability of Free Space $\mu_0 = 4\pi \times 10^{-7}\ (NA^{-2})$

Mass of an Electron $m_e = 9.109 \times 10^{-31}\ (kg)$
Mass of a Proton $m_p = 1.673 \times 10^{-27}\ (kg)$
Mass of a Neutron $m_n = 1.675 \times 10^{-27}\ (kg)$

Charge of an Electron $q_e = -1.602 \times 10^{-19}\ (C)$
Charge of a Proton $q_e = +1.602 \times 10^{-19}\ (C)$

Equations

Electrostatics

$\vec{F} = q\vec{E}$

The Force on a charged particle due to an electric field. The direction is given by the direction of the field however if the charge is negative, it experiences a force in the opposite direction to the direction of the field.

$$E = \frac{V}{d}$$

The formula for Electric field as a function of the scalar voltage field.

$W = qV = q\vec{E} \cdot \vec{d} = qE_{\parallel}d$

Work done on a charged particle in an electric field is the change in electric potential energy.

$$E_1 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1}{r^2}$$

The electric field at a radius due to a charged particle q_1.

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$$

The magnitude of force on a pair of charge particles at a radius (Coulomb's Law).

Circuits

$$V = IR$$

Ohm's Law

$$I = \frac{q}{t}$$

Current is the number of charges passing a point per second.

$$P = IV = I^2 R = \frac{V^2}{R}$$

The power output in a circuit is the number of charges passing per second (current) multiplied by the energy lost by each particle across a component in the circuit (voltage).

Magnetism

$$B = \frac{\mu_0 I}{2\pi r}$$

The magnitude of the electric field at a perpendicular radius from a current carrying conductor.

$$B = \mu_0 I \frac{N}{L}$$

The magnitude of the magnetic field inside a solenoid with N turns and length L.

$$\vec{F} = q\vec{v} \times \vec{B} = q\vec{v}_\perp \vec{B} = qvB\sin\theta$$

The force on a charged particle moving through a magnetic field. Due to the cross product, the force is always perpendicular to the velocity and, as such, will always induce some form of circular motion.

$$\vec{F} = l\vec{I} \times \vec{B} = l\vec{I}_\perp \vec{B} = lIB\sin\theta$$

The Motor Effect: The force on a current carrying conductor of length l in a magnetic field.

$$\frac{F}{l} = \frac{\mu_0}{2\pi} \cdot \frac{I_1 I_2}{r}$$

The force per length on a pair of current carrying conductors. Note that for net force, l is the length of the two wires which are parallel (the common length).

$\phi_B = \vec{B} \cdot \vec{A} = \vec{B}_{\parallel}\vec{A} = BA\cos\theta$

Magnetic Flux through an area. Note that $\vec{A}$ *denotes the area vector which is at* 90° *to the surface.*

Electromagnetism

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

The net force on a charged particle due to both a magnetic and electric field.

$$\varepsilon = -N\frac{\Delta\phi_B}{\Delta t}$$

E.M.F. induced in a coil with N *turns (Faraday-Lenz Law).*

$\vec{\tau} = NI\vec{A} \times \vec{B} = NI\vec{A}_{\perp}\vec{B} = NIAB\sin\theta$

The torque on a rectangular current carrying coil with N turns in a magnetic field.

Transformers

$$\frac{V_P}{N_P} = \frac{V_S}{N_S}$$

Voltage per turn ratio for a transformer is constant.

$V_P I_P = V_S I_S$

The power output for the primary and secondary coils in an ideal transformer is the same.

Extension Equations

Electrostatics

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2} \hat{r}$$

The force on a charged particle due to another charged particle. $\hat{r}$ denotes the unit radius vector. To find the force on particle q_1 the radius begins at particle q_2 and points to particle q_1.

$$W_E = \int_A^B \vec{F}_E \cdot d\vec{r} = q \int_A^B \vec{E} \cdot d\vec{r}$$

$$V = \frac{W_E}{q} = \int_A^B \vec{E} \cdot d\vec{r}$$

Integral form of work done on a charged particle in an electric field and Voltage. Note that the formulae use the dot product.

$$\oint\!\!\!\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

Gauss' Law. The double integral with a circle denotes the fact that the integral is performed across a 3D surface in space.

$$\vec{E} = -\vec{\nabla}\mathrm{V}$$

The formula for electric field as a function of a voltage field.

Magnetism

$$\oiint \vec{B} \cdot d\vec{A} = 0$$

Due to there being no magnetic monopoles, the magnetic flux through any 3D surface is zero.

$$\phi_B = \oint \vec{B} \cdot d\vec{A}$$

$$\vec{\phi}_B = \hat{A} \oint \vec{B} \cdot d\vec{A}$$

Magnetic Flux through any 2D area outlined by some perimeter. Note that the area of a coil is not the physical area but rather the area the coil forms the perimeter of.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Ampere's Law: The current through a conductor is proportional to the magnetic field around it.
(This equation only applies to current carrying conductors and an extra term was added by Maxwell to account for the displacement current between capacitors)

$$\vec{\mu} = nI\vec{A}$$

Magnetic dipole moment of a current carrying coil. The direction obeys the right-hand rule with respect to current.

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

$$U = -\vec{\mu} \cdot \vec{B}$$

Torque and potential energy for a magnetic dipole in a magnetic field.

Electromagnetism

$$\varepsilon = \int_0^l (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$$

E.M.F. is the same as the net work done per charge.

$$\varepsilon = -\frac{\partial}{\partial t}\oint \vec{B} \cdot d\vec{A} = -\frac{\partial \phi_B}{\partial t}$$

E.M.F. induced due to a change in flux through the area outlined by a loop.

$$\varepsilon = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

The E.M.F. generated in a current carrying conductor, in general when moving through a magnetic field.

$$\vec{I} = \frac{dQ}{dt}$$

Differential form for current.

Maxwell-Heaviside Equations

Integral Form	**Vector Form**
$\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$
$\oiint \vec{B} \cdot d\vec{A} = 0$	$\vec{\nabla} \cdot \vec{B} = \vec{0}$
$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{\partial \phi_E}{\partial t}$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

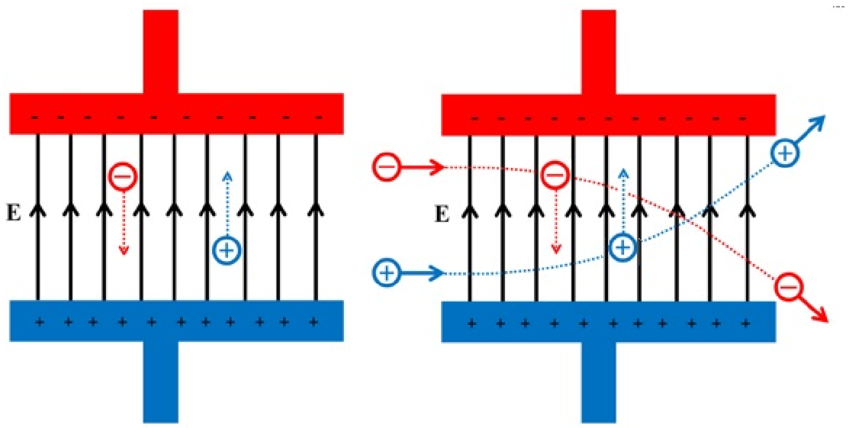
*$\rho = \frac{dq}{dV}$ (charge per volume), $J = \frac{dI}{dA}$ (current per area)

*See Appendices for notes on Del (∇) notation

Course Notes

Charges Moving in an Electric Field

In these types of problems, the same logic from projectile motion is required but instead of gravity acting as the acceleration, it is the electric field. As such, the problem should be broken up such that the y-direction is parallel to the field and the x-direction is perpendicular to the field.



Voltage

Voltage takes many forms in electromagnetism and can often be difficult to conceptualise. Voltage is often considered the 'pushing force' in electrical circuits but, really, it is a way of describing the potential for charges in a field to do work. This work done is why the voltage decreases across components of a circuit, the work was done therefore the potential to do work decreases.

Put simply, Voltage is the measure of the potential of the Electric field and can also be considered the potential energy per charge due to a field. As a result:

$$E = \frac{V}{d}$$

The Magnetic Field

The Magnetic Field is an interesting field to consider as it only exists in certain reference frames (it is fictitious). By performing a Lorentz transform on charged particle moving next to a current carrying conductor, you will find that there is a force on it due to an increased density of charges, caused by length contraction within the wire. The same is not true for a permanent magnet as the magnetism of atoms is due to the magnetic moments of the electrons in the shells.

Charges Moving in a Magnetic Field

For charges to experience a force in a magnetic field, they must be moving. This is due to the effects of special relativity and you can find more details on this on the Veritasium YouTube channel: (https://www.youtube.com/watch?v=1TKSfAkWWN0). This also means that the magnetic force is fictitious, as in, it only exists in certain frames of reference.

The force, because it is a result of the cross product of velocity and the field, will always result in a force which is perpendicular to both the field and velocity, the direction is given by the right-hand rule. As a result of it being perpendicular to the velocity, it can never produce energy (the force can never do work) and will always result in circular motion. (i.e. $\vec{F}_B = \frac{mv^2}{r}$)

Flux

There are two forms of flux in electromagnetism, electric and magnetic. However, the HSC course is only concerned with **magnetic flux** (ϕ_B). Flux is a measure of the amount of field which passes through the defined area and, as such, can be defined as the field which is perpendicular to the area, or the field which is parallel to the area vector.

The way of making sense of this is imagining you are looking at a piece of paper on a table from above. Now you begin to look at the piece of paper from a lower angle, slowly beginning to look at it so you can only see the very thin edge. The apparent decrease in size of the paper is the same as *effective area* and is analogous to amount of area a magnetic field can pass through.

The Area Vector ($\vec{A}$)

The area vector for any 2D surface has the magnitude of the area, and points perpendicular to the surface. Which side the vector points in is arbitrary, but it is important the definition remains consistent in the scenario i.e. if it is defined as pointing up at the start of a rotation, after half a revolution, it should be pointed down.

Induction

Induction is the process through which a Voltage or E.M.F. is induced across a circuit. This can happen due to two different effects:

1) A conductor moving in a magnetic field
2) A changing magnetic flux through a conductive loop.

In the first form of induction, the E.M.F. is produced by the force on each of the charged particles as the conductor moves. Because a conductor is made of protons and electrons, as it moves through the magnetic field, the charges experience a force. The E.M.F. generated is the work done by the force per charge and, for a rod, is given by the equation (where θ is the angle between v and B):

$$\varepsilon = vlB\sin\theta$$

In the second form of induction, it is slightly more abstract as to what causes the voltage as it is a **change in magnetic flux**. Due to Maxwell's equations defining flux through a 3D object as zero, the flux must instead be through a 2D surface. As a result, the area which the flux is generated by is the area traced by a conductive coil and the E.M.F. can be defined as follows for a coil with N turns:

$$\varepsilon = -N\frac{\Delta\phi_B}{\Delta t}$$

The change in flux can be generated in a few ways:

- Turning on/off a magnetic field source
- Moving a magnet or solenoid through the coil
- Moving the coil away/towards a magnet

Why does this effect exist?

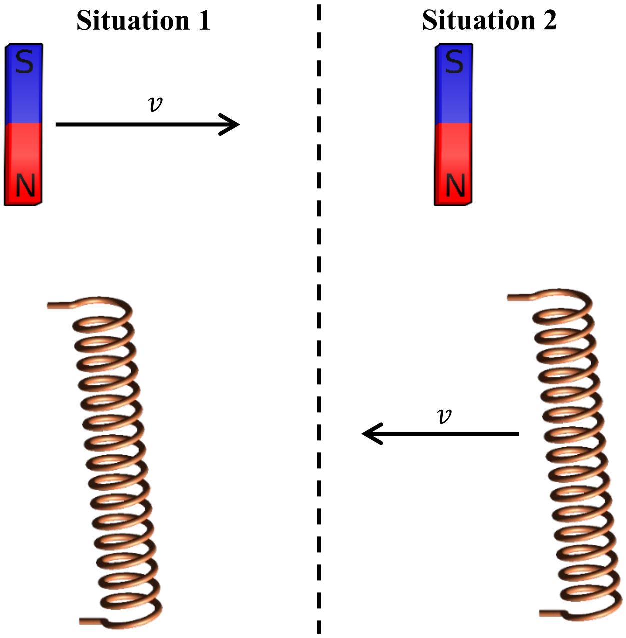
It's not entirely clear why this effect occurs, even to experienced physicists. However, there is a more fundamental law which may help to explain where this comes from.

A Note on the Difference Between Relative Motion and Changing Flux

Many people like to speculate that movement of a magnet across a coil is equivalent to movement of the coil across a magnet. Although these both result in the same phenomenon, they are not mathematically equivalent (i.e. situation 1 and 2 below are equivalent).

Many people like to say that when the coil is moved that the electrons in the coil are moving in a magnetic field and the magnet being moved across the coil can be treated as such. This is not completely true. If you shift into the coil's frame of reference, then yes this is true, and you can achieve the same result by doing so however in the lab frame only the magnet is moving. However, in the lab frame the magnetic field's value at each point in space is changing but the field itself cannot 'move' and as such, treating it this way from the lab's frame of reference is incorrect.

If you are unconvinced, try to prove that these two equations are, in the general case, equivalent as this is what is required to show that the two frames of references are mathematically equivalent.
(1 is the equation which describes situation 1 and 2 for situation 2)



1

$$\varepsilon = -\frac{d\phi_B}{dt}$$

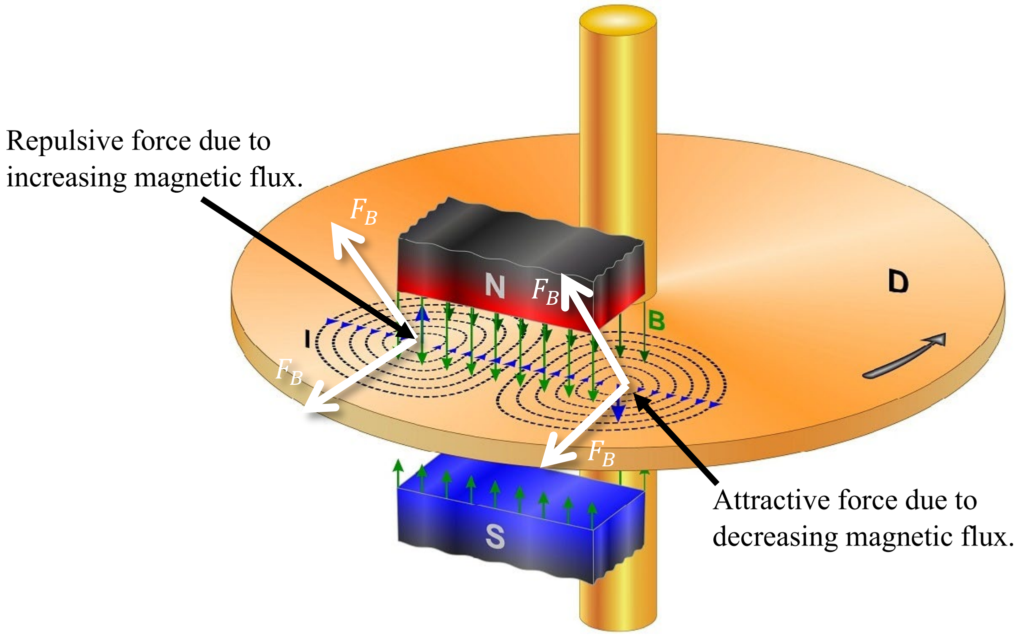
2

$$\varepsilon = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

Electromagnetic Beaks

Electromagnetic breaks use changing flux to create eddy currents which are attracted back towards the magnet which caused them.

The eddy currents created make their own magnetic fields, allowing them to be treated as a magnet. Due to Lenz's law the little areas of the wheel 'want' to retain the same magnetic flux that they had, so the parts of the wheel getting closer to the magnet will create a field which repels them from the magnet and the parts getting further away will create a field which pulls them back.



The Motor Effect

This is the application of moving charges in a magnetic field. The motor effect is merely the notion that charges moving through a current carrying conductor will experience a force. As the charges experience a force, they will move, attracting the conductor with them. This is the mechanism through which the entire wire, not just the charges experience a force.

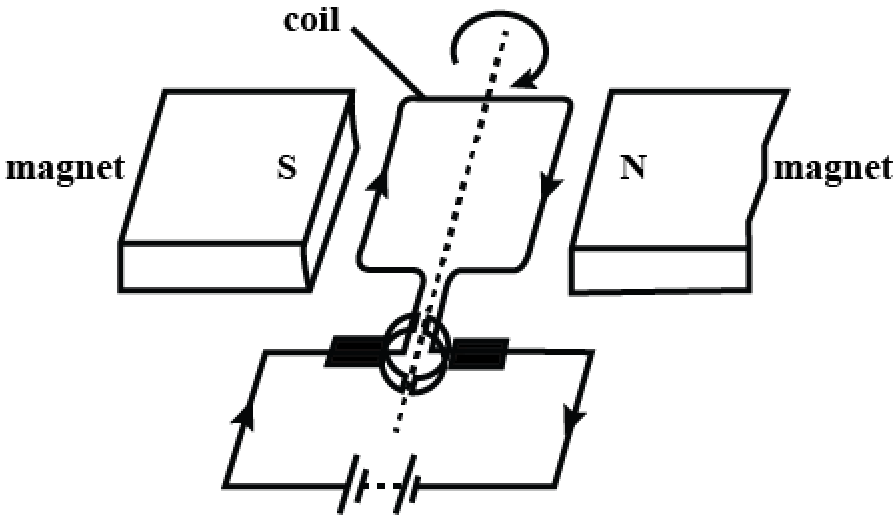
Deriving the Motor Effect

$$q\vec{v} \times \vec{B} = \frac{d}{dt} q\vec{s} \times \vec{B} = \left(\frac{dq}{dt}\vec{s} + q\frac{d\vec{s}}{dt}\right) \times \vec{B} = \left(\frac{dq}{dt}\vec{s} + \vec{0}\right) \times \vec{B} = s\overrightarrow{\frac{dq}{dt}} \times \vec{B} = l\vec{I} \times \vec{B}$$

Motors

DC Motors

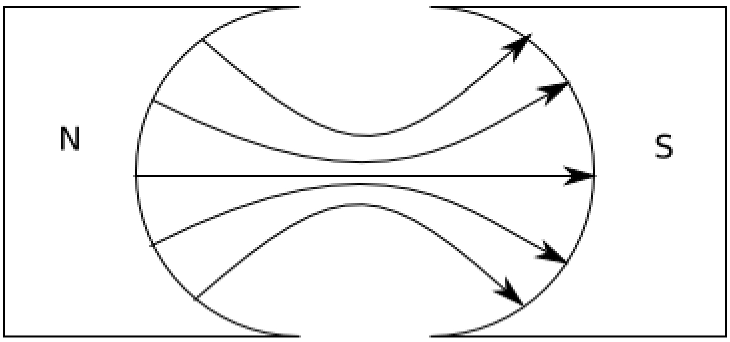
Motors are devices which take advantage of the Motor Effect to produce rotation.



There are two common questions which arise when considering the design of a motor:

1. How does one keep the torque uniform?
2. How does one keep the torque in the same direction?

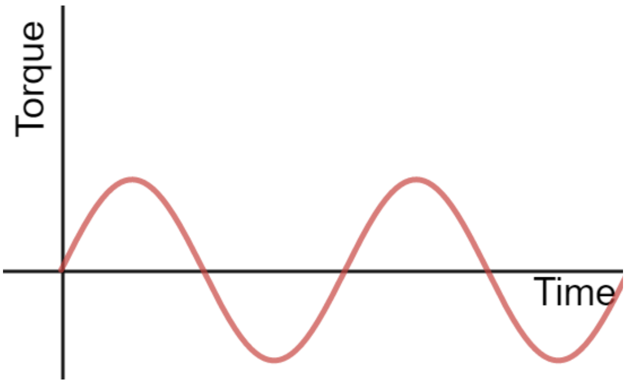
The first problem of keeping the torque at a uniform strength across the rotation has two solutions. The first way to improve this is by using curved magnets. The curved magnets produce a magnetic field which, at the outer edge, is pointed towards or away from the centre. This type of field is known as a **radial** magnetic field.



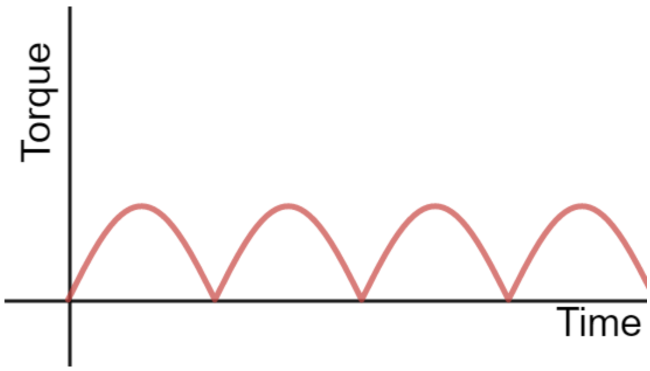
Because the field points in the same direction as the radius at the location of the arms, the angle between the force and the radius is a constant 90° at almost all times during the rotation. The end result is the torque produced is at a maximum value for all of the rotation (except when vertical).

Although this keeps the torque at a maximum for most of the rotation, the torque is not uniform due to the drop when the armature is vertical. To ensure the torque is completely uniform, there need to be more coils. If there are more coils, this drop of one armature as it reaches the vertical is counteracted by another coil at an offset. The more coils added to a motor, the more uniform the torque.

The second commonly considered problem with a motor is what happens when an arm of the armature (loop) changes from one side of the motor to the other (i.e. it becomes closer to the side with the North pole of the magnet).
This would normally result in a change to the direction of the torque on the armature due to the direction of each arm changing from slightly upwards to slightly downwards or vice versa.
This change in direction can be counteracted by a change in the current direction.



Torque where there is no change in current direction.



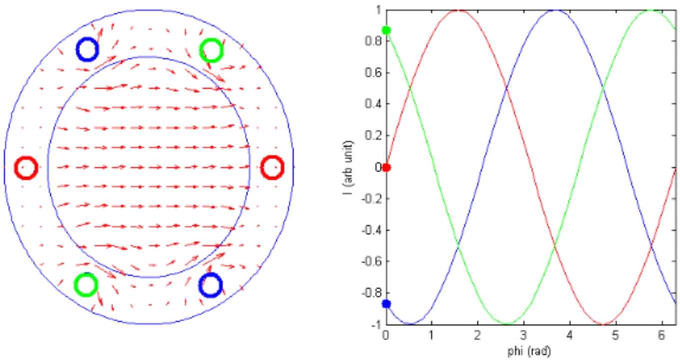
Torque where there is a change in current direction.

In a DC Motor, this change in current direction is achieved through a split-ring commutator, present in the image above. A split ring consists of semi-circular plates attached to each ending of the armature. The plates rotate with the motor and make contact with brushes, which are in turn connected to a voltage supply. As the motor rotates to this point, each plate breaks contact with a brush and connects to the other brush, changing the direction of the voltage across the armature. This then reverses the direction of the current and therefore the force. This acts as a sort of double reversal, keeping the torque in the same direction.

AC Induction Motors

An AC Induction motor works by utilising 3-phase AC power to create a rotating magnetic field inside the motor. This is analogous to the stator (external magnets) being rotated around the coil in a regular motor.

This induces a current in the coil which causes it to be 'dragged' by the field. The coil (or squirrel cage) inside the motor will be dragged with some torque due to the changing magnetic flux until it is rotating at the same speed as the field. The Squirrel Cage is a special type of coil which maximises magnetic flux and torque in this setup.



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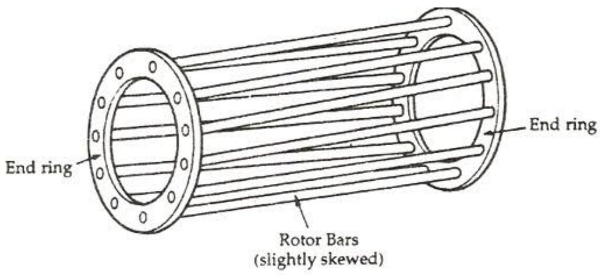


Generators

Generators work in reverse to motors, so instead of supplying power to turn a coil, a coil is turned, generating power.

The act of spinning the coil through a magnetic field creates a force on the electrons in the moving wires, generating a voltage. The voltage generated by this is the same as the back E.M.F. created when the motor is spinning and is in the same direction as when the motor spins in a given direction.

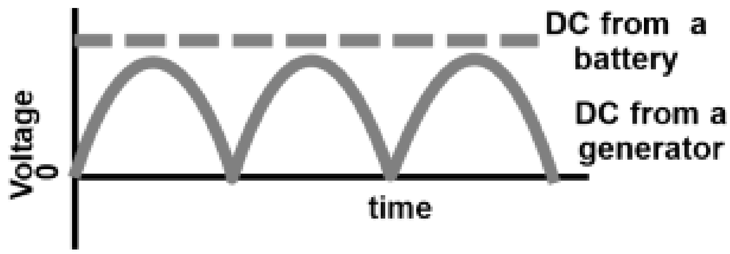
Due to a current being generated in the coil as it is rotated, there is also a force acting as a torque in the opposite direction, resisting the rotation of the generator. This force is described by the equation:

$$\vec{F} = l\vec{I} \times \vec{B}$$

This can be rationalised by considering that if there was no resisting torque, or the torque acted in the other direction, there would be an infinite amount of energy generated (i.e. once it was started it would never stop or it would infinitely generate more energy).

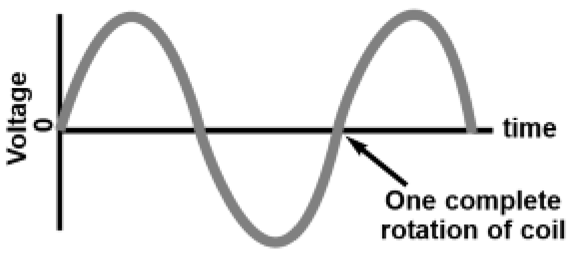
DC Generators

A DC Generator makes use of the same equipment as a DC motor. As with motors, the current direction flips every half turn and, as a result, the current generated would be AC.
To stop this from happening, a split-ring commutator is used, reversing the current again every half turn, keeping the current direction the same. In an ideal DC generator, the graph follows an $|\sin \omega t|$ graph but in reality, there is a sudden drop before it reaches its minima each time (as the commutator loses contact slightly before the transition).



AC Generators

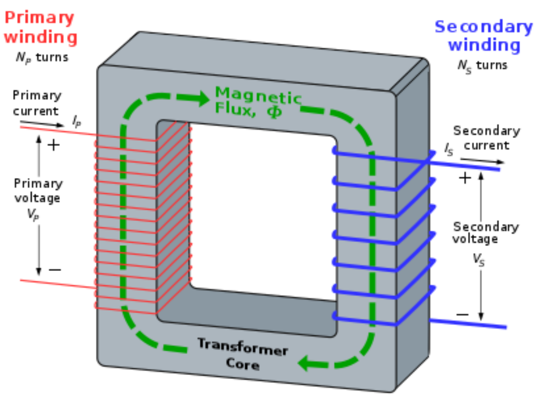
AC generators are actually rather simple as they use the same setup as a DC motor except without a commutator (use a slip ring instead). As already discussed, the direction of force in a DC motor swaps every half turn. In a generator, because the rotation is in a constant direction and being created by an external entity, this reversal still occurs but instead it is the direction of the voltage being generated which swaps. As a result, the direction of the current swaps every half turn which, by definition, is the generation of an Alternating Current.
This is carried away for use through a slip-ring commutator, which doesn't swap the direction of the current. A slip-ring commutator can be thought of as though you attached an alligator clip to each end of the coil and magically made them unable to tangle



Transformers

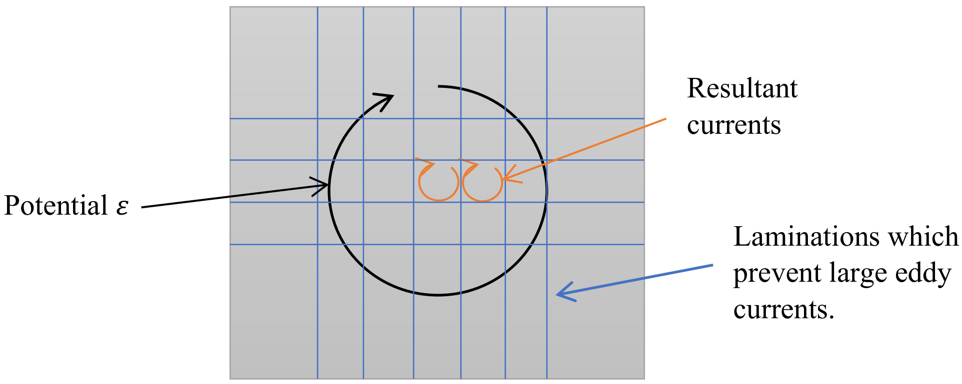
Transformers perform a simple but important task: changing the voltage in an AC circuit.
Transformers are made from two coils and a circular or rectangular iron core which both coils are wound around.
One coil, called the primary coil, is connected to power and has an alternating current put through it (often 50Hz for Australia) which then generates an alternating magnetic field in the solenoid. This magnetic field then induces a magnetic field in the iron core, which then channels the magnetic field around the iron loop and through the other coil, called the secondary coil.
The changing magnetic flux through the secondary coil caused by this process creates an E.M.F. which also alternates at the same frequency. By changing the number of turns of the secondary coil, the amount of E.M.F. produced changes. By increasing the number of coils, the voltage increases from the primary to secondary coils (step up) and be decreasing the number of turns the voltage drops (step down).

Transformer equations rely on the assumption that energy is conserved in the voltage transformation from coil to coil. As a result, when a voltage step-up occurs, there must be a current step-down and vice versa.



Laminations

Laminations are used in transformers to prevent eddy currents from being induced in the iron core. Eddy currents can be a big problem for the efficiency of transformers due to the fact that they use alternating current at high frequencies, which creates a changing magnetic field. Due to the frequency at which this changing magnetic field oscillates, the E.M.F. it would create is massive, which would normally create large eddy currents which would ruin the efficiency of the transformer. To counteract this, plastic laminations are used, parallel to the direction of the magnetic field within the metal core. The laminations reduce the area of the core to small sections, which prevents large currents from forming by reducing the maximum E.M.F. in each section to a smaller value. This increases the effectiveness of the core at channelling the $\vec{B}$ field.



Extension Notes

Coulomb's Law

Coulomb's Law is typically written as $F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$, however, it is a well-known fact that this force has a direction. This direction can be denoted by the unit radius vector which follows the conventions listed in the Vector Conventions section in the appendices.
This new formula is written as follows:

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2} \hat{r}$$

Gauss' Law

Gauss has many laws to his name, some in physics, some in pure mathematics. However, the law we consider "Gauss' Law" in Electromagnetism is:

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

or

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

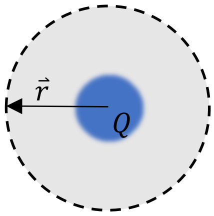
Symmetry

Symmetry is a key part of Gauss' law as it allows for the construction of equations where many of the variables are constant. Symmetry is the notion that for a 3D point in space, any point on a sphere around that point is indistinguishable from another point on that sphere. For a 1D line, any point on a circle around that line is indistinguishable from another point on that circle. This allows the fields in the integrals to be considered constant. This concept, although a neat mathematical trick, also tells us something about the nature of our universe. Because the universe obeys symmetry, we are allowed to use Gauss' Laws in these ways or, in other words, Gauss' Laws are the logical derivatives of what should occur if the universe is symmetric in certain ways.

Gauss' Law to Derive Coulomb's Law

Integral Form

To derive the formula for the field around a point charge, we must begin by considering a point charge in space with some charge Q. In 3D space, around a single point there is what is known as spherical symmetry. Put more simply, because of the nature of the universe all points at a constant radius from a point are indistinguishable from the perspective of that point.
Because of this, we must consider an imaginary sphere of radius r around that point charge.



We must then consider the equation $\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$. Because we have constructed this scenario based on symmetry, we know that $\vec{E}$ is constant and parallel to the area vector at all points on the surface of the sphere and can, therefore, be taken out of the integral as a constant.

$$E \oiint dA = E\ A = \frac{Q_{enc}}{\varepsilon_0}$$

We know the area of a sphere is $4\pi r^2$ and now all we have to do is rearrange the formula for $\vec{E}$.

$$E\ 4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

To derive Coulomb's Law, all we have to do is multiply by q, which will find the equation for force.

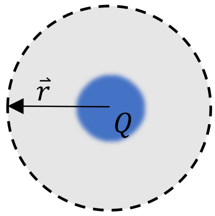
$$F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$$

Vector Form

We shall perform a similar operation except with the vector form of Gauss' Law

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \frac{dQ}{dV}$$

$$\vec{\nabla} \cdot \vec{E} = \begin{bmatrix} \frac{\partial \vec{E}}{\partial x} & \frac{\partial \vec{E}}{\partial y} & \frac{\partial \vec{E}}{\partial z} \end{bmatrix}$$



By assuming spherical symmetry, we can shorten this to merely a radial derivative.

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E}{\partial r} \hat{r}$$

$$\frac{\partial E}{\partial r} \hat{r} = \frac{1}{\varepsilon_0} \frac{dQ}{dV}$$

Since we know the direction of LHS, we can assign a direction to the RHS and integrate

$$\vec{E} = \int \frac{\hat{r}}{\varepsilon_0} \frac{dQ}{dV} dr = \int \frac{\hat{r}}{\varepsilon_0} \frac{dr}{dV} dQ$$

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\vec{E} = \int \frac{\hat{r}}{\varepsilon_0} \frac{1}{4\pi r^2} \, dQ$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}$$

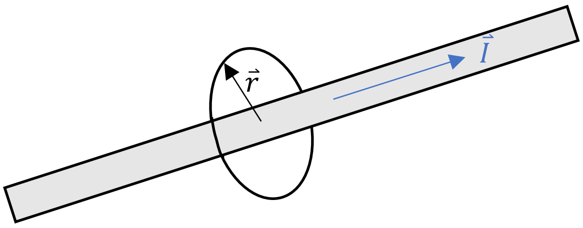
Maxwell's Equations to Derive Magnetic Field Around A Wire

To derive this equation, we consider Ampere's Law for magnetism: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{\partial \phi_E}{\partial t}$

In this case, we are considering the case where there is no changing electric flux as a constant electric field in the wire has already been established. As such, the equation to be considered is the following:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

To consider this scenario, we must construct a scenario with a conductor carrying some current I and an imaginary circle of some constant radius r around the wire.



The reason for this circle is that around a point on a wire, all points on this constructed circle are indistinguishable. If we rotate the circle, from the wire's perspective it will look identical. This allows us to consider B as a constant in the integral which results in the following:

$$B \oint dl = B\, l = \mu_0 I$$

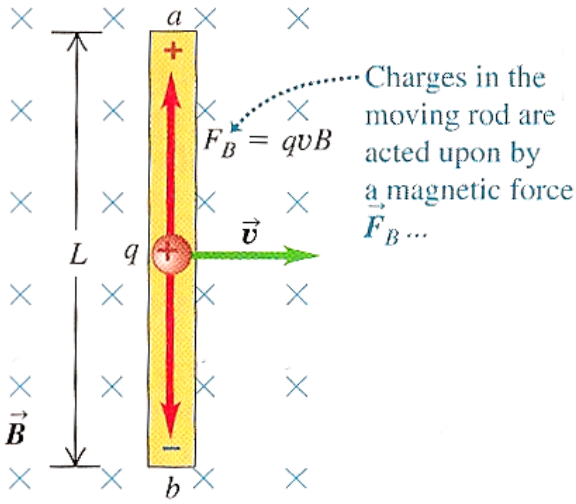
To solve for B all that must be done is establish what l is. In this case, because our shape was a circle, the length is the circumference of the circle, $2\pi r$.

$$B\, 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

What you will notice about the equation we have just derived is that it is the formula given for the magnetic field around a wire.

Derivation of the induced EMF on a rod moving through a Magnetic Field



Since we know that there is a force on all charges in the rod, and that they are dispersed evenly along the rod we can say the rod has a charge density per length of each type of charge:

$$q_+ = \lambda_+ L$$

$$q_- = \lambda_- L$$

$$q_- = q_+$$

The force due to motion on the positive charges is up the page and the force on the negative charges is down the page. Work is the force done along the distance:

$$W = \int_0^L \vec{F} \cdot d\vec{L}$$

The total work is the work done on the positive charges and negative charges:

$$W_{net} = \int_0^L q_+ vB \cdot dL + \int_0^L q_- vB \cdot dL$$

$$= \int_0^L \lambda_+ L\, vB \cdot dL + \int_0^L \lambda_- L\, B \cdot dL$$

$$= \lambda_+\, vB \int_0^L L\, dL + \lambda_-\, vB \int_0^L L\, dL$$

$$= (\lambda_+\, vB + \lambda_-\, vB)\left[\frac{L^2}{2}\right]_0^L$$

$$= \frac{(\lambda_+ + \lambda_-)vBL^2}{2}$$

$$\lambda_+ + \lambda_- = \frac{q_+}{L} + \frac{q_-}{L} = \frac{q_+ + q_-}{L}$$

$$\frac{q_+ + q_-}{L}\frac{vBL^2}{2}$$

$$|q_+| = |q_-| = q$$

$$= qvBL$$

$$\varepsilon = vBL$$

$$V = \frac{W}{q} = \varepsilon$$

The Electric Field Inside A Conductor

The electric field inside an ideal conductor in a circuit is always zero. The reason for this is that when an electric field is created at the end of a conductor, the electrons at that end begin to move. But when the electrons move, there is a net positive field where they used to be and a net negative field where they moved to. As a result, the other electrons in the conductor are pulled towards where the electrons used to be and pushed away from where they are now. This is then the reason why currents occur at a constant rate throughout the circuit rather than being different the further from the voltage supply you are (they are conductors because they conduct the field without causing it to decay).
Furthermore, it is the ratio of the efficiency of this chain reaction to the supplied field strength that gives rise to resistance. The more efficient a conductor the closer to zero the field inside it is and it is the inconsistencies in the field which give rise to resistive effects such as heat.

The Electric Field as A Function of a Voltage Field

$$\vec{E} = -\vec{\nabla}V = -\begin{bmatrix}\frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z}\end{bmatrix}$$

(See Appendices for more on gradient the Del operator)

All this definition states is that the electric field is the negative gradient in space of the voltage (i.e. if voltage increases in one direction, the electric field points in the other direction). In this case, the Electric field vector points in the direction of greatest decrease in voltage.

Solenoids and Inductors

In an electric circuit a solenoid is called an inductor and are commonly used in AC circuits. An inductor makes use of the Faraday-Lenz Law to store energy in the magnetic field and use that energy when the circuit is turned off. When a circuit is turned off, the current through the circuit quickly stops. As the current stops, the magnetic field produced in the inductor decreases at the same rate. The decrease in flux (with respect to time) induces a voltage and therefore a current in the inductor, maintaining the current for a small time after the circuit was turned off.
It is important to note this doesn't last forever as the inductor can only induce a current while there is a rate of change of current. As soon as there is no current from the source at all, the magnetic flux will also experience no change and the induced current will stop. The purpose is not to maintain the current forever but to increase the efficiency of the circuit so that the maximum current is maintained for longer. This effect is especially important in AC Circuits.

Why Metal Rods?

Putting iron rods through the centre of solenoids is common practice for most physics departments and the comment is often made that it amplifies the magnetic field, but why?

What is happening in this scenario is that the iron is a material which rearranges its internal atomic magnetic fields very easily. When iron is placed in even a weak magnetic field, its own internal magnetic domains rearrange to align with the external magnetic field. The iron then creates its own magnetic field due to this alignment.

This effect is why putting iron inside a solenoid magnifies the magnetic field.

MODULE 7: THE NATURE OF LIGHT

Base Units

Length (l) – Metres (m)
Time (t) – Seconds (s)
Speed (v) – Metres per second (ms^{-1} **or** m/s)
Electric Field ($\vec{E}$) – Newtons per Coulomb (NC^{-1}) **or** Volts per Meter (Vm^{-1})
Magnetic Field ($\vec{B}$) – Tesla (T)
Wavelength (λ) – Metres (m)
Frequency (f) – Hertz (Hz **or** s^{-1})
Energy (E) – Joules (J **or** $kg\, m^2\, s^{-2}$)
Intensity (I) – Power per area ($J\, m^{-2}\, s^{-1}$ **or** $kg\, s^{-3}$)

Constants

The Speed of Light $c = 3.00 \times 10^8$ ($m\, s^{-1}$)
Permittivity of Free Space $\varepsilon_0 = 8.854 \times 10^{-12}$ ($A^2 s^4 kg^{-1} m^{-3}$)
Permeability of Free Space $\mu_0 = 4\pi \times 10^{-7}$ (NA^{-2})

Planck Constant $h = 6.626 \times 10^{-34}$ ($kg\, m^2\, s^{-1}$)

Wein's Displacement Constant $b = 2.898 \times 10^{-3}$ ($m^2 s^{-2} K^{-1}$)

Equations

$d \sin\theta = m\lambda$

Describes the angular location of the m^{th} bright spot on a wall due to double slit interference (or dark spots for single slit interference).

$$d \sin\theta = \left(m \pm \frac{1}{2}\right)\lambda$$

Describes the angular location of the m^{th} bright spot due to single slit interference (+); and the m^{th} dark spot on a wall due to double slit interference (−).

$$\lambda_{max} = \frac{b}{T}$$

Wein's Law for Blackbody radiation.

$$I = I_0 \cos^2\theta$$

Malus' Law - The intensity of light that has passed through a polarising filter with θ the difference between the angle of the polarizing filter's axis and the polarization direction of the light.

$$K_{max} = hf - \phi$$

$$\phi = hf_0$$

The maximum amount of kinetic energy an electron can receive from a photon of some frequency f where ϕ is the work function of the material and f_0 the critical frequency of the material.

Extension Equations

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi$$

The differential wave equation. All waves are a solution to this equation.

$$I = \frac{{E_0}^2}{2c\mu_0} = \frac{c{B_0}^2}{2\mu_0}$$

The intensity of a light ray as a function of its maximum Electric and Magnetic field magnitudes.

$$P = \oint \vec{I} \cdot d\vec{A}$$

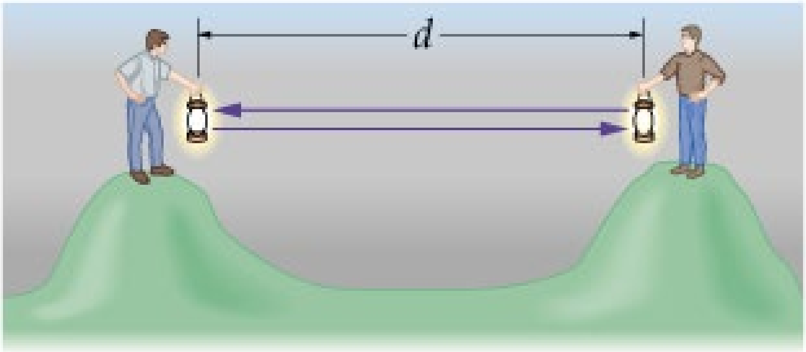
The power output of a wave is the 2D surface integral of the intensity over the area.

Course Notes

Attempts to Measure the Speed of Light

Galileo

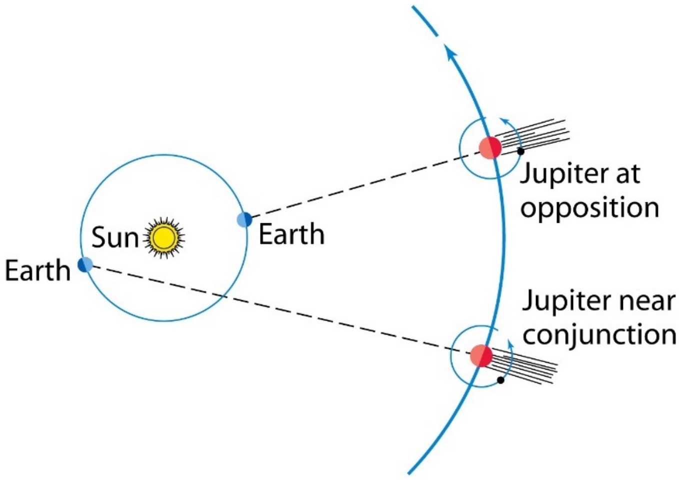
Galileo and his assistant stood on top of hills a reasonable distance (a few kilometres) apart. Galileo uncovered his lamp. His assistant uncovered his lamp once he saw Galileo's light and Galileo noted the time it took for him to see the light again.
He compared this to the time measured at a very small distance (i.e. human reaction time in his lab) and noticed that there was almost no difference. From this he concluded that the speed of light was a minimum speed of the distance between the hills over human reaction time measured in the lab and could be anywhere from this speed to infinite.



Romer

Romer was making measurements of the time at which the eclipse of one of Jupiter's moons Io occurred. Romer noticed that the time at which it appeared to occur followed a sine curve, with a period of one year.
He reasoned that this curve was occurring due to the orbit of the earth around the sun and that this delay must therefore be due to the speed of light travelling longer distances as earth moved to the other side of the sun.

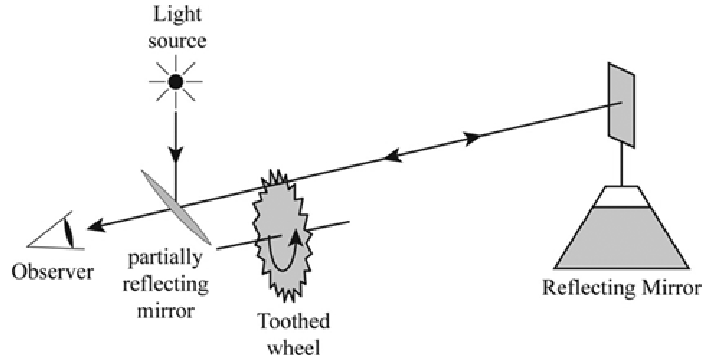
Using the known radius of earth's orbit, he estimated the speed of light to be $220,000,000\ m\ s^{-1}$

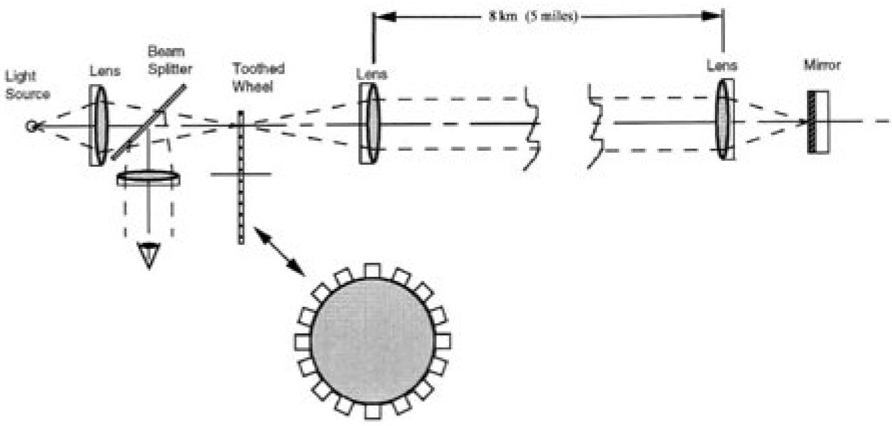


Fizeau

Fizeau used a toothed wheel with known angular speed and angular distance between the teeth. Fizeau passed light pulses through the holes in the teeth and then sped the gear up until the light pulses did not make it through. This meant that the next tooth was getting in the way of the pulse. Because he knew the angular separation of the teeth and the angular speed of the wheel, he calculated the minimum and maximum time the light could be taking after passing through the gap to make it back and hit the tooth. $\left(t = \frac{1}{nf},\ n = teeth\right)$

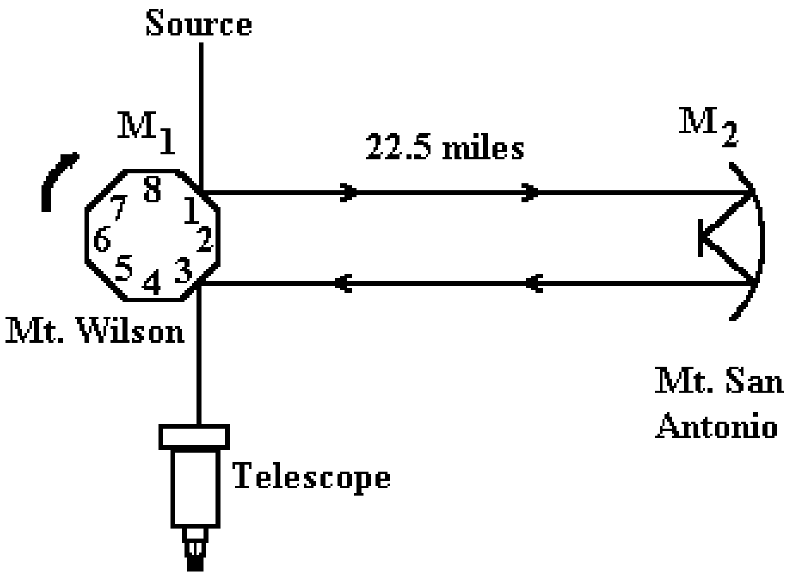
Using the known (large) distance between the mirror and the wheel, he calculated the speed of light using $c = \frac{d}{t}$ to be $3.15 \times 10^8\ m\,s^{-1}$





Michelson

Michelson's rotating mirror experiment is very similar to Fizeau's experiment in that it relied on the precise lining up of rotating objects with a ray of light. Michelson's experiment was, however, more precise than Fizeau's as it could use a continuous beam of light rather than pulses.



At first, the mirror is stationary such that the beam perfectly reflects towards the observer. If the mirror rotates even a little bit, the beam will not reach the observer. The key part of this is that the mirror must be in this orientation for the light to reach the observer.

Once the mirror starts rotating, the beam doesn't reach the observer because by the time the light has travelled the distance, the mirror has rotated to a different orientation.

The next time the beam will be seen is when the mirror is rotating such that:

- The beam reflects off side 1
- As the beam travels the distance, the mirror continues to rotate
- When the beam reaches the mirror, it has done exactly $\frac{1}{8}$ of a rotation and is back in the ideal state, with the light reflecting to the observer from side 2

This means the time taken for the beam to travel the large distance is $t = \frac{1}{8f}$ and the speed is $c = \frac{2d}{t}$

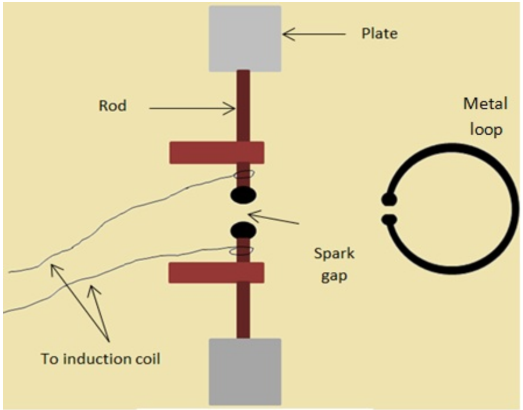
Michelson's measured the speed of light to be $299,979,000\ m\ s^{-1}$ (withing 0.1%)

The Hertz Experiment

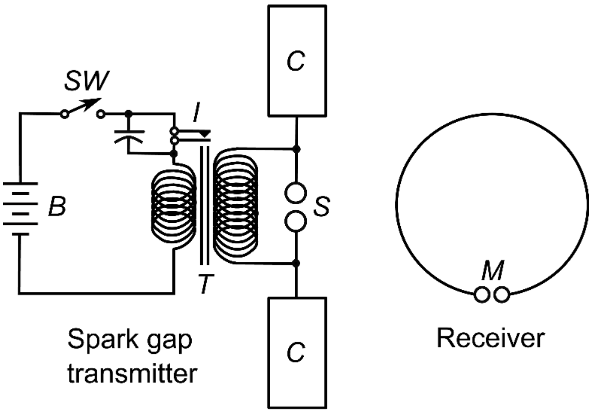
Hertz set out to experimentally prove two of Maxwell's predictions. As such he did two main things:

1. He assumed that light was an electromagnetic wave
2. He used the know relationships for waves to find the speed

If the assumption could be shown to be true and then the speed measured closely matched the prediction, then he could verify Maxwell's prediction.

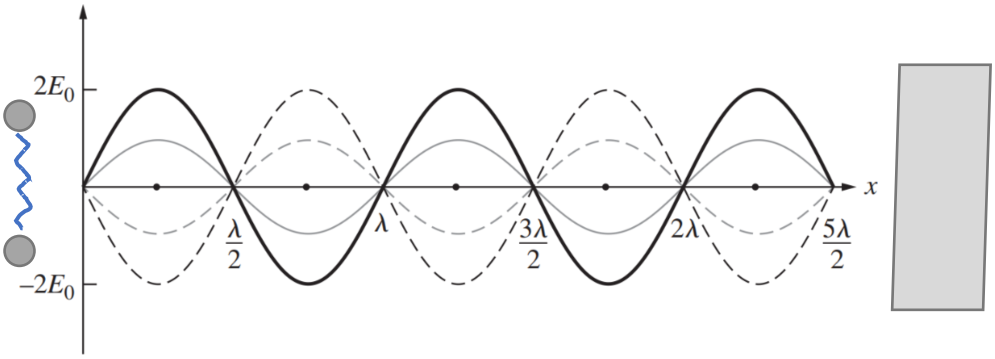


Hertz experiment (1887)



The experiment above cause a spark of electrons undergoing a high acceleration to pass between the spark gap. This created radio waves which were emitted in all directions (which were already known to be a type of light). The setup was created in a way that Hertz already knew the frequency of the wave.

By placing a metal reflector some distance from the emitter, he created a standing wave pattern between the emitter and the plate, where the gap between nodes is half the wavelength.



Now comes the assumption. If light is an electromagnetic wave, then the oscillating magnetic field of this standing wave will induce a current (and therefore a spark) in the loop detector at all points except the nodes. This was indeed the case.

By measuring the distance between points where there was no induced spark, Hertz measured the wavelength of the wave (which he already knew the frequency for).

Due to experimental error in calculating the frequency output of the device and error induced by an inversion of the wave upon reflection, Hertz could only conclude that the speed of light was finite and approximately $3 \times 10^8\ m\,s^{-1}$ (but with a large degree of uncertainty). This was enough to confirm Maxwell's hypothesis.

More recent attempts at his experiment with better data (but the same experimental setup) give a value of $c \approx 3.4 \times 10^8\ m\,s^{-1}$.

Waves need a medium – so what about light?

Once physicists discovered that light was a wave, they began to hypothesise about what is the medium through which it travels?

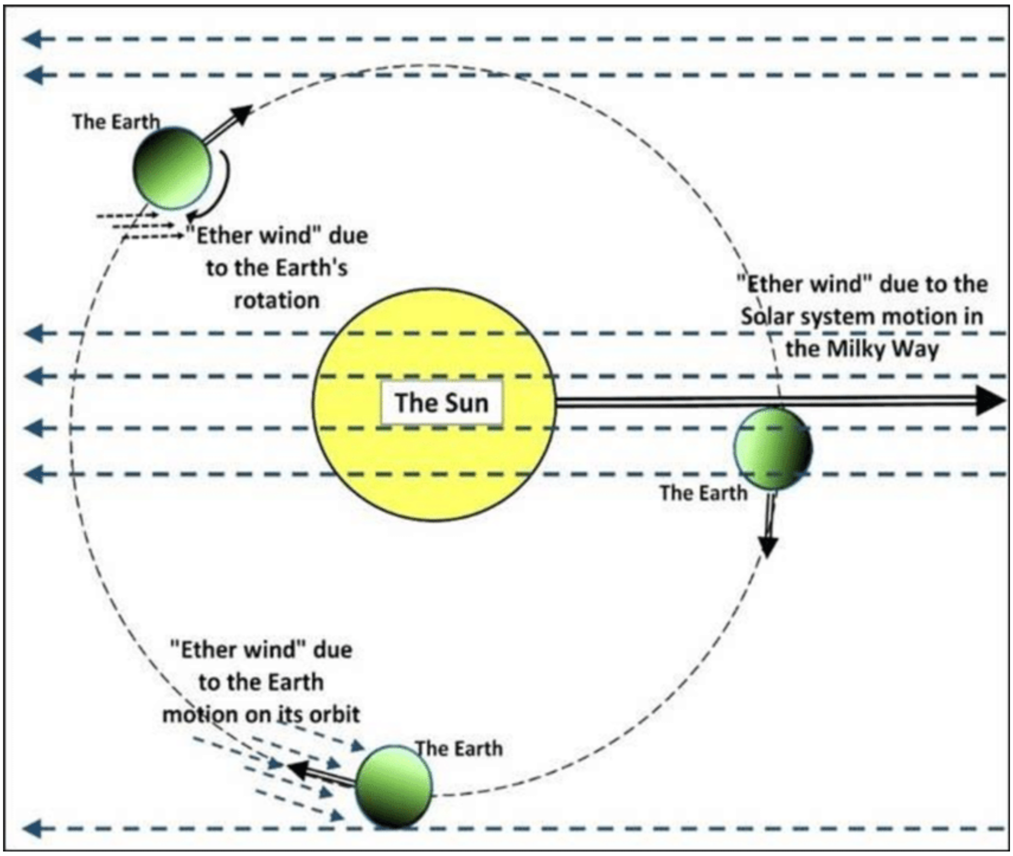
Absolute Space and Time

When Newton was developing his theories, he reasoned that there must be some universal clock and universal zero point. This seemed particularly reasonable since all phenomena observed at the time appeared to happen at the same rate and across the same distances.

The Luminiferous Aether

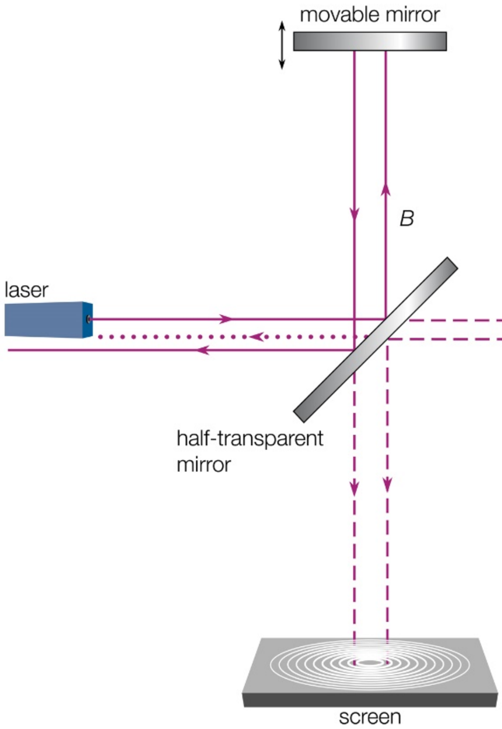
Based on Newton's idea of absolute space informed the construction of a theoretical medium called the aether. The aether was meant to be fixed to the universe's zero point and was the medium through which light travelled with speed c.

This meant that as Earth travelled through the universe, it should have some velocity relative to the aether. This would create 'aether winds' which would slow down or speed up light depending on the relative velocity of the Earth to the respective light ray.



Michelson-Morley Experiment

The Michelson-Morley experiment set out to detect these aether winds. They hypothesised that at a certain point in time and certain spot on Earth there is a given aether wind direction. Using their setup (below), should they rotate the apparatus, the relative speeds of each light ray will differ (they should bend depending on the angle to the aether wind) and a variable diffraction pattern should be produced as the apparatus is rotated.



The apparatus was floating on a mercury bath (because basically everything floats on mercury) allowing for it to be rotated smoothly while also isolating it from any vibrations from footsteps and the like.

The experiment yielded a null result (meaning nothing could be concluded). Due to the great precision of the experiment, and verification through repetition by other labs, it meant that the theory of the aether had to be revisited.

Although it was a null result, the discrepancy between the actual result and the theorised result meant that the theory had to be adjusted.

The Actual Medium for Light

The most current model of light says that there are magnetic and electric fields which permeate through all of space and it is the fields themselves which are the medium for light.

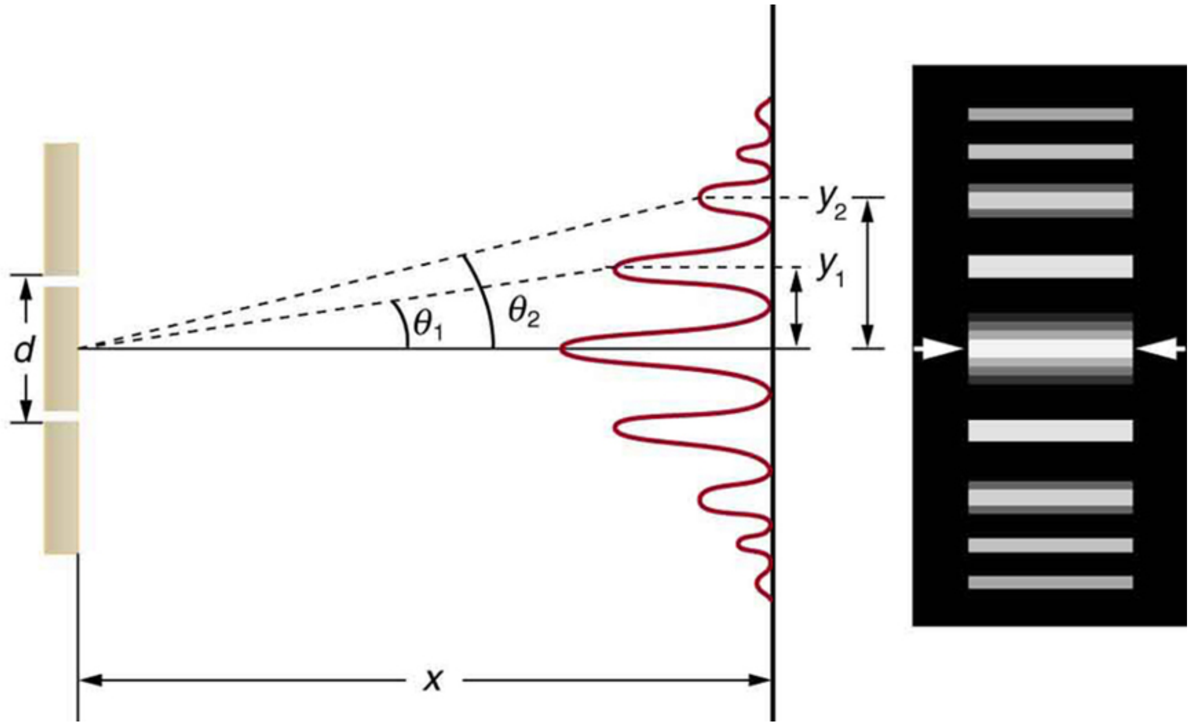
Waves need a medium because they need something to wave however it is the fields which wave, so light doesn't need a physical medium. The medium is the fields as they are what do the waving.

Diffraction

Double Slit

Double slit interference is somewhat easy to understand if you think about how waves work.

The assumptions of a double slit setup are that $X \gg d$ and therefore that $\sin\theta = \tan\theta$

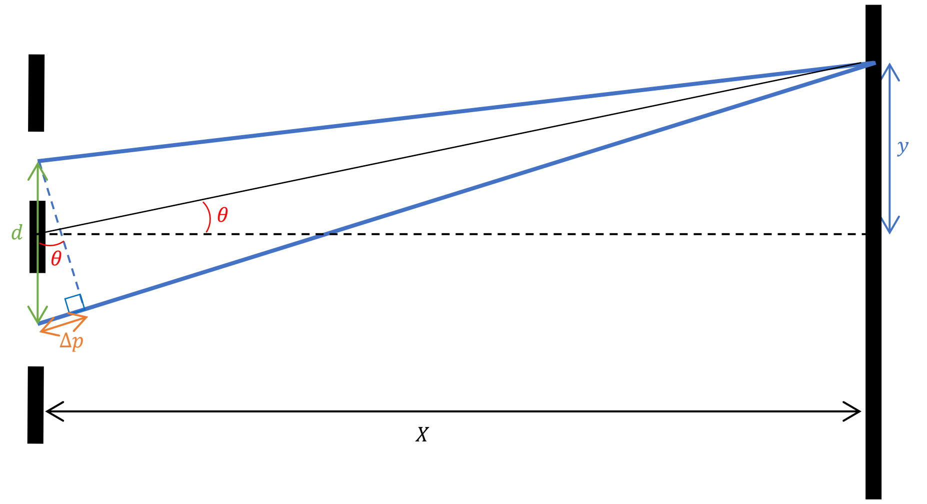


Formulae:

Maxima	Minima
$d\sin\theta_m = m\lambda$	$d\sin\theta_m = \left(m - \frac{1}{2}\right)\lambda$

Distance from Centre of Maxima / Minima		Gap Between Consecutive Maxima / Minima
$y = \dfrac{m\lambda X}{d}$	$y = \dfrac{\left(m-\frac{1}{2}\right)\lambda X}{d}$	$\Delta y = \dfrac{\lambda X}{d}$

Derivation of Formulae



Maxima

Condition for constructive interference is when the waves are in phase. Therefore, the distance covered by one wave must be an integer multiple of the wavelength (since they begin in phase).

Since $\Delta p = m\lambda,\ m \in \mathbb{Z}$

$$\sin\theta = \frac{\Delta p}{d}$$

$$d\sin\theta = m\lambda$$

Minima

Similarly, deconstructive interference occurs where the waves are out of phase and therefore where the path difference is half a wavelength. The first minima will occur where the path difference is half a wavelength.

Since $\Delta p = m\lambda - \frac{1}{2}\lambda,\ m \in \mathbb{Z}$

$$\sin\theta = \frac{\Delta p}{d}$$

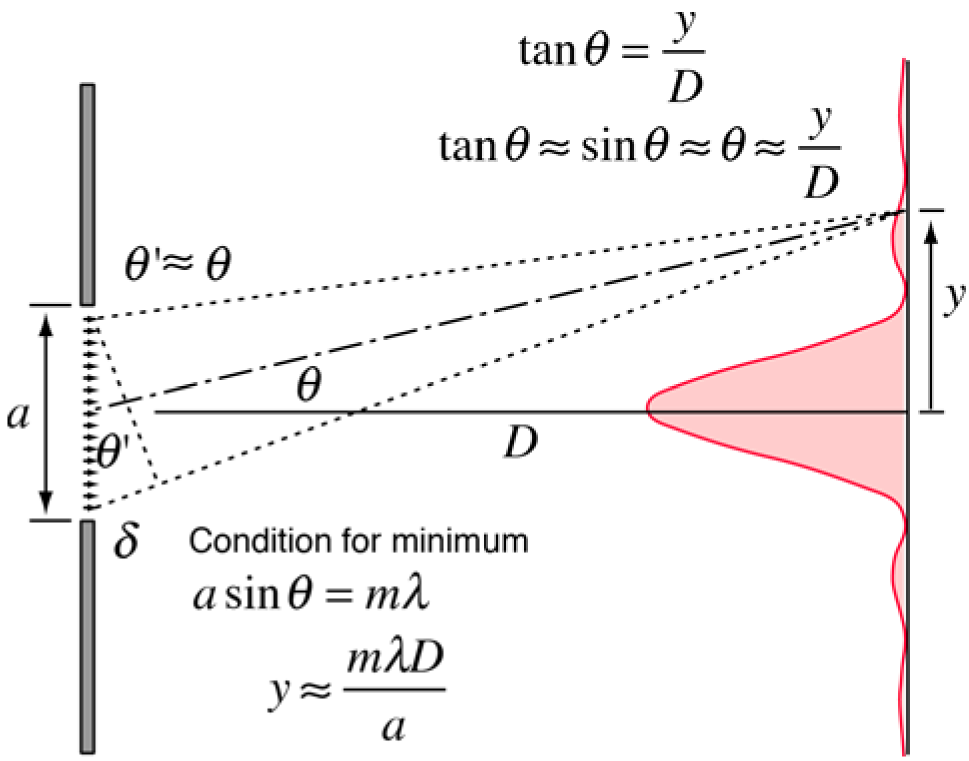
$$d\sin\theta = \left(m - \frac{1}{2}\right)\lambda$$

Height and Width of Peaks on Wall

Height	***Width***
$\sin\theta = \frac{m\lambda}{d} = \tan\theta = \frac{y}{X}$ $y = \frac{m\lambda X}{d}$	$\Delta y = \frac{(m+1)\lambda X}{d} - \frac{m\lambda X}{d}$ $\Delta y = \frac{\lambda X}{d}$

Single Slit

Single slit interference is only qualitatively assessable in Yr. 12 HSC Physics however it isn't terribly difficult. The main difference is that while in single slit interference the width of the centre peak is $2\,\Delta y$.



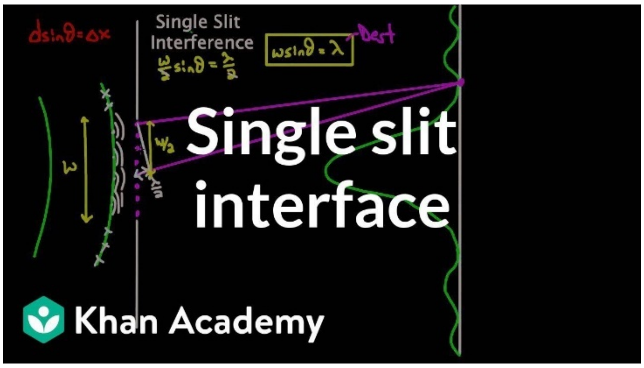
Formulae:

Maxima*	Minima
$a\sin\theta_m = \left(m + \frac{1}{2}\right)\lambda$	$a\sin\theta_m = m\lambda$

Distance from Centre of Maxima / Minima		Gap Between Consecutive Maxima / Minima
$y = \frac{\left(m + \frac{1}{2}\right)\lambda X}{a}$	$y = \frac{m\lambda X}{a}$	$\Delta y = \frac{\lambda X}{a}$

**There is technically no formula for maxima, but this gives the halfway point between two minima*

For Derivations see Khan Academy's video:



https://www.khanacademy.org/science/physics/light-waves/interference-of-light-waves/v/single-slit-interference

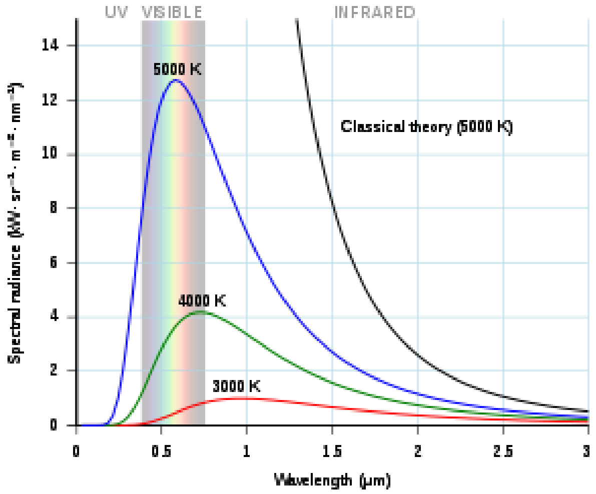
The Quantum Nature of Light

Blackbody Radiation

A blackbody is an object which does not reflect light. All objects of mass emit heat as a function of their temperature (i.e. infrared temperature sensors) and a black body is just something which physicists can analyse (because physicists are lazy and can't be bothered adjusting for light from the environment).
The sun can be considered a black body because it gives off so much more light than it receives from the rest of the universe. It is possible that the sun is actually a perfect mirror but it's about 1000000000000000000000000000000000000 times brighter than anything else, so no one cares.

The Ultraviolet Catastrophe

'The Ultraviolet Catastrophe' is so named due to what is possibly the worst prediction to come from physics. Classical thermodynamics and electromagnetism predicted that a hot black body should emit an exponentially large amount of light in the ultraviolet, x-ray and gamma-ray side of the light spectrum. This violated both common sense and conservation of energy and was the first hint at a quantum nature of light.



Thermodynamics is also known as statistical mechanics and as such, the heat and light output of a blackbody is probabilistic (or statistical) in nature (its why the curve is a curve and not a bar graph). The peak light emission wavelength is derived by calculating the highest probability interaction between subatomic particles at a certain temperature and therefore all other wavelengths will have a lower probability of being produced.

The Mechanism for Blackbody radiation

The way blackbody radiation is produced is by the acceleration of charges when atoms and molecules collide and change velocity. When this occurs, the accelerating charges produced light waves and the principle of probability and temperature distributions still holds.

The current theory is defined by Wein's law:

$$\lambda_{max} = \frac{b}{T}, \qquad b = 2.898 \times 10^{-3}\ (m^2 s^{-2} K^{-1})$$

The Photoelectric Effect

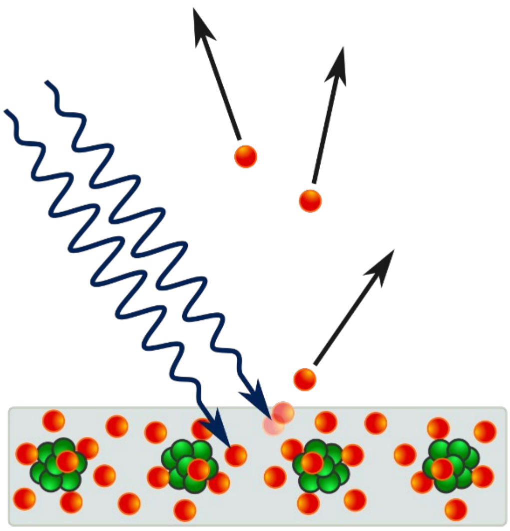
The Photoelectric effect was the first definitive evidence for the quantum nature of light as it completely contradicted classical electromagnetism. Although Blackbody radiation was unsolvable by classical physics, it did not completely contradict it, at least on a fundamental level.

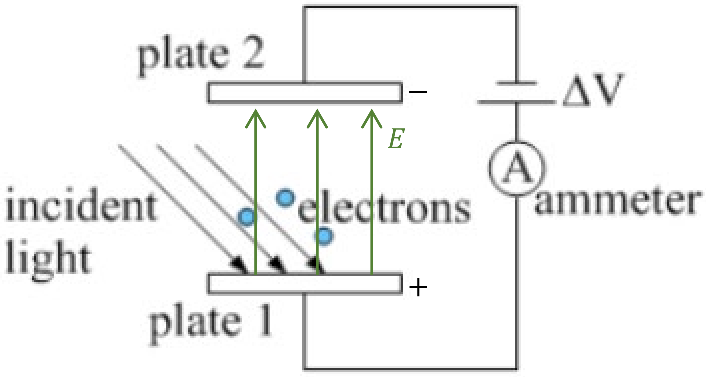
Classical Electromagnetism says that the power output (energy per second) of a wave is proportional to its intensity:

$$P = \oint \vec{I} \cdot d\vec{A} = IA$$

It was therefore completely reasonable to assume that the greater the intensity of light, the greater the amount of energy the light would give electrons. Therefore, the energy electrons possess after being vibrated by a light wave should be proportional to the power of the light wave. This was not the case.

Instead, the energy absorbed by an electron was a function of the frequency (intensity is not a function of frequency at all). This eventually led Einstein to conclude that this must be due to light coming in packets with energy $E = hf$ and intensity being due to an increase in the number of packets (eventually called photons) striking the surface.
This was the beginnings of the quantised understanding of light.





The Work Function (ϕ)

The work function is the lowest amount of potential energy which an electron possesses while part of the material. This is what gives rise to K_{max} rather than just K as it is the lowest amount of resistance the material provides to the electron being removed.

The Experimental Setup

The experiment is setup such that light is shone onto one of two parallel plates. A voltage can be applied to the plates so that there is an electric field between the two plates. This electric field can be setup to oppose the velocity of the emitted electrons.

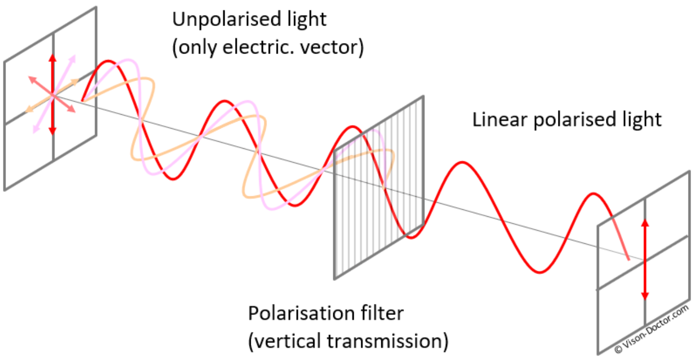
Since the work done by the field is $W = qV$, if the work done by the field is greater than the maximum kinetic energy of the electrons, then the electrons will stop before they hit the other plate. So, if $V = \frac{K_{max}}{q}$ then no current will be detected. As such, if an electric field is established such that it just stops all electrons then $qV = hf - \phi$

$\therefore \phi = hf - qV$

Polarisation

Polarised light is light with an electric field which only oscillates along one axis.

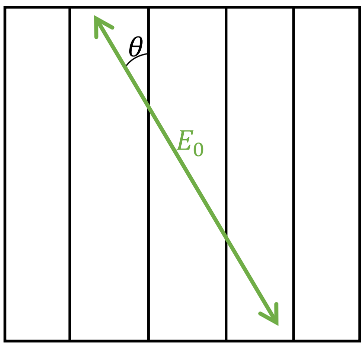
Polarisation is typically understood using a wave model of light and splitting the field vector into components. In reality, it is a quantum phenomenon, with the polarisation of a photon being a function of its spin, something which is based on probabilities.



Wave Model – Derivation of Malus' Law

It is somewhat easy to derive Malus' Law when considering the assumptions of polarisation. If we consider light of a single polarisation direction. If we assume that a polariser blocks all electric field components of a light ray which are perpendicular to its axis, then we can consider the following:

A light ray is shone onto a polariser with its electric field E_0 oscillating on an axis at some angle θ from the axis of the polariser.



The (peak) component of the electric field which will be let through is the parallel component.

$$E = E_0 \cos\theta$$

$$E^2 = E_0^2 \cos^2\theta$$

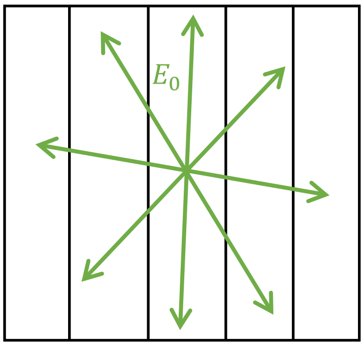
$$I = \frac{E^2}{2c\mu_0}$$

$$\frac{E^2}{2c\mu_0} = \frac{E_0^2}{2c\mu_0}\cos^2\theta$$

$$I = I_0 \cos^2\theta$$

Unpolarised Light passing through a polariser

The general rule is that 50% of the intensity is let through the polariser if the original intensity is for unpolarised light.



We can prove this by summing the intensities for all angles then taking the average (since unpolarised light has light at all angles to the polariser). Since we must account for all angles we have to integrate across all angles however in doing so we will multiply by an angle $(d\theta)$ so to maintain our units we have to divide by the sum of our bounds (2π) with dimension angle such that our units are still correct.

$$I = \frac{\int_0^{2\pi} I_0 \cos^2\theta \;\, d\theta}{2\pi}$$

$$= \frac{\int_0^{2\pi} \frac{I_0}{2}(1 + \cos 2\theta)\, d\theta}{2\pi}$$

$$= \frac{\frac{I_0}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}}{2\pi}$$

$$= \frac{I_0}{2}\frac{(2\pi + 0 - 0 - 0)}{2\pi}$$

$$I = \frac{I_0}{2}$$

Quantum Model

The quantum interpretation of light makes polarisation much harder to understand, but so is the nature of quantum. In reality, intensity is proportional to the number of photons which strike a surface. Photons possess a property called spin and this determines the direction of their electric field oscillation. When a photon is travelling through space, there is a degree of uncertainty in its spin. When a photon hits a polariser, it is interacting with it and therefore, its wave functions will collapse, and its spin will lose its uncertainty. The probability that the photon's spin aligns with the polariser is:

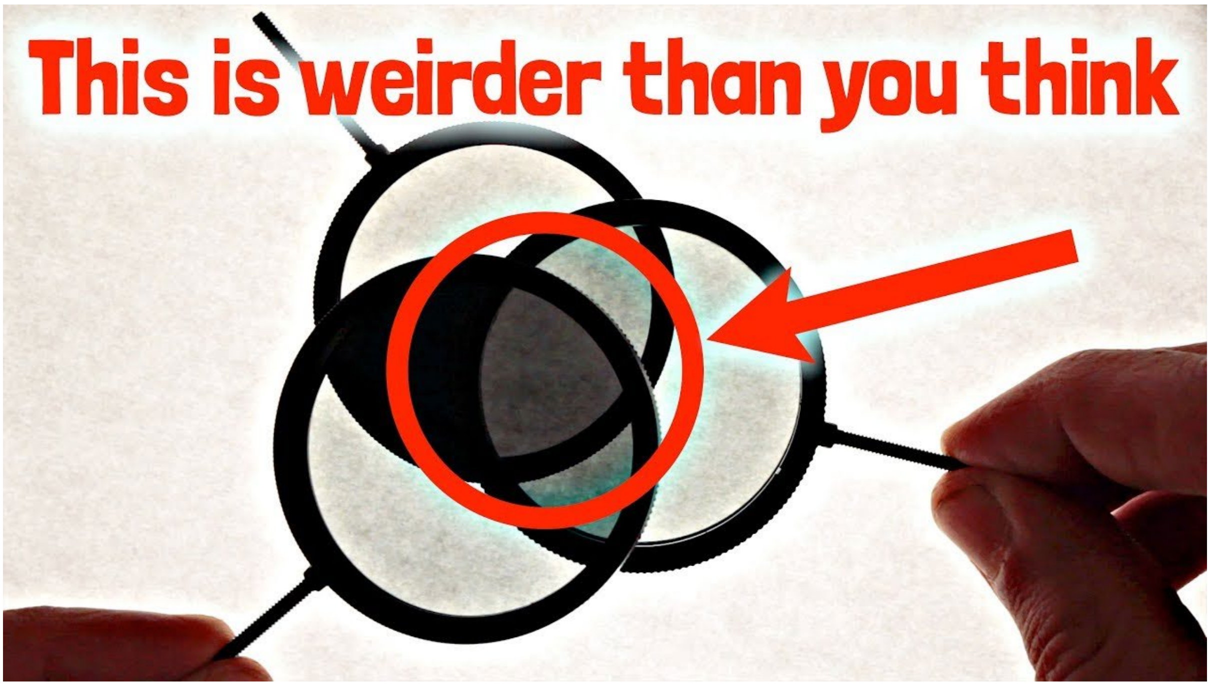
$$P(align) = 50\%$$

After this, the photons which pass through are all 100% aligned with the polariser. Should they come into contact with an analyser at an angle θ to their polarisation axis, the probability that they will change their spin axis to the new spin axis is given by:

$$P(align) = \cos^2 \theta$$

(There are some very complicated calculations that can be done to figure this out with quantum physics but eventually you get this result).

This is particularly evident with three polarisers which are at different angles to each other (see minutephysics' video on the topic):



https://www.youtube.com/watch?v=zcqZHYo7ONs

Light has Momentum?

Many teachers will mention that light has momentum and can move objects very slightly. This is due to the fact that $E = mc^2$ is only true for an object at rest.

The actual equation is:

$$E^2 = (mc^2)^2 + (pc)^2$$

Where p is the momentum.

Light has no mass, so the first term goes to zero.

$$E^2 = (pc)^2$$

$$E = pc$$

And since we know light has energy, it must have momentum. This is why light can cause small metal fans to move in vacuum chambers.

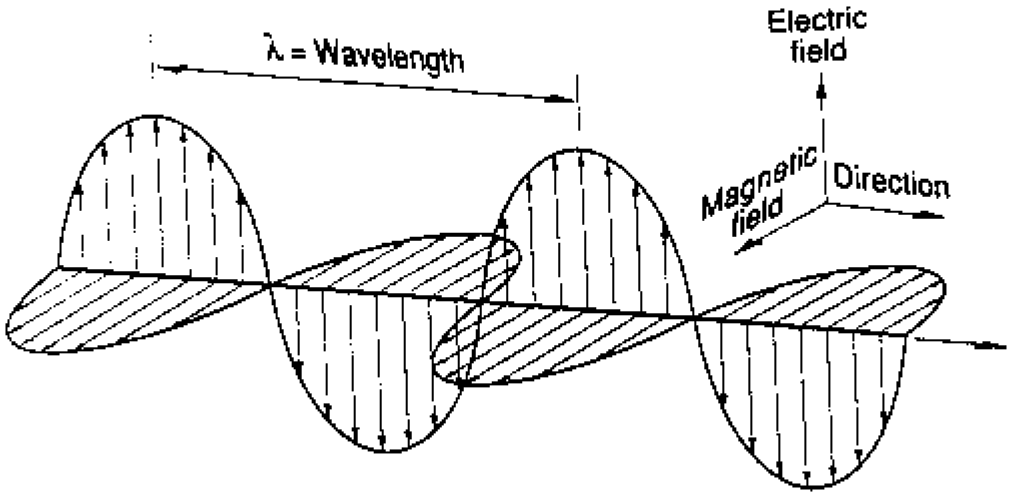
Extension Notes

Maxwell's Prediction for the Speed of Light

Though there are typically four equations known as 'Maxwell's Equations', Maxwell himself actually had around 20 equations and it was Heaviside who unified them into the four. Using his stupid number of equations, Maxwell was able to rearrange them to show that an accelerating charge (which generates a changing magnetic field) could induce a changing electric field which would induce a changing magnetic field, both of which obeyed the wave equation (where y is the amplitude in arbitrary units and x is the direction it is travelling in with speed v):

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

It was from this that he deduced the speed of light squared to be $c^2 = \frac{1}{\mu_0 \varepsilon_0}$



A note for sketching light waves

The rule for light is that $\hat{E} \times \hat{B} = \hat{v}$

Derivation by example of the wave equation

The simplest wave is given by $y = A\sin(2\pi f\ t)$

First let's define a few properties and variables for waves:

$$v = f\lambda$$

$$k = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f$$

If Amplitude y, $v = \frac{dx}{dt}$

$$\omega = vk$$

$$(y \perp v)$$

Since a wave travels at speed v, the time take to cover some distance x is given by $t = \frac{x}{v}$

First, we can re-write the wave equation:

$$y(t) = A\sin(2\pi f\ t) = A\sin(\omega t)$$

Now, we include the x factor in the time

$$y(x,t) = A\sin\left(\omega\left(t - \frac{x}{v}\right)\right)$$

$$\frac{x}{v} = \frac{x}{f\lambda} = \frac{2\pi\ x}{\omega\lambda} = \frac{kx}{\omega}$$

$$y(x,t) = A\sin(\omega t - kx)$$

$\frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx)$	$\frac{\partial y}{\partial x} = -kA\cos(\omega t - kx)$
$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx)$	$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(\omega t - kx)$

$$\frac{\left(\frac{\partial^2 y}{\partial t^2}\right)}{\left(\frac{\partial^2 y}{\partial x^2}\right)} = \frac{\omega^2}{k^2} = v^2$$

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

If instead of just x, the wave is a function of all of space, then we can re-write the equation using the Laplacian such that a wave propagating in all directions will also obey the equation.
(ψ shall replace y for the sake of simplicity)

$$\frac{\partial^2\psi}{\partial t^2} = v^2\ \nabla^2\psi = v^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)$$

Derivation that Light is a Wave with Speed $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \vec{0}, \qquad \rho = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad\qquad \vec{\nabla} \cdot \vec{B} = \vec{0}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\frac{\partial\left(\vec{\nabla} \times \vec{B}\right)}{\partial t} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\nabla^2 \vec{E}$$

$$-\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\nabla^2 \vec{E}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \nabla^2 \vec{E}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0}\left(\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2}\right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \mu_0 \varepsilon_0 \frac{\partial\left(\vec{\nabla} \times \vec{E}\right)}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{B}\right) - \nabla^2 \vec{B}$$

$$\vec{\nabla} \cdot \vec{B} = \vec{0}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = -\nabla^2 \vec{B}$$

$$-\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = -\nabla^2 \vec{B}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \nabla^2 \vec{B}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0}\left(\frac{\partial^2 \vec{B}}{\partial x^2} + \frac{\partial^2 \vec{B}}{\partial y^2} + \frac{\partial^2 \vec{B}}{\partial z^2}\right)$$

Both the $\vec{E}$ and $\vec{B}$ fields satisfy the wave equation and are therefore waves (except the trivial case where $\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{\partial^2 \vec{E}}{\partial t^2} = 0$) such that $v^2 = \frac{1}{\mu_0 \varepsilon_0}$

MODULE 7: RELATIVITY

Base Units

Mass (m) – Kilograms (kg)
Length (l) – Metres (m)
Time (t) – Seconds (s)
Speed (v) – Metres per second (ms^{-1} **or** m/s)
Momentum ($\vec{p}$) – Kilogram metres per second ($kg\ ms^{-1}$ **or** $kg\ m/s$)

Constants

The Speed of Light $c = 3.00 \times 10^8\ (m\ s^{-1})$

Equations

Galilean / Newtonian Relativity

$$\vec{v}_{A\ rel.\ to\ B} = \vec{v}_A - \vec{v}_B$$

The classical velocity transform for inertial reference frames. The equation relies on the velocities $\vec{v}_A$ and $\vec{v}_B$ both being taken from a third inertial reference frame.

Einsteinian / Special Relativity

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$$

Known as the Lorentz Factor. It is used in Space-Time diagrams as part of the Lorentz Transform and appears frequently in Special Relativity. β is the fraction of the speed of light the object is travelling at (i.e. if $v = 0.8c,\ \beta = 0.8$).

$$t = \gamma t_0$$

The formula for time dilation. This transforms the time taken for some event in a stationary reference frame (t_0) into the time taken in a reference frame in motion with respect to the event (t).

The effect of this is that t is always larger than t_0.

$$l = \frac{l_0}{\gamma}$$

Describes the contraction of the space that a moving object inhabits. Space contracts along the axis of motion to a length of l, where the length of the stationary object is l_0.

It is therefore important to note that a moving object will observe all of space around it, including the distances between objects as shrinking.

$\vec{p} = \gamma \, m_0 \vec{v}$

The apparent momentum of an object with some velocity $\vec{v}$ with respect to another object.

$E^2 = (mc^2)^2 + (pc)^2$

$E = \gamma mc^2$

The total energy of a particle or object, where m is the rest mass, p the momentum, and c the speed of light. In a situation where there is no relative motion, p is 0 and the equation simplifies to $E = mc^2$.

Course Notes

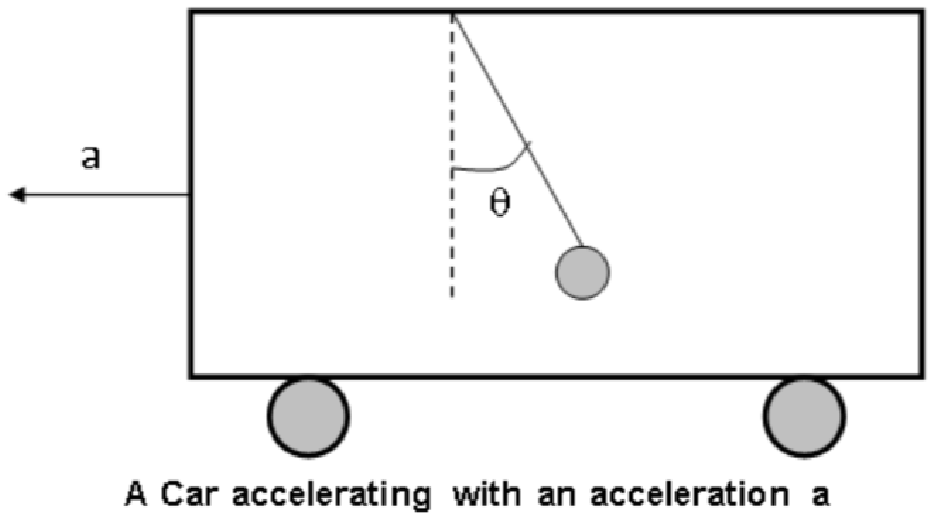
Inertial Reference Frames

An inertial reference frame is the frame of reference of any object moving at a constant speed. It is impossible to determine from within an inertial reference frame whether you are moving.

Non-Inertial Reference Frames

A non-inertial reference frame is any reference frame which is undergoing acceleration. This could be an accelerating rocket, or an object being spun in a circle. It is possible to tell whether you are in a non-inertial reference frame due to pseudoforces, for example the centrifugal force a rotating object appears to feel, or the backwards force felt within an accelerometer.

Special Relativity only applies instantaneously to a non-inertial reference frame as the velocity of the frame is constantly changing. To properly account for accelerating frames, General Relativity must be invoked.



A basic accelerometer, where from inside the ball appears to feel a backward force identical to the forwards force on the car.

Einstein’s Postulates

1. That the Laws of Physics are constant in all inertial reference frames
2. That the Speed of Light is constant in all inertial reference frames (this is also a consequence of postulate 1)

1) The Laws of Physics

Although this may seem like a trivial assumption, there was also a large amount of evidence that it is impossible to tell the difference between inertial reference frames (or in other words there is no experiment that can be done in an inertial reference frame to determine whether you are moving).

Note that the Laws of Physics do change when in a non-inertial frame of reference.

2) The Speed of Light

This is a consequence of Maxwell’s Equations that $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ (where μ_0 and ε_0 are properties of the universe) . It follows that since the laws of physics (and therefore properties of the universe) remain unchanged in inertial frames, the speed of light is constant in all inertial reference frames.

Events in Relativity

In Relativity, any event which occurs in one reference frame will occur in another reference frame (provided they are able to causally affect each other), though they may disagree about when and where they occur.

The Lorentz Factor

The Lorentz Factor, γ, shows up frequently in Special Relativity and can be used to transform space-time coordinates between inertial reference frames. The Lorentz Factor is always equal to or greater than one ($\gamma \geq 1$).

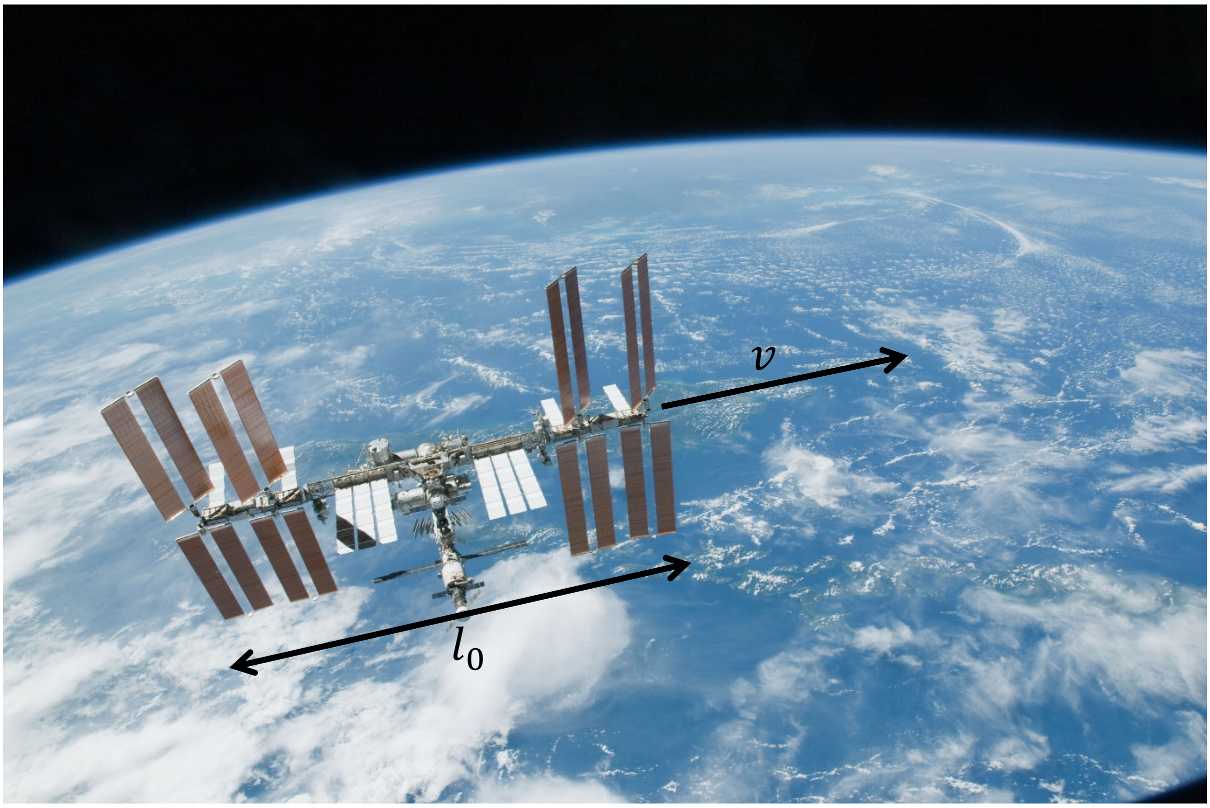
Two things to remember

Two primary intuitions should be used when considering a special relativity problem:

1. Moving objects shrink along the direction of motion
2. Moving clocks run slow

An Introduction to Special Relativity

To understand Special Relativity, first imagine a ball being thrown from one end of the international space station to the other, being viewed from within the space station and from Earth.
From within the space station the ball takes some time t_0 to travel to the other end and the station has some length l_0 along its direction of motion.



For an observer watching on Earth, they will observe the ball taking some time t to reach the end of the station where the space station is of length l.

The rest time between the two events t_0 (the ball being thrown and the ball reaching the other end) and the rest length between the two ends of the station l_0 will be affected by the relative velocity of the space station to an observer on earth[1]:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

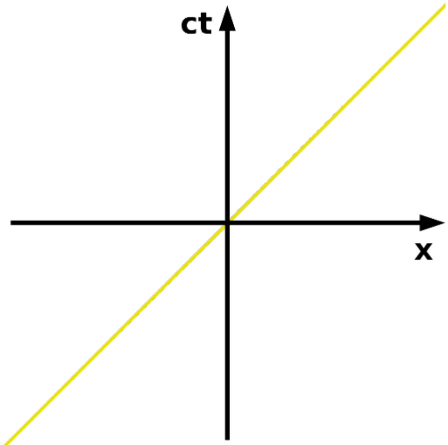
$$t = \gamma t_0$$

$$l = \frac{l_0}{\gamma}$$

[1] This example is not exactly accurate to reality as the space station is undergoing centripetal acceleration and is therefore in a non-inertial reference frame (i.e. we need general relativity to describe the situation). However, this example should otherwise provide a plausible situation where an object is moving at a relatively fast speed.

Spacetime Diagrams

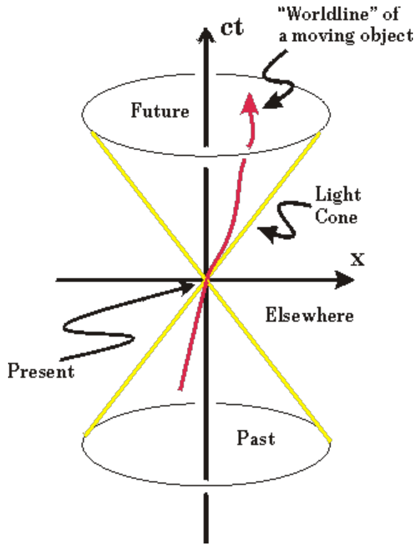
Minkowski Spacetime diagrams are common in Special Relativity as they help to explain many of the paradoxes which can appear when merely considering the equations of relativity.



In this spacetime diagram the horizontal axis, x, represents the distance along the direction of motion of an object. The vertical axis represents time but in a different manner to normal time. Instead of measuring time in seconds it is measured in light-seconds (i.e. the distance light travels in a second). This means that light is graphed as a line with gradient 1 and angle of 45° to both axes (because light travels one light-second in a second and every second that passes its time coordinate increases by one light-second).

Light Cones and Causality

Because light travels at the fastest speed in the universe, we know that the minimum time it takes for one object to interact with another is the time it takes light to reach that object. This can be represented in spacetime diagrams with a light cone. Any event in space and time within the past light cone could have had an effect on the present and anything outside it could not have had an effect on the present.
Similarly, any event within the future cone could have an effect on the future or could be affected in the future by an event in the present.



The Lorentz Transforms

For a point travelling along the x axis with speed $v = \beta c$

$$t' = \frac{\gamma}{c}(ct - \beta x)$$

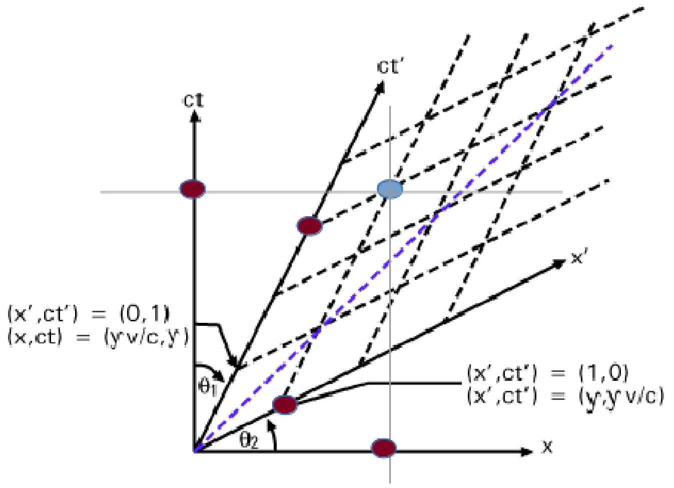
$$x' = \gamma(x - \beta ct)$$

$$y' = y$$

$$z' = z$$

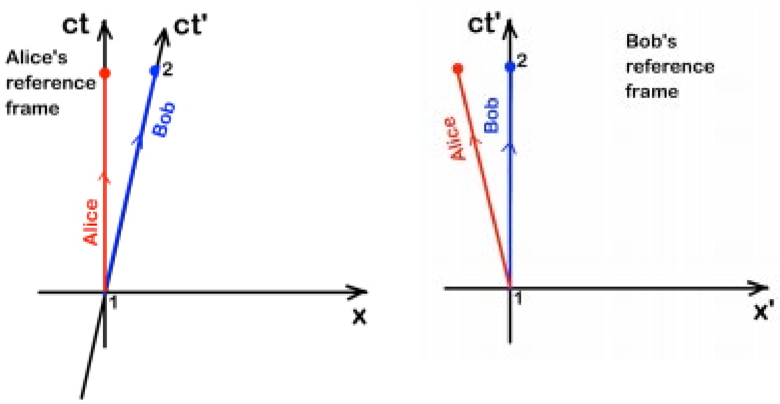
Lorentz Transforms

The Lorentz Transforms are mathematical transforms which demonstrate the apparent positions and times of events relative to different observers. They can be represented graphically on a spacetime diagram by the rotation of coordinate systems.



The above diagram represents an event occurring in spacetime (the blue dot) which for a stationary observer has coordinates (x, ct) and for a moving observer has coordinates (x', ct'). It is worth noting that just as with normal graphing, these coordinates are found by drawing lines from the time axis which are parallel to the space axis and vice versa (hence the seemingly wonky lines for the moving observers' axes).

It is possible however, to go into the moving observer's reference frame with a Lorentz Transform. Below is a Lorentz Transform from the Alice's perspective where Bob is moving, to Bob's perspective where Alice is moving.



Deriving Length Contraction from the Lorentz Transforms

An object with length l has some point x_1 and x_2 such that $l_0 = x_2 - x_1$

$$l' = x_2' - x_1'$$

$$= (\gamma x_2 - \gamma\beta ct) - (\gamma x_1 - \gamma\beta ct) = \gamma x_2 - \gamma x_1 = \gamma l_0$$

BUT

$$t_1' = \gamma t = 0$$

$$t_2' = \frac{\gamma\beta l}{c} \neq t_1'$$

So the times at which the observer is measuring the positions of these points is not the same (but when you measure length you measure the positions at the same time).

This shows that events which are simultaneous for a stationary observer are not simultaneous for a moving observer (this is kind of the whole point of relativity). Let's instead reuse the equations but setting our time to T so that the position of the points can be measure simultaneously.

$$l' = x_2' - x_1'$$

$$T' = t_2' - t_1'$$

$$(x_2' - x_1') = \gamma\big((x_2 - x_1) + v(t_2 - t_1)\big)$$

$$(t_2' - t_1') = \gamma\left((t_2 - t_1) + \frac{\beta}{c}(x_2 - x_1)\right)$$

To measure the stick ends at the same time we need $t_2' = t_1'$ so $(t_2' - t_1') = 0$

$$(t_2 - t_1) = -\frac{\beta}{c}(x_2 - x_1)$$

$$l' = (x_2' - x_1') = \gamma\big((x_2 - x_1) - \beta^2(x_2 - x_1)\big)$$

$$l' = \gamma(x_2 - x_1)(1 - \beta^2)$$

$$\frac{1}{\gamma} = \sqrt{1 - \beta^2}$$

$$l' = \frac{\gamma(x_2 - x_1)}{\gamma^2} = \frac{l}{\gamma}$$

$$\therefore l = \frac{l_0}{\gamma}$$

The Consequences of this Derivation

What this means is that when an observer sees a length contracted object the ends of the object are not actually at those respective positions (because when you actually want to measure the length of the stick you need to measure the end positions at different times). This means a person seeing a length contracted stick is not seeing the stick how it is but instead, where then ends were.

Even stranger is that the times at which each end was at each of its respective positions is different, so the front end may have been where you see it a second ago, but the back end was where you see it half a second ago. This is important for later.

See Fermilab's video on the topic if it is still unclear:

https://www.youtube.com/watch?v=-Poz_95_0RA

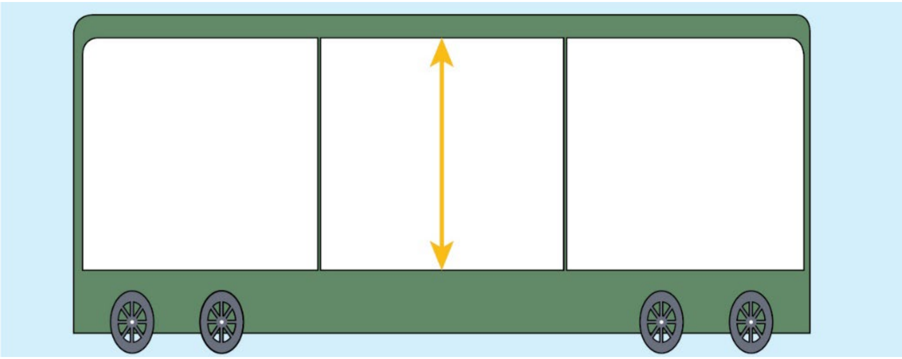
Time Dilation

'A moving clock runs slow'

In the ball on the space station scenario, the time measured for the ball to reach the end of the space station will be shorter for the observer on the space station compared to the time measured by an observer on Earth. Alternatively, the observer on Earth will observe a longer time taken for a single event compared to the stationary observer.

Why does Time Dilation Occur? – The Photon Clock

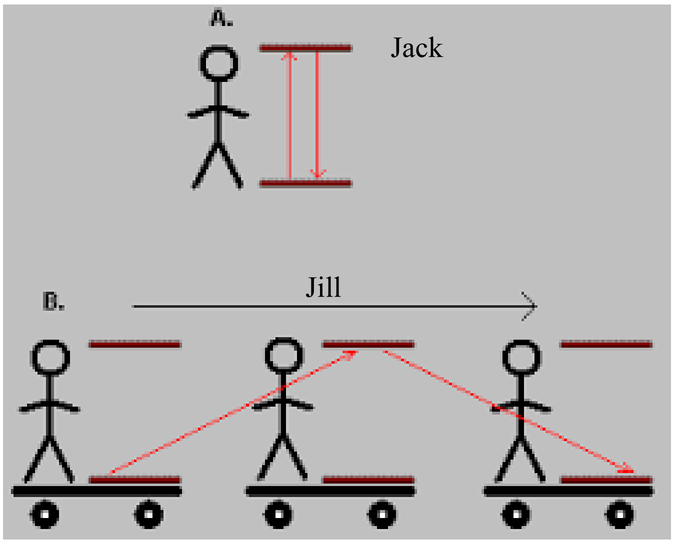
Consider a situation where Jack is onboard a train moving at some fraction of the speed of light β, and Jill is beside the tracks on a platform. Inside the train is an evacuated vertical tube with mirrors on either end (at the top and bottom). Inside the tube is a single photon which bounces back and forth between the mirrors.



Let the distance between the two mirrors be d and the time taken for the light to travel from the bottom mirror to the top mirror be t. As a result, $c = \frac{d}{t}$

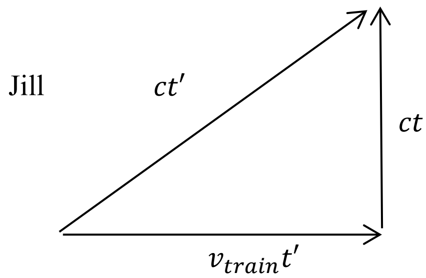
From now on the bottom mirror shall be *Mirror A* and the top mirror is *Mirror B*. Therefore: the photon leaving Mirror A is *Event A* and the photon striking Mirror B is *Event B*.

Now we consider this from Jill's perspective on the platform. While in Jack's perspective Event A and B occur at the same horizontal coordinate, in Jill's perspective they occur at different horizontal locations.



As can be seen in the diagram above, both Jack and Jill will observe events A and B and the distance the photon convers is $speed * time$. Since Jill observes the photon travelling a longer distance and c is constant in both reference frames, Jills time (t') must be longer than Jacks. In other words, Jill's clock ticks faster than Jacks.

To calculate this by completing Jill's diagram, using the speed of the train to calculate the horizontal distance. ($v_{train} = \beta c$)



$$(ct')^2 = (ct)^2 + (vt')^2$$

$$(ct')^2 - (vt')^2 = (ct)^2$$

$$c^2t^2 = (t')^2(c^2 - v^2)$$

$$t^2 = (t')^2\left(1 - \frac{v^2}{c^2}\right)$$

$$t = t'\sqrt{1 - \frac{v^2}{c^2}}$$

$$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma t$$

Formula Sheet:

$$t = \frac{t_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

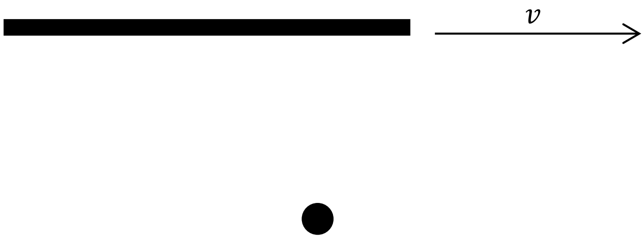
Length Contraction

In the above scenario, the observer on Earth will observe the space station to be shorter along the axis of motion. This is because when an object moves relative to an observer, the space that object inhabits appears to shrink. In essence, all points along the direction of motion within that space get closer together.

For an observer in the moving space station, all objects in the universe appear to be moving relative to it and, therefore, will length contract. The distance between the space station and an object it is moving towards will appear to shrink and the Earth will seem thinner.

Why does Length Contraction Occur?

Consider a moving 1-metre ruler with some speed and an observer watching from some distance.



When the observer measures where the ends of the ruler are, the light rays take a different amount of time to reach the observer. The difference in time between the position measurements and the apparent simultaneity of them to the observer causes length contraction. (See length contraction derivation above).

The Momentum Equation and the Universal Speed Limit

The equation for momentum (which can be derived from the Lorentz Transforms) is

$$p_v = \gamma m_0 v$$

Although it may seem like just an equation, the result of this equation is drastic.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

As $v \to c, \ \gamma \to \infty$

But when $v = c$

$$\gamma = \frac{1}{\sqrt{0}}$$

As a result, nothing in the universe with mass can travel at the speed of light as it would have undefined momentum.

The Ladder Paradox

Setup

A farmer wants to fit his 20-metre ladder in his 10-metre shed. He knows that if his friend runs at some fraction β of the speed of light he will observe the ladder as being length contracted to 10 metres, just short enough to fit into the shed.

But for his friend, the shed appears to be moving and will shrink from 10 metres to 5 metres long, while the ladder remains at 20 metres.

Who is right?

Explanation

They are both correct and both wrong. Length contraction of the ladder occurs due to the difference in the actual times at which the ends of the ladder are at their apparent positions. If we adjust for the time delay, the observers can both agree on where each end of the ladder was but must make their measurements at different times. If we do this, it can be made clearer how the observers disagree (but that's hard so you're just going to have to believe it).

The delay is important as it is the speed of information. The speed of information is typically the speed of light but in the ladder the speed of information is the speed of sound. So, when the ladder inevitably shatters on impact, the tail end won't "know" that it needs to stop moving until the vibration (sound) has travelled to that end.

Similarly, when the farmer sees the front end of the ladder at one side of the shed, it has already moved past that end but the delay in the information leads him to believe that it is in the shed.

The Twins Paradox

Setup

Two twins say goodbye to each other as one boards a large space rocket which will travel to a star 10 lightyears away and then travel back at a reasonable fraction of the speed of light. The twin remaining on Earth notes that since moving clocks run slow, the twin on Earth will have aged much more when they reunite, as the slowed clock of the twin on the rocket will mean they have aged more slowly during the trip.

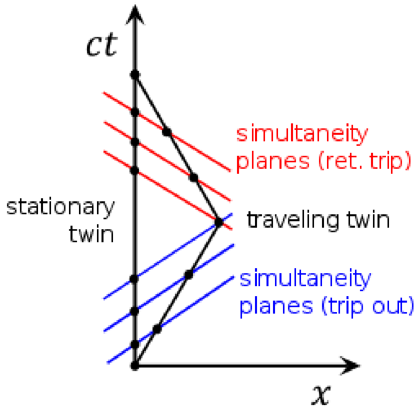
The twin going on the trip counters this by noting that to them, the twin on Earth will appear to be moving and therefore it is the twin on Earth who will age more slowly.

Who is right?

To measure this, the travelling twin agrees to send a beam of light to Earth every time a year passes.

Explanation

Ignoring any curvature due to acceleration, a spacetime diagram would look like this.

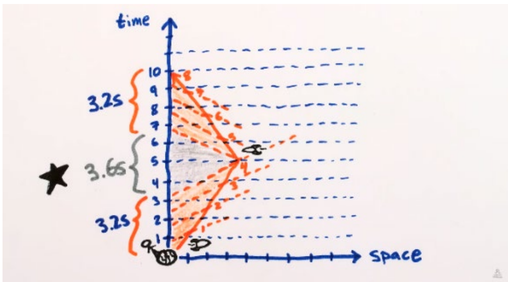


The actual explanation for the time discrepancy is, in fact, acceleration and the effect it has by changing the reference frame of the traveller.

On the first half of the trip the light pulses are angled down on the diagram because they are travelling away from the twin. On the return they are angled up because they are travelling with the twin. When the travelling twin turns around, their velocity rapidly changes and so does their perception of time. This can be seen in the gap between the light pulses being strangely large at the point of acceleration.

Notice that when the twin is travelling with a uniform speed, the gap between years of the stationary twin is smaller than for the travelling twin (i.e. time runs slowly for them), meaning that time dilation is occurring as the twin predicted. However, there is a large gap where the twin turns around and this adds a large period of time where the travelling twin observes no time passing but the stationary twin ages by many years.

For a more graphical explanation, see [minutephysics](https://www.youtube.com/watch?v=0iJZ_QGMLD0)' video on the paradox:



https://www.youtube.com/watch?v=0iJZ_QGMLD0

Experimental Verification for Special Relativity

Special Relativity proves a very difficult topic to verify as one must travel at a significant fraction of the speed of light for its effects to be noticeable. As a result, a precise experiment was required, capable of measuring effects to fractions of a second.

The Hafele-Keating Experiment - 1972

The story

One day Hafele (an assistance professor at the time) was sitting down writing notes on relativity when he did a 'back of the envelope' calculation showing that an atomic clock had enough precision to test the effects of Special and General relativity. He was unable to get the funding to do the test.

Eventually Hafele and Keating met after one of Keating's lectures he was doing on astronomy. Keating had access to atomic clocks, and this was enough to get the pair $8000 dollars of funding, most of which went into plane tickets for the two and 'Mr Clock'.

The experiment involved taking 'Mr Clock', who had been calibrated to his friend 'Mrs Clock' on the ground and shipping him on a long return trip plane ride. The disagreement between Mr and Mrs Clock was measured when they returned and was found to be exactly as Einstein predicted.

Muons from the Sun

Muons are Leptons from the standard model and are theoretically described by the associated quantum mechanics. The maximum time a muon should exist or the maximum distance it should be able to travel is given by Heisenberg's uncertainty principle and can be shown to be too short for Muons from the sun to reach Earth. And yet we still detect them.

The reason for this can either be viewed as length contraction from the Muon's perspective or time dilation from Earth's perspective.

From the Muon's perspective the distance between Earth and the Sun shrinks as it speeds up so at high enough velocities, the distance becomes shorter than the length given by the Heisenberg Uncertainty principle.

Similarly, from Earth's perspective the Muon's clock ticks more slowly so the time taken to reach Earth becomes shorter (for the Muon) to the point where it is shorter than the maximum lifetime of a muon.

Particle Accelerators

Particle accelerators provide similar evidence, with fast moving particles taking longer to decay than slower moving particles due to slower clocks.

MODULE 8: FROM THE UNIVERSE TO THE ATOM

Base Units

Mass (m) – Kilograms (kg)
Displacement ($\vec{s}$) – Metres (m)
Time (t) – Seconds (s)
Speed (v) – Metres per second (ms^{-1} **or** m/s)
Momentum (p) – Kilogram metres per second ($kg\ ms^{-1}$ **or** $kg\ m/s$)
Wavelength (λ) – Metres (m)
Frequency (f) – Hertz (Hz **or** s^{-1})
Energy (E) – Joules (J **or** $kg\ m^2\ s^{-2}$)
Luminosity (L) – Power ($J\ s^{-1}$)
Intensity (I) – Power per area ($J\ m^{-2}\ s^{-1}$ **or** $kg\ s^{-3}$)
Angular Momentum (L) – Kilogram square-metres per second ($kg\ m^2\ s^{-1}$)

Constants

The Speed of Light $c = 3.00 \times 10^8\ m\ s^{-1}$
Planck Constant $h = 6.626 \times 10^{-34}\ kg\ m^2\ s^{-1}$

Rydberg Constant (Hydrogen) $R = 1.097 \times 10^7\ m^{-1}$

Wein's Displacement Constant

Equations

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

The de Broglie wavelength of an object with mass.

$$\frac{1}{\lambda} = R\left(\frac{1}{{n_f}^2} - \frac{1}{{n_i}^2}\right)$$

Rydberg equation for the wavelength of a photon ejected or absorbed by a hydrogen atom.

$$E = mc^2$$

The rest energy of an object with mass.

$$N_t = N_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{0.5}}} = N_0 e^{-\lambda t}$$

$$\lambda = \frac{\log_e 2}{t_{0.5}}$$

The Nuclear decay equations.

$$\hbar = \frac{h}{2\pi}$$

Another notation/form of the Plank Constant (it shows up a lot).

$$L = mr^2\omega = mvr = \frac{nh}{2\pi}$$

$$mvr = n\hbar, \qquad \hbar = \frac{h}{2\pi}$$

Bohr's postulate that Angular Momentum is quantised.

Extension Equations

$$\frac{v}{D} = H_0$$

Hubble's Law (though the Hubble constant (H_0) *has been shown to be different at different distances.*

$$L = \sigma A T^4, \qquad \sigma = 5.7 \times 10^{-8}\, W\, m^{-2}\, T^{-4}$$

Stefan-Boltzmann Law for stars.

$$L = \oiint \vec{I} \cdot d\vec{A}$$

The light flux emitted by an object is a constant and is its luminosity.

$$E = \gamma m c^2, \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The total Energy, including kinetic, of an object or particle with mass.

$$E = \gamma m c^2 - m c^2 = m c^2(\gamma - 1)$$

Relativistic kinetic energy of a particle or object with mass.

$$-\frac{\hbar^2}{2\mathrm{m}} \nabla^2 \Psi + V\Psi = i\hbar \frac{\partial \Psi}{\partial \mathrm{t}}$$

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial \mathrm{t}}$$

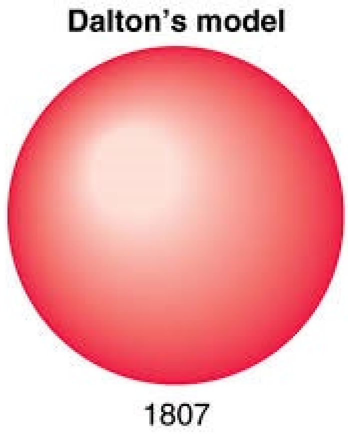
The General form for Schrödinger's Wave Equation. $\hbar$ *is a constant,* $i^2 = -1$, m *is the object's mass,* ∇^2 *is the Laplacian operator and* V *is the potential energy as a function of time and position.*

Course Notes

Models of The Atom

Dalton – 1808

Dalton expanded on the model proposed by the Greeks where he hypothesised that the Atom was made of a solid 'billiard ball' and was uncuttable. He explained different elements by proposing that each element had its own ball.



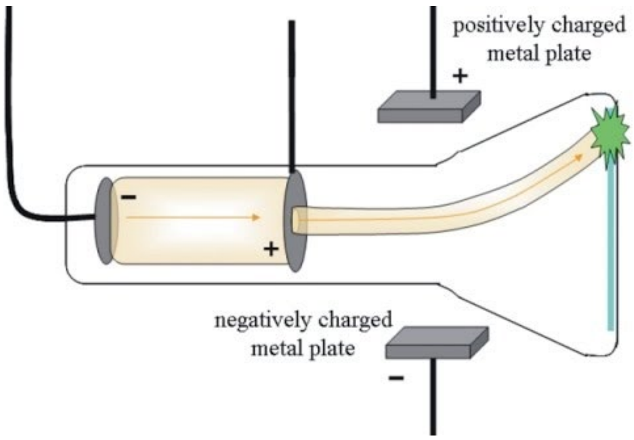
Thomson – 1904

Thomson experimented with cathode ray tubes from different materials, using electric and magnetic fields to determine its properties:

- The ray was negatively charged
- The thing which made up the ray could be forced out of any material

He was also able to balance the effects of the electric and magnetic fields to determine their charge to mass ratio but was unable to calculate either separately. In doing so he determined that the ray was made of small particles with some mass and charge.

From this he developed the 'Plumb Pudding' model which was an amendment to the Dalton model, with each atom being made of a different positively charged ball structure of uniform charge distribution and with negatively charged particles (electrons) distributed randomly throughout.



The Cathode Ray Experiment

The cathode ray experiment allowed the properties of what is now known as the electron to be deduced. The first setup involved:

- an electron gun (parallel plates with a hole for the accelerated electrons to pass through).
- Electric and Magnetic fields setup so they oppose and equal each other aka. a velocity selector: so named because $qvB = qE \Rightarrow v = \frac{E}{B}$

From this he deduced the velocity of the particles.

The next setup involved just using

- A magnetic field
- A detection plate

This allowed the radius of curvature given by $qvB = \frac{mv^2}{r}$ to be found. This allowed for the rearrangement of the above equation, using the velocity from the previous setup:

$$\frac{q}{m} = \frac{v}{Br}$$

$$= \frac{E}{B^2 r}$$

Millikan Oil Drop Experiment – 1909

The oil drop experiment allowed for the determination of the charge of the electron.

Since the density ρ of the oil was known, the equation $m = V\rho$ where V is volume could be used to determine a given drop's mass. When drops were injected into the apparatus, they formed spheres and since the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ and the radius of a given drop could be measured, the mass of a drop could be experimentally derived.

Millikan established an electric field such that it opposed gravity and used the equivalence of the forces and the known mass to find the charge of each drop:

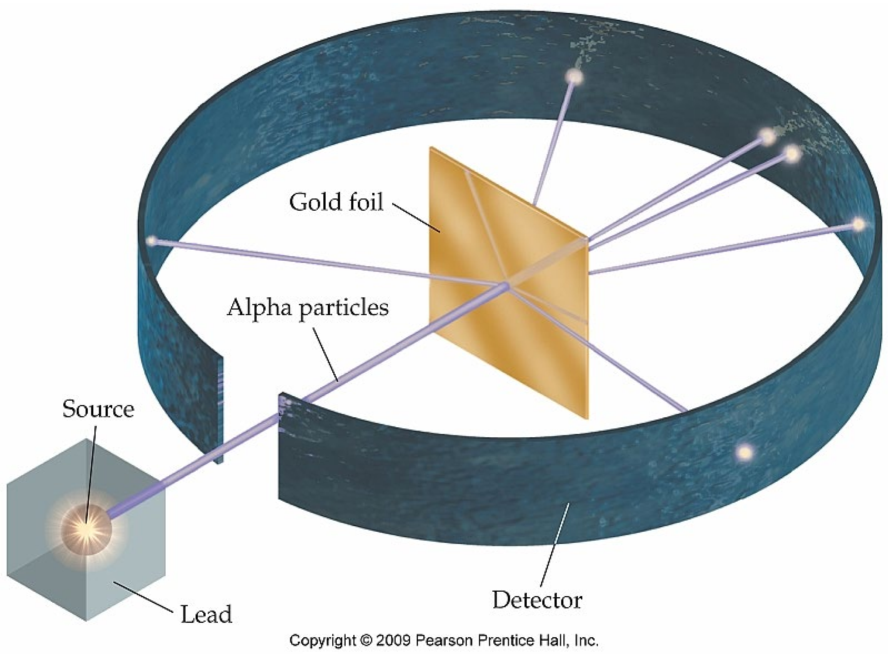
$$qE = mg$$

$$q = \frac{mg}{E}$$

Millikan then noticed that the charge of each drop was always an integer multiple of 1.602×10^{-19} and noted that this must be the fundamental charge unit or the electron's charge.

Rutherford – 1911

Rutherford discovered, with the gold foil experiment (Geiger-Marsden), that the atom is made of mostly empty space but has a centre with an extreme density of positive charge (a nucleus). This was concluded from the fact that most of the alpha particles passed straight through the atoms, some were deflected slightly, and a few were deflected back in the direction they came from.



The fact that most particles went straight through meant that they did not come near another positive charge and experienced nearly no force. However, the fact that ones that did get deflected got deflected by a large amount implied a large charge density where the alpha particle travelled (and a large force).

Bohr – 1913

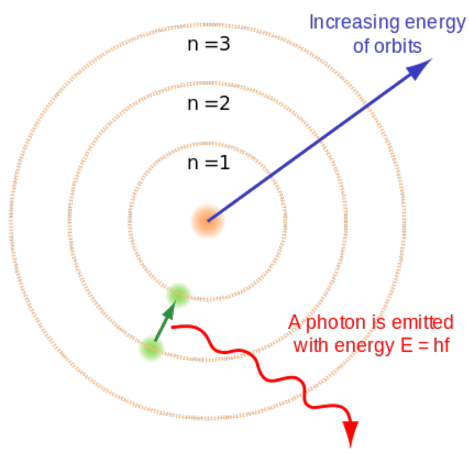
Bohr noted that, by Maxwell's equations, a charge in circular motion was accelerating and would therefore create light. This light should therefore be proportional to the acceleration and would cause the electron to fall into the nucleus emitting a smooth spectrum of light. **Neither of these occur.**

Bohr then went on to develop his quantised orbital theory of electron orbits where the electrons orbited at set radii and would 'jump' or 'fall' between them. To go along with this, he formalised 3 postulates:

1. Electrons orbit around a nucleus with a high density of positive charge per volume with centripetal acceleration from the coulomb force: $\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} = \frac{mv^2}{r}$

2. Electrons orbit in a quantised number of orbitals where the angular momentum of the electrons exists in quantised states with $L = mr^2\omega = mvr = \frac{nh}{2\pi}$ where n is the integer orbit.

3. While in these orbits the electrons are in a 'stationary state', meaning they will not lose a small amount of energy as a photon and fall to some radius between the orbits.

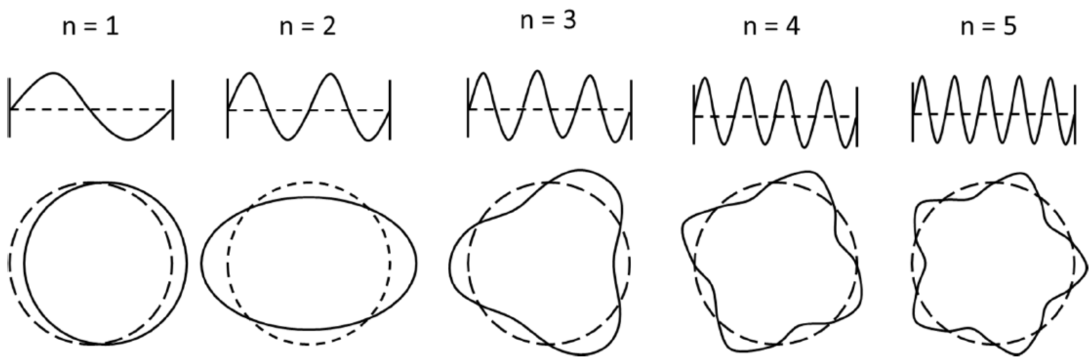


Theorisation of the Proton and Neutron – 1920s

Throughout the 1920s it was discovered that the mass of atoms was approximately integer multiples of the mass of the Hydrogen nucleus and that the atomic number was proportional to the charge on the nucleus. This knowledge, combined with the discovery of isotopes (atoms with different masses but the same atomic number), led to the theorisation of the Proton and Neutron (believed to have the same mass).

De Broglie – 1924

He explained Bohr's model by proposing that the electron could be a wave and a particle (invoking Einstein's wave-particle duality theory). De Broglie postulated that if the electron were a wave then it would reason that it would have an integer number of wavelengths. This explained why the electron only existed at certain radii, as these were the radii where the electron could inhabit an orbit where the circumference is an integer multiple of the wavelength.



See https://www.desmos.com/calculator/xww8n1r3kt for a de Broglie wave generator.

Schrödinger – 1926

Schrödinger took de Broglie's proposed wave theory and generalised it for all fundamental particles (particles which exhibit wave properties). He invented the wave function which stores all properties about that particle including its position, momentum, energy etc. This wave function however represents the probabilities of each of these properties being a certain value, not the values themselves.

Schrodinger's Wave Equation is written in general form, where Ψ is the wave function:

$$-\frac{\hbar^2}{2\mathrm{m}}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial \mathrm{t}}$$

V is the sum of all potentials on the particle (has units of energy) m is its mass, ∇^2 is the Laplacian, $\hbar = \frac{\mathrm{h}}{2\pi}$ and $i^2 = -1$

This theory is currently the accepted theory of subatomic particles.

But isn't wave motion accelerated motion?

Yes, wave oscillation is a form of acceleration. It wasn't until a year or so after Schrödinger published his equation that people began to realise that the thing doing the 'waving' was in fact probabilities, not the particle. As such, Bohr's 'stationary state' idea is correct (just not in the way he thought).

Chadwick – 1932

He found the first experimental evidence for the Neutron (in essence he discovered it, though he did not theorise it).

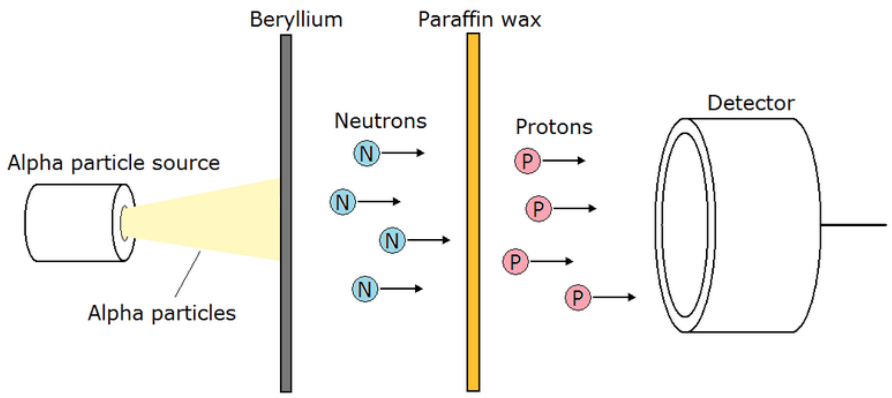
Neutrons had been theorised for a long time and theoretical physicists had just kind of accepted they existed. They provided a great explanation for why the mass of particles on the periodic table increased at a greater rate than their charge.

The Neutron Experiment

Alpha particles were emitted onto a Beryllium plate. A sheet of paraffin wax was then placed after the Beryllium and a charge detector detected what were found to be protons being emitted from the wax. However, the charge detector did not detect anything when placed between the beryllium and wax.

By putting a thick lead plate between the beryllium and the wax, the release of protons from the wax was stopped. Using this, Chadwick reasoned that there must be some neutral charge being emitted from the beryllium.

Using conservation of mass and momentum, Chadwick determined the ratio of the neutral particles mass to the proton and found that the masses were almost identical (the neutral particle was just 0.1% heavier). This allowed Chadwick to conclude that he had found the theorised Neutron.



The Standard Model

Terminology

Quark – A particle which, among other properties, possesses a colour charge (red, green or blue). As a result, they interact with Gluons and are affected by the strong interaction.

Lepton – A particle which does not interact via the strong interaction (i.e. it has no colour charge)

Boson – A particle responsible for force interactions.

Virtual Particle – A particle which is not directly observable but can still interact with other particles.

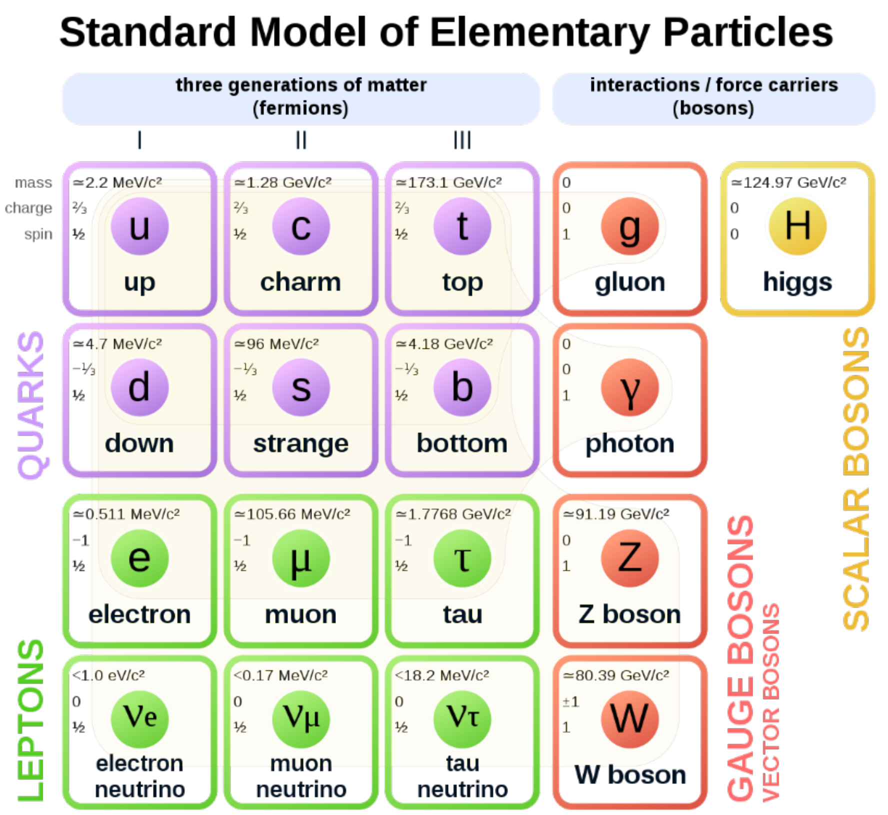
Hadrons – Particles made of two or more quarks with integer charges.

Baryons – Particles made of three quarks

Anti-Matter – A particle which is the complete opposite of another (opposite electric charge, opposite colour charge etc.)

Annihilation – Occurs when a particle collides with its anti-particle. The process creates an immense amount of energy in the form of photons.

Fundamental Forces – The minimum number of forces required to describe all phenomena in the universe (Strong, Weak, Electromagnetic, Gravity)

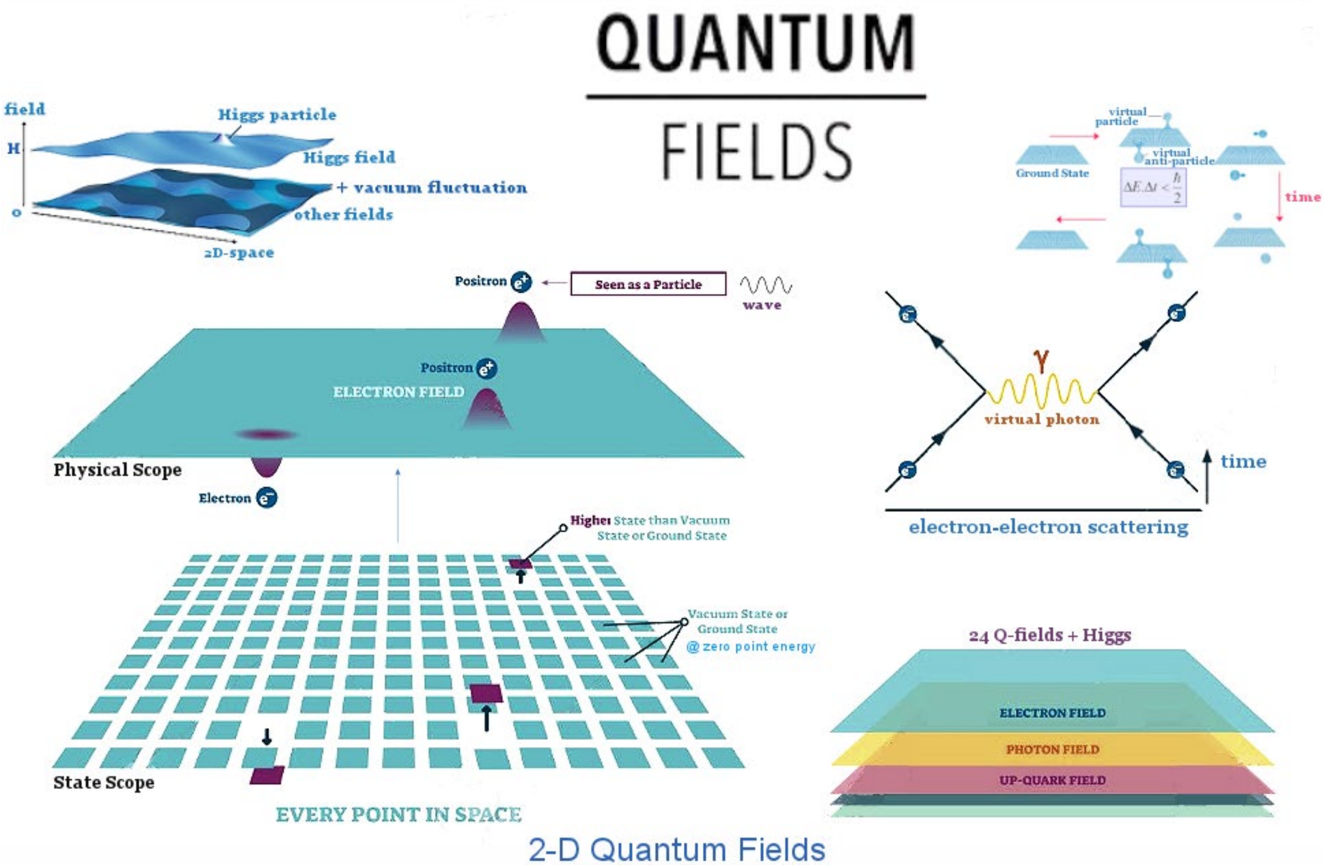


What is the Standard Model?

The standard model is the current model of particle physics which has been experimentally verified.

The standard model consists of fundamental particles with certain properties such as charge, mass etc.

Each of these fundamental particles is a little wiggle (wave) in its respective field (e.g. the electron is a small disturbance in the electron field). The reason particles have certain properties is a function of how much one field affects another field.



Anti-Particles

Anti-particles such as the anti-electron (positron) come up a lot in quantum physics. Although they have been observed in experiment, they were first predicted by Dirac when he was formulating his equation for the electron, where he found his equations always had two solutions. His two solutions consisted of opposite energies and opposite charges and he interpreted this as being a particle and its anti-particle.

Although the maths he performed was a little more complex, a basic understanding of this can be gleaned from the energy equation:

$$E^2 = {m_0}^2c^4 + p^2c^2$$

If we assume the velocity and momentum are zero, we find $E^2 = m^2c^4$

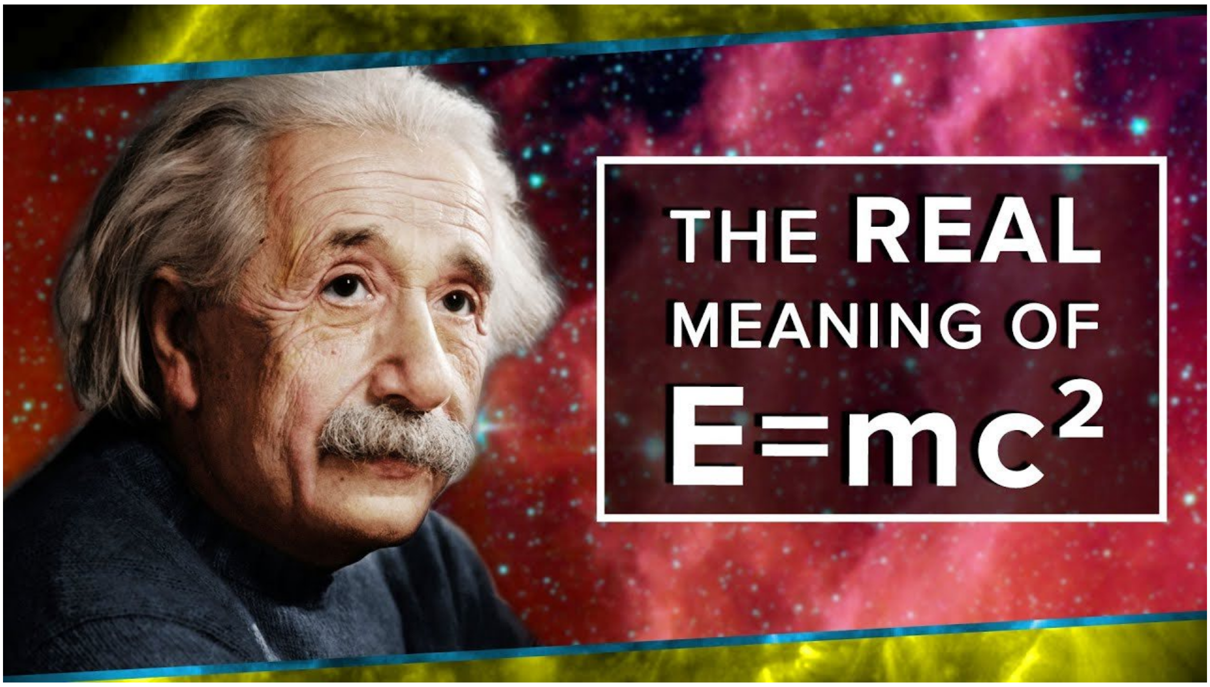
This does not rearrange to $E = mc^2$, rather it rearranges to $E = \pm mc^2$ and the negative solution is what Dirac interpreted as the anti-particle.

What does $E = mc^2$ mean?

When Einstein wrote his paper, he showed that due to special relativity an object which loses some kinetic energy E lost mass $\Delta m = \frac{\Delta E}{c^2}$

Although it is written as $E = mc^2$ it should actually written $m = \frac{E}{c^2}$
What this shows is that mass is actually just a way we measure the net energy content of an object. The inherent mass an object has actually comes from the energy the Higgs field gives the particles it is made of. Furthermore, mass is not a real property, rather it is a manifestation of energy and is how we measure it.

Go watch this video for a good explanation:



https://youtu.be/Xo232kyTsO0

Conservation Laws

All conservation laws are due to invariances in some value. Where there is an invariance in some value during an interaction, some value is conserved (some are given below).

Conservation of Charge

In any interaction where there is Gauge symmetry, the total charge of the universe must be conserved where:

$$(+1) + (-1) = 0$$

Conservation of Colour

In any interaction, the total colour charge must be conserved where:

$$(B) + (G) + (R) = 0$$

$$(B) + \left(\overline{B}\right) = 0$$

Conservation of Momentum

In all interactions where there is translational symmetry (location of the event occurred does not change the interaction between the particles), momentum must be conserved.

Conservation of Angular Momentum (Spin)

In all interactions where there is rotational symmetry (the event would be the same if space were rotated), intrinsic angular momentum is conserved.

Conservation of Lepton Number

This is empirical and has never not been observed. This is an additive law where normal leptons such as electrons (e^-) and electron neutrinos (ν_e) have $+1$ lepton number and their anti-particles such as the positron (e^+) and anti-neutrino $(\overline{\nu}_e)$ have a lepton number of -1.

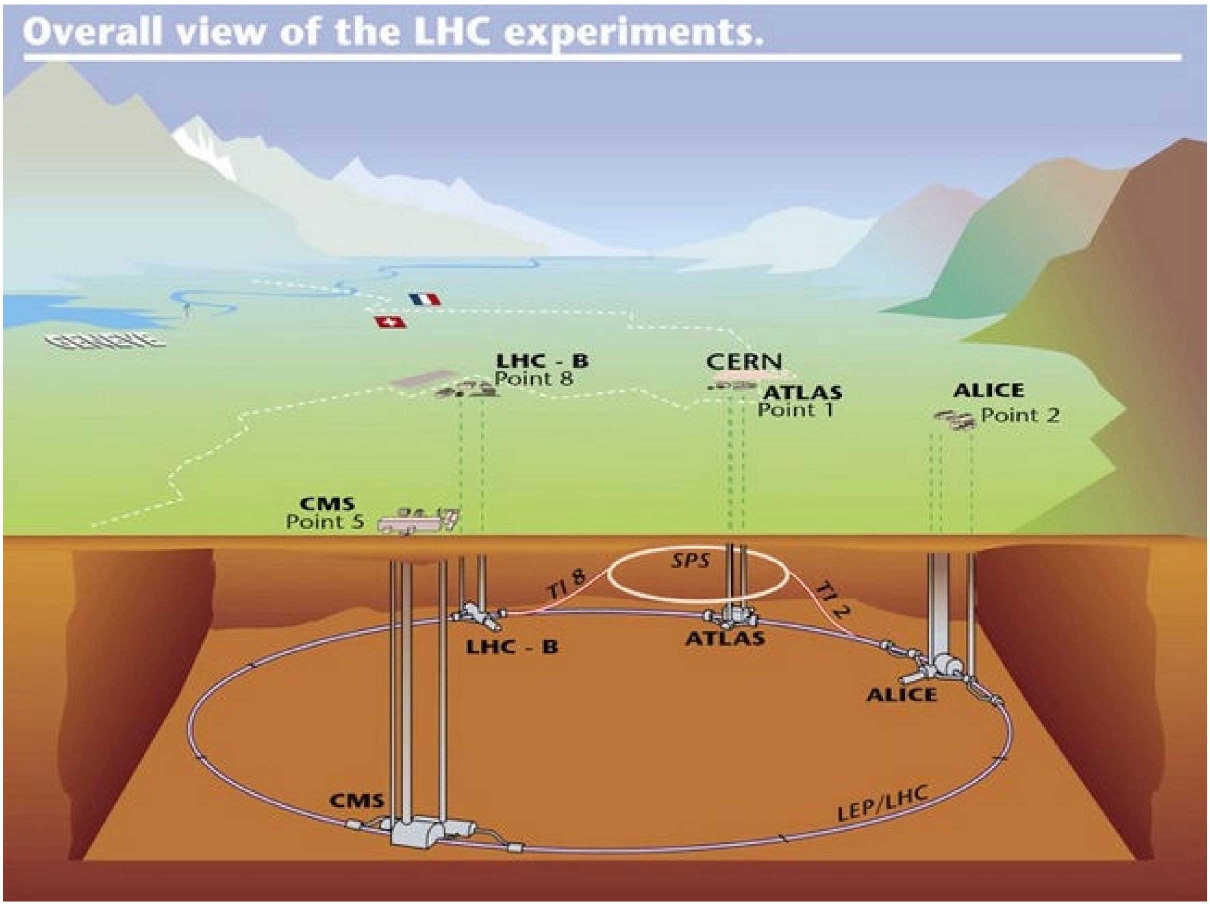
The LHC and other Evidence for the Standard Model

The LHC collides protons and other Hadrons together at very high speeds ($0.999\,999\,99c$).

The protons slam together at such high energies that they break apart and their constituent particles decay into lower energy particles.

The particles produced are random (to an extent) so they slam protons together many millions of times per second. They produce too much data to store so most of it is thrown out and only the possibly interesting ones are stored.

The detectors at the LHC are able to detect certain types of particles depending on the detector (there are 4 main detectors). The detected particles and their energies can then be used to reverse engineer the collision which occurred, and it has been shown that the collisions match what would be expected from protons made of three quarks colliding.



Quarks, Gluons & Colour Charge

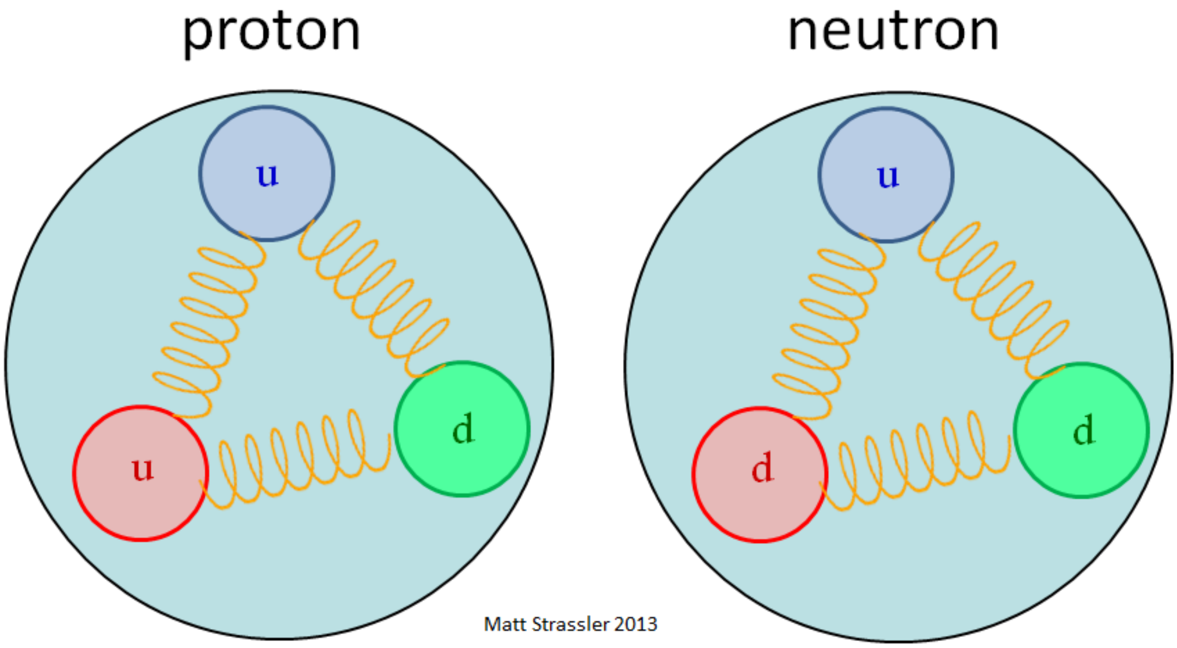
Gluons and Quarks possess a property called colour charge (it is not colourful but there are three types instead of two so physicists needed something more than $+\ -$) the colours are Red, Green and Blue (with a corresponding Anti-Red, Anti-Green and Anti-Blue for anti-particles). The charges are named as such since red + green + blue = white, so the charges cancel in sets of three. The study of these charges and the corresponding interactions is called Quantum Chromodynamics.

The 'flavour' of a quark is its classification (i.e. Up, Down, Charm, Strange etc.).

Though Quarks possess electric charge, the primary force which bonds them is mediated by their colour charge and the force particle corresponding to it: the gluon.

The gluon has no mass or electric charge, but it does have a colour charge and an anti-colour charge (never the same like blue and anti-blue though) and so it is one of the few particles which can interact with quarks without interacting with other particles.

Quarks are held together by flux tubes of gluons which attract the quarks together. The main property of these flux tubes that differs from lone gluons is that the colour charge of the gluons in the flux tubes cancel to be zero, keeping the net colour charge of the proton and neutron zero.



The analogy of springs is used for the flux tubes since the attractive force created by the gluons increases with the distance of the particles as with a spring.

Nuclear Physics

Terminology

Bonding Energy – The potential energy in the bond such that:

$$E_{bond} = -U$$

$$F_{bond} = -\nabla U = -\frac{dU}{dr}\hat{r}$$

Atomic Mass Unit ($\boldsymbol{u}$) – The average mass of the nucleons in a Carbon 12 nucleus such that:

$$u = \frac{m_{C-12}}{12}$$

Decay – The process by which one particle becomes another particle by emitting another particle.

Nucleon – A proton or neutron.

Half Life – The time a sample of some material takes to transmute into another material.

Control Rod – A material used to absorb neutrons in a nuclear reaction.

LHC – The Large Hadron Collider located under Switzerland and run by CERN.

Decay

Nuclear decay is a random (but statistically predictable) process whereby energetically unstable nuclei decay into more stable nuclei and release certain particles such that:

- Charge is conserved
- Energy is conserved
    - $U + E_K$
- Momentum is conserved

Alpha (α) Decay

Alpha decay is a result of the strong force between neighbouring nucleons being weaker than the electrostatic repulsion of the protons. The exact mechanism is hard to explain but fundamentally it is when the potential energy binding an alpha particle (α or ${}^{4}_{2}He$) to the nucleus is near zero. There is still a large potential bond between neighbouring nucleons (due to the strong interaction) however the chunk is only loosely held so it is ejected.

The leftover potential energy is lost as a photon and can be measured as a loss in mass.

The nuclear equation can be written as:

$$ {}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}He + {}^{0}_{0}\gamma $$

Or

$$ {}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}\alpha + {}^{0}_{0}\gamma $$

Beta (β) Decay

Beta decay occurs in two forms

1. β^-
2. β^+

β^- decay is where a neutron becomes a proton through the weak interaction and, by conservation of charge, an electron is released. During this process, an anti-neutrino is also given off.

$$ {}^{A}_{Z}X \rightarrow {}_{Z+1}{}^{A}Y + {}^{0}_{-1}e^- + {}^{0}_{0}\bar{\nu}_e + {}^{0}_{0}\gamma $$

β^+ decay is where a proton becomes a neutron through the weak interaction and, by conservation of charge, a positron is released. During this process, a neutrino is also given off.

$$ {}^{A}_{Z}X \rightarrow {}_{Z-1}{}^{A}Y + {}^{0}_{1}e^- + {}^{0}_{0}\nu_e + {}^{0}_{0}\gamma $$

The potential energy lost by this transmutation is the emitted as a photon.

Why neutrinos are emitted

Neutrinos are emitted in these interactions to conserve spin and lepton number. The neutrino conserves spin because it spins the opposite way to the electron (is spin down if the electron is spin up) making the net spin zero.

Neutrinos and electrons have lepton numbers of $+1$ while anti-neutrinos and positrons have lepton numbers of -1. The total sum of the products must equal the initial (which was zero).

Gamma (γ) Emission

Gamma emission is the emission of a photon (typically in the gamma spectrum but could be in any spectrum) and occurs due to a loss of potential energy (also known as an increase in binding energy_ inside the nucleus. This can occur during any type of radioactive decay and is a result of protons and neutrons swapping places inside the nucleus (a process which is made possible due to the nuclear force interaction).
When the protons move, they move in the direction such that their potential energy decreases. So, by conservation of energy, this energy must be regained. In this case it occurs as a photon.

This can occur during all types of radioactive decay but typically occurs during β decay.

Penetrating Distance and Ionisation

Different forms of radiation are able to penetrate materials to varying distances. Generally, the further radiation can penetrate, the lower its ionisation potential. This is not true when comparing gamma radiation to X-rays as gamma rays possess both a greater penetration distance and ionisation energy.

α particles are most likely to cause ionisation as they strongly attract electrons that they are near to.

β^- (electrons) cause ionisation because they collide with the electrons in outer shells of atoms, transferring momentum and knocking the electron off.

β^+ (positrons) cause ionisation because they annihilate electrons in a collision.

γ rays have much greater energies ($E = hf$) than is required to ionise an atom and so, on collision with an electron, transfer their energy to the electron as kinetic energy, ionising the atom.

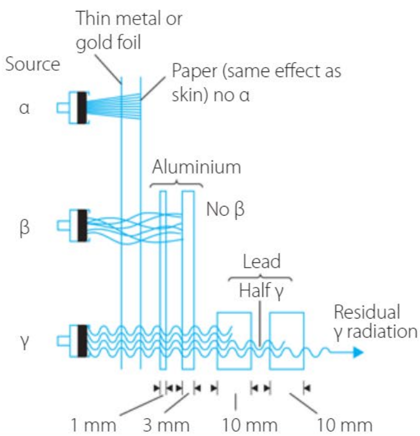


FIGURE 16.6 Gamma rays are much more penetrating than α or β particles. In turn, β particles are more penetrating than α particles.

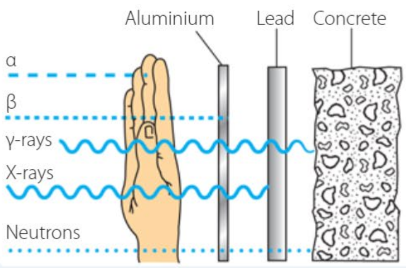


FIGURE 16.7 Penetrating power for different radiations is indicated by their relative absorptions in materials.

Half Life

Half Life describes the statistical nature of nuclear decay in one easy concept. All unstable nuclei (prone to either form of decay) will have some time after which half of a large sample will have decayed.

Since after every integer multiple of the half life time ($t_{0.5}$) the sample has halved in size, we can write that mathematically where N_t is the amount after some time and N_0 is the original amount:

$$N_t = N_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{0.5}}}$$

It is easy to see when $t = t_{0.5},\ \ N_t = \frac{N_0}{2}$ and when $t = 2t_{0.5},\ \ N_t = \frac{N_0}{4}$ etc.

Now we rearrange because the formula sheet is unnecessarily specific…

$$N_t = N_0 e^{\ln\left(\left(\frac{1}{2}\right)^{\frac{t}{t_{0.5}}}\right)}$$

$$N_t = N_0 e^{\frac{t}{t_{0.5}}\ln\left(\frac{1}{2}\right)}$$

$$N_t = N_0 e^{\frac{-t}{t_{0.5}}\ln(2)}$$

$$N_t = N_0 e^{-t\frac{\ln(2)}{t_{0.5}}}$$

Now we let $\frac{\ln(2)}{t_{0.5}} = \lambda$

$$N_t = N_0 e^{-\lambda t}$$

$$\lambda = \frac{\ln(2)}{t_{0.5}}$$

Fission

Fission is the process by which a nucleus is split into two smaller parts. Typically, this is done with heavy nuclei such as Uranium-235 $\left({}^{235}_{92}U\right)$ whereby the atom is made unstable by shooting low velocity neutrons at it and it splits into two parts, one a little heavier than the other. As a result of this, neutrons are also released. E.g.

$$^{1}_{0}n + {}^{235}_{92}U \rightarrow {}^{92}_{36}Kr + {}^{141}_{56}Ba + 3{}^{1}_{0}n$$

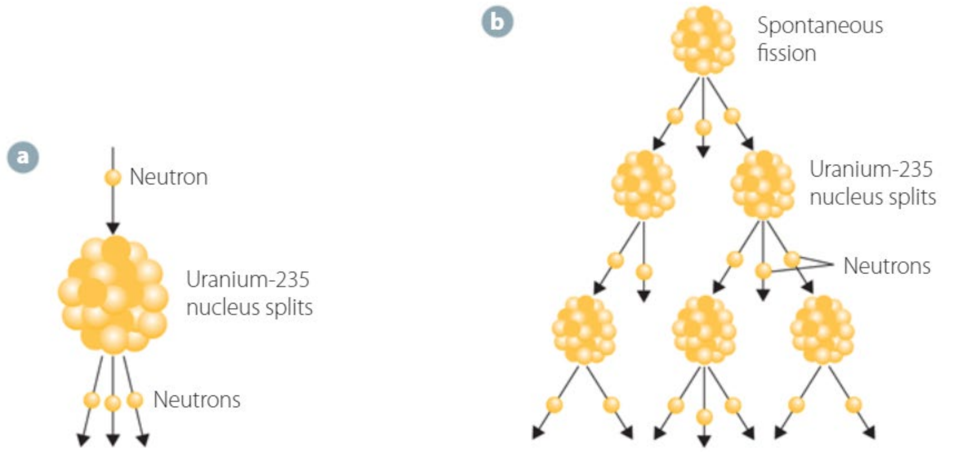


FIGURE 16.11 Nuclear fission. **a** A slow neutron causes a uranium-235 nucleus to split, releasing three fast neutrons. **b** A chain reaction occurs, if, for example, two of the released neutrons cause further nuclear fission in other uranium nuclei. Vast amounts of energy can be released.

A nuclear reaction such as that in a bomb occurs when more neutrons are produced per second than reactions are occurring (i.e. more than one neutron is produced per fission on average), this is a runaway reaction.

A nuclear reaction such as ones used in nuclear reactors is one where the number of neutrons produced per reaction is less than or approximately one, so it is 'controlled'.

The Moderator

This is made of a material with a slightly higher mass than the neutron such as Hydrogen $({}^{1}_{1}H)$, Deuterium $({}^{2}_{1}H)$ or Tritium $({}^{3}_{1}H)$. The neutrons collide with these atoms and share their energy. This slows down the neutron so that it can be absorbed for more reactions.

Control Rods

If a reaction such as one in a nuclear reactor begins producing more neutrons than desired, control rods are inserted. Control rods are made of materials such as boron which more freely absorb neutrons (known as neutron poisons).

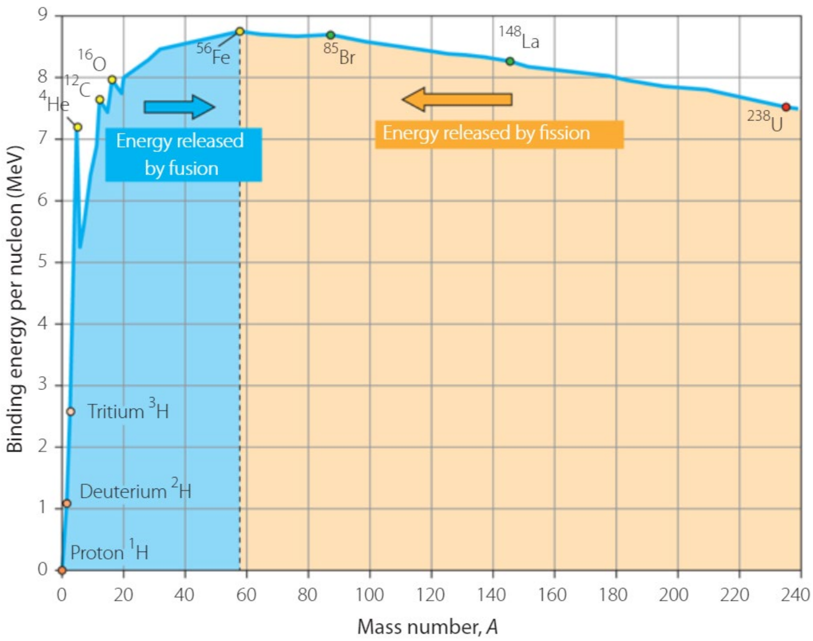
Enrichment

Since Uranium-235 is one of the few very reactive substances for nuclear reactions, it often needs to be separated out from the less reactive Uranium isotopes if a faster reaction is desired (the concentration of Uranium-235 required for a nuclear bomb is around 97%). Other Uranium isotopes can even absorb neutrons without reacting as they are more stable.

Fusion

Fusion is the process by which protons and neutrons are brought together with enough energy that they overcome their electric repulsion and are able to bond via the strong force. In the sun this is done by the pressure of gravity (and with a little help from quantum tunnelling) and in fusion reactors it occurs by colliding particles with enough kinetic energy that they bond but not so much that they obliterate each other.

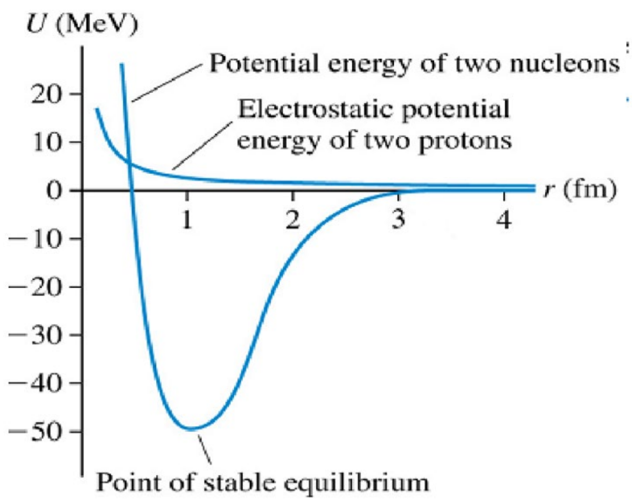
Below is a graph of overall binding energies of nuclei.



Getting Energy from Fusion

Remembering that in actuality, the bonding energy is the negative potential energy, an increase in bonding energy is a net decrease in energy:

As atoms before iron are bonded, the net energy inside the nucleus decreases. By conservation of energy, an equivalent amount of energy must be released. The released energy is given by $E = -\Delta U$ (the change in energy between the particles). If E is negative, then extra energy is needed to bond them.



The attractive strong force appears at close ranges and makes the potential energy negative while the electric potential remains positive.

Forces, Energies and Emitted Particles

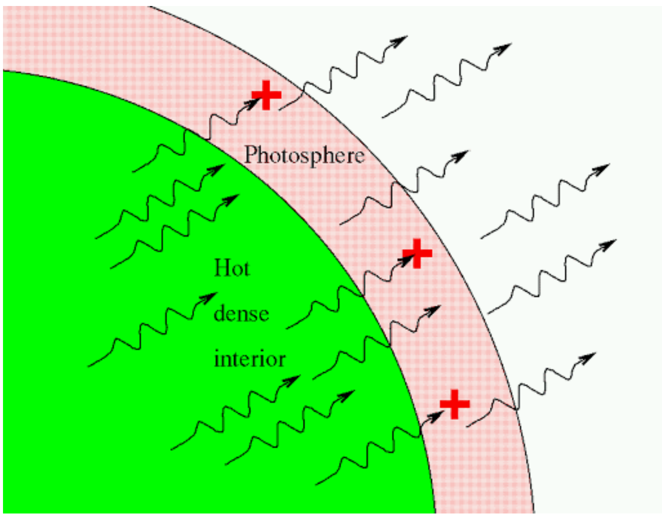
As a general rule, attractive forces produce negative potentials and repulsive forces produce positive potentials. When the net energy between particles decreases (bonding energy increases) there has been an increase in strength of the attractive force and vice versa.

When the energy between two quantum particles decreases, the energy must be emitted. The energy is often emitted as a photon. Sometimes, a particle can acquire a lower energy state (greater attractive force) by transmuting into another particle (like proton to neutron) and if this occurs, conservation of charge dictates that the equivalent charge must be emitted. Therefore, a positron and anti-neutrino will be released in such process such that they carry away the energy and charge.

Mass Defect in Stars

It is often said that stars lose mass when they fuse matter, and that this mass is lost as energy is the form of photons. This is not entirely correct; it is more accurate to say that when we measure mass, we are measuring the net energy content of stars. So, when stars lose energy, their observed mass drops but the particles which make it up do not lose mass, we are just observing the loss of energy in a strange way.

Really, we are saying that $m = \frac{E}{c^2}$ and as E decreases, the apparent mass decreases.



Fusion Inside a Star – Hydrogen to Helium and the CNO Cycle

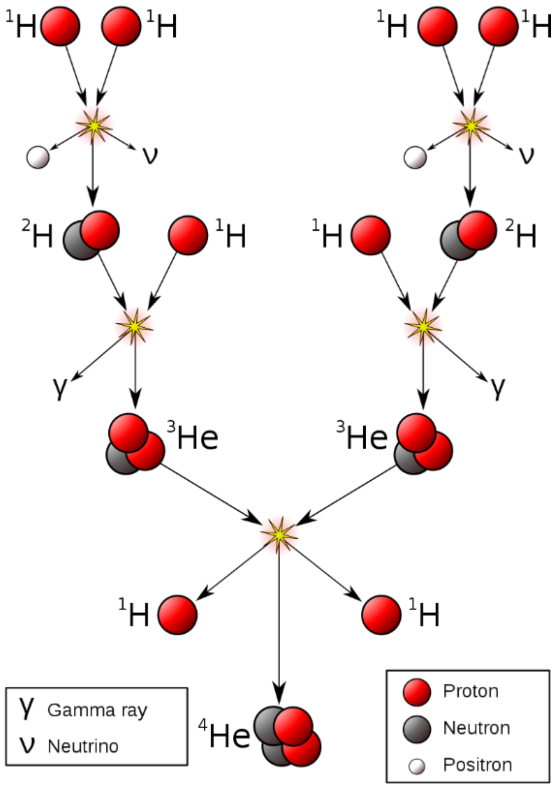
Proton-Proton Fusion – 26.73 MeV

Here's how helium can be made in the sun from 4 protons.

$${}^{1}_{1}H + {}^{1}_{1}H \rightarrow {}^{2}_{1}H + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e$$

$${}^{2}_{1}H + {}^{1}_{1}H \rightarrow {}^{3}_{2}He + {}^{0}_{0}\gamma$$

$${}^{3}_{2}He + {}^{3}_{2}He \rightarrow {}^{4}_{2}He + 2{}^{1}_{1}H$$



The CNO Cycle – 25 MeV

The CNO cycle is another way that stars take 4 protons and make a helium nucleus.

$${}^{12}_{6}C + {}^{1}_{1}H \rightarrow {}^{13}_{7}N + {}^{0}_{0}\gamma$$

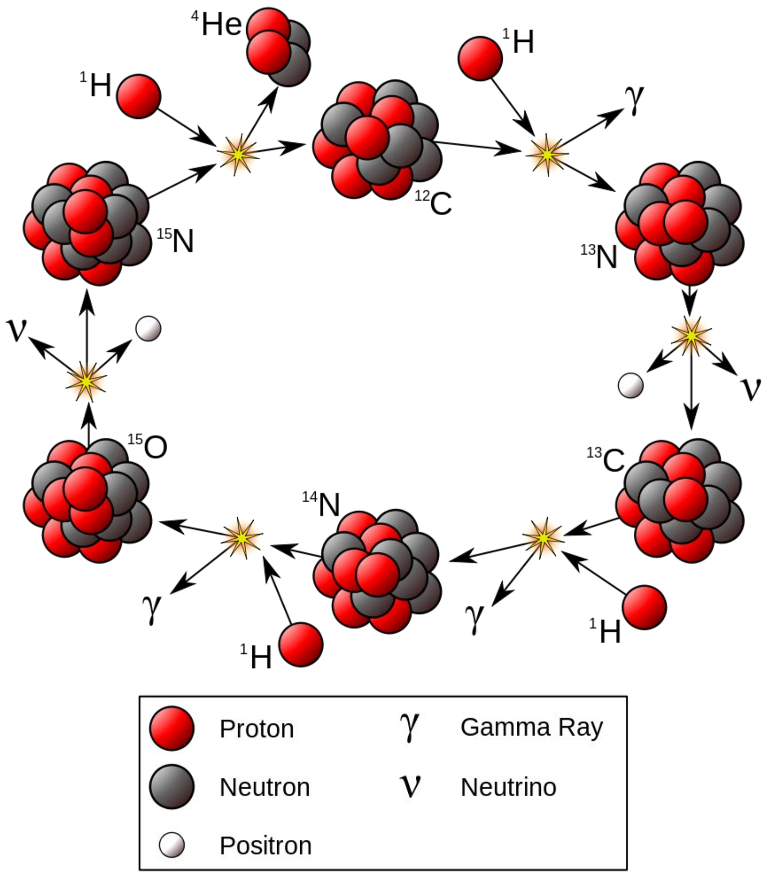
$${}^{13}_{7}N \rightarrow {}^{13}_{6}C + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e$$

$${}^{13}_{6}C + {}^{1}_{1}H \rightarrow {}^{14}_{7}N + {}^{0}_{0}\gamma$$

$${}^{14}_{7}N + {}^{1}_{1}H \rightarrow {}^{15}_{8}O + {}^{0}_{0}\gamma$$

$${}^{15}_{8}O \rightarrow {}^{15}_{7}N + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e$$

$${}^{15}_{7}N + {}^{1}_{1}H \rightarrow {}^{12}_{6}C + {}^{4}_{2}He$$

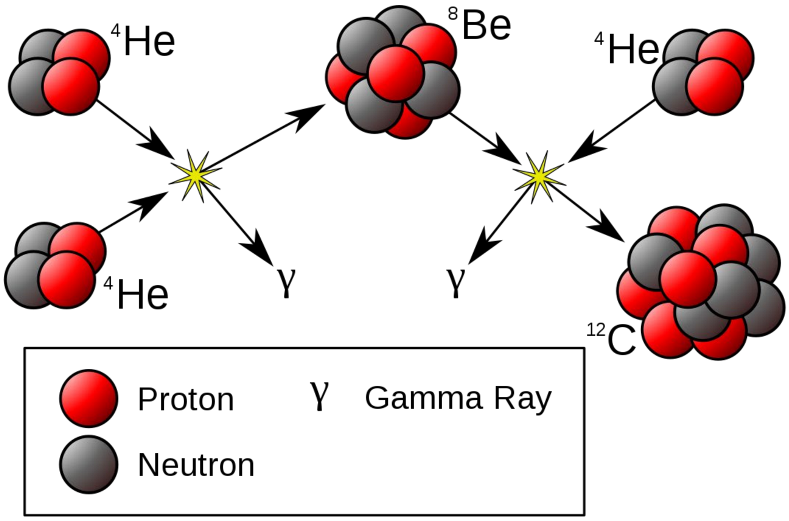


Triple Alpha Fusion – 7.725 MeV

Triple α fusion occurs in old, post main series stars.

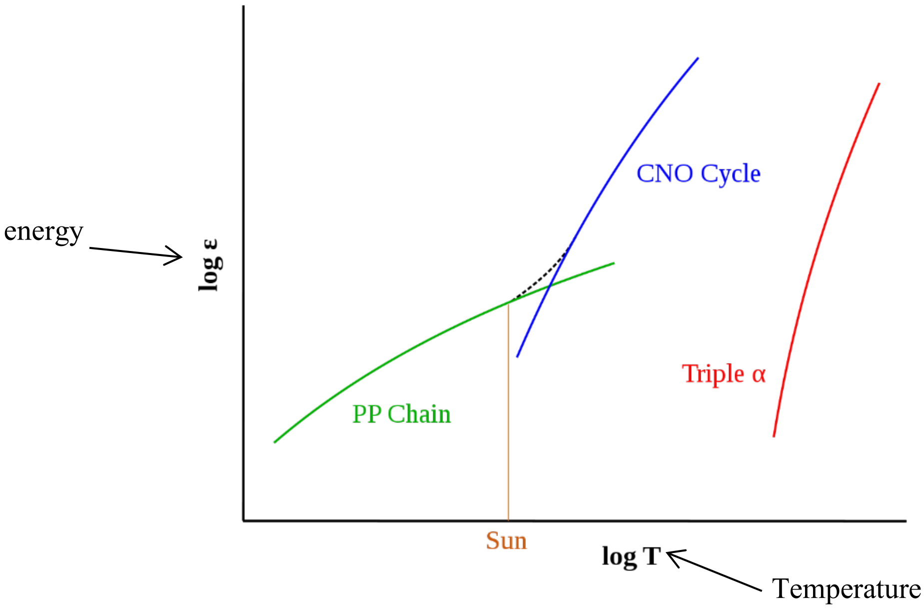
$${}^{4}_{2}He + {}^{4}_{2}He \rightarrow {}^{8}_{4}Be + {}^{0}_{0}\gamma$$

$${}^{8}_{4}Be + {}^{4}_{2}He \rightarrow {}^{12}_{6}C + {}^{0}_{0}\gamma$$



Spectral Classes

Different stars fuse different types of nuclei and, as a result, have certain properties.



PP Chain gives the most energy per reaction but requires a greater amount of gravitational attraction which can only be achieved after the star has made heavier elements. As a result, small stars fuse using PP chain and then hotter stars fuse lots of nuclei using CNO as it is easier to do (particles have more mass so easier to get high momentum collisions).

Big stars with lots of mass and not as much fuel burn using triple α.

Star spectra and what it says about a star

Stars emit light due to Blackbody radiation at their surface. Stars generate the energy to heat up the surface through fusion, however fusion itself does not create light.

The blackbody spectrum is continuous, with a peak in the spectrum occurring at $\lambda_{max} = \frac{b}{T}$. Gaps appear in the spectrum due to the atoms in the upper atmosphere of the star which absorb photons which hit them at certain wavelengths. The atoms then re-emit this energy as photons (not necessarily the same photon) but in a random direction. The result is that light of the wavelengths where it can be absorbed by the atmosphere of the star are much dimmer than the rest of the blackbody spectrum of the star. This is the star's absorption spectra.

What can it the absorption spectra tell us?

Temperature

The brightest part of the spectrum tells us the temperature.

Atmospheric Elements

By comparing the shape and position of the spectral lines to known elements on Earth, we can determine what elements are in the atmosphere of the star (it is rare that elements have the same absorption line, so it is easy to guess and check elements).

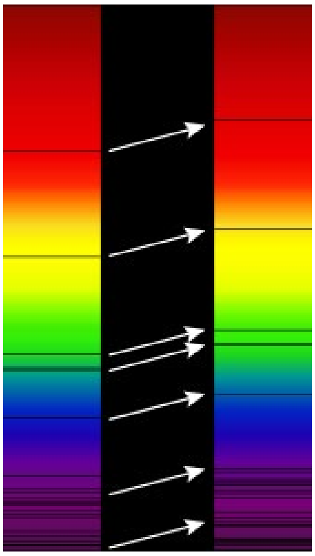
Density

The density of the star's atmosphere is a result of the star's density because more dense stars have a greater gravitational attraction and therefore hold a denser atmosphere. A denser atmosphere will result in darker absorption lines (more stuff to absorb photons).

Rotational and Translational Velocity

The velocity of points on a star will result in a Doppler effect. If a star is rotating, one side of the star will redshift the spectrum and one will blueshift it. This can be measured and used to calculate the rotational velocity.

Translational velocity is similar but instead we compare the theoretical colour of the star based on its temperature to its apparent colour (still a Doppler effect).



The Big Bang and the Origins of the Elements

The Big Bang theory originates from General Relativity, where the solution to Einstein's equations shows that the universe must have a beginning.

The Big Bang Theory is the currently accepted theory that the universe began as infinitesimal point and expanded from that point. The theory describes the very expansion of space itself, that is if space is made of 4 dimensions $(x,\ y,\ z,\ ct)$ then these dimensions are what expanded.

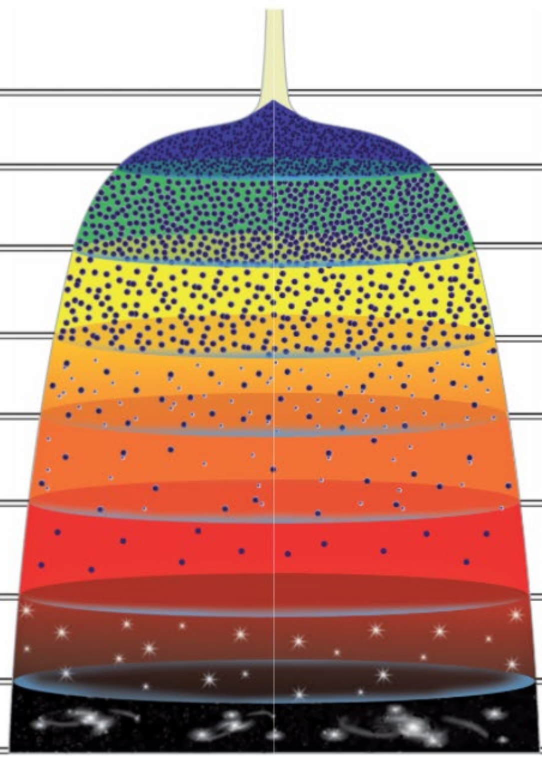
This early universe was very dense in energy and so matter was able to spring out of the energy of the vacuum in the form particle anti-particle pairs. Initially, the temperature of the universe was very great, so the particles were moving too fast to combine and quickly collided with their respective anti-particles.
It is measurable that the amount of matter produced in this process was slightly greater than antimatter, though the cause is unknown.

The energy from these collisions was initially released as gamma rays. As the universe expanded doppler shift occurred, eventually lowering the energy of the photons.

As the universe expanded, the temperature of the particles decreased, and they were able to combine to form protons and neutrons.

Although particles and anti-particles were created in equal parts, the asymmetry of the weak nuclear interaction is attributed as one of the possible reasons there is far less anti-matter in the universe today. This is still one of the great mysteries of physics and is yet to be fully modelled.

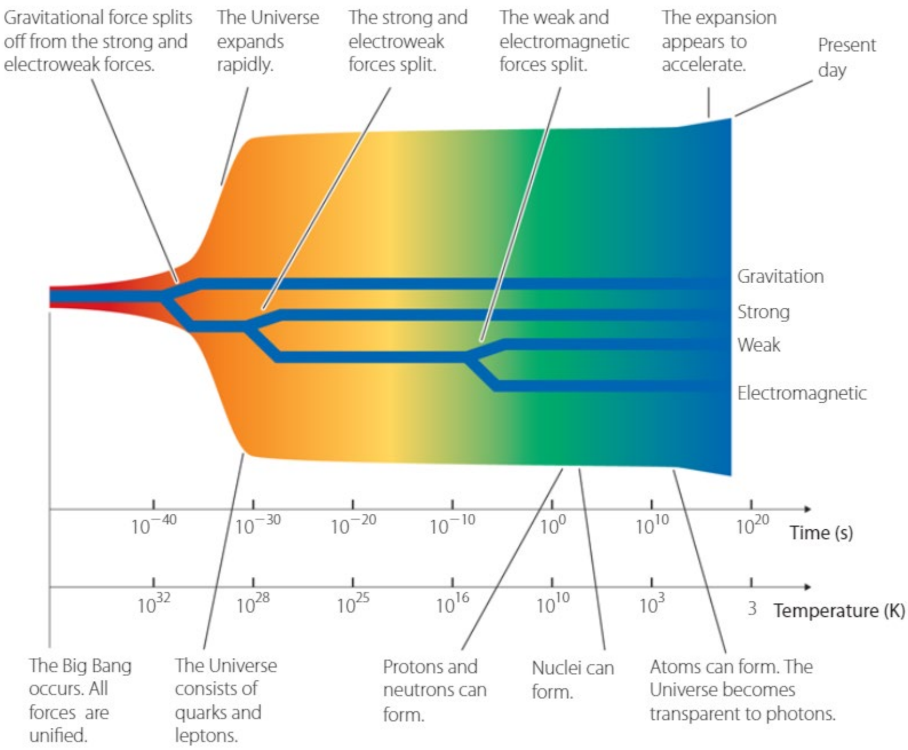
Time since Big Bang	Era	Temperature (K)	Description	Major events in the era
10^{-43} s	Planck	$>10^{32}$	Incredibly high temperature	Space and time established
$10^{-35}-10^{-32}$ s	Inflation	10^{14}	Fundamental particles form and unform	Strong force becomes distinct; cosmic inflation
10^{-10} s	Force separation	10^{13}	Fundamental particles last long enough to interact; explosion of photons as matter and antimatter collide	Separation of strong force, weak force and electromagnetism
10^{-3} s (1 ms)	Particles	10^{12}	Fundamental particles	Fusion begins
3–20 min	• Nuclear fusion • Nucleosynthesis	10^{9}	Protons, neutrons, electrons and neutrinos stabilise	Fusion ceases; primordial element ratios established
Up to 380 000 years	Atoms	Falls to 3000	Plasma of hydrogen, helium and electrons	Photons are trapped, but freed when atoms are formed
380 000 to 200 million years	Dark ages	Falls to 100	Universe is transparent but there are no stars to shine	Cosmic background radiation is unimpeded; only other light is faint hydrogen glow
200 million to 380 million years	First stars	Falls below 100	First stars, 50–500 solar masses, appear	Light from first stars begins to re-ionise gas clouds
400 million years to present	Galaxies	Falls to 2.7	Galaxies appear and evolve	Universe becomes as we know it today



Timeline of the Big Bang

It is believed that, originally, Gravity; Electromagnetism; the Strong Force; and the Weak Force were all one force. It is believed that Gravity was the first to split as its own force, then the strong force, leaving the Electroweak force (the united electromagnetic and weak force which has been mathematically proven).

This is why Physicists would like to find a theory of everything so they can finally prove or disprove the theory.



Inflation

This is an un-proven theory which says that if the universe underwent a brief period where space doubled 100 times (underwent exponential growth) and then returned to a constant expansion, then the homogeneity of the universe can be explained. There is currently no evidence for inflation, though it is widely accepted.

The Cosmic Microwave Background and Cosmological Redshift

Initially, large amounts of radiation was emitted due to matter anti-matter annihilation in the form of gamma rays. As the universe expanded the space taken up by the photons expanded, increasing the wavelength of the photons. This is known as cosmological redshift and is also what redshifts the light coming from distant galaxies.

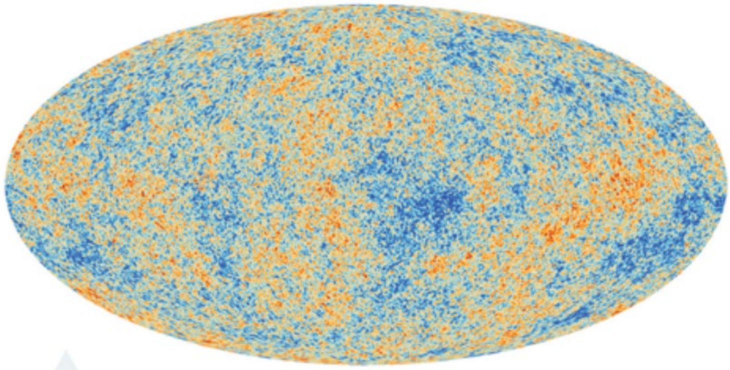


FIGURE 13.5 The remnant radiation from the Big Bang has been stretched by the expansion of the Universe. It is now observed in the microwave region of the electromagnetic spectrum.

The Hubble Constant and Spatial Expansion

Alexander Friedmann was the first to show that solutions to Einstein's field equations had possible solutions for both an expanding and contracting universe. Einstein did not readily accept this idea but with empirical evidence, eventually admitted that Friedmann was correct (after Friedmann had died from typhoid).

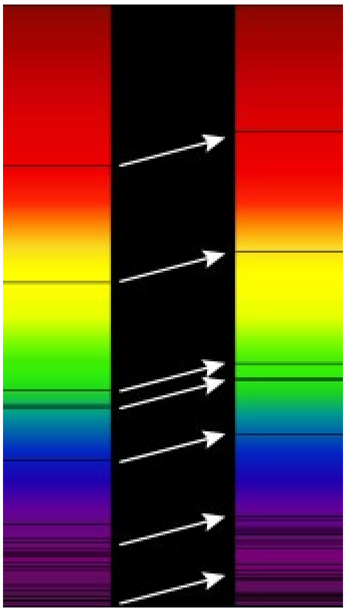
Georges Lemaitre found a similar solution and used empirical astronomical data to show that the universe was expanding. Due to the fact that all points in space are expanding away from each other, objects which are further away move away at a faster speed (since there is more points in space moving away from each other). Using the solution that the universe is expanding at a constant rate, Lemaitre concluded that $\frac{v}{D} = constant$.

Hubble was able to later use empirical evidence to show this relationship and measure the constant. We now call this relationship Hubble's Law $\frac{v}{D} = H_0$ where H_0 is Hubble's constant.

In reality the Hubble constant changes with distance, showing an accelerating speed of expansion.

Proof for Spatial Expansion

The evidence for this expansion is the emission spectra of stars, where spectral lines of similar stars are shifted, and the further the star is, the more it is shifted.



Absorption Spectra as described by Atomic Theory

The absorption spectrum of a star is given by the gaseous particles in its atmosphere. Photons which are emitted by the star that can be absorbed by atoms in the atmosphere are absorbed by those atoms, the re-emitted later with a random direction. The result of this is that a much lower intensity of the light reaches the earth.

Accelerating Spatial Expansion

It was believed that the most probable solution to Einstein's field equations was that space may be expanding, but that it was slowing down due to the negative potential energy of gravity. It turns out that it is possible (and allowed in the field equations) for the universe to have a positive energy content which would cause the universe to expand at an accelerating rate. This energy can be measured by making this assumption but is not directly predicted by any current theories.

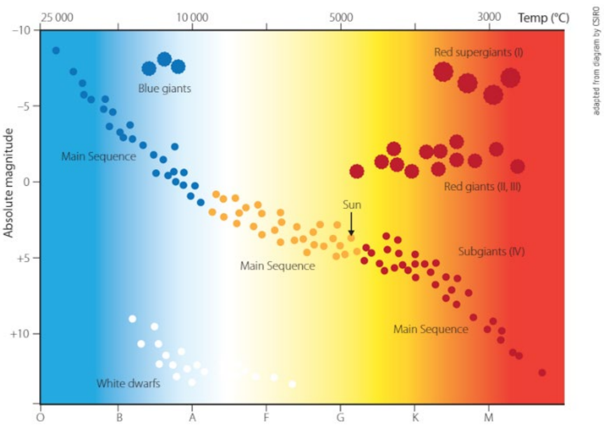
The accelerating rate of spatial expansion was measured by a physicist at ANU. What they found was that the Hubble constant for distant supernovae increased the further away you look, showing that the velocity at which objects move away from each other changes with time and with a positive gradient (i.e. $\frac{\Delta v}{\Delta t} > 0$).

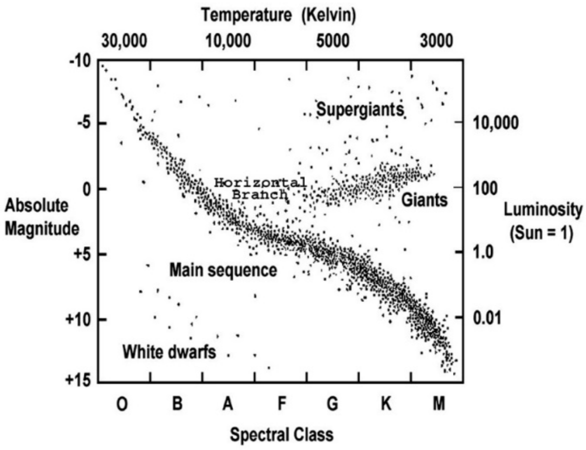
For more, you can watch the video from the man himself:
https://www.youtube.com/watch?v=55pcpTjd3BY&ab_channel=ANUTV

The mechanism for this expansion is called Dark Energy, though nothing is known about it.

The Hertzsprung-Russel (H-R) diagram

The colour, size and mass of a star can be associated with its temperature and luminosity.





The Axes

- **Absolute Magnitude**: How bright the star would appear if it were 10 parsecs away (on a logarithmic scale)
- **Luminosity**: The power output of the star
- **Temperature**: The surface temperature of the star which causes blackbody radiation
- **Spectral Class**: The colour of the star defined at certain cut-off points.

High Mass stars are typically towards the top of the diagram since high mass stars can fuse more atoms per second and therefore are more luminous.

All stars emit light as a function of blackbody radiation (where the wavelength emitted depends on their surface temperature).

The red giants are giant because they are fusing lots of atoms producing a large outward pressure. However, the energy produced is spread across many layers, so the outer layer is cooler and therefore more red.

White dwarfs are the remains of the cores of stars which emit light due to blackbody radiation but no longer fuse nuclei. They have been compressed to the point where the only thing keeping them at a fixed radius is the outwards pressure of the electrons in their shells. The Pauli Exclusion Principle keeps them from compressing further so they maintain their size.

Neutron stars are white dwarfs where the gravity was so strong it allowed the electron shells to shrink and the electrons to combine with protons to make neutrons. Now the only thing keeping them apart is the pressure of the exclusion principle between the neutrons.

Extension Notes

Derivation of the Properties of a Star for the H-R Diagram

Measuring the properties of stars is a difficult process. The properties that are directly measurable from a star are:

- It's apparent brightness (the intensity of the light which reaches the observer)
- The angle it takes up in the sky
- The apparent colour (the redshifted peak wavelength which reaches the observer)
- The star's redshifted absorption spectrum

From these pieces of data it is possible to determine the star's temperature, actual colour, power output (luminosity), radius and its distance away from us. These are the properties which appear on the H-R diagram so it is particularly curious that these are not directly measurable.

The most accurate way of doing this is by comparing the absorption spectrum's shape to known element combinations and logically deriving how redshifted the light is and how hot it must be for certain elements to be in the atmosphere. However, this data has already been collected and used to form the Stefan-Boltzmann equation and Hubble's Law so the derivation below shows how these equations can be used to determine the properties of any star mathematically.

Conservation of Luminosity (Light Flux)

The light flux through a 3D are or luminosity of a star is conserved at any distance. This means that the light flux through a sphere of radius r outside the star is the luminosity:

$$\oiint \vec{I} \cdot d\vec{A} = L$$

$$I(4\pi R^2) = L$$

$$I = \frac{L}{4\pi R^2}$$

This is what gives rise the r^2 law in Year 11 physics ($I_1 {r_1}^2 = I_2 {r_2}^2$).

The Intensity of light which reaches the Earth determines the apparent brightness.

Identifying the Temperature of a Star without Doppler Shift

Due to the effects of cosmological redshift, the colour of stars can change, making Wein's law pretty useless if you're observing the colour of the star directly.

$$\lambda_{max} = \frac{b}{T}$$

The Stefan-Boltzmann Law is then used to determine the temperature of a star, allowing for the assessment of the doppler shift through Wein's law.

$$L = \sigma A_s T^4, \qquad \sigma = 5.7 \times 10^{-8}\ W\ m^{-2}\ T^{-4}$$

Where L is luminosity and A_s the surface area of the star.

Measuring the Temperature of stars

Now we know the theory behind light intensity, we can combine the known Stefan-Boltzmann Law (which was determined experimentally) and the Luminosity law to calculate the temperature of the star.

Since the surface area of the star is given by $A_s = 4\pi r^2$ where r is the radius of the star, we put this into the Stefan-Boltzmann Law.

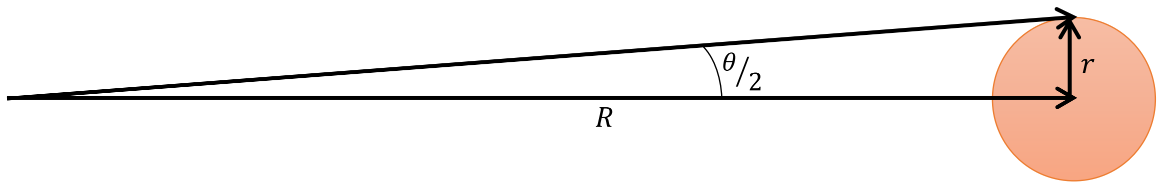
$$L = 4\pi r^2 \sigma T^4$$

We now take that and substitute it into our intensity equation:

$$I = \frac{4\pi r^2 \sigma T^4}{4\pi R^2}$$

$$I = \frac{r^2}{R^2} \sigma T^4$$

Now, we also know that the following is true, where $\frac{\theta}{2}$ is half the angle which the star takes up in the night sky:



$$\tan\frac{\theta}{2} = \frac{R}{r}$$

$$I = \sigma T^4 \tan^2\frac{\theta}{2}$$

$$T = \sqrt[4]{\frac{I}{\sigma \ \tan\frac{\theta}{2}}}$$

Since the intensity can be measured and the angle can be measured, the temperature of the star can be measured.

Applying the known temperature to find the degree of redshift

The following equations will be used

$$\lambda_{max} = \frac{b}{T}, \qquad b = 2.898 \times 10^{-3}\ Km$$

$$\frac{v_s}{H_0} = R, \qquad H_0 = 2.265 \times 10^{-18}\ s^{-1}$$

$$f' = f\frac{|v_w - v_o|}{|v_w - v_s|}, \qquad v_o = 0, \qquad v_w = c$$

$$v = f\lambda$$

It can therefore be derived that

$$\lambda' = \lambda\frac{|v_w - v_s|}{|v_w - v_o|}$$

$$\lambda' = \lambda\frac{|v_w - v_s|}{v_w}$$

$$\lambda' = \lambda\frac{c - v_s}{c}$$

$$\frac{\lambda'}{\lambda} = \left(1 - \frac{v_s}{c}\right)$$

$$\frac{v_s}{c} = 1 - \frac{\lambda'}{\lambda}$$

$$v_s = c\left(1 - \frac{\lambda'}{\lambda}\right)$$

λ' is the peak wavelength of the star which we can directly measure and by knowing T from before we can figure out the theoretical colour if the star were not moving $\left(\lambda = \frac{b}{T}\right)$ so now we know how fast the star is moving.

$$\frac{v_s}{H_0} = R$$

Now we know R we can use the earlier relationship $\tan\frac{\theta}{2} = \frac{r}{R}$

$$r = R\tan\frac{\theta}{2}$$

So now we know the star's size, its colour and its temperature, the last thing to find is its luminosity or its power output. We can do this two ways:

$$L = 4\pi\, I\, R^2 \qquad\qquad L = 4\pi\sigma r^2 T^4$$

Now we know all properties of the star and we can plot it on the H-R diagram, all by measuring its colour, brightness and size in the night sky.

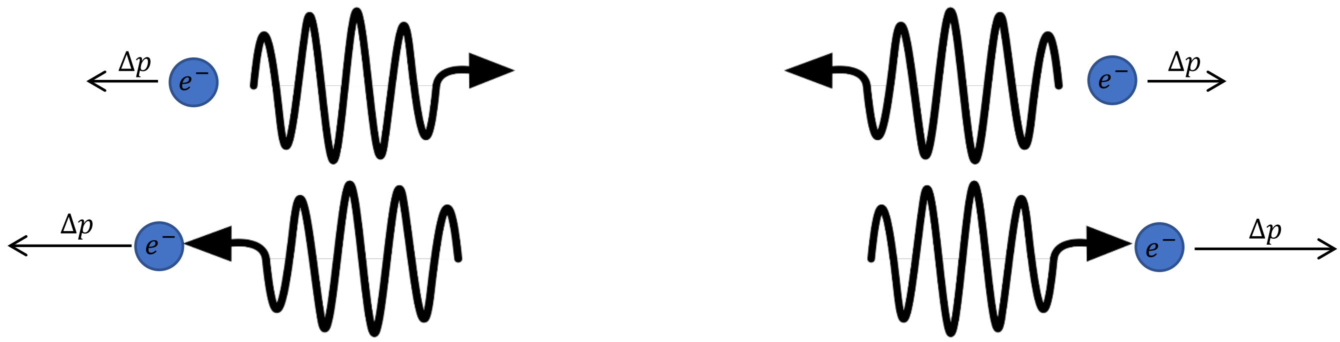
Quantum Forces

In Quantum Physics (specifically Quantum Electrodynamics and Quantum Chromodynamics) particles create a force by exchanging virtual particles. These particles determine the strength of the force and the range.

Virtual particles obey the rule $\Delta E \Delta t < \frac{\hbar}{2}$ where ΔE is the particles uncertainty in energy, Δt is its lifetime. Virtual particles such as electrons and positrons are able to 'pop' in and out of existence from the vacuum of space so long as they obey this rule and the conservation laws.

The Electromagnetic Force

The Particle Theory of electromagnetism is described by Quantum Electrodynamics. This theory predicted that photons which are undetectable are passed between electric particles and they carry the electromagnetic force. For repulsive forces, the photons leave the particles in the direction of the other, due to conservation of momentum, the particles must move apart. When the photons collide with the opposite particle, they again must move away.



The same is apparently true attractive forces except the photons carry negative momentum making the particles move towards each other. Such properties are allowed for virtual particles.

The Weak and Strong Force

The same is apparently true for the weak and strong force as with the electromagnetic force. The only difference is that the force carrying boson is different. In the case of the strong force and the weak force the boson has a mass and therefore its range is reduced.

By rearranging the uncertainty principle with $E = \gamma mc^2$, we find $\Delta t < \frac{\hbar}{2\Delta\gamma mc^2}$

Therefore, with a large mass, a larger uncertainty is allowed and therefore Δt must be smaller.

This is an (albeit lacking) explanation of why the Strong and Weak nuclear forces have a small range.

Evidence for Virtual Particles

The primary evidence for virtual particles is the Casimir Effect, where very close parallel plates (in a vacuum) will move together due to a greater number of virtual particles being able to 'pop' in and out of existence outside the plates than inside the plates. This pushes the plates together due to a greater pressure outside than inside the plates.

The Strong Force

The Strong Force is one of the four fundamental forces in the Standard Model of particle physics. The strong force is what binds three Quarks together into the proton (Up + Up + Down) or the neutron (Up + Down + Down). The strong force is mediated by gluons which 'glue' the quarks together with immense force and Mesons which are made up by Quark – Anti-Quark pairs.

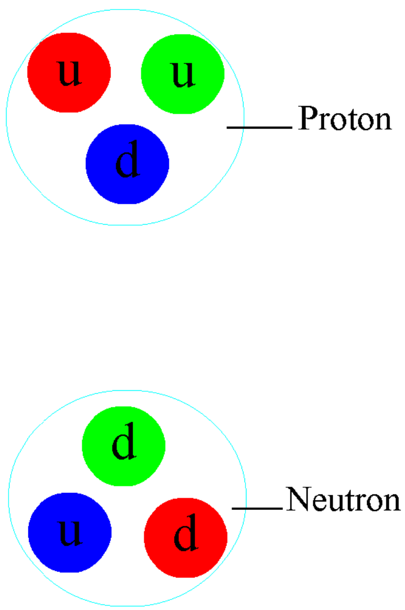
The strong force is what binds quarks. The nuclear force as it is sometimes called is what binds nearby Hadrons. This is a side effect of the strong interaction but is not the true strong force.

The nuclear force occurs when any two Hadrons (such as protons and neutrons) are near each other (less than the diameter of a proton). When a quark is pulled from one of the Hadrons, the potential energy created generates a new quark, which is sucked back into the Hadron, and an anti-quark which attaches to the detached quark to form a meson. This meson is then exchanged with the other Hadron, with the anti-quark annihilating and the new quark replacing the annihilated one. This same process occurs over and over.
This process is sort of like a covalent bond between atoms except the Hadrons share quarks not electrons and it is this process which bonds protons and neutrons.

In the process of sharing quarks (through the intermediary meson), protons and neutrons can swap places. This can happen through an up and down quark transmuting into the other during the swap.

The strong force interaction between nucleons is sometimes called the nuclear force (as it is a side effect of the strong force between quarks). There is a limit in the range of this interaction of about the diameter of a proton due to the limited range that the intermediary meson can travel before decaying.

The strong (nuclear) force is around 100 times stronger than the electric force between protons which may serve as an explanation as to why atoms with around 100 protons are unstable.



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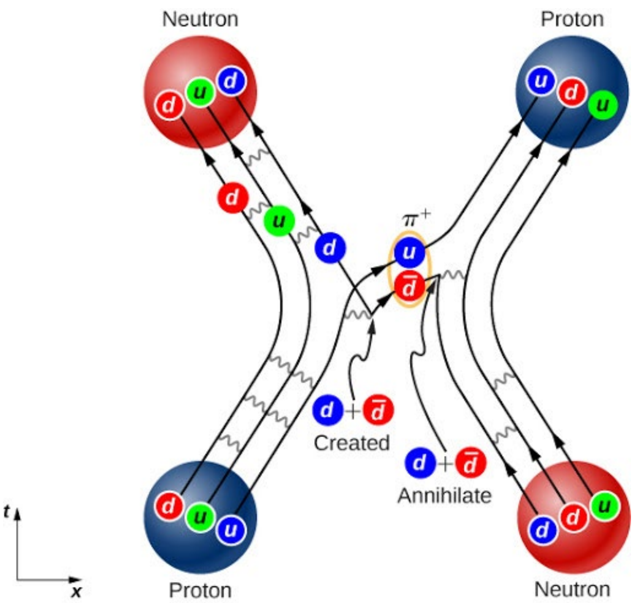


The Weak Force

The Weak Force is more probabilistic in nature than the Strong force and describes the likelihood for a particle such as a proton or neutron to change the 'flavour' of one of its quarks. Quantum physics allows for slight deviations in mass of particles and when a particle such as an up quark $\left(U^{+2/3}\right)$ has too much mass (such as from a positive potential energy or kinetic energy), it can decay into a W^+ boson and down quark $\left(D^{-1/3}\right)$ – remembering that while W bosons have mass, they are only force carrying particles for the Weak force and exist for short periods of time.

The time that a W boson exists for can be calculated using Heisenberg's uncertainty principle for virtual particles: $\Delta E \Delta t < \frac{\hbar}{2} \Rightarrow \Delta t < 10^{-24}\, s$.

The W^+ boson then decays into a positron (e^+) and neutrino (ν_e).

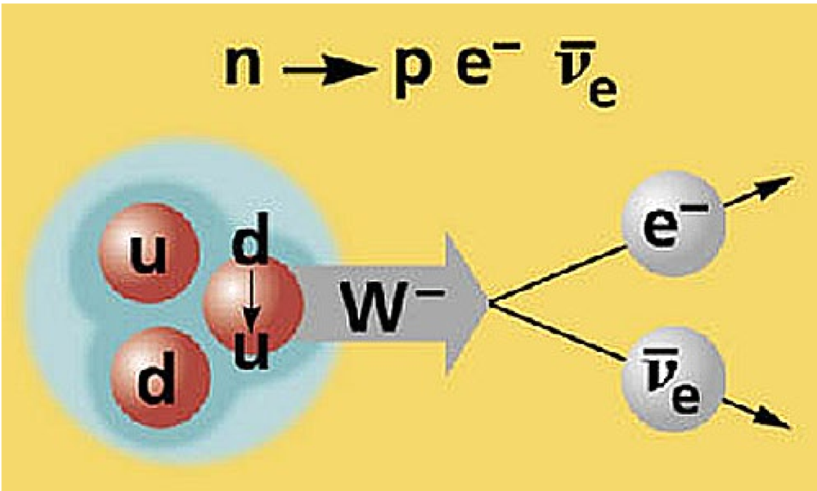
A similar process occurs with neutrons transmuting into protons except a down quark becomes an up quark and a W^- boson becomes an electron (e^-) and anti-neutrino. $(\bar{\nu}_e)$

The remaining mass of the W boson is lost in the form of energy as a photon (γ).

The transmutation from an Up to Down quark is what characterises the Weak force.

The reason a mass difference large enough to create a W boson might arise can be explained by the increased potential energy that arises when many nucleons (protons and neutrons) are near each other.

It is also worth noting that the weak force only acts on particles



APPENDICES

The Laws of Thermodynamics

The 0^{th} Law: Thermal Equilibrium

Fundamentally, this law states that if System A is in contact with System B and neither is transferring heat, then they are in thermal equilibrium. It also expands on this to say that if System A is in thermal equilibrium with System C, then System B is also in thermal equilibrium with System C.

Although this law may seem trivial, it does fundamentally define thermal equilibrium. The reason it exists is because it specifies that there is no difference between perception of temperature from one system to another, i.e. 20 Kelvin is 20 Kelvin for all materials.

The 1^{st} Law: Energy

The 1^{st} Law of Thermodynamics states that, in a closed System consisting of parts A and B, where A is hot and B is cold, the total heat energy lost by System A is equal to the increase in heat of System B and the work done to the environment.

$$Q = \Delta U + W$$

This is fundamentally just the definition of conservation of energy, except it helps because it also details how heat energy can be lost as something other than heat. The work done to the system can be seen in a car engine, where the increase in temperature causes a piston to move, doing work. This work reduces the temperature increase of the gas inside.

The 2^{nd} Law: Entropy

This law is a very broad and confusing law as it becomes less intuitive the harder you think about it. Fundamentally, this law states that "*the average disorder in a closed system cannot decrease*". Disorder in part of a system can be decreased, such as in a fridge, however the whole system increases in disorder due to the increase in heat produced by burning the coal which supplies the electricity.

In the most ideal system, the entropy can remain the same, provided the production of electrical work is 100% efficient, however, even in such an imaginary device, entropy cannot decrease, only remain constant.

The 3^{rd} Law: Absolute Zero and Zero Entropy

The third law can be stated multiple ways; however, it is fundamentally the idea that *it is impossible to reach Absolute Zero* **or** *It requires an infinite amount of energy to reach Absolute Zero*.
The third law also states that *the entropy of a perfect crystal at Absolute Zero, would be zero*.

The last bit about a perfect crystal merely refers to the fact that if it is imperfect, then there is a level of disorder or entropy in the system.

Zero Entropy and Electrons:
The first strange phenomena with this law is that even in a substance with zero entropy, particles such as electrons will still move in a disordered probabilistic manor, due to their quantum wave function. It is easy to logically understand this if you think about what would happen if electrons stopped orbiting the nuclei of atoms, well they would no longer be atoms. As such, even though the whole atom may have zero kinetic energy, its constituents still possess kinetic energy.

Vectors

Scalar Multiplication – Vectors and Scalars

Traditional vector multiplication involves multiplying a vector by a scalar number (such as mass multiplied by acceleration, where mass is scalar, and acceleration is a vector). This multiplication produces a new vector which has an increased value from the original vector but in the same direction. A good example of this is momentum. The momentum vector is always in the same direction as the velocity vector however, because it is multiplied by mass, it has different units and magnitude.

Dot Product – Vectors and Vectors (∥)

The dot product is a special type of multiplication which is used for when two vectors are parallel to each other, such as in Work, where the parallel components of Force and Displacement are multiplied. The dot product, unlike Standard Multiplication, produces a scalar value. This is because of the specific method used to multiply the components of the vector using vector notation and because this method produces an impossible vector. This method, while not producing a useful vector, does produce a single output value.

The stand in for the vector notation method is to multiply the magnitudes of each vector, multiplied by cos of the angle between them, e.g. $W = Fs\cos\theta$

Kinetic Energy is also an example of this since velocity is multiplied by itself.

The Dot Product follows all standard conventions of multiplication except it cannot be undone (there is no reverse Dot Product).

The Dot Product Formula in $\mathbb{R}^3$

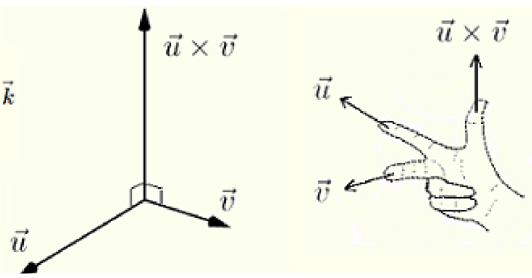
$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}|\cos\theta$

Cross Product – Vectors and Vectors (⊥)

The cross product is another method used to multiply vectors. However, it operates on the components of the vectors perpendicular to each other. Unlike the dot product, the cross product produces a third vector. It is rather complicated to explain how this works without full vector notation, so for now the standard rule is to take the two vectors and multiply them by *sine* of the angle between them. This is used commonly in rotational mechanics, with the most common example being Torque. The formula for Torque is $\vec{\tau} = \vec{r} \times \vec{F} = \vec{r}_{\perp}\vec{F}$, with the magnitude $\tau = rF\sin\theta$.
The torque vector produced by this multiplication, as with all cross multiplications, is perpendicular to both of the vectors used. The direction of the resultant vector is given simply through the right-hand rule.

For further explanation of this, videos such as [Khan Academy]’s using vector notation may be useful.

$$\vec{u} \times \vec{v} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} u_y & u_z \\ v_y & v_z \end{bmatrix}\vec{i} - \begin{bmatrix} u_x & u_z \\ v_x & v_z \end{bmatrix}\vec{j} + \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\vec{k}$$



Cross Product Rules and Properties in $\mathbb{R}^3$

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

$$\lambda(\vec{b} \times \vec{a}) = (\lambda\vec{b}) \times \vec{a} = \vec{b} \times (\lambda\vec{a})$$

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

Like the Dot Product, there is no reverse cross product.

The Cross-Product Formula for $\mathbb{R}^3$

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} = (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$$

The $\vec{\nabla}$ Operator

The Del or Nabla (∇) operator (more commonly known as the upside-down triangle thing) is used to describe properties of Vector Fields. An electric vector field is created when a charged particle is placed somewhere in space and as such, a vector describing the electric field is given to every point in space due to that charge.

The ∇ vector operator is like a vector and can be written sort of like a vector, with a partial derivative in each component. In reality it is still an operator and as such requires another vector to exist, but you'll get the idea.

Formally:

$$\vec{\nabla} = \sum_{j=1}^{m} \hat{u}_j \frac{\partial}{\partial x_j} = \sum_{j=1}^{m} \hat{u}_j \, \partial_j$$

Where $\hat{u}_j$ is the unit vector in the j^{th} dimension and x_j is the j^{th} spatial dimension.

So, in $\mathbb{R}^3$:

$$\vec{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$\vec{\nabla}$ has units m^{-1}

What is curious about the Del operator is that it creates a new vector field which describes the field it is operating on. This is particularly useful in Physics with Electric and Magnetic fields but the Laplacian (∇^2 or $\vec{\nabla} \cdot \vec{\nabla}$) is also used to describe both vector and scalar fields in Lagrangian Mechanics.

Gradient

The gradient operation is the easiest to perform and it is done on scalar fields (like potential energy) and vector fields (like the gravitational field). The gradient operation produces a vector field, with each point in space being assigned a vector which points in the direction of greatest increase in value. The gradient of a scalar or vector field U in $\mathbb{R}^3$:

$$Grad\ U = \vec{\nabla}U = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{bmatrix} = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}$$

Divergence

Divergence is the measure of how the strength or size of the vector field is changing with respect to space. If the field is increasing or decreasing as you get away from a point, then it is diverging and, therefore, the divergence is non-zero.

To find the divergence of a vector field $\left(\vec{F}\right)$, you take the dot product of the Del (∇) operator with the vector field (the only difference being this dot product still produces a vector):

$$Div\ \vec{F} = \vec{\nabla} \cdot \vec{F}$$

So, for a field that exists in $\mathbb{R}^3$:

$$Div\ \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} F_x & F_y & F_z \end{bmatrix} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

The Laplacian

The Laplacian (∇^2) can be seen in many equations in Physics but not very frequently in the topics covered so far. The Laplacian is akin to the second derivative function for a scalar (or vector) field and is defined as follows in $\mathbb{R}^3$:

$$\nabla^2 = div(grad) = \vec{\nabla} \cdot \vec{\nabla} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

The Laplacian functions similarly to the divergence or gradient operator and the Laplacian of a scalar or vector field (U) is:

$$Laplacian\ U = div(grad\ U)$$

$$\nabla^2 U = \vec{\nabla} \cdot \left(\vec{\nabla}U\right) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

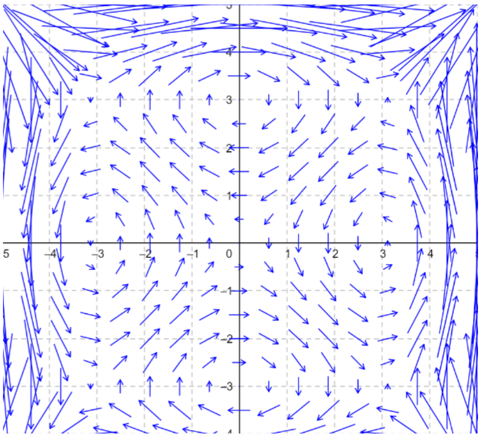
The Laplacian identity for a vector field:

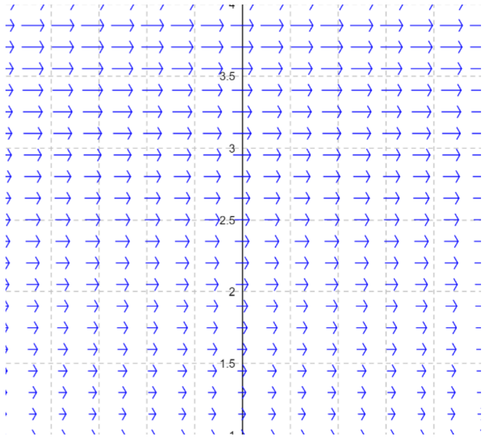
$$\nabla^2 \vec{F} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right)$$

Curl

Curl is a measure of how much the vector field's direction is changing with respect to space. The vector field produced to represent curl obeys the right-hand rule (i.e. the field curls anti-clockwise around the vector if looking down towards the 'arrowhead' of the vector).

Curl can occur where a field is actually curling or where the strength of a field changes perpendicular to its direction:





Both of these fields have curl, the left one is sort of obvious but the right one has curl because as you move up the page the strength changes, despite the field not pointing in that direction. In a sense, curl is actually saying what is the gradient of the field perpendicular to its direction.

To find the curl of a vector field $(\vec{F})$, you take the cross product of the Del (∇) operator and the field.

$$Curl\ \vec{F} = \vec{\nabla} \times \vec{F}$$

So, for a field that exists in $\mathbb{R}^3$:

$$Curl\ \vec{F} = \vec{\nabla} \times \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} F_x & F_y & F_z \end{bmatrix}$$

$$Curl\ \vec{F} = \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \quad \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \quad \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \right] = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$$

The Hamiltonian $(\hat{H})$

The Hamiltonian operator is used in Hamiltonian mechanics and is similar to the Del operator. The Hamiltonian of a system or particle is the sum of all of its energy where $\hat{T}$ is Kinetic energy and $\hat{V}$ is potential energy:

$$\hat{H} = \hat{T} + \hat{V}$$

This is used in Quantum mechanics with a slightly different notation:

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(r,t)$$

Vector Calculus Identities

$$\vec{\nabla} \cdot (f\vec{F}) = f(\vec{\nabla} \cdot \vec{F}) + (\vec{F} \cdot \vec{\nabla})f$$

$$\vec{\nabla} \times (f\vec{F}) = f(\vec{\nabla} \times \vec{F}) + (\vec{\nabla}f) \times \vec{F}$$

$$\vec{\nabla} \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\vec{\nabla} \times \vec{F}) - \vec{F} \cdot (\vec{\nabla} \times \vec{G})$$

$$\vec{\nabla} \times (\vec{F} \times \vec{G}) = F(\vec{\nabla} \cdot \vec{G}) - G(\vec{\nabla} \cdot \vec{F}) + (\vec{G} \cdot \vec{\nabla})\vec{F} - (\vec{F} \cdot \vec{\nabla})\vec{G}$$

$$\vec{\nabla}(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \vec{\nabla})\vec{G} + (\vec{G} \cdot \vec{\nabla})\vec{G} + \vec{F} \times (\vec{\nabla} \times \vec{G}) + \vec{G} \times (\vec{\nabla} \times \vec{F})$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla}f) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2\vec{F}$$

"These identities are not difficult to prove" – Cambridge University Vector Calculus Textbook

For a given 3D surface with normal vectors $\vec{S}$ and volume V and a respective vector function $\vec{F}$ or scalar function U:

$$\oiint \vec{F} \cdot d\vec{S} = \iiint (\vec{\nabla} \cdot \vec{F})\, dV$$

$$\oiint U\, d\vec{S} = \iiint \vec{\nabla}U\, dV$$

$$\oiint \vec{F} \times d\vec{S} = -\iiint (\vec{\nabla} \times \vec{F})\, dV$$

For a given 2D surface S with perimeter l and normal vector $\vec{S}$ and a respective vector function $\vec{F}$:

$$\oint \vec{F} \cdot \mathrm{d}\vec{l} = \iint (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

Vector Conventions

Unit Vectors ($\hat{v}$)

Unit vectors are interesting as they contain the direction of the vector however, they have a magnitude of 1 and are unitless. As such, they can be used in formulae to give the value a direction without having to divide by some arbitrary value. A unit vector is mathematically obtained by dividing a vector by its modulus (length).

Formally they are written as the vector divided by its length, except the case where the vector has a length of zero. In this case it is just zero.

$$\hat{v} = \begin{cases} \dfrac{\vec{v}}{|\vec{v}|}, & |\vec{v}| \neq 0 \\ 0, & |\vec{v}| = 0 \end{cases}$$

This case scenario can be avoided if we write it as:

$$\vec{v} = |\vec{v}|\hat{v}$$

The Radius Vector

Many laws in physics make use of the unit radius vector to denote a direction. As a result, there are certain conventions associated with constructing a radius vector so that the laws can be applied appropriately. *The radius vector is defined as beginning at the object exerting the force and ending at the object experiencing the force.*

This is because the radius vector is the radius of the object with respect to the field and the origin of the field is the location of the field-creating (and force causing) object.

Generalised Potentials for Fields

Let some divergent field $\vec{\mathbb{F}}$ have units f with the field causing a force on particles with property p (units p) such that force is given by $\vec{F} = p\vec{\mathbb{F}}$.

Potential Energy of a divergent field $\left(\vec{\nabla} \times \vec{\mathbb{F}} = 0\right)$ is given by

$$U = -W = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} = \int_{r}^{\infty} p\vec{\mathbb{F}} \cdot d\vec{r}$$

The Voltage is the potential energy per unit property p

$$V = \frac{U}{p} = \frac{-W}{p} = \int_{r}^{\infty} \vec{\mathbb{F}} \cdot d\vec{r}$$

Potentials in Gravitational Fields

Potentials around Point Masses (Divergent Gravitational Fields)

$$U = -W = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} = \int_{r}^{\infty} -\frac{GMm}{r^2}\hat{r} \cdot d\vec{r} = \int_{\infty}^{r} \frac{GMm}{r^2}\hat{r} \cdot d\vec{r}$$

$$V = \frac{U}{m} = \frac{-W}{m} = \int_{\infty}^{r} \frac{GM}{r^2}\hat{r} \cdot d\vec{r}$$

Potentials in Electric Fields

Potentials around Point Charges (Divergent Electric Fields)

$$U = -W = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} = \int_{r}^{\infty} q\vec{E} \cdot d\vec{r}$$

$$V = \frac{U}{q} = \frac{-W}{q} = \int_{r}^{\infty} \vec{E} \cdot d\vec{r}$$

Potentials around Curling Electric Fields

$$U = -W = -\oint \vec{F} \cdot d\vec{l} = \oint q\vec{E} \cdot d\vec{l}$$

$$V = \frac{U}{q} = \frac{-W}{q} = \oint \vec{E} \cdot d\vec{l}$$

Potentials in Magnetic Fields

Potentials around Curling Magnetic Fields

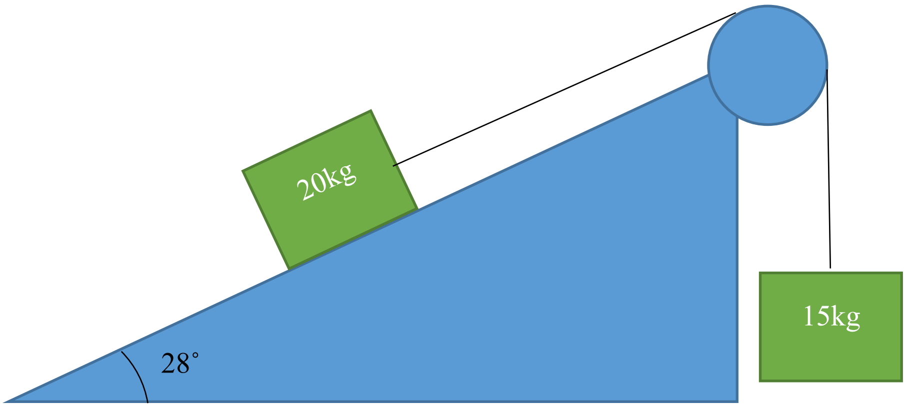
$$U = -W = -\oint \vec{F} \cdot d\vec{l} = \oint q\left(\vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

$$V = \frac{U}{q} = \frac{-W}{q} = \oint \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

Bonus Question

Tension

This type of question can pull a sneaky one on you in the final exam.



a) What is the Force rotating the pulley clockwise?
b) What is the Force rotating the pulley anti-clockwise?
c) Given the coefficient of Static Friction, 0.3, what direction will the system rotate?
d) Given the coefficient of Kinetic Friction, 0.2, calculate the acceleration of the system.
e) Calculate the tension in the string holding the masses.

a) $\vec{F}_{\circlearrowright} = 15kg \cdot 9.8ms^{-2} = 147N$

b) $\vec{F}_{\circlearrowleft} = 20kg \cdot 9.8ms^{-2} \cdot \sin 28 = 92.0N$

c) If the system would rotate clockwise, then the resisting force will be the force down the slope + friction.

$\therefore \vec{F}_{\circlearrowright} = 147N - \left(92N + \vec{F}_f\right)$

$\vec{F}_f = \mu_s N \rightarrow N = 20kg \cdot 9.8ms^{-2} \cdot \cos 28 = 173N$

$\vec{F}_f = 0.3 \cdot 173N = 51.9N$

$\vec{F}_{\circlearrowright} = 147N - (92N + 51.9N) = 147N - 143.9N = 3.1N$

$\therefore$The system will rotate clockwise

d) $\vec{F}_{\circlearrowright} = 147N - (92N + \vec{F}_f)$

$\vec{F}_f = \mu_k N = 0.2 \cdot 173 = 34.6N$

$\vec{F}_{\circlearrowright} = 147N - (92N + 34.6N) = 147N - 126.6N = 20.4N$

$\vec{F} = m\vec{a} = \Sigma m \cdot \vec{a}_{\circlearrowright}$

$$\vec{a}_{\circlearrowright} = \frac{\vec{F}_{\circlearrowright}}{\Sigma m} = \frac{20.4}{20 + 15} = \frac{20.4}{35} = 0.58\,ms^{-2}$$

e) Given we know the acceleration clockwise is $0.58ms^{-2}$, we can say that the Net Force down on the 15kg block must result in this acceleration. Therefore:

$$\frac{\vec{F}_\downarrow - \vec{F}_\uparrow}{m} = \vec{a}_\downarrow$$

$\vec{F}_\downarrow - \vec{F}_\uparrow = 15\,kg \cdot 0.58\,ms^{-2}$

$147N - \vec{F}_\uparrow = 8.7N$

$\vec{F}_\uparrow = 147 - 8.7 = 138.3N$

$\therefore$The tension in the string is this upwards force (138.3N up)