THE NATURE OF LIGHT

Base Units

Length (l) – Metres (m)
Time (t) – Seconds (s)
Speed (v) – Metres per second (ms^{-1} **or** m/s)
Electric Field ($\vec{E}$) – Newtons per Coulomb (NC^{-1}) **or** Volts per Meter (Vm^{-1})
Magnetic Field ($\vec{B}$) – Tesla (T)
Wavelength (λ) – Metres (m)
Frequency (f) – Hertz (Hz **or** s^{-1})
Energy (E) – Joules (J **or** $kg\, m^2\, s^{-2}$)
Intensity (I) – Power per area ($J\, m^{-2}\, s^{-1}$ **or** $kg\, s^{-3}$)

Constants

The Speed of Light $c = 3.00 \times 10^8$ ($m\, s^{-1}$)
Permittivity of Free Space $\varepsilon_0 = 8.854 \times 10^{-12}$ ($A^2 s^4 kg^{-1} m^{-3}$)
Permeability of Free Space $\mu_0 = 4\pi \times 10^{-7}$ (NA^{-2})

Planck Constant $h = 6.626 \times 10^{-34}$ ($kg\, m^2\, s^{-1}$)

Wein's Displacement Constant $b = 2.898 \times 10^{-3}$ ($m^2 s^{-2} K^{-1}$)

Equations

$$d \sin\theta = m\lambda$$

Describes the angular location of the m^{th} bright spot on a wall due to double slit interference (or dark spots for single slit interference).

$$d \sin\theta = \left(m \pm \frac{1}{2}\right)\lambda$$

Describes the angular location of the m^{th} bright spot due to single slit interference ($+$); and the m^{th} dark spot on a wall due to double slit interference ($-$).

$$\lambda_{max} = \frac{b}{T}$$

Wein's Law for Blackbody radiation.

$$I = I_0 \cos^2\theta$$

Malus' Law - The intensity of light that has passed through a polarising filter with θ the difference between the angle of the polarizing filter's axis and the polarization direction of the light.

$K_{max} = hf - \phi$

$\phi = hf_0$

The maximum amount of kinetic energy an electron can receive from a photon of some frequency f where ϕ is the work function of the material and f_0 the critical frequency of the material.

Extension Equations

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \, \nabla^2 \psi$$

The differential wave equation. All waves are a solution to this equation.

$$I = \frac{{E_0}^2}{2c\mu_0} = \frac{c{B_0}^2}{2\mu_0}$$

The intensity of a light ray as a function of its maximum Electric and Magnetic field magnitudes.

$$P = \oint \vec{I} \cdot d\vec{A}$$

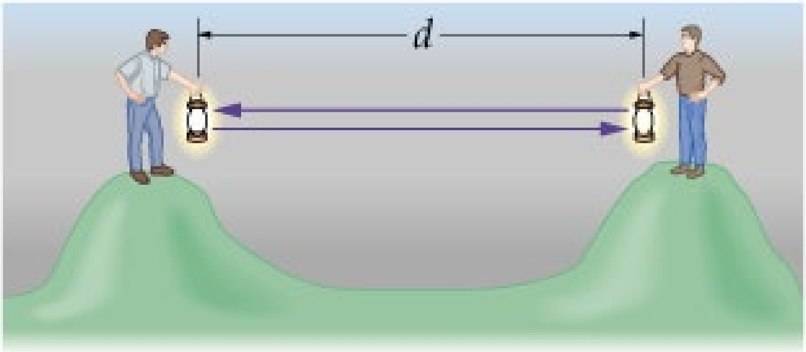
The power output of a wave is the 2D surface integral of the intensity over the area.

Course Notes

Attempts to Measure the Speed of Light

Galileo

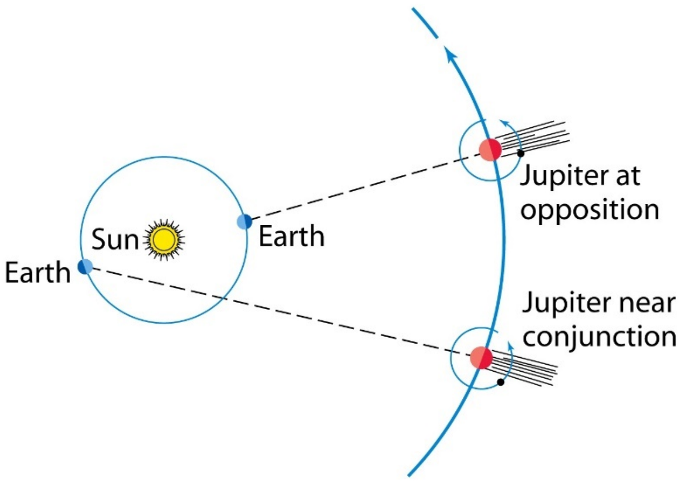
Galileo and his assistant stood on top of hills a reasonable distance (a few kilometres) apart. Galileo uncovered his lamp. His assistant uncovered his lamp once he saw Galileo's light and Galileo noted the time it took for him to see the light again.
He compared this to the time measured at a very small distance (i.e. human reaction time in his lab) and noticed that there was almost no difference. From this he concluded that the speed of light was a minimum speed of the distance between the hills over human reaction time measured in the lab and could be anywhere from this speed to infinite.



Romer

Romer was making measurements of the time at which the eclipse of one of Jupiter's moons Io occurred. Romer noticed that the time at which it appeared to occur followed a sine curve, with a period of one year.
He reasoned that this curve was occurring due to the orbit of the earth around the sun and that this delay must therefore be due to the speed of light travelling longer distances as earth moved to the other side of the sun.

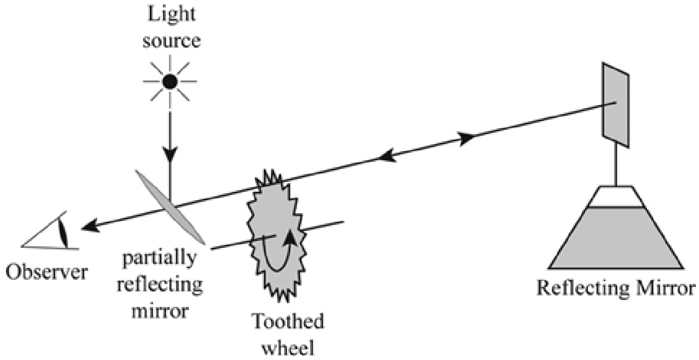
Using the known radius of earth's orbit, he estimated the speed of light to be $220,000,000\ m\ s^{-1}$

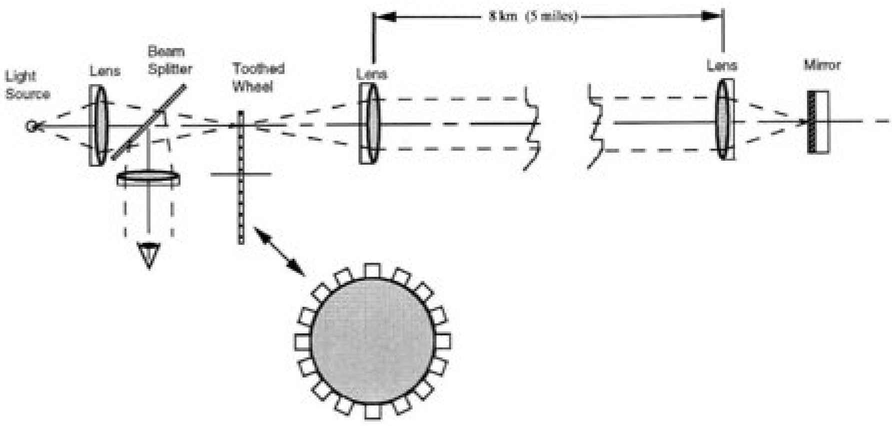


Fizeau

Fizeau used a toothed wheel with known angular speed and angular distance between the teeth. Fizeau passed light pulses through the holes in the teeth and then sped the gear up until the light pulses did not make it through. This meant that the next tooth was getting in the way of the pulse. Because he knew the angular separation of the teeth and the angular speed of the wheel, he calculated the minimum and maximum time the light could be taking after passing through the gap to make it back and hit the tooth. $\left(t = \frac{1}{nf},\ n = teeth\right)$

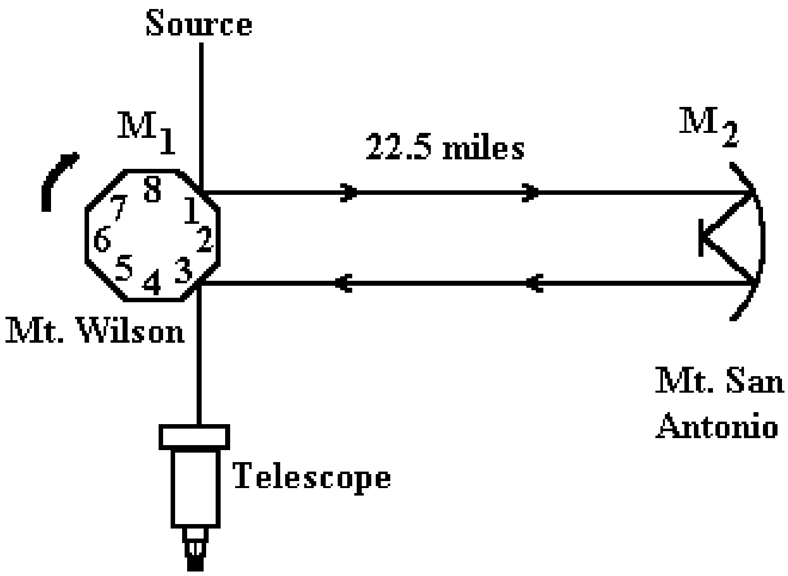
Using the known (large) distance between the mirror and the wheel, he calculated the speed of light using $c = \frac{d}{t}$ to be $3.15 \times 10^{8}\ m\,s^{-1}$





Michelson

Michelson's rotating mirror experiment is very similar to Fizeau's experiment in that it relied on the precise lining up of rotating objects with a ray of light. Michelson's experiment was, however, more precise than Fizeau's as it could use a continuous beam of light rather than pulses.



At first, the mirror is stationary such that the beam perfectly reflects towards the observer. If the mirror rotates even a little bit, the beam will not reach the observer. The key part of this is that the mirror must be in this orientation for the light to reach the observer.

Once the mirror starts rotating, the beam doesn't reach the observer because by the time the light has travelled the distance, the mirror has rotated to a different orientation.

The next time the beam will be seen is when the mirror is rotating such that:

- The beam reflects off side 1
- As the beam travels the distance, the mirror continues to rotate
- When the beam reaches the mirror, it has done exactly $\frac{1}{8}$ of a rotation and is back in the ideal state, with the light reflecting to the observer from side 2

This means the time taken for the beam to travel the large distance is $t = \frac{1}{8f}$ and the speed is $c = \frac{2d}{t}$

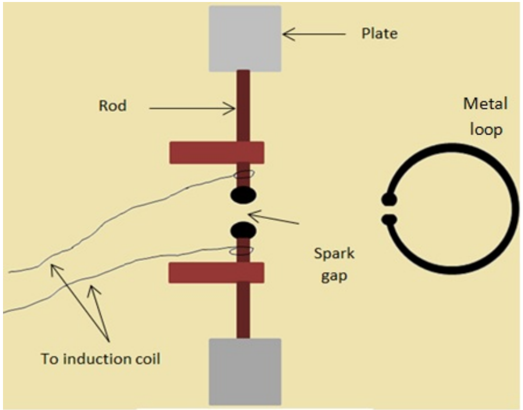
Michelson's measured the speed of light to be $299,979,000\ m\ s^{-1}$ (withing 0.1%)

The Hertz Experiment

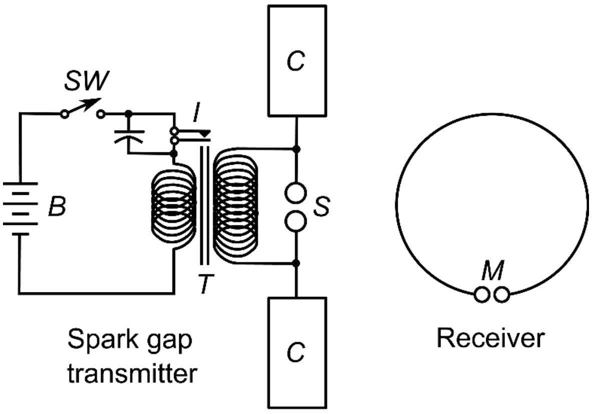
Hertz set out to experimentally prove two of Maxwell's predictions. As such he did two main things:

1. He assumed that light was an electromagnetic wave
2. He used the know relationships for waves to find the speed

If the assumption could be shown to be true and then the speed measured closely matched the prediction, then he could verify Maxwell's prediction.

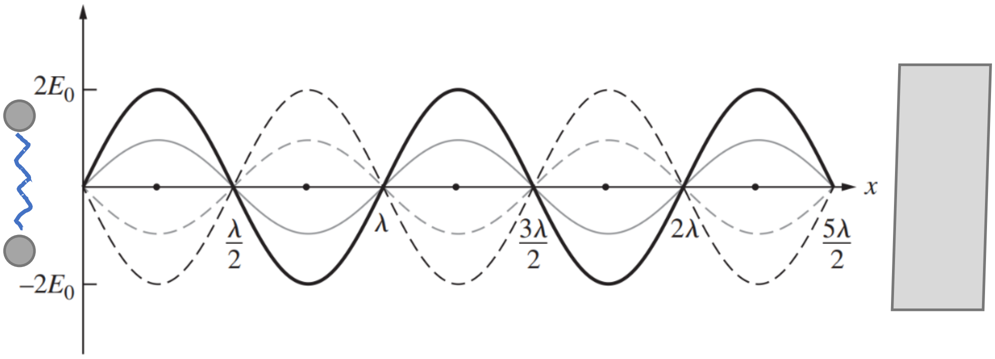


Hertz experiment (1887)



The experiment above cause a spark of electrons undergoing a high acceleration to pass between the spark gap. This created radio waves which were emitted in all directions (which were already known to be a type of light). The setup was created in a way that Hertz already knew the frequency of the wave.

By placing a metal reflector some distance from the emitter, he created a standing wave pattern between the emitter and the plate, where the gap between nodes is half the wavelength.



Now comes the assumption. If light is an electromagnetic wave, then the oscillating magnetic field of this standing wave will induce a current (and therefore a spark) in the loop detector at all points except the nodes. This was indeed the case.

By measuring the distance between points where there was no induced spark, Hertz measured the wavelength of the wave (which he already knew the frequency for).

Due to experimental error in calculating the frequency output of the device and error induced by an inversion of the wave upon reflection, Hertz could only conclude that the speed of light was finite and approximately $3 \times 10^8\ m\ s^{-1}$ (but with a large degree of uncertainty). This was enough to confirm Maxwell's hypothesis.

More recent attempts at his experiment with better data (but the same experimental setup) give a value of $c \approx 3.4 \times 10^8\ m\ s^{-1}$.

Waves need a medium – so what about light?

Once physicists discovered that light was a wave, they began to hypothesise about what is the medium through which it travels?

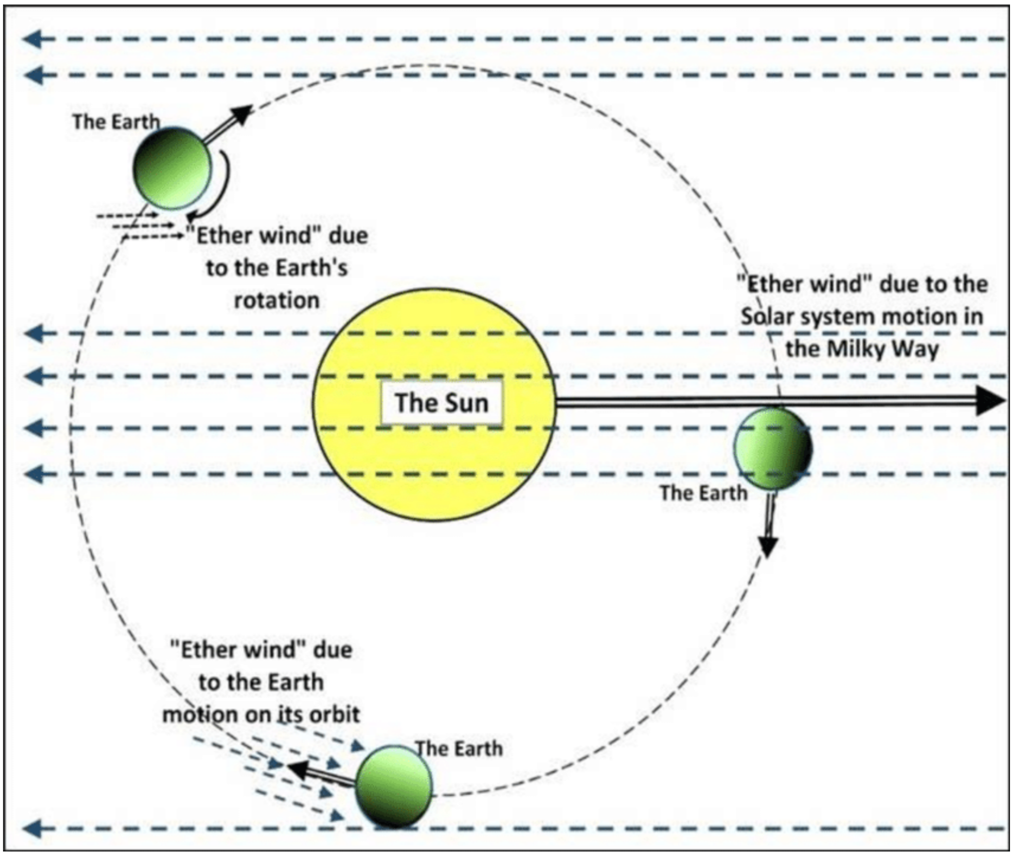
Absolute Space and Time

When Newton was developing his theories, he reasoned that there must be some universal clock and universal zero point. This seemed particularly reasonable since all phenomena observed at the time appeared to happen at the same rate and across the same distances.

The Luminiferous Aether

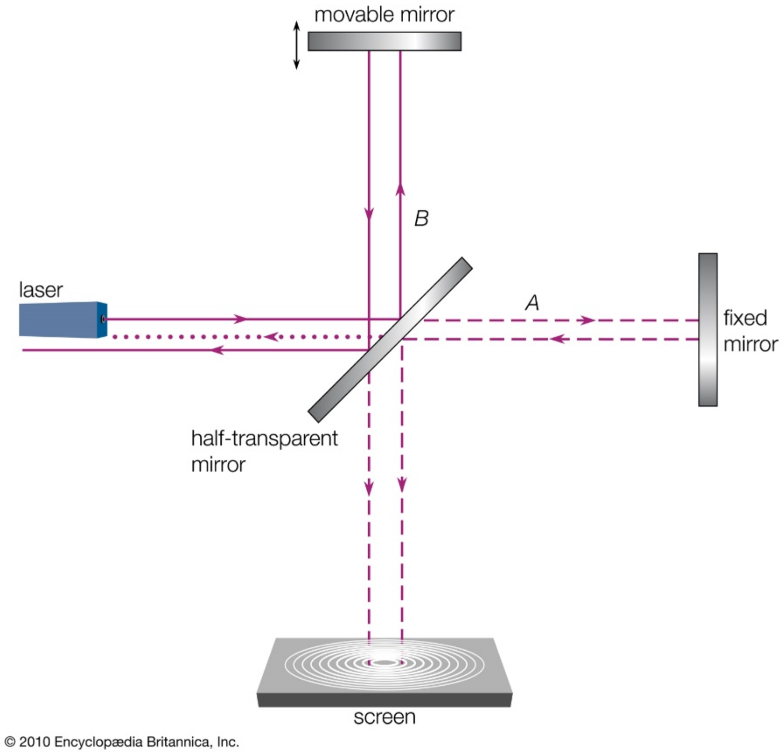
Based on Newton's idea of absolute space informed the construction of a theoretical medium called the aether. The aether was meant to be fixed to the universe's zero point and was the medium through which light travelled with speed c.

This meant that as Earth travelled through the universe, it should have some velocity relative to the aether. This would create 'aether winds' which would slow down or speed up light depending on the relative velocity of the Earth to the respective light ray.



Michelson-Morley Experiment

The Michelson-Morley experiment set out to detect these aether winds. They hypothesised that at a certain point in time and certain spot on Earth there is a given aether wind direction. Using their setup (below), should they rotate the apparatus, the relative speeds of each light ray will differ (they should bend depending on the angle to the aether wind) and a variable diffraction pattern should be produced as the apparatus is rotated.



The apparatus was floating on a mercury bath (because basically everything floats on mercury) allowing for it to be rotated smoothly while also isolating it from any vibrations from footsteps and the like.

The experiment yielded a null result (meaning nothing could be concluded). Due to the great precision of the experiment, and verification through repetition by other labs, it meant that the theory of the aether had to be revisited.

Although it was a null result, the discrepancy between the actual result and the theorised result meant that the theory had to be adjusted.

The Actual Medium for Light

The most current model of light says that there are magnetic and electric fields which permeate through all of space and it is the fields themselves which are the medium for light.

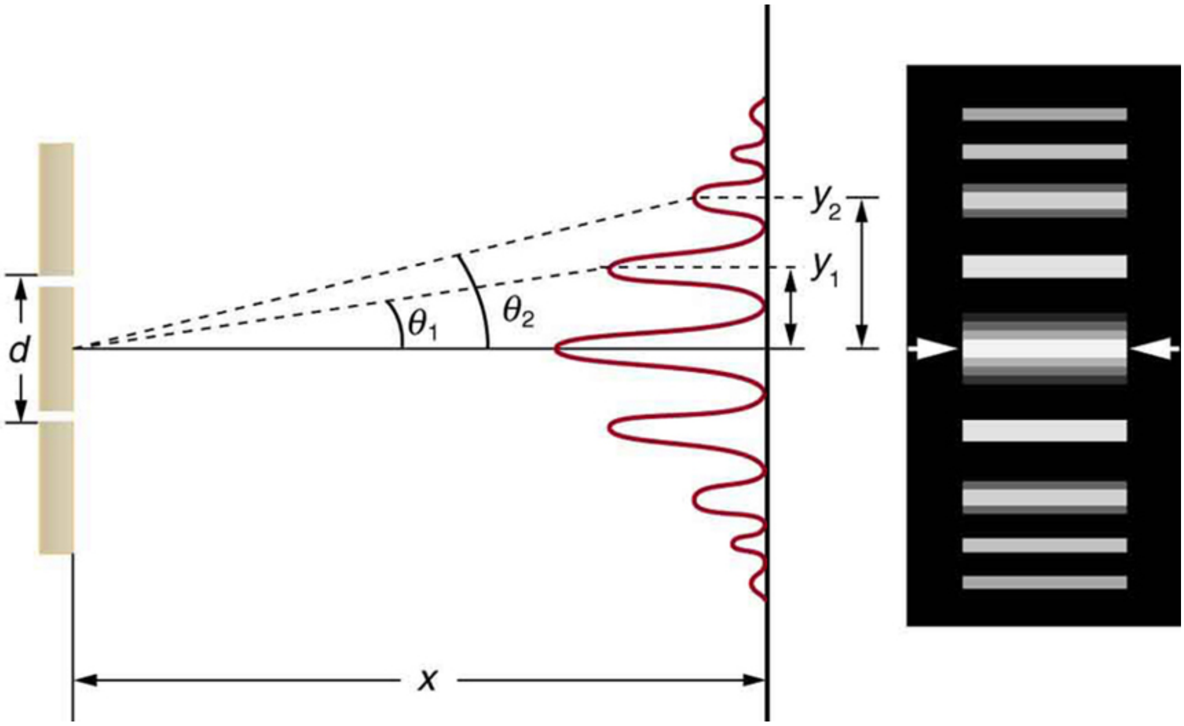
Waves need a medium because they need something to wave however it is the fields which wave, so light doesn't need a physical medium. The medium is the fields as they are what do the waving.

Diffraction

Double Slit

Double slit interference is somewhat easy to understand if you think about how waves work.

The assumptions of a double slit setup are that $X \gg d$ and therefore that $\sin\theta = \tan\theta$

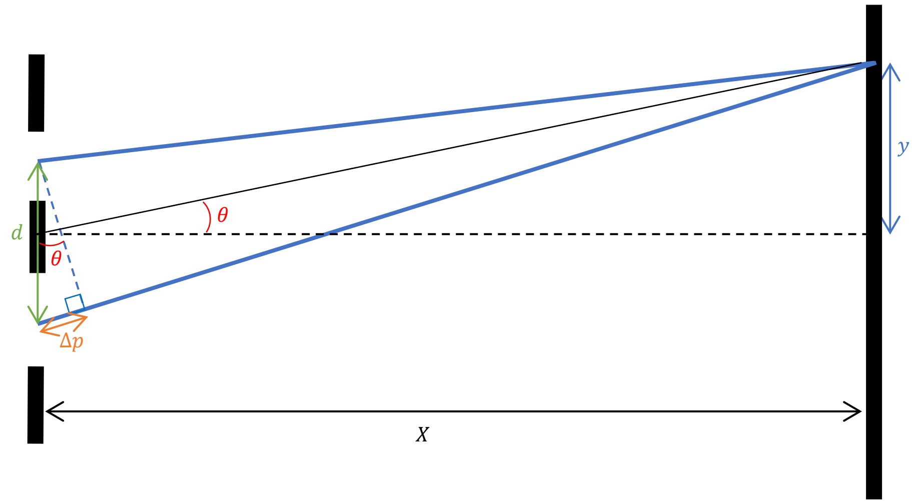


Formulae:

Maxima	Minima
$d\sin\theta_m = m\lambda$	$d\sin\theta_m = \left(m - \frac{1}{2}\right)\lambda$

Distance from Centre of Maxima / Minima		Gap Between Consecutive Maxima / Minima
$y = \frac{m\lambda X}{d}$	$y = \frac{\left(m-\frac{1}{2}\right)\lambda X}{d}$	$\Delta y = \frac{\lambda X}{d}$

Derivation of Formulae



Maxima

Condition for constructive interference is when the waves are in phase. Therefore, the distance covered by one wave must be an integer multiple of the wavelength (since they begin in phase).

Since $\Delta p = m\lambda,\ m \in \mathbb{Z}$

$$\sin\theta = \frac{\Delta p}{d}$$

$$d\sin\theta = m\lambda$$

Minima

Similarly, deconstructive interference occurs where the waves are out of phase and therefore where the path difference is half a wavelength. The first minima will occur where the path difference is half a wavelength.

Since $\Delta p = m\lambda - \frac{1}{2}\lambda,\ m \in\ \mathbb{Z}$

$$\sin\theta = \frac{\Delta p}{d}$$

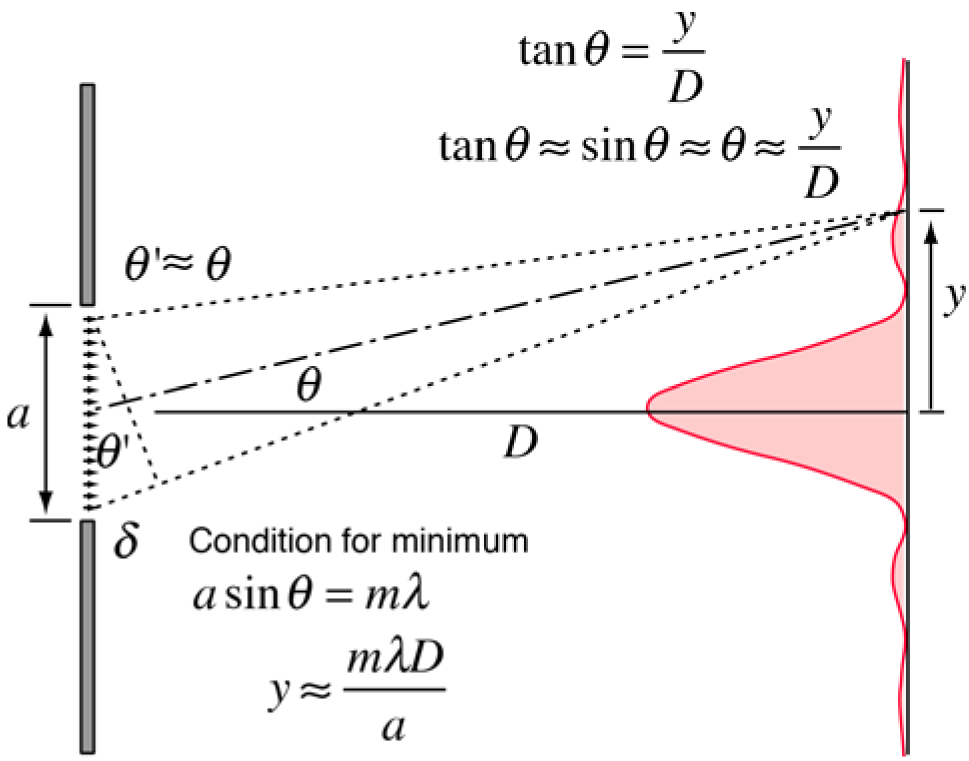
$$d\sin\theta = \left(m - \frac{1}{2}\right)\lambda$$

Height and Width of Peaks on Wall

Height	***Width***
$\sin\theta = \frac{m\lambda}{d} = \tan\theta = \frac{y}{X}$	$\Delta y = \frac{(m+1)\lambda X}{d} - \frac{m\lambda X}{d}$
$y = \frac{m\lambda X}{d}$	$\Delta y = \frac{\lambda X}{d}$

Single Slit

Single slit interference is only qualitatively assessable in Yr. 12 HSC Physics however it isn't terribly difficult. The main difference is that while in single slit interference the width of the centre peak is $2\,\Delta y$.

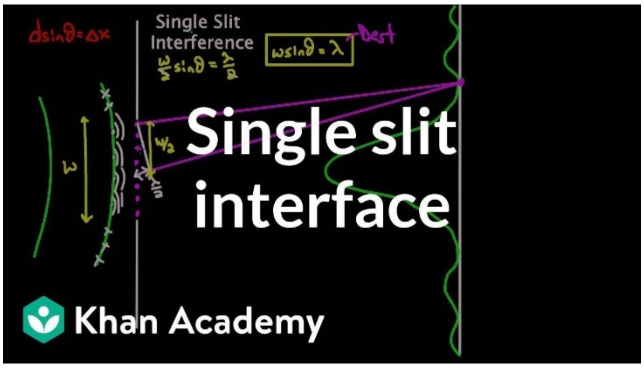


Formulae:

Maxima*	Minima
$a\sin\theta_m = \left(m + \frac{1}{2}\right)\lambda$	$a\sin\theta_m = m\lambda$
Distance from Centre of Maxima / Minima	**Gap Between Consecutive Maxima / Minima**
$y = \frac{\left(m + \frac{1}{2}\right)\lambda X}{a}$ $y = \frac{m\lambda X}{a}$	$\Delta y = \frac{\lambda X}{a}$

**There is technically no formula for maxima, but this gives the halfway point between two minima*

For Derivations see Khan Academy's video:



https://www.khanacademy.org/science/physics/light-waves/interference-of-light-waves/v/single-slit-interference

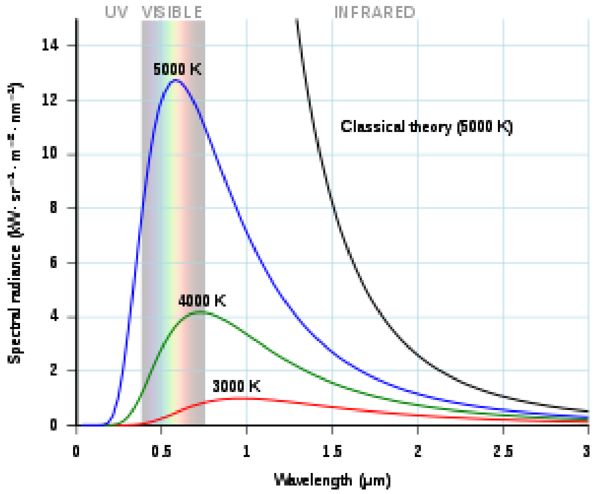
The Quantum Nature of Light

Blackbody Radiation

A blackbody is an object which does not reflect light. All objects of mass emit heat as a function of their temperature (i.e. infrared temperature sensors) and a black body is just something which physicists can analyse (because physicists are lazy and can't be bothered adjusting for light from the environment).
The sun can be considered a black body because it gives off so much more light than it receives from the rest of the universe. It is possible that the sun is actually a perfect mirror but it's about 1000000000000000000000000000000000000 times brighter than anything else, so no one cares.

The Ultraviolet Catastrophe

'The Ultraviolet Catastrophe' is so named due to what is possibly the worst prediction to come from physics. Classical thermodynamics and electromagnetism predicted that a hot black body should emit an exponentially large amount of light in the ultraviolet, x-ray and gamma-ray side of the light spectrum. This violated both common sense and conservation of energy and was the first hint at a quantum nature of light.



Thermodynamics is also known as statistical mechanics and as such, the heat and light output of a blackbody is probabilistic (or statistical) in nature (its why the curve is a curve and not a bar graph). The peak light emission wavelength is derived by calculating the highest probability interaction between subatomic particles at a certain temperature and therefore all other wavelengths will have a lower probability of being produced.

The current theory is defined by Wein's law:

$$\lambda_{max} = \frac{b}{T}, \qquad b = 2.898 \times 10^{-3}\ (m^2 s^{-2} K^{-1})$$

The Photoelectric Effect

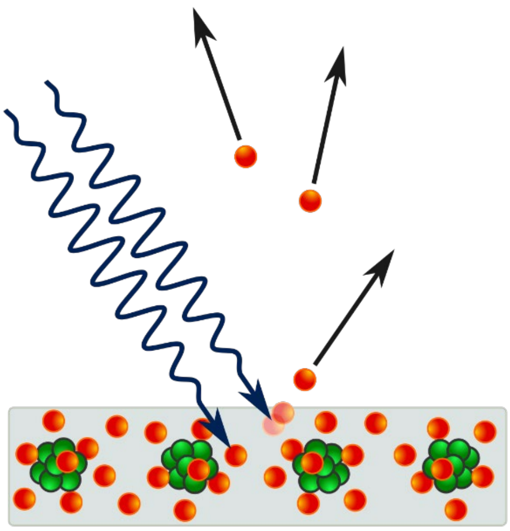
The Photoelectric effect was the first definitive evidence for the quantum nature of light as it completely contradicted classical electromagnetism. Although Blackbody radiation was unsolvable by classical physics, it did not completely contradict it, at least on a fundamental level.

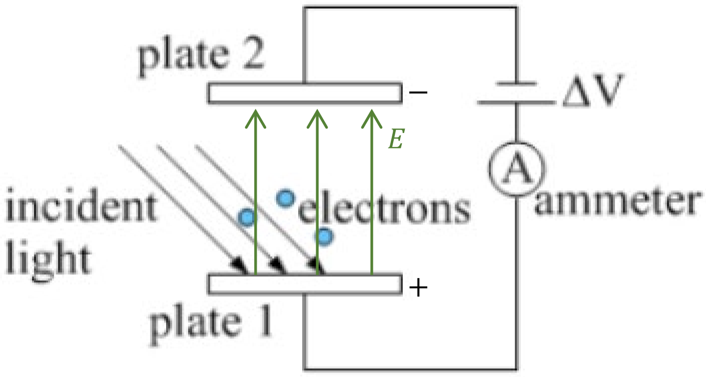
Classical Electromagnetism says that the power output (energy per second) of a wave is proportional to its intensity:

$$P = \oint \vec{I} \cdot d\vec{A} = IA$$

It was therefore completely reasonable to assume that the greater the intensity of light, the greater the amount of energy the light would give electrons. Therefore, the energy electrons possess after being vibrated by a light wave should be proportional to the power of the light wave. This was not the case.

Instead, the energy absorbed by an electron was a function of the frequency (intensity is not a function of frequency at all). This eventually led Einstein to conclude that this must be due to light coming in packets with energy $E = hf$ and intensity being due to an increase in the number of packets (eventually called photons) striking the surface.
This was the beginnings of the quantised understanding of light.





The Work Function (ϕ)

The work function is the lowest amount of potential energy which an electron possesses while part of the material. This is what gives rise to K_{max} rather than just K as it is the lowest amount of resistance the material provides to the electron being removed.

The Experimental Setup

The experiment is setup such that light is shone onto one of two parallel plates. A voltage can be applied to the plates so that there is an electric field between the two plates. This electric field can be setup to oppose the velocity of the emitted electrons.

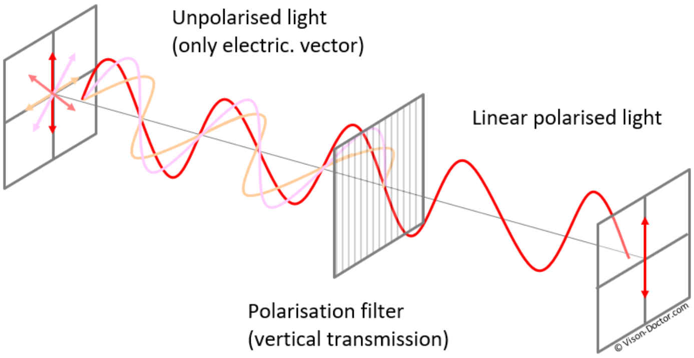
Since the work done by the field is $W = qV$, if the work done by the field is greater than the maximum kinetic energy of the electrons, then the electrons will stop before they hit the other plate. So, if $V = \frac{K_{max}}{q}$ then no current will be detected. As such, if an electric field is established such that it just stops all electrons then $qV = hf - \phi$

$\therefore \phi = hf - qV$

Polarisation

Polarised light is light with an electric field which only oscillates along one axis.

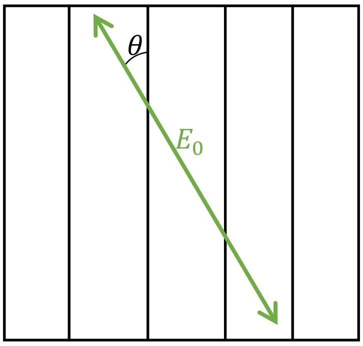
Polarisation is typically understood using a wave model of light and splitting the field vector into components. In reality, it is a quantum phenomenon, with the polarisation of a photon being a function of its spin, something which is based on probabilities.



Wave Model – Derivation of Malus' Law

It is somewhat easy to derive Malus' Law when considering the assumptions of polarisation. If we consider light of a single polarisation direction. If we assume that a polariser blocks all electric field components of a light ray which are perpendicular to its axis, then we can consider the following:

A light ray is shone onto a polariser with its electric field E_0 oscillating on an axis at some angle θ from the axis of the polariser.



The (peak) component of the electric field which will be let through is the parallel component.

$$E = E_0 \cos\theta$$

$$E^2 = E_0^2 \cos^2\theta$$

$$I = \frac{E^2}{2c\mu_0}$$

$$\frac{E^2}{2c\mu_0} = \frac{E_0^2}{2c\mu_0}\cos^2\theta$$

$$I = I_0 \cos^2\theta$$

Quantum Model

The quantum interpretation of light makes polarisation much harder to understand, but so is the nature of quantum. In reality, intensity is proportional to the number of photons which strike a surface. Photons possess a property called spin and this determines the direction of their electric field oscillation. When a photon is travelling through space, there is a degree of uncertainty in its spin. When a photon hits a polariser, it is interacting with it and therefore, its wave functions will collapse, and its spin will lose its uncertainty. The probability that the photon's spin aligns with the polariser is:

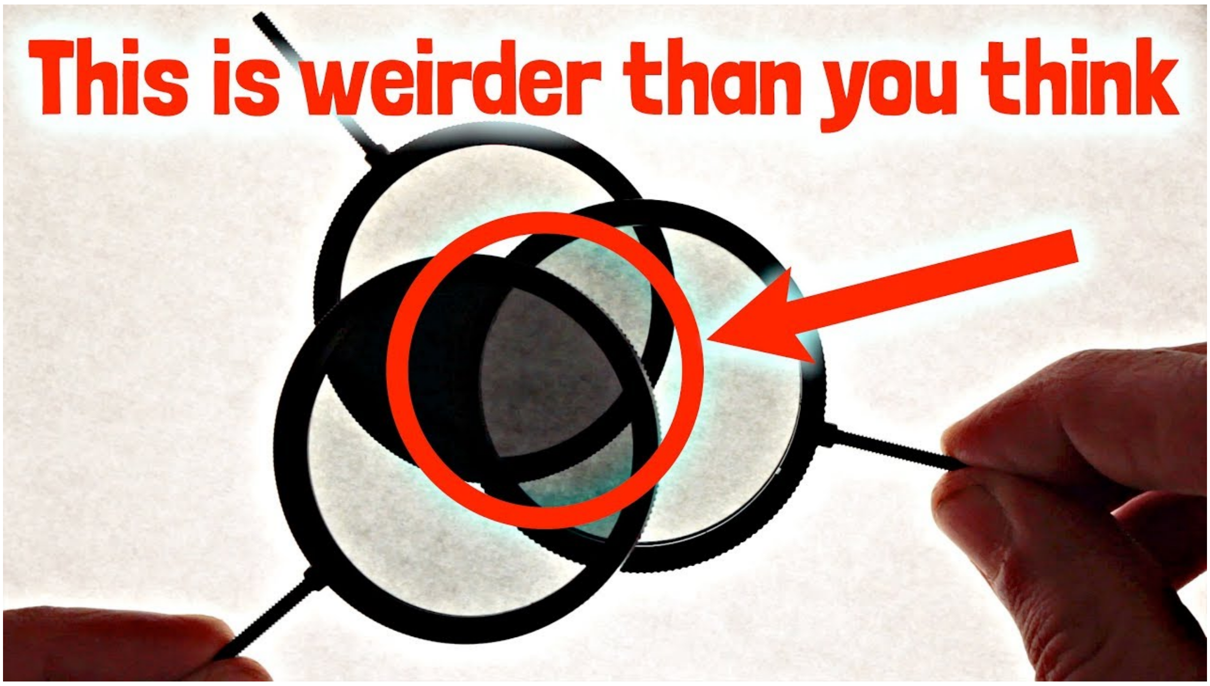
$$P(align) = 50\%$$

After this, the photons which pass through are all 100% aligned with the polariser. Should they come into contact with an analyser at an angle θ to their polarisation axis, the probability that they will change their spin axis to the new spin axis is given by:

$$P(align) = \cos^2 \theta$$

(There are some very complicated calculations that can be done to figure this out with quantum physics but eventually you get this result).

This is particularly evident with three polarisers which are at different angles to each other (see minutephysics' video on the topic):



https://www.youtube.com/watch?v=zcqZHYo7ONs

Light has Momentum?

Many teachers will mention that light has momentum and can move objects very slightly. This is due to the fact that $E = mc^2$ is only true for an object at rest.

The actual equation is:

$$E^2 = (mc^2)^2 + (pc)^2$$

Where p is the momentum.

Light has no mass, so the first term goes to zero.

$$E^2 = (pc)^2$$

$$E = pc$$

And since we know light has energy, it must have momentum. This is why light can cause small metal fans to move in vacuum chambers.

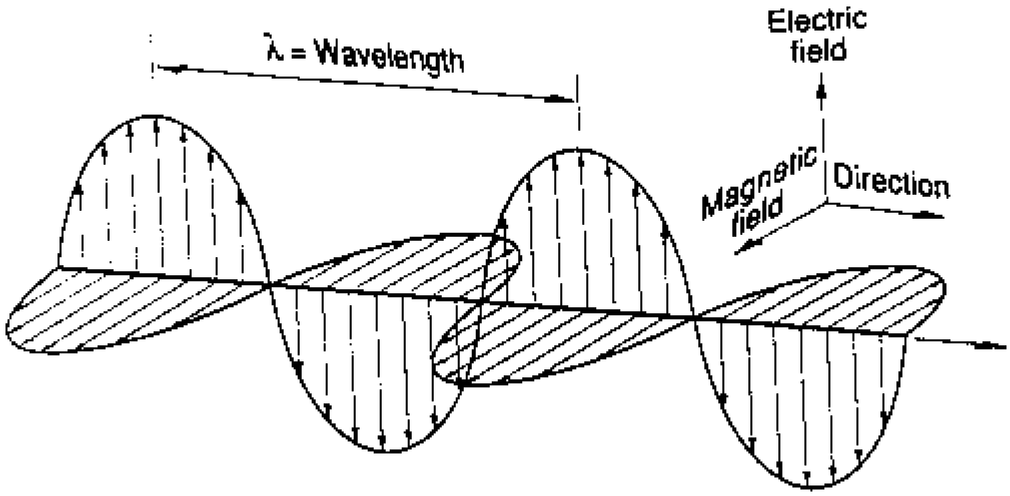
Extension Notes

Maxwell's Prediction for the Speed of Light

Though there are typically four equations known as 'Maxwell's Equations', Maxwell himself actually had around 20 equations and it was Heaviside who unified them into the four. Using his stupid number of equations, Maxwell was able to rearrange them to show that an accelerating charge (which generates a changing magnetic field) could induce a changing electric field which would induce a changing magnetic field, both of which obeyed the wave equation (where y is the amplitude in arbitrary units and x is the direction it is travelling in with speed v):

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

It was from this that he deduced the speed of light to be $c^2 = \frac{1}{\mu_0 \varepsilon_0}$



A note for sketching light waves

The rule for light is that $\hat{E} \times \hat{B} = \hat{v}$

Derivation by example of the wave equation

The simplest wave is given by $y = A\sin(2\pi f\ t)$

First let's define a few properties and variables for waves:

$$v = f\lambda \qquad\qquad k = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f$$

$$\text{If Amplitude } y,\ v = \frac{dx}{dt} \qquad\qquad \omega = vk$$

$$(y \perp v)$$

Since a wave travels at speed v, the time take to cover some distance x is given by $t = \frac{x}{v}$

First, we can re-write the wave equation:

$$y(t) = A\sin(2\pi f\ t) = A\sin(\omega t)$$

Now, we include the x factor in the time

$$y(x,t) = A\sin\left(\omega\left(t - \frac{x}{v}\right)\right)$$

$$\frac{x}{v} = \frac{x}{f\lambda} = \frac{2\pi\ x}{\omega\lambda} = \frac{kx}{\omega}$$

$$y(x,t) = A\sin(\omega t - kx)$$

$\frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx)$	$\frac{\partial y}{\partial x} = -kA\cos(\omega t - kx)$
$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx)$	$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(\omega t - kx)$

$$\frac{\left(\frac{\partial^2 y}{\partial t^2}\right)}{\left(\frac{\partial^2 y}{\partial x^2}\right)} = \frac{\omega^2}{k^2} = v^2$$

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

If instead of just x, the wave is a function of all of space, then we can re-write the equation using the Laplacian such that a wave propagating in all directions will also obey the equation.
(ψ shall replace y for the sake of simplicity)

$$\frac{\partial^2 \psi}{\partial t^2} = v^2\,\nabla^2\psi = v^2\left[\frac{\partial^2 \psi}{\partial x^2}\quad\frac{\partial^2 \psi}{\partial y^2}\quad\frac{\partial^2 \psi}{\partial z^2}\right]$$

Derivation that Light is a Wave with Speed $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \vec{0}, \qquad \rho = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \vec{\nabla} \cdot \vec{B} = \vec{0}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\frac{\partial\left(\vec{\nabla} \times \vec{B}\right)}{\partial t} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\nabla^2 \vec{E}$$

$$-\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\nabla^2 \vec{E}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \nabla^2 \vec{E}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0}\left[\frac{\partial^2 \vec{E}}{\partial x^2} \quad \frac{\partial^2 \vec{E}}{\partial y^2} \quad \frac{\partial^2 \vec{E}}{\partial z^2}\right]$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \mu_0 \varepsilon_0 \frac{\partial\left(\vec{\nabla} \times \vec{E}\right)}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{B}\right) - \nabla^2 \vec{B}$$

$$\vec{\nabla} \cdot \vec{B} = \vec{0}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = -\nabla^2 \vec{B}$$

$$-\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = -\nabla^2 \vec{B}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \nabla^2 \vec{B}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0}\left[\frac{\partial^2 \vec{B}}{\partial x^2} \quad \frac{\partial^2 \vec{B}}{\partial y^2} \quad \frac{\partial^2 \vec{B}}{\partial z^2}\right]$$

Both the $\vec{E}$ and $\vec{B}$ fields satisfy the wave equation and are therefore waves (except the trivial case where $\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{\partial^2 \vec{E}}{\partial t^2} = 0$) such that $v^2 = \frac{1}{\mu_0 \varepsilon_0}$

RELATIVITY

Base Units
Mass (m) – Kilograms (kg)
Length (l) – Metres (m)
Time (t) – Seconds (s)
Speed (v) – Metres per second (ms^{-1} **or** m/s)
Momentum ($\vec{p}$) – Kilogram metres per second ($kg\ ms^{-1}$ **or** $kg\ m/s$)

Constants
The Speed of Light $c = 3.00 \times 10^8\ (m\ s^{-1})$

Equations
Galilean / Newtonian Relativity
$\vec{v}_{A\ rel.\ to\ B} = \vec{v}_A - \vec{v}_B$

The classical velocity transform for inertial reference frames. The equation relies on the velocities $\vec{v}_A$ and $\vec{v}_B$ both being taken from a third inertial reference frame.

Einsteinian / Special Relativity
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$$

Known as the Lorentz Factor. It is used in Space-Time diagrams as part of the Lorentz Transform and appears frequently in Special Relativity. β is the fraction of the speed of light the object is travelling at (i.e. if $v = 0.8c,\ \beta = 0.8$).

$t = \gamma t_0$

The formula for time dilation. This transforms the time taken for some event in a stationary reference frame (t_0) into the time taken in a reference frame in motion with respect to the event (t).

The effect of this is that t is always larger than t_0.

$$l = \frac{l_0}{\gamma}$$

Describes the contraction of the space that a moving object inhabits. Space contracts along the axis of motion to a length of l, where the length of the stationary object is l_0.

It is therefore important to note that a moving object will observe all of space around it, including the distances between objects as shrinking.

$\vec{p} = \gamma\, m_0 \vec{v}$

The apparent momentum of an object with some velocity $\vec{v}$ with respect to another object.

$E^2 = (mc^2)^2 + (pc)^2$

$E = \gamma mc^2$

The total energy of a particle or object, where m is the rest mass, p the momentum, and c the speed of light. In a situation where there is no relative motion, p is 0 and the equation simplifies to $E = mc^2$.

Course Notes

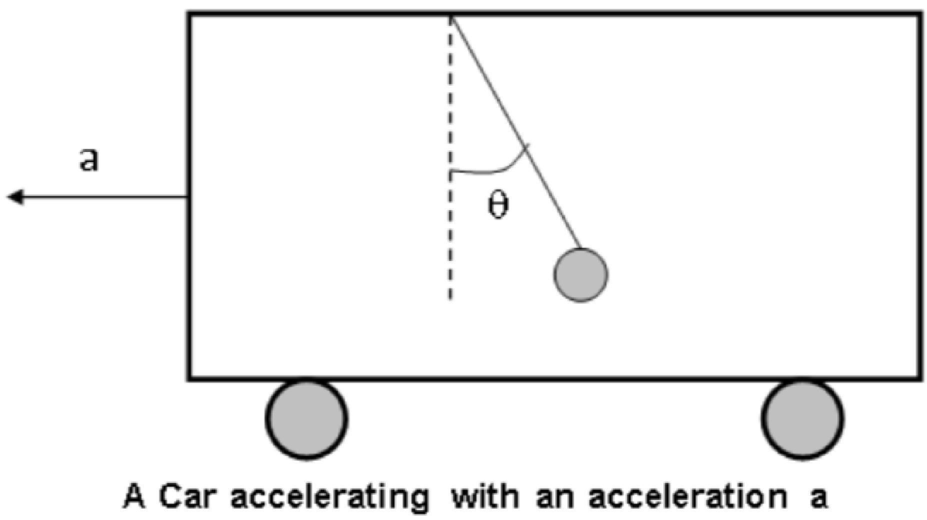
Inertial Reference Frames

An inertial reference frame is the frame of reference of any object moving at a constant speed. It is impossible to determine from within an inertial reference frame whether you are moving.

Non-Inertial Reference Frames

A non-inertial reference frame is any reference frame which is undergoing acceleration. This could be an accelerating rocket, or an object being spun in a circle. It is possible to tell whether you are in a non-inertial reference frame due to pseudoforces, for example the centrifugal force a rotating object appears to feel, or the backwards force felt within an accelerometer.

Special Relativity only applies instantaneously to a non-inertial reference frame as the velocity of the frame is constantly changing. To properly account for accelerating frames, General Relativity must be invoked.



A basic accelerometer, where from inside the ball appears to feel a backward force identical to the forwards force on the car.

Einstein's Postulates

1. That the Laws of Physics are constant in all inertial reference frames
2. That the Speed of Light is constant in all inertial reference frames (this is also a consequence of postulate 1)

1) The Laws of Physics

Although this may seem like a trivial assumption, there was also a large amount of evidence that it is impossible to tell the difference between inertial reference frames (or in other words there is no experiment that can be done in an inertial reference frame to determine whether you are moving).

Note that the Laws of Physics do change when in a non-inertial frame of reference.

2) The Speed of Light

The consequence of this is that the derivation from Maxwell's Equations that $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ remains unchanged in any reference frames (since μ_0 and ε_0 are properties of the universe) . As a result, the speed of light is constant in all inertial reference frames.

Events in Relativity

In Relativity, any event which occurs in one reference frame will occur in another reference frame (provided they are able to causally affect each other), though they may disagree about when and where they occur.

The Lorentz Factor

The Lorentz Factor, γ, shows up frequently in Special Relativity and can be used to transform space-time coordinates between inertial reference frames. The Lorentz Factor is always equal to or greater than one ($\gamma \geq 1$).

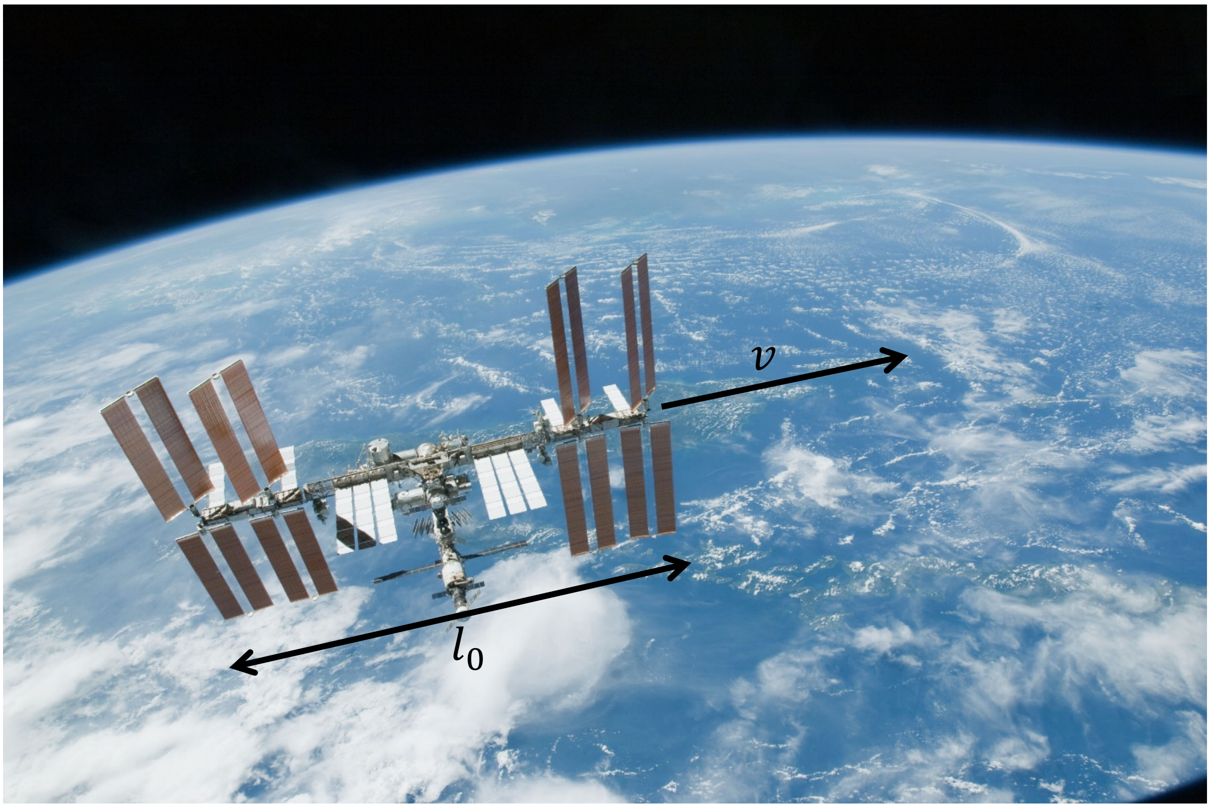
Two things to remember

Two primary intuitions should be used when considering a special relativity problem:

1. Moving objects shrink along the direction of motion
2. Moving clocks run slow

An Introduction to Special Relativity

To understand Special Relativity, first imagine a ball being thrown from one end of the international space station to the other, being viewed from within the space station and from Earth.
From within the space station the ball takes some time t_0 to travel to the other end and the station has some length l_0 along its direction of motion.



For an observer watching on Earth, they will observe the ball taking some time t to reach the end of the station where the space station is of length l.

The rest time between the two events t_0 (the ball being thrown and the ball reaching the other end) and the rest length between the two ends of the station l_0 will be affected by the relative velocity of the space station to an observer on earth[1]:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

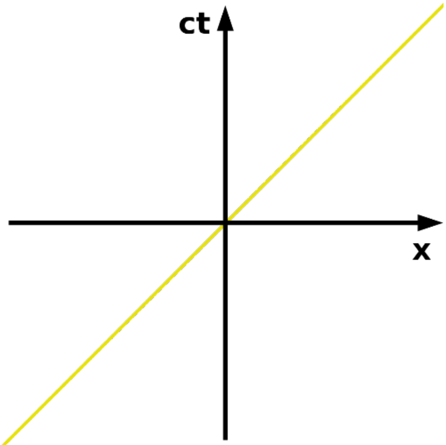
$$t = \gamma t_0$$

$$l = \frac{l_0}{\gamma}$$

[1] This example is not exactly accurate to reality as the space station is undergoing centripetal acceleration and is therefore in a non-inertial reference frame (i.e. we need general relativity to describe the situation). However, this example should otherwise provide a plausible situation where an object is moving at a relatively fast speed.

Spacetime Diagrams

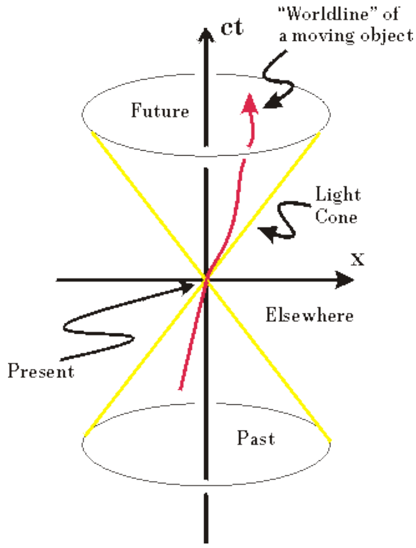
Minkowski Spacetime diagrams are common in Special Relativity as they help to explain many of the paradoxes which can appear when merely considering the equations of relativity.



In this spacetime diagram the horizontal axis, x, represents the distance along the direction of motion of an object. The vertical axis represents time but in a different manner to normal time. Instead of measuring time in seconds it is measured in light-seconds (i.e. the distance light travels in a second). This means that light is graphed as a line with gradient 1 and angle of 45° to both axes (because light travels one light-second in a second and every second that passes its time coordinate increases by one light-second).

Light Cones and Causality

Because light travels at the fastest speed in the universe, we know that the minimum time it takes for one object to interact with another is the time it takes light to reach that object. This can be represented in spacetime diagrams with a light cone. Any event in space and time within the past light cone could have had an effect on the present and anything outside it could not have had an effect on the present.
Similarly, any event within the future cone could have an effect on the future or could be affected in the future by an event in the present.



The Lorentz Transforms

For a point travelling along the x axis with speed $v = \beta c$

$$t' = \frac{\gamma}{c}(ct - \beta x)$$

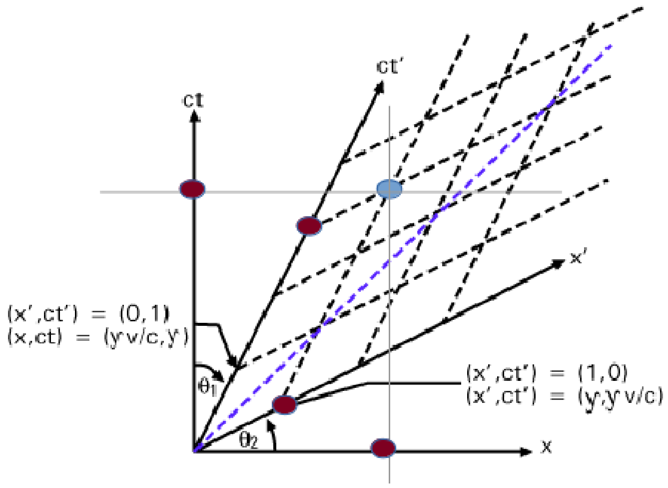
$$x' = \gamma(x - \beta ct)$$

$$y' = y$$

$$z' = z$$

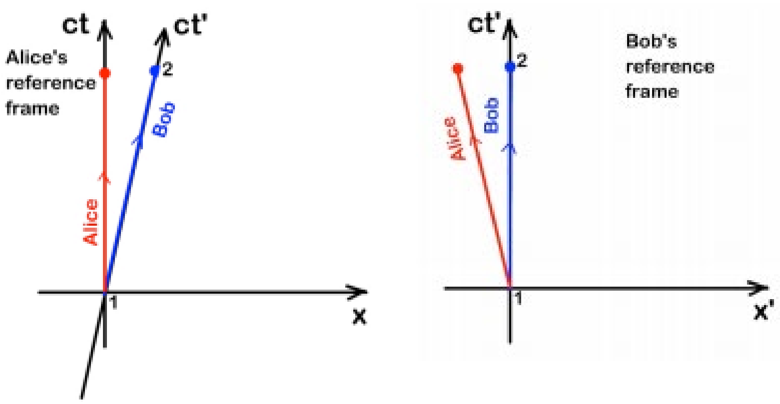
Lorentz Transforms

The Lorentz Transforms are mathematical transforms which demonstrate the apparent positions and times of events relative to different observers. They can be represented graphically on a spacetime diagram by the rotation of coordinate systems.



The above diagram represents an event occurring in spacetime (the blue dot) which for a stationary observer has coordinates (x, ct) and for a moving observer has coordinates (x', ct'). It is worth noting that just as with normal graphing, these coordinates are found by drawing lines from the time axis which are parallel to the space axis and vice versa (hence the seemingly wonky lines for the moving observers' axes).

It is possible however, to go into the moving observer's reference frame with a Lorentz Transform. Below is a Lorentz Transform from the Alice's perspective where Bob is moving, to Bob's perspective where Alice is moving.



Deriving Length Contraction from the Lorentz Transforms

An object with length l has some point x_1 and x_2 such that $l_0 = x_2 - x_1$

$(x_1, t_1) = (0,0)$ $\qquad\qquad$ $(x_2, t_2) = (l, 0)$

$$l' = x_2' - x_1'$$

$$= (\gamma x_2 - \gamma\beta ct) - (\gamma x_1 - \gamma\beta ct) = \gamma x_2 - \gamma x_1 = \gamma l_0$$

BUT

$$t_1' = \gamma t = 0$$

$$t_2' = \frac{\gamma\beta l}{c} \neq t_1'$$

So the times at which the observer is measuring the positions of these points is not the same (but when you measure length you measure the positions at the same time).
This shows that events which are simultaneous for a stationary observer are not simultaneous for a moving observer (this is kind of the whole point of relativity). Let's instead reuse the equations but setting our time to T so that the position of the points can be measure simultaneously.

$$l' = x_2' - x_1'$$

$$T' = t_2' - t_1'$$

$$(x_2' - x_1') = \gamma\big((x_2 - x_1) + v(t_2 - t_1)\big)$$

$$(t_2' - t_1') = \gamma\left((t_2 - t_1) + \frac{\beta}{c}(x_2 - x_1)\right)$$

To measure the stick ends at the same time we need $t_2' = t_1'$ so $(t_2' - t_1') = 0$

$$(t_2 - t_1) = -\frac{\beta}{c}(x_2 - x_1)$$

$$l' = (x_2' - x_1') = \gamma\big((x_2 - x_1) - \beta^2(x_2 - x_1)\big)$$

$$l' = \gamma(x_2 - x_1)(1 - \beta^2)$$

$$\frac{1}{\gamma} = \sqrt{1 - \beta^2}$$

$$l' = \frac{\gamma(x_2 - x_1)}{\gamma^2} = \frac{l}{\gamma}$$

$$\therefore l = \frac{l_0}{\gamma}$$

The Consequences of this Derivation

What this means is that when an observer sees a length contracted object the ends of the object are not actually at those respective positions (because when you actually want to measure the length of the stick you need to measure the end positions at different times). This means a person seeing a length contracted stick is not seeing the stick how it is but instead, where then ends were.

Even stranger is that the times at which each end was at each of its respective positions is different, so the front end may have been where you see it a second ago, but the back end was where you see it half a second ago. This is important for later.

See Fermilab's video on the topic if it is still unclear:

https://www.youtube.com/watch?v=-Poz_95_0RA

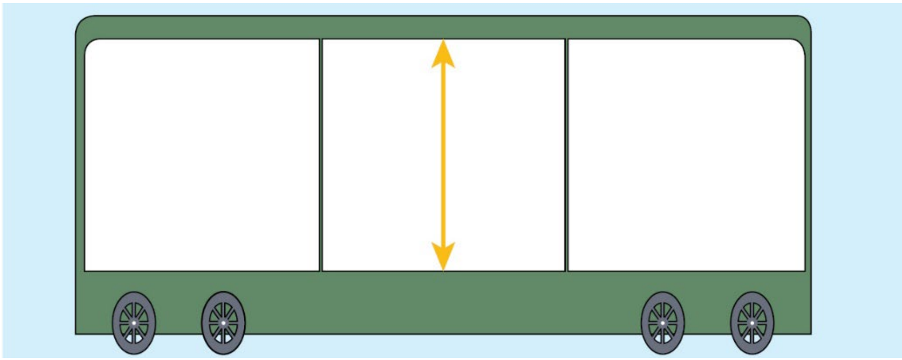
Time Dilation

'A moving clock runs slow'

In the ball on the space station scenario, the time measured for the ball to reach the end of the space station will be shorter for the observer on the space station compared to the time measured by an observer on Earth. Alternatively, the observer on Earth will observe a longer time taken for a single event compared to the stationary observer.

Why does Time Dilation Occur? – The Photon Clock

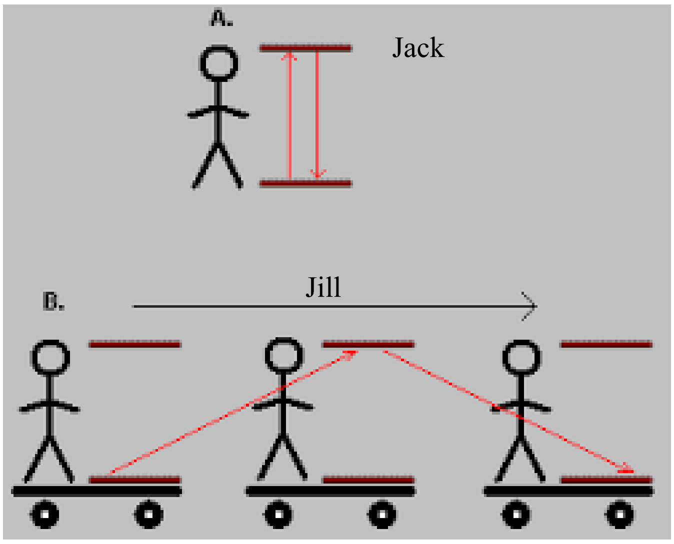
Consider a situation where Jack is onboard a train moving at some fraction of the speed of light β, and Jill is beside the tracks on a platform. Inside the train is an evacuated vertical tube with mirrors on either end (at the top and bottom). Inside the tube is a single photon which bounces back and forth between the mirrors.

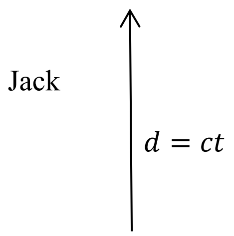


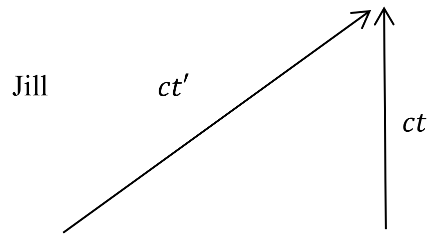
Let the distance between the two mirrors be d and the time taken for the light to travel from the bottom mirror to the top mirror be t. As a result, $c = \frac{d}{t}$

From now on the bottom mirror shall be *Mirror A* and the top mirror is *Mirror B*. Therefore: the photon leaving Mirror A is *Event A* and the photon striking Mirror B is *Event B*.

Now we consider this from Jill's perspective on the platform. While in Jack's perspective Event A and B occur at the same horizontal coordinate, in Jill's perspective they occur at different horizontal locations.

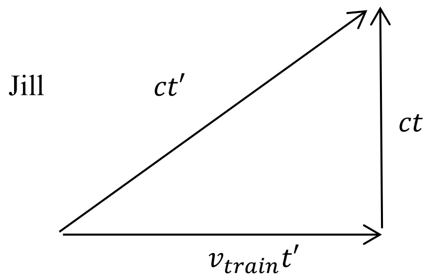






As can be seen in the diagram above, both Jack and Jill will observe events A and B and the distance the photon convers is $speed * time$. Since Jill observes the photon travelling a longer distance and c is constant in both reference frames, Jills time (t') must be longer than Jacks. In other words, Jill's clock ticks faster than Jacks.

To calculate this by completing Jill's diagram, using the speed of the train to calculate the horizontal distance. ($v_{train} = \beta c$)



$$(ct')^2 = (ct)^2 + (vt')^2$$

$$(ct')^2 - (vt')^2 = (ct)^2$$

$$c^2t^2 = (t')^2(c^2 - v^2)$$

$$t^2 = (t')^2\left(1 - \frac{v^2}{c^2}\right)$$

$$t = t'\sqrt{1 - \frac{v^2}{c^2}}$$

$$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma t$$

Formula Sheet:

$$t = \frac{t_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Conclusion: Time passes faster in a frame of reference in motion with respect to an event (i.e. the event takes less time).

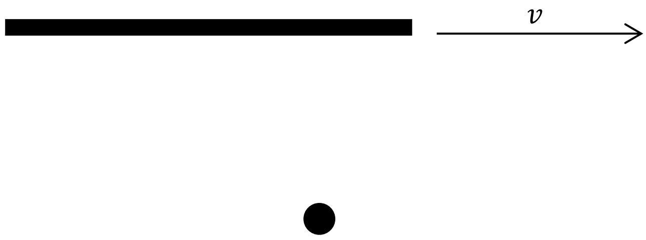
Length Contraction

In the above scenario, the observer on Earth will observe the space station to be shorter along the axis of motion. This is because when an object moves relative to an observer, the space that object inhabits appears to shrink. In essence, all points along the direction of motion within that space get closer together.

For an observer in the moving space station, all objects in the universe appear to be moving relative to it and, therefore, will length contract. The distance between the space station and an object it is moving towards will appear to shrink and the Earth will seem thinner.

Why does Length Contraction Occur?

Consider a moving 1-metre ruler with some speed and an observer watching from some distance.



When the observer measures where the ends of the ruler are, the light rays take a different amount of time to reach the observer. The difference in time between the position measurements and the apparent simultaneity of them to the observer causes length contraction. (See length contraction derivation above).

The Ladder Paradox

Setup

A farmer wants to fit his 20-metre ladder in his 10-metre shed. He knows that if his friend runs at some fraction β of the speed of light he will observe the ladder as being length contracted to 10 metres, just short enough to fit into the shed.

But for his friend, the shed appears to be moving and will shrink from 10 metres to 5 metres long, while the ladder remains at 20 metres.

Who is right?

Explanation

They are both correct and both wrong. Length contraction of the ladder occurs due to the difference in the actual times at which the ends of the ladder are at their apparent positions. If we adjust for the time delay, the observers can both agree on where each end of the ladder was but must make their measurements at different times. If we do this, it can be made clearer how the observers disagree (but that's hard so you're just going to have to believe it).

The delay is important as it is the speed of information. The speed of information is typically the speed of light but in the ladder the speed of information is the speed of sound. So, when the ladder inevitably shatters on impact, the tail end won't "know" that it needs to stop moving until the vibration (sound) has travelled to that end.

Similarly, when the farmer sees the front end of the ladder at one side of the shed, it has already moved past that end but the delay in the information leads him to believe that it is in the shed.

The Twins Paradox

Setup

Two twins say goodbye to each other as one boards a large space rocket which will travel to a star 10 lightyears away and then travel back at a reasonable fraction of the speed of light. The twin remaining on Earth notes that since moving clocks run slow, the twin on Earth will have aged much more when they reunite, as the slowed clock of the twin on the rocket will mean they have aged more slowly during the trip.

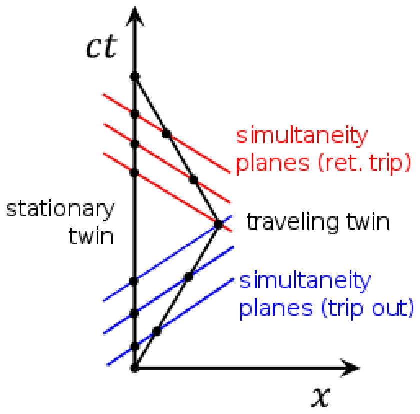
The twin going on the trip counters this by noting that to them, the twin on Earth will appear to be moving and therefore it is the twin on Earth who will age more slowly.

Who is right?

To measure this, the travelling twin agrees to send a beam of light to Earth every time a year passes.

Explanation

Ignoring any curvature due to acceleration, a spacetime diagram would look like this.

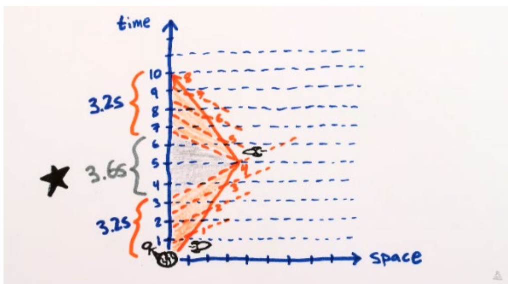


The actual explanation for the time discrepancy is, in fact, acceleration and the effect it has by changing the reference frame of the traveller.

On the first half of the trip the light pulses are angled down on the diagram because they are travelling away from the twin. On the return they are angled up because they are travelling with the twin. When the travelling twin turns around, their velocity rapidly changes and so does their perception of time. This can be seen in the gap between the light pulses being strangely large at the point of acceleration.

Notice that when the twin is travelling with a uniform speed, the gap between years of the stationary twin is smaller than for the travelling twin (i.e. time runs slowly for them), meaning that time dilation is occurring as the twin predicted. However, there is a large gap where the twin turns around and this adds a large period of time where the travelling twin observes no time passing but the stationary twin ages by many years.

For a more graphical explanation, see [minutephysics](https://www.youtube.com/watch?v=0iJZ_QGMLD0)' video on the paradox:



https://www.youtube.com/watch?v=0iJZ_QGMLD0