Proof of the binomial theorem by mathematical induction

In this section, we give an alternative proof of the binomial theorem using mathematical induction. We will need to use Pascal's identity in the form

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}, \qquad \text{for} \quad 0 < r \leq n.$$

We aim to prove that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + > b^n.$$

We first note that the result is true for $n = 1$ and $n = 2$.

Let k be a positive integer with $k \geq 2$ for which the statement is true. So

$$(a+b)^k = a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k-1}ab^{k-1} + b^k.$$

Now consider the expansion

$$\begin{aligned}
&(a+b)^{k+1} \\
&= (a+b)(a+b)^k \\
&= (a+b)\left(a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k-1}ab^{k-1} + b^k\right) \\
&= a^{k+1} + \left[1 + \binom{k}{1}\right]a^k b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 + \cdots \\
&\cdots + \left[\binom{k}{r-1} + \binom{k}{r}\right]a^{k-r+1}b^r + \cdots + \left[\binom{k}{k-1} + 1\right]ab^k + b^{k+1}.
\end{aligned}$$

From Pascal's identity, it follows that

$$(a+b)^{k+1} = a^{k+1} + \binom{k+1}{1}a^k b + \cdots + \binom{k+1}{r}a^{k-r+1}b^r + \cdots + \binom{k+1}{k}ab^k + b^{k+1}.$$

Hence the result is true for $k+1$. By induction, the result is true for all positive

integers n.