Forces in rod

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Chapter 1

1D rod system

Object of work is system of 1D rods. Object has been divided to discrete elements n1, n2 and n3. All elements are connected to each other through nodes n2, n3 and to special points through n1 and n4 (figure 1). Mathematical model of discrete system is expressed by equations of motion for nodes. As elements is 1D, nodes will be 1D as well. All system acting in global coordinate system $\{X, Y, Z\}$.

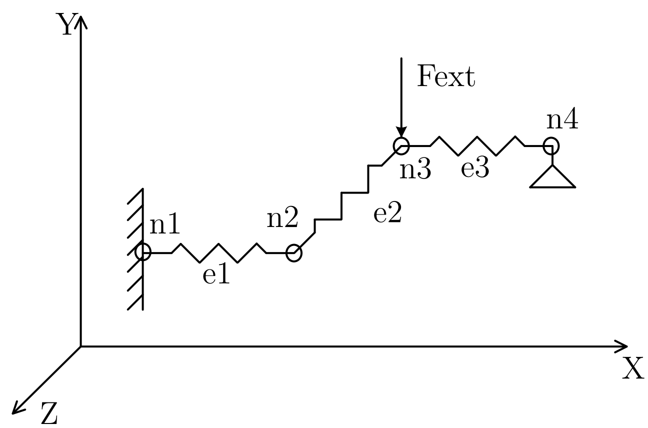

Figure 1.1: 1D Rod system in global coordinate system

The motion of nodes can be expressed by Newton's equation of motion. If only the normal component of the translational motion is considered, the equation reduces to

$$F(x) - m\ddot{x} = 0 \tag{1.1}$$

where $F(x)$ is axial force, m – mass of node and $\ddot{x}$ is acceleration, initial conditions are: $x(0) = 0$ and $\dot{x}(0) = V_0$.

Mass of each node can be calculated, like sum of half mass of each element, which acting in node. $m_n = \sum_e m_e/2$

All vector varibles of node should be calculated in global coordinate system and of element in own local coordinate system(figure 1). For trasformation between coordinate systems direction cosine matrix (DCM)(1.2) can be used.

$$DCM = \begin{bmatrix} cos(Xx) & cos(Xy) & cos(Xz) \\ cos(Yx) & cos(Yy) & cos(Yz) \\ cos(Zx) & cos(Zy) & cos(Zz) \end{bmatrix} \tag{1.2}$$

where X, Y, Z is global coordinate system and x, y, z is local coordinate system.

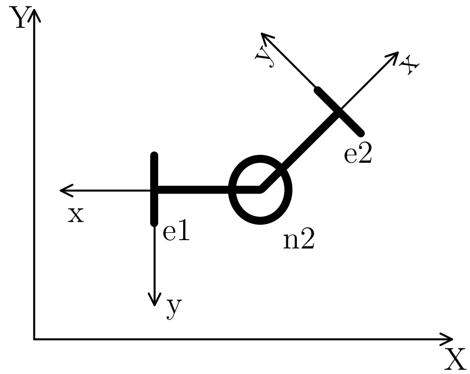

Figure 1.2: Extracting node from system

For investigating motion of system need to integrate (1.1). For this propose need to express all possible to act forces in node(1.3) for each node in relation to their application place.

$$\overrightarrow{F_n} = \sum \overrightarrow{F} = \overrightarrow{F_{ext}} + \overrightarrow{F_{elem}} \times [DCM] + \overrightarrow{F_{press}} \times [DCM] \tag{1.3}$$

1.1 Linear case

F_{ext} is external load force, applied to node in global coordinates. Value of this force for each time step is loaded from list of loads.

F_{elem} is sum of internal forces of each elemen, which acting in node. From each element becomes only half of force to node, other half going to neigourh node. In case of 1D element system, internal force of each element can be express like axial force and it is equal to integral of stress over area:

$$N(x) = \int_A \sigma dA \tag{1.4}$$

For 1D rod system F_{elem} can be expressed like:

$$F_{elem} = \sum_e N_e(x)/2 \tag{1.5}$$

F_{press} is external pressure, can be described like force applied to element in local coordinates. Value of this force for each time step is loaded from list of loads.

Element force becomes from physical deformation of element. Geometry equation(1.6) describing relation between initial length of element and lenght in Δt state, by other words its describing deformation of element over time.

$$\varepsilon = \frac{dU}{dx} = \frac{l - l_0}{l_0} \tag{1.6}$$

According to Hook law $\sigma = \varepsilon E$ and geometry equation (1.6), inner force can be changed to

$$N(x) = \int_A \varepsilon E dA = EA \int \varepsilon = \frac{EA}{l_0} * (l - l_0) \tag{1.7}$$

where l is current length of element, l_0 length of element at $t = 0$, E – Young's modulus for element material. To be able to integrate equation of motion, need to express deformation in equation (1.7) by differenses between displacements of nodes, to which element is connected:

$$N(x) = \frac{EA}{l_0} * (U_i - U_j) \tag{1.8}$$

Equations of motion for integrating can be described like:

$$\ddot{U}(\Delta t) = Fn/m \tag{1.9}$$

$$\dot{U}(\Delta t) = \dot{U}(t) + \ddot{U}(\Delta t)\Delta t \tag{1.10}$$

$$U(\Delta t) = U(t) + \dot{U}(\Delta t)\Delta t \tag{1.11}$$

1.2 Nonlinear case

As in liner case there would act same list of forces : external load force F_{ext}, internal forces of each element F_{elem}, and external pressure F_{press} as well. In nonlinear case all of these forces could get nonlinear changing over time. F_{ext} and F_{press} is loaded from list of loads and their nonlinearity will be predicted out of this work. Equation of F_{elem} would change to nonlinear form:

$$N(x) = \int\limits_{t} \int\limits_{A} \sigma dA dt \tag{1.12}$$

In this case cross sectional area and stiffness coefficient get nonlinearity.

1.2.1 Nonlinear cross sectional area

1.2.2 Nonlinear stiffness coefficient

Geometry equation stays same as in linear case, but Hook Law doesnt work anymore. Mooney-Rivlin model would replace it in nonlinear case.

Mooney-Rivlin models are popular for modeling the large strain nonlinear behavior of incompressible materials. According to thermodynamics laws, the 2nd Piola-Kirchhoff stress is the partial derivative of the Helmholtz free energy with respect to the elastic part of the Green strain tensor (with a density thrown in).

$$\boldsymbol{\sigma}^{\mathrm{PK2}} = \rho_o \frac{\partial \Psi}{\partial \mathbf{E}^{\mathrm{el}}}$$

The Helmholtz free energy contains thermal energy and mechanical strain energy. But in most every discussion of Mooney-Rivlin coefficients, the thermal part is neglected, leaving only the mechanical part, W. (Actually, W is declared to represent $\rho_o\Psi$, not just Ψ). Second, since all of the deformation of a hyperelastic material is elastic by definition, it is sufficient to write $\mathbf{E}^{\mathrm{el}}$ simply as $\mathbf{E}$. This gives

$$\boldsymbol{\sigma}^{\mathrm{PK2}} = \frac{\partial W}{\partial \mathbf{E}}$$

But there is a challenge with this general approach. It is the determination of off-diagonal (shear) terms. As with the shear terms in Hooke's Law, they are not independent of the normal terms, but must be consistent with coordinate transformations that transform normal components into shears and vice-versa. And as with Hooke's Law, the resolution is to define the material behavior for the principal values and rely on coordinate transformations to give the appropriate corresponding behavior of the shear terms.

$$\sigma_i^{\mathrm{PK2}} = \frac{\partial W}{\partial E_i}$$

But alas, even this is not quite what is done in Mooney-Rivlin models. Instead, derivatives are taken with respect to ¡i¿stretch ratios¡/i¿, λ, which are the ratios of initial and final lengths in the principal directions, (L_F/L_o). So the stretch ratio is "one plus engineering strain," $\lambda = 1 + \epsilon_{Eng}$, and therefore $\lambda - 1 = \epsilon_{Eng} = \Delta L\,/L_o$.

According to Mooney-Rivlin model and geometry equation (1.6), inner force can be changed to

$$N(x) = \int\limits_A \varepsilon E dA = EA \int \varepsilon = \frac{EA}{l_0} * (l - l_0) \tag{1.13}$$

where l is current length of element, l_0 length of element at $t = 0$, $E -$ Young's modulus for element material. To be able to integrate equation of motion, need to express deformation in equation (1.13) by differenses between displacements of nodes, to which element is connected:

$$N(x) = \frac{EA}{l_0} * (U_i - U_j) \tag{1.14}$$

Equations of motion for integrating (1.9), (1.10),(1.11) would be same as in linear case Picault et al. [1].

List of literature

[1] E. Picault et al. "A planar rod model with flexible cross-section for the folding and the dynamic deployment of tape springs: Improvements and comparisons with experiments". In: *International Journal of Solids and Structures* (2014). ISSN: 00207683. DOI: 10.1016/j.ijsolstr.2014.05.020.