Biological Tissue Movement

Oleksandr Hubanov
Vilnius Gediminas Technical University

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Mathematical model

The motion of chordae as motion of physical object can be expressed by Newton's equation of motion:

$$m\ddot{\vec{x}} + k\vec{x} + c\dot{\vec{x}} = \vec{F} \tag{1}$$

where F is internal force of object, m – mass of object and $\ddot{x}$ is acceleration, k – stiffness coefficient of object, x is displacement of object and c – damping coefficient of object, $\dot{x}$ is velosity of object.

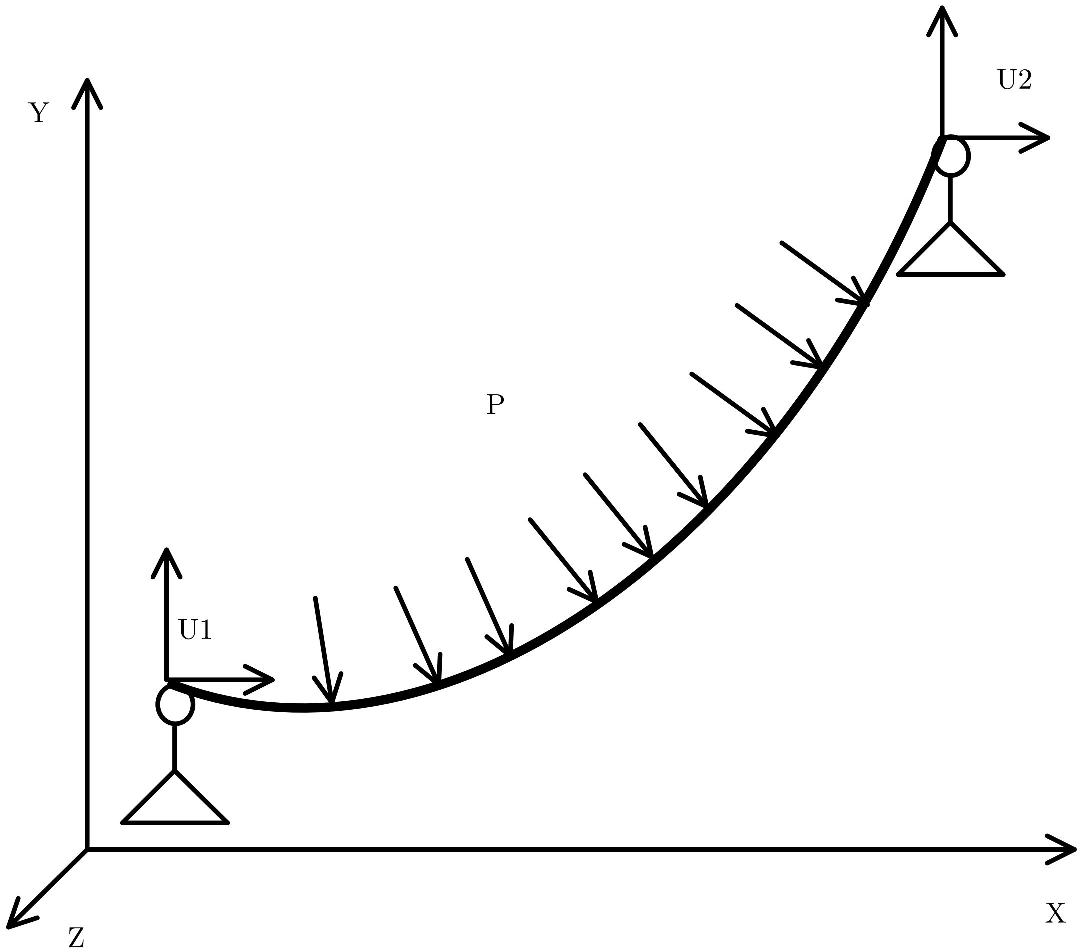

Figure 1: Chordae tindae in global coordinate system

Initial conditions are: $x(0) = 0$ and $\dot{x}(0) = V_0$. Acceleration could be known from (1) as:

$$\ddot{\vec{x}} = \vec{F}/m \tag{2}$$

Than Euler's scheme of integration can be described like:

$$\vec{x}(t + \Delta t) = \vec{x}(t) + \dot{\vec{x}}(t)\Delta t \tag{3}$$

$$\vec{x}(t+\Delta t) = \vec{x}(t) + \vec{\dot{x}}(t)\Delta t \tag{4}$$

Linear deformation

Internal object force becomes from physical deformation of object it self. In linear case of study, deformation of element much less compare to element dimensions. It is expressed by linear geometry equation(5), which showing relation between initial length of element and length in Δt state.

$$\varepsilon = \frac{dU}{dx} = \frac{l(\Delta t) - l(t)}{l(t)} \tag{5}$$

According to Hook law $\sigma = \varepsilon E$ and linear geometry equation (5), inner force can be described as:

$$\begin{aligned} N(x) &= \int\limits_{Al} \varepsilon E dl dA = EA \int\limits_{l} \varepsilon dl \\ &= \frac{EA}{l(t)} * (l(t+\Delta t) - l(t)) \end{aligned} \tag{6}$$

where $l(\Delta t)$ is current length of chordae, $l(t)$ length of chordae at previuous time moment, E – Young's modulus for chordae, A – volume of chordae. To be able to integrate equation of motion, need to express deformation in equation (6) by differences between displacements of nodes, to which chordae is connected:

$$N(x) = \frac{EA}{l(t)} * (x_i(t) - x_j(t)) \tag{7}$$

Nonlinear deformation

Nonlinearity in main mean that chordae can get huge deformation compare to its demesions. Equation of F_{elem} in this case would change to nonlinear form:

$$N(x) = \int\limits_{t} \int\limits_{A} \sigma dA dt \tag{8}$$

From this equantion comes that cross sectional area and stiffness coefficient will get nonlinearity.

Changing of cross sectional area over time of chordae is changing its length over time. Linear geometry equation(5), showing linear relations between length, because difference in Δt state takes according initial length of chordae. In case of huge deformation need to recalculate length of chordae on each time step and take difference of displacement according to previous time step. In end of geometry equation become to nonlinear form:

$$\varepsilon = \frac{dU}{dx} = \frac{l(\Delta t) - l(t-\Delta t)}{l(\Delta t)} \tag{9}$$

Inner force(6) also become to nonlinear form:

$$\begin{aligned} N(x) &= \int\limits_{t} \int\limits_{A} \varepsilon E dA = EA \int\limits_{t} \varepsilon \\ &= \frac{EA}{l(\Delta t)} * (l(\Delta t) - l(t-\Delta t)) \end{aligned} \tag{10}$$

where l is current length of element, l_0 length of element at $t = 0$, E – Young's modulus for element material.

And nonlinear equantion of inner force for integration:

$$N(x) = \frac{EA}{l_0} * (U_i - U_j) \tag{11}$$

Discrete model

Investigating motion of any mechanical system comes to integration motion equation of discrete parts of this system. Chordae tindae presents itself as fibrous tissue and could be described as system of 1D rods, in case of getting mechanical equivalent of system. Example of such flat 2D system are shown on figure 3.

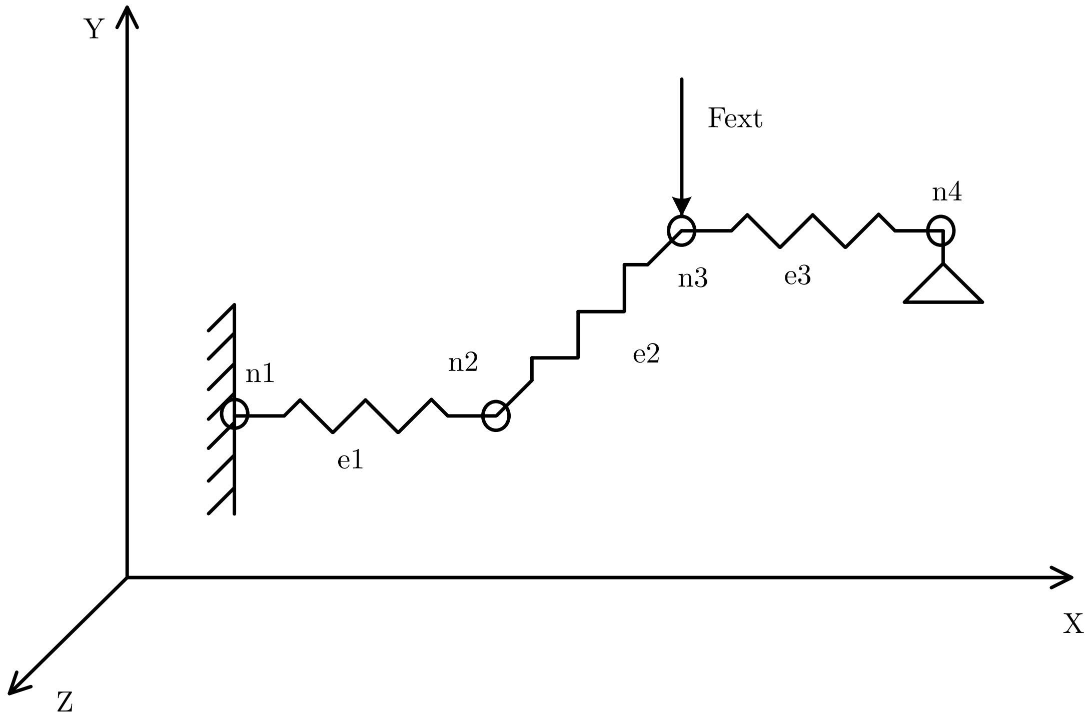

Figure 2: 1D Rod system in global coordinate system

System consists of discrete elements e_1, e_2 and e_3. All elements are connected to each other through nodes n_2, n_3 and to special points through n_1 and n_4. Each element e_n of system has own orientation in global coordinate system. From schematic representation of node comes that all vector variables of node should be calculated in global coordinate system and element variables in own local coordinate system(figure 3). For transformation between coordinate systems direction cosine matrix (DCM)(12) can be used.

$$DCM = \begin{bmatrix} cos(X,x) & cos(X,y) & cos(X,z) \\ cos(Y,x) & cos(Y,y) & cos(Y,z) \\ cos(Z,x) & cos(Z,y) & cos(Z,z) \end{bmatrix} \tag{12}$$

where $\{X, Y, Z\}$ is global coordinate system and $\{x, y, z\}$ is local coordinate system.

According to primitive scheme of node 3, mass of each node can be calculated, like sum of half mass of each element, which acting in node. $m_n = \sum_e m_e/2$

In case that node does not have external interrupt, like pressure or other applied force, schematic represent of node can be as on figure 3.

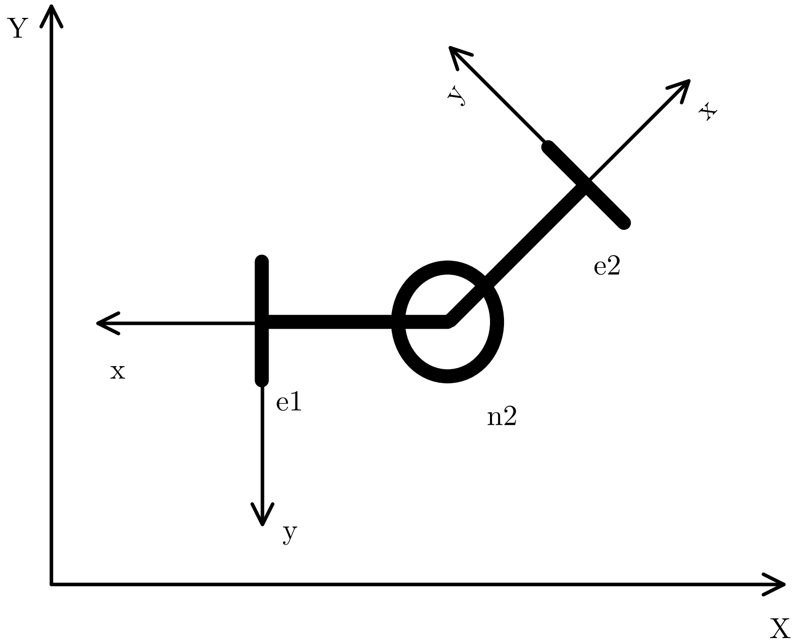

Figure 3: Extracted node from system

Mathematical model of discrete system is expressed by equations of nodes motion. All system acting in global coordinate system $\{X, Y, Z\}$ and each element acting in own local coordinate system $\{x, y, z\}$.

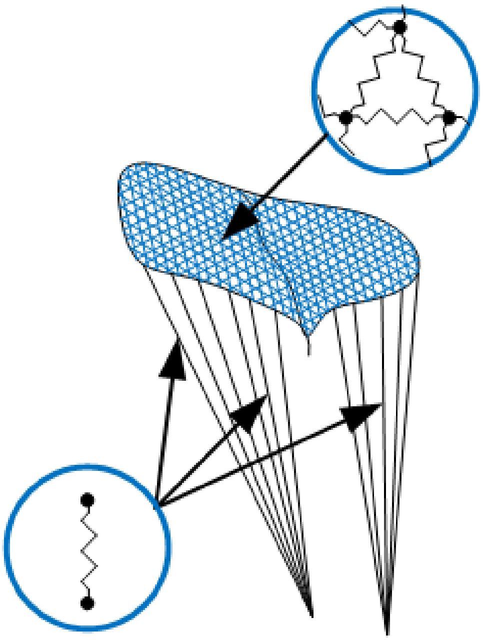

Figure 4: Displacement of papillary muscle over cardiac cycle

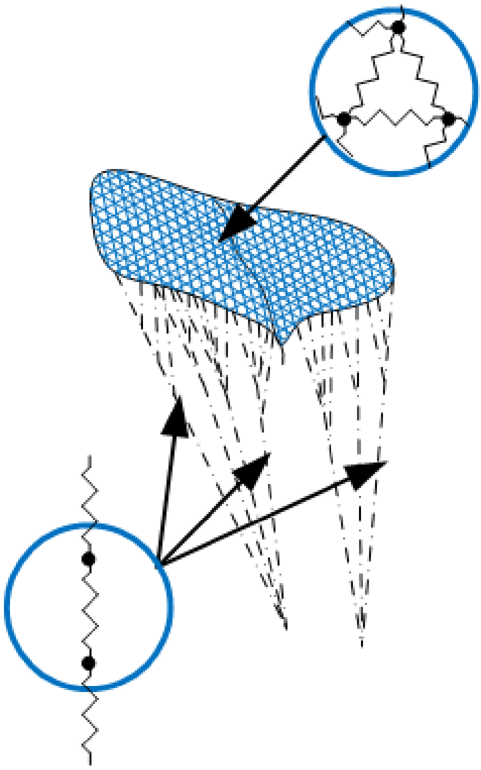

Figure 5: Displacement of papillary muscle over cardiac cycle

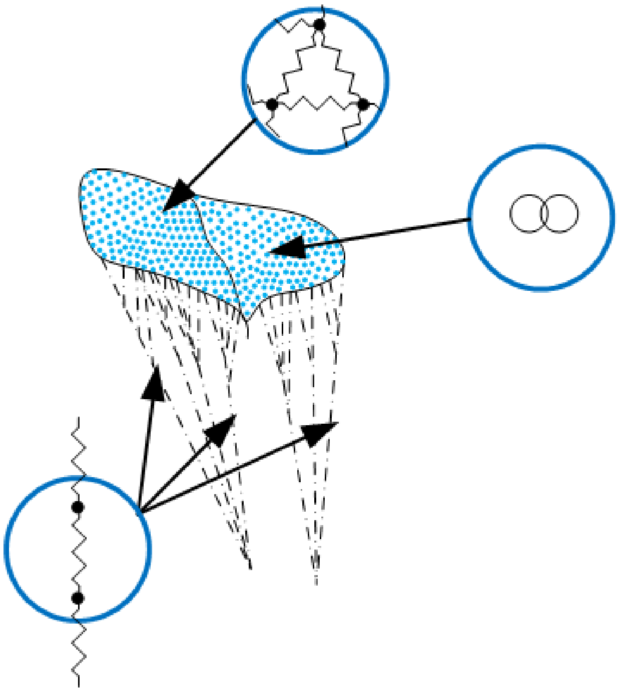

Figure 6: Displacement of papillary muscle over cardiac cycle

research aim is to compare 1 rod and rope of rods

If express kx and $c\dot{x}$ by $\sum F_{elem}$, then motion equantion will become to:

$$\vec{F}_{inertia} = \vec{F}_{ext} - \sum \vec{F}_{elem} \tag{13}$$

F_{ext} is external load force, applied to node in global coordinates. Value of this force for each time step is loaded from list of loads. F_{elem} is sum of internal forces of each element, which acting in node. From each element counts only half of force to node, other half going to neighbour node. For 1D rod system F_{elem} can be expressed like:

$$F_{elem} = \sum_{e} N_e(x) \tag{14}$$

To words about mechanical properties of valve components, poisson ratio ν is same, 0.49 for all. Mean value of chordae length L $0,025m$ and diameter d $0,001m$. Chordae has density $\rho = 1040$ kg/m3 and stiffness $E = 2000N/m$, when leaflets has $\rho = 1.06$E3 kg/m3 and $E = 2MPa$ (Anterior leaflet), $E = 1MPa$ (Posterior leaflet)