ME 513 HMWK 1

Alex Hagen

9/11/15

1.2.1 Given two springs of stiffness s and two bodies of mass M, find the natural frequencies of the systems sketched below.

(a) Two parallel springs First, we must perform a force balance on the mass. In the upward direction, the force is applied by the two springs, giving us

$$F_{up} = 2sx$$

and there is no force in the downward direction. Applying downward as the positive direction, and also applying the equastion of motion in one cartesian dimension, we get

$$ma = m\ddot{x} = -2sx$$

or

$$\ddot{x} + \frac{2s}{m}x = 0$$

The general solution to this differential equation is

$$x(t) = A\sin\left(\sqrt{\frac{2s}{m}}t\right) + B\cos\left(\sqrt{\frac{2s}{m}}t\right)$$

which shows us that the natural frequency is

$$\omega_0 = \frac{2s}{m}$$

(b) Double mass The force balance is again performed to start, but this time there is only one spring, giving us

$$F_{up} = sx$$

and there is no force in the downward direction. Again, down is regarded as positive and the equation of motion is applied, giving us

$$2ma = 2m\ddot{x} = -sx$$
$$\ddot{x} + \frac{s}{2m}x = 0$$

with a general solution of

$$x(t) = A\sin\left(\sqrt{\frac{s}{2m}}t\right) + B\cos\left(\sqrt{\frac{s}{2m}}t\right)$$

and the natural frequency

$$\omega_0 = \frac{s}{2m}$$

(c) Two series springs For this problem, we must perform two mass balances, the first on the point between the springs

$$F_{up1} = sx_1 - s\left(x_2 - x_1\right)$$

and now at the mass

$$F_{up} = s\left(x_2 - x_1\right)$$

and now we can apply the equation of motion and set down as the positive direction, getting

$$m\ddot{x}_2 = -sx_1 + s\left(x_2 - x_1\right) - s\left(x_2 - x_1\right) = -sx_1$$

and we can apply that as

$$m\ddot{x}_2 + sx_1 = 0$$

but now we have two variables, so we must convert into all x_2. Knowing the force balance from the top point is equal to 0, we have

$$\begin{aligned} s\left(x_2 - x_1\right) &= sx_1 \\ sx_2 &= 2sx_1 \\ x_2 &= 2x_1 \end{aligned}$$

giving us

$$m\ddot{x}_2 + s\frac{x_2}{2} = 0$$

and therefore

$$\ddot{x}_2 + \frac{s}{2m}x_2 = 0$$

with a general solution

$$x\left(t\right) = A\sin\left(\sqrt{\frac{s}{2m}}t\right) + B\cos\left(\sqrt{\frac{s}{2m}}t\right)$$

which shows the natural frequency as

$$\omega_0 = \frac{s}{2m}$$

(d) Mass between two springs For this problem, the mass balance is similar to the first, with

$$F_{up} = \underbrace{sx}_{\substack{\text{spring force}\\ \text{from top spring}}} - \underbrace{sx}_{\substack{\text{spring force}\\ \text{from bottom}\\ \text{spring}}} = 0$$

And because there is no net force on the mass (because we are neglecting gravity), there is no motion.

1.3.2 A simple oscillator whose natural frequency is $5\,{}^{rad}\!/\!{}_{s}$ is dispaced a distance $0.03\,\mathrm{m}$ from its equilibrium position and released. Find

(a) the initial acceleration, Using the equation of motion to set up the general solution, we get

$$x = x_0\cos\omega_0 t + \frac{u_0}{\omega_0}\sin\omega_0 t$$

Because the problem statement says "release", we assume that there is no initial velocity, i.e. $u_0 = 0$, therefore

$$x = x_0\cos\omega_0 t$$

with $\omega_0 = 5\,{}^{rad}\!/\!{}_{s}$ and $x_0 = 0.03\,\mathrm{m}$. To determine acceleration, we must take two derivatives and calculate the initial acceleration

$$a = \frac{d^2x}{dt^2} = -x_0\omega_0^2\cos\omega_0 t = -\left(0.03\,\mathrm{m}\right)\left(5\,{}^{rad}\!/\!{}_{s}\right)^2\cancelto{1}{\cos\left(5\,{}^{rad}\!/\!{}_{s}\right)\left(0.0\,\mathrm{s}\right)} = 0.75\,{}^{m}\!/\!{}_{s^2}$$

(b) the amplitude of the resulting motion, and The amplitude of the resulting motion is simply the coefficient of the cosine term, which is

$$|x| = |x_0 \cos \omega_0 t| = x_0 = 0.03\,\mathrm{m}$$

This intuitively makes sense as the string cannot stretch past the point that it was initially stretched (or it would violate energy conservation).

(c) the maximum speed obtained. To find the maximum speed, we can take the derivative of the equation of motion to find velocity, and take the derivative of that (acceleration, taken above) and set it equal to zero, finally evaluating for velocity at the value of t which makes a go to zero

$$v = \frac{dx}{dt} = -x_0 \omega_0 \sin \omega_0 t$$

$$\begin{aligned} a\left(t\right) &= 0 \\ -x_0 \omega_0^2 \cos \omega_0 t &= 0 \\ \omega_0 t &= \frac{n\pi}{2} \quad n = 1, ... \\ t &= \frac{\pi}{2\omega_0} \end{aligned}$$

$$v_{max} = v\left(\frac{\pi}{2\omega_0}\right) = -\left(0.03\,\mathrm{m}\right)\left(5\,{}^{rad}\!/\!{}_{\mathrm{s}}\right)\sin \cancel{\left(5\,{}^{rad}\!/\!{}_{\mathrm{s}}\right)}\left(\frac{\pi}{2\cancel{\left(5\,{}^{rad}\!/\!{}_{\mathrm{s}}\right)}}\right) = -0.15\,{}^{m}\!/\!{}_{\mathrm{s}}$$

1.5.2 Find the real part, magnitude, and phase of

(a) $\sqrt{x + jy}$ In order to calculate the square root of $\sqrt{x+jy}$, we will use the formulas

$$\gamma = \sqrt{\frac{a + \sqrt{a^2+b^2}}{2}}$$

$$\delta = \mathrm{sgn}\left(b\right)\sqrt{\frac{-a + \sqrt{a^2+b^2}}{2}}$$

where

$$\sqrt{a+jb} = \pm\left(\gamma + \delta i\right)$$

Using the argument of the square root $a + jb = x + jw$, we get that

$$\sqrt{x+jy} = \pm\left(\sqrt{\frac{x + \sqrt{x^2+y^2}}{2}} + j\sqrt{\frac{-x + \sqrt{x^2+y^2}}{2}}\right)$$

with the real part being

$$\mathrm{Re}\left\{\sqrt{x+jy}\right\} = \sqrt{\frac{x + \sqrt{x^2+y^2}}{2}}$$

the magnitude

$$\left|\sqrt{x+jy}\right| = \sqrt{\frac{x + \sqrt{x^2+y^2}}{2} + \frac{-x + \sqrt{x^2+y^2}}{2}} = \sqrt[4]{x^2+y^2}$$

and the phase

$$\begin{aligned} \arg\left(\sqrt{x+jy}\right) = \arctan\left(\frac{b}{a}\right) &= \arctan\left(\frac{\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}}{\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}}\right) \\ &= \arctan\left(\sqrt{\frac{-x+\sqrt{x^2+y^2}}{x+\sqrt{x^2+y^2}}}\right) \end{aligned}$$

(b) $A\exp\left[j\left(\omega t+\phi\right)\right]$ **and**

$$\begin{aligned}
\operatorname{Re}\left\{A\exp\left[j\left(\omega t+\phi\right)\right]\right\} &= A\cos\left(\omega t+\phi\right) \\
\left|A\exp\left[j\left(\omega t+\phi\right)\right]\right| &= A \\
\arg\left(A\exp\left[j\left(\omega t+\phi\right)\right]\right) &= 2\pi\left\lceil -\frac{\phi+\omega t-\pi}{2\pi}\right\rceil+\omega t+\phi
\end{aligned}$$

(c) $\left[1+\exp\left(-2j\phi\right)\right]\exp\left(j\phi\right)$**.**

$$\begin{aligned}
\operatorname{Re}\left\{\left[1+\exp\left(-2j\phi\right)\right]\exp\left(j\phi\right)\right\} &= 2\cos\left(\phi\right) \\
\left|\left[1+\exp\left(-2j\phi\right)\right]\exp\left(j\phi\right)\right| &= 2\left|\cos\left(\phi\right)\right| \\
\arg\left(\left[1+\exp\left(-2j\phi\right)\right]\exp\left(j\phi\right)\right) &= \arg\left[\cos\left(\phi\right)\right]
\end{aligned}$$

1.6.1 A mass of $0.5\,\mathrm{kg}$ hangs on a spring. When an additional mass of $0.2\,\mathrm{kg}$ is attached to the spring, the spring stretches an additional $0.04\,\mathrm{m}$. When the $0.2\,\mathrm{kg}$ mass is abruptly removed, the amplitude of the ensuing oscillations of the $0.5\,\mathrm{kg}$ mass is observed to decrease to $1/e$ of its initial value in $1.0\,\mathrm{s}$. Compute values for R_m, ω_d, A, and ϕ. Using the general equation

$$x = A\exp\left[-\beta t\right]\cos\left(\omega_d t+\phi\right)$$

where

$$\begin{aligned}
\beta &= \frac{R_m}{2m} \\
\omega_d &= \sqrt{\omega_0^2-\beta^2} \\
\omega_0 &= \sqrt{\frac{s}{m}}
\end{aligned}$$

giving us

$$x = A\exp\left[-\frac{R_m}{2m}t\right]\cos\left(\sqrt{\frac{s}{m}-\frac{R_m^2}{4m^2}}t+\phi\right)$$

and we can then apply boundary conditions, of which we need several. The first is that we know that after $1.0\,\mathrm{s}$, the oscillation amplitude will decrease to $1/e$, so

$$\begin{aligned}
\left|x\left(1.0\,\mathrm{s}\right)\right| &= \frac{1}{e}\left|x\left(0.0\,\mathrm{s}\right)\right| \\
\cancel{A}\exp\left[-\frac{R_m}{2m}\right] &= \frac{1}{e}\cancel{A}\cancelto{1}{\exp\left[0\right]} \\
\exp\left[\frac{-R_m}{2m}\right] &= \frac{1}{e} \\
\frac{R_m}{2m} &= 1 \\
R_m &= 2m \\
R_m &= 2\left(0.5\,\mathrm{kg}\right) = 1.0\,\mathrm{kg}
\end{aligned}$$

The second is that we know that with the additional mass, the spring stretches an additional $0.04\,\mathrm{m}$, so using a two force balances, we can get the spring constant

$$\begin{aligned}
m_1 g - s x_1 &= m_2 g - s x_2 \\
s\left(x_2-x_1\right) &= g\left(m_2-m_1\right) \\
s &= g\frac{m_2-m_1}{x_2-x_1} \\
s &= \left(9.81\,\mathrm{m/s^2}\right)\frac{0.2\,\mathrm{kg}}{0.04\,\mathrm{m}} = 49.05\,\mathrm{kg/s^2}
\end{aligned}$$

Now that we know s, m, and R_m, we can determine ω_d

$$\omega_d = \sqrt{\frac{s}{m} - \frac{R_m^2}{4m^2}} = \sqrt{\frac{49.05\,{}^{kg}/_{s^2}}{0.5\,\mathrm{kg}} - \frac{(1.0\,\mathrm{kg})^2}{4\,(0.5\,\mathrm{kg})^2}} = 9.85\,{}^{rad}/_{s}$$

And now we can attempt to solve for A and ϕ. Knowing that at time 0, the displacement is 0 by our coordinate system, that sets the phase equal to zero (since setting the amplitude A to zero would provide the trivial solution).

$$\phi = 0$$

Now, we must apply the initial condition for A and since we don't know any absolute displacements, only relative displacements, we cannot compute A.