PHYS 556 HMWK 3

Alex Hagen

9/23/15

[Q1] Shown above is the cross-section for absorption of photons by protons and deuterons as a function of photon energy up to $1300\,\mathrm{MeV}$. The peaks are due to photons being absorbed to produce an excited state when the photon energy matches the excitation energy of the state. The subtraction of the proton cross-section from the deuteron cross-section yields the neutron cross section.

[a] Find the energy associated with the 1st and 2nd resonance peaks. See answer to part [b].

[b] Estimate the energy widths of 1st and 2nd resonance peaks and estimate the life times of these resonances using the uncertainty principle

$$\Delta E \Delta t \sim \hbar$$

The peak energies and energy widths were calculated graphically, and are shown in Table 1. Then the lifetime was simply calculated using the uncertainty principle above, in the form:

$$\Delta t \sim \frac{\hbar}{\Delta E}$$

Table 1: Peak energy, energy width, and lifetimes for proton and deuteron photon cross sections [1].

Particle	**Resonance**	**Peak Energy**	**Energy Width**	**Lifetime**
p	**1st**	$314\,\mathrm{MeV}$	$136\,\mathrm{MeV}$	$4.8 \times 10^{-24}\,\mathrm{s}$
p	**2nd**	$719\,\mathrm{MeV}$	$196\,\mathrm{MeV}$	$3.4 \times 10^{-24}\,\mathrm{s}$
d	**1st**	$332\,\mathrm{MeV}$	$190\,\mathrm{MeV}$	$3.5 \times 10^{-24}\,\mathrm{s}$
d	**2nd**	$719\,\mathrm{MeV}$	$180\,\mathrm{MeV}$	$3.7 \times 10^{-24}\,\mathrm{s}$

[c] Since these resonances are the absorption of a gamma ray by a nucleon estimate some of their nuclear quantum numbers. Give your reasoning behind your QM selections. The two reactions we are interested in are

$$p + \gamma \rightarrow \text{hadron}$$

and

$$d + \gamma \rightarrow \text{hadron}$$

Table 2: Nuclear quantum numbers estimated from gamma ray absorption resonances

Resonance	**1st Resonance**	**2nd Resonance**	**Comment**
Mass	$1252\,{}^{\mathrm{MeV}}\!/_{c^2}$	$1657\,{}^{\mathrm{MeV}}\!/_{c^2}$	mass given is equal to the energy center of the resonance plus the rest mass of the proton
Baryon #	1	2	All nucleons have to be baryons, thus their numbers are 1, and this must be conserved
Spin	$^3\!/_2$	2	conservation of spin with $p = {}^1\!/_2$, $d = 1$, and $\gamma = 1$
Isospin	$\lvert{}^1\!/_2, {}^1\!/_2\rangle$	$\lvert 1, 0\rangle$	The hadron created cannot have $I = 0$, because $I_3 = 0$ for both the d and the γ, thus the isospin selection rule prohibits $\Delta I = 0$
Parity	$+$	-	Applying the parity operator shows that there is no change for the proton $\mathscr{P}(p)$.
Related Charge States	–	–	–

[d] To check your ideas go to the Particle Data Group (PDG) web site http://pdg.lbl.gov/2014/tables/contents_tables_baryons.html and print out a summary: All Baryons. From the table make your best guess at the resonances being observed. Then look up in detail their measured properties. Found a state

$$\Delta\,(1232)$$

which is close to the 1252 MeV resonance and has a spin-parity of $^3\!/_2^+$ with a similar width as 120 MeV. Also found a state

$$N\,(1650)$$

which is close to the 1657 MeV resonance and has a spin-parity of $^1\!/_2^-$ with similar width at 170 MeV.

[Q2] Starting from Coulomb's Law show that the potential function for a uniformly charged sphere is given by

$$V\,(r) = -\frac{Ze^2}{4\pi\epsilon_0 R}\left(\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right)$$

Where R is the charge radius of the sphere. Find the difference between this potential function and a point like nucleus on the energy of an electron in the ground state electron wave function from the hydrogen atom. Coulomb's Law gives us

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

To determine the potential, we must integrate over a spherical volume element, giving us

$$\begin{aligned}
\int F_C dV &= \int \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} dV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_R^r \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\sin\phi r^2 dr d\phi d\theta \\
&= \int_R^r \frac{1}{4\pi\epsilon_0} Ze\cdot e dr \\
&= \frac{Ze^2}{2\pi\epsilon_0}\int_R^r dr \\
&= \frac{Ze^2}{2\pi\epsilon_0}\left[r - R\right] + c_1 \\
&= \frac{Ze^2}{2\pi\epsilon_0 R}\left[\frac{r}{R} - 1\right] + c_1 \\
&= -\frac{Ze^2}{2\pi\epsilon_0 R}\left[1 - \frac{r}{R}\right] + c_1
\end{aligned}$$

and we know that when $r = R$, the potential function is equal to

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0}$$

meaning that

$$c_1 = -\frac{Ze^2}{4\pi\epsilon_0}$$

and finally

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 R}\left(\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right)$$

[Q3] The spherical harmonics are given by:

$$Y_{lm}(\theta,\phi) = \varepsilon\left\{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right\}^{1/2} P_l^m(\cos\theta)\exp(im\phi)$$

such that

$$\varepsilon = \begin{cases}(-1)^m & m \geq 0 \\ 1 & m < 0\end{cases}$$

where

$$P_l^m(x) = \left(1 - x^2\right)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2 - 1\right)^l$$

$P_l(x)$ **is the Legendre Functions, and** $P_l^m(x)$ **is the associated Legendre Functions.**

[a] Starting with the associated Legendre Functions, write out $Y_{3,+1}$. To find $Y_{3,+1}$, we must first find the third Legendre Polynomial, which can be performed by

$$\begin{aligned}
P_3(x) &= \frac{1}{2^3 3!}\frac{d^3}{dx^3}\left(x^2-1\right)^3 \\
&= \frac{1}{48}\frac{d^3}{dx^3}\left(x^2-1\right)^3 \\
&= \frac{1}{48}\frac{d^2}{dx^2}\left[3\left(x^2-1\right)^2(2x)\right] \\
&= \frac{1}{8}\frac{d^2}{dx^2}\left[x\left(x^2-1\right)^2\right] \\
&= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)^2+2\left(x^2-1\right)(2x)\right] \\
&= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)^2+4x\left(x^2-1\right)\right] \\
&= \frac{1}{8}\left[\left(x^2-1\right)(2x)+4\left(x^2-1\right)+4x(2x)\right] \\
&= \frac{1}{8}\left[2x^3-2x+4x^2-4+8x^2\right] \\
&= \frac{1}{8}\left(2x^3+12x^2-2x-4\right)
\end{aligned}$$

Wolfram alpha states that this derivative is

$$P_3(x) = 24x\left(5x^2-3\right)$$

Then to find the associated polynomial, we must apply the +1 condition to $|m|$, giving:

$$\begin{aligned}
P_3^{+1}(x) &= \left(1-x^2\right)^{+1/2}\frac{d}{dx}P_3(x) \\
&= \sqrt{1-x^2}\frac{d}{dx}\left[24x\left(5x^2-3\right)\right] \\
&= \sqrt{1-x^2}\left[24\left(5x^2-3\right)+24x(10x)\right] \\
&= \sqrt{1-x^2}\left(360x^2-72\right)
\end{aligned}$$

Now we have to apply this to the spherical harmonic, $Y(\theta,\phi)$.

$$\begin{aligned}
Y_{3,+1}(\theta,\phi) &= (-1)\left\{\frac{7}{4\pi}\frac{2!}{4!}\right\}^{1/2} P_3^{+1}(\cos\theta)\exp(i\phi) \\
&= -\sqrt{\frac{14}{96\pi}}\sqrt{1-\cos^2\theta}\left(360\cos^2\theta-72\right)\exp(i\phi)
\end{aligned}$$

[b] Check the normalization of your results by using the orthogonality condition. The dot product of any orthogonal harmonic must be zero, giving us the requirement that

$$\int Y_{3,+1}Y_{3,+1}d\Omega = 0$$

By substituting in the actual spherical harmonic and expanding the integrals, we get

$$\begin{aligned}
\int Y_{3,+1}Y_{3,+1}d\Omega &= \int_0^{\pi}\int_0^{2\pi}\left(-\sqrt{\frac{14}{96\pi}}\sqrt{1-\cos^2\theta}\left(360\cos^2\theta-72\right)\exp\left(i\phi\right)\right)^2 d\phi d\theta \\
&= \frac{14}{96\pi}\int_0^{\pi}\int_0^{2\pi}\left(1-\cos^2\theta\right)\left(360\cos^2\theta-72\right)^2\exp\left(2i\phi\right)d\phi d\theta \\
&= \frac{14}{96\pi}\int_0^{\pi}\left(1-\cos^2\theta\right)\left(360\cos^2\theta-72\right)^2\int_0^{2\pi}\exp\left(2i\phi\right)d\phi d\theta \\
&= \frac{14}{96\pi}\int_0^{\pi}\left(1-\cos^2\theta\right)\left(360\cos^2\theta-72\right)^2\left[\frac{1}{2i}\exp\left(2i\phi\right)\right]_0^{2\pi}d\theta \\
&= \frac{14}{96\pi}\int_0^{\pi}\left(1-\cos^2\theta\right)\left(360\cos^2\theta-72\right)^2\cancelto{0}{\left[\frac{1}{2i}-\frac{1}{2i}\right]}d\theta \\
&= 0
\end{aligned}$$

So the normalization does work as far as orthogonality is concerned.

[c] Show how any $Y_{lm}(\theta,\phi)$ transforms under parity. For a parity transformation, we have the conversion of

$$\begin{aligned}
r &\to r \\
\theta &\to \pi-\theta \\
\phi &\to \pi+\phi
\end{aligned}$$

and we can apply this to $Y_{lm}(\theta,\phi)$. To determine the effect of the change in ϕ, we need only look at the exponential term, which changes

$$\exp\left(i\phi\right)\to\exp\left(i\left(\pi+\phi\right)\right)$$

In the complex plane, this simply changes the angle of the complex vector by 180^o. This is a transformation that has odd (negative parity)

$$\exp\left(i\phi\right)=-\exp\left(i\left(\pi+\phi\right)\right)$$

For the effect of the change in θ, we must look at the legendre polynomial, which takes $\cos\theta$ as an argument. Because of the phase shift,

$$\cos\left(\pi-\theta\right)=-\cos\left(\theta\right)$$

which shows that there is -1 parity for the legendre polynomial term.

Thus, the entire function has

$$\begin{aligned}
\mathscr{P}\left(Y_{lm}\left(\theta,\phi\right)\right) &= \mathscr{P}\left(P_l^m\left(\cos\theta\right)\right)\mathscr{P}\left(\exp\left(i\phi\right)\right) \\
&= \left(-1\right)\left(-1\right) \\
&= +1
\end{aligned}$$

[d] How is $P_l^m(x)$ related to $P_l^{-m}(x)$? The formula for the associated legendre polynomial is restated below

$$P_l^m\left(x\right)=\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_l\left(x\right)$$

By inspection, it can be seen that P_l is not affected by any change in m, so essentially, we are looking at how the expression

$$\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}$$

changes with parity of m.

[e] For the Rodrigues' formula for the Legendre Functions prove orthogonality for any value of l. That is

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \delta_{nm}$$

We can use integration by parts to determine this, giving us a table such as

$$\begin{array}{ccc} & \text{D} & \text{I} \\ 0 & \frac{d^n}{dx^n}\left(x^2-1\right)^n & \frac{d^m}{dx^m}\left(x^2-1\right)^m \\ 1 & \frac{d^{n+1}}{dx^{n+1}}\left(x^2-1\right)^n & \frac{d^{m-1}}{dx^{m-1}}\left(x^2-1\right)^m \\ 2 & \frac{d^{n+2}}{dx^{n+2}}\left(x^2-1\right)^n & \frac{d^{m-2}}{dx^{m-2}}\left(x^2-1\right)^m \\ 3 & \frac{d^{n+3}}{dx^{n+3}}\left(x^2-1\right)^n & \frac{d^{m-3}}{dx^{m-3}}\left(x^2-1\right)^m \\ \vdots & \vdots & \vdots \\ p & \frac{d^{n+p}}{dx^{n+p}}\left(x^2-1\right)^n & \frac{d^{m-p}}{dx^{m-p}}\left(x^2-1\right)^m \end{array}$$

Because the terms in the right column (the integrated column) become zero on the boundary, we don't have to look beyond the same order of polynomial as the same order we are trying to work with.

[Q4] The wave function of the two-nucleon state may be written:

$$\Psi_{total} = \Phi_{total}\alpha_{spin}\chi_{isospin}$$

In this case we consider the proton and neutron to be identical particles with just different projections in isospin space. This leads to the extended Pauli Exclusion Principle that two nucleons must have an overall wave function that is anti-symmetric w.r.t. interchange of the two particles.

[a] Discuss the space part of the wave function and symbolically write out the spatial wave function. The space part of the wave function for a spherical potential can be written as the spherical harmonics discussed above, which provide a function for the polar and azimuthal angular dependence. There will also be a radial dimensionality, which is probably related to the potential, but at this level of abstraction, we cannot say much more about it. This function, by applying parity transformations to it, is symmetrical. Thus, symbolically, we can write this function as

$$\Phi_{total} = \mathscr{R}(r) Y_{lm}(\theta, \varphi)$$

[b] Discuss the spin part of the wave function and symbolically write out the spin wave function. The spin wave function compared between the two particles, a proton and a neutron, is symmetric. This function is the sum of the quarks in the neutron and proton, the neutron having *udd* and the proton having *uud*. Thus, the third component of the spin vector is the only requirement here, and as such, our spin can be written as only a function of this factor.

$$\alpha_{spin} = \mathscr{S}(s_z)$$

[c] Given our knowledge of the space and spin of the deuteron what must the symmetry of the isospin wave function be? Write explicitly the isospin wave function of the deuteron. The symmetry must be antisymmetric because of the symmetry of the spin and spatial wavefunctions, to garner a totally asymmetric wave function, Ψ_{total}. The isospin wave function can be found by the four additions of the $|p, n\rangle$. These states are

$$|p, n\rangle = |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$$

And we need to then normalize these to 1

$$\begin{aligned} N^2 |p, n\rangle^2 &= N^2 \left(\langle\downarrow\uparrow| + \langle\uparrow\downarrow|\right)\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) = 1 \\ &= N^2 (1 + 0 + 0 + 1) = 1 \\ N^2 &= \frac{1}{2} \end{aligned}$$

So then, with our normalization $N = 1/\sqrt{2}$ and in one dimension using the possible values of m, which are $-1 \leq m \leq 1$, we have a triplet state (with $j = 1$) of

$$|\uparrow\uparrow\rangle$$

$$\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

$$|\downarrow\downarrow\rangle$$

and a single state of

$$\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right)$$

Unfortunately, two of the states of the triplet state (the $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$) cannot exist (two protons or two neutrons cannot be bound). Thus, the entire triplet state cannot exist. Then, the singlet state must exist, and the property of the singlet state, with $j = 0$, $m = 0$ is that it is antisymmetric. We can prove it's antisymmetry by applying parity to it

$$\begin{aligned}\mathscr{P}\left(\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right)\right) &= \frac{1}{\sqrt{2}}\left(-|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) \\ &= -\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right)\end{aligned}$$

[Q5] Check if the following reaction violates any of the conservations laws you know including isospin, both I and $\mathbf{I}_z$.

$$d + d \to {}^4\mathrm{He} + \pi^0$$

Checking the conservations laws for spin number, charge, baryon number, and isospin number in table 3.

Table 3: Conservation Laws Checks for $d + d$ fusion reaction

Law	d	$+$	d	$\to$	^{4}He	$+$	π^o	**Conserved?**	**Method**
Charge	1	$+$	1	$=$	2	$+$	0	Yes	Charges were looked up.
Spin	1	$+$	1	$=$	0	$+$	0	Yes	The total angular momentum must be conserved, so the product must have spin between $\lvert J_1 + J_2 \rvert \ldots \lvert J_1 - J_2 \rvert = 2 \ldots 0$
Baryon	2	$+$	2	$=$	4	$+$	0	Yes	Baryon numbers were looked up.
Isospin ($\lvert I, I_z\rangle$	$\lvert 0,0\rangle$	$+$	$\lvert 0,0\rangle$	$=$	$\lvert 0,0\rangle$	$+$	$\lvert 1,0\rangle$	No	Using ladder operators to check for "brothers" of α, we found that the states ^{4}Li and ^{4}H do not exist, which means that the isospin of α has to be zero. Also, the isospin selectron criteria restricts a $I = 0$, $I_3 = 0$ from transitioning to an $I = 0$ nucleus [2, p. 391], as is attempted in $d + d \to \alpha + \pi^0$

[Q6] Given the Pauli Spin Matrices

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

evaluate

[a] $\tau_3 \left| p \right\rangle$

$$\begin{aligned}
\tau_3 \left| p \right\rangle &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left| p \right\rangle \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}$$

[b] $\tau_3 \left| n \right\rangle$

$$\begin{aligned}
\tau_3 \left| n \right\rangle &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left| n \right\rangle \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 \\ -1 \end{pmatrix}
\end{aligned}$$

[c] $\vec{\tau}^2 = \vec{\tau} \circ \vec{\tau}$

$$\begin{aligned}
\vec{\tau}^2 &= \vec{\tau} \circ \vec{\tau} \\
&= \tau_1\tau_1 + \tau_2\tau_2 + \tau_3\tau_3 \\
&= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

[d] $\vec{\tau}^2 \left| p \right\rangle$

$$\begin{aligned}
\vec{\tau}^2 \left| p \right\rangle &= \left[\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{8}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}$$

[e] $\vec{\tau}^2 \left| n \right\rangle$

$$\begin{aligned}
\vec{\tau}^2 \left| n \right\rangle &= \left[\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{8}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

[f] $\tau_+ = \tau_1 + i\tau_2$

$$\tau_+ = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

[g] $\tau_- = \tau_1 - i\tau_2$

$$\tau_- = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

[h] Operate $\tau_{\pm}$ on $|p\rangle$ and $|n\rangle$

$$\begin{aligned}
\tau_+ |p\rangle &= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} |p\rangle \\
&= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\tau_- |p\rangle &= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} |p\rangle \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 \\ 2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\tau_+ |p\rangle &= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} |n\rangle \\
&= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 2 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\tau_- |p\rangle &= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} |n\rangle \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

References

[1] W. N. Cottingham and D. A. Greenwood. *An Introduction to Nuclear Physics.* Cambridge University Press, Cambridge, MA, USA, 2nd edition, 2001.

[2] Kenneth S. Krane. *Introductory Nuclear Physics.* John Wiley & Sons, Inc., Hoboken, NJ, 2nd edition, 1988.