ME 513 HMWK 2

Alex Hagen

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2.4.1 By direct substitution show that each of the following is a solution of the wave equation:

(a) $f_1(x-ct)$

$$\frac{\partial^2 f_1(x-ct)}{\partial t^2} = c^2 \frac{\partial^2 f_1(x-ct)}{\partial t^2}$$

Therefore

$$\frac{\partial^2 f_1(x-ct)}{\partial x^2} - \frac{1}{c^2}\left(c^2 \frac{\partial^2 f_1(x-ct)}{\partial t^2}\right) = 0$$

(b) $\ln[a(ct-x)]$

$$\frac{\partial \ln[a(ct-x)]}{\partial x} = -\frac{1}{ct-x}$$

$$\frac{\partial^2 \ln[a(ct-x)]}{\partial x^2} = -\frac{1}{(ct-x)^2}$$

$$\frac{\partial \ln[a(ct-x)]}{\partial t} = \frac{c}{ct-x}$$

$$\frac{\partial^2 \ln[a(ct-x)]}{\partial t^2} = -\frac{c^2}{(ct-x)^2}$$

Therefore

$$-\frac{1}{(ct-x)^2} - \frac{1}{c^2}\left(-\frac{c^2}{(ct-x)^2}\right) = 0$$

$$-\frac{1}{(ct-x)^2} + \frac{1}{(ct-x)^2} = 0$$

$$0 = 0$$

(c) $a(ct-x)^2$, and

$$\frac{\partial a(ct-x)^2}{\partial x} = 2a(x-ct)$$

$$\frac{\partial^2 a(ct-x)^2}{\partial x^2} = 2a$$

$$\frac{\partial a(ct-x)^2}{\partial t} = 2ac(ct-x)$$

$$\frac{\partial^2 a(ct-x)^2}{\partial t^2} = 2ac^2$$

Therefore

$$2a - \frac{1}{c^2}\left(2ac^2\right) = 0$$

$$2a - 2a = 0$$

$$0 = 0$$

(d) $\cos\left[a\left(ct-x\right)\right]$

$$\frac{\partial \cos\left[a\left(ct-x\right)\right]}{\partial x} = a\sin\left[a\left(ct-x\right)\right]$$
$$\frac{\partial^2 \cos\left[a\left(ct-x\right)\right]}{\partial x^2} = -a^2\cos\left[a\left(ct-x\right)\right]$$
$$\frac{\partial \cos\left[a\left(ct-x\right)\right]}{\partial t} = -ac\sin\left[a\left(ct-x\right)\right]$$
$$\frac{\partial^2 \cos\left[a\left(ct-x\right)\right]}{\partial t^2} = -a^2c^2\cos\left[a\left(ct-x\right)\right]$$

Therefore

$$\begin{aligned} -a^2\cos\left[a\left(ct-x\right)\right] - \frac{1}{c^2}\left(-a^2c^2\cos\left[a\left(ct-x\right)\right]\right) &= 0 \\ -a^2\cos\left[a\left(ct-x\right)\right] + a^2\cos\left[a\left(ct-x\right)\right] &= 0 \\ 0 &= 0 \end{aligned}$$

Similarly, show that each of the following is not a solution of the wave equation:

(e) $a\left(ct-x^2\right)$ **and**

$$\frac{\partial a\left(ct-x^2\right)}{\partial x} = -2ax$$
$$\frac{\partial^2 a\left(ct-x^2\right)}{\partial x^2} = -2a$$
$$\frac{\partial a\left(ct-x^2\right)}{\partial t} = ac$$
$$\frac{\partial^2 a\left(ct-x^2\right)}{\partial t^2} = 0$$

Therefore

$$\begin{aligned} -2a - \frac{1}{c^2}\left(0\right) &= 0 \\ -2a &\neq 0 \end{aligned}$$

(f) $at\left(ct-x\right)$

$$\frac{\partial at\left(ct-x\right)}{\partial x} = -at$$
$$\frac{\partial^2 at\left(ct-x\right)}{\partial x^2} = 0$$
$$\frac{\partial at\left(ct-x\right)}{\partial t} = a\left(2ct-x\right)$$
$$\frac{\partial^2 at\left(ct-x\right)}{\partial t^2} = 2ac$$

Therefore

$$\begin{aligned} 0 - \frac{1}{c^2}\left(2ac\right) &= 0 \\ -\frac{2a}{c} &\neq 0 \end{aligned}$$

2.8.2 An infinite string ($-\infty < x \leq 0$) of linear density ρ_L and under tension T is attached at $x = 0$ to a second infinite string ($0 < x < \infty$) under the same tension but of linear density $2\rho_L$. If a wave of angular frequency ω and amplitude A is traveling in the $+x$ direction on the first string, find the amplitude of the wave traveling on the second string. With the two strings, we can define the mechanical impedance as

$$z_{mo,1} = \rho_l c$$

and

$$z_{mo,2} = 2\rho_l c$$

and because the definition of mechanical impedance is

$$z_{mo} = \frac{f}{u\left(0,t\right)}$$

we then have

$$\begin{aligned} 2z_{mo,1} &= z_{mo,2} \\ 2\frac{f}{u_1\left(0,t\right)} &= \frac{f}{u_2\left(0,t\right)} \\ u_2\left(0,t\right) &= \frac{1}{2}u_1\left(0,t\right) \\ A_2 &= \frac{1}{2}A_1 \end{aligned}$$

Therefore, the amplitude of the disturbance in the second string is one half of that in the first string.

2.9.2 Evaluate the mechanical impedance seen by the applied force driving an infinite string at a distance L from a fixed end. Interpret the individual terms in the mechanical impedance. The input mechanical impedance can be described as

$$z_{mo} = \frac{\text{complex applied driving force at } x = 0}{\text{velocity at the drive point}}$$

at the drive point, so if the force is applied at point $x = L$, then this goes to

$$z_{mL} = \frac{\text{complex applied driving force at } x = L}{\text{velocity at the drive point, } x = L} = \frac{F\exp\left(j\omega t\right)\rho_l c}{F\exp\left(j\omega t\right)} = \rho_l c$$

and refers to the dissipation of energy from the drive point to infinity, which is the other end of the string.

2.9.3a A string is streched between rigid supports a distance L apart. It is driven by a force $F\cos\omega t$ located at its midpoint.

(a) What is the mechanical impedance at the midpoint? (Assume that the driving force is $F\exp j\omega t$, and that it is applied at $x = L/4$ rather than at the midpoint of the string: i.e., calculate the input mechanical impedance at $x = L/4$.) We must treat this problem as two separate strings with boundary conditions to connect them, giving us

$$\frac{d^2y_1}{dx^2} - \frac{1}{c^2}\frac{d^2y_1}{dt^2} = 0$$

and

$$\frac{d^2y_2}{dx^2} - \frac{1}{c^2}\frac{d^2y_2}{dt^2} = 0$$

so, using the harmonic solutions, we have

$$y_1\left(x,t\right) = A\exp\left[j\left(\omega t - k_1 x\right)\right] + B\exp\left[j\left(\omega t + k_1 x\right)\right]$$

and

$$y_2\left(x,t\right) = C\exp\left[j\left(\omega t - k_2 x\right)\right] + D\exp\left[j\left(\omega t + k_2 x\right)\right]$$

with the boundary conditions relating to no displacements at the ends, matching displacement at the drive point, and matching force at the drive point.

$$y_1 = 0 \text{ at } x = 0$$
$$y_2 = 0 \text{ at } x = L$$
$$y_1 = y_2 \text{ at } x = {}^{L}\!/_{4}$$

Applying the first boundary condition, we find that

$$\begin{aligned} y_1(0,t) = 0 &= A\exp[j\omega t] + B\exp[j\omega t] \\ 0 &= A + B \\ B &= -A \end{aligned}$$

and-

$$\begin{aligned} y_1(x,t) &= A\exp[j(\omega t - k_1 x)] - A\exp[j(\omega t + k_1 x)] \\ &= -2jA\sin(k_1 x)\exp(j\omega t) \end{aligned}$$

Applying the second boundary condition, we have

$$\begin{aligned} y_2(L,t) = 0 &= C\exp[j(\omega t - k_2 L)] + D\exp[j(\omega t + k_2 L)] \\ 0 &= \exp(j\omega t)[C\exp(-jk_2 L) + D\exp(jk_2 L)] \\ -C\exp(-jk_2 L) &= D\exp(jk_2 L) \\ -C &= D\exp(2jk_2 L) \\ C &= -D\exp(2jk_2 L) \end{aligned}$$

and finally, the third boundary contition gives

$$\begin{aligned} y_1({}^{L}\!/_{4}, t) &= y_2({}^{L}\!/_{4}, t) \\ -2jA\sin\left(k_1\frac{L}{4}\right)\exp(j\omega t) &= \exp(j\omega t)\left[-D\exp(2jk_2 L)\exp\left(-jk_2\frac{L}{4}\right) + D\exp\left(jk_2\frac{L}{4}\right)\right] \\ -2jA\sin\left(k_1\frac{L}{4}\right) &= \left[-D\exp\left(2jk_2 L - jk_2\frac{L}{4}\right) + D\exp\left(jk_2\frac{L}{4}\right)\right] \\ -2jA\sin\left(k_1\frac{L}{4}\right) &= -D\exp\left(7jk_2\frac{L}{4}\right) + D\exp\left(jk_2\frac{L}{4}\right) \\ -2jA\sin\left(k_1\frac{L}{4}\right) &= D\left[\exp\left(jk_2\frac{L}{4}\right) - \exp\left(7jk_2\frac{L}{4}\right)\right] \\ A &= 2jD\frac{\exp\left(jk_2\frac{L}{4}\right) - \exp\left(7jk_2\frac{L}{4}\right)}{\sin\left(k_1\frac{L}{4}\right)} \end{aligned}$$

So finally, we can find the wavefunction

$$\begin{aligned} y_1 &= 2jD\frac{\exp\left(jk_2\frac{L}{4}\right) - \exp\left(7jk_2\frac{L}{4}\right)}{\sin\left(k_1\frac{L}{4}\right)}\exp[j(\omega t - k_1 x)] - 2jD\frac{\exp\left(jk_2\frac{L}{4}\right) - \exp\left(7jk_2\frac{L}{4}\right)}{\sin\left(k_1\frac{L}{4}\right)}\exp[j(\omega t + k_1 x)] \\ &= -2j2jD\frac{\exp\left(jk_2\frac{L}{4}\right) - \exp\left(7jk_2\frac{L}{4}\right)}{\sin\left(k_1\frac{L}{4}\right)}\sin(k_1 x)\exp(j\omega t) \\ &= 4D\frac{\exp\left(jk_2\frac{L}{4}\right) - \exp\left(7jk_2\frac{L}{4}\right)}{\sin\left(k_1\frac{L}{4}\right)}\sin(k_1 x)\exp(j\omega t) \end{aligned}$$

So the velocity is

$$\begin{aligned} u_1 &= \frac{\partial y_1}{\partial t} \\ &= 4j\omega D\frac{\exp\left(jk_2\frac{L}{4}\right) - \exp\left(7jk_2\frac{L}{4}\right)}{\sin\left(k_1\frac{L}{4}\right)}\sin(k_1 x)\exp(j\omega t) \end{aligned}$$

and to find the input mechanical impedance, we would have

$$
\begin{aligned}
z_{m^{L/4}} &= \frac{\text{applied driving force}}{\text{velocity at the drive point}} \\
&= \frac{F\exp\left(j\omega t\right)}{\frac{\partial y}{\partial t}} \\
&= \frac{F\cancel{\exp\left(j\omega t\right)}}{4j\omega D\frac{\exp\left(jk_2\frac{L}{4}\right)-\exp\left(7jk_2\frac{L}{4}\right)}{\sin\left(k_1\frac{L}{4}\right)}\sin\left(k_1 x\right)\cancel{\exp\left(j\omega t\right)}}
\end{aligned}
$$

2.11.1 Find kL for the normal modes of a fixed, spring-loaded string when $T = sL$. Sketch the waveforms for the fundamental and the first overtone. We can assume the harmonic solution of

$$y\left(x,t\right) = A\exp\left[j\left(\omega t - kx\right)\right] + B\exp\left[j\left(\omega t + kx\right)\right]$$

with

$$k = {}^{\omega}/_{c}$$

and

$$c = \sqrt{{}^{T}/_{\rho_l}} = \sqrt{{}^{sL}/_{\rho_l}}$$

Applying the fixed boundary condition at $x = 0$, we find that $B = -A$, and therefore

$$y\left(x,t\right) = -2jA\sin\left[kx\right]\exp\left[j\omega t\right]$$

and so we need one more boundary condition. To find this, we can find that, at the end

$$T\sin\theta\big|_{x=L} - s\ y\big|_{x=L} = 0$$

$$\frac{dy}{dx} = -2jkTA\cos\left(kx\right)\exp\left(j\omega t\right)$$

and placing this into the expression for y from before, we have

$$2jkTA\exp\left(j\omega t\right)\cos\left(kL\right) + 2jsA\exp\left(j\omega t\right)\sin\left(kL\right) = 0$$

and therefore

$$k_n T\cos\left(k_n L\right) = -s\sin\left(k_n L\right)$$

, then putting in the definition of T here, we can have

$$
\begin{aligned}
k_n sL\cos\left(k_n L\right) &= -s\sin\left(k_n L\right) \\
k_n L &= -\tan\left(k_n L\right)
\end{aligned}
$$

which has no closed form solution. But the node shape can be sketched as below for the successive k_n's when a value for L is chosen and fixed.

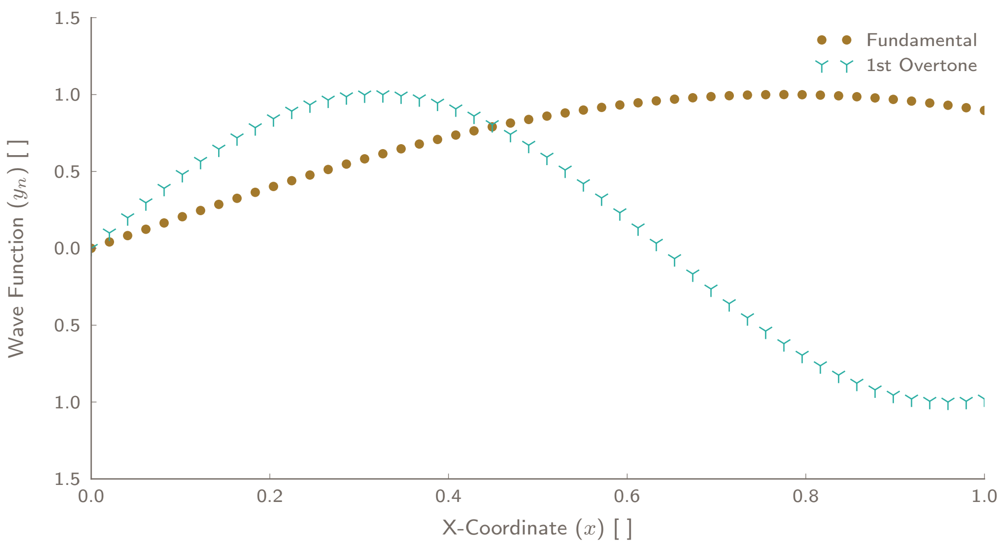

Figure 1: Wave Mode Shapes of Fundamental and First Overtone