ME 513 HMWK 3

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1 A $0.01\,\mathrm{m}^2$ piston oscillates at one end of a $1\,\mathrm{m}$ long tube that is terminated at the other end by a partially absorbing surface. At a single frequency, ω, the spatial dependence of the plane wave within the tube has the form

$$p\left(x\right) = \exp\left(-jkx\right) + 0.8\exp\left(jkx\right)$$

where $x = 0$ is at the surface of the absorbing materials, and the positive x-direction is *into* the absorbing surface. The drawing of the coordinate system and other described properties of this problem as drawing below:

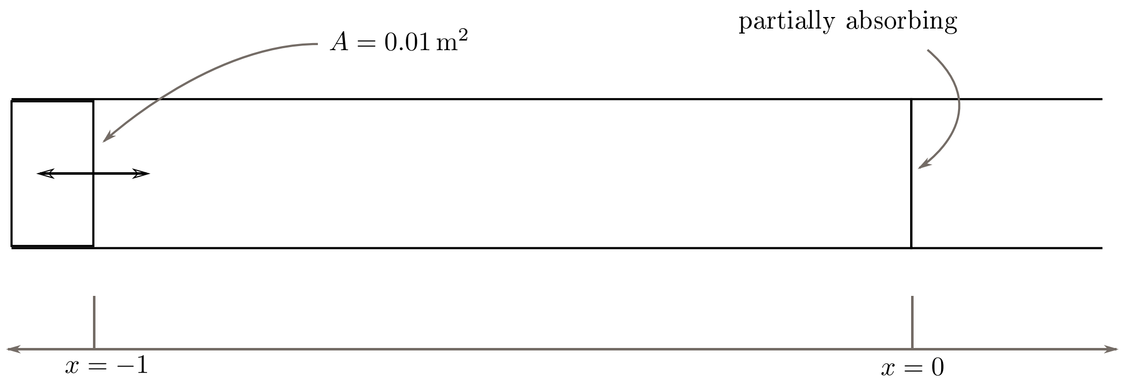

Figure 1: Figure of setup for problem 1

(a) Derive by using the linearized Euler equation an expression for the spatial dependence of the particle velocity field within the tube, and calculate the velocity of the piston. The linear Euler's equation is given as

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p$$

and since we are working in one dimension, we have

$$\rho_0 \frac{du}{dt} = -\frac{dp}{dx}$$

. And, by assumption, the time dependence of u is $\propto \exp\left(-\omega jt\right)$. Integrating the above expression, then we get.

$$u = \frac{1}{\rho_0 \omega j}\frac{dp}{dx}$$

By applying the given expression for pressure, first we can the spatial derivative of this function

$$\frac{dp}{dx} = -jk\exp\left(-jkx\right) + 0.8jk\exp\left(jkx\right)$$

which is therefore equal to the negated density multiplied by the acceleration, thus

$$\frac{du}{dt} = \frac{jk\exp\left(-jkx\right) - 0.8jk\exp\left(jkx\right)}{\rho_0 \omega j}$$
$$\frac{du}{dt} = \frac{jk}{\rho_0 \omega j}\left[\exp\left(-jkx\right) - 0.8\exp\left(jkx\right)\right]$$

and therefore, integrating

$$u = \frac{k}{\rho_0}\left[\exp\left(-jkx\right) - 0.8\exp\left(jkx\right)\right] + u_0$$

and applying the condition at the piston face ($x = -1$) and assuming incompressible (air velocity at piston face will equal exactly the piston face velocity, we have

$$u\left(0\right) = \frac{k}{\rho_0}\left[\exp\left(jk\right) - 0.8\exp\left(-jk\right)\right] + u_0$$

. Finally, assuming that the air was stationary at time zero, we have $u_0 = 0$ and thus

$$u\left(0\right) = \frac{k}{\rho_0}\left[\exp\left(jk\right) - 0.8\exp\left(-jk\right)\right]$$

(b) The time-averaged acoustic intensity at any point within the tube can be calculated using the expression $I\left(x\right) = \frac{1}{2}\mathrm{Re}\left\{p\left(x\right)u\left(x\right)\right\}$ where Re denotes the real part, and $u\left(x\right)$ is the acoustic particle velocity. Use this expression to show that the acoustic intensity does not depend on position within the tube, and then to determine the sound power delivered to the tube by the piston. Applying the expression to find the acoustic intensity, we have

$$\begin{aligned}
I\left(x\right) &= \frac{1}{2}\mathrm{Re}\left\{p\left(x\right)u\left(x\right)\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\left[\exp\left(-jkx\right) + 0.8\exp\left(jkx\right)\right]\left[\frac{k}{\rho_0}\left[\exp\left(-jkx\right) - 0.8\exp\left(jkx\right)\right]\right]\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\frac{k}{\rho_0}\left[\exp\left(-2jkx\right) + 0.8 - 0.8 - 0.64\exp\left(2jkx\right)\right]\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\frac{k}{\rho_0}\left[\exp\left(-2jkx\right) - 0.64\exp\left(2jkx\right)\right]\right\} \\
&= \frac{1}{2}\frac{k}{\rho_0}\left\{\cos(2kx) - 0.64\cos\left(-2kx\right)\right\}
\end{aligned}$$

and operating on the argument of the real function, it is easy to see that the real part will be a periodic function in x, which is contrary to what the problem is asking.

(c) What is the specific acoustic impedcance at the piston surface and at the absorbing surface? The specific acoustic impedance is defined as

$$\begin{aligned}
z &= \frac{p\left(x\right)}{u\left(x\right)} \\
&= \frac{\exp\left(-jkx\right) + 0.8\exp\left(jkx\right)}{\frac{k}{\rho_0}\left[\exp\left(-jkx\right) - 0.8\exp\left(jkx\right)\right]} \\
&= -\frac{\rho_0}{k}\frac{\exp\left(-jkx\right) + 0.8\exp\left(jkx\right)}{\exp\left(-jkx\right) - 0.8\exp\left(jkx\right)}
\end{aligned}$$

and at the pistion ($x = -1$), this is:

$$z_{piston} = -\frac{\rho_0}{k}\frac{\exp\left(jk\right) + 0.8\exp\left(-jk\right)}{\exp\left(jk\right) - 0.8\exp\left(-jk\right)}$$

and at the absorbing surface this is

$$\begin{aligned}
z_{surf} &= -\frac{\rho_0}{k}\frac{1 + 0.8}{1 - 0.8} \\
&= -\frac{\rho_0}{k}\frac{1.8}{0.2} \\
&= -\frac{9\rho_0}{k}
\end{aligned}$$

2 A unit amplitude plane wave strikes a surface at $y = 0$ as shown. A reflected plane wave, having complex amplitude R is generated at the surface and it propagates in the direction shown.

(a) Write an expression for the sound pressure in region $y < 0$ (defining quantities as necessary). The sound pressure in the region below the surface is defined through continuity, with

$$\overset{1}{\cancel{P_i}} \exp\left(-jk_1 y \sin\theta\right) + \overset{R}{\cancel{P_r}} \exp\left(-jk_1 y \sin\theta\right) = P_t \exp\left(-jk_2 y \sin\theta_t\right)$$

And we can find the angle of transmission with

$$\frac{\sin\theta}{c_1} = \frac{\sin\theta_t}{c_2}$$

or

$$\theta_t = \arccos\left(\sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}\right)$$

and therefore

$$\begin{aligned} p_{y<0} &= P_t \exp\left(-jk_2 y \sin\left[\arccos\left(\sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}\right)\right]\right) \\ &= P_t \exp\left(-jk_2 y \sqrt{1 - \sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}}\right) \end{aligned}$$

(b) By using the linearized Euler equation, derive an expression for the y-component of the particle velocity. The particle velocity of this is

$$\begin{aligned} u &= -\frac{1}{j\omega\rho_0}\frac{dp}{dy} \\ &= \frac{1}{j\omega\rho_0} jk_2 \sqrt{1 - \sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}} P_t \exp\left(-jk_2 y \sqrt{1 - \sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}}\right) \\ &= \frac{k_2}{\omega\rho_0} \sin\theta_t p_{y<0} \end{aligned}$$

(c) Based solely on the field in the region $y < 0$, derive an experession for the time averaged sound power per unit area flowing into the surface at $y = 0$. The time averaged sound per unit area is the acoustic intensity, which is

$$\begin{aligned} I &= \frac{1}{2}\mathrm{Re}\left\{pu\right\} \\ &= \frac{1}{2}\mathrm{Re}\left\{\frac{k_2}{\omega\rho_2}\sin\theta_t p_{y<0}^2\right\} \\ &= \frac{k_2}{2\omega\rho_2}\sqrt{1 - \sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}}\, \mathrm{Re}\left\{P_t^2 \exp\left(-2jk_2 y \sqrt{1 - \sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}}\right)\right\} \\ &= \frac{k_2}{2\omega\rho_2} P_t^2 \sqrt{1 - \sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}} \cos\left(2k_2 y \sqrt{1 - \sqrt{1 - \left({}^{c_2}/_{c_1}\right)^2 \sin^2\theta}}\right) \end{aligned}$$

3 A sound field has the form

$$p(x,t) = A\exp\left(-\alpha x\right)\exp\left(-j\beta x\right)\exp\left(j\omega t\right)$$

where A is complex and α and β are real.

(a) Sketch the spatial variation of the sound field. What is the significance of α and β? What is the wavelength of the sound field in terms of the parameters of the sound field? The above equation can be simplified into the form

$$p(x,t) = A\exp(-jkx)\exp(j\omega t)$$

when

$$k = \beta - \alpha j$$

and knowing the relationship between wavenumber, frequency, and wavelength, we can find that

$$k = \frac{\omega}{c} = f\frac{2\pi}{c}$$

and putting this in terms of frequency, f

$$f = \frac{kc}{2\pi}$$

Then, with the definition of wavelength

$$\lambda = \frac{c}{f}$$

we can subsitute in the wavenumber and see that

$$\lambda = \frac{2\pi c}{kc} = \frac{2\pi}{\beta - \alpha j}$$

in this case.

(b) Derive an expression for the acoustic particle velocity. The particle velocity is given by the linearized wave equation, i.e.

$$u = \frac{1}{-j\omega\rho_0}\frac{dp}{dx}$$

and so we need to know the value of $\frac{dp}{dx}$, which is

$$\begin{aligned}\frac{dp}{dx} &= \frac{d}{dx}\left[A\exp(-\alpha x)\exp(-j\beta x)\exp(j\omega t)\right]\\ &= A\alpha j\beta\exp(-\alpha x)\exp(-j\beta x)\exp(j\omega t)\end{aligned}$$

and then

$$\begin{aligned}u &= -\frac{A\alpha j\beta}{j\omega\rho_0}\exp(-\alpha x)\exp(-j\beta x)\exp(j\omega t)\\ &= -\frac{\alpha\beta}{\omega\rho_0}p(x,t)\end{aligned}$$

(c) Evaluate the acoustic intensity, and show that it is a function of position in this case. Finally, the acoustic intensity is given by

$$\begin{aligned}I_y &= \frac{1}{2}\mathrm{Re}\left\{pu^*\right\}\\ &= \frac{1}{2}\mathrm{Re}\left\{p(x,t)\left[-\frac{\alpha\beta}{\omega\rho_0}p(x,t)\right]\right\}\\ &= -\frac{\alpha\beta}{\omega\rho_0}\mathrm{Re}\left\{p^2\right\}\\ &= -\frac{\alpha\beta}{2\omega\rho_0}\mathrm{Re}\left\{\exp(-2\alpha x)\exp(-2j\beta x)\exp(j\omega t)\right\}\\ &= -\frac{\alpha\beta}{2\omega\rho_0}\exp(-2\alpha x)\cos(2\beta x)\cos(-\omega t)\end{aligned}$$

4 A two-dimensional sound field in the farfield of a cylindrical source can be expressed in cylindrical coordinates as:

$$p(r,\theta) = \frac{A}{r^{1/2}} \sin(\theta) \exp(-jkr)$$

(a) By using the linearized Euler equation in cylindrical coordinates, find the vector particle velocity field associated with this pressure field. Note that the vector particle velocity field has two components, in this case. The linearized wave equation is

$$\rho_0 \frac{\partial u}{\partial t} = -\nabla p$$

and applying cylindrical coordinates gives

$$\rho_0 \frac{\partial u}{\partial t} = -\left[\vec{r}\frac{\partial p}{\partial r} + \vec{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \cancel{\frac{\partial f}{\partial z}}\right]$$

and now we have to find the partial with respect to radius and to angle, giving us

$$\frac{\partial p}{\partial r} = -\frac{Aj(2kr-j)}{2r^{3/2}} \sin(\theta)\exp(-jkr) = \frac{j(2kr-j)}{2r}p$$

$$\frac{\partial p}{\partial \theta} = \frac{A}{\sqrt{r}}\cos(\theta)\exp(-jkr)$$

and so, we have

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\left[\vec{r}\frac{j(2kr-j)}{2r}p + \vec{\theta}\frac{1}{r}\frac{A}{\sqrt{r}}\cos(\theta)\exp(-jkr)\right]$$

and integrating, we then get

$$u = -\frac{t}{\rho_0}\left[\vec{r}\frac{j(2kr-j)}{2r}p + \vec{\theta}\frac{1}{r}\frac{A}{\sqrt{r}}\cos(\theta)\exp(-jkr)\right] + u_0$$

(b) Find the radial component of the acoustic intensity. The acoustic intensity is given as

$$\begin{aligned}
I_r &= \frac{1}{2}\mathrm{Re}\{pu^*\} \\
&= \frac{1}{2}\mathrm{Re}\left\{-p\frac{t}{\rho_0}\frac{j(2kr-j)}{2r}p\right\} \\
&= \frac{t}{2\rho_0}\mathrm{Re}\left\{-p^2\left(jk+\frac{1}{2r}\right)\right\} \\
&= \frac{t}{2\rho_0}\mathrm{Re}\left\{-jk\frac{A^2}{r}\sin^2(\theta)\exp(-2jkr) - \frac{A^2}{2r^2}\sin^2(\theta)\exp(-2jkr)\right\} \\
&= \frac{t}{2\rho_0}\left[\frac{A^2k}{r}\sin^2(\theta)\sin^2(2kr) - \frac{A^2}{2r^2}\sin^2(\theta)\cos(2kr)\right]
\end{aligned}$$

(c) Find the farfield specific acoustic impedance based on the radial particle velocity - does it limit to the plane wave impedance in the farfield? The specific acoustic impedance is the pressure divided by the velocity, which we can see is

$$\begin{aligned}
z &= \frac{p}{u} \\
&= \frac{\cancel{p}}{-\frac{t}{\rho_0}\left[\vec{r}\frac{j(2kr-j)}{2r}\cancel{p} + \vec{\theta}\frac{1}{r}\cancel{p}\tan(\theta)\right]} \\
&= -\frac{\rho_0}{t\left[\vec{r}\frac{j(2kr-j)}{2r} + \vec{\theta}\frac{1}{r}\tan(\theta)\right]}
\end{aligned}$$

Now, finding the limit of this as $r \to \infty$, we can see that

$$
\begin{aligned}
\lim_{r\to\infty} z &= \lim_{r\to\infty} -\frac{\rho_0}{t\left[\vec{r}\frac{j(2kr-j)}{2r} + \cancel{\vec{\theta}\frac{1}{r}\tan(\theta)}\right]} \\
&= \lim_{r\to\infty} -\frac{\rho_0}{t\left[\vec{r}\left(jk + \cancel{\frac{1}{2r}}\right)\right]} \\
&= -\frac{\rho_0}{t\left[\vec{r}\left(jk + \cancel{\frac{1}{2r}}\right)\right]} \\
&= -\frac{\rho_0}{jkt}
\end{aligned}
$$

which is similar to the plane wave impedance of

$$z = \frac{\rho c}{A}$$

45.12.3

4 A plane sound wave in air of $100\,\mathrm{Hz}$ has a peak acoustic pressure amplitude of $2\,\mathrm{Pa}$.

(a) What is its intensity and its intensity level? I will assume a harmonic sound wave, with the equation

$$p(t) = (2\,\mathrm{Pa})\exp(-jkx)\exp(-j\omega t)$$

and thus the intensity is

$$I = \pm\frac{P^2}{2\rho_0 c} = \frac{4\,\mathrm{Pa}^2}{2\,(415\,\mathrm{Rayls})} = 0.04819\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

and the intensity level is

$$
\begin{aligned}
IL &= 10\log\left({}^{I}/_{I_{ref}}\right) \\
&= 10\log\left({}^{0.004819\,\frac{\mathrm{W}}{\mathrm{m}^2}}/_{10^{-12}\,\frac{\mathrm{W}}{\mathrm{m}^2}}\right) \\
&= 96.83\,\mathrm{dB}
\end{aligned}
$$

(b) What is its peak particle displacement amplitude? The displacement amplitude is

$$\xi = \frac{u}{\omega}$$

and we know that the particle velocity is

$$u = \frac{p}{\rho_0 c} = \frac{2\,\mathrm{Pa}}{415\,\mathrm{Rayls}} = 0.004819\,{}^{m}/_{\mathrm{s}}$$

so the displacement is

$$\xi = \frac{0.004819\,{}^{m}/_{\mathrm{s}}}{628\,\mathrm{s}} = 7.674\,\mu\mathrm{m}$$

(c) What is its peak particle speed amplitude? See the calculation from part (b)., $0.004819\,{}^{m}/_{\mathrm{s}}$

(d) What is its effective or rms pressure? The rms pressure is

$$P_e = \frac{P}{\sqrt{2}}$$

with

$$P = \sqrt{2\rho_0 c I}$$

which is defined in the problem as $2\,\mathrm{Pa}$, so the rms pressure is

$$P_e = \frac{2\,\mathrm{Pa}}{\sqrt{2}} = 1.414\,\mathrm{Pa}$$

(e) What is its sound pressure level $re\,20\,\mu$Pa**?** The sound pressure level with reference to $20\,\mu$Pa is

$$\begin{aligned} SPL &= 20\log\left({}^{P_e}\!/\!{}_{P_{ref}}\right) \\ &= 20\log\left({}^{1.414\,\text{Pa}}\!/\!{}_{2.0\times10^{-5}\,\text{Pa}}\right) \\ &= 96.9884\,\text{dB}\,re\,20\,\mu\text{Pa} \end{aligned}$$