NUCL 510 Nuclear Reactor Theory I
Fall 2011

Homework #9

Due December 8

1. Homework problem #6 of Ch. 5

   An approximate expression for the resonance integral in a homogeneous reactor is given as $I_{eff} = 3.8(\Sigma_s / N_U)^{0.42}$ b with Σ_s as the total macroscopic scattering cross section and N_U as the number density of uranium atoms (use cm in Σ_s and N_U). Using $\sigma_s^U = 8.3$ b, $\sigma_s^C = 4.7$ b, and $\sigma_s^{H_2O} = 44.8$ b, calculate the resonance escape probability for:

   a. A natural-uranium-fueled graphite reactor where the number of carbon atoms (N_C) to uranium (N_U) is 500.

$$\frac{\Sigma_s}{N_U} = \frac{\sigma_s^U N_U + \sigma_s^C N_C}{N_U} = \sigma_s^U + \sigma_s^C \frac{N_C}{N_U} = 8.3 + 4.7 \times 500 = 2358 \text{ b}$$

$$I_{eff} = 3.8\left(\frac{\Sigma_s}{N_U}\right)^{0.42} = 3.8 \times 2358^{0.42} = 99 \text{ b}$$

$$\alpha_C = \left(\frac{12-1}{12+1}\right)^2 = 0.716; \quad \xi_C = 0.158; \quad \alpha_U = \left(\frac{238-1}{238+1}\right)^2 = 0.983; \quad \xi_U = 0.00838$$

$$\bar{\xi} = \frac{N_C \sigma_C \xi_C + N_U \sigma_U \xi_U}{N_C \sigma_C + N_U \sigma_U} = \frac{\sigma_C \xi_C + (N_U / N_C)\sigma_U \xi_U}{\sigma_C + (N_U / N_C)\sigma_U} = 0.157$$

$$p = \exp\left(-\frac{N_U}{\bar{\xi}\Sigma_s} I_{ff}\right) = \exp\left(-\frac{99}{0.157 \times 2358}\right) = 0.765$$

   b. A natural uranium light water reactor where the number of water to uranium atom is 5.

$$\frac{\Sigma_s}{N_U} = \frac{\sigma_s^U N_U + \sigma_s^{H_2O} N_{H_2O}}{N_U} = \sigma_s^U + \sigma_s^{H_2O} \frac{N_{H_2O}}{N_U} = 8.3 + 44.8 \times 5 = 232.3 \text{ b}$$

$$I_{eff} = 3.8\left(\frac{\Sigma_s}{N_U}\right)^{0.42} = 3.8 \times 232.3^{0.42} = 37.5 \text{ b}$$

$$\alpha_O = \left(\frac{16-1}{16+1}\right)^2 = 0.779; \quad \xi_O = 1 + \frac{0.779}{1-0.779} \ln(0.779) = 0.120$$

$$\bar{\xi} = \frac{N_H \sigma_H \xi_H + N_H \sigma_O \xi_O + N_U \sigma_U \xi_U}{N_H \sigma_H + N_H \sigma_O + N_U \sigma_U} = \frac{2\sigma_H \xi_H + \sigma_O \xi_O + (N_U / N_{H_2O})\sigma_U \xi_U}{2\sigma_H + \sigma_O + (N_U / N_{H_2O})\sigma_U} = 0.889$$

$$p = \exp\left(-\frac{N_U}{\bar{\xi}\Sigma_s} I_{ff}\right) = \exp\left(-\frac{37.5}{0.889 \times 232}\right) = 0.834$$

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<u>Resonance integral and resonance escape probability</u>

Probability of neutrons reaching an energy E_{i-1} (just above a resonance i) to be absorbed in the resonance i

$$\pi_i = \frac{1}{q_{i-1}} \int_{E_i}^{E_{i-1}} \Sigma_a(E)\varphi(E)dE = \frac{1}{\bar{\xi}\Sigma_t(E_{i-1})E_{i-1}\phi(E_{i-1})} \int_{E_i}^{E_{i-1}} \Sigma_a(E)\varphi(E)dE$$

$$= \frac{1}{\bar{\xi}\Sigma_s} \int_{E_i}^{E_{i-1}} \Sigma_a(E)\varphi(E)dE \quad [\text{with normalization such that } E_{i-1}\varphi(E_{i-1}) = 1]$$

$$= \frac{N_F}{\bar{\xi}\Sigma_s} \int_{E_i}^{E_{i-1}} \sigma_a(E)\varphi(E)dE = \frac{N_F}{\bar{\xi}\Sigma_s} I_i$$

Resonance escape probability

$$p = \prod_{i=1}^{N}(1-\pi_i) \approx \prod_{i=1}^{N} e^{-\pi_i} = \exp\left[-\sum_{i=1}^{N}\pi_i\right] = \exp\left[-\frac{N_F}{\bar{\xi}\Sigma_s}\sum_{i=1}^{N} I_i\right] = \exp\left[-\frac{N_F}{\bar{\xi}\Sigma_s} I\right]$$

2. Homework problem #7 of Ch. 5

Using the expression for the homogeneous effective resonance integral given in problem 6, show it by concentrating all of the uranium into one half of a reactor's volume that the resonance absorption is reduced to ~75% of the value for a uniformly distributed homogeneous reactor. (Hint: Assume that most of the scattering is due to the moderator even in the volume that contains the uranium.) In addition to the physical impracticability, why is this example, although illustrative of the effect of "lumping," grossly inaccurate?

Ans.) If the uranium is concentrated into one half of the reactor volume, the uranium number density N_U is increased by a factor of 2. If most of the scattering is due to the moderator even in the volume that contains the uranium, however, Σ_s would not change significantly. As a result, the effective resonance integral would be reduced by

$$\frac{I_{eff}^{het}}{I_{eff}^{hom}} = \frac{1}{2^{0.42}} = 0.747$$

<u>This example is grossly inaccurate since the spatial self-shielding effect is completely neglected</u>. Due to the spatial self-shielding effect, the average flux in the fueled region would be much lower than that in the moderator region, and the resonance integral would be reduced further.

3. Homework problem #8 of Ch. 5

For widely spaced absorber elements in a heterogeneous system, the resonance integral has been shown to approximate the following geometric dependency:

$$I = a + b\sqrt{S/M}$$

where a and b are constants depending on the absorber, S is the surface area in cm^2, and M is the specific mass of the absorber in gram. Using the following integral data, calculate the resonance integrals for typical high-temperature gas-cooled reactor (ThO_2) and pressurized water reactor (UO_2) fuel *S/M* ratios:

ThO_2 : $a = 3.4, \quad b = 17.3$

and

UO_2 : $a = 3.0, \quad b = 28.0$

Discuss the corrections needed and the implications of assuming closely packed lattices on the above widely spaced lattice results.

Ans.) HTGR: ThO_2 density = 10.0 g/cm^3, fuel compact radius = 0.62 cm, surface area per unit length = 3.896 cm^2, volume per unit length = 1.208 cm^3

$$\frac{S}{M} = \frac{3.895}{10.0 \times 1.208} = 0.323, \quad I = 3.4 + 17.3\sqrt{0.323} = 13.2 \text{ b}$$

PWR: UO_2 density = 10.97 g/cm^3, fuel pellet radius = 0.41 cm, surface area per unit length = 2.576 cm^2, volume per unit length = 0.528 cm^3

$$\frac{S}{M} = \frac{2.576}{10.97 \times 0.528} = 0.445, \quad I = 3.0 + 28.0\sqrt{0.445} = 21.7 \text{ b}$$

In a closely packed lattice, the neutrons escaping from a fuel rod can be absorbed in other fuel rods, and the escape probability from the fuel lump is reduced relative to an isolated fuel rod (or widely spaced lattice). This results in increased resonance absorption. This Dancoff correction needs to be made for a closely packed lattice.

4. Homework problem #9 of Ch. 5

Show that the area under a Doppler-broadened resonance is the sale regardless of temperature.

Ans.) With the single level Breit-Wigner formula, the Doppler broadened resonance can be approximated by the following ψ function:

$$\psi(x,\xi) = \frac{\xi}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1+w^2} \exp\left[-\frac{\xi^2}{4}(x-w)^2 \right] dw$$

where $x = \dfrac{2(E-E_0)}{\Gamma_t}$, $\xi = \dfrac{\Gamma_t}{\Delta} = \left(\dfrac{A}{4kTE} \right)^{1/2} \Gamma_t$. Thus the area under the Doppler broadened resonance is given by

$$\int_{-\infty}^{\infty} dx \psi(x,\xi) = \int_{-\infty}^{\infty} \frac{dw}{1+w^2} \int_{-\infty}^{\infty} dx \frac{\xi}{2\sqrt{\pi}} \exp\left[-\frac{\xi^2}{4}(x-w)^2 \right]$$

However, the inner integral over x is unity since

$$\int_{-\infty}^{\infty} dx \frac{\xi}{2\sqrt{\pi}} \exp\left[-\frac{\xi^2}{4}(x-w)^2 \right] = \frac{\xi}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dx \exp\left(-\frac{\xi^2}{4} x^2 \right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy\, e^{-y^2} = 1$$

Therefore, the area under the Doppler broadened resonance can be obtained as

$$\int_{-\infty}^{\infty} dx \psi(x,\xi) = \int_{-\infty}^{\infty} \frac{dw}{1+w^2} = \tan^{-1} w \Big|_{-\infty}^{\infty} = \pi \quad \text{(independent of temperatue)}$$

5. Homework problem #10 of Ch. 5

Show for any normalized Gaussian function, $\psi(x)$, that the integral over the square, i.e.,

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$$\int_{-\infty}^{\infty} \psi^2(x)dx$$

always decreases with increasing width s. Suggestion: use $\psi(x) \propto \exp(-x^2/s^2)$.

Ans.) A normalized Gaussian distribution can be written as

$$\psi(x) = \frac{1}{\sqrt{\pi s}} \exp(-x^2/s^2)$$

since

$$\frac{1}{\sqrt{\pi s}} \int_{-\infty}^{\infty} dx \exp\left(-\frac{x^2}{s^2}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-y^2} = 1$$

Thus, the derivative of the integral over its square with respect to the width s is given by

$$\begin{aligned} \frac{d}{ds}\left[\frac{1}{\pi s^2}\int_{-\infty}^{\infty} dx \exp\left(-\frac{2x^2}{s^2}\right)\right] &= -\frac{2}{\pi s^3}\int_{-\infty}^{\infty} dx \exp\left(-\frac{2x^2}{s^2}\right) + \frac{1}{\pi s^2}\int_{-\infty}^{\infty} dx \left(\frac{4x^2}{s^3}\right)\exp\left(-\frac{2x^2}{s^2}\right) \\ &= -\frac{\sqrt{2}}{\pi s^2}\int_{-\infty}^{\infty} dy \exp(-y^2) + \frac{\sqrt{2}}{\pi s^2}\int_{-\infty}^{\infty} dy y^2 \exp(-y^2) \qquad , \\ &= -\frac{\sqrt{2}}{\pi s^2}\sqrt{\pi} + \frac{\sqrt{2}}{\pi s^2}\frac{\sqrt{\pi}}{2} = -\frac{1}{\sqrt{2\pi} s^2} \end{aligned}$$

which is always negative. Therefore, the integral over the square always decreases with increasing width.