NUCL 510 Nuclear Reactor Theory I
Fall 2011

Homework #7

Due November 1

1. Homework problem #2 of Ch. 3

   Consider a H_2O slab that is four diffusion lengths thick with a uniform infinite plane neutron source S neutrons/cm^{2}s located at one diffusion length from one side and three diffusion lengths from the other side. What fraction of source neutrons escapes from the sides? Does this ratio change if the source is placed at the center line of the slab?

   Ans.) Consider a coordinate system of which origin is located at the center line of the slab and let the plane source is located at $x = L$. Then, the diffusion equation and the associated boundary conditions become

$$-D\frac{d^2}{dx^2}\phi(x) + \Sigma_a\phi(x) = 0, \quad x \neq L; \quad \phi(\pm 2L) = 0; \quad \lim_{x\to L^-} D\frac{d\phi}{dx} = \frac{S}{2}; \quad \lim_{x\to L^+}\left[-D\frac{d\phi}{dx}\right] = \frac{S}{2}$$

$$\frac{d^2}{dx^2}\phi(x) - \frac{1}{L^2}\phi(x) = 0, \quad x \neq L \quad \Rightarrow \quad \phi(x) = A\cosh\frac{x}{L} + C\sinh\frac{x}{L}$$

   Applying the boundary condition $\phi(-2L) = 0$, the solution for the left side of the plane source can be written as $\phi(x) = C\sinh[(x/L+2)]$. Thus the source condition at $x = L^-$ becomes $D\left.\frac{d\phi}{dx}\right|_{x=L} = C\frac{D}{L}\cosh 3 = \frac{S}{2}$. Thus the flux distribution in the left side of the plane source is obtained as

$$\phi(x) = \frac{SL}{2D\cosh(3)}\sinh(x/L+2), \quad -2L \le x < L.$$

   Similarly, the solution in the right side of the plane source is obtained as

$$\phi(x) = \frac{SL}{2D\cosh(1)}\sinh(2 - x/L), \quad L < x \le 2L.$$

   As a result, the source neutrons escaping from the sides can be obtained as

$$n_L = D\left.\frac{d\phi}{dx}\right|_{x=-2L} - D\left.\frac{d\phi}{dx}\right|_{x=2L} = \frac{S}{2\cosh(3)} + \frac{S}{2\cosh(1)}.$$

   Thus the fraction of source neutrons that escape from the sides is given by

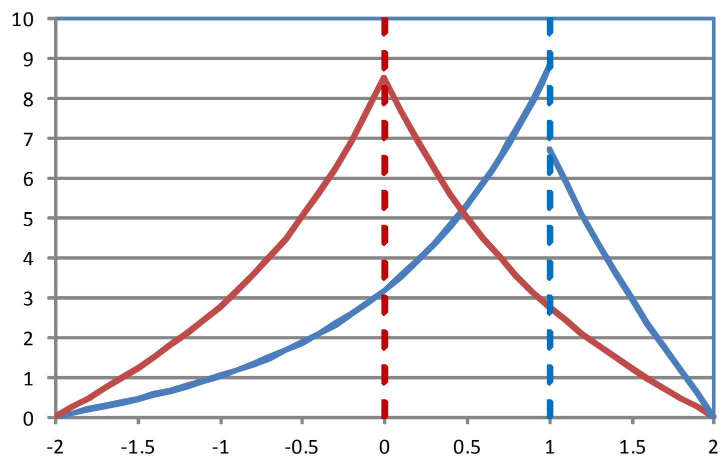
$$f_L = \frac{1}{2\cosh(3)} + \frac{1}{2\cosh(1)} = 0.3737.$$

   If the source placed on the centerline, the flux distribution would be

$$\phi(x) = \frac{SL}{2D\cosh(2)}\sinh(2 - |x|/L).$$

As a result, the leakage fraction would become

$$f_L = \frac{1}{\cosh(2)} = 0.2658\,.$$



2. Homework problem #3 of Ch. 3

Derive the peak-to-average flux ratios for bare critical reactors in the shape of

a) a finite cylinder of minimum volume

The average flux in a finite cylinder is given by

$$\bar{\phi} = \frac{\phi_0}{\pi R^2 H}\int_{-H/2}^{H/2} dz\cos\left(\pi\frac{z}{H}\right)\int_0^R dr(2\pi r)J_0\left(2.405\frac{r}{R}\right)$$

$$= \frac{2\phi_0}{2.405^2}\int_{-1/2}^{1/2} d\tau\cos(\pi\tau)\int_0^{2.405} d\rho\rho J_0(\rho) \quad (\tau = z/H, \quad \rho = 2.405r/R)$$

$$\int_0^{2.405} d\rho\rho J_0(\rho) = \rho J_1(\rho)\Big|_0^{2.405} = 2.405J_1(2.405) = 2.405\times 0.519$$

$$f_\phi = \frac{\phi_0}{\bar{\phi}} = \frac{2.405^2}{2}\times\frac{\pi}{2}\times\frac{1}{2.405J_1(2.405)} = \frac{2.405\pi}{4J_1(2.405)} = 3.639$$

b) a cube

$$\bar{\phi} = \frac{\phi_0}{a^3}\left[\int_{-a/2}^{a/2} dx\cos\left(\pi\frac{x}{a}\right)\right]^3 = \phi_0\left[\int_{-1/2}^{1/2} d\tau\cos(\pi\tau)\right]^3 = \phi_0\left(\frac{2}{\pi}\right)^3$$

$$f_\phi = \frac{\phi_0}{\bar{\phi}} = \left(\frac{\pi}{2}\right)^3 = 3.876$$

NUCL 510 Nuclear Reactor Theory I

Fall 2011

3. Homework problem #4 of Ch. 3

   Show that the fraction of neutrons that leaks from the surface of a rectangular parallelepiped reactor to those lost due to leakage and absorption is $L^2B^2/(1+L^2B^2)$

   Ans.) The flux and current shapes in a rectangular parallelepiped reactor are given by

$$\phi(x,y,z)=\phi_0\cos\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\cos\left(\frac{\pi}{c}z\right)$$

$$\begin{aligned}\vec{J}(x,y,z)&=-D\nabla\phi(x,y,z)\\ &=\frac{\pi D\phi_0}{a}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\cos\left(\frac{\pi}{c}z\right)\vec{e}_x\\ &\quad+\frac{\pi D\phi_0}{b}\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)\cos\left(\frac{\pi}{c}z\right)\vec{e}_y\\ &\quad+\frac{\pi D\phi_0}{c}\cos\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{c}z\right)\vec{e}_z\end{aligned}$$

   Thus the absorption rate is obtained as

$$\begin{aligned}A&=\int_V\Sigma_a\phi dV=\Sigma_a\phi_0\int_{-a/2}^{a/2}dx\cos\left(\pi\frac{x}{a}\right)\int_{-b/2}^{b/2}dy\cos\left(\pi\frac{y}{b}\right)\int_{-c/2}^{c/2}dz\cos\left(\pi\frac{z}{c}\right)\\ &=\Sigma_a\phi_0abc\left[\int_{-1/2}^{1/2}d\tau\cos(\pi\tau)\right]^3=\Sigma_a\phi_0abc\left(\frac{2}{\pi}\right)^3\end{aligned}$$

   The leakage through the surface at $x=a/2$ can be determined as

$$\begin{aligned}L_x^+&=\int_A(\vec{e}_x\cdot\vec{J})dA=\frac{\pi D\phi_0}{a}\sin\left(\frac{\pi}{2}\right)\int_{-b/2}^{b/2}dy\cos\left(\pi\frac{y}{b}\right)\int_{-c/2}^{c/2}dz\cos\left(\pi\frac{z}{c}\right)\\ &=\frac{\pi D\phi_0}{a}bc\left[\int_{-1/2}^{1/2}d\tau\cos(\pi\tau)\right]^2=\frac{\pi D\phi_0}{a}bc\left(\frac{2}{\pi}\right)^2\end{aligned}$$

   Similarly, the leakages through the other surfaces can be determined as

$$L_x^-=\frac{\pi D\phi_0}{a}bc\left(\frac{2}{\pi}\right)^2,\quad L_y^+=L_y^-=\frac{\pi D\phi_0}{b}ca\left(\frac{2}{\pi}\right)^2,\quad L_z^+=L_z^-=\frac{\pi D\phi_0}{c}ab\left(\frac{2}{\pi}\right)^2$$

   Thus the total leakage is given by

$$L=L_x^++L_x^-+L_y^++L_y^-+L_z^++L_z^-=\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2+\left(\frac{\pi}{c}\right)^2\right]D\phi_0abc\left(\frac{2}{\pi}\right)^3=B^2D\phi_0abc\left(\frac{2}{\pi}\right)^3$$

   Consequently, the leakage fraction of the total loss can be determined by

$$\frac{L}{A+L}=\frac{DB^2}{\Sigma_a+DB^2}=\frac{L^2B^2}{1+L^2B^2}$$

NUCL 510 Nuclear Reactor Theory I
Fall 2011

4. Homework problem #5 of Ch. 3

Consider a finite slab of a non-multiplying medium of thickness H (infinite in the other two directions) containing a uniform plane source of neutrons S neutrons/cm^{2}s located on its left side at $x = 0$.

a. Give the boundary conditions for this asymmetrical problem.

$$\phi(H) = 0; \quad \lim_{x\to 0^+}\left[-D\frac{d\phi}{dx}\right] = \frac{S}{2}$$

b. Show that the flux is given by $\phi(x) = \frac{SL}{2D}\frac{\sinh[(H-x)/L]}{\cosh(H/L)}$

$$\frac{d^2}{dx^2}\phi(x) - \frac{1}{L^2}\phi(x) = 0, \quad x \neq 0 \quad \Rightarrow \quad \phi(x) = A\cosh\frac{x}{L} + C\sinh\frac{x}{L}$$

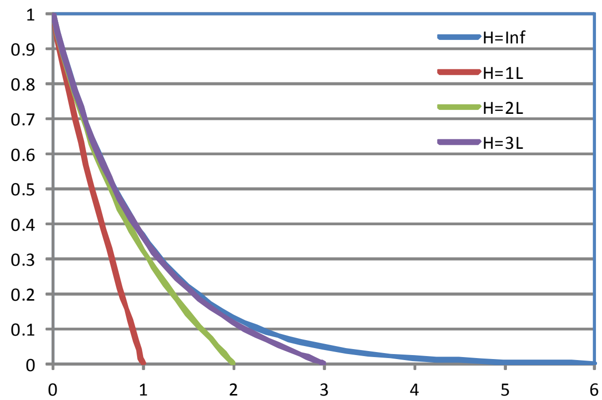
Applying the boundary condition $\phi(H) = 0$, $\phi(x) = C\sinh[(H-x)/L]$.

Thus the source condition at $x = 0^+$ becomes $-D\frac{d\phi}{dx}\bigg|_{x=0} = C\frac{D}{L}\cosh\frac{H}{L} = \frac{S}{2}$.

As a result, the flux distribution is obtained as

$$\phi(x) = \frac{SL}{2D\cosh(H/L)}\sinh\frac{H-x}{L}.$$

c. Compare the flux expression obtained in part (b) with that for an infinite slab (Eq. 3.83) by plotting the relative flux versus the distance from the source measured in diffusion lengths. Consider the effect for several slab thickness expressed in diffusion lengths.



d. What can one conclude concerning the representation of a finite slab flux distribution with the infinite slab equation?

Ans.) When the slab thickness is bigger than three diffusion lengths, the infinite slab equation approximates the finite one reasonably.

NUCL 510 Nuclear Reactor Theory I
Fall 2011

5. Homework problem #7 of Ch. 3

   Prove that the dimensions of the rectangular parallelepiped for a given value of B^2 are those for a cube when the minimum critical reactor volume is achieved. What is the total power of a cubical reactor 4 m on a side containing 80 kg of uniformly distributed ^{235}U if the maximum thermal neutron flux is 5×10^{13} neutrons/cm^{2}s? Use $\sigma_f = 575$ b for ^{235}U.

   Ans.) The problem is to minimize the volume $V = abc$ for a fixed buckling

$$B^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2$$

   This problem can be solved easily using the Lagrange multiplier, that is, from the conditions that the following augmented function has an extreme value:

$$V_a(a,b,c,\lambda) = abc + \lambda\left[B^2 - \left(\frac{\pi}{a}\right)^2 - \left(\frac{\pi}{b}\right)^2 - \left(\frac{\pi}{c}\right)^2\right]$$

   By requiring the derivatives of V_a with respect to *a*, *b*, *c*, and λ to vanish, we have

$$\frac{\partial V_a}{\partial a} = bc - \frac{2\pi^2\lambda}{a^3} = 0 \quad \Rightarrow \quad abc = 2\lambda\frac{\pi^2}{a^2}$$

$$\frac{\partial V_a}{\partial b} = ac - \frac{2\pi^2\lambda}{b^3} = 0 \quad \Rightarrow \quad abc = 2\lambda\frac{\pi^2}{b^2}$$

$$\frac{\partial V_a}{\partial c} = ab - \frac{2\pi^2\lambda}{c^3} = 0 \quad \Rightarrow \quad abc = 2\lambda\frac{\pi^2}{c^2}$$

$$\frac{\partial V_a}{\partial \lambda} = B^2 - \left(\frac{\pi}{a}\right)^2 - \left(\frac{\pi}{b}\right)^2 - \left(\frac{\pi}{c}\right)^2 = 0$$

   From the first three equations, we have $a = b = c$. Using these equations, $V = abc = a^3$ and $B^2 = 3(\pi / a)^2$

   As shown in Problem 2, the peak-to-average flux ratio for a cubical reactor is 3.876. Thus the average flux is $5\times10^{13} / 3.786 = 1.29\times10^{13}$. As a result, the total power can be determined as

$$P = \kappa\Sigma_f\bar{\phi}V = \kappa NV\sigma_f\bar{\phi} = \kappa\frac{\rho N_A}{M}V\sigma_f\bar{\phi} = \kappa\frac{mN_A}{M}\sigma_f\bar{\phi}$$

$$= (200 \text{ MeV/fission})\times(1.6\times10^{-13} \text{ J/MeV})\times\frac{(80 \text{ kg})\times(0.6022\times10^{24})}{235 \text{ g}}\times(575\times10^{-24} \text{ cm}^2)$$

$$\times(1.29\times10^{13} \text{ \#/cm}^2\text{s}) = 4.87\times10^7 \text{ J/s} = 48.7 \text{ MW}$$

NUCL 510 Nuclear Reactor Theory I
Fall 2011

6. Homework problem #8 of Ch. 3

   Determine the reduction in the critical dimensions of a slab reactor consisting of a uniform mixture of uranium and graphite if a 0.5 m thick reflector is placed on both faces. The material buckling of the bare core is 2×10^{-4} cm^{-2}. For both the core and the reflector, use the diffusion coefficient for graphite (D_G=0.85 cm).

   Ans.) The given diffusion coefficient is for thermal neutrons. The corresponding absorption cross section is 2.4×10^{-4} cm^{-1}. Thus the diffusion length is 59 cm. Using Eq. (3.141b) with the critical condition $(\lambda = 1)$,

$$s = \frac{1}{B_m}\tan^{-1}\left[\frac{D_c B_m}{D_r \kappa}\tanh(\kappa\tau)\right]$$

   the reflector saving can be obtained as

$$s = \frac{1}{\sqrt{2\times10^{-4}}}\tan^{-1}\left[\sqrt{2\times10^{-4}}\times59\tanh\left(\frac{50}{59}\right)\right] =\sim 37 \text{ cm}$$

   Note that the critical dimension for the corresponding bare reactor is given by

$$a = \frac{\pi}{B_m} = \frac{\pi}{\sqrt{2\times10^{-4}}} =\sim 222 \text{ cm}$$