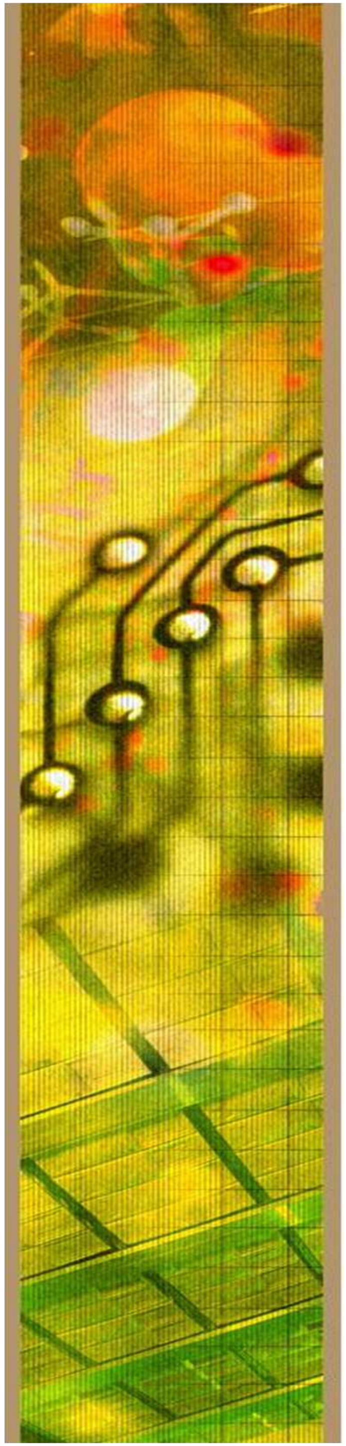

NUCL 510
Nuclear Reactor Theory

Fall 2011
Lecture Note 4

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Integral Equation for Angular Flux (1)

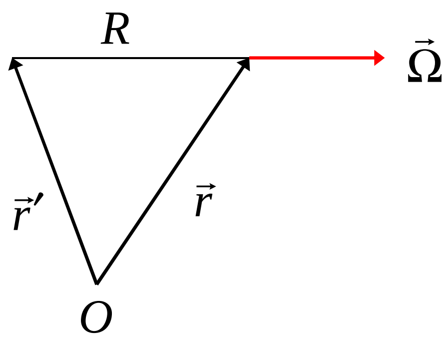
- Steady-state Boltzmann equation for angular flux

$$\Omega \cdot \nabla \psi(\vec{r},\vec{\Omega},E) + \Sigma_t(\vec{r},E)\psi(\vec{r},\vec{\Omega},E) = Q(r,\vec{\Omega},E)$$

$$Q(\vec{r},\vec{\Omega},E) = \int dE' \int d\Omega' \Sigma_s(\vec{r},E' \to E,\vec{\Omega}' \to \vec{\Omega})\psi(\vec{r},\vec{\Omega}',E') + \frac{1}{4\pi}\int dE' \chi(E' \to E)\nu(E')\Sigma_f(\vec{r},E')\phi(\vec{r},E') + S(\vec{r},\vec{\Omega},E)$$

- Integration along the direction of neutron travel
  - The streaming operator is just the directional derivative along the direction of neutron travel

$$-\frac{d}{dR}\psi(\vec{r}-R\vec{\Omega},\vec{\Omega},E) + \Sigma_t(\vec{r}-R\vec{\Omega},E)\psi(\vec{r}-R\vec{\Omega},\vec{\Omega},E) = Q(\vec{r}-R\vec{\Omega},\vec{\Omega},E) \quad \text{(neutron balance at } \vec{r}' = \vec{r}-R\vec{\Omega})$$



$$-\frac{d}{dR}\left\{\psi(\vec{r}-R\vec{\Omega},\vec{\Omega},E)\exp\left[-\int_0^R dR'\Sigma_t(\vec{r}-R'\vec{\Omega},E)\right]\right\} = Q(\vec{r}-R\vec{\Omega},\vec{\Omega},E)\exp\left[-\int_0^R dR'\Sigma_t(\vec{r}-R'\vec{\Omega},E)\right]$$

$$\exp\left[-\int_0^R dR'\Sigma_t(\vec{r}-R'\vec{\Omega},E)\right]$$

(integrating factor)

Integral Equation for Angular Flux (2)

- Integrating back along the path of neutron travel from 0 to R

$$\psi(\vec{r},\vec{\Omega},E)-\psi(\vec{r}-R\vec{\Omega},\vec{\Omega},E)\exp\left[-\int_0^R dR'\Sigma_t(\vec{r}-R'\vec{\Omega},E)\right]$$

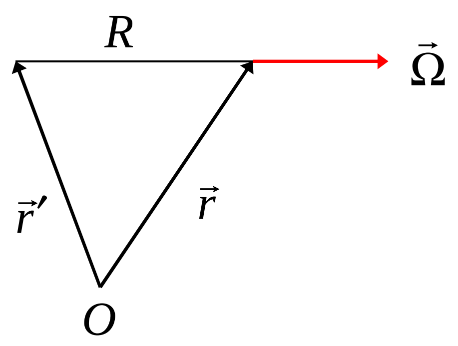
$$=\int_0^R dR' Q(\vec{r}-R'\vec{\Omega},\vec{\Omega},E)\exp\left[-\int_0^{R'} dR''\Sigma_t(\vec{r}-R''\vec{\Omega},E)\right]$$

$$\psi(\vec{r},\vec{\Omega},E)=\int_0^R dR' Q(\vec{r}-R'\vec{\Omega},\vec{\Omega},E)e^{-\tau(\vec{r},\vec{r}-R'\vec{\Omega};E)}+\psi(\vec{r}-R\vec{\Omega},\vec{\Omega},E)e^{-\tau(\vec{r},\vec{r}-R\vec{\Omega};E)}$$

$$\tau(\vec{r},\vec{r}-R\vec{\Omega};E)=\int_0^R dR'\Sigma_t(\vec{r}-R'\vec{\Omega},E)\quad \text{(optical path)}$$

  – If Σ_t is constant,

$$\tau(\vec{r},\vec{r}-R\vec{\Omega};E)=\Sigma_t(E)R=\Sigma_t(E)\,|\,\vec{r}-\vec{r}'\,|$$



- Assuming the angular flux vanishes as R goes to infinity

$$\psi(\vec{r},\vec{\Omega},E)=\int_0^\infty dR' Q(\vec{r}-R'\vec{\Omega},\vec{\Omega},E)e^{-\tau(\vec{r},\vec{r}-R'\vec{\Omega};E)}$$

Integral Equation for Angular Flux (3)

- Since only the neutrons passing r in the direction Ω contribute to $\psi(r,\Omega,E)$, this can be represented as a double integral as

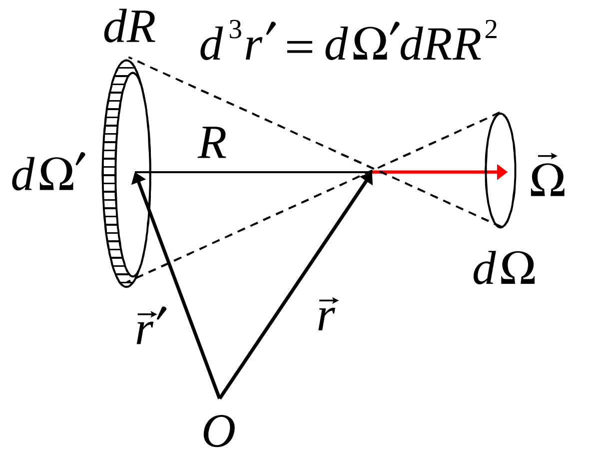
$$\psi(\vec{r},\vec{\Omega},E)=\int d\Omega'\int_0^\infty dR\,Q(\vec{r}-R\vec{\Omega}',\vec{\Omega},E)e^{-\tau(\vec{r},\vec{r}-R\vec{\Omega}';E)}\delta(\vec{\Omega}'-\vec{\Omega})$$

$$\psi(\vec{r},\vec{\Omega},E)=\int d\Omega'\int_0^\infty dR\,R^2 Q(\vec{r}-R\vec{\Omega}',\vec{\Omega},E)\frac{e^{-\tau(\vec{r},\vec{r}-R\vec{\Omega}';E)}}{|\vec{r}-\vec{r}'|^2}\delta(\vec{\Omega}'-\vec{\Omega})$$

$$\vec{r}'=\vec{r}-R\vec{\Omega}'$$
$$R=|\vec{r}-\vec{r}'|$$

- If we take a spherical coordinate system with r' as the origin, the incremental volume centered about r' is given by

$$dV'=d^3r'=d\Omega' dR R^2$$



- Thus the angular flux can be written as a volume integral

$$\psi(\vec{r},\vec{\Omega},E)=\int d^3r' Q(\vec{r}',\vec{\Omega},E)\frac{e^{-\tau(\vec{r},\vec{r}';E)}}{|\vec{r}-\vec{r}'|^2}\delta\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}-\vec{\Omega}\right)$$

Integral Equation for Scalar Flux

- If scattering is isotropic, Q is isotropic and thus the angular flux $\psi(r,\Omega,E)$ becomes

$$\psi(\vec{r},\vec{\Omega},E)=\frac{1}{4\pi}\int d^3r' Q(\vec{r}\,',E)\frac{e^{-\tau(\vec{r},\vec{r}\,';E)}}{\left|\vec{r}-\vec{r}\,'\right|^2}\delta\left(\frac{\vec{r}-\vec{r}\,'}{\left|\vec{r}-\vec{r}\,'\right|}-\vec{\Omega}\right)$$

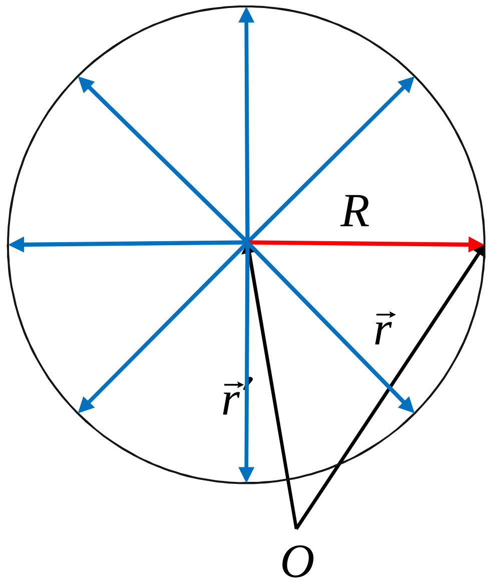
  – Linearly anisotropic scattering source is often take into account by the use of transport corrected total cross section

- Integrating this equation over the angle, we obtained the integral equation for the scalar flux

$$\phi(\vec{r},E)=\int d^3r' Q(\vec{r}\,',E)\frac{e^{-\tau(\vec{r},\vec{r}\,';E)}}{4\pi\left|\vec{r}-\vec{r}\,'\right|^2}$$

$$\tau(\vec{r},\vec{r}\,';E)=\int_0^{|\vec{r}-\vec{r}\,'|} dR'\Sigma_t\left(\vec{r}-R'\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|},E\right)$$

  – Q depends only on the scalar flux



Integral Equation for Collision Density

- Define collision density and transport operator as

$$F(\vec{r},\vec{\Omega},E)=\Sigma_t(\vec{r},E)\psi(\vec{r},\vec{\Omega},E)$$

$$T(\vec{r}',\vec{r}\,|\,\vec{\Omega},E)=\Sigma_t(\vec{r},E)\frac{e^{-\tau(\vec{r},\vec{r}';E)}}{\left|\vec{r}-\vec{r}'\right|^2}\delta\left(\frac{\vec{r}-\vec{r}'}{\left|\vec{r}-\vec{r}'\right|}-\vec{\Omega}\right)$$

- The integral equation for angular flux can be written for the collision density as

$$F(\vec{r},\vec{\Omega},E)=\int d^3r'T(\vec{r}',\vec{r}\,|\,\vec{\Omega},E)Q(\vec{r}',\vec{\Omega},E)$$

$$Q(\vec{r},\vec{\Omega},E)=\int dE'\int d\Omega'C(\vec{\Omega}',E'\to\vec{\Omega},E\,|\,\vec{r})F(\vec{r},\vec{\Omega}',E')+S(\vec{r},\vec{\Omega},E)$$

  – Collision operator

$$C(\vec{\Omega}',E'\to\vec{\Omega},E\,|\,\vec{r})=\frac{\Sigma_s(\vec{r},\vec{\Omega}'\to\vec{\Omega},E'\to E)}{\Sigma_t(\vec{r},E')}$$

$$=\frac{\Sigma_s(\vec{r},E')}{\Sigma_t(\vec{r},E')}\frac{\Sigma_s(\vec{r},\vec{\Omega}'\to\vec{\Omega},E'\to E)}{\Sigma_s(\vec{r},E')}=P_s(\vec{r},E')C_s(\vec{\Omega}',E'\to\vec{\Omega},E\,|\,\vec{r})$$

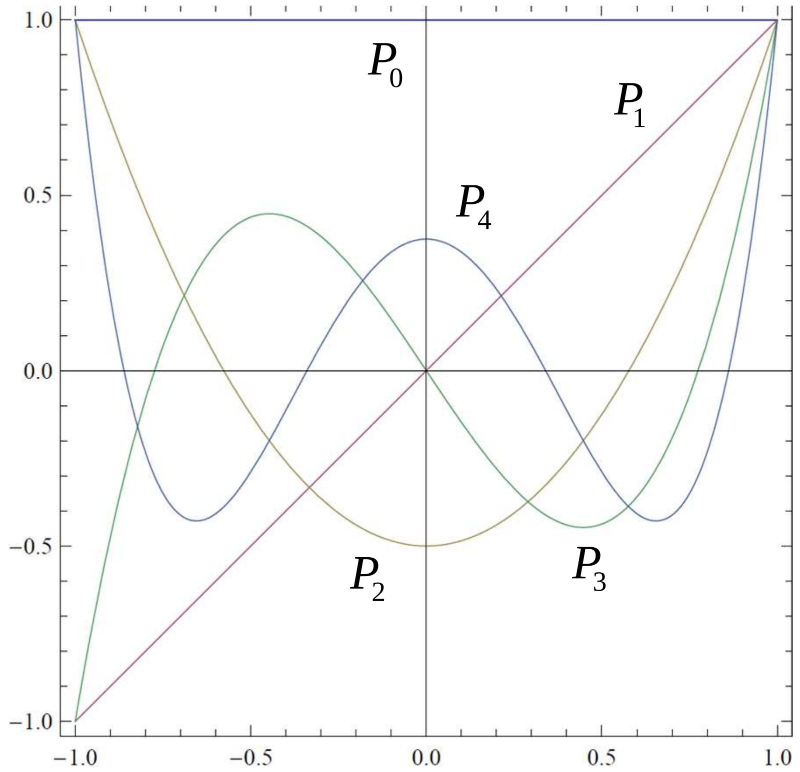
Legendre Polynomial

- Recurrence relation

$$(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu)$$

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu$$

$$P_2(\mu) = \frac{1}{2}(3\mu^2 - 1), \quad P_3(\mu) = \frac{1}{2}\mu(5\mu^2 - 3)$$



- Orthogonality

$$\int_{-1}^{1} P_l(\mu) P_m(\mu) d\mu = \frac{2}{2l+1}\delta_{lm}$$

- Legendre polynomial expansion

$$f(\mu) = \sum_{l=0}^{L} a_l P_l(\mu)$$

$$\int_{-1}^{1} d\mu\, f(\mu) P_m(\mu) = \sum_{l=0}^{L} a_l \int_{-1}^{1} d\mu P_l(\mu) P_m(\mu) = \sum_{l=0}^{L} a_l \frac{2}{2l+1}\delta_{lm} = \frac{2}{2m+1} a_m$$

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} d\mu\, f(\mu) P_m(\mu)$$

Spherical Harmonics

- Spherical harmonics

$$Y_{lk}(\vec{\Omega}) = Y_{lk}(\theta,\varphi) = \left[\frac{(2l+1)(l-k)!}{4\pi(l+k)!}\right]^{1/2} P_l^k(\cos\theta)e^{ik\varphi}, \quad -l \le k \le l$$

$$P_l^k(\mu) = (1-\mu^2)^{k/2}\frac{d^k}{d\mu^k}P_l(\mu) \quad \text{(associate Legendre polynomial)}$$

- Orthogonality

$$\int_{4\pi} \overline{Y}_n^m(\vec{\Omega})Y_l^k(\vec{\Omega})d\Omega = \delta_{nl}\delta_{mk}$$

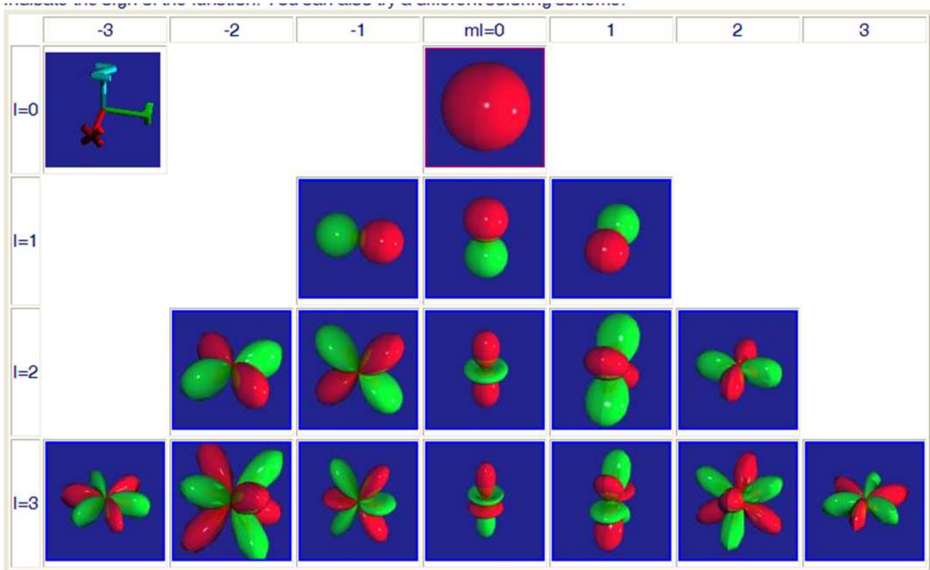
$$d\Omega = \sin\theta d\theta d\varphi = d\mu d\varphi$$

$$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \overline{Y}_n^m(\theta,\varphi)Y_l^k(\theta,\varphi) = \delta_{nl}\delta_{mk}$$

- Spherical harmonics expansion

$$\psi(\vec{r},E,\vec{\Omega}) = \sum_{l=0}^{L}\sum_{k=-l}^{l}\psi_{lk}(\vec{r},E)Y_{lk}(\vec{\Omega})$$

$$\psi_{lk}(\vec{r},E) = \int_{4\pi} d\Omega\, \psi(\vec{r},E,\vec{\Omega})\overline{Y}_{lk}(\vec{\Omega})$$

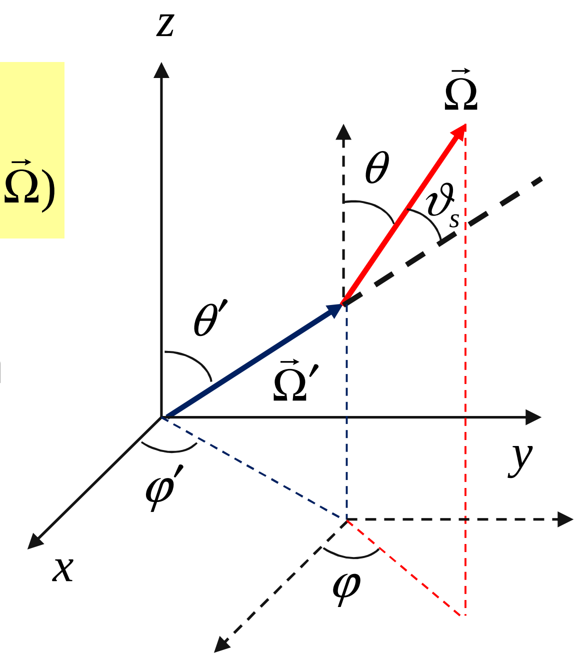


Legendre Polynomial Expansion of Scattering Kernel

- Steady-state Boltzmann equation

$$\vec{\Omega}\cdot\nabla\psi(\vec{r},E,\vec{\Omega})+\Sigma_t(\vec{r},E)\psi(\vec{r},E,\vec{\Omega})$$
$$=\int dE'\int d\Omega'\Sigma_s(\vec{r},E'\to E,\vec{\Omega}'\to\vec{\Omega})\psi(\vec{r},E',\vec{\Omega}')+S(\vec{r},E,\vec{\Omega})$$

- The scattering kernel is commonly represented by a Legendre polynomial expansion in the form



$$\sigma_s^i(E'\to E,\vec{\Omega}'\to\vec{\Omega})=\sigma_s^i(E'\to E,\vec{\Omega}'\cdot\vec{\Omega})$$
$$=\sum_{l=0}^{L}\frac{(2l+1)}{4\pi}\sigma_{sl}^i(E'\to E)P_l(\vec{\Omega}'\cdot\vec{\Omega})$$

$$\sigma_{sl}^i(E'\to E)=2\pi\int_{-1}^{1}d\mu_s\sigma_s^i(E'\to E,\mu_s)P_l(\mu_s)$$

$$\vec{\Omega}\cdot\vec{\Omega}'=\cos\vartheta_s=\mu_s$$

$$\vec{\Omega}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta),\quad \vec{\Omega}'=(\sin\theta'\cos\varphi',\sin\theta'\sin\varphi',\cos\theta')$$

$$\begin{aligned}\cos\vartheta_s &=\sin\theta\cos\varphi\sin\theta'\cos\varphi'+\sin\theta\sin\varphi\sin\theta'\sin\varphi'+\cos\theta\cos\theta'\\ &=\cos\theta\cos\theta'+\sin\theta\sin\theta'(\cos\varphi\cos\varphi'+\sin\varphi\sin\varphi')\\ &=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')\end{aligned}$$

Legendre Polynomial Moments of Scattering Kernel

- For elastic and discrete inelastic scattering cross sections, the energy transfer and the deflection angle are strictly correlated.

$$\sigma_s^i(E' \to E, \mu_s) = \frac{1}{2\pi}\sigma_s^i(E' \to E)\delta[\mu_s - \mu_s(E', E)]$$

- The differential scattering cross section from E' to E can be obtained as

$$\sigma_s^i(E' \to E) = 2\pi\sigma_s^i(E', \mu_c)\left|\frac{d\mu_c}{dE}\right| = \frac{4\pi}{(1-\alpha^i)E'}\sigma_s^i(E', \mu_c)$$

$$\alpha^i = \left(\frac{A^i - 1}{A^i + 1}\right)^2$$

- Using the differential scattering cross section given in terms of μ_c

$$\sigma_s^i(E', \mu_c) = \frac{\sigma_s^i(E')}{2\pi}\sum_{n=0}^{N}\frac{2n+1}{2} f_n^i(E')P_n(\mu_c)$$

- Thus the Legendre polynomial moments can be obtained as

$$\sigma_{sl}^i(E' \to E) = 2\pi\int_{-1}^{1} d\mu_s \sigma_s^i(E' \to E, \mu_s)P_l(\mu_s) = \sigma_s^i(E' \to E)P_l[\mu_s(E', E)]$$

$$\sigma_{sl}^i(E' \to E) = \frac{\sigma_s^i(E')P_l[\mu_s(E', E)]}{(1-\alpha^i)E'}\sum_{n=0}^{N}(2n+1) f_n^i(E')P_n[\mu_c(E', E)]$$

Time-Independent Boltzmann Equation

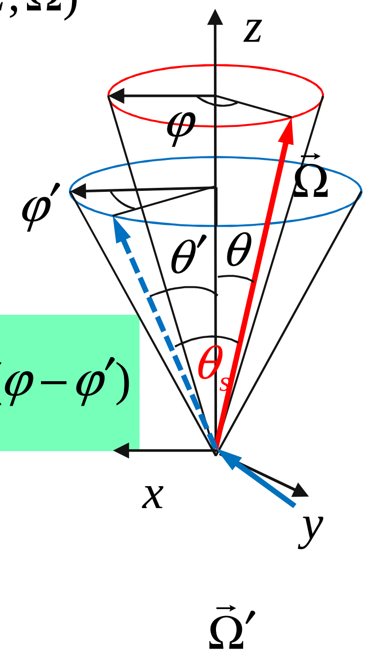
- With this Legendre polynomial expansion of scattering kernel, we have

$$\vec{\Omega}\cdot\nabla\psi(\vec{r},E,\vec{\Omega})+\Sigma_t(\vec{r},E)\psi(\vec{r},E,\vec{\Omega})$$
$$=\sum_i\sum_{l=0}^{L}\frac{2l+1}{4\pi}\int_{E'}dE'\Sigma_{sl}^i(\vec{r},E'\to E)\int_{4\pi}d\Omega' P_l(\vec{\Omega}'\cdot\vec{\Omega})\psi(\vec{r},E',\vec{\Omega}')+S(\vec{r},E,\vec{\Omega})$$

- By the use of the addition theorem of spherical harmonics

$$P_l(\vec{\Omega}'\cdot\vec{\Omega})=\frac{4\pi}{2l+1}\sum_{k=-l}^{l}\bar{Y}_{lk}(\vec{\Omega}')Y_{lk}(\vec{\Omega})$$

$$P_n(\cos\vartheta_s)=P_n(\cos\theta)P_n(\cos\theta')+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi-\varphi')$$



- Boltzmann equation can be written as

$$\vec{\Omega}\cdot\nabla\psi(\vec{r},E,\vec{\Omega})+\Sigma_t(\vec{r},E)\psi(\vec{r},E,\vec{\Omega})$$
$$=\sum_i\sum_{l=0}^{L}\sum_{k=-l}^{l}Y_{lk}(\vec{\Omega})\int_{E'}dE'\Sigma_{sl}^i(\vec{r},E'\to E)\psi_{lk}(\vec{r},E')+S(\vec{r},E,\vec{\Omega})$$

$$\psi_{lk}(\vec{r},E)=\int_{4\pi}d\Omega\bar{Y}_{lk}(\Omega)\psi(\vec{r},E,\vec{\Omega})$$

P_N Equations in Plane Geometry (1)

- In the 1-D plane geometry, the symmetry in the azimuthal angle yields

$$\psi(\vec{r},E,\vec{\Omega}) = \frac{1}{2\pi}\psi(x,E,\mu) \qquad \psi_{lk}(\vec{r},E) = \delta_{k0}\sqrt{\frac{2l+1}{4\pi}}\psi_l(r,E)$$

- Thus Boltzmann equation becomes

$$\mu\frac{d}{dx}\psi(x,E,\mu) + \Sigma_t(x,E)\psi(x,E,\mu) = \sum_i\sum_{l=0}^{L}\frac{2l+1}{2}P_l(\mu)\int_{E'} dE'\Sigma_{sl}^i(x,E'\to E)\psi_l(x,E') + S(x,E,\mu)$$

- Legendre series expansion

$$\psi(x,E,\mu) = \sum_{n=0}^{\infty}\frac{2n+1}{2}\psi_n(x,E)P_n(\mu) \qquad S(x,E,\mu) = \sum_{n=0}^{\infty}\frac{2n+1}{2}S_n(x,E)P_n(\mu)$$

$$\sum_{n=0}^{\infty}\frac{2n+1}{2}[\mu\frac{d}{dx}\psi_n(x,E) + \Sigma_t(x,E)\psi_n(x,E)]P_n(\mu) = \sum_{l=0}^{L}\frac{2l+1}{2}P_l(\mu)\sum_i\int_{E'} dE'\Sigma_{sl}^i(x,E'\to E)\psi_l(x,E') + \sum_{n=0}^{\infty}\frac{2n+1}{2}S_n(x,E)P_n(\mu)$$

P_N Equations in Plane Geometry (2)

- Using the recurrence relation

$$(2n+1)\mu P_n(\mu) = (n+1)P_{n+1}(\mu) + nP_{n-1}(\mu)$$

- Previous equation can be rewritten as

$$\sum_{n=0}^{\infty}[(n+1)\frac{d}{dx}\psi_{n+1}(x,E) + n\frac{d}{dx}\psi_{n-1}(x,E) + (2n+1)\Sigma_t(x,E)\psi_n(x,E)]P_n(\mu)$$

$$= \sum_{l=0}^{L}(2l+1)P_l(\mu)\sum_i \int_{E'} dE'\Sigma_{sl}^i(x,E'\to E)\psi_l(x,E') + \sum_{n=0}^{\infty}(2n+1)S_n(x,E)P_n(\mu)$$

- Using the orthogonality of the Legendre polynomials, the spherical harmonics equations are obtained as

$$\frac{n+1}{2n+1}\frac{d}{dx}\psi_{n+1}(x,E) + \frac{n}{2n+1}\frac{d}{dx}\psi_{n-1}(x,E) + \Sigma_t(x,E)\psi_n(x,E)$$

$$= \sum_i \int_{E'} dE'\Sigma_{sn}^i(x,E'\to E)\psi_n(x,E') + S_n(x,E), \quad n = 0,1,2,\cdots$$

$$\psi_{-1} = 0$$

$$\Sigma_{sn}^i = 0 \text{ for } n > L$$

- The P_N approximation is obtained by considering the first $N+1$ equations of this set and neglecting the derivative of ψ_{N+1}

P_1 Theory (1)

- P_1 expansion of angular flux

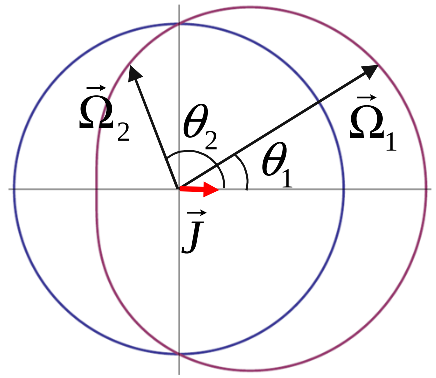
$$\psi(\vec{r},E,\vec{\Omega}) = \frac{1}{4\pi}[\phi(\vec{r},E) + 3\vec{\Omega}\cdot\vec{J}(\vec{r},E)]$$

$$\psi(x,E,\mu) = \frac{1}{2}[\phi(x,E) + 3\mu J(x,E)]$$

$$\int d\Omega\,\Omega_x^{n_1}\Omega_y^{n_2}\Omega_z^{n_3} = \begin{cases} \dfrac{4\pi}{n_1+n_2+n_3+1}\dfrac{(n/2)!}{(n_1/2)!(n_2/2)!(n_3/2)!}\dfrac{n_1!n_2!n_3!}{n!}, & \text{if } n_1,\ n_2,\ \text{and } n_3 \text{ are all even} \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{4\pi} d\Omega = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi = 4\pi \qquad \int_{4\pi} d\Omega\,\vec{\Omega} = \vec{0}$$

$$\int_{4\pi} d\Omega\,\vec{\Omega}\vec{\Omega} = \frac{4\pi}{3}\mathbf{I} \qquad \int_{4\pi} d\Omega\,\vec{\Omega}\vec{\Omega}\vec{\Omega} = 0$$



$$\int_{4\pi} d\Omega\,\psi(\vec{r},E,\vec{\Omega}) = \phi(\vec{r},E) \qquad \int_{4\pi} d\Omega\,\vec{\Omega}\,\psi(\vec{r},E,\vec{\Omega}) = \vec{J}(\vec{r},E)$$

$$\Omega_x = \sqrt{\frac{2\pi}{3}}(\overline{Y}_{1,-1} - \overline{Y}_{1,1}), \quad \Omega_y = i\sqrt{\frac{2\pi}{3}}(\overline{Y}_{1,1} + \overline{Y}_{1,-1}), \quad \Omega_z = \sqrt{\frac{4\pi}{3}}\overline{Y}_{1,0}$$

P_1 Theory (2)

- Partial currents

$$J^+(\vec{r},E)=\int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega\vec{n}\cdot\vec{\Omega}\psi(\vec{r},E,\vec{\Omega})=\frac{1}{4\pi}\int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega\vec{n}\cdot\vec{\Omega}[\phi(\vec{r},E)+3\vec{\Omega}\cdot\vec{J}(\vec{r},E)]$$

$$=\frac{1}{4\pi}\int_0^{2\pi} d\varphi\int_0^1 d\mu[\mu\phi(\vec{r},E)+3\mu^2 J_n(\vec{r},E)]=\frac{1}{4}\phi(\vec{r},E)+\frac{1}{2}J_n(\vec{r},E)$$

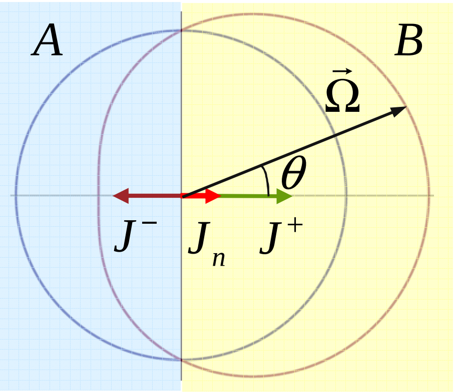
$$J^-(\vec{r},E)=\int_{\vec{n}\cdot\vec{\Omega}<0} d\Omega\,|\vec{n}\cdot\vec{\Omega}|\,\psi(\vec{r},E,\vec{\Omega})=\frac{1}{4\pi}\int_{\vec{n}\cdot\vec{\Omega}<0} d\Omega\,|\vec{n}\cdot\vec{\Omega}|\,[\phi(\vec{r},E)+3\vec{\Omega}\cdot\vec{J}(\vec{r},E)]$$

$$=-\frac{1}{4\pi}\int_0^{2\pi} d\varphi\int_{-1}^0 d\mu[\mu\phi(\vec{r},E)+3\mu^2 J_n(\vec{r},E)]=\frac{1}{4}\phi(\vec{r},E)-\frac{1}{2}J_n(\vec{r},E)$$

$$\phi(\vec{r},E)=2[J^+(\vec{r},E)+J^-(\vec{r},E)]$$

$$J_n(\vec{r},E)=J^+(\vec{r},E)-J^-(\vec{r},E)$$

$$\mu=\vec{n}\cdot\vec{\Omega}=\cos\theta$$



- Albedo at an interface

$$\alpha=\frac{J^-(\vec{r}_i,E)}{J^+(\vec{r}_i,E)}$$

Reflection coefficient of region A with respect to B

P_1 Theory (3)

- Integration of Boltzmann transport equation

$$\int_{4\pi} d\Omega \vec{\Omega}\cdot\nabla\psi(\vec{r},E,\vec{\Omega}) + \Sigma_t(\vec{r},E)\int_{4\pi} d\Omega\,\psi(\vec{r},E,\vec{\Omega})$$
$$= \sum_i \sum_{l=0}^{L} \sum_{k=-l}^{l} \left[\int_{4\pi} d\Omega Y_{lk}(\vec{\Omega})\right] \int_{E'} dE' \Sigma_{sl}^i(\vec{r},E'\to E)\psi_{lk}(\vec{r},E') + \int_{4\pi} d\Omega S(\vec{r},E,\vec{\Omega})$$

$$\int_{4\pi} d\Omega \vec{\Omega}\cdot\nabla\psi = \int_{4\pi} d\Omega(\nabla\cdot\vec{\Omega}\psi) = \nabla\cdot\left[\int_{4\pi} d\Omega\vec{\Omega}\psi\right] = \nabla\cdot\vec{J}(\vec{r},E)$$

$$\int_{4\pi} d\Omega\,\psi(\vec{r},E,\vec{\Omega}) = \phi(\vec{r},E)$$

$$\int_{4\pi} d\Omega Y_{lk}(\vec{\Omega}) = \sqrt{4\pi}\int_{4\pi} d\Omega \bar{Y}_{00}(\vec{\Omega})Y_{lk}(\vec{\Omega}) = \sqrt{4\pi}\delta_{l0}\delta_{k0}, \quad \sqrt{4\pi}\psi_{00} = \phi$$

$$\int_{4\pi} d\Omega S(\vec{r},E,\vec{\Omega}) = S_0(\vec{r},E)$$

- Flux or neutron balance equation

$$\nabla\cdot\vec{J}(\vec{r},E) + \Sigma_t(\vec{r},E)\phi(\vec{r},E) = \sum_i \int_{E'} dE' \Sigma_s^i(\vec{r},E'\to E)\phi(\vec{r},E') + S_0(\vec{r},E)$$

P_1 Theory (4)

- Integration of (Boltzmann transport equation $\times\Omega$)

$$\int_{4\pi} d\Omega\vec{\Omega}\vec{\Omega}\cdot\nabla\psi(\vec{r},E,\vec{\Omega})+\Sigma_t(\vec{r},E)\int_{4\pi} d\Omega\vec{\Omega}\psi(\vec{r},E,\vec{\Omega})$$
$$=\sum_i\sum_{l=0}^{L}\sum_{k=-l}^{l}\left[\int_{4\pi} d\Omega\vec{\Omega}Y_{lk}(\vec{\Omega})\right]\int_{E'} dE'\Sigma_{sl}^i(\vec{r},E'\to E)\psi_{lk}(\vec{r},E')+\int_{4\pi} d\Omega\vec{\Omega}S(\vec{r},E,\vec{\Omega})$$

$$\int_{4\pi} d\Omega\vec{\Omega}\vec{\Omega}\cdot\nabla\psi=\frac{1}{4\pi}\int_{4\pi} d\Omega\vec{\Omega}\vec{\Omega}\cdot\nabla\phi+\frac{3}{4\pi}\int_{4\pi} d\Omega\vec{\Omega}\vec{\Omega}\cdot\nabla(\vec{\Omega}\cdot\vec{J})=\frac{1}{3}\nabla\phi \quad (\mathrm{P}_1 \text{ expansion})$$

$$\int_{4\pi} d\Omega\vec{\Omega}\psi(\vec{r},E,\vec{\Omega})=\vec{J}(\vec{r},E)$$

$$\int_{4\pi} d\Omega\Omega_x Y_{lk}=\sqrt{\frac{2\pi}{3}}\int_{4\pi} d\Omega(\bar{Y}_{1,-1}-\bar{Y}_{1,1})Y_{lk}=\sqrt{\frac{2\pi}{3}}\delta_{l1}(\delta_{k,-1}-\delta_{k,1}),\quad \sqrt{\frac{2\pi}{3}}(\psi_{11}-\psi_{1,-1})=J_x$$

$$\int_{4\pi} d\Omega\Omega_y Y_{lk}=i\sqrt{\frac{2\pi}{3}}\int_{4\pi} d\Omega(\bar{Y}_{1,1}+\bar{Y}_{1,-1})Y_{lk}=i\sqrt{\frac{2\pi}{3}}\delta_{l1}(\delta_{k1}+\delta_{k,-1}),\quad i\sqrt{\frac{2\pi}{3}}(\psi_{11}+\psi_{1,-1})=J_y$$

$$\int_{4\pi} d\Omega\Omega_z Y_{lk}=\sqrt{\frac{4\pi}{3}}\int_{4\pi} d\Omega\bar{Y}_{1,0}Y_{lk}=\sqrt{\frac{4\pi}{3}}\delta_{l1}\delta_{k0},\quad \sqrt{\frac{4\pi}{3}}\psi_{10}=J_z$$

$$\int_{4\pi} d\Omega\vec{\Omega}S(\vec{r},E,\vec{\Omega})=\vec{S}_1(\vec{r},E)$$

P_1 Theory (5)

- Current or P_1 equation

$$\frac{1}{3}\nabla\phi(\vec{r},E)+\Sigma_t(\vec{r},E)\vec{J}(\vec{r},E)=\sum_i\int_{E'}dE'\Sigma_{s1}^i(\vec{r},E'\rightarrow E)\vec{J}(\vec{r},E')+\vec{S}_1(\vec{r},E)$$

  – If there is no anisotropic independent source, $S_1=0$

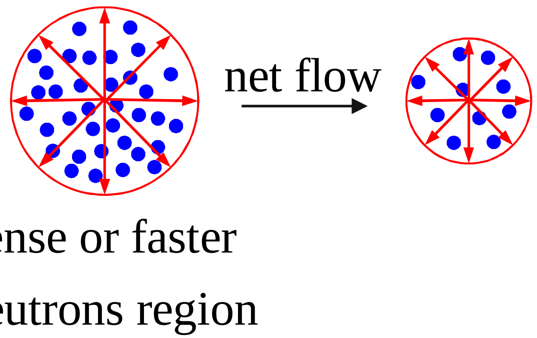
$$\frac{1}{3}\nabla\phi(\vec{r},E)+\Sigma_t(\vec{r},E)\vec{J}(\vec{r},E)=\sum_i\int_{E'}dE'\Sigma_{s1}^i(\vec{r},E'\rightarrow E)\vec{J}(\vec{r},E')$$

  – If the scattering is isotropic in LS, $\Sigma_{s1}=0$ (Fick's law of diffusion)

$$\frac{1}{3}\nabla\phi(\vec{r},E)+\Sigma_t(\vec{r},E)\vec{J}(\vec{r},E)=0$$

$$\vec{J}(\vec{r},E)=-D(\vec{r},E)\nabla\phi(\vec{r},E)$$

$$D(\vec{r},E)=\frac{1}{3\Sigma_t(\vec{r},E)}$$



    • *The current is due to more neutrons moving from the areas of high neutron density to the areas of low density than the other way around*

P_1 Theory (6)

- Transport correction
  - If no energy loss in anisotropic scattering is assumed

$$\Sigma_{s1}^{i}(\vec{r}, E' \to E) = \Sigma_{s1}^{i}(\vec{r}, E')\delta(E - E')$$

$$\frac{1}{3}\nabla\phi(\vec{r}, E) + \left[\Sigma_t(\vec{r}, E) - \sum_i \Sigma_{s1}^{i}(\vec{r}, E)\right]\vec{J}(\vec{r}, E) = 0 \qquad \vec{J}(\vec{r}, E) = -D(\vec{r}, E)\nabla\phi(\vec{r}, E)$$

$$D(\vec{r}, E) = \frac{1}{3\Sigma_{tr}(\vec{r}, E)}, \quad \Sigma_{tr}(\vec{r}, E) = \Sigma_t(\vec{r}, E) - \Sigma_{s1}(\vec{r}, E)$$

  - Transport (transport corrected total) cross section

$$\sigma_{tr}^{i}(E) = \sigma_t^{i}(E) - \sigma_{s1}^{i}(E) = \sigma_t^{i}(E) - \bar{\mu}^{i}(E)\sigma_s^{i}(E)$$

$$\sigma_{s1}^{i}(E \to E') = 2\pi\int_{-1}^{1} d\mu_s \sigma_s^{i}(E \to E', \mu_s) P_1(\mu_s)$$

$$= \int_{-1}^{1} d\mu_s \sigma_s^{i}(E \to E')\delta[\mu_s - \mu_s(E, E')]\mu_s = \mu_s(E', E)\sigma_s^{i}(E \to E')$$

$$\sigma_{s1}^{i}(E) = \int_{E'} dE' \mu_s(E, E')\sigma_s^{i}(E \to E') = \bar{\mu}^{i}(E)\sigma_s^{i}(E)$$

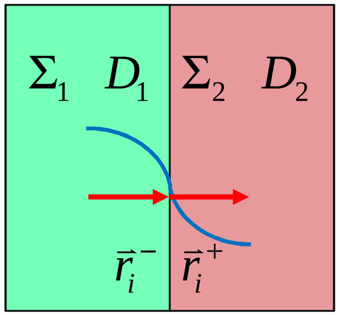
Diffusion Equation (1)

- Inserting the current equation into the neutron balance equation yields

$$-\nabla\cdot D(\vec{r},E)\nabla\phi(\vec{r},E)+\Sigma_t(\vec{r},E)\phi(\vec{r},E)=\sum_i\int_{E'}dE'\Sigma_s^i(\vec{r},E'\to E)\phi(\vec{r},E')+S_0(\vec{r},E)$$

  – Diffusion equation is a differential equation of second order; thus, it requires two boundary conditions and two conditions on each interface.

- Interface conditions



$$\psi(\vec{r}_i^{\,-},E,\vec{\Omega})=\psi(\vec{r}_i^{\,+},E,\vec{\Omega})\quad\text{(continuity of angular flux)}$$

$$\int_{4\pi}\psi(\vec{r}_i^{\,-},E,\vec{\Omega})d\Omega=\int_{4\pi}\psi(\vec{r}_i^{\,+},E,\vec{\Omega})d\Omega$$

$$\phi(\vec{r}_i^{\,-},E)=\phi(\vec{r}_i^{\,+},E)\quad\text{(continuity of flux)}$$

$$\int_{4\pi}\vec{n}\cdot\vec{\Omega}\,\psi(\vec{r}_i^{\,-},E,\vec{\Omega})d\Omega=\int_{4\pi}\vec{n}\cdot\vec{\Omega}\,\psi(\vec{r}_i^{\,+},E,\vec{\Omega})d\Omega$$

$$J_n(\vec{r}_i^{\,-},E)=J_n(\vec{r}_i^{\,+},E)\quad\text{(continuity of normal component of current)}$$

$$\vec{n}\cdot D(\vec{r}_i^{\,-},E)\nabla\phi(\vec{r}_i^{\,-},E)=\vec{n}\cdot D(\vec{r}_i^{\,+},E)\nabla\phi(\vec{r}_i^{\,+},E)$$

  – Flux derivative is discontinuous at an interface of different materials

Diffusion Equation (2)

- Vacuum boundary condition
  - Since the detailed angular dependence is not incorporated, the correct boundary condition should be replaced by an appropriate approximation

$$\psi(\vec{r}_v, E, \vec{\Omega}_{in}) = 0 \quad \text{for } \vec{\Omega}_{in} \cdot \vec{n} < 0 \quad \text{(no incoming angular flux)}$$

  - Zero net incoming current condition

$$J^-(\vec{r}_v, E) = \frac{1}{4}\phi(\vec{r}_v, E) - \frac{1}{2}J_n(\vec{r}_v, E) = 0$$

$$\phi(\vec{r}_v, E) + 2D(\vec{r}_v, E)\frac{d\phi(\vec{r}_v, E)}{dn} = 0, \quad D = \frac{\lambda_{tr}}{3}$$

First order Taylor expansion of

$$\phi(\vec{r}_v + \frac{2}{3}\lambda_{tr}\vec{n}, E) = 0$$

  - Zero flux at an extrapolated location
    - *With more accurate transport theoretical derivation, the factor 2/3 is replaced by 0.711*

$$\phi(\vec{r}_v + 0.711\lambda_{tr}\vec{n}, E) = 0$$