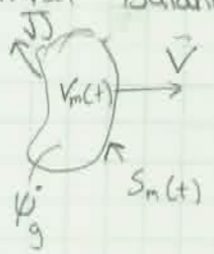
General Balance equation (Hilbert)



- ψ some property enclosed in $V(t)$ per unit volume

$\int_V \psi \, dV$ = total amount enclosed in $V(t)$ (of ψ)

$\mathbb{J}$: flux of property across the surface (intrinsic) (efflux)

$-\oint \mathbb{J} \cdot \vec{n} \, dS$ = net flow in across the surface

$\dot{\psi}_g$ generation of ψ per unit volume

$\int_V \dot{\psi} \, dV$ = total internal generation of ψ

Total change in vol = influx through surface + generation in vol

$$\frac{D}{Dt}\int_{V_m} \psi \, dV = -\oint_{S_m} \mathbb{J}\cdot\vec{n}\, dS + \int_{V_m} \dot{\psi}_g \, dV \qquad (1)$$

Integral balance $\rightarrow$ Differential balance

- Reynolds transport volume

$$\frac{D}{Dt}\int_{V_m} \psi \, dV = \int_V \left[\frac{\partial \psi}{\partial t} + \nabla\cdot(\psi \vec{V})\right] dV \qquad (2)$$

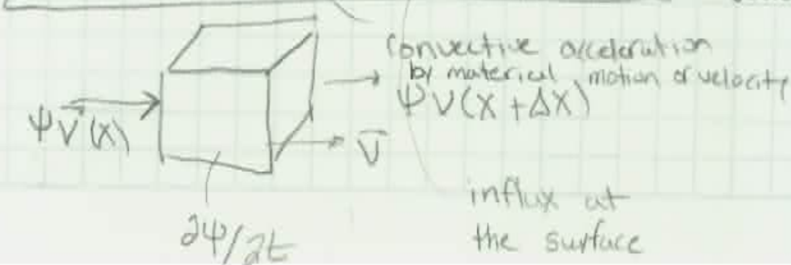
- Greens theorem

$$-\oint \mathbb{J}\cdot\vec{n}\, dS = -\int_V \nabla\cdot\mathbb{J}\, dV \qquad (3)$$

Convert (2)(3) into (1)

$$\int_V \left[\frac{\partial \psi}{\partial t} + \nabla\cdot(\psi \vec{J})\right] dV = -\int_V \nabla\cdot\mathbb{J}\, dV + \int_V \dot{\psi}_g \, dV$$

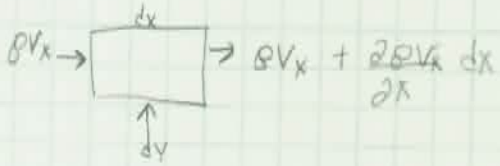
$$\boxed{\frac{\partial \psi}{\partial t} + \nabla\cdot(\psi\vec{V}) = -\nabla\cdot\mathbb{J} + \dot{\psi}_g}$$ Local differential balance



Mass Balance Eqn: Continuity

Quantity to balance $^{mass}/_{unit\ vol}$ → ρ density

$$\left\{\begin{array}{l} \psi = \rho \\ \vec{J} = 0 \\ \dot{\psi}_g = 0 \end{array}\right. \rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{v} = 0}$$



$$\frac{\partial}{\partial t}(\rho\, dx\, dy\, dz)$$

$$= dy\, dz\left[\rho V_x - \left(\rho V_x + \frac{\partial \rho}{\partial x} dx\right)\right]$$

$$+ dx\, dy\left[\quad\right]$$

$$+ dx\, z\left[\quad\right]$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_x}{\partial x} + \frac{\partial \rho V_y}{\partial y} + \frac{\partial \rho V_z}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{v} = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho + \rho\nabla\cdot\vec{v} = 0$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\vec{v}$$

* incompressible

$\frac{D\rho}{Dt} = 0$ divergence of velocity is 0 ⇒ ρ = constant

$\nabla\cdot\vec{v} = 0$

- inviscid or irrotational

$$\vec{v} = \nabla\phi$$

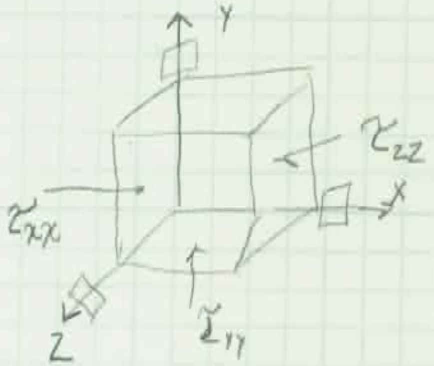
$\nabla^2\phi = 0$ ← now solvable

Momentum Balance Eqn : Eqn. of motion

$\vec{a} M = \Sigma \vec{F}$

Forces
- gravity
- surface

shear stress



τ_{xx}: x component force acting on x surface

τ_{yx}: x component force action on y surface

τ_{zx}: " " " z surface

3 Directions of forces
3 Directions of surfaces
} 9 components $\underline{\underline{\tau}}$

Quantity to balance: momentum $\rho\vec{v}$

$\vec{\psi} = \rho\vec{v}$

$\mathbb{J} = \Pi = P\,\mathbb{I} + \underline{\underline{\tau}}$

pressure, shear stress

$\dot{\psi}_g = \rho\hat{F}$ (Body force)
$= \rho\vec{g}$ (gravitational force)

$$\boxed{\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = -\nabla P - \nabla\cdot\underline{\underline{\tau}} + \rho\vec{g}}$$

$$LHS = \rho\frac{\partial\vec{v}}{\partial t} + \vec{v}\frac{\partial\rho}{\partial t} + \rho\vec{v}\cdot(\nabla\vec{v}) + \vec{v}(\nabla\cdot\rho\vec{v})$$

$$= \rho\left[\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right] + \vec{v}\,\cancel{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v}\right]}$$ continuity equation

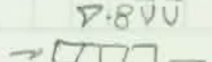
$$= \rho\frac{D\vec{v}}{Dt} = \rho\vec{a}$$

$\rho\vec{a} = \Sigma$ Forces = pressure forces + viscous forces + body forces
(surface)

$$\boxed{\rho\frac{D\vec{v}}{Dt} = -\nabla P - \nabla\cdot\underline{\underline{\tau}} + \rho\vec{g}}$$ Equation of motion conservative form

$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}$

$\nabla\cdot\rho\vec{v}\vec{v}$



numerical losses
i.e. non-conservative

Natural Convection

Boussinesq Assumption
- Density change due to thermal expansion
- Density change only impacts gravity term
- Thermal expansion coefficient

$$\beta = \left(\frac{1}{v}\right)\frac{\partial v}{\partial t}\Big)_p \qquad v = \text{specific volume}$$

$$= -\frac{1}{\rho}\frac{\partial \rho}{\partial t}\Big)_p$$

$$d\rho = -\rho\beta dT \qquad \text{Reference density, temp } \bar{\rho}, \bar{T}$$

$$\rho - \bar{\rho} = -\bar{\rho}\beta(T-\bar{T}) \xrightarrow[\text{insert}]{} \text{momentum eqn}$$

$$\bar{\rho}\frac{D\vec{v}}{Dt} = -\nabla P - \nabla\cdot\tau + [\bar{\rho} - \bar{\rho}\beta(T-\bar{T})]\vec{g}$$

Mechanical Energy Equation

$$\vec{v}\cdot\left[\rho\frac{D\vec{v}}{Dt} = -\nabla P - \nabla\cdot\tau + \rho\vec{g}\right]$$

$$\rho\frac{D v^2/2}{Dt} = -\vec{v}\cdot\nabla P - \vec{v}\cdot(\nabla\cdot\tau) + \rho\vec{v}\cdot\vec{g}$$

$$\underbrace{\frac{\partial \rho v^2/2}{\partial t}}_{\text{rate of change of kinetic energy}} + \underbrace{\nabla\cdot\left(\frac{1}{2}\rho v^2\vec{v}\right)}_{\text{rate of kinetic energy conversion}} = \underbrace{-\nabla\cdot(p\vec{v})}_{\text{work done by pressure}} + \underbrace{p\nabla\vec{v}}_{\text{reversible conversion to internal energy}}$$

$$\underbrace{-\nabla\cdot(\tau\cdot\vec{v})}_{\text{work done by viscous}} + \underbrace{\tau:\nabla\vec{v}}_{\text{irreversible conversion to internal energy}}$$

$$+ \underbrace{\rho(\vec{v}\cdot\vec{g})}_{\text{work done by gravity}}$$