NUCL 510 Nuclear Reactor Theory I
Fall 2011

Homework #4

Due September 22

1. A spherical target of radius R is placed in a mono-directional beam of neutrons of intensity I. The area of the beam is larger than the cross sectional area of the sphere. Show that the total reaction rate within the sphere in such a beam is equal to the reaction rate when the sphere is completely immersed in an isotropic flux of magnitude equal to I.

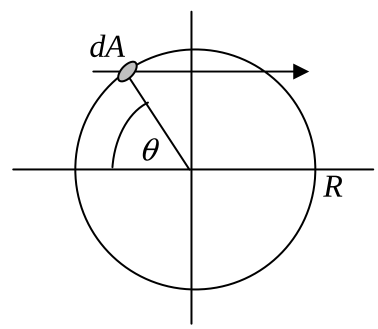
   (Hint) For the mono-directional beam, consider the number of neutrons coming into the sphere through an incremental area dA at an angle θ with respect to the beam direction and then the number of interactions made by these neutrons over the traveling path $2R\cos\theta$ within the spherical target. Similarly, for the isotropic flux case, consider the number of neutrons passing through an incremental area dA into an incremental solid angle $d\Omega$ about $\vec{\Omega}$ and then the number of interactions mae by these neutrons within the spherical target.

   Mono-directional beam

   Number of neutrons coming into the sphere through the area dA: $I\left|\vec{e}_x \cdot \vec{n}\right| dA$

   Number of interactions made by these neutrons over the distance $2R\cos\theta$

$$R_A = \int_0^{2R\cos\theta} \Sigma_t I(x)dx = \int_0^{2R\cos\theta} \Sigma_t I e^{-\Sigma_t x} dx$$
$$= I(1 - e^{-2\Sigma_t R\cos\theta})$$



   Total reaction rate within the sphere

$$R = \int_A I(1 - e^{-2\Sigma_t R\cos\theta})\vec{e}_x \cdot \vec{n} dA$$
$$= I\int_{\theta=0}^{\pi/2}\int_{\varphi=0}^{2\pi}(1 - e^{-2\Sigma_t R\cos\theta})\cos\theta R^2 \sin\theta d\varphi d\theta$$
$$= 2\pi R^2 I\int_0^1 (1 - e^{-2\Sigma_t R\mu})\mu d\mu$$
$$= \pi R^2 I\left[1 + \frac{1}{\Sigma_t R}e^{-2\Sigma_t R} + \frac{1}{2(\Sigma_t R)^2}(e^{-2\Sigma_t R} - 1)\right]$$

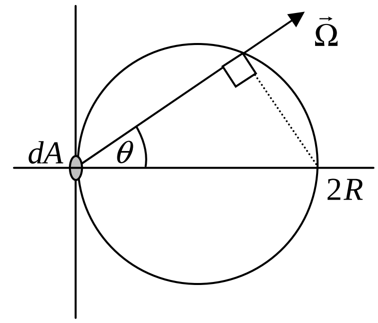
   Isotropic flux

   Number of neutrons passing through the area dA into a solid angle $d\Omega$ about $\vec{\Omega}$:

$$\frac{I}{4\pi}\left|\vec{e}_x \cdot \vec{\Omega}\right| d\Omega dA$$

   Number of interactions made by these neutrons within the sphere

$$R_A = \frac{I}{4\pi}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(1-e^{-2\Sigma_t R\cos\theta})\cos\theta\sin\theta d\phi d\theta$$
$$= \frac{I}{2}\int_0^1 (1-e^{-2\Sigma_t R\mu})\mu d\mu$$
$$= \frac{I}{4}\left[1+\frac{1}{\Sigma_t R}e^{-2\Sigma_t R}+\frac{1}{2(\Sigma_t R)^2}(e^{-2\Sigma_t R}-1)\right]$$



Total reaction rate within the sphere

$$R = \int_A R_A dA = R_A(4\pi R^2)$$
$$= \pi R^2 I\left[1+\frac{1}{\Sigma_t R}e^{-2\Sigma_t R}+\frac{1}{2(\Sigma_t R)^2}(e^{-2\Sigma_t R}-1)\right]$$

2. Let f_k be an eigenfunction of an operator A corresponding to eigenvalue a_k, and let g_l be an eigenfunction of the adjoint operator A^* with eigenvalue b_l. Show that either b_l is the complex conjugate of a_k the eigenvector g_l is orthogonal to f_k.

$$Af_k = a_k f_k \quad \Rightarrow \quad (g_l, Af_k) = (g_l, a_k f_k) = a_k(g_l, f_k)$$
$$A^* g_l = b_l g_l \quad \Rightarrow \quad (A^* g_l, f_k) = (g_l, Af_k) = (b_l g_l, f_k) = \overline{b_l}(g_l, f_k)$$
$$\Rightarrow \quad (a_k - \overline{b_l})(g_l, f_k) = 0 \quad \Rightarrow \quad a_k = \overline{b_l} \text{ or } (g_l, f_k) = \delta_{kl}$$

3. Represent $x^5 - x^3$ in terms of Legendre polynomials $P_n(x)$.

$$f(x) = x^5 - x^3 = \sum_{n=0}^{5} a_n P_n(x); \quad a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx = \frac{2n+1}{2}\int_{-1}^{1}(x^5-x^3)P_n(x)dx$$
$$a_0 = \frac{1}{2}\int_{-1}^{1}(x^5-x^3)dx = 0;$$
$$a_1 = \frac{3}{2}\int_{-1}^{1}(x^5-x^3)xdx = -\frac{6}{35};$$
$$a_2 = \frac{5}{2}\int_{-1}^{1}(x^5-x^3)\frac{1}{2}(3x^2-1)dx = 0;$$
$$a_3 = \frac{7}{2}\int_{-1}^{1}(x^5-x^3)\frac{1}{2}(5x^3-3x)dx = \frac{4}{315}$$
$$a_4 = \frac{9}{2}\int_{-1}^{1}(x^5-x^3)\frac{1}{8}(35x^4-30x^2+3)dx = 0;$$
$$a_5 = \frac{11}{2}\int_{-1}^{1}(x^5-x^3)\frac{1}{8}(63x^5-70x^3-15x)dx = \frac{16}{693}$$

4. Represent the following functions in terms of spherical harmonics functions $Y_{lk}(\theta,\varphi)$, where

θ and φ denote the polar and azimuthal angles, respectively. (a) $\sin\theta\cos\varphi$, (b) $\sin\theta\sin\varphi$ and (c) $\cos\theta$.

$$Y_{0,0}(\theta,\varphi)=\frac{1}{2\sqrt{\pi}};\quad Y_{1,-1}(\theta,\varphi)=\frac{1}{2}e^{-i\varphi}\sqrt{\frac{3}{2\pi}}\sin\theta$$

$$Y_{1,0}(\theta,\varphi)=\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta;\quad Y_{1,1}(\theta,\varphi)=-\frac{1}{2}e^{i\varphi}\sqrt{\frac{3}{2\pi}}\sin\theta$$

$$\Omega_x=\sin\theta\cos\varphi=\sqrt{\frac{2\pi}{3}}[Y_{1,-1}(\theta,\varphi)-Y_{1,1}(\theta,\varphi)]$$

$$\Omega_y=\sin\theta\sin\varphi=-i\sqrt{\frac{2\pi}{3}}[Y_{1,-1}(\theta,\varphi)+Y_{1,1}(\theta,\varphi)]$$

$$\Omega_z=\cos\theta=2\sqrt{\frac{\pi}{3}}Y_{1,0}(\theta,\varphi)$$

5. The angular flux for mono-energetic neutrons at a point $\vec{r}$ is given by $\psi(\vec{r},\vec{\Omega})=a+b\cos\theta$ where *a* and *b* are constants, and θ is the angle between $\vec{\Omega}$ and the z-axis. Compute at $\vec{r}$ (a) the flux, (b) the current, (c) the partial current in the positive *z* direction, and (d) the partial current in the negative *z* direction.

$$\phi(\vec{r})=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta(a+b\cos\theta)=2\pi\int_{-1}^{1}d\mu(a+b\mu)=4\pi a$$

$$\vec{J}(\vec{r})=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\vec{\Omega}(a+b\cos\theta)$$

$$J_x(\vec{r})=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\Omega_x(a+b\cos\theta)=\int_0^{2\pi}d\varphi\cos\varphi\int_0^{\pi}d\theta\sin^2\theta(a+b\cos\theta)=0$$

$$J_y(\vec{r})=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\Omega_y(a+b\cos\theta)=\int_0^{2\pi}d\varphi\sin\varphi\int_0^{\pi}d\theta\sin^2\theta(a+b\cos\theta)=0$$

$$J_z(\vec{r})=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\Omega_z(a+b\cos\theta)=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\cos\theta(a+b\cos\theta)$$

$$=2\pi\int_{-1}^{1}d\mu\mu(a+b\mu)=\frac{4\pi}{3}b$$

$$J_z^+(\vec{r})=\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\theta\sin\theta\Omega_z(a+b\cos\theta)=\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\theta\sin\theta\cos\theta(a+b\cos\theta)$$

$$=2\pi\int_0^{1}d\mu\mu(a+b\mu)=2\pi\left(\frac{1}{2}a+\frac{1}{3}b\right)=\frac{1}{4}\phi(\vec{r})+\frac{1}{2}J_z(\vec{r})$$

$$J_z^-(\vec{r})=\int_0^{2\pi}d\varphi\int_{\pi/2}^{\pi}d\theta\sin\theta\,|-\Omega_z|\,(a+b\cos\theta)=\int_0^{2\pi}d\varphi\int_{\pi/2}^{\pi}d\theta\sin\theta\cos\theta(a+b\cos\theta)$$

$$=2\pi\int_0^{-1}d\mu\mu(a+b\mu)=2\pi\left(\frac{1}{2}a-\frac{1}{3}b\right)=\frac{1}{4}\phi(\vec{r})-\frac{1}{2}J_z(\vec{r})$$