NUCL 511 Nuclear Reactor Theory and Kinetics

Homework #4

Due February 13

1. Consider the following 2-group diffusion equation for a one-region slab reactor of thickness a:

$$\begin{cases} -D_1 \dfrac{d^2}{dx^2}\phi_1(x) + \Sigma_{r1}\phi_1(x) = \dfrac{1}{k}[\nu\Sigma_{f1}\phi_1(x) + \nu\Sigma_{f2}\phi_2(x)] \\ -D_2 \dfrac{d^2}{dx^2}\phi_2(x) + \Sigma_{a2}\phi_2(x) = \Sigma_{s1\to 2}\phi_1(x) \end{cases}, \quad -\frac{a}{2} \le x \le \frac{a}{2}$$

$$\phi_1(-a/2) = \phi_1(a/2) = 0, \quad \phi_2(-a/2) = \phi_2(a/2) = 0$$

(1) Assuming that the neutron flux is separable in its space and energy dependence, i.e., $\phi_1(x) = \varphi_1\phi(x)$ and $\phi_2(x) = \varphi_2\phi(x)$ with the same spatial flux shape $\phi(x)$, determine the fundamental mode solution of this system of equations. Normalize the flux such that the total fission rate, i.e., the integral over energy and space, is unity.

Answer) Inserting $\phi_1(x) = \varphi_1\phi(x)$ and $\phi_2(x) = \varphi_2\phi(x)$ in the 2-group diffusion equation, we have

$$\begin{cases} -D_1\varphi_1 \dfrac{d^2}{dx^2}\phi(x) + \Sigma_{r1}\varphi_1\phi(x) = \dfrac{1}{k}[\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2]\phi(x) \\ -D_2\varphi_2 \dfrac{d^2}{dx^2}\phi(x) + \Sigma_{a2}\varphi_2\phi(x) = \Sigma_{s1\to 2}\varphi_1\phi(x) \end{cases}$$

Dividing these equations by $\phi(x)$ yields

$$\begin{cases} -D_1\varphi_1 \left[\dfrac{1}{\phi(x)}\dfrac{d^2}{dx^2}\phi(x)\right] + \Sigma_{r1}\varphi_1 = \dfrac{1}{k}[\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2] \\ -D_2\varphi_2 \left[\dfrac{1}{\phi(x)}\dfrac{d^2}{dx^2}\phi(x)\right] + \Sigma_{a2}\varphi_2 = \Sigma_{s1\to 2}\varphi_1 \end{cases} \tag{1}$$

Since all the other terms are constant, the ratio $\phi''(x)/\phi(x)$ should be constant. Denoting this constant as $-B^2$, we have

$$\frac{d^2}{dx^2}\phi(x) + B^2\phi(x) = 0$$

With the given zero flux boundary conditions, this equation can be solved as

$$\phi(x) = \cos(Bx), \quad B = \frac{\pi}{a} \tag{2}$$

Inserting Eq. (2) to Eq. (1), we have

$$\begin{cases} (D_1B^2 + \Sigma_{r1})\varphi_1 = \dfrac{1}{k}[\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2] \\ (D_2B^2 + \Sigma_{a2})\varphi_2 = \Sigma_{s1\to 2}\varphi_1 \end{cases} \tag{3}$$

From the second equation of Eq. (3), the fast flux can be written in terms of the thermal flux as

$$\varphi_1 = \frac{D_2 B^2 + \Sigma_{a2}}{\Sigma_{s1\to 2}} \varphi_2 \tag{4}$$

Inserting Eq. (4) into the second equation of Eq. (3), the multiplication factor can be determined as

$$k = \frac{\nu\Sigma_{f1}}{D_1 B^2 + \Sigma_{r1}} + \frac{\Sigma_{s1\to 2}}{D_1 B^2 + \Sigma_{r1}} \frac{\nu\Sigma_{f2}}{D_2 B^2 + \Sigma_{a2}} \tag{5}$$

Using Eq. (2) and Eq. (4), the fast and thermal fluxes can be written as

$$\begin{cases} \phi_1(x) = \dfrac{D_2 B^2 + \Sigma_{a2}}{\Sigma_{s1\to 2}} \varphi_2 \cos\left(\dfrac{\pi x}{a}\right) \\ \phi_2(x) = \varphi_2 \cos\left(\dfrac{\pi x}{a}\right) \end{cases} \tag{6}$$

Using Eq. (6), the total fission rate can be obtained as

$$\int_{-a/2}^{a/2} [\nu\Sigma_{f1}\phi_1(x) + \nu\Sigma_{f2}\phi_2(x)]dx = \varphi_2 \frac{(D_2 B^2 + \Sigma_{a2})\nu\Sigma_{f1} + \Sigma_{s1\to 2}\nu\Sigma_{f2}}{\Sigma_{s1\to 2}} \int_{-a/2}^{a/2} \cos\left(\frac{\pi x}{a}\right) dx$$

$$= \varphi_2 \frac{(D_2 B^2 + \Sigma_{a2})\nu\Sigma_{f1} + \Sigma_{s1\to 2}\nu\Sigma_{f2}}{\Sigma_{s1\to 2}} \frac{2a}{\pi} = 1$$

Thus, φ_2 is given by

$$\varphi_2 = \frac{\pi}{2a} \frac{\Sigma_{s1\to 2}}{(D_2 B^2 + \Sigma_{a2})\nu\Sigma_{f1} + \Sigma_{s1\to 2}\nu\Sigma_{f2}} \tag{7}$$

Inserting Eq. (7) into Eq. (6), the fast and thermal fluxes can be obtained as

$$\begin{cases} \phi_1(x) = \dfrac{\pi}{2a} \dfrac{D_2 B^2 + \Sigma_{a2}}{(D_2 B^2 + \Sigma_{a2})\nu\Sigma_{f1} + \Sigma_{s1\to 2}\nu\Sigma_{f2}} \cos\left(\dfrac{\pi x}{a}\right) \\ \phi_2(x) = \dfrac{\pi}{2a} \dfrac{\Sigma_{s1\to 2}}{(D_2 B^2 + \Sigma_{a2})\nu\Sigma_{f1} + \Sigma_{s1\to 2}\nu\Sigma_{f2}} \cos\left(\dfrac{\pi x}{a}\right) \end{cases} \tag{8}$$

(2) Write the adjoint 2-group diffusion equation.

Answer) The 2-group diffusion equation can be written in a matrix equation form as

$$\begin{bmatrix} -D_1 \dfrac{d^2}{dx^2} + \Sigma_{r1} & 0 \\ -\Sigma_{s1\to 2} & -D_2 \dfrac{d^2}{dx^2} + \Sigma_{a2} \end{bmatrix} \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \frac{1}{k} \begin{bmatrix} \nu\Sigma_{f1} & \nu\Sigma_{f2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix}$$

Using the self-adjoint property of the second order differential and the multiplication operator and Taking the transpose of the matrices, the adjoint 2-group equation and its boundary conditions can be written as

$$\begin{bmatrix} -D_1\dfrac{d^2}{dx^2}+\Sigma_{r1} & -\Sigma_{s1\to 2} \\ 0 & -D_2\dfrac{d^2}{dx^2}+\Sigma_{a2} \end{bmatrix}\begin{bmatrix}\phi_1^*(x)\\ \phi_2^*(x)\end{bmatrix}=\frac{1}{k^*}\begin{bmatrix}\nu\Sigma_{f1} & 0\\ \nu\Sigma_{f2} & 0\end{bmatrix}\begin{bmatrix}\phi_1^*(x)\\ \phi_2^*(x)\end{bmatrix}$$

$$\phi_1^*(-a/2)=\phi_1^*(a/2)=0,\quad \phi_2^*(-a/2)=\phi_2^*(a/2)=0$$

(3) Assuming that the adjoint flux is separable in its space and energy dependence, i.e., $\phi_1^*(x)=\varphi_1^*\phi^*(x)$ and $\phi_2^*(x)=\varphi_2^*\phi^*(x)$, determine the adjoint group fluxes. Normalize the total adjoint flux, i.e., the integral over energy and space to unity.

Answer) Inserting $\phi_1^*(x)=\varphi_1^*\phi^*(x)$ and $\phi_2^*(x)=\varphi_2^*\phi^*(x)$ in the 2-group adjoint diffusion equation and dividing the resulting equations by $\phi^*(x)$, we have

$$\begin{cases} -D_1\varphi_1^*\left[\dfrac{1}{\phi^*(x)}\dfrac{d^2}{dx^2}\phi^*(x)\right]+\Sigma_{r1}\varphi_1^*-\Sigma_{s1\to 2}\varphi_2^*=\dfrac{1}{k^*}\nu\Sigma_{f1}\varphi_1^* \\ -D_2\varphi_2^*\left[\dfrac{1}{\phi^*(x)}\dfrac{d^2}{dx^2}\phi^*(x)\right]+\Sigma_{a2}\varphi_2^*=\dfrac{1}{k^*}\nu\Sigma_{f2}\varphi_1^* \end{cases} \tag{9}$$

Since all the other terms are constant, the ratio $[\phi^*(x)]''/\phi^*(x)$ should be constant. Denoting this constant as $-B^2$, we have

$$\frac{d^2}{dx^2}\phi^*(x)+B^2\phi^*(x)=0$$

With the given zero adjoint flux boundary conditions, this equation can be solved as

$$\phi^*(x)=\cos(Bx),\quad B=\frac{\pi}{a} \tag{10}$$

That is, in this particular problem with the separability assumption, the spatial shape of the adjoint flux is identical to that of the real flux. Inserting Eq. (10) to Eq. (9), we have

$$\begin{cases} (D_1B^2+\Sigma_{r1})\varphi_1^*-\Sigma_{s1\to 2}\varphi_2^*=\dfrac{1}{k^*}\nu\Sigma_{f1}\varphi_1^* \\ (D_2B^2+\Sigma_{a2})\varphi_2^*=\dfrac{1}{k^*}\nu\Sigma_{f2}\varphi_1^* \end{cases} \tag{11}$$

Eq. (11) can be rewritten in a matrix equation form as

$$\begin{bmatrix} D_1B^2+\Sigma_{r1}-\nu\Sigma_{f1}/k^* & -\Sigma_{s1\to 2} \\ -\nu\Sigma_{f2}/k^* & D_2B^2+\Sigma_{a2} \end{bmatrix}\begin{bmatrix}\varphi_1^*(x)\\ \varphi_2^*(x)\end{bmatrix}=0 \tag{12}$$

For Eq. (12) to have a non-trivial solution, the determinant of the matrix in Eq. (12) should be zero. Thus, the adjoint eigenvalue can be obtained as

$$k^*=\frac{\nu\Sigma_{f1}}{D_1B^2+\Sigma_{r1}}+\frac{\Sigma_{s1\to 2}}{D_1B^2+\Sigma_{r1}}\frac{\nu\Sigma_{f2}}{D_2B^2+\Sigma_{a2}}=k \tag{13}$$

Inserting Eq. (13) into the second equation of Eq. (11), the adjoint fast flux can be written in terms of the adjoint thermal flux as

$$\varphi_1^* = \frac{k(D_2B^2+\Sigma_{a2})}{\nu\Sigma_{f2}}\varphi_2^* = \frac{(D_2B^2+\Sigma_{a2})\nu\Sigma_{f1}+\Sigma_{s1\to2}\nu\Sigma_{f2}}{(D_1B^2+\Sigma_{r1})\nu\Sigma_{f2}}\varphi_2^* \tag{14}$$

Using Eq. (10) and Eq. (14), the fast and thermal adjoint fluxes can be written as

$$\begin{cases} \phi_1^*(x) = \dfrac{(D_2B^2+\Sigma_{a2})\nu\Sigma_{f1}+\Sigma_{s1\to2}\nu\Sigma_{f2}}{(D_1B^2+\Sigma_{r1})\nu\Sigma_{f2}}\varphi_2^*\cos\left(\dfrac{\pi x}{a}\right) \\ \phi_2^x(x) = \varphi_2^*\cos\left(\dfrac{\pi x}{a}\right) \end{cases} \tag{15}$$

Using Eq. (15), the total adjoint flux can be obtained as

$$\int_{-a/2}^{a/2}[\phi_1^*(x)+\phi_2^*(x)]dx = \varphi_2^*\left[\frac{(D_2B^2+\Sigma_{a2})\nu\Sigma_{f1}+\Sigma_{s1\to2}\nu\Sigma_{f2}}{(D_1B^2+\Sigma_{r1})\nu\Sigma_{f2}}+1\right]\int_{-a/2}^{a/2}\cos\left(\frac{\pi x}{a}\right)dx$$

$$= \varphi_2^*\frac{(D_2B^2+\Sigma_{a2})\nu\Sigma_{f1}+[(D_1B^2+\Sigma_{r1})+\Sigma_{s1\to2}]\nu\Sigma_{f2}}{(D_1B^2+\Sigma_{r1})\nu\Sigma_{f2}}\frac{2a}{\pi} = 1$$

Thus, φ_2^* is given by

$$\varphi_2^* = \frac{\pi}{2a}\frac{(D_1B^2+\Sigma_{r1})\nu\Sigma_{f2}}{(D_2B^2+\Sigma_{a2})\nu\Sigma_{f1}+[(D_1B^2+\Sigma_{r1})+\Sigma_{s1\to2}]\nu\Sigma_{f2}} \tag{16}$$

Inserting Eq. (7) into Eq. (6), the fast and thermal fluxes can be obtained as

$$\begin{cases} \phi_1^*(x) = \dfrac{\pi}{2a}\dfrac{(D_2B^2+\Sigma_{a2})\nu\Sigma_{f1}+\Sigma_{s1\to2}\nu\Sigma_{f2}}{(D_2B^2+\Sigma_{a2})\nu\Sigma_{f1}+\Sigma_{s1\to2}\nu\Sigma_{f2}}\cos\left(\dfrac{\pi x}{a}\right) \\ \phi_2^*(x) = \dfrac{\pi}{2a}\dfrac{(D_1B^2+\Sigma_{r1})\nu\Sigma_{f2}}{(D_2B^2+\Sigma_{a2})\nu\Sigma_{f1}+[(D_1B^2+\Sigma_{r1})+\Sigma_{s1\to2}]\nu\Sigma_{f2}}\cos\left(\dfrac{\pi x}{a}\right) \end{cases} \tag{17}$$

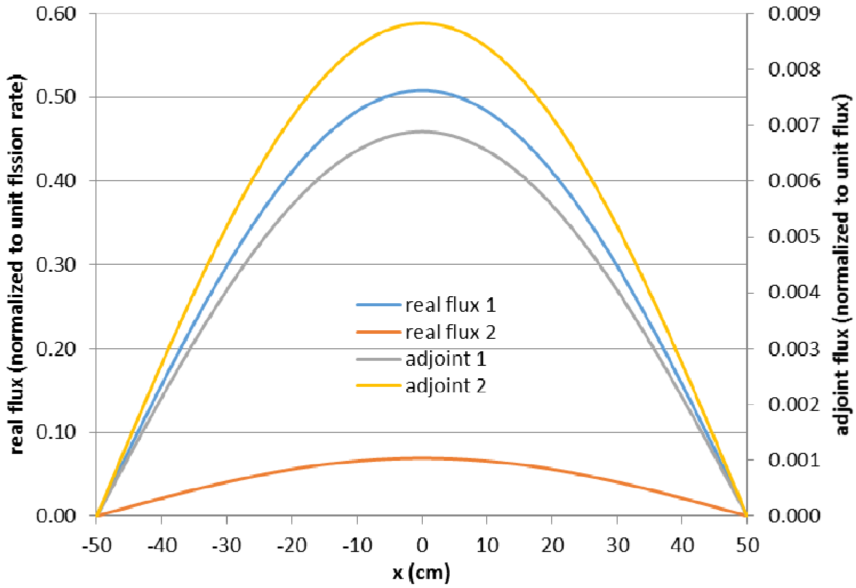
(4) Using the following cross sections and $a = 100$ cm, plot the forward and adjoint group fluxes.

Group	D	Σ_a	Σ_{aF}	$\nu\Sigma_f$	Σ_{s12}
1	1.44	0.01	0.01	0.008	0.017
2	0.366	0.125	0.09	0.169	

Answer) Using these data, the real and adjoint fluxes can be evaluated as

$$\begin{cases} \phi_1(x) = 0.50806\cos\left(\dfrac{\pi x}{a}\right) \\ \phi_2(x) = 0.06890\cos\left(\dfrac{\pi x}{a}\right) \end{cases} \tag{18}$$

$$\begin{cases} \phi_1^*(x) = 0.00688\cos\left(\dfrac{\pi x}{a}\right) \\ \phi_2^*(x) = 0.00883\cos\left(\dfrac{\pi x}{a}\right) \end{cases} \tag{19}$$



2. For the perturbations of thermal absorption cross section by 1%, 2%, 5% and 10%, estimate the reactivity change using the first order perturbation theory formula. Also determine the exact reactivity change by calculating the multiplication factors of the base and perturbed cases and compare the results with those of FOP.

Answer) The first order perturbation theory formula for the reactivity change is given by

$$\Delta\rho = \frac{<\phi^*,[\lambda\Delta F - \Delta M]\phi>}{<\phi^*,F\phi>} \tag{20}$$

where

$$\begin{aligned} &<\phi^*,F\phi> = \int_{-a/2}^{a/2}\phi_1^*(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2)dx \\ &<\phi^*,\Delta F\phi> = \int_{-a/2}^{a/2}\phi_1^*[\Delta(\nu\Sigma_{f1})\phi_1 + \Delta(\nu\Sigma_{f2})\phi_2]dx \\ &<\phi^*,\Delta M\phi> = \int_{-a/2}^{a/2}\left[\phi_1^*\left\{\Delta D_1\frac{d^2\phi_1}{dx^2} - \Delta\Sigma_{r1}\phi_1\right\} + \phi_2^*\left\{-\Delta\Sigma_{s1\to2}\phi_1 + \Delta D_2\frac{d^2\phi_2}{dx^2} - \Delta\Sigma_{a2}\phi_2\right\}\right]dx \end{aligned} \tag{21}$$

For the thermal absorption cross section change only, Eq. (20) is reduced to

$$\Delta\rho = \frac{\int_{-a/2}^{a/2}\phi_2^*\Delta\Sigma_{a2}\phi_2 dx}{\int_{-a/2}^{a/2}\phi_1^*(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2)dx} = -5.6267\Delta\Sigma_{a2} \tag{22}$$

The exact reactivity change can be evaluated by calculating the base and perturbed multiplication factors using Eq. (5). The results are given in the following table and figure.

$\Delta\Sigma_{a2}$ (%)	k-eff	Exact $\Delta\rho$	FOP $\Delta\rho$
0	1.05087	0.00000	0.00000
1	1.04318	-0.00702	-0.00703
2	1.03564	-0.01399	-0.01407
5	1.01388	-0.03472	-0.03517
10	0.98024	-0.06856	-0.07033
20	0.92135	-0.13377	-0.14067
30	0.87151	-0.19584	-0.21100
40	0.82877	-0.25502	-0.28133
50	0.79171	-0.31149	-0.35167