NUCL 510 Nuclear Reactor Theory I
Fall 2011

Homework #8

Due November 29

1. Homework problem #4 of Ch. 4

   How many elastic scattering events are required to slow a neutron down from 2.0 MeV to 0.025 eV at room temperature in graphite and in hydrogen if s-wave scattering with constant σ_s is assumed in the entire energy range?

   Ans.) A=12 for graphite

   $$\alpha = \left(\frac{A-1}{A+1}\right)^2 = 0.716$$

   $$\xi = 1 + \frac{\alpha}{1-\alpha}\ln\alpha = 1 + \frac{0.716}{0.384}\ln(0.716) = 0.158$$

   $$\bar{n} = \frac{1}{\xi}\ln\frac{E_0}{E} = \frac{1}{0.158}\ln\frac{2\times10^6}{0.025} \approx 115$$

   A=1 for hydrogen

   $$\xi = 1$$

   $$\bar{n} = \ln\frac{E_0}{E} = \ln\frac{2\times10^6}{0.025} = 18.2 \approx 19$$

2. Homework problem #5 of Ch. 4

   In presenting neutron flux spectra from reactor design calculations, usually $\varphi(u)$ or $E\varphi(E)$ versus log E is plotted rather than $\varphi(E)$ versus log E. Explain why this former method of presenting spectra is preferred.

   Ans.) It is generally perceived that the area under a curve is proportional to the quantity represented by the curve.

   $$\int_{E_1}^{E_2}[E\varphi(E)]d(\log E) = \int_{E_1}^{E_2}[E\varphi(E)]\frac{dE}{E} = \int_{E_1}^{E_2}\varphi(E)dE \quad \Rightarrow \quad \text{flux integral from } E_1 \text{ to } E_2$$

   $$\int_{E_1}^{E_2}\varphi(u)d(\log E) = \int_{E_1}^{E_2}[E\varphi(E)]\frac{dE}{E} = \int_{E_1}^{E_2}\varphi(E)dE \quad \Rightarrow \quad \text{flux integral from } E_1 \text{ to } E_2$$

   $$\int_{E_1}^{E_2}\varphi(E)d(\log E) = \int_{E_1}^{E_2}\frac{\varphi(E)}{E}dE \neq \int_{E_1}^{E_2}\varphi(E)dE \quad \Rightarrow \quad \text{not the flux integral from } E_1 \text{ to } E_2$$

3. Homework problem #9 of Ch. 4

   a. Present the main steps of the derivation for the slowing down spectrum for hydrogen with

$\sigma_s = \sigma_s(E)$ and no absorption.

$$\Sigma_s^H(E)\varphi(E) - \int_E^\infty dE' \frac{\Sigma_s^H(E')\varphi(E')}{E'} = \chi(E)s_0$$

$$F(E) - \int_E^\infty dE' \frac{F(E')}{E'} = \chi(E)s_0, \quad F(E) = \Sigma_s^H(E)\varphi(E) \quad \text{(scattering density)}$$

$$\frac{dF(E)}{dE} + \frac{F(E)}{E} = \frac{d\chi(E)}{dE}s_0 \quad \Rightarrow \quad E\frac{dF(E)}{dE} + F(E) = Es_0\frac{d\chi(E)}{dE}$$

$$\frac{d}{dE}[EF(E)] = Es_0\frac{d\chi(E)}{dE}$$

$$E'F(E')\Big|_E^\infty = s_0\int_E^\infty dE'E'\frac{d\chi(E')}{dE'} = s_0\left[E'\chi(E')\Big|_E^\infty - \int_E^\infty dE'\chi(E')\right]$$

$$-EF(E) = s_0\left[-E\chi(E) - \int_E^\infty dE'\chi(E')\right] \quad \Rightarrow \quad F(E) = s_0\chi(E) + \frac{s_0}{E}\int_E^\infty dE'\chi(E')$$

$$E\varphi(E) = \frac{s_0}{\Sigma_s^H(E)}E\chi(E) + \frac{s_0}{\Sigma_s^H(E)}\int_E^\infty dE'\chi(E')$$

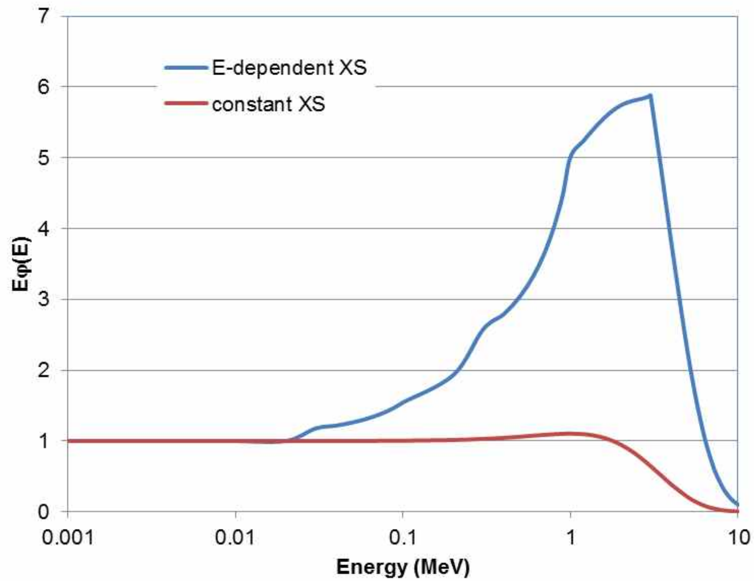
b. Derive the analytical solution for the simple χ approximation: $\chi(E) = Ee^{-E}$ with E in mega-electron volts.

$$\int_E^\infty dE'\chi(E') = \int_E^\infty dE'E'e^{-E'} = (1+E)e^{-E}$$

$$E\varphi(E) = \frac{s_0}{\Sigma_s^H(E)}E\chi(E) + \frac{s_0}{\Sigma_s^H(E)}\int_E^\infty dE'\chi(E') = \frac{s_0}{\Sigma_s^H(E)}(1+E+E^2)e^{-E}$$

c. Plot the resulting spectrum for σ_s = 20 b = constant.

Ans.) By normalizing the neutron spectrum so that $s_0/\Sigma_s(E) = 1$ at energies below 20 keV, we obtain the spectra for the problems 9c and 9d as in the figure below.



d. Plot several points of this spectrum for σ_s = 20 b below E = 20 keV; and for σ_s (E) = 17, 13, 8, 4.4, 2.2, and 1 b for E = 30, 100, 300 keV, 1, 3, and 10 MeV, respectively.

e. Find the average energy of the emitted neutrons, E_f, for the approximation given in problem 9b.

$$E_f = \int_0^\infty dE E\chi(E) = \int_0^\infty dE E^2 e^{-E} = 2 \text{ MeV}$$

4. Homework problem #11 of Ch. 4

A cylindrical BF_3 counter is placed in a neutron flux with a Maxwellian distribution at a temperature of 25°C. The BF_3 in the counter is at a pressure of 25 cm of mercury (at 20°C) and physically is 0.6 cm in diameter and 25 cm long. The counter has an efficiency of 1 % (i.e., detects l out of every 100 incident neutrons). It is placed in an isotropic neutron flux it registers 10,000 count/min.

a. What is the magnitude of the incident total neutron flux?

Ans.) The BF_3 proportional counter employs $^{10}B(n,\alpha)^7Li$ reaction for the detection of thermal neutrons. Most counters are filled with BF_3 gas enriched in the ^{10}B isotope to typically 96%. Using the ideal gas approximation with the given data,

p = 25 cmHg / 76 cmHg = 0.329 atm

$V = \pi(0.6/2 \text{ cm})^2 \times 25 \text{ cm} = 7.069\times10^{-3}$ liters,

T = 20 °C = 293.15 K

R = 0.08206 liter-atm/K-mol

the amount of BF_3 gas in the counter is obtained as

$$n = \frac{pV}{RT} = \frac{(0.329\text{ atm}) \times (7.069\times 10^{-3})\text{ liter}}{(0.08206\text{ liter-atm/K-mol}) \times (293.15\text{ K})} = 9.67\times 10^{-5}\text{ moles}$$

Assuming 96% enrichment in ^{10}B, the number of ^{10}B atoms in the counter is given by

$$N_{B10} = (9.67\times 10^{-5}\text{ moles}) \times (6.022\times 10^{23}\text{ atoms/mole}) \times 0.96 = 5.59\times 10^{19}\text{ atoms}$$

The (n,α) cross section of ^{10}B at 0.025 eV in the ENDF/B-VII libraries is 3843 barns and inversely proportional to the relative speed between neutron and target nuclide in thermal energy region, so the average thermal cross section is obtained as

$$\bar{\sigma}_a^{B10}(T) = \frac{\sqrt{\pi}}{2}\sigma_0\sqrt{\frac{T_0}{T}} = \frac{\sqrt{\pi}}{2}\times 3843\times\sqrt{\frac{293.15}{298.15}} = 3377\text{ barns}$$

Since the counter efficiency is 1%, the (n,α) reaction rates of ^{10}B is given by

$$R_a^{B10}(T) = \frac{10,000\text{ cpm}}{0.01} = 1.667\times 10^4\text{ \#/sec}$$

Therefore, the neutron flux can be obtained as

$$\phi_{th}(T) = \frac{R_a^{B10}(T)}{N_{B10}\bar{\sigma}_a^{B10}(T)} = \frac{1.667\times 10^4\text{ \#/sec}}{(5.59\times 10^{19})\times(3377\times 10^{-24}\text{ cm}^2)} = 8.83\times 10^4\text{ \#/cm}^2\text{s}$$

b. If the Maxwellian neutron distribution is at 200°C, what would the neutron flux level with the same 10,000 count/min counting rate be?

Ans.) At 200°C, the average thermal cross section is reduced to

$$\bar{\sigma}_a^{B10}(T) = \frac{\sqrt{\pi}}{2}\sigma_0\sqrt{\frac{T_0}{T}} = \frac{\sqrt{\pi}}{2}\times 3843\times\sqrt{\frac{293.15}{473.15}} = 2681\text{ barns}$$

Therefore, for the same counting rate, the thermal flux is increased to

$$\phi_{th}(T) = \frac{R_a^{B10}(T)}{N_{B10}\bar{\sigma}_a^{B10}(T)} = \frac{1.667\times 10^4\text{ \#/sec}}{(5.59\times 10^{19})\times(2681\times 10^{-24}\text{ cm}^2)} = 1.11\times 10^5\text{ \#/cm}^2\text{s}$$

c. Assuming the same total neutron flux, what would the relative count rate between neutrons at 25 and 200°C be?

Ans.) For the same neutron flux, the reduced average cross section decreases the counting rate to

$$R_a^{B10}(250°C) = R_a^{B10}(25°C)\times\frac{\bar{\sigma}_a(250°C)}{\bar{\sigma}_a(25°C)} = 10000\times\frac{2681}{3377} = 7938\text{ cpm}$$

5. Homework problem #1 of Ch. 5

a. Derive the energy-dependence of $\sigma_c(E)$ and $\sigma_f(E)$ for small E (limit $E\rightarrow 0$) from the Breit-Wigner formula for the lowest s-wave resonance. Note, and Γ_γ and Γ_f are independent of *E*.

Ans.) Using the single level Breit-Wigner formula, the capture or fission cross section of the lowest s-wave resonance can be written as

$$\sigma_a(E_r) = \frac{\pi g_J}{k^2}\frac{\Gamma_n \Gamma_a}{(E-E_R)^2+\Gamma^2/4} \quad (a=\gamma, f)$$

As $E \to 0$, $(E-E_R)^2 \to E_R^2$, but $E_R >> \Gamma$. Thus we have

$$\sigma_a(E) \approx \frac{\pi g_J}{k^2}\frac{\Gamma_n \Gamma_a}{E_R^2}$$

The capture and fission widths Γ_γ and Γ_f are independent of E, and the neutron width Γ_n is proportional to $\sqrt{E}$ for the low-energy s-wave resonance. The wave number $k = 2\pi/\lambda$ is also proportional to $\sqrt{E}$ since the wave length λ is inversely proportional to the neutron momentum. Therefore the capture and fission cross section at low energies below the first s-wave resonance can be written as

$$\sigma_a(E) \approx \frac{\pi g_J}{c_1 E}\frac{c_2\sqrt{E}\Gamma_a}{E_R^2} = \frac{c}{\sqrt{E}}$$

b. Express the result in terms of the neutron velocity.

$$v(E) = \sqrt{2mE}$$

$$\sigma_a(E) \approx \frac{c}{\sqrt{E}} = \frac{\sigma(E_0)v(E_0)}{v(E)}$$

c. In the same way, find the energy dependence of $\sigma_s(E)$ for small *E*.

Ans.) Using the single level Breit-Wigner formula, the scattering cross section of the lowest s-wave resonance can be written as

$$\sigma_n(E) \approx 4\pi a^2 + \frac{\pi g_J}{k^2}\frac{\Gamma_n^2}{(E-E_R)^2+\Gamma^2/4} + \frac{4\pi g_J a}{k}\frac{\Gamma_n(E-E_R)}{(E-E_R)^2+\Gamma^2/4}$$

$$\approx 4\pi a^2 + \frac{\pi g_J}{k^2}\frac{\Gamma_n^2}{E_R^2} - \frac{4\pi g_J a}{k}\frac{\Gamma_n}{E_R}$$

where $a = 0.123A^{1/3} + 0.08$ is the channel radius in units of 10^{-12} cm. Since both the neutron width Γ_n and the wave number k are proportional to $\sqrt{E}$, the second and third terms are independent of E. In addition, they become much smaller than the first term since $E_R >> \Gamma_n$. Thus we have

$$\sigma_n(E) \approx 4\pi a^2$$

6. Homework problem #5 of Ch. 5

a. Calculate the *J* function for the natural line shape.

Ans.) The *J* function is defined with the Doppler broadened symmetric line shape ψ as

NUCL 510 Nuclear Reactor Theory I

Fall 2011

$$J(\xi,\beta)=\int_0^\infty \frac{\psi(x,\xi)}{\beta+\psi(x,\xi)}dx$$

where

$$\psi(x,\xi)=\frac{\xi}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1}{1+w^2}\exp\left[-\frac{\xi^2}{4}(x-w)^2\right]dw$$

$$x=\frac{2(E-E_0)}{\Gamma_t},\quad \xi=\frac{\Gamma_t}{\Delta}=\left(\frac{A}{4kTE}\right)^{1/2}\Gamma_t,\ \beta=\frac{\Sigma_p}{\Sigma_m}=\frac{\Sigma_p}{N\sigma_m}=\frac{\sigma_b}{\sigma_m},\quad \sigma_m=\frac{4\pi}{k^2}g_J\frac{\Gamma_n}{\Gamma}$$

The natural line shapes of ψ at 0K is the Lorentzian $L=1/(1+x^2)$, so the J function for the natural line shape becomes

$$J(\infty,\beta)=\int_0^\infty \frac{\psi(x,\infty)}{\beta+\psi(x,\infty)}dx=\frac{1}{\beta}\int_0^\infty \frac{dx}{(1+\beta^{-1})+x^2}=\frac{\pi}{2\beta(1+\beta^{-1})^{1/2}}$$

b. On the basis of problem 5a, calculate the ratio of the capture reaction rates with and without self-shielding for the first resonance in ^{238}U, assuming uranium metal. Use $\sigma_p = 9$ b; $\Gamma_\gamma = 26$ meV; $\Gamma_n = 5.0$ meV; and $E_R = 6.67$ eV.

Ans.) If the resonance absorber is infinitely dilute in a scattering medium, then the self-shielding effect can be neglected. The J function without self-shielding is given by

$$\lim_{\beta\to\infty}J(\xi,\beta)=\frac{1}{\beta}\int_0^\infty \psi(x,\xi)dx=\frac{1}{2\beta}\int_{-\infty}^{\infty}\psi(x,\xi)dx=\frac{\pi}{2\beta}$$

The self-shielding factor is the ratio of self-shielded reaction rate to un-shielded reaction rate

$$f(\xi,\beta)=\frac{J(\xi,\beta)}{J(\xi,\infty)}=\frac{2\beta}{\pi}J(\xi,\beta)$$

For the natural line shape, the self-shielding factor becomes

$$f(\infty,\beta)=\left(\frac{\beta}{1+\beta}\right)^{1/2}$$

Using the given data, this self-shielding factor can be obtained as

$$k=\frac{2\pi}{\lambda}=\frac{2\pi}{h}\sqrt{2mE}=\frac{2\pi\sqrt{(1.68\times10^{-27}\ \text{kg})\times(6.67\ \text{eV})\times(1.60\times10^{-19}\ \text{J/eV})}}{6.63\times10^{-34}\ \text{J}\cdot\text{sec}}$$

$$=2.20\times10^{9}\ /\text{cm}$$

$$g=1$$

$$\sigma_m=\frac{4\pi g\Gamma_n}{k^2\Gamma}=\frac{4\pi\times5\ \text{meV}}{(2.20\times10^9\ /\text{cm})^2\times31\ \text{meV}}=4.20\times10^5\ \text{barn}$$

$$\beta=\frac{\sigma_b}{\sigma_m}=\frac{9}{4.20\times10^5}=2.14\times10^{-5}$$

$$f(\infty,\beta)=\left(\frac{\beta}{1+\beta}\right)^{1/2}=4.63\times10^{-3}$$

c. Calculate as problem 5b but for ^{238}U in a mixture in which ^{238}U contributes only 11.1% to the potential cross section.

Ans.) Since ^{238}U contribution is 11.1%, the potential cross section increases to

$$\sigma_p = \frac{9}{0.111} = 81 \text{ barn}$$

$$\beta = \frac{\sigma_b}{\sigma_m} = \frac{81}{4.20 \times 10^5} = 1.93 \times 10^{-4}$$

$$f(\infty, \beta) = \left(\frac{\beta}{1+\beta} \right)^{1/2} = 1.39 \times 10^{-2}$$