NUCL 511 HMWK 4

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1 Consider the following 2-group diffusion equation for a one-region slab reactor of thickness a:

$$\begin{cases} -D_1 \frac{d^2}{dx^2}\phi_1(x) + \Sigma_{r1}\phi_1(x) = \frac{1}{k}\left[\nu\Sigma_{f1}\phi_1(x) + \nu\Sigma_{f2}\phi_2(x)\right] \\ -D_2 \frac{d^2}{dx^2}\phi_2(x) + \Sigma_{a2}\phi_2(x) = \Sigma_{s1\to 2}\phi_1(x) \end{cases} , -\frac{a}{2} \leq x \leq \frac{a}{2}$$

$$\phi_1(-a/2) = \phi_1(a/2) = 0, \;\; \phi_2(-a/2) = \phi_2(a/2) = 0$$

(1) Assuming that the neutron flux is separable in its space and energy dependence, i.e., $\phi_1(x) = \varphi_1\phi(x)$ and $\phi_2(x) = \varphi_2\phi(x)$ with the same spatial flux shape $\phi(x)$, determine the fundamental mode solution of this system of equations. Normalize the flux such that the total fission rate, i.e., the integral over energy and space, is unity. First, we apply the separability

$$\begin{cases} -D_1\varphi_1 \frac{d^2}{dx^2}\phi(x) + \Sigma_{r1}\varphi_1\phi(x) = \frac{1}{k}\left[\nu\Sigma_{f1}\varphi_1\phi(x) + \nu\Sigma_{f2}\varphi_2\phi(x)\right] \\ -D_2\varphi_2 \frac{d^2}{dx^2}\phi(x) + \Sigma_{a2}\varphi_2\phi(x) = \Sigma_{s1\to 2}\varphi_1\phi(x) \end{cases} , -\frac{a}{2} \leq x \leq \frac{a}{2} \tag{1}$$

Then, we can apply buckling, with the equation

$$-B^2 = \frac{\frac{d^2}{dx^2}\phi(x)}{\phi(x)} \tag{2}$$

and divide through, getting

$$\begin{cases} D_1 B^2\varphi_1 + \Sigma_{r1}\varphi_1 = \frac{1}{k}\left[\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2\right] \\ D_2 B^2\varphi_2 + \Sigma_{a2}\varphi_2 = \Sigma_{s1\to 2}\varphi_1 \end{cases} , -\frac{a}{2} \leq x \leq \frac{a}{2}$$

and put into a matrix form

$$\begin{bmatrix} D_1 B^2 + \Sigma_{r1} - \lambda\nu\Sigma_{f1} & -\lambda\nu\Sigma_{f2} \\ -\Sigma_{s1\to 2} & D_2 B^2 + \Sigma_{a2} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

After we find λ using the determinant of the above matrix and setting it equal to 0, we can solve this for φ is the nullspace of $\mathbf{M} - \lambda\mathbf{F}$ which is the matrix given above.

To find the fundamental mode of this solution means that we take the fundamental mode solution of the spatial dependent equation [1], which is given above

$$\frac{d^2\phi}{dx^2} + B^2\phi = 0$$

Solving this, we get

$$\phi(x) = C_1 \cos(Bx) + C_2 \sin(Bx)$$

Using the fact that the maximum of the distribution is at zero ($\frac{d\phi}{dx} = 0$) we get that C_2 must be zero. Using the full thickness as the extrapolated thickness, we find that

$$0 = C_1 \cos\left(B\frac{a}{2}\right)$$

and C_1 cannot be 0 as that would leave the trivial solution. Thus,

$$\frac{Ba}{2} = \phi\left(n - \frac{1}{2}\right)$$

, and the fundamental solution means we take the smallest of that ($n = 1$). For higher order solutions (harmonics), we would have to take $n > 1$. So our fundamental spatial solution is

$$\phi(x) = \phi_0 \cos\left(\frac{\pi x}{a}\right) \tag{4}$$

To find ϕ_0, we must use the condition that the total fission rate over energy and space is unity. This means that

$$\int_V \nu\Sigma_{f1}\varphi_1\phi(x) + \nu\Sigma_{f2}\varphi_2\phi(x)dV = 1$$

$$\int_{-a/2}^{a/2} \nu\Sigma_{f1}\varphi_1\phi_0 \cos\left(\frac{\pi x}{a}\right) + \nu\Sigma_{f2}\varphi_2\phi_0 \cos\left(\frac{\pi x}{a}\right) dx = 1$$

$$\phi_0\left(\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2\right)\int_{-a/2}^{a/2} \cos\left(\frac{\pi x}{a}\right) dx = 1$$

$$\phi_0\left(\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2\right)\left[\frac{a\sin\left(\frac{\pi x}{a}\right)}{\pi}\right]_{-a/2}^{a/2} = 1$$

$$\phi_0\left(\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2\right)\frac{2a}{\pi} = 1$$

$$\phi_0 = \frac{\pi}{2a\left(\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2\right)} \tag{5}$$

(2) Write the adjoint 2-group diffusion equation. With the above equation in the form

$$\mathbf{M}\phi(x) = \lambda\mathbf{F}\phi(x)$$

the operators $\mathbf{M}$ and $\mathbf{F}$ can be used in the adjoint flux by determining the adjoint of those (the adjoint of a real matrix is the transpose). So

$$\mathbf{M}^*\phi^*(x) = \lambda^*\mathbf{F}^*\phi^*(x) \tag{6}$$

$$\begin{bmatrix} D_1B^2 + \Sigma_{r1} & -\Sigma_{s1\to 2} \\ 0 & D_2B^2 + \Sigma_{a2} \end{bmatrix}\begin{bmatrix} \varphi_1^* \\ \varphi_2^* \end{bmatrix} = \lambda^*\begin{bmatrix} \nu\Sigma_{f1} & 0 \\ \nu\Sigma_{f2} & 0 \end{bmatrix}\begin{bmatrix} \varphi_1^* \\ \varphi_2^* \end{bmatrix}$$

(3) Assuming that the adjoint flux is separable in its space and energy dependence, i.e., $\phi_1^*(x) = \varphi_1^*\phi^*(x)$ and $\phi_2^*(x) = \varphi_2^*\phi^*(x)$, the fundamental mode adjoint fluxes. Normalize the total adjoint flux, i.e., the integral over energy and space to unity. To solve the two group diffusion equation as given above, we first must move the fission operator to one side, getting

$$\begin{bmatrix} D_1B^2 + \Sigma_{r1} - \lambda^*\nu\Sigma_{f1} & -\Sigma_{s1\to 2} \\ -\lambda^*\nu\Sigma_{f2} & D_2B^2 + \Sigma_{a2} \end{bmatrix}\begin{bmatrix} \varphi_1^* \\ \varphi_2^* \end{bmatrix} = 0 \tag{7}$$

It should be seen that this is exactly the same as equation Assuming that the neutron flux is separable in its space and energy dependence, i.e., $\phi_1(x) = \varphi_1\phi(x)$ and $\phi_2(x) = \varphi_2\phi(x)$ with the same spatial flux shape $\phi(x)$, determine the fundamental mode solution of this system of equations. Normalize the flux such that the total fission rate, i.e., the integral over energy and space, is unity [2]. There are two equations and three unknowns, as λ^* isn't known and neither are the fluxes. After finding λ^* by setting the above determinant equal to zero, we can solve where φ^* is the nullspace of the matrix $\mathbf{M}^* - \lambda^*\mathbf{F}^*$.

It should be noted that spatially the adjoint flux is the same, because we have used the same buckling to solve the spatial dependence. To normalize the total flux to unity, we see that

$$\int_V \varphi_1^*\phi^*(x) + \varphi_2^*\phi^*(x)dV = 1$$

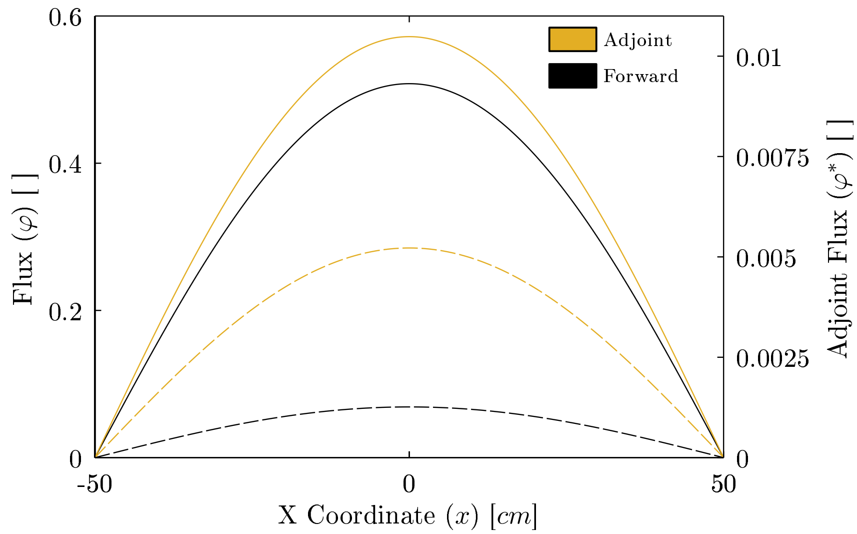

Figure 1: Flux Shape for Adjoint and Forward Fluxes using Two Group Diffusion Theory

$$\phi_0^* \left(\varphi_1^* + \varphi_2^*\right) \frac{2a}{\pi} = 1$$

Thus, the adjoint normalization is

$$\phi_0^* = \frac{\pi}{2a\left(\varphi_1^* + \varphi_2^*\right)} \tag{8}$$

(4) Using the following cross sections and $a = 100\,cm$, plot the forward and adjoint group fluxes.

Group	D	Σ_a	Σ_{aF}	$\nu\Sigma_f$	$\Sigma_{s1\to 2}$
1	1.44	0.01	0.01	0.008	0.017
2	0.366	0.125	0.09	0.169	

Using the solutions to equations 3 for the forward flux, equation 7 for the adjoint flux, then the normalization 5 for the forward flux and normalization 8 for the adjoint flux, and finally using the spatial distribution 4 for both fluxes, we can get the following profiles. Assuming that $\Sigma_{r1} = \Sigma_{a1} + \Sigma_{s1\to 2}$ and that $\Sigma_{r2} = \Sigma_{a2}$.

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0.9909 \\ 0.1344 \end{bmatrix}$$

$$\phi_0 = 0.5127$$

$$\begin{bmatrix} \varphi_1^* \\ \varphi_2^* \end{bmatrix} = \begin{bmatrix} 0.8951 \\ 0.4456 \end{bmatrix}$$

$$\phi_0^* = 0.0117$$

2 For the perturbations of thermal absorption cross section by 1%, 2%, 5% and 10%, estimate the reactivity change using the first order perturbation theory formula. Also determine the exact reactivity change by calculating the multiplication factors of the base and perturbed cases and compare the results with those of FOP.

We are assuming this is a one group point model, and thus we can use the first order perturbation equation given by

$$\Delta\rho = -\Delta\lambda = \frac{\left\langle \phi^*, \left[\lambda \mathbf{F}^{'} - \mathbf{M}^{'}\right] \phi \right\rangle}{\langle \phi^*, \mathbf{F}\phi \rangle} = \frac{\Delta\left(\nu\Sigma_f\right) - \nu\Sigma_f \Delta\Sigma_a}{\left(\nu\Sigma_f\right)^2}$$

Because the fission cross section or yield has not changed ($\Delta\nu\Sigma_f = 0$), we can find out the change in reactivity by

$$\Delta\rho = \frac{\nu\Sigma_f \Delta\Sigma_a}{\left(\nu\Sigma_f\right)^2} = \frac{\Delta\Sigma_a}{\nu\Sigma_f}$$

This means that the reactivity changes by for 1%, 2%, 5%, and 10% change in absorption cross section by that amount over the fission cross section times the fission yield. Given the value of $\nu\Sigma_f$, we could then calculate the reactivity differences.

The exact difference is given by

$$\Delta\rho_e = \frac{\Delta\left(\nu\Sigma_f\right)\Sigma_a - \nu\Sigma_f \Delta\Sigma_a}{\nu\Sigma_f\left[\nu\Sigma_f + \Delta\left(\nu\Sigma_f\right)\right]} = \frac{\Delta\Sigma_a}{\nu\Sigma_f}$$

showing no change.

We can also determine the change in multiplication factor using the one group point formula [1]. For one group,

$$k = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a}$$

$$\rho = 1 - \frac{1}{k} = 1 - \frac{DB^2 + \Sigma_a}{\nu\Sigma_f} = \frac{\nu\Sigma_f - DB^2 - \Sigma_a}{\nu\Sigma_f}$$

Now if we perturb $\Sigma_a^{'} = \Sigma_a + \delta\Sigma_a$

$$k^{'} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a + \delta\Sigma_a}$$

$$\rho^{'} = 1 - \frac{1}{k^{'}} = 1 - \frac{DB^2 + \Sigma_a + \delta\Sigma_a}{\nu\Sigma_f} = \frac{\nu\Sigma_f - DB^2 - \Sigma_a - \delta\Sigma_a}{\nu\Sigma_f}$$

with $\rho^{'} = \rho + \delta\rho$

$$\delta\rho = \rho^{'} - \rho = \frac{\nu\Sigma_f - DB^2 - \Sigma_a - \delta\Sigma_a}{\nu\Sigma_f} - \frac{\nu\Sigma_f - DB^2 - \Sigma_a}{\nu\Sigma_f}$$

Simplifying, we get

$$\delta\rho = \frac{\delta\Sigma_a}{\nu\Sigma_f}$$

This result is exactly the same as the first order perturbation theory result, and the reactivity will again change by the amount of absorption perturbation divided by the fission cross section times the fission yield.

References

[1] K Ott and W Bezella. *Introductory Nuclear Reactor Statics*. American Nuclear Society, La Grange Park, Illinois, 1989.

[2] K Ott and R Neuhold. *Introductory Nuclear Reactor Dynamics*. American Nuclear Society, La Grange Park, Illinois, 1985.