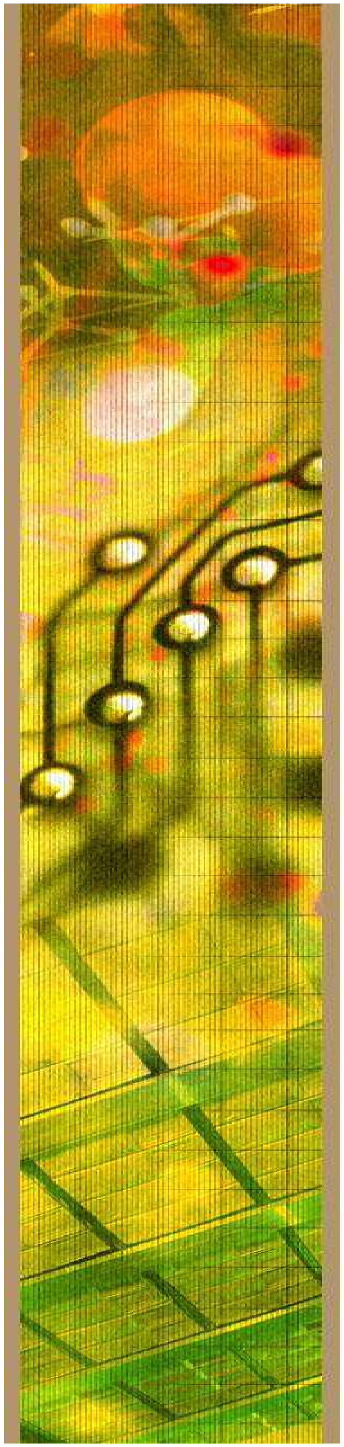


NUCL 511
Nuclear Reactor Theory and Kinetics

Lecture Note 4

Prof. Won Sik Yang

Purdue University
School of Nuclear Engineering

Weakness of Intuitive and One-Group Point Kinetics

- No distinction between prompt and delayed neutron spectra after integration

$$\int_0^\infty \chi_p(E)dE = 1, \quad \int_0^\infty \chi_{dk}(E)dE = 1$$

- Spatial and energy distributions of neutrons are neglected
  - Regardless of the birth position and energy, all the source neutrons have the same effect

$$\hat{\psi} = \int_V \int_0^\infty \psi(r,E)dEdV, \quad \hat{S}(t) = \int_V \int_0^\infty S(r,E,t)dEdV$$

  - For example, however, the neutrons born at the core center would have higher contributions to the neutron multiplication than those born at the core periphery
- Need to differentiate the neuron importance depending on the birth position and energy
  - This can be done employing adjoint functions

Linear Space and Operator

- Operators express certain mathematical operations or prescriptions to be carried out with a function or a vector
  - Applying an operator to a function (vector) yields another function (vector)

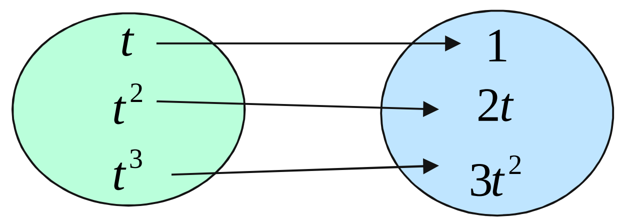
$$\mathbf{D}f(t) = \frac{d}{dt} f(t) \Rightarrow f'(t) \quad \text{(differntial operator)}$$

$$\mathbf{K}f(E) = \int_0^\infty dE' K(E' \to E) f(E') \Rightarrow g(E) \quad \text{(integral operator)}$$

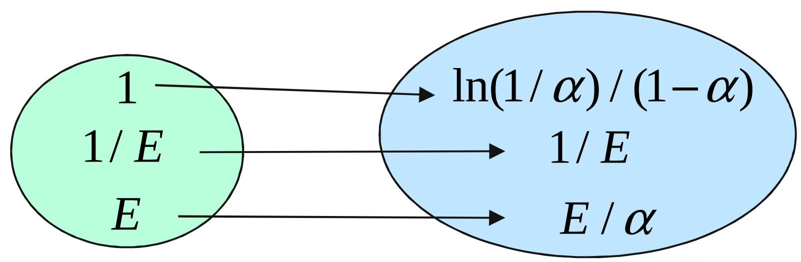
$$\mathbf{A}\vec{u} \Rightarrow \vec{v} \quad \text{(matrix operator)}$$

  - Mapping from a vector (linear) space to another vector (linear) space

$$\mathbf{D}f(t) = \frac{d}{dt} f(t) = g(t)$$



$$\mathbf{K}\phi(E) = \int_E^{E/\alpha} \frac{\phi(E')}{(1-\alpha)E'} dE'$$

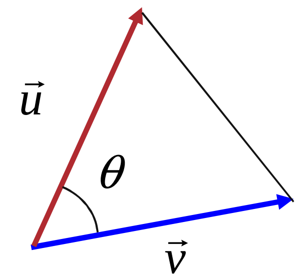


Inner Product

- Space of n-dimensional real vectors

$$\vec{u} = (u_1, u_2, \cdots, u_n)^T, \quad \vec{v} = (v_1, v_2, \cdots, v_n)^T, \quad (\vec{u}, \vec{v}) = \vec{u}^T \vec{v} = \sum_{i=1}^{n} u_i v_i$$

$$\|\vec{u}\| = (\vec{u}, \vec{u}), \quad \|\vec{v}\| = (\vec{v}, \vec{v}), \quad (\vec{u}, \vec{v}) = \|\vec{u}\| \|\vec{v}\| \cos\theta$$



- Space of n-dimensional complex vectors

$$< \vec{u}, \vec{v} > = \vec{u}^H \vec{v} = (\bar{\vec{u}})^T \vec{v}, \quad \vec{u}^H = (\bar{\vec{u}})^T = (\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n) \text{ (conjugate transpose)}$$

- Space of real functions

$$< f, g > = \int_a^b f(x) g(x) dx$$

$$\| f \| = < f, f > = \int_a^b [f(x)]^2 dx, \quad \| g \| = < g, g > = \int_a^b [g(x)]^2 dx$$

$$\cos\theta = \frac{< f, g >}{\| f \| \| g \|}$$

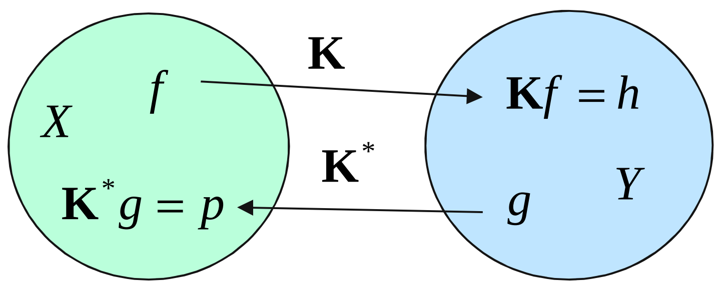
- Space of complex functions

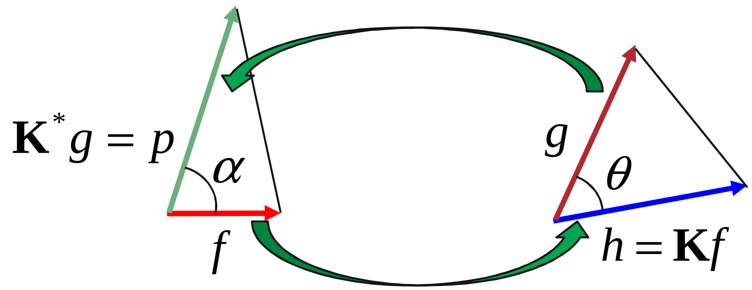
$$< f, g > = \int_a^b \overline{f(x)} g(x) dx$$

Adjoint Operators

- Adjoint operator

$$< g,\mathbf{K}f > = < \mathbf{K}^{*}g, f >, \quad \forall f \in X, \quad \forall g \in Y, \quad \mathbf{K}: X \to Y, \quad \mathbf{K}^{*}: Y \to X$$





$$< p, f > = \|p\|\|f\|\cos\vartheta \qquad < g,h > = \|g\|\|h\|\cos\theta$$

- Matrix operator
  - Real space

$$< \vec{u}, \mathbf{A}\vec{v} > = \vec{u}^{T}\mathbf{A}\vec{v} = (\mathbf{A}\vec{v})^{T}\vec{u} = \vec{v}^{T}\mathbf{A}^{T}\vec{u} = < \mathbf{A}^{T}\vec{u}, \vec{v} > \quad \Rightarrow \quad \mathbf{A}^{*} = \mathbf{A}^{T}$$

If $\mathbf{A} = \mathbf{A}^{T}$ (symmetric), then $\mathbf{A} = \mathbf{A}^{*}$ (self-adjoint)

  - Complex space

$$< \vec{u}, \mathbf{A}\vec{v} > = (\bar{\vec{u}})^{T}\mathbf{A}\vec{v} = (\mathbf{A}\vec{v})^{T}\bar{\vec{u}} = \vec{v}^{T}\mathbf{A}^{T}\bar{\vec{u}} = < \bar{\mathbf{A}}^{T}\vec{u}, \vec{v} > \quad \Rightarrow \quad \mathbf{A}^{*} = \bar{\mathbf{A}}^{T} = \mathbf{A}^{H}$$

If $\mathbf{A} = \mathbf{A}^{H}$ (Hermitian), then $\mathbf{A} = \mathbf{A}^{*}$ (self-adjoint)

Adjoint Operators

- Differential operator
  - Real space

$$\left\langle f, \frac{d}{dt} g \right\rangle = \int_a^b f \frac{dg}{dt} dt = fg\Big|_a^b + \int_a^b \left(-\frac{df}{dt} \right) g dt = \left\langle -\frac{d}{dt} f, g \right\rangle \quad \Rightarrow \quad \left(\frac{d}{dt} \right)^* = -\frac{d}{dt}$$

$$\text{if} \quad fg\Big|_a^b = f(b)g(b) - f(a)g(a) = 0 \quad \text{(bilinear concomitant)}$$

  - The function spaces for a differential operator are defined with the boundary conditions that make the bilinear concomitant vanish
- Integral operators
  - Real space

$$\mathbf{K}g(E) = \int dE' K(E' \to E) g(E')$$

$$< f, \mathbf{K}g > = \int dx f(x) \int dx' K(x' \to x) g(x') = \int dx \int dx' f(x) K(x' \to x) g(x')$$

$$= \int dx' \int dx f(x') K(x \to x') g(x) = \int dx g(x) \int dx' K(x \to x') f(x') = < \mathbf{K}^* f, g >$$

$$\Rightarrow \quad \mathbf{K}^* f(E) = \int dE' K(E \to E') f(E')$$

Adjoint Diffusion Equation

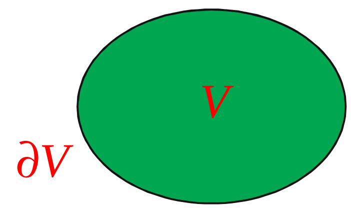
- Continuous-energy neutron diffusion equation

$$\mathbf{M}\phi(r,E)=\lambda\mathbf{F}\phi(r,E)$$

$$\mathbf{M}\phi(r,E)=-\nabla\cdot D(r,E)\nabla\phi(r,E)+\Sigma_t(r,E)\phi(r,E)-\int_0^\infty\Sigma_s(r,E'\to E)\phi(r,E')dE'$$

$$\mathbf{F}\phi(r,E)=\chi(E)\int_0^\infty\nu\Sigma_f(r,E')\phi(r,E')dE'$$

$$a\mathbf{n}\cdot D(r,E)\nabla\phi(r,E)+b\phi(r,E)=0,\quad r\in\partial V$$



- Multi-group diffusion equation

$$\mathbf{M}\vec{\phi}(r)=\lambda\mathbf{F}\vec{\phi}(r)$$

$$\vec{\phi}(r)=\begin{bmatrix}\phi_1\\\phi_2\\\vdots\\\phi_G\end{bmatrix}$$

$$\mathbf{M}=\begin{bmatrix}-\nabla\cdot D_1\nabla+\Sigma_{r1} & -\Sigma_{s21} & \cdots & -\Sigma_{sG1}\\ -\Sigma_{s12} & -\nabla\cdot D_2\nabla+\Sigma_{r2} & \cdots & -\Sigma_{sG2}\\ \vdots & \vdots & \ddots & \vdots\\ -\Sigma_{s1G} & -\Sigma_{s2G} & \cdots & -\nabla\cdot D_G\nabla+\Sigma_{rG}\end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix}\chi_1\nu\Sigma_{f1} & \chi_1\nu\Sigma_{f2} & \cdots & \chi_1\nu\Sigma_{fG}\\ \chi_2\nu\Sigma_{f1} & \chi_2\nu\Sigma_{f2} & \cdots & \chi_2\nu\Sigma_{fG}\\ \vdots & \vdots & \ddots & \vdots\\ \chi_G\nu\Sigma_{f1} & \chi_G\nu\Sigma_{f2} & \cdots & \chi_G\nu\Sigma_{fG}\end{bmatrix}$$

Adjoint Diffusion Operator

- Diffusion term

$$\nabla\cdot(f\vec{g}) = \nabla f\cdot\vec{g} + f\nabla\cdot\vec{g} \quad \Rightarrow$$

$$\phi^*\nabla\cdot(D\nabla\phi) = \nabla\cdot(\phi^* D\nabla\phi) - \nabla\phi^*\cdot(D\nabla\phi)$$

$$\phi\nabla\cdot(D\nabla\phi^*) = \nabla\cdot(\phi D\nabla\phi^*) - \nabla\phi\cdot(D\nabla\phi^*)$$

  – Subtracting the second equation from the first one yields (for simplicity, integration over energy is omitted)

$$\int_V \phi^*\nabla\cdot(D\nabla\phi)dV = \int_V \phi\nabla\cdot(D\nabla\phi^*)dV + \int_V \nabla\cdot(\phi^* D\nabla\phi - \phi D\nabla\phi^*)dV$$

$$= \int_V \phi\nabla\cdot(D\nabla\phi^*)dV + (\phi^*\mathbf{n}\cdot D\nabla\phi - \phi\mathbf{n}\cdot D\nabla\phi^*)_{\partial V}$$

  – If the boundary condition for the adjoint flux is defined as

$$a\mathbf{n}\cdot D(r,E)\nabla\phi^*(r,E) + b\phi^*(r,E) = 0, \quad r\in\partial V$$

  – Adjoint diffusion operator can be obtained as

$$\int_V \phi^*\nabla\cdot(D\nabla\phi)dV = \int_V \phi\nabla\cdot(D\nabla\phi^*)dV \quad \Rightarrow$$

$$\{-\nabla\cdot[D\nabla(\bullet)]\}^* = -\nabla\cdot[D\nabla(\bullet)] \quad (\text{self-adjoint})$$

Boundary Conditions and Bilinear Concomitant

- Forward equation

$$a\mathbf{n}\cdot D(r,E)\nabla\phi(r,E)+b\phi(r,E)=0, \quad r\in\partial V$$

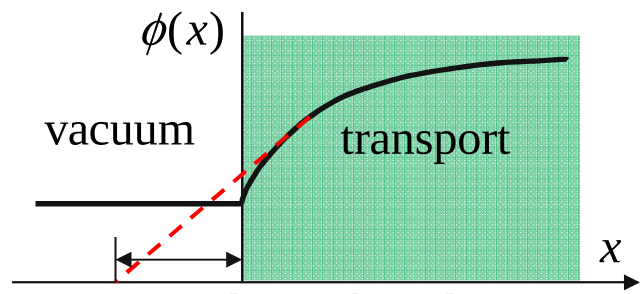
$$a=0 \quad \Rightarrow \quad \phi(r_b,E)=0 \quad \text{(zero flux)}$$

$$b=0 \quad \Rightarrow \quad \mathbf{n}\cdot D(r_b,E)\nabla\phi(r_b,E)=J_n(r_b,E)=0$$

$$\text{(zero current)}$$

$$b=1,\ a=2.133 \quad \Rightarrow \quad \phi(r_b+0.711\lambda_{tr},E)=0$$

$$\text{(extrapolated bc)}$$



- Adjoint boundary condition

$$a\mathbf{n}\cdot D(r,E)\nabla\phi^*(r,E)+b\phi^*(r,E)=0, \quad r\in\partial V$$

- Bilinear concomitant

$$\phi^*(a\mathbf{n}\cdot D\nabla\phi+b\phi)-\phi(a\mathbf{n}\cdot D\nabla\phi^*+b\phi^*)=0 \quad \Rightarrow \quad a(\phi^*\mathbf{n}\cdot D\nabla\phi-\phi\mathbf{n}\cdot D\nabla\phi^*)=0$$

$$a\neq 0 \quad \Rightarrow \quad (\phi^*\mathbf{n}\cdot D\nabla\phi-\phi\mathbf{n}\cdot D\nabla\phi^*)=0$$

$$a=0 \quad \Rightarrow \quad \phi(r_b,E)=\phi^*(r_b,E)=0 \quad \Rightarrow \quad (\phi^*\mathbf{n}\cdot D\nabla\phi-\phi\mathbf{n}\cdot D\nabla\phi^*)=0$$

  – Bilinear concomitant vanishes if the forward and adjoint fluxes satisfy the these boundary conditions

Adjoint Reaction Operators

- Total reaction term

$$\int_V \int_0^\infty \phi^*(\Sigma_t \phi)dEdV = \int_V \int_0^\infty \phi(\Sigma_t \phi^*)dEdV \quad \Rightarrow \quad (\Sigma_t)^* = \Sigma_t \quad \text{(self-adjoint)}$$

Total number of reactions per unit time $\Rightarrow$ ϕ^* is a dimensionless weighting function

- Scattering source term (for simplicity, integration over space is omitted)

$$\int_0^\infty \phi^*(r,E)\left[\int_0^\infty \Sigma_s(r,E'\to E)\phi(r,E')dE'\right]dE$$

$$= \int_0^\infty \phi(r,E')\left[\int_0^\infty \Sigma_s(r,E'\to E)\phi^*(r,E)dE\right]dE' \quad \Rightarrow$$

$$= \int_0^\infty \phi(r,E)\left[\int_0^\infty \Sigma_s(r,E\to E')\phi^*(r,E')dE'\right]dE$$

$$\left[\int_0^\infty \Sigma_s(r,E'\to E,t)\phi(r,E')dE'\right]^* = \int_0^\infty \Sigma_s(r,E\to E',t)\phi^*(r,E')dE'$$

- Fission source term (for simplicity, integration over space is omitted)

$$\int_0^\infty \phi^*(r,E)\left[\chi(E)\int_0^\infty \nu\Sigma_f(r,E')\phi(r,E')dE'\right]dE$$

$$= \int_0^\infty \phi(r,E')\left[\nu\Sigma_f(r,E')\int_0^\infty \chi(E)\phi^*(r,E)dE\right]dE' \quad \Rightarrow$$

$$= \int_0^\infty \phi(r,E)\left[\nu\Sigma_f(r,E)\int_0^\infty \chi(E')\phi^*(r,E')dE'\right]dE$$

$$\left[\chi(E)\int_0^\infty \nu\Sigma_f(r,E')\phi(r,E')dE'\right]^* = \nu\Sigma_f(r,E)\int_0^\infty \chi(E')\phi^*(r,E')dE'$$

Adjoint Diffusion Equation

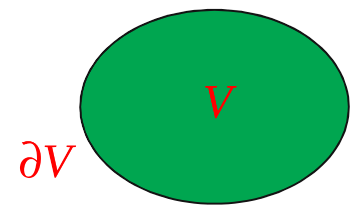
- Adjoint continuous energy neutron diffusion equation

$$\mathbf{M}^*\phi^*(r,E) = \lambda \mathbf{F}^*\phi^*(r,E)$$

$$\mathbf{M}^*\phi^*(r,E) = -\nabla\cdot D(r,E)\nabla\phi^*(r,E) + \Sigma_t(r,E)\phi^*(r,E) - \int_0^\infty \Sigma_s(r,E\to E')\phi^*(r,E')dE'$$

$$\mathbf{F}^*\phi^*(r,E) = \lambda \nu\Sigma_f(r,E)\int_0^\infty \chi(E')\phi^*(r,E')dE'$$

$$a\mathbf{n}\cdot D(r,E)\nabla\phi^*(r,E) + b\phi^*(r,E) = 0, \quad r\in\partial V$$



- Adjoint multi-group diffusion equation

$$\mathbf{M}^*\vec{\phi}^*(r) = \lambda \mathbf{F}^*\vec{\phi}^*(r)$$

$$\vec{\phi}^*(r) = \begin{bmatrix} \phi_1^* \\ \phi_2^* \\ \vdots \\ \phi_G^* \end{bmatrix}$$

$$\mathbf{M}^* = \begin{bmatrix} -\nabla\cdot D_1\nabla + \Sigma_{r1} & -\Sigma_{s12} & \cdots & -\Sigma_{s1G} \\ -\Sigma_{s21} & -\nabla\cdot D_2\nabla + \Sigma_{r2} & \cdots & -\Sigma_{s2G} \\ \vdots & \vdots & \ddots & \vdots \\ -\Sigma_{sG1} & -\Sigma_{sG2} & \cdots & -\nabla\cdot D_G\nabla + \Sigma_{rG} \end{bmatrix}$$

$$\mathbf{F}^* = \begin{bmatrix} \nu\Sigma_{f1}\chi_1 & \nu\Sigma_{f1}\chi_2 & \cdots & \nu\Sigma_{f1}\chi_G \\ \nu\Sigma_{f2}\chi_1 & \nu\Sigma_{f2}\chi_2 & \cdots & \nu\Sigma_{f2}\chi_G \\ \vdots & \vdots & \ddots & \vdots \\ \nu\Sigma_{fG}\chi_1 & \nu\Sigma_{fG}\chi_2 & \cdots & \nu\Sigma_{fG}\chi_G \end{bmatrix}$$

Physical Interpretation of Adjoint Flux

- Consider a system characterized by the operators **M** and **F**
  - The fundamental mode adjoint flux is determined by the following eigenvalue problem

$$(\mathbf{M}^* - \lambda \mathbf{F}^*)\phi^*(r,E) = 0$$

  - If a unit source of energy E_0 is introduced at position r_0, the corresponding flux is determined by the following source problem

$$(\mathbf{M} - \mathbf{F})\phi(r,E) = \delta(r-r_0)\delta(E-E_0)$$

  - Taking an inner product with the adjoint flux, we have

$$\left\langle \phi^*, (\mathbf{M}-\mathbf{F})\phi \right\rangle = \left\langle \phi^*, \delta(r-r_0)\delta(E-E_0) \right\rangle$$

$$\left\langle \phi^*, \delta(r-r_0)\delta(E-E_0) \right\rangle = \int_V \int_0^\infty \phi^*(r,E)\delta(r-r_0)\delta(E-E_0)\,dE\,dV = \phi^*(r_0,E_0)$$

$$\left\langle \phi^*, (\mathbf{M}-\mathbf{F})\phi \right\rangle = \left\langle \mathbf{M}^*\phi^*, \phi \right\rangle - \left\langle \phi^*, \mathbf{F}\phi \right\rangle = \left\langle \lambda \mathbf{F}^*\phi^*, \phi \right\rangle - \left\langle \phi^*, \mathbf{F}\phi \right\rangle = (\lambda - 1)\left\langle \phi^*, \mathbf{F}\phi \right\rangle$$

$$\phi^*(r_0,E_0) = (\lambda-1)\left\langle \phi^*, \mathbf{F}\phi \right\rangle$$

  - Thus the adjoint flux represents the relative contribution to the fission source by the unit source introduced in the specified position and energy

Adjoint Flux for Detector Response

- Fixed source problem

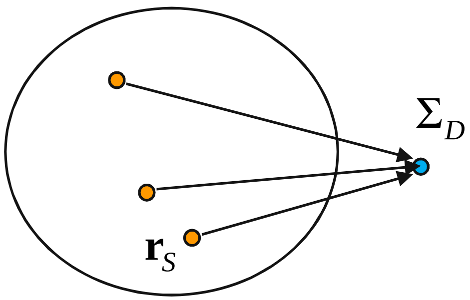
$$\mathbf{M}\phi = \mathbf{F}\phi + S \quad \text{or} \quad (\mathbf{M}-\mathbf{F})\phi = S \quad (\text{neutrons/cm}^3\text{s})$$

- Corresponding adjoint source problem for the detector response

$$(\mathbf{M}^* - \mathbf{F}^*)\phi^* = S^* \quad (1/\text{cm} \Rightarrow \text{unit of macroscopic cross section})$$

$$S^* = \Sigma_D \quad (\text{detector cross section})$$

- Consider a source located at $\mathbf{r}_s$ and a detector located at $\mathbf{r}_D$



  - The source problem is then specified as

$$(\mathbf{M}-\mathbf{F})\phi = S\delta(r - r_s)$$

  - The corresponding adjoint problem can be specified as

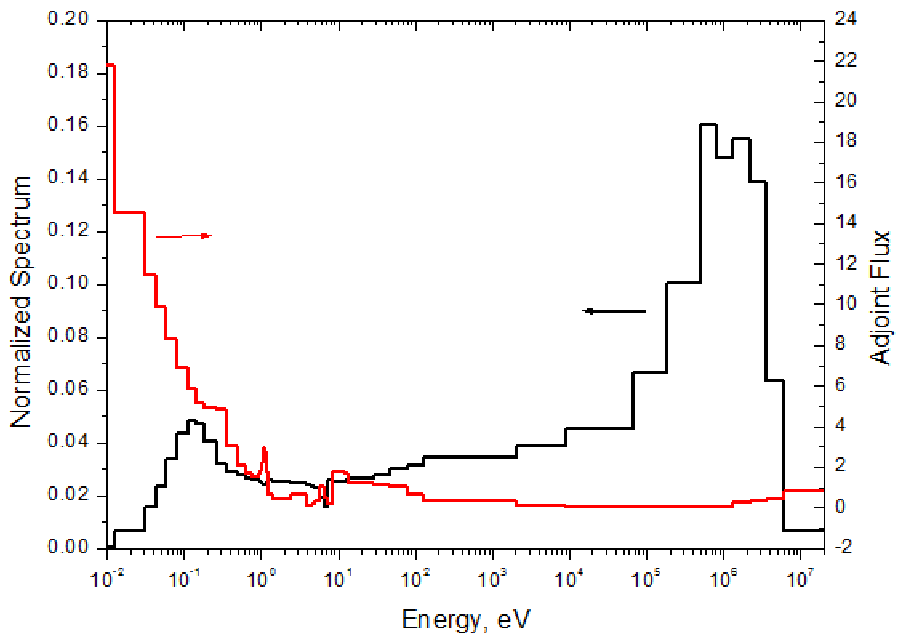
$$(\mathbf{M}^* - \mathbf{F}^*)\phi^* = \Sigma_D \delta(r - r_D)$$

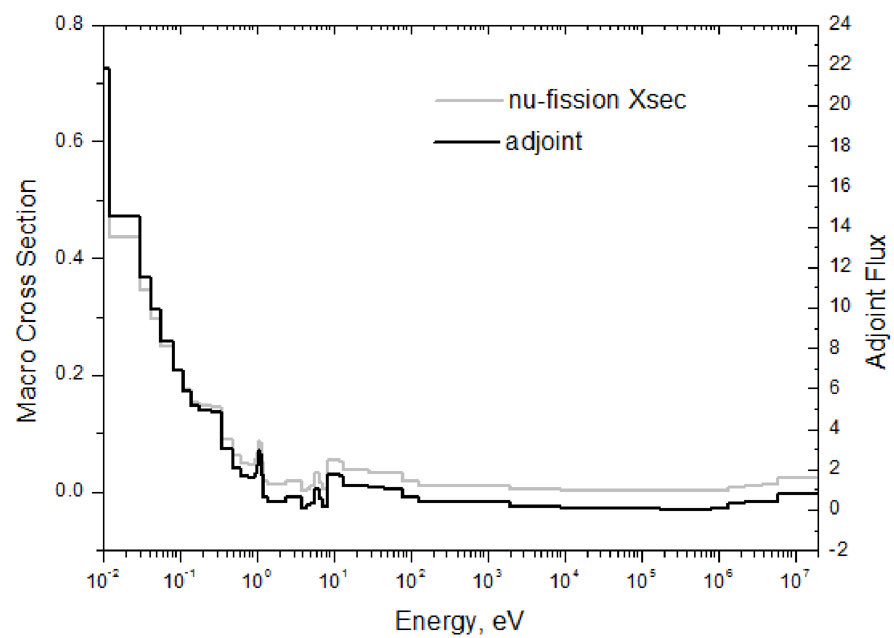
$$0 = <\phi^*, (\mathbf{M}-\mathbf{F})\phi> - <\phi, (\mathbf{M}^* - \mathbf{F}^*)\phi^*> = <\phi^*, S\delta(r-r_s)> - <\phi, \Sigma_D\delta(r - r_D)>$$

$$\Rightarrow \quad \phi^*(r_s)S = \phi(r_D)\Sigma_D \quad \Rightarrow \quad \phi^*(r_s) = \frac{\phi(r_D)\Sigma_D}{S}$$

  - Detector response (i.e. reaction rate at the detector) per unit source at $\mathbf{r}_s$

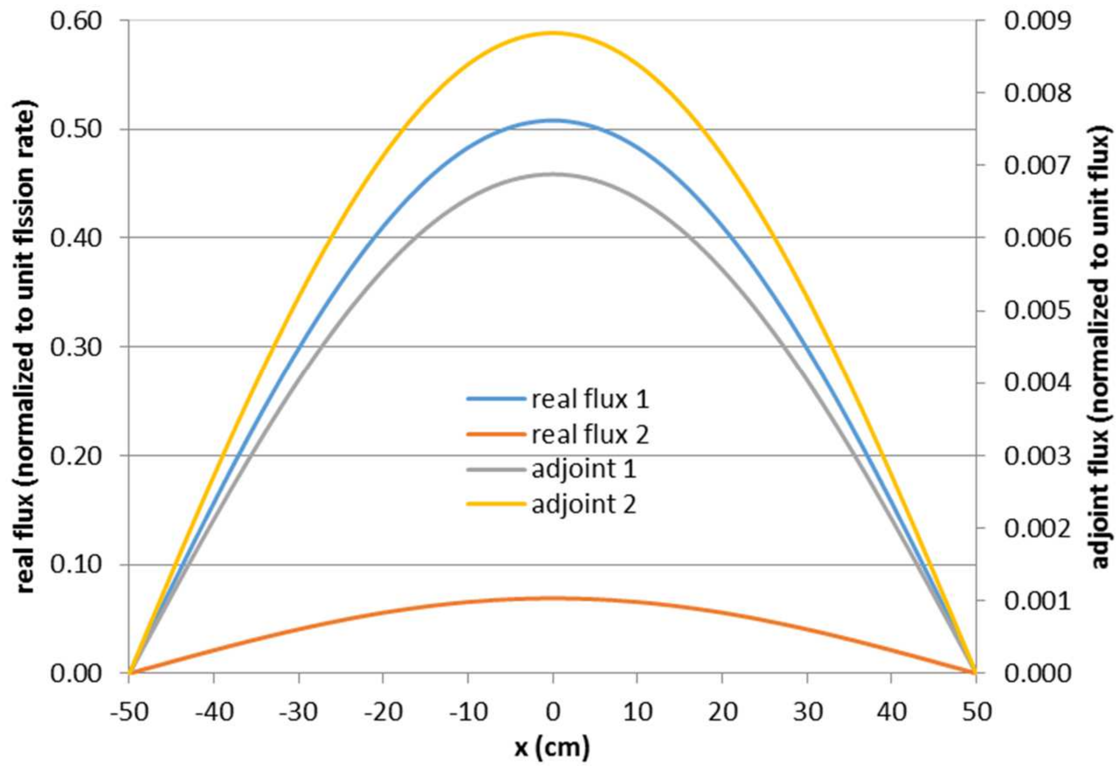
Adjoint Flux in PWR Pin Cell





$$-\nabla \cdot D(r,E)\nabla\phi^*(r,E) + \Sigma_t(r,E)\phi^*(r,E)$$
$$-\int_0^\infty \Sigma_s(r,E \to E')\phi^*(r,E')dE'$$
$$= \lambda \nu \Sigma_f(r,E)\int_0^\infty \chi(E')\phi^*(r,E')dE'$$

Two-Group Real and Adjoint Fluxes in Slab Reactor



Group	D	Σ_a	Σ_{aF}	$\nu\Sigma_f$	χ	Σ_{s12}
1	1.44	0.01	0.01	0.008	1.0	0.017
2	0.366	0.125	0.09	0.169	0.0	

First Order Perturbation Theory

- Eigensystem of the adjoint operator

$$\begin{aligned}\mathbf{A}\phi_i &= \lambda_i\phi_i \\ \mathbf{A}^*\phi_j^* &= \lambda_j^*\phi_j^*\end{aligned} \quad \Rightarrow \quad \begin{aligned}&<\phi_j^*, \mathbf{A}\phi_i> = <\phi_j^*, \lambda_i\phi_i> = \lambda_i <\phi_j^*, \phi_i> \\ &<\mathbf{A}^*\phi_j^*, \phi_i> = <\lambda_j^*\phi_j^*, \phi_i> = \overline{\lambda}_j^* <\phi_j^*, \phi_i>\end{aligned} \quad \Rightarrow \quad \begin{aligned}&(\lambda_i - \overline{\lambda}_j^*) <\phi_j^*, \phi_i> = 0 \\ &\lambda_j^* = \overline{\lambda}_i \text{ or } <\phi_j^*, \phi_i> = 0\end{aligned}$$

  – If the eigenfunctions are labeled in such a way that the eigenvalues associated with ϕ_i are the complex conjugates of the eigenvalues associated with ϕ_i^*

$$<\phi_j^*, \phi_i> = 0, \quad i \neq j \quad \text{(i.e., biorthogonal)}$$

- The aim of the first order perturbation theory is to evaluate the eigenvalue perturbation without solving the perturbed equation
  – A perturbation introduced in the operator changes the eigenvalue and eigenfunction to satisfy the perturbed equation

$$\mathbf{A}'\phi' = \lambda'\phi' \quad \Rightarrow \quad (\mathbf{A} + \Delta\mathbf{A})(\phi + \Delta\phi) = (\lambda + \Delta\lambda)(\phi + \Delta\phi)$$

$$\mathbf{A}\phi = \lambda\phi \quad \Rightarrow \quad \mathbf{A}\Delta\phi + \Delta\mathbf{A}\phi + \Delta\mathbf{A}\Delta\phi = \lambda\Delta\phi + \Delta\lambda\phi + \Delta\lambda\Delta\phi$$

  – Neglecting the second order terms yields

$$A\Delta\phi + \Delta A\phi = \lambda\Delta\phi + \Delta\lambda\phi$$

First Order Perturbation Theory

- First order estimation of eigenvalue perturbation

$$\mathbf{A}\Delta\phi + \Delta\mathbf{A}\phi = \lambda\Delta\phi + \Delta\lambda\phi \quad \text{(first order equation)}$$

  - Inner product with the adjoint function

$$<\phi^*, \cancel{\mathbf{A}\Delta\phi}> + <\phi^*, \Delta\mathbf{A}\phi> = <\phi^*, \cancel{\lambda\Delta\phi}> + <\phi^*, \Delta\lambda\phi> \quad \Rightarrow \quad \boxed{\Delta\lambda = \frac{<\phi^*, \Delta\mathbf{A}\phi>}{<\phi^*, \phi>}}$$

$$<\phi^*, \mathbf{A}\Delta\phi> = <\mathbf{A}^*\phi^*, \Delta\phi> = <\bar{\lambda}\phi^*, \Delta\phi> = \lambda <\phi^*, \Delta\phi>$$

$$<\phi^*, \lambda\Delta\phi> = \lambda <\phi^*, \Delta\phi>$$

  - Eigenvalue perturbation can be evaluated only with the operator change (without knowing the eigenfunction perturbation)

- Forward and adjoint neutron balance equations

$$\boxed{\mathbf{M}\phi = \lambda\mathbf{F}\phi, \quad \mathbf{M}^*\phi^* = \lambda\mathbf{F}^*\phi^*}$$

  - First order estimation of eigenvalue perturbation

$$\mathbf{M}'\phi' = \lambda'\mathbf{F}'\phi' \quad \Rightarrow \quad (\mathbf{M} + \Delta\mathbf{M})(\phi + \Delta\phi) = (\lambda + \Delta\lambda)(\mathbf{F} + \Delta\mathbf{F})(\phi + \Delta\phi)$$

$$\mathbf{M}\phi = \lambda\mathbf{F}\phi \quad \Rightarrow \quad \mathbf{M}\Delta\phi + \Delta\mathbf{M}\phi = \lambda\mathbf{F}\Delta\phi + \lambda\Delta\mathbf{F}\phi + \Delta\lambda\mathbf{F}\phi \quad \text{(first order equation)}$$

$$\mathbf{M}^*\phi^* = \lambda\mathbf{F}^*\phi^* \quad \Rightarrow \quad <\phi^*, \cancel{\mathbf{M}\Delta\phi}> + <\phi^*, \Delta\mathbf{M}\phi>$$

$$= \lambda <\phi^*, \cancel{\mathbf{F}\Delta\phi}> + \lambda <\phi^*, \Delta\mathbf{F}\phi> + \Delta\lambda <\phi^*, \mathbf{F}\phi>$$

First Order Perturbation Theory

- First order estimation of eigenvalue perturbation

$$\boxed{\Delta\lambda = \frac{<\phi^*,[\Delta\mathbf{M}-\lambda\Delta\mathbf{F}]\phi>}{<\phi^*,\mathbf{F}\phi>}}$$

$$(\mathbf{M}-\lambda\mathbf{F})\phi = 0$$

$$\Delta\lambda = \frac{<\phi^*,[\Delta\mathbf{M}-\lambda\Delta\mathbf{F}]\phi>}{<\phi^*,\mathbf{F}\phi>} = \frac{<\phi^*,[(\mathbf{M}+\Delta\mathbf{M})-\lambda(\mathbf{F}+\Delta\mathbf{F})]\phi>}{<\phi^*,\mathbf{F}\phi>} = \frac{<\phi^*,[\mathbf{M}'-\lambda\mathbf{F}']\phi>}{<\phi^*,\mathbf{F}\phi>}$$

- First order estimation of reactivity change

$$\rho = 1-\frac{1}{k} = 1-\lambda \quad \text{(static reactivity)}$$

$$\boxed{\Delta\rho = -\Delta\lambda = \frac{<\phi^*,[\lambda\Delta\mathbf{F}-\Delta\mathbf{M}]\phi>}{<\phi^*,\mathbf{F}\phi>} = \frac{<\phi^*,[\lambda\mathbf{F}'-\mathbf{M}']\phi>}{<\phi^*,\mathbf{F}\phi>}}$$

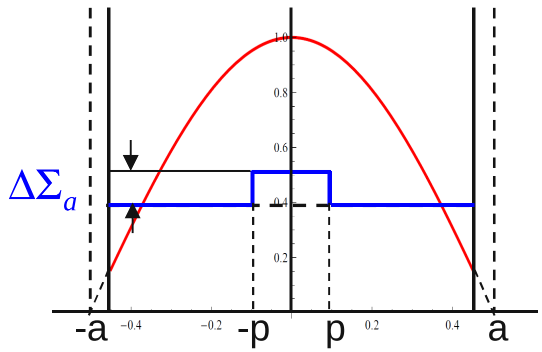
  – Simple one-group point model

$$\Sigma_a\phi = \lambda\nu\Sigma_f\phi, \quad \lambda = \frac{\Sigma_a}{\nu\Sigma_f} \quad \Rightarrow \quad \Delta\rho = \frac{\lambda\Delta(\nu\Sigma_f)-\Delta\Sigma_a}{\nu\Sigma_f} = \frac{\Sigma_a\Delta(\nu\Sigma_f)-\nu\Sigma_f\Delta\Sigma_a}{(\nu\Sigma_f)^2} \quad \text{(FOP)}$$

$$\text{Exact difference:} \quad \Delta\rho = \frac{\Sigma_a}{\nu\Sigma_f}-\frac{\Sigma_a'}{(\nu\Sigma_f)'} = \frac{(\nu\Sigma_f)'\Sigma_a-\nu\Sigma_f\Sigma_a'}{\nu\Sigma_f(\nu\Sigma_f)'} = \frac{\Delta(\nu\Sigma_f)\Sigma_a-\nu\Sigma_f\Delta\Sigma_a}{\nu\Sigma_f[\nu\Sigma_f+\Delta(\nu\Sigma_f)]}$$

One-Dimensional, One-Group Problem

- Increased absorption XS at the core center by $\Delta\Sigma_a$ for $-p \le x \le p$
- 1-D, 1-G diffusion equation



$$-D\frac{d^2\phi}{dx^2} + \Sigma_a\phi = \lambda\nu\Sigma_f\phi, \quad \phi(\pm a) = 0$$

$$\frac{d^2\phi}{dx^2} + \frac{\lambda\nu\Sigma_f - \Sigma_a}{D}\phi = 0, \quad B^2 = \frac{\lambda\nu\Sigma_f - \Sigma_a}{D}$$

$$\frac{d^2\phi}{dx^2} + B^2\phi = 0 \quad \Rightarrow \quad \phi(x) = A\cos(Bx) \quad (\because \text{ symmetric})$$

$$\phi(\pm a) = 0 \quad \Rightarrow \quad A\cos(Ba) = 0 \quad \Rightarrow \quad B = \frac{\pi}{2a} \qquad \phi(0) = \phi_0 \quad \Rightarrow \quad \phi(x) = \phi(0)\cos(Bx)$$

$$\lambda = \frac{DB^2 + \Sigma_a}{\nu\Sigma_f}, \quad \rho = 1 - \lambda = \frac{\nu\Sigma_f - (DB^2 + \Sigma_a)}{\nu\Sigma_f}$$

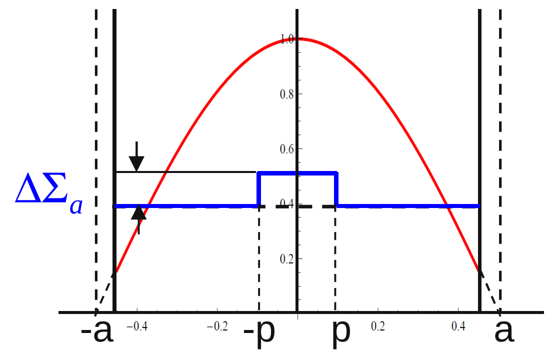
- Adjoint equation

$$-D\frac{d^2\phi^*}{dx^2} + \Sigma_a\phi^* = \lambda\nu\Sigma_f\phi^*, \quad \phi^*(\pm a) = 0 \quad \text{(self-adjoint)}$$

$$\phi^*(x) = \phi_0^*\cos(Bx)$$

One-Dimensional, One-Group Problem

- Reactivity change by FOP



$$\Delta\rho = \frac{<\phi^*,[\lambda\Delta\mathbf{F}-\Delta\mathbf{M}]\phi>}{<\phi^*,F\phi>} = -\frac{\int_{-p}^{p}\Delta\Sigma_a\cos^2(Bx)dx}{\int_{-a}^{a}\nu\Sigma_f\cos^2(Bx)dx}$$

$$\int_{-a}^{a}\nu\Sigma_f\cos^2(Bx)dx = 2\nu\Sigma_f\int_0^a\frac{1+\cos(2Bx)}{2}dx \cong \nu\Sigma_f a$$

$$\int_{-p}^{p}\Delta\Sigma_a\cos^2(Bx)dx = 2\Delta\Sigma_a\int_0^p\frac{1+\cos(2Bx)}{2}dx \cong \Delta\Sigma_a(2p) \quad \Leftarrow \quad \cos(2Bx)\simeq 1 \text{ near } x=0$$

$$\Delta\rho = -\frac{2p}{a}\frac{\Delta\Sigma_a}{\nu\Sigma_f}$$

- Reactivity change by reaction rates without using adjoint flux

$$\Delta\rho = \frac{\text{change in net production rate}}{\text{production rate}} = \frac{-\int_{-p}^{p}\Delta\Sigma_a\cos Bxdx}{\int_{-a}^{a}\nu\Sigma_f\cos Bxdx}$$

$$\cong -\frac{p\Delta\Sigma_a}{\nu\Sigma_f/B} = -\frac{\pi}{2}\frac{p}{a}\frac{\Delta\Sigma_a}{\nu\Sigma_f} \cong -\frac{1.7p}{a}\frac{\Delta\Sigma_a}{\nu\Sigma_f}$$

Exact Perturbation Theory

- Exact perturbation theory expresses the exact eigenvalue differences in a form similar to FOP formula using the solution to the perturbed problem

$$\boxed{\mathbf{M}'\phi' = \lambda'\mathbf{F}'\phi', \quad \mathbf{M}^*\phi^* = \lambda\mathbf{F}^*\phi^*}$$

$$<\phi^*, \mathbf{M}'\phi'> = \lambda' <\phi^*, \mathbf{F}'\phi'> \quad \Rightarrow \quad <\phi^*, \cancel{\mathbf{M}}\phi'> + <\phi^*, \Delta\mathbf{M}\phi'>$$

$$= \lambda <\phi^*, \mathbf{F}'\phi'> + \Delta\lambda <\phi^*, \mathbf{F}'\phi'>$$

$$= \lambda <\phi^*, \cancel{\mathbf{F}}\phi'> + \lambda <\phi^*, \Delta\mathbf{F}\phi'> + \Delta\lambda <\phi^*, \mathbf{F}'\phi'>$$

$$\boxed{\Delta\lambda = \frac{<\phi^*, [\Delta\mathbf{M} - \lambda\Delta\mathbf{F}]\phi'>}{<\phi^*, \mathbf{F}'\phi'>}}$$

$$\lambda = 1 \quad \Rightarrow \quad \rho^{st} = 1 - \lambda = \frac{<\phi^*, [\Delta\mathbf{F} - \Delta\mathbf{M}]\phi'>}{<\phi^*, \mathbf{F}'\phi'>} = \frac{<\phi^*, [\mathbf{F}' - \mathbf{M}']\phi>}{<\phi^*, \mathbf{F}'\phi'>} \quad \text{(static reactivity)}$$

- Alternative formula based on the perturbed adjoint flux

$$\boxed{\mathbf{M}\phi = \lambda\mathbf{F}\phi, \quad \mathbf{M}'^*\phi'^* = \lambda'\mathbf{F}'^*\phi'^*}$$

$$<\phi, \mathbf{M}'^*\phi'^*> = \lambda' <\phi, \mathbf{F}'^*\phi'^*> \quad \Rightarrow \quad <\phi, \cancel{\mathbf{M}^*}\phi'^*> + <\phi, \Delta\mathbf{M}^*\phi'^*>$$

$$= \lambda <\phi, \mathbf{F}'^*\phi'^*> + \Delta\lambda <\phi, \mathbf{F}'^*\phi'^*> = \lambda <\phi, \cancel{\mathbf{F}^*}\phi'^*> + \lambda <\phi, \Delta\mathbf{F}^*\phi'^*> + \Delta\lambda <\phi, \mathbf{F}'^*\phi'^*>$$

$$\boxed{\Delta\lambda = \frac{<\phi, [\Delta\mathbf{M}^* - \lambda\Delta\mathbf{F}^*]\phi'^*>}{<\phi, \mathbf{F}'^*\phi'^*>} = \frac{<\phi'^*, [\Delta\mathbf{M} - \lambda\Delta\mathbf{F}]\phi>}{<\phi'^*, \mathbf{F}'\phi>}}$$