NUCL 510 Nuclear Reactor Theory I
Fall 2011

Homework #6

Due October 13

1. Suppose an angular flux in a slab geometry is given by

   $$\psi(z,\mu) = \phi_0(\cos Bz + A\mu \sin Bz).$$

   Here, μ is the cosine of the polar angle measured from the z axis.

   (a) Find the flux $\phi(z)$ and the z-directional current $J(z)$.

   $$\phi(z) = \phi_0 \int_{-1}^{1} d\mu(\cos Bz + A\mu \sin Bz) = 2\phi_0 \cos Bz$$

   $$J(z) = \phi_0 \int_{-1}^{1} d\mu\mu(\cos Bz + A\mu \sin Bz) = \frac{2}{3}\phi_0 A \sin Bz$$

   (b) Find the partial currents $J^+(z)$ and $J^-(z)$ in the upper and lower half of the solid angle.

   $$J^+(z) = \phi_0 \int_0^1 d\mu\mu(\cos Bz + A\mu \sin Bz) = \frac{1}{2}\phi_0 \cos Bz + \frac{1}{3}\phi_0 A \sin Bz$$

   $$J^-(z) = -\phi_0 \int_{-1}^{0} d\mu\mu(\cos Bz + A\mu \sin Bz) = \frac{1}{2}\phi_0 \cos Bz - \frac{1}{3}\phi_0 A \sin Bz$$

   (c) Rewrite the angular flux in terms of $\phi(z)$ and $J(z)$.

   $$\psi(z,\mu) = \frac{1}{2}\phi(z) + \frac{2}{3}\mu J(z)$$

   (d) Rewrite $\phi(z)$ and $J(z)$ in terms of the partial currents.

   $$\phi(z) = 2\phi_0 \cos Bz = 2[J^+(z) + J^-(z)]$$

   $$J(z) = \frac{2}{3}\phi_0 A \sin Bz = J^+(z) - J^-(z)$$

2. Derive the one-group integral transport equation for the angular flux $\psi(r,\vec{\Omega})$ for a spherical geometry consisting of two media (a center sphere in a spherical shell with a vacuum outside). Assume isotropic scattering and uniformly distributed independent source.

   Ans.) A general expression for the angular flux at a position $\vec{r}$ in a direction $\vec{\Omega}$ is given by

   $$\psi(\vec{r},\vec{\Omega}) = \int_0^R dR' Q(\vec{r} - R'\vec{\Omega}, \vec{\Omega}) e^{-\tau(\vec{r},\vec{r}-R'\vec{\Omega})} + \psi(\vec{r} - R\vec{\Omega}, \vec{\Omega}) e^{-\tau(\vec{r},\vec{r}-R\vec{\Omega})}$$

   where the optical path is defined as

   $$\tau(\vec{r},\vec{r} - R\vec{\Omega}) = \int_0^R dR' \Sigma_t(\vec{r} - R'\vec{\Omega})$$

   For the assumed isotropic scattering and uniformly distributed independent source, the source becomes

   $$Q(\vec{r} - R'\vec{\Omega}, \vec{\Omega}) = \frac{1}{4\pi}\left[\Sigma_s(\vec{r} - R'\vec{\Omega})\phi(\vec{r} - R'\vec{\Omega}) + S(\vec{r} - R'\vec{\Omega})\right]$$

   Since the in-coming flux is zero at a vacuum boundary, the boundary flux becomes

   $$\psi(\vec{r} - R\vec{\Omega}, \vec{\Omega}) = 0$$

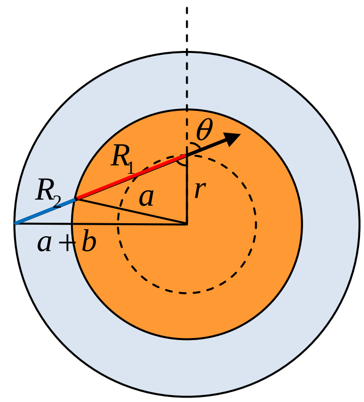
Thus the angular flux at a position $\vec{r}$ in a direction $\vec{\Omega}$ becomes

$$\psi(\vec{r},\vec{\Omega}) = \frac{1}{4\pi}\int_0^R dR'\left[\Sigma_s(\vec{r}-R'\vec{\Omega})\phi(\vec{r}-R'\vec{\Omega}) + S(\vec{r}-R'\vec{\Omega})\right]e^{-\tau(\vec{r},\vec{r}-R'\vec{\Omega})}$$

Let the radius of the center sphere be a and the thickness of the outer shell be b. Consider a direction of a polar angle θ at a point of radius r in the center sphere. Then, the path lengths in the center sphere and in the outer shell are given by

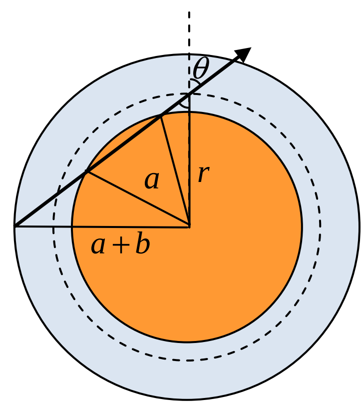
$$R_1 = r\cos\theta + \sqrt{a^2 - r^2\sin^2\theta}$$

$$\begin{aligned} R_2 &= \left[r\cos\theta + \sqrt{(a+b)^2 - r^2\sin^2\theta}\right] \\ &\quad -\left[r\cos\theta + \sqrt{a^2 - r^2\sin^2\theta}\right] \\ &= \sqrt{(a+b)^2 - r^2\sin^2\theta} - \sqrt{a^2 - r^2\sin^2\theta} \end{aligned}$$



As a result, the angular flux can be written as

$$\begin{aligned} \psi(\vec{r},\vec{\Omega}) &= \frac{1}{4\pi}\int_0^{R_1} dR'\left[\Sigma_{s1}\phi(\vec{r}-R'\vec{\Omega}) + S_1\right]e^{-\Sigma_{t1}R'} \\ &+ \frac{e^{-\Sigma_{t1}R_1}}{4\pi}\int_0^{R_2} dR'\left[\Sigma_{s2}\phi(\vec{r}-(R_1+R')\vec{\Omega}) + S_2\right]e^{-\Sigma_{t2}R'} \end{aligned}$$



Similarly, consider a point in the outer shell. If the ray passes through the center sphere, the ray can be divided into three segments with lengths

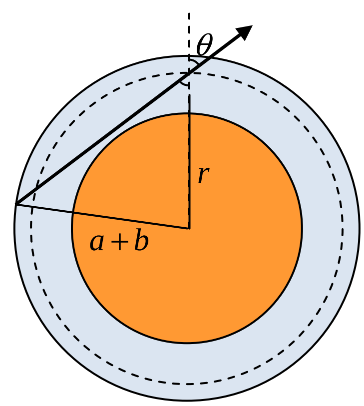
$$R_1 = r\cos\theta - \sqrt{a^2 - r^2\sin^2\theta}$$

$$R_2 = 2\sqrt{a^2 - r^2\sin^2\theta}$$

$$R_2 = \sqrt{(a+b)^2 - r^2\sin^2\theta} - \sqrt{a^2 - r^2\sin^2\theta}$$

and the angular flux can be written as

$$\begin{aligned} \psi(\vec{r},\vec{\Omega}) &= \frac{1}{4\pi}\int_0^{R_1} dR'\left[\Sigma_{s2}\phi(\vec{r}-R'\vec{\Omega}) + S_2\right]e^{-\Sigma_{t2}R'} \\ &+ \frac{e^{-\Sigma_{t2}R_1}}{4\pi}\int_0^{R_2} dR'\left[\Sigma_{s1}\phi(\vec{r}-(R_1+R')\vec{\Omega}) + S_1\right]e^{-\Sigma_{t1}R'} \\ &+ \frac{e^{-(\Sigma_{t2}R_1+\Sigma_{t1}R_2)}}{4\pi}\int_0^{R_3} dR'\left[\Sigma_{s2}\phi(\vec{r}-(R_1+R_2+R')\vec{\Omega}) + S_2\right]e^{-\Sigma_{t2}R'} \end{aligned}$$



If the ray does not pass through the center sphere, the path length in the outer shell becomes

$$R_1 = r\cos\theta + \sqrt{(a+b)^2 - r^2\sin^2\theta}$$

and the angular flux is given by

$$\psi(\vec{r},\vec{\Omega}) = \frac{1}{4\pi}\int_0^{R_1} dR'\left[\Sigma_{s2}\phi(\vec{r}-R'\vec{\Omega}) + S_2\right]e^{-\Sigma_{t2}R'}$$

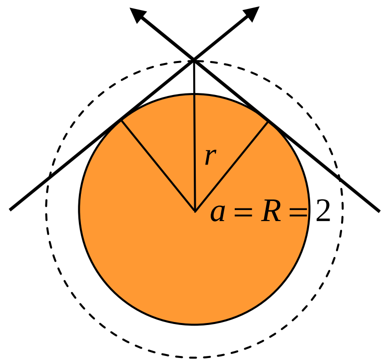
3. Investigate the iterative computational procedure for finding the flux $\phi(r)$ in the spherical problem described in problem 2. Assume the inner sphere of radius $R = 2$ cm is black (i.e., its absorption cross section is infinite); take $\Sigma_s = 0.2$ cm^{-1} and $\Sigma_a = 0$ in the outer medium,

assumed to be infinitely large, with $\phi(r \to \infty) = \phi_\infty$. (Homework problem #5 of Ch. 6)

a. Calculate the first iterate $\phi^{(1)}(r)$ from an assumed flux distribution as the initial guess $\phi^{(0)}(r) = \phi_\infty$, in the outer medium.

Ans.) Since the absorption cross section of the center sphere is infinite, the flux inside the center sphere is zero. In addition, only the rays that do not pass through the center sphere contribute to the flux in the outer shell. For simplicity, neglect the independence source. Then, the first iterate of the angular flux in a direction $\vec{\Omega}$ can be obtained as

$$\psi^{(1)}(\vec{r},\vec{\Omega}) = \frac{1}{4\pi}\int_0^\infty dR'(\Sigma_{s2}\phi_\infty)e^{-\Sigma_{s2}R'} = \frac{\phi_\infty}{4\pi}$$

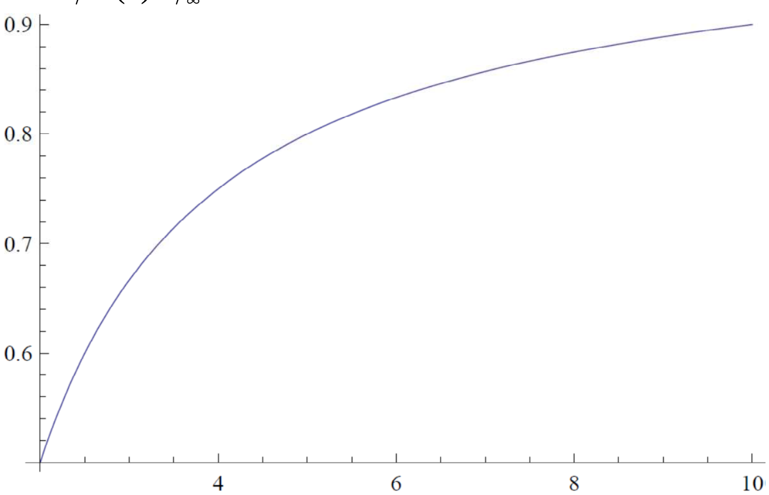


The scalar flux at a radius $r \geq 2$ can be obtained by integrating this angular flux. Noting that the neutrons meeting the center sphere would be absorbed completely, the scalar flux can be obtained as

$$\phi^{(1)}(r) = \frac{\phi_\infty}{4\pi}\times\left(4\pi - 2\pi\frac{R}{r}\right) = \phi_\infty\left(1 - \frac{R}{2r}\right)$$

$$= \phi_\infty\left(1 - \frac{1}{r}\right), \quad r \geq 2$$

b. Plot $\phi^{(1)}(r)/\phi_\infty$ and discuss the result.

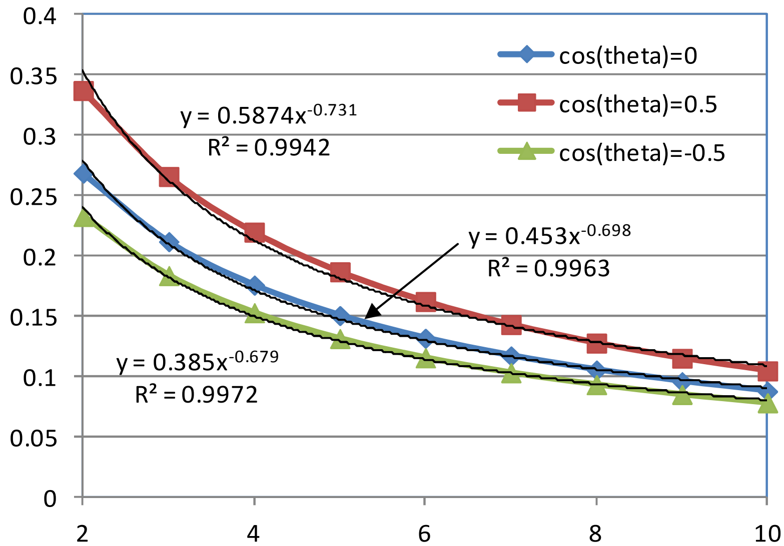


c. Discuss qualitatively the changes to be expected in the next iterate $\phi^{(2)}(r)$ as well as in the converged solution.

$$\psi^{(2)}(\vec{r},\vec{\Omega}) = \frac{1}{4\pi}\int_0^\infty dR'\Sigma_{s2}\phi^{(1)}(R')e^{-\Sigma_{s2}R'} = \frac{\Sigma_{s2}\phi_\infty}{4\pi}\int_0^\infty dR'\left[1 - \frac{1}{|\vec{r} - R'\vec{\Omega}|}\right]e^{-\Sigma_{s2}R'}$$

$$= \frac{\Sigma_{s2}\phi_\infty}{4\pi}\left[\frac{1}{\Sigma_{s2}} - \int_0^\infty dR'\frac{e^{-\Sigma_{s2}R'}}{|\vec{r} - R'\vec{\Omega}|}\right] = \frac{\phi_\infty}{4\pi}\left[1 - \Sigma_{s2}\int_0^\infty dR'\frac{e^{-\Sigma_{s2}R'}}{\sqrt{R'^2 - 2r\cos\theta R' + r^2}}\right]$$

The integral in the bracket decreases with increasing r and decreasing $\cos\theta$ as shown in the figure below.



Thus $\psi^{(2)}(r,\vec{\Omega})$ increases with increasing r and decreasing $\cos\theta$. Thus the scalar flux can be approximated as

$$\phi^{(2)}(r) = 4\pi\left(1-\frac{1}{r}\right)\times\frac{\phi_\infty}{4\pi}\left(1-\frac{\alpha}{r^\beta}\right) = \phi_\infty\left(1-\frac{1}{r}\right)\left(1-\frac{\alpha}{r^\beta}\right)$$

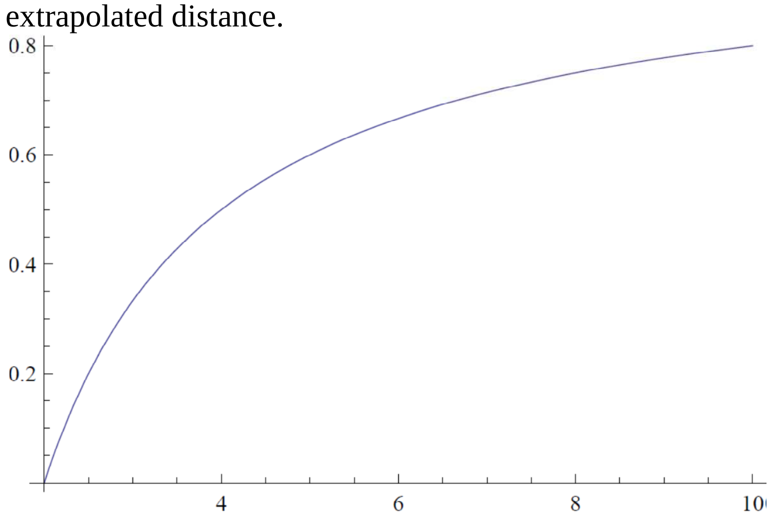
d. Solve the one-group diffusion equation for this case; plot, compare, and discuss the results.

$$-D\nabla^2\phi + \Sigma_a\phi = S \quad \Rightarrow \quad \nabla^2\phi = 0 \quad (\Sigma_a = 0, \quad S = 0)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0 \quad \Rightarrow \quad \phi(r) = C - \frac{A}{r}$$

$$\phi(r\to\infty) = \phi_\infty, \quad \phi(R=2) = 0 \quad \Rightarrow \quad \phi(r) = \phi_\infty\left(1-\frac{2}{r}\right), \quad r \ge 2$$

The zero flux boundary condition for the black absorber is generally applied at the extrapolated distance.



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4. By applying the separation of energy and spatial variables to the energy-dependent diffusion equation, the following balance equations are obtained:

$$\nabla^2\phi(r) + B^2\phi(r) = 0$$

$$[D(E)B^2 + \Sigma_t(E)]\varphi(E) - \int_0^\infty dE'\Sigma_s(E' \to E)\varphi(E') = \lambda\chi(E)\int_0^\infty dE'\nu\Sigma_f(E')\varphi(E')$$

Simplify the above equations so that they represent the corresponding pair of equations for plane geometry [$\phi(x)$] and one energy group. Assuming $\phi(\pm a) = 0$ as boundary conditions and using the one-group values of $D = 0.4$ cm, $\Sigma_a = 0.0044$ cm^{-1}, and $\nu\Sigma_f = 0.00584$ cm^{-1}, answer the following questions. (Homework problem #5 of Ch. 2)

Ans.) For plane geometry, the Helmholtz equation becomes

$$\frac{d^2}{dx^2}\phi(x) + B^2\phi(x) = 0$$

Integrating the slowing-down equation over the energy yields

$$(DB^2 + \Sigma_t)\varphi - \Sigma_s\varphi = \lambda\nu\Sigma_f\varphi$$

$$DB^2 + \Sigma_a = \lambda\nu\Sigma_f$$

a. Find the material buckling and describe the general procedure.

$$DB^2 + \Sigma_a = \lambda\nu\Sigma_f, \quad DB_m^2 + \Sigma_a = \nu\Sigma_f \quad (\lambda = 1)$$

$$B_m^2 = \frac{\nu\Sigma_f - \Sigma_a}{D} = \frac{k_\infty - 1}{L^2} = 0.0036 \text{ cm}^{-2}$$

$$k_\infty = \frac{\nu\Sigma_f}{\Sigma_a} = \frac{0.00584}{0.0044} = 1.3273, \quad L^2 = \frac{D}{\Sigma_a} = 90.91 \text{ cm}^2$$

b. Find $k = 1/\lambda$ for $a = 50$ cm and describe the general procedure.

$$B_g^2 = \left(\frac{\pi}{2a}\right)^2 = \left(\frac{\pi}{2\times 50}\right)^2 = 0.000987 \text{ cm}^{-2}$$

$$k = \frac{1}{\lambda} = \frac{\nu\Sigma_f}{DB_g^2 + \Sigma_a} = \frac{k_\infty}{1 + B_g^2L^2} = 1.218$$

c. Find the critical dimension a_c and describe the general procedure.

$$B_g^2 = B_m^2 \quad \Rightarrow \quad B_c^2 = \left(\frac{\pi}{a_c}\right)^2 = 0.0036 \text{ cm}^{-2} \quad \Rightarrow \quad a_c = 52.36 \text{ cm (full thickness)}$$

d. Increase the critical dimension by 5%, i.e., $a' = 1.05a_c$, and find the required increase in absorption (i.e., $\delta\Sigma_a$) to make the system critical. Describe the general procedure.

$$B'^2 = \left(\frac{\pi}{1.05a_c}\right)^2 = 0.003265 \text{ cm}^{-2}$$

$$k = \frac{\nu\Sigma_f}{DB'^2 + \Sigma_a'} = 1 \quad \Rightarrow \quad \Sigma_a' = \nu\Sigma_f - DB'^2 = 0.004534 \text{ cm}^{-1}$$

$$\delta\Sigma_a = \Sigma_a' - \Sigma_a = 0.000134 \text{ cm}^{-1}$$

e. Take the critical dimension and modify the original composition by increasing $\nu\Sigma_f$ such that the resulting k equals 1.05. Describe the general procedure.

$$B_c^2 = \left(\frac{\pi}{a_c}\right)^2 = 0.0036 \text{ cm}^{-2}$$

$$k = \frac{\nu\Sigma_f'}{DB_c^2 + \Sigma_a} = 1.05 \quad \Rightarrow \quad \nu\Sigma_f' = 1.05 \times (DB_c^2 + \Sigma_a) = 0.006132 \text{ cm}^{-1}$$