NUCL 510
Nuclear Reactor Theory

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Lecture Note 8

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Separation of Space and Energy Variables

- Separation of variables in one-region diffusion equation

$$-D(E)\nabla^2\phi(\vec{r},E)+\Sigma_t(E)\phi(\vec{r},E)$$
$$=\int_{E'} dE'\Sigma_s(E'\to E)\phi(\vec{r},E)+\lambda\chi(E)\int_{E'} dE'\nu\Sigma_f(E')\phi(\vec{r},E)$$

$$\Leftarrow \quad \phi(\vec{r},E)=\phi(\vec{r})\varphi(E)$$

- Helmholtz equation for spatial flux shape

$$\nabla^2\phi(\vec{r})+B^2\phi(\vec{r})=0$$

- Slowing-down equation for spectrum (λ=1)

$$[D(E)B^2+\Sigma_t(E)]\varphi(E)-\int_0^\infty dE'\Sigma_s(E'\to E)\varphi(E')=\chi(E)\int_0^\infty dE'\nu\Sigma_f(E')\varphi(E')$$

  - One-group

$$B_m^2=\frac{\nu\Sigma_f-\Sigma_a}{D}=\frac{k_\infty-1}{L^2}$$

  - Two-group

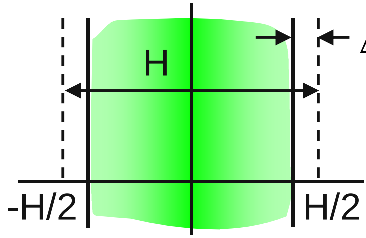
$$B^2=\frac{-(L_1^2+L_2^2)\pm\left[(L_1^2+L_2^2)^2-4L_1^2L_2^2(1-k_\infty)\right]^{1/2}}{2L_1^2L_2^2}$$

$$k_\infty>1,\quad B_m^2>0\quad B_2^2<0$$
$$k_\infty<1,\quad B_m^2<0\quad B_2^2<0$$

Flux Shape in Infinite Slab (1)

- Helmholtz equation

$$\frac{d^2\phi}{dx^2} + B^2\phi = 0, \quad \phi(\pm H/2) = 0$$



$$\Delta H = 0.711\lambda_{tr} = \frac{0.711}{\Sigma_t - \bar{\mu}\Sigma_s}$$

1) $B^2 = 0 \quad (k_\infty = 1)$

$$\frac{d^2\phi}{dx^2} = 0 \quad \Rightarrow \quad \phi(x) = Ax + C \qquad \phi\left(\pm\frac{H}{2}\right) = 0 \quad \Rightarrow \quad \begin{bmatrix} H/2 & 1 \\ -H/2 & 1 \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix} = 0$$

$$\det \mathbf{M} = H \neq 0 \quad \Rightarrow \quad A = C = 0 \qquad \phi(x) = 0$$ trivial solution only!

2) $B^2 < 0 \quad (k_\infty < 1)$

$$\frac{d^2\phi}{dx^2} - |B|^2\phi = 0 \quad \Rightarrow \quad \phi(x) = A\cosh(|B|x) + C\sinh(|B|x)$$

$$\phi\left(\pm\frac{H}{2}\right) = 0 \quad \Rightarrow \quad \begin{bmatrix} \cos(BH/2) & \sin(BH/2) \\ \cos(BH/2) & -\sin(BH/2) \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix} = 0$$

$$\det \mathbf{M} = -2\cosh(|B|H/2)\sinh(|B|H/2) = -\sinh(2|B|H) \neq 0 \quad \forall |B| \in \mathbb{R}$$

$$\Rightarrow \quad A = C = 0 \qquad \phi(x) = 0$$ trivial solution only!

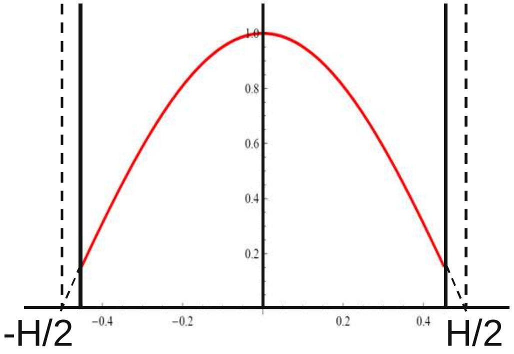
Flux Shape in Infinite Slab (2)

- Helmholtz equation

3) $B^2 > 0 \quad (k_\infty > 1)$

$$\phi(x) = A\cos(Bx) + C\sin(Bx)$$

$$\phi\left(\pm\frac{H}{2}\right) = 0 \quad \Rightarrow \quad \begin{bmatrix} \cos(BH/2) & \sin(BH/2) \\ \cos(BH/2) & -\sin(BH/2) \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0$$



$$\det \mathbf{M} = -2\cos(BH/2)\sin(BH/2) = -\sin(BH) = 0 \quad \Rightarrow \quad BH = n\pi$$

$$n = 4m; \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0 \quad \Rightarrow \quad A = 0 \qquad n = 4m+1; \quad \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0 \quad \Rightarrow \quad C = 0$$

$$n = 4m+2; \quad \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0 \quad \Rightarrow \quad A = 0 \quad n = 4m+3; \quad \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0 \quad \Rightarrow \quad C = 0$$

$$B_n = \frac{n\pi}{H}, \quad \phi(x) = \begin{cases} C_n \cos B_n x & \text{for } n = 1, 3, 5, \cdots \\ C_n \sin B_n x & \text{for } n = 2, 4, 6, \cdots \end{cases}$$

non-negative only for $n = 1$!

$$B_1 = \frac{\pi}{H}, \quad \phi(x) = \phi_0 \cos\frac{\pi x}{H} \quad \text{(fundamental mode)}, \quad \phi_0 = \phi(0)$$

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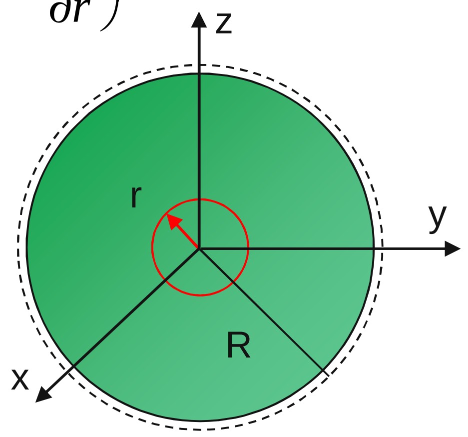
Flux Shape in Bare Spherical Reactor (1)

- Laplacian in spherical coordinate system

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

polar and azimuthal symmetry $\Rightarrow \dfrac{\partial}{\partial\theta} = 0, \dfrac{\partial}{\partial\alpha} = 0 \Rightarrow \nabla^2 = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right)$



- Helmholtz equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) + B^2\phi = 0$$

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + B^2\phi = 0 \Rightarrow \quad r\frac{d^2\phi}{dr^2} + 2\frac{d\phi}{dr} + B^2 r\phi = 0$$

$$\frac{d^2(r\phi)}{dr^2} = \frac{d}{dr}(\phi + r\phi') = r\phi'' + 2\phi' \quad \Rightarrow \quad \boxed{\frac{d^2(r\phi)}{dr^2} + B^2(r\phi) = 0}$$

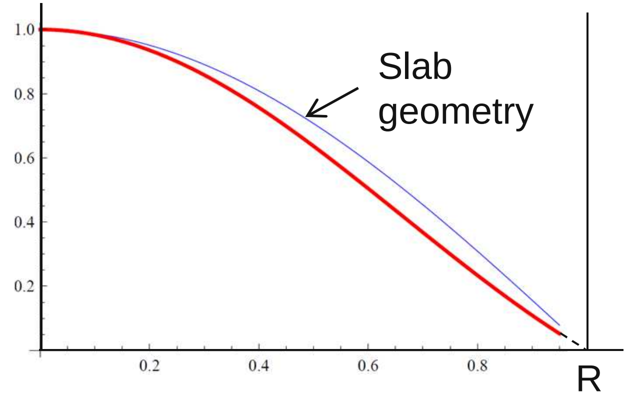
$$r\phi = A\cos Br + C\sin Br \qquad \boxed{\phi(r) = A\frac{\cos Br}{r} + C\frac{\sin Br}{r}}$$

Flux Shape in Bare Spherical Reactor (2)

- Boundary conditions
  - Finite at the core center $\Rightarrow A = 0$

$$\phi(r) = C\frac{\sin Br}{r}$$

  - Zero flux at the extrapolated outer boundary



$$\phi(R) = C\frac{\sin BR}{R} = 0 \quad \Rightarrow \quad BR = \pi \quad \Rightarrow \quad B = \frac{\pi}{R} \quad \text{(fundamental mode)}$$

$$\phi(r) = C\frac{\sin(\pi r / R)}{r}$$

- Normalization condition at the core center

$$\phi(0) = \lim_{r\to 0} C\frac{\sin Br}{r} = C\lim_{r\to 0}\frac{d(\sin Br)/dr}{d(r)/dr} \quad \text{(L'Hopital's rule)}$$

$$= CB = \phi_0 \quad \Rightarrow \quad C = \frac{\phi_0}{B}$$

$$\phi(r) = \phi_0\frac{\sin Br}{Br}, \quad B = \frac{\pi}{R}$$

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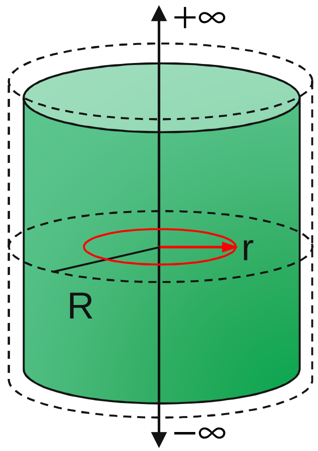
Flux Shape in Infinite Cylinder (1)

- Laplacian in cylindrical coordinate system



$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

axial and azimuthal symmetry $\Rightarrow \dfrac{\partial}{\partial z} = 0, \dfrac{\partial}{\partial \alpha} = 0 \Rightarrow \nabla^2 = \dfrac{1}{r}\dfrac{\partial}{\partial r} r \dfrac{\partial}{\partial r}$

- Helmholtz equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + B^2\phi = 0$$

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + B^2\phi = 0 \quad \Rightarrow \quad r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + B^2 r^2\phi = 0$$

$$Br = \tau \quad \Rightarrow \quad \tau^2\frac{d^2\phi}{d\tau^2} + \tau\frac{d\phi}{d\tau} + \tau^2\phi = 0$$ (zeroth order Bessel equation)

$$\phi(r) = AJ_0(Br) + CY_0(Br)$$

$$x^2\frac{d^2}{dx^2}y(x) + x\frac{d}{dx}y(x) + (x^2 - n^2)y(x) = 0$$

Flux Shape in Infinite Cylinder (2)

- Boundary conditions
  - Finite at the core center $\Rightarrow$ $C = 0$

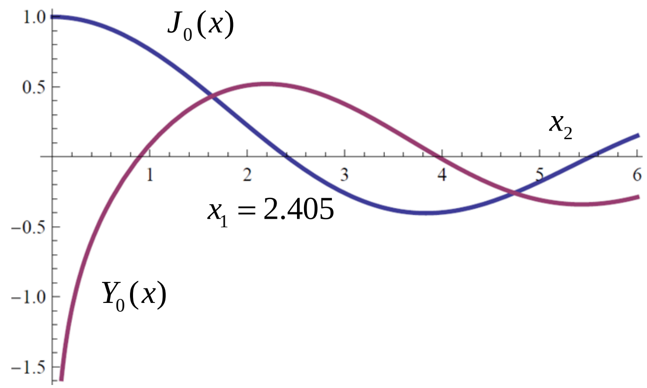
$$\phi(r) = AJ_0(Br)$$

  - Zero flux at the extrapolated outer boundary

$$\phi(R) = AJ_0(BR) = 0 \quad \Rightarrow \quad BR = 2.405$$

$$B = \frac{2.405}{R} \quad \text{(fundamental mode)}$$

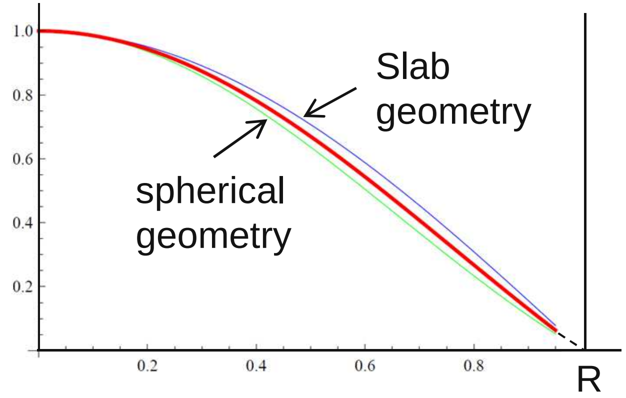
$$\phi(r) = AJ_0\left(\frac{2.405r}{R}\right)$$



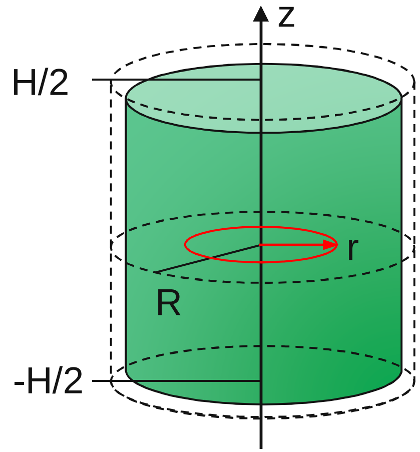
- Normalization condition at the core center

$$\phi(0) = AJ_0 = \phi_0 \quad \Rightarrow \quad A = \phi_0$$

$$\phi(r) = \phi_0 J_0\left(\frac{2.405r}{R}\right)$$



Flux Shape in Finite Cylinder



- Helmholtz equation in cylindrical coordinate with azimuthal symmetry

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial z^2}+B^2\phi=0$$

- Separation of variables

$$\phi(r,z)=R(r)Z(z)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)Z+\frac{\partial^2 Z}{\partial z^2}R+B^2RZ=0 \quad\Rightarrow\quad \frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+B^2=0$$

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+B_r^2=0,\quad \frac{d^2Z}{dr^2}+B_z^2Z=0,\quad B^2=B_r^2+B_z^2$$

$$R(r)=R_0J_0\left(B_r r\right),\quad B_r=\frac{2.405}{R};\quad Z(z)=Z_0\cos\left(B_z z\right),\quad B_z=\frac{\pi}{H}$$

$$\phi(r,z)=\phi_0J_0\left(\frac{2.405}{R}r\right)\cos\left(\frac{\pi}{H}z\right)$$

$$B^2=B_r^2+B_z^2=\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2$$

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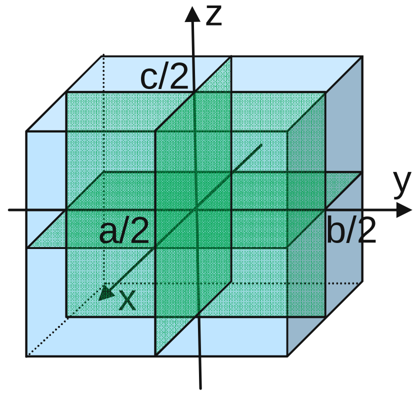
Flux Shape in Rectangular Parallelepiped

- Helmholtz equation in Cartesian geometry

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+B^2\phi=0$$



- Separation of variables

$$\phi(x,y,z)=X(x)Y(y)Z(z)$$

$$\frac{\partial^2 X}{\partial x^2}YZ+\frac{\partial^2 Y}{\partial y^2}ZX+\frac{\partial^2 Z}{\partial z^2}XY+B^2XYZ=0 \quad\Rightarrow\quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+B^2=0$$

$$\frac{\partial^2 X}{\partial x^2}+B_x^2X=0,\quad \frac{\partial^2 Y}{\partial y^2}+B_y^2Y=0,\quad \frac{\partial^2 Z}{\partial z^2}+B_z^2Z=0,\quad B^2=B_x^2+B_y^2+B_z^2$$

$$X=X_0\cos(B_xx),\quad B_x=\frac{\pi}{a};\quad Y=Y_0\cos(B_yy),\quad B_y=\frac{\pi}{b};\quad Z=Z_0\cos(B_zz),\quad B_z=\frac{\pi}{c}$$

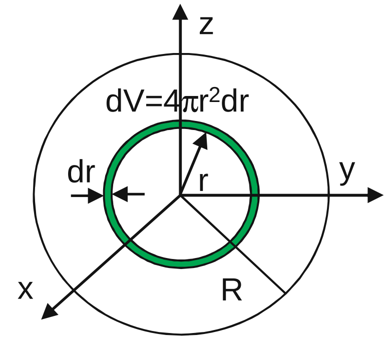
$$\phi(x,y,z)=\phi_0\cos\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\cos\left(\frac{\pi}{c}z\right)$$

$$B^2=\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2+\left(\frac{\pi}{c}\right)^2$$

Flux Peaking Factor (1)

- Average flux and flux peaking factor

$$\bar{\phi} = \frac{1}{V}\int_V \phi(\vec{r})dV, \quad f_\phi = \frac{\text{peak flux}}{\text{average flux}} = \frac{\phi_0}{\bar{\phi}}$$



- Sphere

$$\bar{\phi} = \frac{\phi_0}{4\pi R^3/3}\int_0^R dr(4\pi r^2)\frac{\sin(\pi r/R)}{\pi r/R} = \frac{3}{\pi}\phi_0\int_0^1 d\tau\tau\sin(\pi\tau) \quad (\tau = r/R)$$

$$\int_0^1 \tau\sin(\pi\tau)d\tau = \tau\frac{\cos(\pi\tau)}{\pi}\Big|_0^1 - \frac{1}{\pi}\int_0^1 \cos(\pi\tau)d\tau = \frac{1}{\pi} \quad \Rightarrow \quad \boxed{f_\phi = \frac{\phi_0}{\bar{\phi}} = \frac{\pi^2}{3} = 3.290}$$

- Rectangular parallelepiped

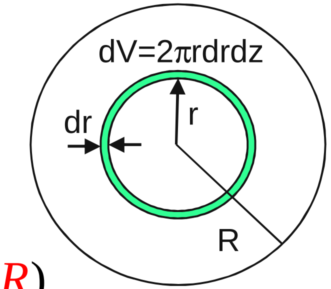
$$\bar{\phi} = \frac{\phi_0}{abc}\int_{-a/2}^{a/2} dx\cos\left(\pi\frac{x}{a}\right)\int_{-b/2}^{b/2} dy\cos\left(\pi\frac{y}{b}\right)\int_{-c/2}^{c/2} dz\cos\left(\pi\frac{z}{c}\right)$$

$$= \phi_0\left[\int_{-1/2}^{1/2} d\tau\cos(\pi\tau)\right]^3 \quad (\tau = x/a = y/b = z/c)$$

$$\int_{-1/2}^{1/2} d\tau\cos(\pi\tau) = \frac{1}{\pi}\sin(\pi\tau)\Big|_{-1/2}^{1/2} = \frac{2}{\pi} \quad \Rightarrow \quad \boxed{f_\phi = \left(\frac{\pi}{2}\right)^3 = 3.876}$$

Flux Peaking Factor (2)

- Cylinder



$$\bar{\phi} = \frac{\phi_0}{\pi R^2 H}\int_{-H/2}^{H/2} dz \cos\left(\pi \frac{z}{H}\right)\int_0^R dr(2\pi r)J_0\left(2.405\frac{r}{R}\right)$$

$$= \frac{2\phi_0}{2.405^2}\int_{-1/2}^{1/2} d\tau \cos(\pi\tau)\int_0^{2.405} d\rho\rho J_0(\rho) \quad (\tau = z/H, \quad \rho = 2.405r/R)$$

$$\int_0^{2.405} d\rho\rho J_0(\rho) = \rho J_1(\rho)\Big|_0^{2.405} = 2.405 J_1(2.405) = 2.405\times 0.519 \quad \Rightarrow$$

$$f_\phi = \frac{2.405^2}{2}\times\frac{\pi}{2}\times\frac{1}{2.405 J_1(2.405)} = \frac{2.405\pi}{4J_1(2.405)} = 3.639$$

- Summary of flux peaking factor

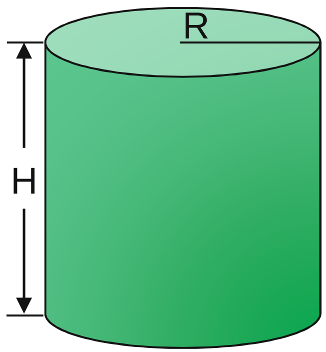
$$f_{\text{sphere}} = 3.29 < f_{\text{cylinder}} = 3.64 < f_{\text{parallelepiped}} = 3.88$$

  – The peaking factor increases as the surface to volume ratio increases

Minimum Critical Volume

- Optimum height-to-diameter ratio for finite cylindrical reactor
  - Minimum critical volume for a given composition (i.e., material buckling)

$$B^2 = \left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 \quad \Rightarrow \quad R^2 = \frac{2.405^2}{B^2 - (\pi/H)^2}$$



$$V = \pi R^2 H = \frac{\pi(2.405)^2}{B^2 - (\pi/H)^2} H$$

$$\frac{dV}{dH} = 2.405^2 \frac{3H^2(B^2H^2 - \pi^2) - H^3 2B^2 H}{(B^2H^2 - \pi^2)^2} = \frac{H^2(B^2H^2 - 3\pi^2)}{(B^2H^2 - \pi^2)^2} = 0$$

$$\Rightarrow \quad H = \frac{\sqrt{3}\pi}{B} \quad \Rightarrow \quad R = \frac{2.405}{B}\sqrt{\frac{3}{2}} \quad \Rightarrow \quad \frac{R}{H} = 0.54$$

$$D = 1.08H \qquad V_{\min} = \frac{148}{B^3}$$

One-Group Source-Sink Problems

- Flux solutions for a localized source in non-multiplying or subcritical system
  - One-group diffusion equation with fixed source

$$-D\nabla^2\phi(\vec{r})+\Sigma_a\phi(\vec{r})=\nu\Sigma_f\phi(\vec{r})+s(\vec{r}) \quad (k<1)$$

$$\nabla^2\phi+\frac{\nu\Sigma_f-\Sigma_a}{D}\phi=-\frac{s}{D} \quad \Rightarrow \quad \nabla^2\phi+B_m^2\phi=-\frac{s}{D}$$

$$B_m^2=\frac{\nu\Sigma_f-\Sigma_a}{D}=\frac{k_\infty-1}{L^2}$$

- Material buckling

1) $\nu\Sigma_f=0$ ($k_\infty=0$; non multiplying medium) $\Rightarrow \quad B_m^2=-1/L^2<0$

2) $\nu\Sigma_f<\Sigma_a$ ($k_\infty<1$; subproductive system) $\Rightarrow \quad B_m^2<0, \quad B_m^2=-|B_m|^2$

3) $\nu\Sigma_f>\Sigma_a$ ($k_\infty>1$; superproductive system but subcritical with leakage)

$\Rightarrow \quad B_m^2>0$

Point Source in Infinite Non-multiplying Medium (1)

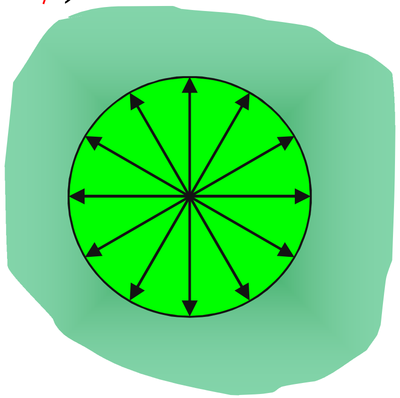
- Neutron balance equation

$$\nabla^2\phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{s_0}{D}\delta(\vec{r}) \quad \text{(source at the origin)}$$

- Solution

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\phi\right) - \frac{1}{L^2}\phi = 0 \quad (r \neq 0) \quad \Rightarrow \quad \frac{d^2 w}{dr^2} - \frac{1}{L^2}w = 0 \quad (w = r\phi)$$

$$w = Ae^{r/L} + Ce^{-r/L} \quad \Rightarrow \quad \phi(r) = A\frac{e^{r/L}}{r} + C\frac{e^{-r/L}}{r}$$



– Flux is zero as $r \to \infty \Rightarrow A=0$

– Source condition as $r \to 0$ $\quad \lim_{r \to 0} 4\pi r^2 J_n(r) = s_0$

$$\vec{J} = -D\frac{d\phi}{dr}\vec{e}_r = DCe^{-r/L}\left(\frac{1}{r^2} + \frac{1}{rL}\right)\vec{e}_r \quad \Rightarrow \quad \lim_{r\to 0} 4\pi r^2 J_n(r) = 4\pi DC = s_0$$

$$C = \frac{s_0}{4\pi D} \qquad \phi(r) = \frac{s_0}{4\pi D}\frac{e^{-r/L}}{r}$$

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Point Source in Infinite Non-multiplying Medium (2)

- Limit of zero absorption

$$\text{As } \Sigma_a \to 0, \quad L \to \infty \quad \Rightarrow \quad \phi(r) = \frac{S_0}{4\pi D r} \quad \text{(geometrical attenuation)}$$

  - Transport solution

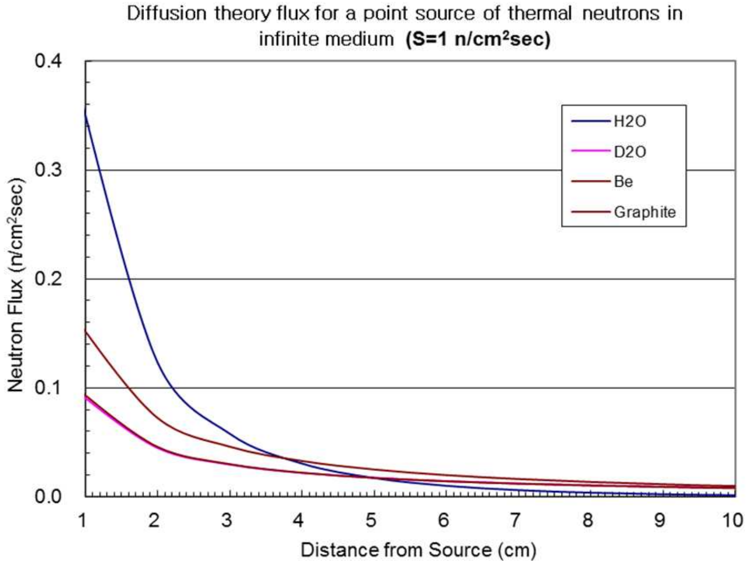
$$\phi(\vec{r}) = \int_{V'} dV' \frac{S(\vec{r}\,')e^{-\Sigma_t |\vec{r}-\vec{r}\,'|}}{4\pi |\vec{r}-\vec{r}\,'|^2} \quad \Rightarrow \quad \phi(r) = \frac{S_0}{4\pi r^2}$$

  - Diffusion results in a smaller decrease of the neutron population

- Physically unrealistic singularity at r=0
  - Due to the mathematical artifact of a "point" source

- Distributed sources in an infinite medium

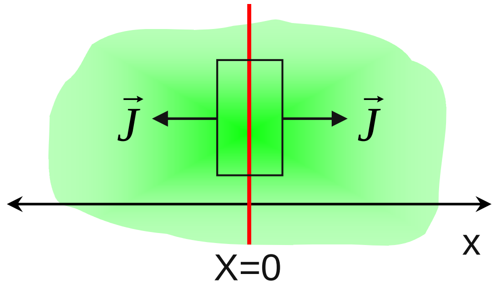
$$\phi(\vec{r}) = \int_{V'} dV' \frac{S(\vec{r}\,')e^{-|\vec{r}-\vec{r}\,'|/L}}{4\pi D |\vec{r}-\vec{r}\,'|}$$



Plane Source in Infinite Medium

- Neutron balance equation

$$\frac{d^2}{dx^2}\phi(x)-\frac{1}{L^2}\phi(x)=-\frac{s_0}{D}\delta(x)$$



- Solution

$$\frac{d^2\phi}{dr^2}-\frac{1}{L^2}\phi=0 \quad\Rightarrow\quad \phi(x)=Ae^{x/L}+Ce^{-x/L}$$

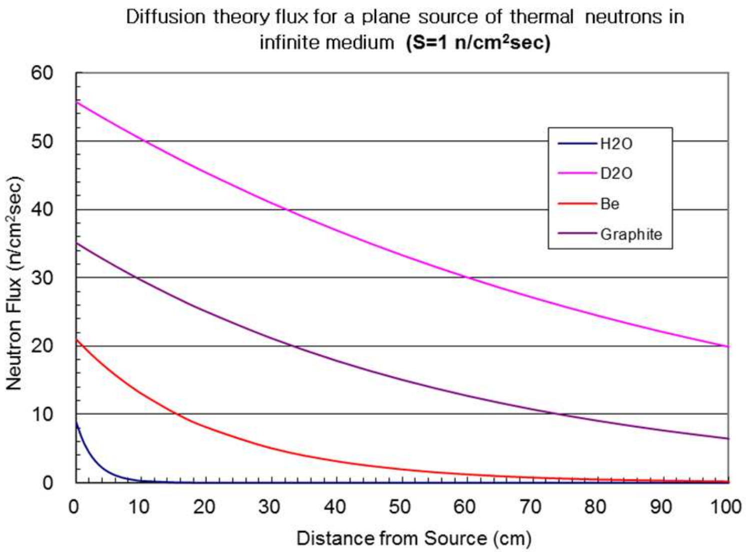
  – Flux is zero as $x\rightarrow\pm\infty \Rightarrow A=0$ for $x>0$ and $C=0$ for $x<0$

  – Source condition as $x\rightarrow 0$

$$\lim_{x\rightarrow 0} J_n(r)=s_0/2$$

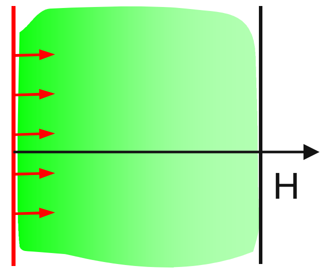
$$\vec{J}=-D\frac{d\phi}{dx}\vec{e}_x=\begin{cases} CDe^{-x/L}/L\vec{e}_x, & x>0 \\ -ADe^{x/L}/L\vec{e}_x, & x<0 \end{cases}$$

$$A=C=\frac{s_0L}{2D} \qquad \phi(x)=\frac{s_0L}{2D}e^{-|x|/L}$$



Sub-productive Subcritical Slab (1)

- Problem conditions
  - $k_\infty < 1$ and thus material buckling is negative
  - Flux is sustained by a plane source placed at the center
  - Flux has its maximum at the center and vanishes at the outer boundary



- Flux distribution

$$\frac{d^2\phi}{dx^2} + B_m^2\phi = 0, \quad B_m^2 = \frac{\nu\Sigma_f - \Sigma_a}{D} = \frac{k_\infty - 1}{L^2} < 0 \quad \Rightarrow \quad \frac{d^2\phi}{dx^2} - |B_m|^2 \phi = 0$$

$$\phi(x) = A\cosh(|B_m|x) + C\sinh(|B_m|x)$$

$$\phi(0) = \phi_0 \quad \Rightarrow \quad A = \phi_0$$

$$\phi(H) = \phi_0\cosh(|B_m|H) + C\sinh(|B_m|H) = 0 \quad \Rightarrow \quad C = -\phi_0\frac{\cosh(|B_m|H)}{\sinh(|B_m|H)}$$

$$\phi(x) = \frac{\phi_0}{\sinh(|B_m|H)}\left[\sinh(|B_m|H)\cosh(|B_m|x) - \cosh(|B_m|H)\sinh(|B_m|x)\right]$$

Sub-productive Subcritical Slab (2)

- Flux solution

$$\phi(x) = \frac{\phi_0}{\sinh(|B_m|H)} \sinh[|B_m|(H-x)]$$

- Limit of infinite thickness

$$\lim_{H\to\infty} \phi(x) = \lim_{H\to\infty} \phi_0 \left[\cosh(|B_m|x) - \frac{\cosh(|B_m|H)}{\sinh(|B_m|H)} \sinh(|B_m|x) \right]$$

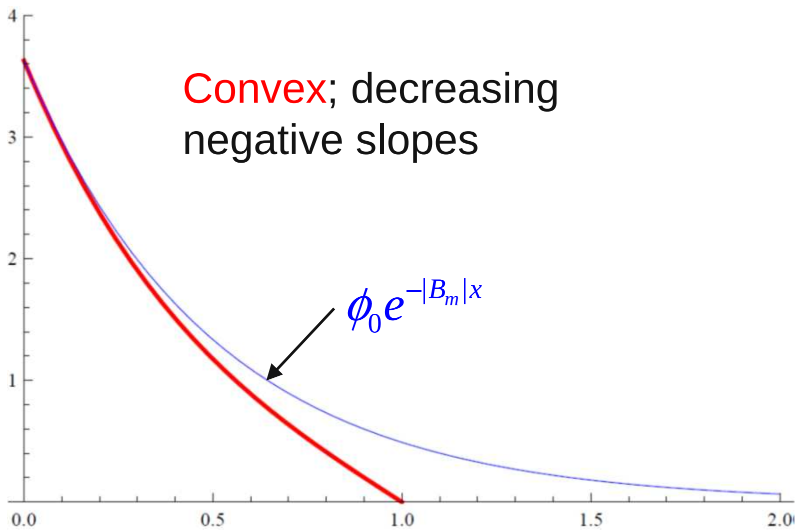
$$= \phi_0 \left[\cosh(|B_m|x) - \sinh(|B_m|x) \right] = \phi_0 e^{-|B_m|x}$$

- Relaxation length

$$B_m = i\frac{\sqrt{1-k_\infty}}{L}, \quad |B_m| = \frac{\sqrt{1-k_\infty}}{L}$$

$$\text{Relaxation length } = \frac{1}{|B_m|} = \frac{L}{\sqrt{1-k_\infty}}$$

Relaxation length is the distance for which the flux decreases by a factor of e



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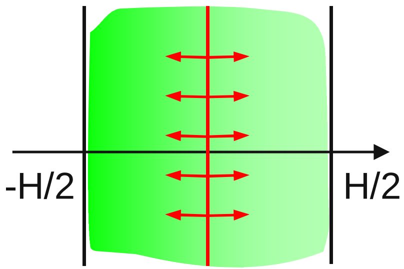
Super-productive Subcritical Slab (1)

- Problem conditions
  - $k_\infty > 1$ and thus material buckling is positive
  - Subcritical with leakage
  - Flux is sustained by a plane source placed at the center
  - Boundary conditions are given by the current at the center and the zero flux condition at the outer boundary



- Flux distribution

$$\frac{d^2\phi}{dx^2} + B_m^2\phi = 0 \quad \Rightarrow \quad \phi(x) = A\cos(B_m x) + C\sin(B_m x)$$

$$\phi\left(\pm\frac{H}{2}\right) = A\cos\left(\frac{B_m H}{2}\right) \pm C\sin\left(\frac{B_m H}{2}\right) = 0 \quad \Rightarrow \quad C = \mp A\frac{\cos(B_m H/2)}{\sin(B_m H/2)}$$

$$\phi(x) = \frac{A}{\sin(B_m H/2)}\sin\left[B_m\left(\frac{H}{2} \mp x\right)\right] = \frac{A}{\sin(B_m H/2)}\sin\left[B_m\left(\frac{H}{2} - |x|\right)\right]$$

$$J(x) = -D\frac{d\phi}{dx} = \pm\frac{ADB_m}{\sin(B_m H/2)}\cos\left[B_m\left(\frac{H}{2} - |x|\right)\right] \quad (+ \text{ for } x > 0 \text{ and } - \text{ for } x < 0)$$

Super-productive Subcritical Slab (2)

- Flux solution

$$J_n(\pm 0) = \left[\pm ADB_m \frac{\cos(B_m H/2)}{\sin(B_m H/2)} \right] \times (\pm 1) = ADB_m \frac{\cos(B_m H/2)}{\sin(B_m H/2)} = \frac{s_0}{2}$$

$$A = \frac{s_0}{2DB_m} \frac{\sin(B_m H/2)}{\cos(B_m H/2)} \quad \Rightarrow \quad \phi(x) = \frac{s_0}{2DB_m \cos(B_m H/2)} \sin\left[B_m \left(\frac{H}{2} - |x| \right) \right]$$

$$\phi(0) = \frac{s_0}{2DB_m} \tan\left(\frac{B_m H}{2} \right)$$

As H increases to the critical dimension π/B_m, $\phi(0) \to \infty$

- Source multiplication factor at $x=0$

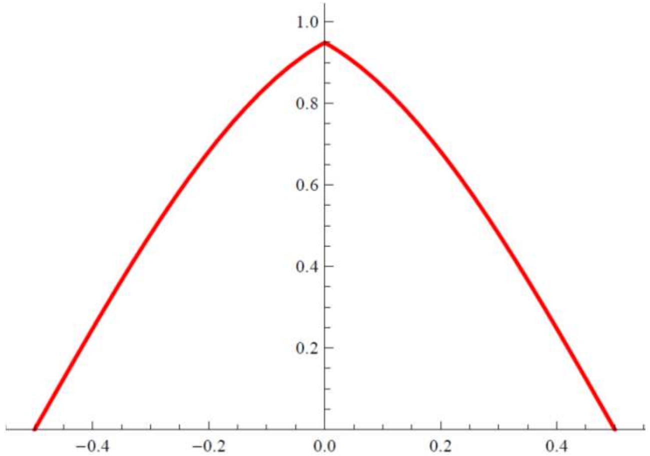
$$M_s(0) = \frac{S_f(0)}{s_0} = \frac{\nu\Sigma_f \phi(0)}{s_0} = \frac{\nu\Sigma_f}{2DB} \tan\left(\frac{BH}{2} \right)$$

  – Inverse source multiplication factor

$$\frac{1}{M_s(0)} \propto \frac{1}{\tan(BH/2)} \to \infty$$

as the reactor approaches the criticality

Concave; increasing negative slopes

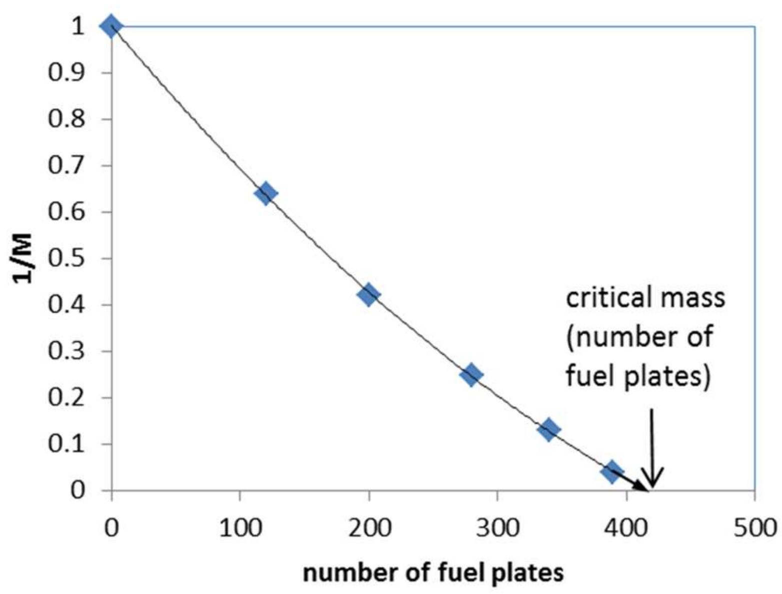


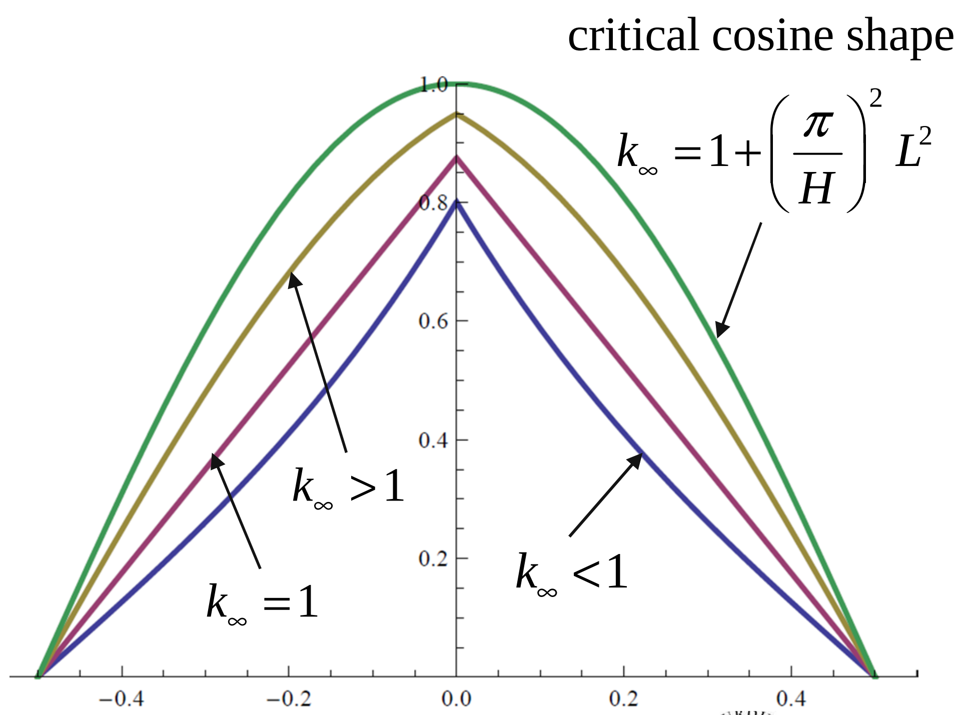
Source Multiplication in Subcritical System

- Source multiplication

$$S + Sk + Sk^2 + Sk^3 + \cdots = \frac{S}{1-k} = MS \quad \text{(total number of neutrons);} \quad M = \frac{1}{1-k}$$

  – Source multiplication is often used during the loading of fuel into a reactor to measure the degree of sub-criticality and to extrapolate to the critical condition





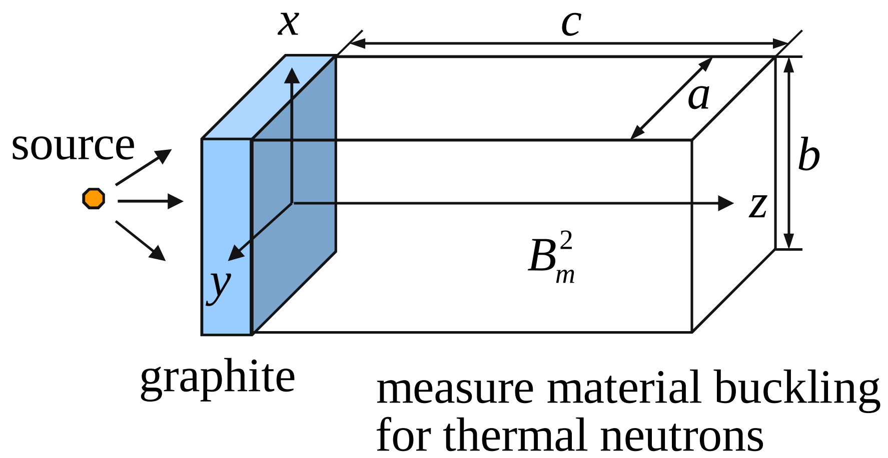
Exponential Pile (1)

- Balance equation and boundary conditions

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+B_m^2\phi=0;$$

$$\phi\left(\pm\frac{a}{2},y,z\right)=0,\quad \phi\left(x,\pm\frac{b}{2},z\right)=0,$$

$$\phi(x,y,0)=\phi_0(x,y),\quad \phi(x,y,c)=0$$



- Separation of variables

$$\phi(x,y,z)=X(x)Y(y)Z(z)\quad\Rightarrow\quad \frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}+B_m^2=0$$

$$\frac{X''}{X}=-B_x^2=\left(\frac{\pi}{a}\right)^2,\quad \frac{X''}{X}=-B_y^2=\left(\frac{\pi}{b}\right)^2,\quad \frac{Z''}{Z}=-B_z^2=B_m^2-B_x^2-B_y^2$$

For sufficiently small a and b, $\quad B_z^2=B_m^2-\left(\frac{\pi}{a}\right)^2-\left(\frac{\pi}{b}\right)^2<0,\quad B_z^2=-\gamma^2$

$$\phi(x,y,z)=\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}C_{lm}\cos\left(\frac{l\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right)\sinh[\gamma_{lm}(c-z)],\quad \gamma_{lm}^2=\left(\frac{l\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-B_m^2$$

Exponential Pile (2)

- Coefficients C_{lm} can be obtained by applying the boundary condition at $z = 0$
  - Far away from the source plane, the higher harmonics die away as γ_{lm} increases with l and m

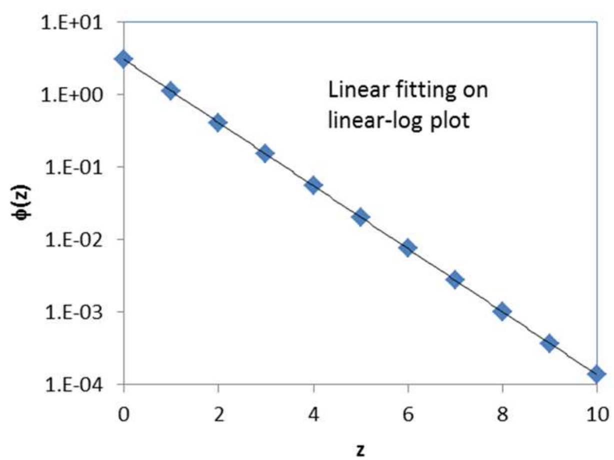
$$\phi(x,y,z) = C\cos\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\sinh[\gamma_{11}(c-z)]$$

- For a large c, the flux shape in the z-direction is essentially an exponential function

$$\phi(x,y,z) = C\cos\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\exp[-\gamma_{11}(z-c)]$$

- Material buckling
  - Measure flux shape along the z-axis
  - Determine γ by the slope in the linear-log plot
  - Determine the material buckling

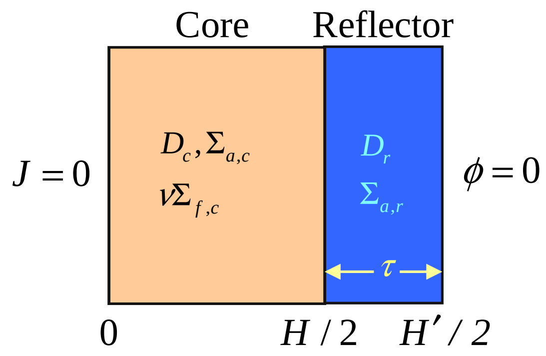
$$B_m^2 = B_x^2 + B_y^2 - \gamma_{11}^2$$



Reflected Reactor

- One-group two-region neutron balance equation

$$\begin{cases} -D_c\nabla^2\phi_c + \Sigma_{a,c}\phi_c = \lambda\nu\Sigma_{f,c}\phi_c \\ -D_r\nabla^2\phi_r + \Sigma_{a,r}\phi_r = 0 \end{cases}$$



$$\begin{cases} \nabla^2\phi_c + B_\lambda^2\phi_c = 0 \\ \nabla^2\phi_r - \kappa^2\phi_r = 0 \end{cases}$$

$$B_\lambda^2 = \frac{\lambda k_\infty - 1}{L_c^2}, \quad \kappa^2 = \frac{1}{L_r^2} = \frac{\Sigma_{a,r}}{D_r} = -B_{m,r}^2$$

- Boundary conditions
  - Reflective at the center $J(0)=0$
  - Zero flux at the outer boundary $\phi(H'/2)=0$

- Interface conditions

$$\phi_c(H/2) = \phi_r(H/2), \quad J_c(H/2) = J_r(H/2)$$

Flux Shape in Reflected Infinite Slab (1)

- In the core

$$\frac{d^2\phi_c}{dx^2}+B_\lambda^2\phi_c=0 \quad\Rightarrow\quad \phi_c(x)=C_1\cos(B_\lambda x)+C_2\sin(B_\lambda x)$$

$$J_c(x)=C_1 D_c B_\lambda \sin(B_\lambda x)-C_2 D_c B_\lambda \cos(B_\lambda x)$$

$$J_c(0)=0 \quad\Rightarrow\quad \phi_c(x)=C_1\cos(B_\lambda x),\quad J_c(x)=C_1 D_c B_\lambda \sin(B_\lambda x)$$

- In the reflector

$$\frac{d^2\phi_r}{dx^2}-\kappa^2\phi_r=0 \quad\Rightarrow\quad \phi_r(x)=C_3\cosh(\kappa x)+C_4\sinh(\kappa x)$$

$$\phi_r(H'/2)=0 \quad\Rightarrow\quad \phi_r(x)=C_3'\sinh[\kappa(H'/2-|x|)]$$

$$J_r(x)=C_3' D_r\kappa\cosh[\kappa(H'/2-|x|)]$$

- Interface conditions

$$\begin{bmatrix}\cos(B_\lambda H/2) & -\sinh(\kappa\tau)\\ D_c B_\lambda\sin(B_\lambda H/2) & -D_r\kappa\cosh(\kappa\tau)\end{bmatrix}\begin{bmatrix}C_1\\ C_3'\end{bmatrix}=0,\quad \tau=\frac{H'}{2}-\frac{H}{2}\ \text{(reflector thickness)}$$

$$D_c B_\lambda\tan(B_\lambda H/2)=D_r\kappa\coth(\kappa\tau)$$

$$C_1=\phi_c(0),\quad C_3'=C_1\frac{\cos(B_\lambda H/2)}{\sinh(\kappa\tau)}$$

Flux Shape in Reflected Infinite Slab (2)

- Criticality equation for a given H
  - B_λ is to be determined

$$(B_\lambda H/2)\tan(B_\lambda H/2) = \frac{D_r H \kappa}{2D_c}\coth(\kappa\tau)$$

$$\xi \tan \xi = \alpha \;\Rightarrow\; \text{first branch } \xi_1 \;\Rightarrow\; B_\lambda = \frac{2\xi_1}{H}$$

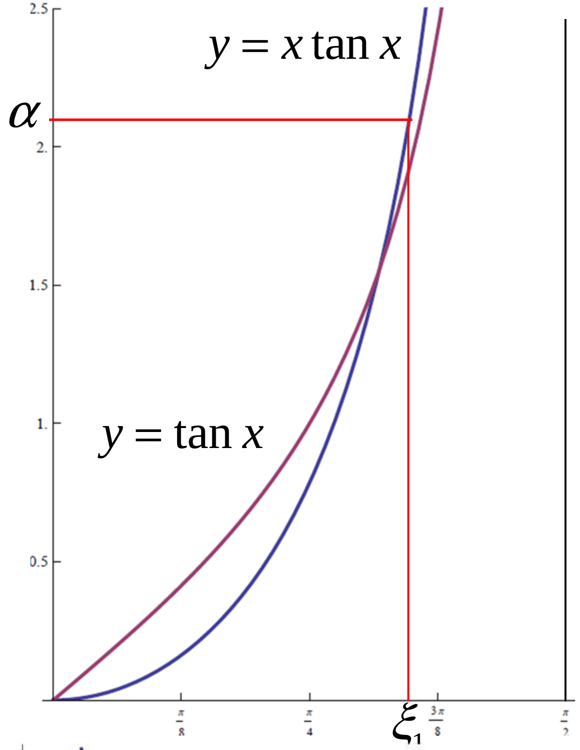
$$B_\lambda^2 = \frac{\lambda k_\infty - 1}{L_c^2} = \left(\frac{2\xi_1}{H}\right)^2 \quad (\text{smallest } B_\lambda^2)$$

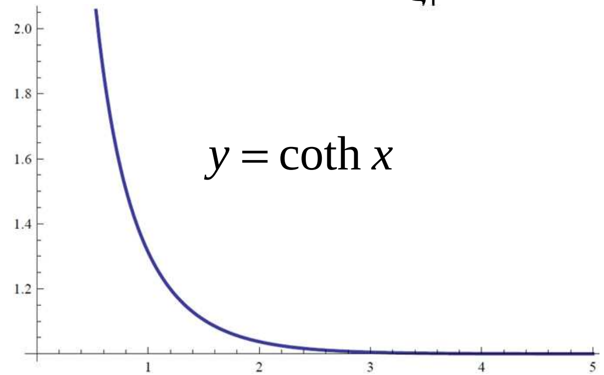
  - The value of B_λ can be established by λ (i.e., k) eigenvalue or composition search

$$k_1 = \frac{\nu\Sigma_{f,c}}{\Sigma_{a,c} + D_c(2\xi_1/H)^2} \quad (\text{largest } k)$$

  - As $\tau \to 0$ (bare reactor), $\coth(\kappa\tau) \to \infty$

$$\xi_1 \to \frac{\pi}{2} \;(\text{largest } \xi_1) \;\Rightarrow\; B_\lambda \to \frac{\pi}{H} \;\Rightarrow\; \text{smallest } k_1$$





Flux Shape in Reflected Infinite Slab (3)

- For $\kappa\tau > 3$ (thick reflector), $\coth(\kappa\tau) \approx 1$

$$\xi \tan\xi = \frac{D_r H \kappa}{2D_c} \quad (\text{smallest } \alpha) \quad \Rightarrow \quad \text{smallest } \xi_1 \quad \Rightarrow \quad \text{largest } k_1$$

- Criticality equation for $\lambda=1$
  - Critical core thickness H for a given material buckling

$$\tan(B_m H/2) = \frac{D_r \kappa}{D_c B_m} \coth(\kappa\tau)$$

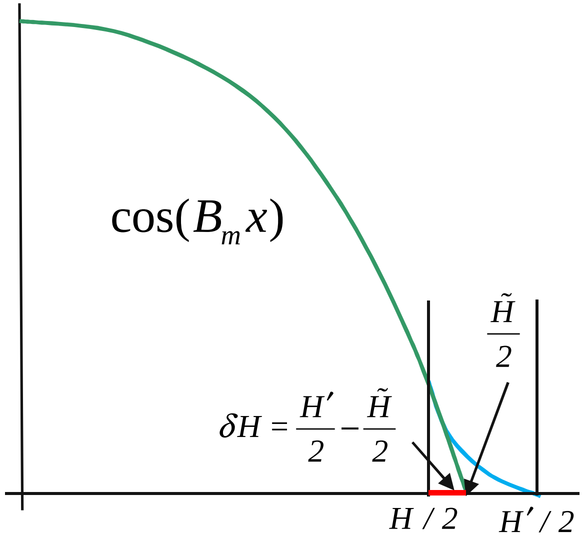
$$H = \frac{2}{B_m} \tan^{-1}\left[\frac{D_r \kappa}{D_c B_m} \coth(\kappa\tau)\right] \quad \Rightarrow \quad H \downarrow \text{ as } \tau \uparrow$$

  - As $\tau \to 0$ (bare reactor), $\coth(\kappa\tau) \to \infty$

$$\cos\left(\frac{B_m \tilde{H}}{2}\right) = 0 \quad \Rightarrow \quad \tilde{H} = \frac{\pi}{B_m} \quad (\text{largest } H)$$

  - For $\kappa\tau > 3$ (thick reflector), $\coth(\kappa\tau) \approx 1$

$$H = \frac{2}{B_m} \tan^{-1}\left[\frac{D_r \kappa}{D_c B_m}\right] \quad (\text{smallest } H)$$



Reflector Saving

- Critical dimension reduced by the presence of reflector

$$s=\frac{\tilde{H}}{2}-\frac{H}{2}=\frac{\pi}{2B_m}-\frac{1}{B_m}\tan^{-1}\left[\frac{D_r\kappa}{D_cB_m}\coth(\kappa\tau)\right]=\frac{1}{B_m}\left\{\frac{\pi}{2}-\tan^{-1}\left[\frac{D_r\kappa}{D_cB_m}\coth(\kappa\tau)\right]\right\}$$

$$\tan\left[\frac{\pi}{2}-\tan^{-1}x\right]=\cot[\tan^{-1}x]=\frac{1}{\tan[\tan^{-1}x]}=\frac{1}{x}\quad\Rightarrow\quad\frac{\pi}{2}-\tan^{-1}x=\tan^{-1}(1/x)$$

$$s=\frac{1}{B_m}\tan^{-1}\left[\frac{D_cB_m}{D_r\kappa}\tanh(\kappa\tau)\right]$$

- For a large core, $B_m << 1$ and $\tan^{-1}x \approx x$ for $x << 1$

$$s=\frac{D_c}{D_r\kappa}\tanh(\kappa\tau)=\frac{D_c}{D_r}L_r\tanh\frac{\tau}{L_r}$$

$$\tau << L_r \quad\Rightarrow\quad s\approx\frac{D_c}{D_r}\tau \quad (\tanh x\approx x \text{ for } x<<1)$$

$$\tau >> L_r \quad\Rightarrow\quad s\approx\frac{D_c}{D_r}L_r \quad (\tanh x\approx 1 \text{ for } x>3)$$