NUCL 511
Nuclear Reactor Theory and Kinetics

Lecture Note 3

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Average Neutron Generation Time and Lifetime

- Average generation time

  $\lambda_f = 1/\Sigma_f =$ the mean free path for fission, i.e., the average distance a neutron travels from its birth to a fission event

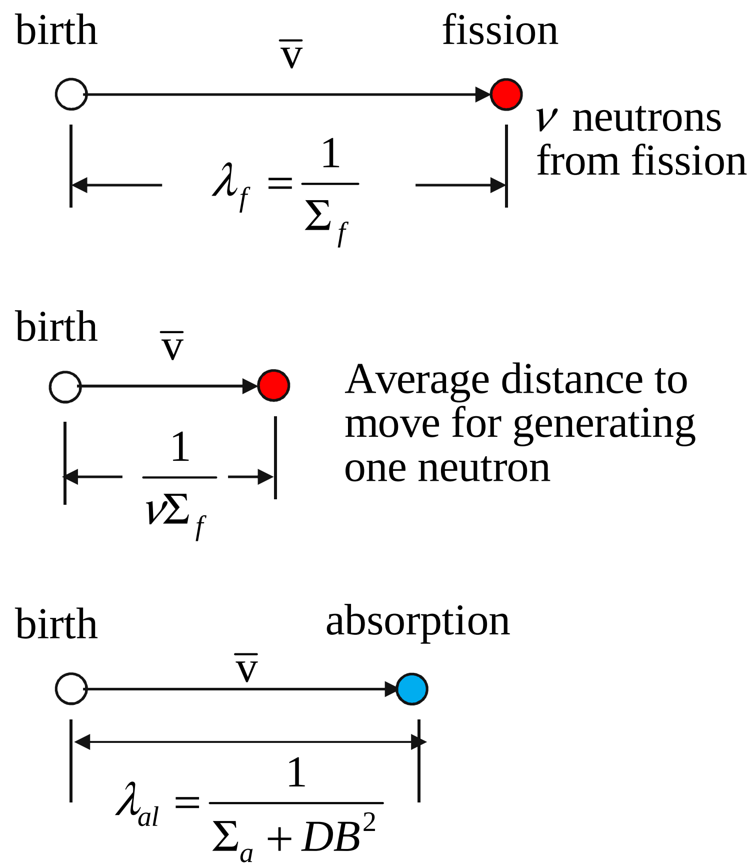
  $\Delta t_f = \lambda_f / \bar{v} =$ the average time between the birth of a neutron and a fission event

  $\Lambda = \Delta t_f / \nu =$ the average time between birth and the birth of a single neutron in the next generation

- Average lifetime

  $\lambda_{al} = 1/(\Sigma_a + DB^2) =$ the average distance a neutron travels from its birth to absorption or leakage from the system

  $l = \lambda_{al} / \bar{v} =$ the average time between the birth and the loss of a neutron



$$\Lambda = \frac{1}{\bar{v}\nu\Sigma_f}, \quad l = \frac{1}{\bar{v}(\Sigma_a + DB^2)}, \quad \frac{l}{\Lambda} = k$$

Prompt Kinetics

- Consider a reactor for which one-group homogenized XS are defined

$$\phi(r) = \int_0^\infty \varphi(r,E)dE: \text{ one group flux}$$

$$\hat{\phi} = \int_V dV \int_0^\infty dE\varphi(r,E): \text{ total flux of the core}$$

$$\Sigma_a, \nu\Sigma_f \quad D,\ B^2$$

$$\nabla^2\phi(\mathbf{r}) + B^2\phi(\mathbf{r}) = 0 \quad \Rightarrow \quad \int_V \nabla\cdot(-D\nabla\phi)dV = D\int_V(-\nabla^2\phi)dV = D\int_V B^2\phi dV = DB^2\hat{\phi}$$

- Steady state balance equation for the entire reactor

$$(DB^2 + \Sigma_a)\hat{\phi} = \nu\Sigma_f\hat{\phi} \qquad k = 1 \quad \Rightarrow \quad DB^2 + \Sigma_a = \nu\Sigma_f$$

- Suppose that the system is perturbed, e.g., by reducing the absorption XS. Then, under the assumption that all delayed neutrons are promptly emitted, one-group neutron balance equation becomes

$$\frac{d}{dt}\hat{n}(t) = \nu\Sigma_f\hat{\phi}(t) - (\Sigma_a' + DB^2)\hat{\phi}(t), \quad \Sigma_a' = \Sigma_a + \delta\Sigma_a$$

$$\hat{\phi}(t) = \bar{v}\hat{n}(t) \quad \Rightarrow \quad \frac{1}{\bar{v}}\frac{d}{dt}\hat{\phi}(t) = (\nu\Sigma_f - \Sigma_a' - DB^2)\hat{\phi}(t)$$

Prompt Kinetics

- One-group neutron balance equation

$$\frac{1}{\bar{v}}\frac{d}{dt}\hat{\phi}(t) = (\nu\Sigma_f - \Sigma_a' - DB^2)\hat{\phi}(t) \quad \Rightarrow \quad \frac{1}{\bar{v}\nu\Sigma_f}\frac{d}{dt}\hat{\phi}(t) = \left(1 - \frac{\Sigma_a' + DB^2}{\nu\Sigma_f}\right)\hat{\phi}(t)$$

$$k = \frac{\nu\Sigma_f}{\Sigma_a' + DB^2} \quad \Rightarrow \quad \frac{1}{\bar{v}\nu\Sigma_f}\frac{d}{dt}\hat{\phi}(t) = \left(1 - \frac{1}{k}\right)\hat{\phi}(t) = \rho\hat{\phi}(t)$$

$$\rho = 1 - \frac{1}{k} = \frac{\Sigma_a + DB^2}{\nu\Sigma_f} - \frac{\Sigma_a' + DB^2}{\nu\Sigma_f} = \frac{\Sigma_a - \Sigma_a'}{\nu\Sigma_f} = -\frac{\delta\Sigma_a}{\nu\Sigma_f} \quad \text{(reactivity)}$$

$$\Lambda = \frac{1}{\bar{v}\nu\Sigma_f} \quad \Rightarrow \quad \Lambda\frac{d}{dt}\hat{\phi}(t) = \rho\hat{\phi}(t)$$

$$\frac{d}{dt}\hat{\phi}(t) = \alpha_p\hat{\phi}(t) \qquad \alpha_p = \frac{\rho}{\Lambda} = \frac{1}{T_p} \quad \text{(prompt inverse period)}$$

$$\Rightarrow \quad \hat{\phi}(t) = \hat{\phi}(0)e^{\alpha_p t} = \hat{\phi}(0)e^{t/T_p}$$

Numerical Example of Prompt Kinetics

- 0.1% reduction in absorption in a large reactor with negligible leakage
  - In a large LWR, leakage loss is ~3.5%

$$\rho = -\frac{\delta\Sigma_a}{\nu\Sigma_f} = -\frac{\Sigma_a}{\nu\Sigma_f}\frac{\delta\Sigma_a}{\Sigma_a} \approx -\frac{\Sigma_a + DB^2}{\nu\Sigma_f}\frac{\delta\Sigma_a}{\Sigma_a} = -\frac{1}{k}\frac{\delta\Sigma_a}{\Sigma_a} = -\frac{\delta\Sigma_a}{\Sigma_a} = 0.001$$

- Average generation time in a large thermal reactor

$$\Lambda = \frac{1}{\overline{v}\nu\Sigma_f} \sim \frac{1}{2200 \text{ m/s} \times 0.05 \text{ cm}^{-1}} \sim 10^{-4}\text{ s}$$

- Prompt inverse period

$$\alpha_p = \frac{\rho}{\Lambda} \sim \frac{10^{-3}}{10^{-4}} = 10 \text{ s}^{-1}$$

- Reactor is not controllable because of too rapid increase of flux

$$\hat{\phi}(0.1) = \hat{\phi}(0)e^{10\times 0.1} = \hat{\phi}(0)e^{1} = 2.7\hat{\phi}(0)$$

$$\hat{\phi}(0.5) = \hat{\phi}(0)e^{10\times 0.5} = \hat{\phi}(0)e^{5} = 148\hat{\phi}(0)$$

Kinetics with Delayed Neutrons

- Suppose a critical reactor with delayed neutron fraction $\beta=0.007$
  - β is slightly larger than the value of U-235 due to U-238 fission contribution
  - $\beta_p\nu=(1-\beta)\nu=0.993\nu$ neutrons are produced promptly per each fission
  - $\beta\nu$ precursors are newly produced per each fission
  - $\beta\nu$ delayed neutrons are produced by the decay of precursors born $1/\lambda$ s ago on average
  - Neutron balance is maintained by these delayed neutrons
- In other words, for each generation of n_0 neutrons
  - βn_0 precursors are produced
  - If the precursor concentration is C_0, the delayed neutron production rate per <u>each generation</u> is $\lambda C_0\Lambda$, where Λ is the average generation time
  - Steady state precursor concentration

$$\lambda C_0\Lambda = \beta n_0 \quad \Rightarrow \quad C_0 = \frac{\beta n_0}{\lambda\Lambda}$$

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Prompt Multiplication of Neutrons (1)

- For 0.1% reduction of absorption cross section results in k=1.001
  - 1000 generations of fission chains in the first 0.1 s
- First generation prompt neutrons during the first 0.1 s

$$n_{1,p} = \beta_p (kn_0), \quad n_{2,p} = \beta_p (kn_{1,p}) = (k\beta_p)^2 n_0, \ \cdots,$$

$$n_{1000,p} = (k\beta_p)^{1000} n_0 = (1.001 \times 0.993)^{1000} n_0 = 0.0024 n_0$$

- Precursor concentration
  - Since the average delay time is ~7 s, new precursors formed during the first 0.1 s make essentially no contribution to the neutron densities at 0.1 s
  - The reduction of precursor concentration during the first 0.1 s is negligible

$$\lambda \sim 1/7 \ s^{-1} \sim 0.15 \ s^{-1}$$

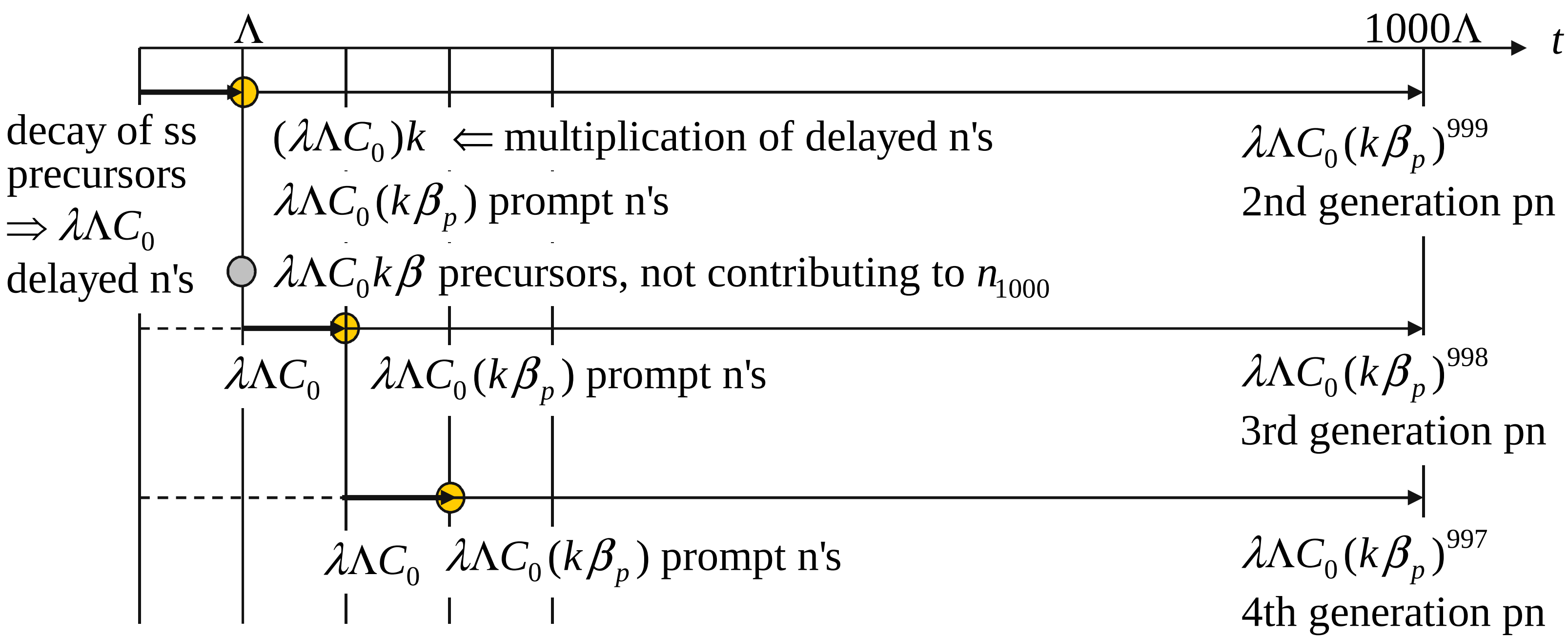
$$e^{-\lambda t} = e^{-0.15 \times 0.1} = 0.985$$

  - Thus the precursor concentration during the first 0.1 s can be approximated by the steady state concentration

$$C(t) \sim C_0$$

Prompt Multiplication of Neutrons (2)

- Multiplication of delayed neutrons



$$n_{1000,d} = \lambda\Lambda C_0 \sum_{i=0}^{999} (k\beta_p)^i = \lambda\Lambda C_0 \frac{1-(k\beta_p)^{1000}}{1-k\beta_p} = 166(\lambda\Lambda C_0) = 166(\beta n_0) = 1.162 n_0$$

- Neutron flux increases only by a factor of ~1.2 (vs. 2.7 times for prompt kinetics), indicating that it is manageable

$$n_{1000} = n_{1000,p} + n_{1000,d} = (0.0024 + 1.162)n_0 = 1.165 n_0$$

Intuitive Point Kinetics Equation

- Separation of fission source considering time delay

$$\nu\Sigma_f\hat{\phi}(t) \;\Rightarrow\; \nu_p\Sigma_f\hat{\phi}(t)+\sum_{k=1}^{6}\lambda_k C_k(t) \quad (\nu_p=\nu-\nu_d)$$

- Neutron flux equation

$$\frac{1}{\bar{v}}\frac{d}{dt}\bar{\phi}(t)=(\nu_p\Sigma_f-\Sigma_a'-DB^2)\hat{\phi}(t)+\sum_{k=1}^{6}\lambda_k C_k(t)$$

$$\nu_p=\nu-\nu_d$$

$$\Rightarrow \frac{1}{\bar{v}\nu\Sigma_f}\frac{d}{dt}\hat{\phi}(t)=\left(1-\frac{\nu_d}{\nu}-\frac{\Sigma_a'+DB^2}{\nu\Sigma_f}\right)\hat{\phi}(t)+\sum_{k=1}^{6}\lambda_k\frac{C_k(t)}{\nu\Sigma_f}$$

$$\varsigma_k(t)=\frac{1}{\nu\Sigma_f}C_k(t)$$

$$\Lambda\frac{d}{dt}\hat{\phi}(t)=(\rho-\beta)\hat{\phi}(t)+\sum_{k=1}^{6}\lambda_k\varsigma_k(t) \;\Rightarrow\; \frac{d}{dt}\hat{\phi}(t)=\frac{\rho-\beta}{\Lambda}\hat{\phi}(t)+\frac{1}{\Lambda}\sum_{k=1}^{6}\lambda_k\varsigma_k(t)$$

- Precursor equation

$$\frac{d}{dt}C_k(t)=-\lambda_k C_k(t)+\nu_{dk}\Sigma_f\hat{\phi}(t)$$

$$\frac{d}{dt}\varsigma_k(t)=-\lambda_k\varsigma_k(t)+\beta_k\hat{\phi}(t)$$

$$\alpha_p=\frac{\rho-\beta}{\Lambda} \text{ (prompt inverse period)}$$

$$\alpha_p=(0.001-0.007)/10^{-4}=-60\text{ s}^{-1}$$

$\alpha_p<0$ for $\rho<\beta \;\Rightarrow\;$ ptompt neutron contribution is decreasing

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Solution with Constant Precursor Approximation

- Neglect the change in the precursor population in a short time interval
  - Steady state precursor concentration

$$\frac{d}{dt}\varsigma_k(t) = -\lambda_k \varsigma_k(t) + \beta_k \hat{\phi}(t) = 0 \quad \Rightarrow \quad \varsigma_{k0} = \frac{\beta_k}{\lambda_k}\hat{\phi}_0$$

  - Transient balance equation

$$\frac{d}{dt}\hat{\phi}(t) = \alpha_p \hat{\phi}(t) + \frac{1}{\Lambda}\sum_{k=1}^{6}\lambda_k \varsigma_k(t) = \alpha_p \hat{\phi}(t) + \frac{1}{\Lambda}\sum_{k=1}^{6}\beta_k \phi_0 \quad \Rightarrow \quad \boxed{\frac{d}{dt}\hat{\phi}(t) = \alpha_p \hat{\phi}(t) + \frac{1}{\Lambda}\beta\phi_0}$$

- Solution of ODE

$$\frac{d}{dt}\hat{\phi}(t) - \alpha_p \hat{\phi}(t) = \frac{1}{\Lambda}\beta\phi_0 \quad \Rightarrow \quad \frac{d}{dt}[\hat{\phi}(t)e^{-\alpha_p t}] = \frac{1}{\Lambda}\beta\phi_0 e^{-\alpha_p t}$$

$$\hat{\phi}(t)e^{-\alpha_p t} - \hat{\phi}_0 = \frac{1}{\Lambda}\beta\phi_0 \int_0^t e^{-\alpha_p t'}dt' = \frac{1}{\Lambda\alpha_p}\beta\phi_0(1 - e^{-\alpha_p t})$$

$$\hat{\phi}(t) = \hat{\phi}_0 e^{\alpha_p t} + \frac{1}{\Lambda\alpha_p}\beta\phi_0(e^{\alpha_p t} - 1) = \hat{\phi}_0 e^{\alpha_p t} + \frac{\beta}{\rho - \beta}\phi_0(e^{\alpha_p t} - 1)$$

$$\frac{\rho}{\beta} = \frac{0.001}{0.007} = 0.143\$$$

$$e^{\alpha_p t} = e^{-60\times 0.1} = 0.0025$$

$$\frac{\beta}{\rho - \beta} = -1.17$$

$$\hat{\phi}(0.1) \approx -\frac{\beta}{\rho - \beta}\phi_0 = 1.17\phi_0$$

Time-Dependent Neutron Balance Equation

- Time-dependent neutron diffusion equation

$$\frac{1}{v(E)}\frac{\partial\phi(r,E,t)}{\partial t}=(\mathbf{F}_p-\mathbf{M})\phi(r,E,t)+S_d(r,E,t)+S(r,E,t)$$

  – Production rates of prompt fission neutrons

$$\mathbf{F}_p\phi(r,E,t)=\sum_i N_i(r,t)\chi_p^i(E)\int_0^\infty \nu_p^i(E')\sigma_f^i(E')\phi(r,E',t)dE'$$
$$=\chi_p(E)\int_0^\infty \nu_p\Sigma_f(r,E',t)\phi(r,E',t)dE'$$

  – Loss rate by absorption and leakage & scattering source

$$\mathbf{M}\phi(r,E,t)=-\nabla\cdot D(r,E,t)\nabla\phi(r,E,t)+\Sigma_t(r,E,t)\phi(r,E,t)$$
$$-\int_0^\infty \Sigma_s(r,E'\to E,t)\phi(r,E',t)dE'$$

  – Delayed neutron source

$$S_d(r,E,t)=\sum_k \chi_{dk}(E)\lambda_k C_k(r,t)$$

- Precursor balance equation

$$\frac{\partial C_k(r,t)}{\partial t}=\int_0^\infty \nu_{dk}\Sigma_f(r,E',t)\phi(r,E',t)dE'-\lambda_k C_k(r,t)$$

One Group Point Kinetics

- Motivation
  - Simplified representation of the reactor convenient for the prediction of transient behavior
- Approach
  - Neglect energy and space dependence of the flux during the transient calculation, but consider only the level (or amplitude) change
  - Integrate over space and energy to yield 1-G, 0-D equations
- Approximations
  - Constant fission cross section $\Sigma_f(r,E,t) = \Sigma_f(r,E)$
  - Time dependence of the flux is separable from its space and energy dependence

$$\phi(r,E,t) = p(t)\psi(r,E)$$

$$P(t) = p(t)\int_V \int_0^\infty \kappa\Sigma_f(r,E)\psi(r,E)dEdV = p(t)P_0$$

shape: $\psi(r,E) = \phi_0(r,E)$

amplitude: $p(t), \quad p(0) = 1$

  - Neutron leakage is determined by buckling

$$-\nabla \cdot D\nabla\phi(r,E,t) = D(r,E)B^2(r,E)\phi(r,E,t) = DB^2\psi(r,E)\,p(t)$$

One-Group Point Kinetics Equation

- Time-dependent neutron diffusion equation

$$\frac{1}{v(E)}\frac{\partial\phi(r,E,t)}{\partial t}=\chi_p(E)\int_0^\infty \nu_p\Sigma_f(r,E')\phi(r,E',t)dE'+\sum_k \chi_{dk}(E)\lambda_k C_k(r,t)+S(r,E,t)$$

$$+\int_0^\infty \Sigma_s(r,E'\to E,t)\phi(r,E',t)dE'+\nabla\cdot D(r,E,t)\nabla\phi(r,E,t)-\Sigma_t(r,E,t)\phi(r,E,t)$$

- With the three approximations

$$\frac{\psi(r,E)}{v(E)}\frac{\partial p(t)}{\partial t}=\Big\{\chi_p(E)\int_0^\infty \nu_p\Sigma_f(r,E')\psi(r,E')dE'+\int_0^\infty \Sigma_s(r,E'\to E,t)\psi(r,E')dE'$$

$$-D(r,E)B^2(r,E)\psi(r,E)-\Sigma_t(r,E,t)\psi(r,E)\Big\}\,p(t)+\sum_k \chi_{dk}(E)\lambda_k C_k(r,t)+S(r,E,t)$$

- Integration over energy yields a one-group equation

$$\left\{\int_0^\infty \frac{1}{v(E)}\psi(r,E)dE\right\}\frac{\partial p(t)}{\partial t}=\Big\{\int_0^\infty \chi_p(E)dE\int_0^\infty \nu_p\Sigma_f(r,E')\psi(r,E')dE'$$

$$+\int_0^\infty\int_0^\infty \Sigma_s(r,E'\to E,t)\psi(r,E',t)dE'dE-\int_0^\infty D(r,E)B^2(r,E)\psi(r,E)dE$$

$$-\int_0^\infty \Sigma_t(r,E,t)\psi(r,E)dE\Big\}\,p(t)+\sum_k \lambda_k C_k(r,t)\int_0^\infty \chi_{dk}(E)dE+\int_0^\infty S(r,E,t)dE$$

One-Group Point Kinetics Equation

- Integration over volume yields zero-dimensional kinetics equations

$$\left\{\int_V\int_0^\infty \frac{\psi(r,E)}{v(E)}dEdV\right\}\frac{\partial p(t)}{\partial t}=\left\{\int_V\int_0^\infty \nu_p\Sigma_f(r,E)\psi(r,E)dEdV\right.$$

$$\left.-\int_V\int_0^\infty D(r,E)B^2(r,E)\psi(r,E)dEdV-\int_V\int_0^\infty \Sigma_a(r,E,t)\psi(r,E)dEdV\right\}p(t)$$

$$+\sum_k \lambda_k\int_V C_k(r,t)dV+\int_V\int_0^\infty S(r,E,t)dEdV$$

$$\frac{\partial}{\partial t}\int_V C_k(r,t)dV=p(t)\int_V\int_0^\infty \nu_{dk}\Sigma_f(r,E',t)\psi(r,E')dE'dV-\lambda_k\int_V C_k(r,t)dV$$

  - Integrated functions and average one-group cross sections

$$\hat{\psi}=\int_V\int_0^\infty \psi(r,E)dEdV,\quad \hat{C}_k(t)=\int_V C_k(r,t)dV,\quad \hat{S}(t)=\int_V\int_0^\infty S(r,E,t)dEdV$$

$$\overline{\left(\frac{1}{v}\right)}=\frac{1}{\hat{\psi}}\int_V\int_0^\infty \frac{\psi(r,E)}{v(E)}dEdV \qquad \nu_p\Sigma_f=\frac{1}{\hat{\psi}}\int_V\int_0^\infty \nu_p\Sigma_f(r,E)\psi(r,E)dEdV$$

$$\Sigma_a=\frac{1}{\hat{\psi}}\int_V\int_0^\infty \Sigma_a(r,E,t)\psi(r,E)dEdV \qquad DB^2=\frac{1}{\hat{\psi}}\int_V\int_0^\infty D(r,E)B^2(r,E)\psi(r,E)dEdV$$

One-Group Point Kinetics Equation

- Zero-dimensional kinetics equations

$$\overline{\left(\frac{1}{v}\right)}\frac{\partial p(t)}{\partial t} = (\nu_p\Sigma_f - \Sigma_a - DB^2)p(t) + \frac{1}{\hat{\psi}}\sum_k \lambda_k \hat{C}_k(t) + \frac{1}{\hat{\psi}}\hat{S}(t)$$

$$\frac{\partial}{\partial t}\hat{C}_k(t) = \nu_{dk}\Sigma_f\hat{\psi}\,p(t) - \lambda_k\hat{C}_k(t)$$

- One-group point kinetics equations are obtained as in the case of the intuitive point kinetics equation

$$\frac{1}{\nu\Sigma_f}\overline{\left(\frac{1}{v}\right)}\frac{\partial p(t)}{\partial t} = \left(1 - \frac{\Sigma_a + DB^2}{\nu\Sigma_f} - \frac{\nu_d\Sigma_f}{\nu\Sigma_f}\right)p(t) + \sum_k \lambda_k \frac{\hat{C}_k(t)}{\nu\Sigma_f\hat{\psi}} + \frac{\hat{S}(t)}{\nu\Sigma_f}$$

$$\nu\Sigma_f\hat{\psi} = \int_V\int_0^\infty \nu\Sigma_f(r,E)\psi(r,E)dEdV = \hat{S}_{f0} \qquad \nu_p = \nu - \nu_d$$

$$\Lambda\frac{\partial p(t)}{\partial t} = (\rho - \beta)p(t) + \sum_k \lambda_k \frac{\hat{C}_k(t)}{\hat{S}_{f0}} + \frac{\hat{S}(t)}{\hat{S}_{f0}}$$

$$\frac{\partial}{\partial t}\frac{\hat{C}_k(t)}{\hat{S}_{f0}} = \frac{\nu_{dk}\Sigma_f}{\nu\Sigma_f}p(t) - \lambda_k\frac{\hat{C}_k(t)}{\hat{S}_{f0}}$$

One-Group Point Kinetics Equation

- One-group point kinetics equation is obtained as in the case of the intuitive point kinetics equation

$$\frac{1}{\nu\Sigma_f}\overline{\left(\frac{1}{v}\right)}\frac{\partial p(t)}{\partial t}=\left(1-\frac{\Sigma_a+DB^2}{\nu\Sigma_f}-\frac{\nu_d\Sigma_f}{\nu\Sigma_f}\right)p(t)+\sum_k\lambda_k\frac{\hat{C}_k(t)}{\nu\Sigma_f\hat{\psi}}+\frac{\hat{S}(t)}{\nu\Sigma_f}$$

$$\nu\Sigma_f\hat{\psi}=\int_V\int_0^\infty\nu\Sigma_f(r,E)\psi(r,E)dEdV=\hat{S}_{f0}$$

$$\Lambda\frac{\partial p(t)}{\partial t}=(\rho-\beta)p(t)+\frac{1}{\hat{S}_{f0}}\sum_k\lambda_k\hat{C}_k(t)+\frac{\hat{S}(t)}{\hat{S}_{f0}}$$

$$\varsigma_k(t)=\frac{\hat{C}_k(t)}{\hat{S}_{f0}}\ \text{(reduced precursor)},\quad s(t)=\frac{\hat{S}(t)}{\hat{S}_{f0}}\ \text{(reduced source)}$$

$$\frac{\partial p(t)}{\partial t}=\frac{\rho-\beta}{\Lambda}p(t)+\frac{1}{\Lambda}\sum_k\lambda_k\varsigma_k(t)+\frac{1}{\Lambda}s(t)$$

$$\frac{\partial\varsigma_k(t)}{\partial t}=-\lambda_k\varsigma_k(t)+\beta_k p(t)$$

$$\beta_k=\frac{\int_V\int_0^\infty\nu_{dk}\Sigma_f(r,E)\psi(r,E)dEdV}{\int_V\int_0^\infty\nu\Sigma_f(r,E)\psi(r,E)dEdV}$$

Conventional Point Kinetics Equation

- Commonly, the $1/\Lambda$ factor in front of sources are combined with these quantities

$$c_k(t) = \frac{1}{\Lambda}\varsigma_k(t) = \frac{\hat{C}_k(t)}{\Lambda \hat{S}_{f0}}$$

$$s_c(t) = \frac{1}{\Lambda}s(t) = \frac{\hat{S}(t)}{\Lambda \hat{S}_{f0}}$$

- This yields the more familiar form of the point kinetics equation

$$\frac{\partial p(t)}{\partial t} = \frac{\rho - \beta}{\Lambda} p(t) + \sum_k \lambda_k c_k(t) + s_c(t)$$

$$\frac{\partial c_k(t)}{\partial t} = -\lambda_k c_k(t) + \frac{\beta_k}{\Lambda} p(t)$$

  – Precursor and independent source variables have no direct physical meaning
  – Prompt jump approximation is not simply derived with $\Lambda \rightarrow 0$