NUCL 510
Nuclear Reactor Theory

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Lecture Note 9

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Separation of Space and Energy Dependencies

- Equation for space dependency

$$\nabla^2 \phi(\vec{r}) + B^2 \phi(\vec{r}) = 0$$

  - Wave or Helmholtz equation to determine the fundamental mode flux shape
  - B^2 represents the geometrical curvature of the flux, and it is an eigenvalue to be determined from boundary conditions
    - *The smallest eigenvalue B^2 is called the geometrical buckling*

- Equation for energy dependency

$$[D(E)B^2 + \Sigma_t(E)]\varphi(E) - \int_0^\infty dE'\Sigma_s(E' \to E)\varphi(E') = \chi(E)\int_0^\infty dE'\nu\Sigma_f(E')\varphi(E')$$

  - Integral equation for fundamental or asymptotic spectrum
    - *$\phi(r,E)$ is not separable around interfaces, but it is separable far away from interfaces (e.g., asymptotic spectrum in a large medium)*
  - The asymptotic spectrum is independent of boundary conditions
    - *It is independent of the size of the core, and it is the same in the reflected core as in an un-reflected core of the same material, provided the regions are large enough*

Equation for Energy Dependency

- Slowing-down equation

$$[D(E)B^2 + \Sigma_t(E)]\varphi(E) - \int_0^\infty dE'\Sigma_s(E' \to E)\varphi(E') = \chi(E)\int_0^\infty dE'\nu\Sigma_f(E')\varphi(E')$$

Energy		1 eV	1 keV	1 MeV
Energy range characterization		Thermalization	Slowing-down	Source
Type of Reaction	Scattering	Up and down scattering	Down scattering CMS isotropic elastic scattering	Inelastic scattering; Elastic scattering with forward peak
	Capture	1/v	Resonances	Smooth cross sections
	Fission (fissile)	1/v	Resonances	Smooth cross sections
	Fission (fissionable)			Threshold cross section

Differential Scattering Cross Section

- Legendre moments of scattering cross section

$$\sigma_{sl}^{i}(E' \to E) = \frac{\sigma_s^i(E')P_l[\mu_s(E',E)]}{(1-\alpha^i)E'} \sum_{n=0}^{N}(2n+1) f_n^i(E') P_n[\mu_c(E',E)]$$

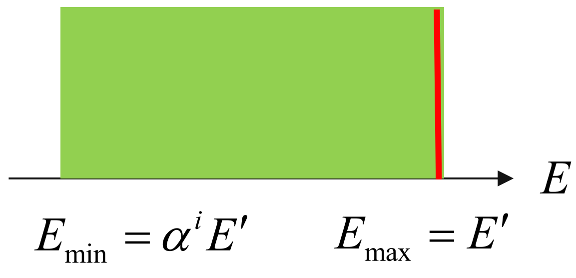
$$\mu_c(E',E) = \frac{1}{1-\alpha}\left[2\frac{E}{E'} - (1+\alpha)\right]$$

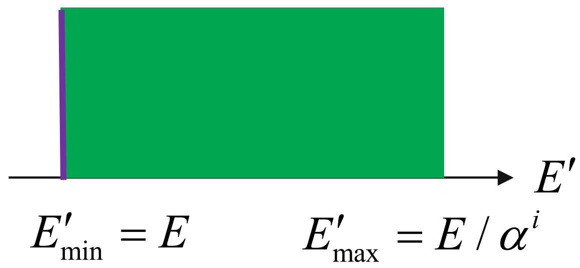
$$\mu_s(E',E) = \frac{1}{2}\left[(A+1)\sqrt{\frac{E}{E'}} - (A-1)\sqrt{\frac{E'}{E}}\right]$$

- CMS isotropic elastic scattering

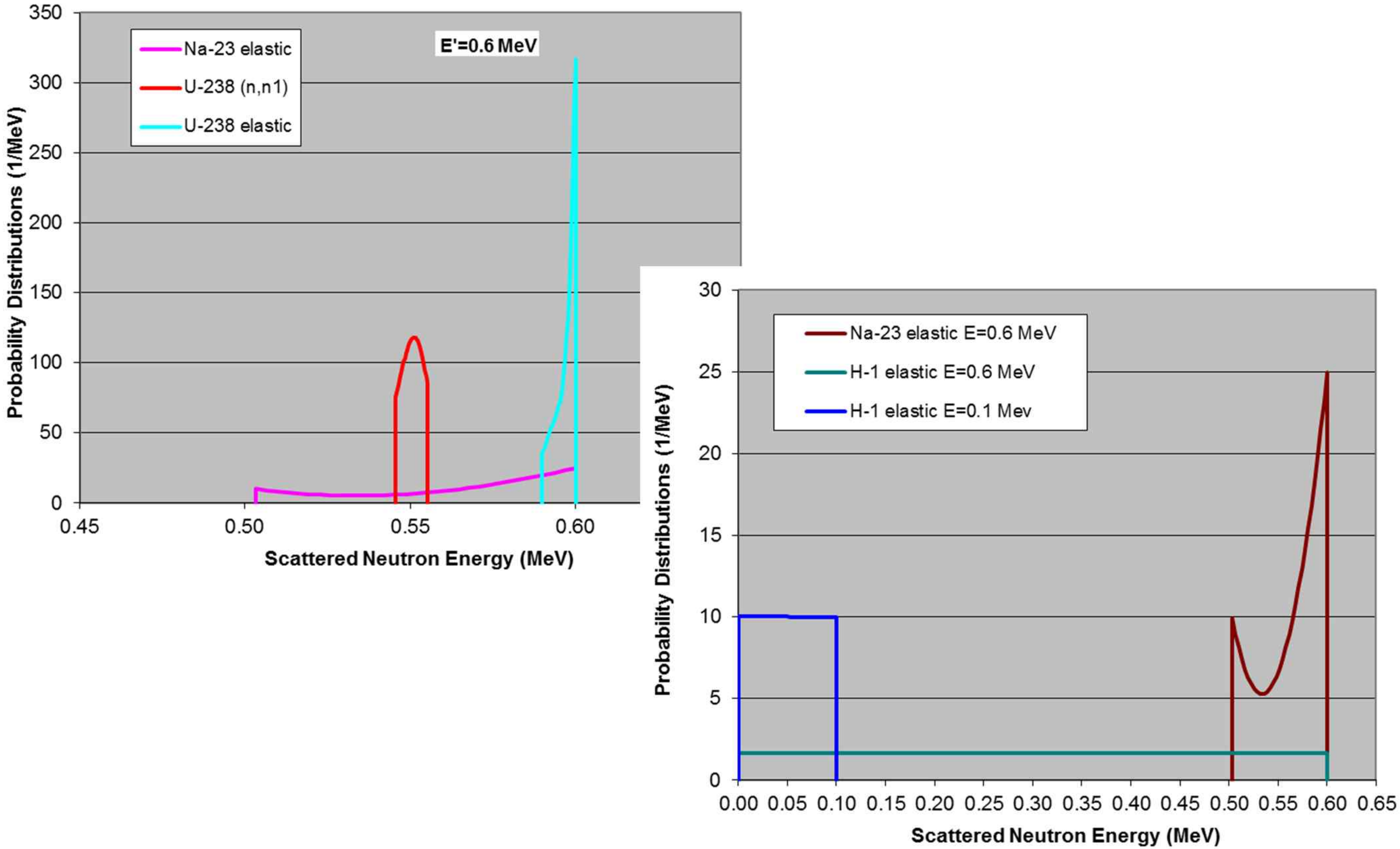
$$\sigma_s^i(E' \to E) = \frac{\sigma_s^i(E')}{(1-\alpha^i)E'}, \quad \alpha^i = \frac{(A_i-1)^2}{(A_i+1)^2}$$

$$P_s^i(E' \to E) = \begin{cases} \dfrac{1}{(1-\alpha^i)E'}, & \alpha^i E' \le E \le E' \\ 0, & \text{otherwise} \end{cases}$$





Sample Energy Transfer Functions



Energy Loss and Lethargy Gain (1)

- Average energy loss by isotropic elastic scattering

$$\bar{E} = \int_{\alpha^i E'}^{E'} dE P(E' \to E) E = \int_{\alpha^i E'}^{E'} dE \frac{E}{(1-\alpha^i)E'} dE = \frac{1}{2}(1+\alpha^i)E' \quad \begin{pmatrix} \text{average energy} \\ \text{after a collision} \end{pmatrix}$$

$$\overline{\Delta E} = E' - \bar{E} = \frac{1}{2}(1-\alpha^i)E' \quad \text{(average energy loss per collision)}$$

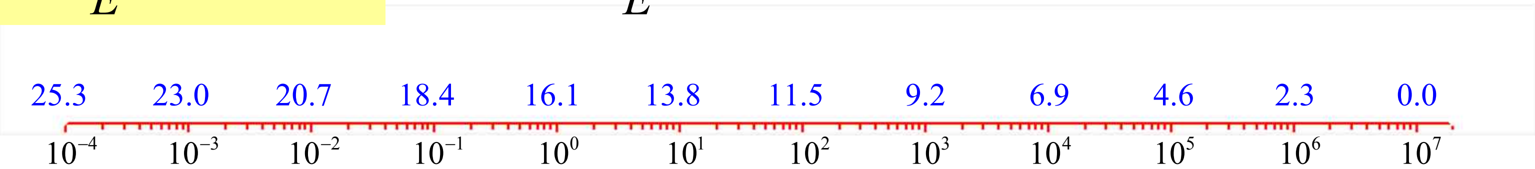
$$f_\ell^i = \frac{\overline{\Delta E}}{E'} = \frac{1}{2}(1-\alpha^i) = \frac{2A_i}{(A_i+1)^2} \quad \text{(average fractional energy loss)}$$

For $A_i = 1, \quad f_\ell^i = 50\%; \quad$ For $A_i = 238, \quad f_\ell^i = 0.83\%$

$$\bar{E}_n = (1-f_\ell^i)^n E' = \left(\frac{1+\alpha^i}{2}\right)^n E' \quad \text{(average energy after } n \text{ collisions)}$$

- Lethargy

$$u = \ln\frac{E_0}{E}, \quad E = E_0 e^{-u} \qquad du = -\frac{dE}{E} \quad \text{(lethargy increases as energy decreases)}$$



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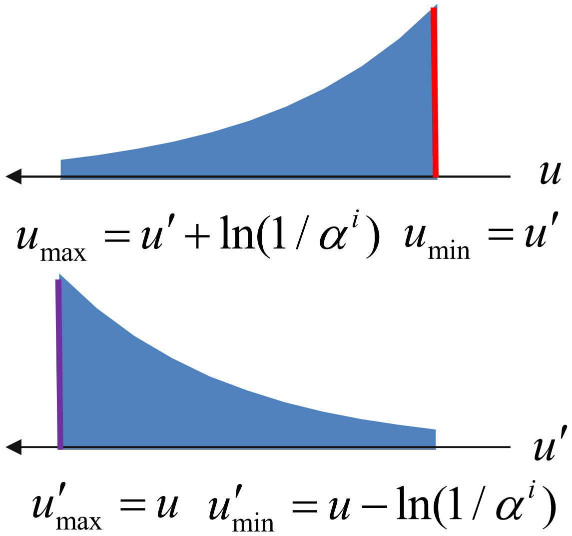
Energy Loss and Lethargy Gain (2)

- CMS isotropic elastic scattering in terms of lethargy

$$\sigma_s^i(u' \to u) = \sigma_s^i(E' \to E)\left|\frac{dE}{du}\right| = \frac{\sigma_s^i(E')}{(1-\alpha^i)E'}E$$

$$= \frac{\sigma_s^i(u')E_0 e^{-u}}{(1-\alpha^i)E_0 e^{-u'}} = \sigma_s^i(u')\frac{e^{-(u-u')}}{1-\alpha^i}$$

$$P_s^i(u' \to u) = \begin{cases} \dfrac{e^{-(u-u')}}{1-\alpha^i}, & u' \le u \le u' + \ln(1/\alpha^i) \\ 0, & \text{otherwise} \end{cases}$$



- Average lethargy gain

$$\xi_i = \overline{\Delta u} = \int_{u'}^{u'+\ln(1/\alpha^i)} (u-u')P(u' \to u)du = \frac{1}{1-\alpha^i}\int_{u'}^{u'+\ln(1/\alpha^i)} (u-u')e^{-(u-u')}du$$

$$= \frac{1}{1-\alpha^i}\int_0^{\ln(1/\alpha^i)} we^{-w}dw = \frac{-1}{1-\alpha^i}(1+w)e^{-w}\Big|_0^{\ln(1/\alpha^i)} = 1 - \frac{\alpha^i}{1-\alpha^i}\ln(1/\alpha^i)$$

$$\xi_i = 1 - \frac{\alpha^i}{1-\alpha^i}\ln(1/\alpha^i) = 1 - \frac{(A_i-1)^2}{2A_i}\ln\left(\frac{A_i+1}{A_i-1}\right) \approx \frac{2}{A_i + 2/3}$$

Energy Loss and Lethargy Gain (3)

- Approximate formula for average lethargy gain

$$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots;\quad \ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\cdots$$

$$\begin{aligned}\xi_i&=1-\frac{(A_i-1)^2}{2A_i}\ln\left(\frac{A_i+1}{A_i-1}\right)=1-\frac{(1-A_i^{-1})^2}{2A_i^{-1}}\ln\left(\frac{1+A_i^{-1}}{1-A_i^{-1}}\right)\\&=1-\frac{(1-A_i^{-1})^2}{2A_i^{-1}}\left[2A_i^{-1}\left(1+\frac{1}{3}A_i^{-1}\right)+O(A_i^{-5})\right]=2A_i^{-1}\left(1-\frac{2}{3}A_i^{-1}\right)+O(A_i^{-3})\\&=\frac{2A_i^{-1}}{1+(2/3)A_i^{-1}}+O(A_i^{-3})=\frac{2}{A_i+2/3}+O(A_i^{-3})\end{aligned}$$

- For hydrogen, $A_i=1$ and thus $\alpha_i=0$

$$\lim_{\alpha_i\to 0}\alpha^i\ln(\alpha^i)=\lim_{\alpha_i\to 0}\frac{\ln(\alpha^i)}{1/\alpha^i}=\lim_{\alpha_i\to 0}\frac{1/\alpha^i}{-1/(\alpha^i)^2}=-\lim_{\alpha_i\to 0}\alpha^i=0$$

$$\lim_{\alpha_i\to 0}\xi_i=\lim_{\alpha_i\to 0}\left[1+\frac{\alpha^i}{1-\alpha^i}\ln(\alpha^i)\right]=1$$

Average Lethargy Gain for Mixture

- For a mixture of scattering isotopes
  - Fraction of scattering by isotope i : Σ_{si}/Σ_s
  - Average lethargy gain

$$\bar{\xi} = \overline{u - u'} = \overline{\Delta u} = \sum_i \frac{\Sigma_{si}}{\Sigma_s} \xi_i = \frac{1}{\Sigma_s} \sum_i \Sigma_{si} \xi_i = \frac{\sum_i \Sigma_{si} \xi_i}{\sum_i \Sigma_{si}}$$

  - H_2O example

$$\bar{\xi} = \frac{1}{\Sigma_s} \sum_i \Sigma_{si} \xi_i = \frac{2N_{H_2O}\sigma_s^H \xi_H + N_{H_2O}\sigma_s^O \xi_O}{2N_{H_2O}\sigma_s^H + N_{H_2O}\sigma_s^O} = \frac{2\sigma_s^H \xi_H + \sigma_s^O \xi_O}{2\sigma_s^H + \sigma_s^O}$$

- Slowing-down power

$$\bar{\xi}\Sigma_s$$

- Slowing-down ratio

$$\frac{\bar{\xi}\Sigma_s}{\Sigma_a}$$

	$\bar{\xi}$	$\bar{\xi}\Sigma_s$	$\frac{\bar{\xi}\Sigma_s}{\Sigma_a}$
H_2O	0.93	1.28	58
D_2O	0.51	0.18	21,000
C	0.158	0.056	200

Slowing Down in Hydrogen

- Slowing down equation

$$[D(E)B^2+\Sigma_t(E)]\varphi(E)-\sum_i\int_E^{E/\alpha_i}dE'\Sigma_s^i(E'\to E)\varphi(E')=\chi(E)\int_0^\infty dE'\nu\Sigma_f(E')\varphi(E')$$

- Balance equation for slowing down in hydrogen

$$[D(E)B^2+\Sigma_t(E)]\varphi(E)-\int_E^\infty dE'\frac{\Sigma_s^H(E')}{E'}\varphi(E')=\chi(E)s_0$$

$$\Sigma_s^i(E'\to E)=\frac{\Sigma_s^i(E')}{(1-\alpha_i)E'}$$

$$\Sigma_t(E)=\Sigma_s^H(E)+\Sigma_a^H(E)+\Sigma_a^{others}(E)$$

- Without leakage

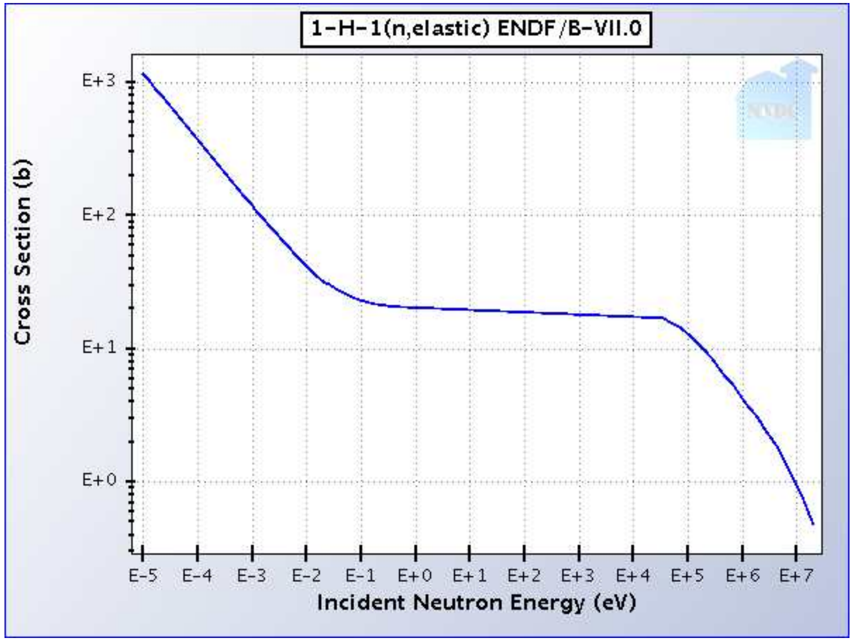
$$\Sigma_t(E)\varphi(E)-\int_E^\infty dE'\frac{\Sigma_s^H(E')}{E'}\varphi(E')=\chi(E)s_0$$

- Without absorption or leakage

$$\Sigma_s^H(E)\varphi(E)-\int_E^\infty dE'\frac{\Sigma_s^H(E')}{E'}\varphi(E')=\chi(E)s_0$$

- Constant scattering cross section

$$\Sigma_s^H\varphi(E)-\Sigma_s^H\int_E^\infty dE'\frac{\varphi(E')}{E'}=\chi(E)s_0$$



Slowing Down in Hydrogen with Constant σ_s (1)

- Slowing down equation

$$\Sigma_s^H \varphi(E) - \Sigma_s^H \int_E^\infty dE' \frac{\varphi(E')}{E'} = \chi(E) s_0 \quad \Rightarrow \quad \varphi(E) - \int_E^\infty dE' \frac{\varphi(E')}{E'} = \frac{s_0}{\Sigma_s^H} \chi(E)$$

  – Volterra integral equation that can be converted into a differential equation

- First order differential equation and its solution

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = f[b(t),t]b'(t) - f[a(t),t]a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx$$

$$\frac{d\varphi(E)}{dE} + \frac{\varphi(E)}{E} = \frac{d\chi(E)}{dE}\frac{s_0}{\Sigma_s^H} \quad \Rightarrow \quad \left[E\frac{d\varphi(E)}{dE} + \varphi(E)\right] = \frac{s_0}{\Sigma_s^H} E \frac{d\chi(E)}{dE}$$

$$\frac{d}{dE}[E\varphi(E)] = \frac{s_0}{\Sigma_s^H} E \frac{d\chi(E)}{dE}$$

$$E'\varphi(E')\Big|_E^\infty = \frac{s_0}{\Sigma_s^H}\int_E^\infty dE' E' \frac{d\chi(E')}{dE'} = \frac{s_0}{\Sigma_s^H}\left[E'\chi(E')\Big|_E^\infty - \int_E^\infty dE' \chi(E')\right]$$

$$-E\varphi(E) = \frac{s_0}{\Sigma_s^H}\left[-E\chi(E) - \int_E^\infty dE' \chi(E')\right] \quad \Rightarrow \quad \varphi(E) = \frac{s_0}{\Sigma_s^H}\chi(E) + \frac{s_0}{\Sigma_s^H E}\int_E^\infty dE' \chi(E')$$

Slowing Down in Hydrogen with Constant σ_s (2)

$$\varphi(E) = \frac{S_0}{\Sigma_s^H}\chi(E) + \frac{S_0}{\Sigma_s^H E}\int_E^{\infty}\chi(E')dE'$$

- Below the fission source range

$$\chi(E) = 0, \quad \int_E^{\infty}\chi(E')dE' = 1 \quad \Rightarrow$$

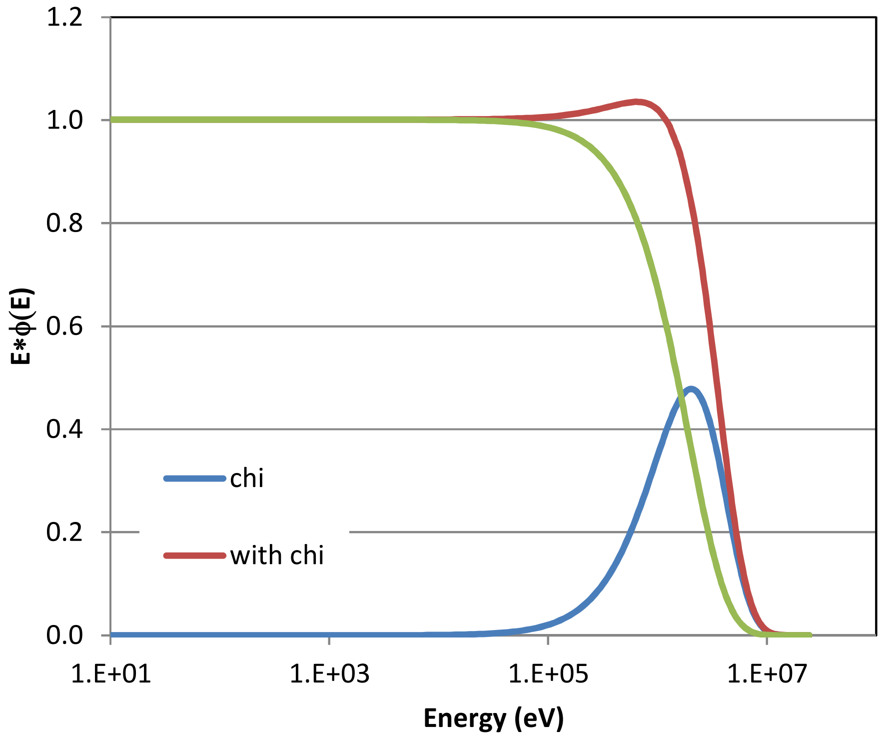
$$\varphi(E) = \frac{S_0}{\Sigma_s^H E} \quad (1/E \text{ spectrum})$$

$$E\varphi(E) = \varphi(u) = \frac{S_0}{\Sigma_s^H} = \text{const}$$

- Within the fission source range

$$E\varphi(E) = \frac{S_0}{\Sigma_s^H}E\chi(E) + \frac{S_0}{\Sigma_s^H}\int_E^{\infty}\chi(E')dE'$$

$$\varphi(u) = \frac{S_0}{\Sigma_s^H}\chi(u) + \frac{S_0}{\Sigma_s^H}\int_u^{\infty}\chi(u')du'$$



Slowing Down with Energy-Dependent XS (1)

- Slowing down equation

$$\Sigma_s^H(E)\varphi(E) - \int_E^\infty dE' \frac{\Sigma_s^H(E')\varphi(E')}{E'} = \chi(E)s_0$$

$$F(E) - \int_E^\infty dE' \frac{F(E')}{E'} = \chi(E)s_0, \quad F(E) = \Sigma_s^H(E)\varphi(E) \quad \text{(scattering density)}$$

- First order differential equation and its solution

$$\frac{dF(E)}{dE} + \frac{F(E)}{E} = \frac{d\chi(E)}{dE}s_0 \quad \Rightarrow \quad E\frac{dF(E)}{dE} + F(E) = Es_0\frac{d\chi(E)}{dE}$$

$$\frac{d}{dE}[EF(E)] = Es_0\frac{d\chi(E)}{dE}$$

$$E'F(E')\Big|_E^\infty = s_0\int_E^\infty dE'E'\frac{d\chi(E')}{dE'} = s_0\left[E'\chi(E')\Big|_E^\infty - \int_E^\infty dE'\chi(E')\right]$$

$$-EF(E) = s_0\left[-E\chi(E) - \int_E^\infty dE'\chi(E')\right] \quad \Rightarrow \quad F(E) = s_0\chi(E) + \frac{s_0}{E}\int_E^\infty dE'\chi(E')$$

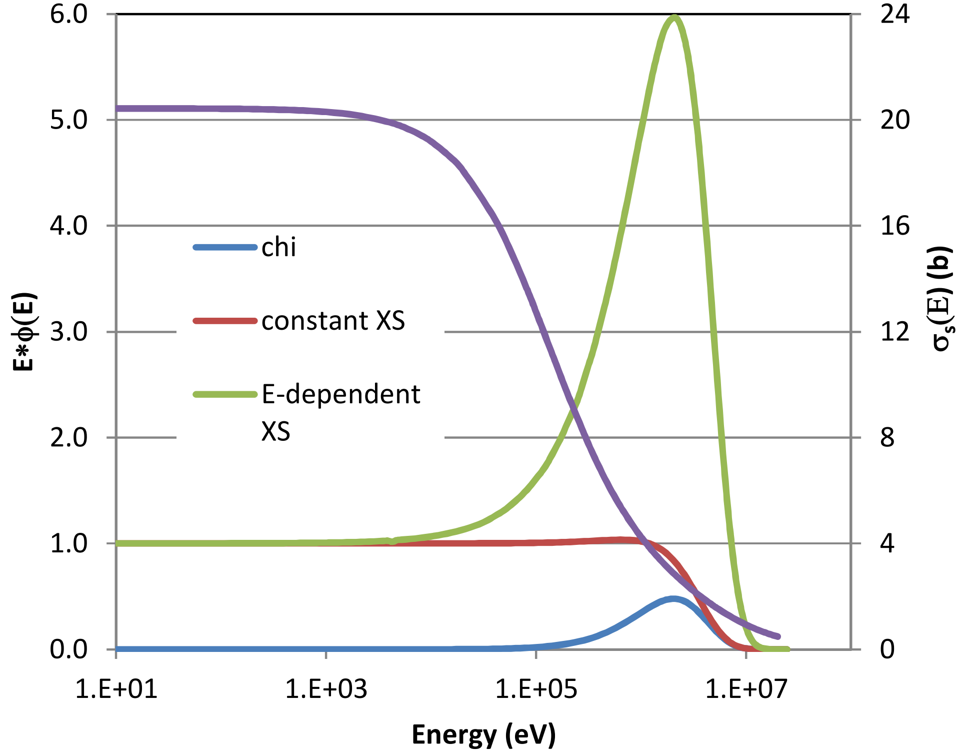
$$E\varphi(E) = \frac{s_0}{\Sigma_s^H(E)}E\chi(E) + \frac{s_0}{\Sigma_s^H(E)}\int_E^\infty dE'\chi(E')$$

Slowing Down with Energy-Dependent XS (2)

$$E\varphi(E) = \frac{s_0}{\Sigma_s^H(E)} E\chi(E) + \frac{s_0}{\Sigma_s^H(E)} \int_E^\infty dE'\chi(E')$$

$$\varphi(u) = \frac{s_0}{\Sigma_s^H(u)} \chi(u) + \frac{s_0}{\Sigma_s^H(u)} \int_u^\infty du'\chi(u')$$

- Increased flux for low cross section region



Slowing Down with Absorption (1)

- Mixture of fuel and hydrogen moderator
  - Neglect the scattering by fuel and the absorption by hydrogen

$$\Sigma_t(E)\varphi(E) - \int_E^\infty dE' \frac{\Sigma_s^H(E')\varphi(E')}{E'} = \chi(E)s_0 \qquad \sigma_{sF}\xi_F << \sigma_{sF}\xi_H \ \text{(scattering power)}$$

$$\Sigma_t(E) = \Sigma_a^F(E) + \Sigma_s^H(E)$$

- First order differential equation

$$F(E) - \int_E^\infty dE' \frac{\Sigma_s^H(E')F(E')}{\Sigma_t(E')E'} = \chi(E)s_0, \quad F(E) = \Sigma_t(E)\varphi(E) \quad \text{(collision density)}$$

$$\frac{dF(E)}{dE} + \frac{\Sigma_s^H(E)F(E)}{\Sigma_t(E)E} = \frac{d\chi(E)}{dE}s_0 \quad \Rightarrow \quad E\frac{dF(E)}{dE} + \left[1 - \frac{\Sigma_a^F(E)}{\Sigma_t(E)}\right]F(E) = Es_0\frac{d\chi(E)}{dE}$$

$$\frac{d}{dE}[EF(E)] - \frac{\Sigma_a^F(E)}{\Sigma_t(E)E}[EF(E)] = E\frac{d\chi(E)}{dE}s_0$$

$$\frac{d\tilde{F}(E)}{dE} - \frac{a(E)}{E}\tilde{F}(E) = E\frac{d\chi(E)}{dE}s_0 \qquad a(E) = \frac{\Sigma_a^F(E)}{\Sigma_t(E)}$$

Slowing Down with Absorption (2)

- Integrating factor

$$\frac{d\tilde{F}(E)}{dE}-\frac{a(E)}{E}\tilde{F}(E)=E\frac{d\chi(E)}{dE}s_0$$

$$I(E)=\exp\left(-\int_{E_0}^{E}\frac{a(E')}{E'}dE'\right)=e^{h(E)},\quad h(E)=\int_{E}^{E_0}a(E')\frac{dE'}{E'}=\int_{E}^{E_0}\frac{\Sigma_a^F(E')}{\Sigma_a^F(E')+\Sigma_s^H(E')}\frac{dE'}{E'}$$

$$e^{h(E)}\frac{d\tilde{F}(E)}{dE}-e^{h(E)}\frac{a(E)}{E}\tilde{F}(E)=e^{h(E)}E\frac{d\chi(E)}{dE}s_0$$

$$\frac{d}{dE}[\tilde{F}(E)e^{h(E)}]=e^{h(E)}E\frac{d\chi(E)}{dE}s_0$$

- Integration from E to ∞

$$E'F(E')e^{h(E')}\Big|_E^{\infty}=\int_E^{\infty}e^{h(E')}E'\frac{d\chi(E')}{dE'}s_0dE' \quad \leftarrow \tilde{F}(E)=EF(E)$$

$$-EF(E)e^{h(E)}=\int_E^{\infty}e^{h(E')}E'\frac{d\chi(E')}{dE'}s_0dE'$$

$$\varphi(E)=-\frac{e^{-h(E)}}{\Sigma_t(E)E}\int_E^{\infty}e^{h(E')}E'\frac{d\chi(E')}{dE'}s_0dE'$$

Slowing Down with Absorption (3)

$$\varphi(E) = -\frac{e^{-h(E)}}{\Sigma_t(E)E}\int_E^{\infty} e^{h(E')}E'\frac{d\chi(E')}{dE'}s_0 dE'$$

- At high energy where $\chi(E) \neq 0$, $\Sigma_a \sim 0$

$$h(E) = \int_E^{E_0} \frac{\Sigma_a^F(E')}{\Sigma_a^F(E') + \Sigma_s^H(E')}\frac{dE'}{E'} \approx 0 \quad \Rightarrow \quad e^{h(E)} \approx 1$$

$$\varphi(E) \approx -\frac{e^{-h(E)}}{\Sigma_t(E)E}s_0\int_E^{\infty} E'\frac{d\chi(E')}{dE'}dE' = -\frac{e^{-h(E)}}{\Sigma_t(E)E}s_0\left(-E\chi(E) - \int_E^{\infty}\chi(E')dE'\right)$$

$$\varphi(E) = \frac{e^{-h(E)}}{E\Sigma_t(E)}s_0\left(E\chi(E) + \int_E^{\infty}\chi(E')dE'\right)$$

- Below the fission source range, $\chi(E) = 0, \quad \int_E^{\infty}\chi(E')dE' = 1$

$$\varphi(E) = \frac{s_0}{E\Sigma_t(E)}e^{-h(E)} = \frac{s_0}{E\Sigma_t(E)}\exp\left(-\int_E^{E_0}\frac{\Sigma_a^F(E')}{\Sigma_a^F(E') + \Sigma_s^H(E')}\frac{dE'}{E'}\right)$$

non absorption probability

Slowing Down with Absorption and Leakage

- Below the fission source range, $\chi(E) = 0, \quad \int_E^\infty \chi(E')dE' = 1$

$$\varphi(E) = \frac{S_0}{E\Sigma_{tl}(E)} e^{-h(E)} e^{-h_l(E)}, \quad \Sigma_{tl}(E) = \Sigma_t(E) + D(E)B^2$$

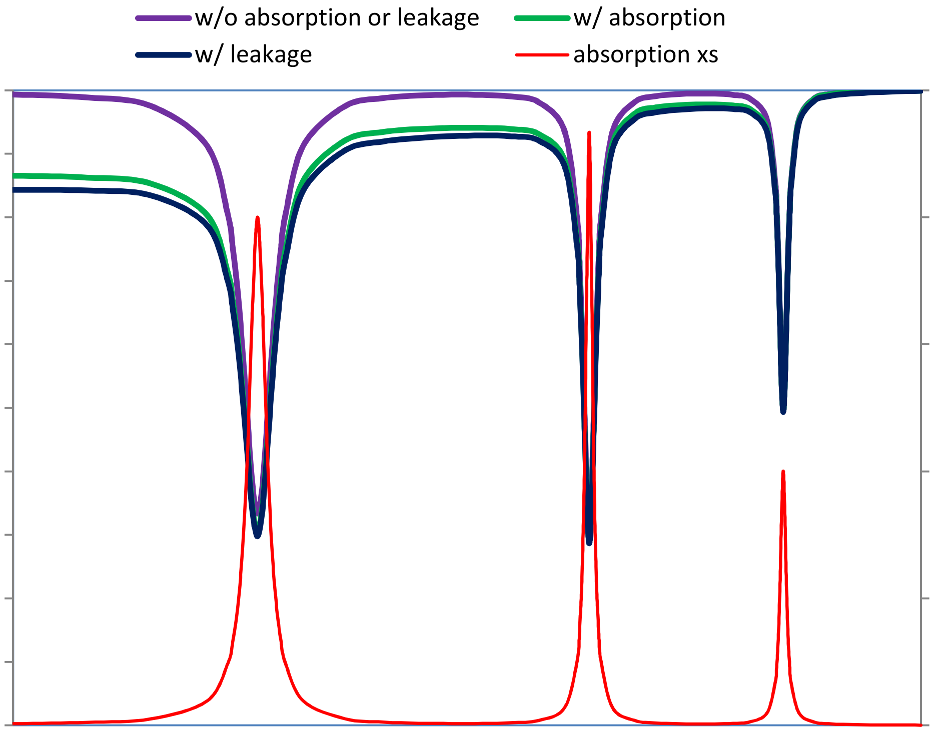
  – Non-absorption probability

$$e^{-h(E)} = \exp\left(-\int_E^{E_0} \frac{\Sigma_a^F(E')}{\Sigma_{tl}(E')} \frac{dE'}{E'} \right)$$

  – Non-leakage probability

$$e^{-h_l(E)} = \exp\left(-B^2 \int_E^{E_0} \frac{D(E')}{\Sigma_{tl}(E')} \frac{dE'}{E'} \right)$$

$$\approx B^2 \tau$$



Slowing Down in Nonhydrogeneous Materials (1)

- Slowing down equation for constant scattering cross section

$$\Sigma_s \varphi(E) - \int_E^{E/\alpha} \frac{\Sigma_s \varphi(E')}{(1-\alpha)E'} dE' = s_0 \chi(E) \qquad \alpha \neq 0$$

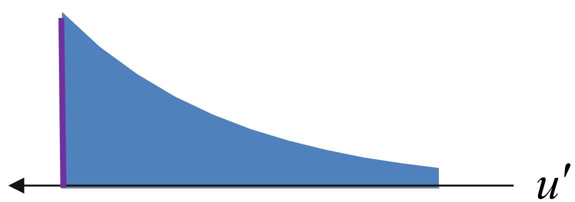
- Differentiation yields a "difference-differential" equation

$$\Sigma_s \frac{d\varphi(E)}{dE} - \frac{\Sigma_s}{1-\alpha}\left[\varphi(E/\alpha) - \varphi(E)\right] = s_0 \frac{d\chi(E)}{dE} \qquad \text{(No known analytic solution)}$$

$$\varphi\left(\frac{E}{\alpha}\right) \approx \varphi(E) + \frac{d\varphi(E)}{dE}\left(\frac{E}{\alpha} - E\right) \qquad \text{(First order Taylor expansion not valid for small } \alpha\text{)}$$

- Taylor series expansion of flux in lethargy

$$P_s^i(u' \to u) = \begin{cases} \dfrac{e^{-(u-u')}}{1-\alpha^i}, & u - \ln(1/\alpha^i) \le u' \le u \\ 0, & \text{otherwise} \end{cases}$$



$$u'_{\max} = u \quad u'_{\min} = u - \ln(1/\alpha^i)$$

$$\Sigma_s \varphi(u) - \Sigma_s \int_{u-\ln(1/\alpha)}^{u} \frac{e^{-(u-u')}}{1-\alpha} \varphi(u') du' = s_0 \chi(u)$$

$$\Sigma_s \varphi(u) - \Sigma_s \int_{u-\ln(1/\alpha)}^{u} \frac{e^{-(u-u')}}{1-\alpha} \left[\varphi(u) + \frac{d\varphi(u)}{du}(u'-u) + \cdots\right] du' = s_0 \chi(u)$$

Slowing Down in Nonhydrogeneous Materials (2)

- First order Taylor expansion of flux in lethargy

$$\Sigma_s\varphi(u)-\Sigma_s\varphi(u)\int_0^{\ln(1/\alpha)}\frac{e^{-w}}{1-\alpha}dw+\Sigma_s\frac{d\varphi(u)}{du}\int_0^{\ln(1/\alpha)}\frac{we^{-w}}{1-\alpha}dw\approx s_0\chi(u)\quad(w=u-u')$$

$$\xi\Sigma_s\frac{d\varphi(u)}{du}\approx s_0\chi(u)$$

$$\xi\Sigma_s\varphi(u)=s_0\int_{-\infty}^{u}\chi(u')du'\quad\Rightarrow\quad\varphi(u)=\frac{s_0}{\xi\Sigma_s}\int_{-\infty}^{u}\chi(u')du'$$

$$\varphi(u)=E\varphi(E),\quad\chi(u)du=-\chi(E)dE\quad\Rightarrow\quad\varphi(E)=\frac{s_0}{\xi\Sigma_s E}\int_E^{\infty}\chi(E')dE'$$

- Below the fission source range

$$\varphi(E)=\frac{s_0}{\xi\Sigma_s(E)E}\qquad\varphi(E)=\frac{s_0}{\overline{\xi\Sigma}_s(E)E}=\frac{s_0}{\sum_i\xi_i\Sigma_s^i(E)E}\quad\text{(for mixture)}$$

  – !/E spectrum satisfies the balance equation below the source

$$\Sigma_s(E)\varphi(E)-\int_E^{E/\alpha}\frac{\Sigma_s(E')\varphi(E')}{(1-\alpha)E'}dE'=\frac{s_0}{\xi E}+\frac{s_0}{(1-\alpha)\xi}\left.\frac{1}{E'}\right|_E^{E/\alpha}=0$$

Scattering Source and Slowing Down Density

- Scattering source below fission source range for 1/E spectrum

$$S_s(E) = \int_E^{E/\alpha} \frac{\Sigma_s(E')\varphi(E')}{(1-\alpha)E'} dE' = \int_E^{E/\alpha} \frac{\Sigma_s(E')}{(1-\alpha)E'} \frac{s_0}{\xi\Sigma_s(E')E'} dE' = \int_E^{E/\alpha} \frac{s_0}{(1-\alpha)\xi(E')^2} dE'$$

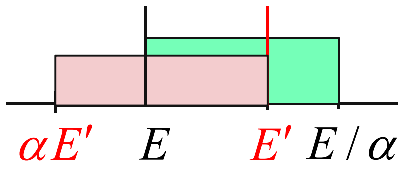
$$= \frac{s_0}{\xi E} = \Sigma_s(E)\frac{s_0}{\xi\Sigma_s(E)E} = \Sigma_s(E)\varphi(E)$$

  – Scattering source at energy E is equal to the out-scattering rate at E

- Slowing down density below fission source range

  – $q_{sd}(E)$: Number of neutrons slowed down below energy E

$$q_{sd}(E) = \int_E^{E/\alpha} \frac{E-\alpha E'}{(1-\alpha)E'} \Sigma_s(E')\varphi(E')dE'$$



$$= \frac{s_0}{\xi}\int_E^{E/\alpha} \frac{E-\alpha E'}{(1-\alpha)E'} \Sigma_s(E') \frac{s_0}{\xi\Sigma_s(E')E'} dE'$$

$$= \frac{s_0}{\xi(1-\alpha)}\left[-\frac{E}{E'}\Big|_E^{E/\alpha} - \alpha \ln E'\Big|_E^{E/\alpha} \right] = \frac{s_0}{\xi}\left(1 - \frac{\alpha}{1-\alpha}\ln\frac{1}{\alpha}\right) = s_0$$

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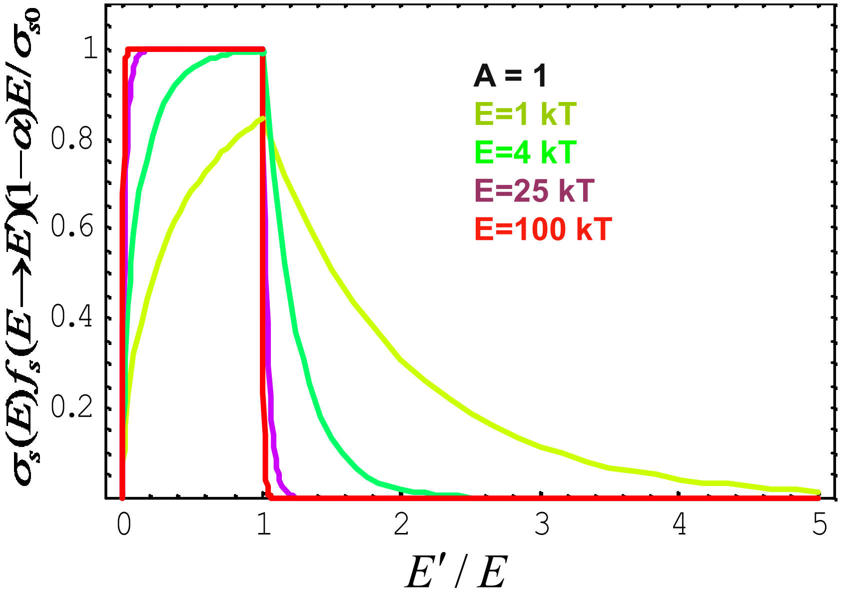
Thermal Neutron Spectrum (1)

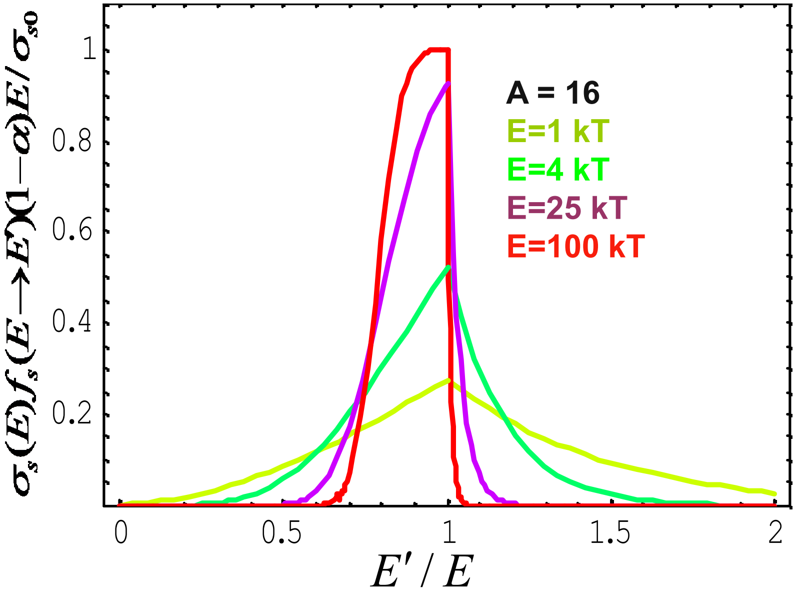
- Energy transfer function

$$\sigma_s(E) f_s(E \to E') = \frac{\sigma_{s0}\eta^2}{2E}\left\{ \operatorname{erf}\left(\eta\sqrt{\frac{E'}{kT}} - \rho\sqrt{\frac{E}{kT}}\right) \mp \operatorname{erf}\left(\eta\sqrt{\frac{E'}{kT}} + \rho\sqrt{\frac{E}{kT}}\right) + e^{(E-E')/kT}\left[\operatorname{erf}\left(\eta\sqrt{\frac{E}{kT}} - \rho\sqrt{\frac{E'}{kT}}\right) \pm \operatorname{erf}\left(\eta\sqrt{\frac{E}{kT}} + \rho\sqrt{\frac{E'}{kT}}\right)\right]\right\}$$

$$\eta = \frac{A+1}{2\sqrt{A}} \qquad \rho = \frac{A-1}{2\sqrt{A}}$$

  – The upper signs are to be used for $E' > E$ and the lower signs for $E' < E$





Thermal Neutron Spectrum (2)

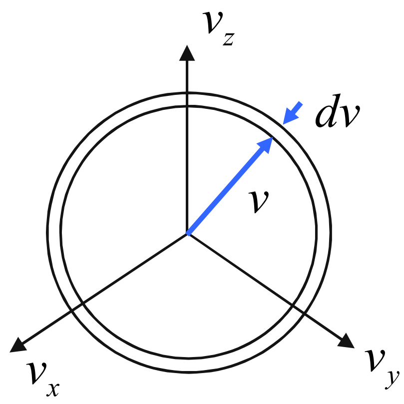
- Thermal neutron spectrum
  - Without absorption or leakage, the neutron population in the thermal range is in equilibrium with the thermal motion of the scattering nuclei
- Maxwellian distribution of neutron velocity at temperature T

$$p(v_x,v_y,v_z)dv_x dv_y dv_z = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(v_x^2+v_y^2+v_z^2)/2kT} dv$$

- Speed distribution

$$p(v)dv = \int_{v_x^2+v_y^2+v_z^2} p(v_x,v_y,v_z)dv_x dv_y dv_z = p(v_x,v_y,v_z)(4\pi v^2 dv)$$

$$p(v)dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} dv$$



- Energy distribution

$$E = \frac{mv^2}{2} \quad \Rightarrow \quad dE = mvdv \quad \Rightarrow \quad dv = \frac{dE}{mv} = \frac{dE}{\sqrt{2mE}}$$

$$p(E)dE = p(v)dv = \frac{2}{\sqrt{\pi}}\frac{\sqrt{E}}{(kT)^{3/2}} e^{-E/kT} dE \qquad \bar{E} = \int_0^\infty Ep(E)dE = \frac{3}{2}kT$$

Thermal Neutron Spectrum (3)

- Maxwellian flux distribution

$$\varphi(E) = nv(E)p(E) = n\sqrt{\frac{2E}{m}}\frac{2}{\sqrt{\pi}}\frac{\sqrt{E}}{(kT)^{3/2}}e^{-E/kT} = \frac{2n}{\sqrt{\pi}}\sqrt{\frac{2}{m}}\frac{E}{(kT)^{3/2}}e^{-E/kT}$$

  – If the energy integral is normalized to unity,

$$\varphi_M(E) = \frac{E}{(kT)^2}e^{-E/kT}$$

- Most probable energy

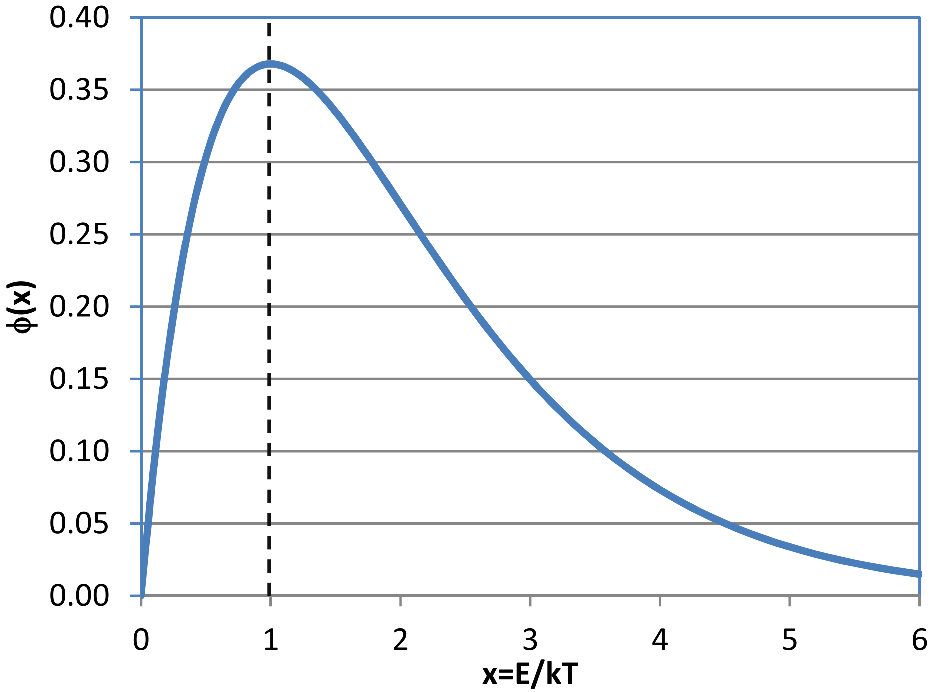
$$\frac{d\varphi_M(E)}{dE} = \frac{1}{(kT)^2}e^{-\frac{E}{kT}}\left(1 - \frac{E}{kT}\right) = 0$$

$$\Rightarrow \quad E_p = kT$$

  – At room temperature (293.6K)

$$kT_0 = 0.0253\, eV$$

$$v_0 = 2,200\, m/\sec$$



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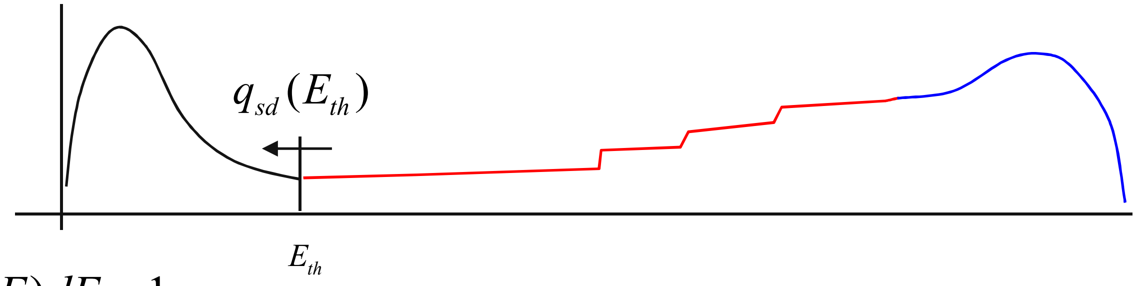
Connection of Thermal and Epithermal Fluxes

- Total thermal flux

$$\varphi_{th} = \int_0^{E_{th}} \varphi(E)dE$$

$$\varphi(E) = \varphi_{th}\varphi_M(E)$$

$$\Leftarrow \quad \int_0^{E_{th}} \varphi_M(E)dE \approx \int_0^{\infty} \varphi_M(E)dE = 1$$



- Total absorption rate in the thermal region

$$R_a = \int_0^{E_{th}} \Sigma_a(E)\varphi(E)dE = \bar{\Sigma}_{a,th}\varphi_{th}$$

- Balance between slowing down density and thermal absorption

$$\bar{\Sigma}_{a,th}\varphi_{th} = q_{sd}(E_{th}) = \bar{\xi}\Sigma_s(E_{th})(E\varphi)_{E=E_{th}} \quad \Rightarrow \quad \varphi_{th} = \frac{\bar{\xi}\Sigma_s(E_{th})}{\bar{\Sigma}_{a,th}} E_{th}\varphi(E_{th}) = \frac{1}{\Delta} E_{th}\varphi(E_{th})$$

- Total thermal flux decreases as Δ increases
  - *Δ:* Ratio of thermal absorption to slowing down power at the thermal cutoff energy

Thermal Fluxes

- Total thermal flux

$$\varphi_{th} = \int_0^{E_{th}} \varphi(E)dE = \frac{2n}{\sqrt{\pi}}\sqrt{\frac{2kT}{m}}\int_0^{\sim 5kT} \frac{E}{kT} e^{-E/kT} \frac{dE}{kT} \approx \frac{2n}{\sqrt{\pi}}\sqrt{\frac{2kT}{m}}\int_0^{\infty} \frac{E}{kT} e^{-E/kT} \frac{dE}{kT}$$

$$\varphi_{th}(T) = \frac{2n}{\sqrt{\pi}}\sqrt{\frac{2kT}{m}}$$

$$E_T = kT, \quad v_T = \sqrt{\frac{2E_T}{m}} = \sqrt{\frac{2kT}{m}} \quad \Rightarrow \quad \varphi_T = \varphi_{th}(T) = \frac{2}{\sqrt{\pi}} n v_T$$

- 2200 m/sec flux

$$\varphi_0 = n v_0 \quad \Rightarrow \quad \frac{\varphi_T}{\varphi_0} = \frac{2}{\sqrt{\pi}}\frac{v_T}{v_0} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{T}{T_0}}$$

- Gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt; \quad \Gamma(x) = (x-1)\Gamma(x-1)$$

– Interger: $\Gamma(1) = 1, \quad \Gamma(n) = (n-1)!$

– Half-interger: $\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(n+1/2) = (n-1/2)(n-3/2)\cdots(3/2)(1/2)\sqrt{\pi}$

Average Thermal Absorption Cross Sections

- Average thermal absorption cross sections

$$\bar{\sigma}_a = \frac{1}{\varphi_{th}} \int_0^{E_{th}} \sigma_a(E)\varphi(E)dE = \int_0^{E_{th}} \sigma_a(E)\varphi_M(E)dE$$

- 1/v absorber

$$\sigma_a(E) = \sigma_0 \frac{v_0}{v} = \sigma_0 \sqrt{\frac{E_0}{E}} = \sigma_0 \sqrt{\frac{kT_0}{E}}$$

$$\bar{\sigma}_a \varphi_{th} = \int_0^{E_{th}} \sigma_0 \frac{v_0}{v(E)} v(E)n(E)dE = \sigma_0 v_0 n = \sigma_0 \varphi_0$$

$$\bar{\sigma}_a = \int_0^{E_{th}} \sigma_a(E)\varphi_M(E)dE \approx \sigma_0 \sqrt{\frac{T_0}{T}} \int_0^{\infty} \sqrt{\frac{E}{kT}} e^{-E/kT} \frac{dE}{kT} = \frac{\sqrt{\pi}}{2} \sigma_0 \sqrt{\frac{T_0}{T}}$$

$$\bar{\sigma}_a = \frac{\sqrt{\pi}}{2} \sigma_0 \sqrt{\frac{T_0}{T}}$$

- Non-1/v absorber

$$\bar{\sigma}_a = \frac{\sqrt{\pi}}{2} \sigma_0 \sqrt{\frac{T_0}{T}} g_a(T)$$

$g_a(T)$: Westcott's non-1/v factor

Absorption Hardening

- Absorption cross sections in the thermal range are approximately 1/v
  - Relatively more low-energy neutrons are absorbed than high-energy neutrons
  - Spectrum is shifted to right

- Equivalent temperature (called “neutron temperature”)

$$T_h = T\left(1+\alpha_h \frac{\Sigma_{a,th}}{\xi\Sigma_s}\right)$$

$$\alpha_h = 1.46$$