NUCL 511 HMWK 7

Alex Hagen

3/13/14

1 ***Transients with Constant Reactivity*. Find numerically (with a suitable code) the solution of the point kinetics equations for the following reactivity insertions in a critical reactor and plot the results (in all cases, use $\Lambda = 10^{-4}$, 10^{-5}, and 10^{-6} s). Use one group of delayed neutrons with $\beta = 0.007$ and $\overline{\lambda} = 0.1\ s^{-1}$:** The general solution of the one-group delayed neutron point kinetics equation is

$$p(t) = \frac{p_0}{1-\rho_{1\$}} \exp(\alpha_1 t) - \frac{\rho_{1\$} p_0}{1-\rho_{1\$}} \exp(\alpha_P t)$$

and we assume that $p_0 = 1$. Now we simply need to find α_1 and α_P. When the reactivity is less than one dollar ($\rho_{1\$} < 1\$$), the roots of the characteristic equation can be found with

$$\alpha_1 = \frac{\rho_{1\$}}{1-\rho_{1\$}} \cdot \overline{\lambda}$$

and

$$\alpha_P = \frac{\rho_1 - \beta}{\Lambda}$$

This means the following gives the power level as a function of time with respect to a constant activity:

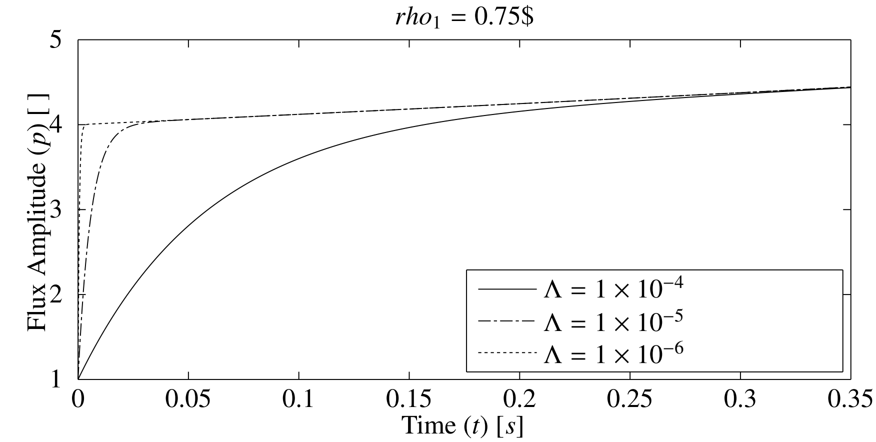
$$p(t) = \frac{1}{1-\rho_{1\$}} \exp\left(\frac{\rho_{1\$}}{1-\rho_{1\$}} \cdot \overline{\lambda} \cdot t\right) - \frac{\rho_{1\$}}{1-\rho_{1\$}} \exp\left(\frac{\rho_1 - \beta}{\Lambda} \cdot t\right)$$

This can be plotted easily. Figure 1 shows the short time phenomena for the reactivities 0.75\$, −1.0\$, and −3.0\$.

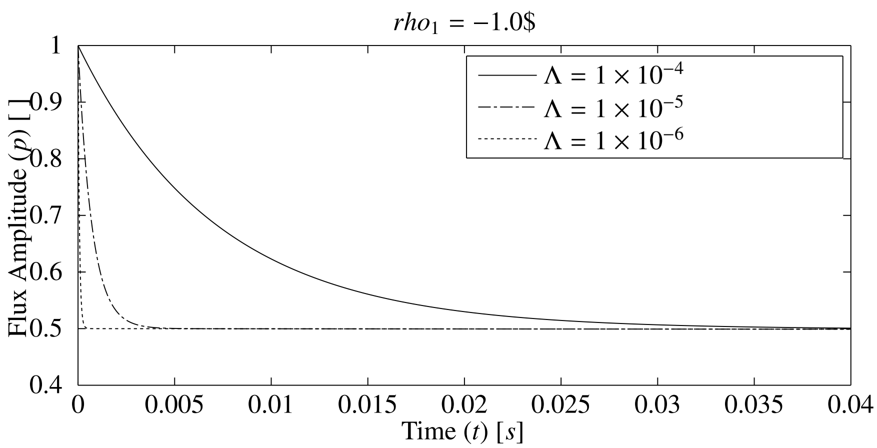
(1) Estimate first the length of time you want to run the transient; present the rationale of your estimate. For each transient, I would run them until they reached their asymptotic level (after the short time behavior had been removed). Notice that for the negative reactivities, times of 0.04 s and 0.02 s would place any of the systems into a reduced power level at steady state. For the positive reactivity, a time of 0.35 s would place any system into a positive reactor period, and thus the power level would increase over time. For each of these cases, a decreased Λ requires decreased time to reach that stable period.

(2) Discuss the transient results, in particular the short-time behavior, asymptotic behavior, and Λ dependence of both. The short time behavior of the positive and negative reactivity insertions is similar. Starting at p_0, the power level changes drastically before slowing down to reach the next stable period (the asymptotic solution). For smaller Λ, the faster the change is made. For example, the decrease in a reactivity insertion of −1.0\$ happens in about 0.04 s with Λ of 10^{-4} s^{-1}, whereas it takes much less than 0.005 s when Λ is 10^{-6} s^{-1}.

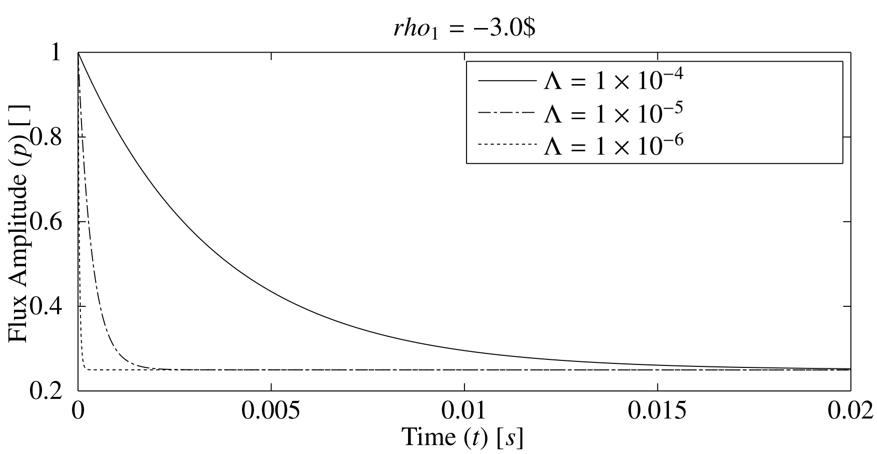
The asymptotic solutions of the reactivity insertions are different. For positive reactivity, the asymptotic solution is a linearly increasing power level. This power level is independent of Λ. This is shown in Figure 1a, as the power level linearly increases towards the end of the time window. For negative reactivity solutions, the asymptotic solution is a constant power level, and the amount of the reactivity insertion determines that power level. In Figure 1b, a −1.0\$ reactivity insertion has an asymptotic power level of 0.5, whereas in Figure 1c, a −3.0\$ reactivity insertion has an asymptotic power level of ∼ 0.25. For negative reactivities, Λ does not alter the asymptotic power level, but instead changes how quickly the power drops to that constant level, as described earlier.



(a) Positive Reactivity Insertion



(b) Negative Reactivity Insertion



(c) Largely Negative Reactivity Insertion

Figure 1: Short Time Phenomena of Different Reactivity Insertions

2 Find numerically the transient that follows a ρ jump from $\rho_0 = -1\$$ to $\rho_1 = 0$, with a source $s_0 = \beta p_0$ present during the transient. Again use the three Λ values of problem 5. Use one group of delayed neutrons with $\beta = 0.007$ and $\overline{\lambda} = 0.1\ s^{-1}$. We have the two group point kinetics equations with an independent source as follows:

$$\begin{cases} \Lambda \dot{p} = [\rho(t) - \beta]\, p(t) + \lambda \zeta(t) + s(t) \\ \dot{\zeta}(t) = \beta p(t) - \lambda \zeta(t) \end{cases}$$

which can be combined into a second order ODE by using the reduced precursor concentration, ζ.

$$\Lambda \ddot{p} + (\lambda \Lambda + \beta - \rho)\, \dot{p} - (\lambda \rho + \dot{\rho})\, p = \lambda s + \dot{s}$$

We can simplify this expression by knowing several values. First, we know that $\lambda = \overline{\lambda}$, since we are using only one group of delayed neutrons. Secondly, we know that the transient includes a jump, which is modeled by the heaviside step function $\rho = \rho_0 + (\rho_1 - \rho_0)\, H(t)$, which will have a derivative of $\dot{\rho} = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$. Finally, we know the constant source, $s_0 = \beta p_0$, which will have a zero derivative. Thus, the ODE we must solve is [2]

$$\Lambda \ddot{p} + \left(\overline{\lambda} \Lambda + \beta - \rho\right) \dot{p} - \left(\overline{\lambda} \rho + \dot{\rho}\right) p = \overline{\lambda} \beta p_0$$

which, for $t \neq 0$ is

$$\Lambda \ddot{p} + \left(\overline{\lambda} \Lambda + \beta - \rho\right) \dot{p} - \left(\overline{\lambda} \rho\right) p = \overline{\lambda} \beta p_0$$

To solve this ODE, we must first place it in the form $L[y] = y'' + p(t)\, y' + q(t)\, y = g(t)$ [1], so we divide by Λ:

$$\ddot{p} + \frac{\left(\overline{\lambda} \Lambda + \beta - \rho\right)}{\Lambda} \dot{p} - \frac{\left(\overline{\lambda} \rho + \dot{\rho}\right)}{\Lambda} p = \frac{\overline{\lambda} \beta p_0}{\Lambda}$$

Then, we find the general solution of the corresponding homogenous equation,

$$\ddot{p} + \frac{\left(\overline{\lambda} \Lambda + \beta - \rho\right)}{\Lambda} \dot{p} - \frac{\left(\overline{\lambda} \rho + \dot{\rho}\right)}{\Lambda} p = 0$$

which has the general form of

$$p = c_1 \exp(r_1 t) + c_2 \exp(r_2 t)$$

so the characteristic equation is

$$r^2 + \frac{\left(\overline{\lambda} \Lambda + \beta - \rho\right)}{\Lambda} r - \frac{\left(\overline{\lambda} \rho\right)}{\Lambda} p = 0$$

when $t > 0$, $\rho = 0$, so this becomes

$$r \left(r + \frac{\left(\overline{\lambda} \Lambda + \beta\right)}{\Lambda} \right) = 0$$

$$r_1 = 0, \quad r_2 = -\frac{\left(\overline{\lambda} \Lambda + \beta\right)}{\Lambda}$$

To find the particular solution, we can use the method of undetermined coefficients, where

$$\ddot{p} + \frac{\left(\overline{\lambda} \Lambda + \beta\right)}{\Lambda} \dot{p} = \frac{\overline{\lambda} \beta p_0}{\Lambda}$$

and we guess a particular solution

$$Y_P = c_3 t, \quad \dot{Y}_P = c_3, \quad \ddot{Y}_P = 0$$

$$\frac{\left(\overline{\lambda} \Lambda + \beta\right)}{\Lambda} c_3 = \frac{\overline{\lambda} \beta p_0}{\Lambda}$$

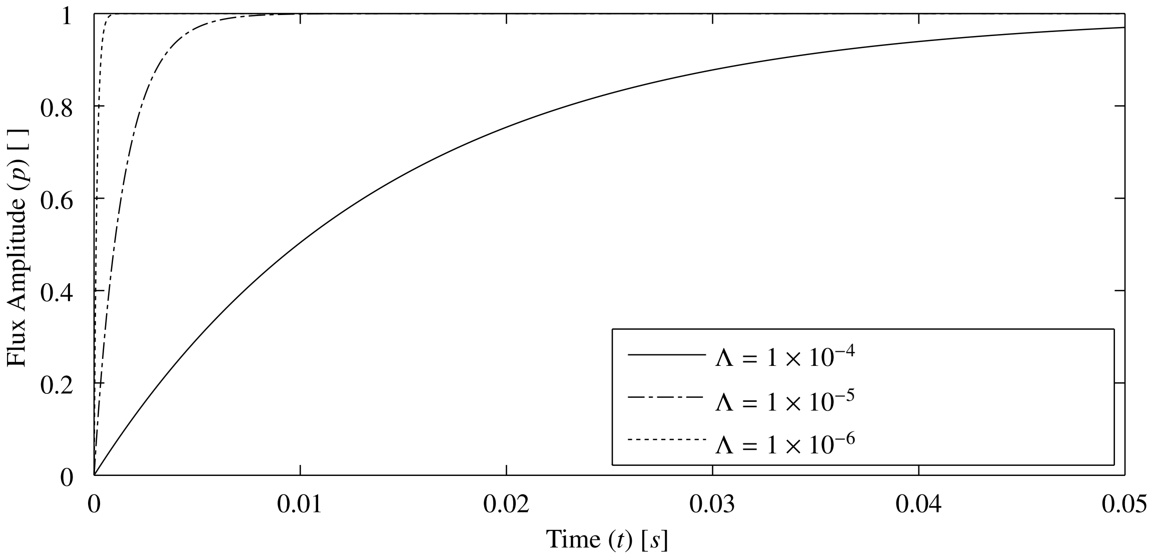


Figure 2: Power Response with Step Reactivity Change from $\rho_0 = 0$ to $\rho_1 = 1$

$$c_3 = \frac{\bar{\lambda}\beta p_0}{\left(\bar{\lambda}\Lambda + \beta\right)}$$

Now we need to find c_1 and c_2, knowing that

$$p = c_1 + c_2 \exp\left(-\frac{\left(\bar{\lambda}\Lambda + \beta\right)}{\Lambda}t\right) + \frac{\bar{\lambda}\beta p_0}{\left(\bar{\lambda}\Lambda + \beta\right)}t$$

We must then determine coefficients c_1 and c_2. We can use time $t = 0^+$ to determine an initial condition, at which we know

$$p\left(0^+\right) = 0 = c_1 + c_2$$

We also know that at time $t = \infty$, the answer must be finite, because the reactor is never super critical. Because of this state, c_3 becomes incorrect, and instead we must take the trivial solution for c_3. Thus,

$$p\left(\infty\right) = c_1 = 1$$

In this way, $c_2 = -p\left(\infty\right) = -1$. Note that depending on our conditions for the asymptotic power and initial power, we could vary the change between these two functions.

$$p = 1 - \exp\left(-\frac{\left(\bar{\lambda}\Lambda + \beta\right)}{\Lambda}t\right)$$

This is plotted in Figure 2. Notice that while the reactivity is negative, that the flux does increase until it reaches a steady state. Note that again, the decreasing Λ causes a decreased time to reach the asymptotic level.

3 ***Interpretation of the Adjoint Flux*****. Prove by application of the point kinetics equations the interpretation of the adjoint flux, $\phi_0^*\left(\mathbf{r}, E\right)$, as the asymptotic relative flux rise δp_{as} (aside from a proportionality constant) that follows a burst-insertion of independent source neutrons in a critical reactor at location r with energy E:**

$$\delta p_{as} = c\phi_0^*\left(\mathbf{r}, E\right)$$

(1) Find the analytical solution for a burst-insertion of independent source neutrons in a critical reactor, $\delta s(t) = s_0 \delta(t)$. Use one group of delayed neutrons, $\lambda = \overline{\lambda}$. Find $p(t)$, and especially δp_{as}. Again, we start with the point kinetics equations with an independent source, which is shown below

$$\begin{cases} \Lambda \dot{p} = [\rho(t) - \beta]\, p(t) + \lambda \zeta(t) + s(t) \\ \dot{\zeta}(t) = \beta p(t) - \lambda \zeta(t) \end{cases}$$

then, we reduce the two first order ODEs into a single second order ODE by using the reduced precursor concentration [2], ζ.

$$\Lambda \ddot{p} + (\lambda\Lambda + \beta - \rho)\,\dot{p} - (\lambda\rho + \dot{\rho})\,p = \lambda s + \dot{s}$$

We know that $s = s_0 \delta(t)$ where $\delta(t)$ is Dirac's delta function at time t. Because this has the delta function in it, we can attempt to solve it using the Laplace transform. The reactivity is not changed in the problem, so $\dot{\rho} = 0$, and $s = s_0 \delta(t)$, so the equation to be solved is:

$$\Lambda \ddot{p} + \left(\overline{\lambda}\Lambda + \beta - \rho\right)\dot{p} - \overline{\lambda}\rho p = \overline{\lambda} s_0 \delta(t)$$

We take the Laplace transform of the equation, obtaining [1]

$$\Lambda \mathcal{L}[\ddot{p}] + \left(\overline{\lambda}\Lambda + \beta - \rho\right)\mathcal{L}[\dot{p}] - \overline{\lambda}\rho \mathcal{L}[p] = \overline{\lambda} s_0 \mathcal{L}[\delta(t)]$$

$$\Lambda t^2 P(t) - \Lambda t p(0) - \Lambda \dot{p}(0) + \left(\overline{\lambda}\Lambda + \beta - \rho\right)[tP(t) - p(0)] + \overline{\lambda}\rho P(t) = \overline{\lambda} s_0$$

with the initial conditions that $p(0) = 1$, and, for lack of a better assumption, $\dot{p} = 0$.

$$P(t) = \frac{\overline{\lambda} s_0 + \Lambda t + \left(\overline{\lambda}\Lambda + \beta - \rho\right)}{\Lambda t^2 + \left(\overline{\lambda}\Lambda + \beta - \rho\right)t + \overline{\lambda}\rho}$$

$$P(t) = \frac{t}{t^2 + \frac{\left(\overline{\lambda}\Lambda + \beta - \rho\right)}{\Lambda} t + \frac{\overline{\lambda}}{\Lambda}\rho} + \frac{\frac{\overline{\lambda} s_0 + \left(\overline{\lambda}\Lambda + \beta - \rho\right)}{\Lambda}}{t^2 + \frac{\left(\overline{\lambda}\Lambda + \beta - \rho\right)}{\Lambda} t + \frac{\overline{\lambda}}{\Lambda}\rho}$$

which can be of the form

$$\frac{s - a}{(s-a)^2 + b^2} + c_1 \frac{b}{(s-a)^2 + b^2}$$

so finally, the power amplitude function is

$$p(t) = \exp(at)\,[\cos(bt) + c_1 sin(bt)]$$

So we have a cosine function of exponentially decreasing amplitude.

(2) Why is it sufficient to use a single group of delayed neutrons for determining δp_{as}. For a burst, the better delayed neutron source yields a better power response using an arithmetic mean, as stated in lecture. This means that, while still analytically possible to solve, the single group of delayed neutrons determines the power response well for the power burst.

(3) Relate s_0 to $\phi_0^*(\mathbf{r}, E)$ and present the desired proof. Starting with the exact PKE for an initial critical reactor, we get

$$\Lambda(t)\,\dot{p} = [\rho(t) - \beta(t)] + s_d(t) + s(t)$$

where

$$\Lambda(t) = \frac{\left\langle \phi_0^*, (1/v)\,\psi \right\rangle}{\left\langle \phi_0^*, \mathbf{F}\psi \right\rangle}$$

$$\rho(t) = \frac{\left\langle \phi_0^*, (\Delta\mathbf{F} - \Delta\mathbf{M})\,\psi \right\rangle}{\left\langle \phi_0^*, \mathbf{F}\psi \right\rangle}$$

$$\beta(t) = \frac{\left\langle \phi_0^*, \mathbf{F}_d \psi \right\rangle}{\left\langle \phi_0^*, \mathbf{F} \psi \right\rangle}$$

$$s_d(t) = \frac{\left\langle \phi_0^*, S_d \psi \right\rangle}{\left\langle \phi_0^*, \mathbf{F} \psi \right\rangle}$$

and

$$s(t) = \frac{\left\langle \phi_0^*, S \psi \right\rangle}{\left\langle \phi_0^*, \mathbf{F} \psi \right\rangle}$$

We can multiply through by $\left\langle \phi_0^*, \mathbf{F}\psi \right\rangle$, obtaining

$$\left\langle \phi_0^*, (1/v)\,\psi \right\rangle \dot{p} = \left[\left\langle \phi_0^*, (\Delta\mathbf{F} - \Delta\mathbf{M})\,\psi \right\rangle - \left\langle \phi_0^*, \mathbf{F}_d \psi \right\rangle \right] p + \left\langle \phi_0^*, S_d \psi \right\rangle + \left\langle \phi_0^*, S \psi \right\rangle$$

In this way, knowing the solution is of the form

$$p(t) = \exp(at)\left[\cos(bt) + c_1 sin(bt)\right]$$

, and

$$S = s_0 \delta(t)$$

s_0 is the related to ϕ_0^* using the weighting scheme

$$\left\langle \phi_0^*, s_0 \delta(t)\,\psi \right\rangle = \iint s_0 \delta(t)\,\psi(\mathbf{r}, E)\,\phi_0^*(\mathbf{r}, E)\,d\mathbf{r}dE$$

References

[1] W. Boyce and R. DiPrima. *Elementary Differential Equations and Boundary Value Problems*. John Wiley & Sons, Inc., 8th edition, 2005.

[2] K Ott and R Neuhold. *Introductory Nuclear Reactor Dynamics*. American Nuclear Society, La Grange Park, Illinois, 1985.