Mass Continuity Eq

Hilbert General Balance Equation

$\Psi \rightarrow$ property to be balanced

$\int \Psi \, dV \rightarrow$ total amount in V

$\mathbb{J} \rightarrow$ Flux property across surface

$-\oint_{S_m} \mathbb{J} \cdot \hat{n} \, ds \rightarrow$ net flow across surface

$\dot{\Psi}_g \rightarrow$ generation of Ψ per unit volume

$$\left\{ \begin{matrix} \text{change of } \Psi \\ \text{in time} \end{matrix} \right\} = \left\{ \begin{matrix} \text{Flux across} \\ \text{surface} \end{matrix} \right\} + \left\{ \text{generation} \right\}$$

$$\frac{D}{Dt} \int_{V_m} \Psi \, dV = -\oint_{S_m} \mathbb{J} \cdot \hat{n} \, ds + \int \dot{\Psi}_g \, dV$$

using Reynolds

$$\frac{D}{Dt} \int_{V_m} \Psi \, dV = \int_V \left[\frac{d\Psi}{dt} + \nabla \cdot (\Psi \vec{v}) \right] dV$$

and green's

$$-\oint_{S_m} \mathbb{J} \cdot \hat{n} \, ds = -\int_V \nabla \cdot \mathbb{J} \, dV$$

we get

$$\underbrace{\frac{d\Psi}{dt}}_{\text{time rate of change}} + \underbrace{\nabla \cdot (\Psi \vec{v})}_{\text{convection}} = \underbrace{-\nabla \cdot \mathbb{J}}_{\text{flux}} + \underbrace{\dot{\Psi}_g}_{\text{generation}}$$

so mass continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot \rho \vec{v} = \cancelto{0}{-\nabla \cdot \mathbb{J}} + \cancelto{0}{\dot{\rho}_g}$$

$$\frac{d\rho}{dt} + \nabla \cdot \rho \vec{v} = \frac{d\rho}{dt} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} = 0$$

set $\frac{D\rho}{Dt} = \left.\frac{\partial \rho}{\partial t}\right)_{\vec{z}} \rightarrow$ fixed mass

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \vec{v}$$

incompressible $\Rightarrow \nabla \cdot \vec{v} = 0$

Cylindrical: $\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r v_r) + \frac{1}{r} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$

Use Hilbert General Balance Eq

$$\psi = \rho\vec{v} \quad \text{(fluid so can't spin)}$$

$$\underline{\underline{J}}_\psi = \underline{\underline{\pi}} = \cancel{p\underline{\underline{\mathbb{I}}}} + \underline{\underline{\tau}}$$

$$\psi_g = \rho F = \rho\vec{g} \quad \text{(only typical volume force unless magnets)}$$

so plugging in:

$$\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \rho\vec{g}$$

$$\frac{D}{Dt}\psi = \frac{d}{dt}\psi + \vec{v}\cdot\nabla\psi \Rightarrow \frac{d}{dt}\psi = \frac{D}{Dt}\psi - \vec{v}\cdot\nabla\psi = \frac{D}{Dt}\psi - \nabla\cdot\psi\vec{v}$$

$$\rho\frac{D\vec{v}}{Dt} = \frac{\partial\rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v})$$

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \rho\vec{g}$$

Cylindrical

$$r: \rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right.$$

$$+\rho g_r$$

$$z: \rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Momentum Eq (Nat. Circ)

Start from momentum Eq (use conservative form: $\nabla \cdot \rho \vec{v}$ not $\rho(\nabla \cdot \vec{v})$)

$$\frac{\partial \rho \vec{v}}{\partial t} + \underbrace{\nabla \cdot (\rho \vec{v}\vec{v})}_{\text{non-linear}} = -\nabla p - \nabla \cdot \tau + \rho \vec{g}$$

Assumption
Density change due to thermal expansion, but it's only important in gravity terms

Use thermal expansion coefficient β

$$\beta \equiv \frac{1}{v} \left.\frac{\partial v}{\partial T}\right)_p = -\frac{1}{\rho} \left.\frac{\partial \rho}{\partial T}\right)_p$$

$$\rho = \rho(T, p) \sim \rho(T)$$

$$d\rho = -\rho \beta \, dT$$

$$\rho - \bar{\rho} = -\bar{\rho}\beta(T - \bar{T})$$

$$\bar{\rho} \frac{D\vec{v}}{Dt} = -\nabla p - \nabla \cdot \tau + [\bar{\rho} - \bar{\rho}\beta(T - \bar{T})]\vec{g}$$

ref dens.

density deviation due to thermal expansion

then, if $\vec{v}$ is small (hydrostatic)

$$-\nabla p + \bar{\rho}\vec{g} = 0$$

so

$$\bar{\rho} \frac{D\vec{v}}{Dt} = -\nabla \cdot \tau - \bar{\rho}\beta(T - \bar{T})\vec{g}$$

Use Hilbert General Balance Equation

$$\vec{v}\cdot[M.E.] \xrightarrow[\text{from Momentum to KE}]{\text{transform}} [KEE]$$

$$\rho\frac{D\left(\frac{\vec{v}^2}{2}\right)}{Dt} = -\vec{v}\cdot\nabla p - \vec{v}\cdot(\nabla\cdot\tau) + \rho\vec{v}\cdot\vec{g} \Rightarrow \begin{cases}\nabla\cdot(p\vec{v}) = p\nabla\cdot\vec{v} + \vec{v}\cdot\nabla p \\ \nabla\cdot(\tau\cdot\vec{v}) = \tau:\nabla\vec{v} + \vec{v}\cdot(\nabla\cdot\tau)\end{cases}$$

$$\underbrace{\frac{\partial\rho\left(\frac{\vec{v}^2}{2}\right)}{\partial t}}_{\text{change in KE}} + \underbrace{\nabla\cdot\left(\frac{1}{2}\rho\vec{v}^2\vec{v}\right)}_{\text{convection}} = \underbrace{-\nabla\cdot(p\vec{v})}_{\text{work by pressure}} + \underbrace{p\nabla\cdot\vec{v}}_{\text{reversible work converted to internal}} \underbrace{-\nabla\cdot(\tau\cdot\vec{v})}_{\text{work by viscous forces}} + \underbrace{\tau:\nabla\vec{v}}_{\text{irreversible work converted to internal}} + \underbrace{\rho\vec{v}\cdot}_{\text{grav wor}}$$

$$\Psi = \rho(\underset{\text{I.E.}}{u} + \underset{\text{K.E.}}{\frac{\vec{V}^2}{2}})$$

$$\vec{J} = \underset{\text{heat flux}}{\vec{q}} + \underset{\text{work done by surface forces}}{\underline{\tau} \cdot \vec{V}}$$

$$\dot{\Psi}_g = \underset{\text{gravity work}}{\rho \vec{V} \cdot \vec{g}} + \dot{q}$$

$\dot{q}$: internal heat gen → chem, electrical, nuclear

Hilbert General Balance Equation

$$\frac{\partial \rho(u + \frac{V^2}{2})}{\partial t} + \nabla \cdot \left[\rho\left(u + \frac{\vec{V}^2}{2}\right)\vec{V}\right] = -\nabla \cdot \vec{q} - \nabla \cdot (p\vec{V}) + \nabla \cdot (\underline{\tau} \cdot \vec{V}) + \rho \vec{V} \cdot \vec{g} + \dot{q}$$

$$\Downarrow$$

Thermal Energy Equation

$$[E\,E] - [K\,E]$$

$$\frac{\partial \rho(u + \frac{V^2}{2})}{\partial t} + \nabla \cdot \left[\rho\left(u + \frac{\vec{V}^2}{2}\right)\vec{V}\right] = -\nabla \cdot \vec{q} - \nabla \cdot (p\vec{V}) + \nabla \cdot (\underline{\tau} \cdot \vec{V}) + \rho \vec{V} \cdot \vec{g} + \dot{q}$$

$$-\left[\rho \frac{\partial \frac{V^2}{2}}{\partial t} = -\vec{V} \cdot \nabla p - \vec{V} \cdot [\nabla \cdot \underline{\tau}] + \rho \vec{V} \cdot \vec{g}\right]$$

$$\rho \frac{Du}{Dt} = \underset{\text{heat flux}}{-\nabla \cdot \vec{q}} - \underset{\text{reversible work}}{p \nabla \cdot \vec{v}} - \underset{\text{dissipation small, not 0}}{\underline{\tau} : \nabla \vec{V}} + \underset{\text{heat gen}}{\dot{q}}$$

$$\Downarrow$$

$$\rho \frac{Du}{Dt} = -\nabla \cdot \vec{q} - p \nabla \cdot \vec{V}$$

for most cases
(small dissipation, no internal heat gen)

Hilbert General Balance Equation
for k is the component

$\psi = \rho_k$

$\mathbb{J} = \rho_k(\vec{v}_k - \vec{v}) = \rho_k \vec{v}_{km} = \vec{J}_k$

$\dot{\psi}_g = r_k$

so $$\underbrace{\frac{\partial \rho_k}{\partial t}}_{\text{change of mass in time}} + \underbrace{\nabla \cdot \rho_k \vec{v}}_{\text{mass convection}} = \underbrace{-\nabla \cdot \vec{J}_k}_{\text{diffusion mass flux}} + \underbrace{r_k}_{\text{creation}}$$ [Mixture CE]

Hilbert GBE

$\psi = \rho \vec{v}$

$\mathbb{J} = p\,\mathbb{I} + \boldsymbol{\tau} + \Sigma \rho_k \vec{v}_k \vec{v}_k = \mathbb{T}$

$\dot{\psi}_g = \Sigma \rho_k \vec{g}$

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \rho \vec{v}\,\vec{v} = -\nabla p - \nabla \cdot \boldsymbol{\tau} - \nabla \Sigma \rho_k \vec{v}_{km} \vec{v}_{km} + \Sigma \rho_k \vec{g}$$ [Mixture ME]

Hilbert GBE

$\psi = \rho(u + \frac{\vec{v}^2}{2})$

$\mathbb{J} = \mathbb{T} \cdot \vec{v}$

$\dot{\psi}_g = \Sigma \rho_k \vec{v}_k \cdot \vec{g} + \dot{q}_k$

Types

Mechanical (τ, $\vec{g}$)

Thermal ($\vec{q}$, $\dot{q}$)

State Eq Examples → Ideal Gas ($p = RT\rho$, $u = u(T)$), Incomp ($\rho(p,T) = \rho$)

Mechanical

Inviscid (98% correct) → $\tau = 0$

Linearly viscous → $\tau_{yx} = \mu \frac{dV_x}{dy}$

could be viscous force, heat flux, mass flux, diffusion

Thermal

heat conduction → $\vec{q} = -k \nabla T$

internal heat gen → $\dot{q} = \dot{q}(x, T)$ fission, electrical resistance, etc.

Mass Diffusion

Mass diffusion flux $\vec{j}_k = \rho_k (\vec{V}_k - \vec{V})$

Diffusion model specified by constitutive eq

$$j_k \equiv \rho_k (\vec{V}_k - \vec{V}) = -\rho D \nabla \left(\frac{\rho_k}{\rho}\right) = -\rho D \nabla w_k$$

Entropy generation check

$$T ds = du - \frac{p}{\rho^2} d\rho \qquad s(u, \rho)$$

$$\rho \left\{ T \frac{Ds}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt} \right\}$$

$$\rho \frac{Ds}{Dt} = -\frac{\nabla \cdot \vec{q}}{T} - \frac{\tau : \nabla \vec{V}}{T} + \frac{\dot{q}}{T}$$

2nd Law

$$\rho \frac{Ds}{Dt} + \nabla \cdot \left(\frac{\vec{q}}{T}\right) - \frac{\dot{q}}{T} \equiv \Delta > 0$$

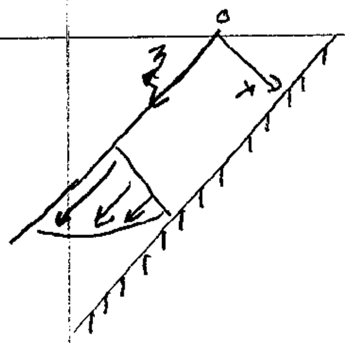
$$\Delta = -\frac{\vec{q} \cdot \nabla T}{T^2} - \frac{\tau : \nabla \vec{u}}{T} \geq 0$$

$T > 0$, $-\vec{q} \cdot \nabla T \geq 0$; $-\tau : \nabla \vec{V} \geq 0$ so satisfies entropy generation

from high to low

dissipation

Falling Laminar Film



Assumptions

→ steady state $\left(\frac{\partial}{\partial t} \to 0\right)$

→ fully developed $\left(\frac{\partial}{\partial z} \to 0\right)$ $\left(\cancel{\frac{\partial v_z(z)}{\partial y}}\right)$ $(v_z = v_z(x),\ v_y = 0$

→ Laminar Flow (use navier-stokes)

→ Adiabatic, Isothermal (no need for E Equation)

→ No y-dependence $\left(2D,\ \frac{\partial}{\partial y} \to 0\right)$

→ Incompressible $(\rho = \text{const},\ \nabla \cdot \vec{v} = 0)$

C.E.

$$\frac{\partial v_x}{\partial x} + \cancelto{0}{\frac{\partial v_y}{\partial y}} + \cancelto{0}{\frac{\partial v_z}{\partial z}} = 0 \implies \frac{\partial v_x}{\partial x} = 0$$

B.C.
$v_x = 0$ at $x = t$ (velocity of wall cannot be finite) $\Rightarrow v_x = 0$ everywhere

M.E.

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p - \nabla \cdot \tau + \rho g$$

$$z \to \rho\left\{\cancelto{0}{\frac{\partial v_z}{\partial t}} + v_x \cancelto{0}{\frac{\partial v_z}{\partial x}} + v_y \cancelto{0}{\frac{\partial v_z}{\partial y}} + v_z \cancelto{0}{\frac{\partial v_z}{\partial z}}\right\} = -\frac{\partial p}{\partial z} + \mu\left\{\frac{\partial^2 v_z}{\partial x^2} + \cancelto{0}{\frac{\partial^2 v_z}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right\} + \rho g_z$$

$$-\frac{\partial p}{\partial z} + \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g_z = 0$$

$$\rho\left\{\cancelto{0}{\frac{Dv_x}{Dt}} + v_x \cancelto{0}{\frac{\partial v_x}{\partial x}} + v_y \cancelto{0}{\frac{\partial v_x}{\partial y}} + v_z \cancelto{0}{\frac{\partial v_x}{\partial z}}\right\} = -\frac{\partial p}{\partial x} + \mu\left\{\cancelto{0}{\frac{\partial^2 v_x}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v_x}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v_x}{\partial z^2}}\right\} + \rho \vec{g}_x$$

$$-\frac{\partial p}{\partial x} + \rho \vec{g}_x = 0$$

$$\rho g_x = \rho \cos\theta\, g$$

B.C. at $x = 0$, no shear $\Rightarrow \frac{\partial v_z}{\partial x} = 0$

B.C. at $x = \delta$, $v_z = 0$

$p = p_\infty$

$$p = \rho g_x x + c_1(z)$$
$$p = p_\infty \text{ at } x_0 \Rightarrow c_1(z) = p_\infty \Rightarrow p = \rho g_x + p_\infty$$

$$-\cancelto{0}{\frac{\partial p}{\partial z}} + \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g_z = 0 \Rightarrow \frac{\partial v_z}{\partial x} = \frac{\rho}{\mu} g_z x + C_1 \Rightarrow v_z = \frac{\rho g_z}{2\mu} x^2 + C_1 x + C_2$$

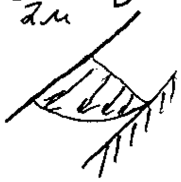
From B.C. 2

$$0 = \frac{\partial v_z(0)}{\partial x} = \frac{\rho}{\mu} g_z(0) + C_1 \Rightarrow C_1 = 0$$

From B.C. 1

$$0 = v_z(\delta) = \frac{\rho g_z}{2\mu}\delta^2 + C_2 \Rightarrow C_2 = -\frac{\rho g_z}{2\mu}\delta^2$$

$$v_z = \frac{\rho g \delta^2 \cos\theta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right] \Rightarrow$$



1) Scaling parameter

Distance

$x^* = \frac{x}{D},\ y^* = \frac{y}{D},\ z^* = \frac{z}{D}$

Time

$\tau = \frac{D}{V},\quad t^* = \frac{t}{\tau} = \frac{tV}{D}$

Velocity

$v^* = \frac{\vec{v}}{V}$

Pressure

$p^* = \frac{P - P_0}{\rho_0 V^2}$

temperature

$T^* = \frac{T - T_0}{\Delta T}$

Density

$\rho^* = \frac{\rho}{\rho_0}$

Diff Operator

$\nabla^* = D\nabla$

$\nabla^{*2} = D^2 \nabla^2$

$\frac{D}{Dt^*} = \frac{D}{V}\frac{D}{Dt}$

$\frac{\delta}{\delta t^*} = \frac{D}{V}\frac{\delta}{\delta t}$

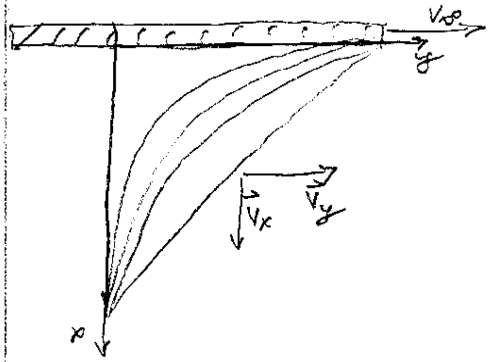
$$Re = \frac{\rho_0 V D}{\mu} = \frac{\rho V^2/D}{\mu(V/D)/D} = \frac{\text{inertia}}{\text{viscous force}}$$

$$Pr = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/\rho c_v} = \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}}$$

$$Ec = \frac{V^2}{c_p \Delta T} = \frac{\rho V^2/D}{c_p \rho \Delta T/D} = \frac{\text{K.E. convection}}{\text{enthalpy convection}}$$

$$Fr = \frac{V^2}{gD} = \frac{\rho V^2/D}{\rho g} = \frac{\text{inertia}}{\text{gravity}}$$

$$Pe = Re\, Pr$$



Assumptions

2D, No z Dependence $\left(\frac{\partial}{\partial z}=0\right)$
No velocity in z-dir $(v_z=0)$
at $t<0$, $v_x=0$, $v_y=0$, $t=0^+ \Rightarrow v_y(0,y)=\vec{V}_\infty$
$v_x=0$ @ $x=0$
incompressible $(\rho=\text{const}, \nabla\cdot\vec{v}=0)$
const properties $(\mu=\text{const})$
isothermal, adiabatic (no E.E.)
Infinite board $\left(\frac{\partial}{\partial y}=0\right)$

C.E.

$$\cancel{\frac{\partial \rho}{\partial t}}^0+\nabla\cdot\rho\vec{v}=0, \quad \nabla\cdot\vec{v}=0$$

$$\cancel{\rho\nabla\cdot\vec{v}}^0+\vec{v}\cdot\nabla\rho=0$$

$$\vec{v}\cdot\nabla\rho=0$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0 \qquad \left.\begin{aligned}\frac{\partial v_x}{\partial x}&=0\\ v_x\big|_x&=0\end{aligned}\right\}\Rightarrow v_x=0$$

$$\frac{\partial v_y}{\partial y}=0$$

$$v_y=v_y(t,x)$$

M.E.

$$y\rightarrow \rho\left\{\frac{\partial v_y}{\partial t}+\cancel{v_x\frac{\partial v_y}{\partial x}}^0+v_y\cancel{\frac{\partial v_y}{\partial y}}^0+\cancel{v_z\frac{\partial v_y}{\partial z}}^0\right\}=-\frac{\partial p}{\partial y}+\mu\left\{\frac{\partial^2 v_y}{\partial x^2}+\cancel{\frac{\partial^2 v_y}{\partial y^2}}^0+\cancel{\frac{\partial^2 v_y}{\partial z^2}}^0\right\}+\cancel{\rho g_y}$$

$$\rho\frac{\partial v_y}{\partial t}=-\frac{\partial p}{\partial y}+\mu\frac{\partial^2 v_y}{\partial x^2}$$

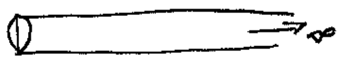
$$p=p(x)\Rightarrow\frac{\partial p}{\partial y}=0$$

$$\rho\frac{\partial v_y}{\partial t}=\mu\frac{\partial^2 v_y}{\partial x^2}$$

use $\nu=\frac{\mu}{\rho}$

$$\frac{\partial v_y}{\partial t}=\nu\frac{\partial^2 v_y}{\partial x^2}$$

becomes similar w/ heat conduction through rod



Energy Eq

$$\rho C_v\left[\frac{\partial T}{\partial t}+\cancel{\vec{v}\cdot\nabla T}^{0\ \text{solid}}\right]=k\nabla^2T+\dot{q}+p\cancel{\nabla\cdot\vec{v}}^{0\ \text{incomp}}-\cancel{\underline{\tau}:\nabla\vec{v}}^{0\ \text{dissipation}}$$

$$\rho C_v\frac{\partial T}{\partial t}=k\nabla^2T\Rightarrow\rho C_v\frac{\partial T}{\partial t}=k\frac{\partial^2T}{\partial x^2}$$

$$a=\frac{k}{\rho c_v}$$

$$\frac{\partial T}{\partial t}=a\frac{\partial^2T}{\partial x^2}$$

Thermal Penetration Depth

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}$$

$$\Theta = T - T_\infty$$

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} \quad \text{and} \quad \Theta(x,0) = T_0 - T_\infty = \Theta_0 \quad \text{and} \quad \theta(0,t)=0 \quad \text{and} \quad \theta(\infty,t)=\theta_0$$

use similarity

$$\theta = \phi(\eta)$$
$$\eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{\partial \theta/\theta_0}{\partial t} = -\frac{1}{2}\frac{\eta}{t}\phi'$$

$$\phi'' + 2\eta\phi' = 0$$

$$\phi' = C_1 \exp(-\eta^2)$$

$$\phi = C_1\int_0^\eta \exp(-\eta^2)\,ds + C_2$$

$$C_1 = \frac{1}{\int_0^\infty \exp(-\eta^2)\,d\eta}, \quad C_2 = 0$$

$$\phi = \frac{\int_0^\eta \exp(-\eta^2)\,d\eta}{\int_0^\infty \exp(-\eta^2)} = \frac{2}{\pi}\int_0^\eta \exp(-\eta^2)\,d\eta$$

$$\phi = \operatorname{erf}(\eta) \Rightarrow \frac{\theta}{\theta_0} = \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

$$t \le 0 \quad T(0,x) = T_0$$

$$t = 0^+ \quad T(0^+,0) = T_\infty$$

Temperature Gradient

$$\frac{\partial T}{\partial x} = \frac{\partial \Theta}{\partial x} = \frac{\partial \phi\Theta}{\partial x} = \Theta_0\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \phi}{\partial \eta} = \frac{\partial}{\partial \eta}\left[\operatorname{erf}(\eta)\right] = \frac{2}{\pi}\left[\int \frac{\partial}{\partial \eta}\exp(-\eta^2)\,d\eta + \frac{\partial \eta}{\partial \eta}\exp(-\eta^2)\right] = \frac{2}{\sqrt{\pi}}\exp(-\eta^2)$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}$$

$$\left.\frac{\partial T}{\partial x}\right|_0 = \Theta_0\frac{2}{\sqrt{\pi}}\exp(-\eta^2)\left.\frac{1}{\sqrt{4\alpha t}}\right|_0 = \frac{\Theta_0}{\sqrt{\pi\alpha t}} = \frac{T_0 - T_\infty}{\sqrt{\pi\alpha t}} \Rightarrow d_T = \sqrt{\pi\alpha t}$$

$$\tau^t = \rho \overline{v'v'}$$

$$\tau^\mu = -\mu\left[\nabla\vec{v} + (\nabla\vec{v})^t\right]$$

$$\tau^T = \tau^\mu + \tau^t$$

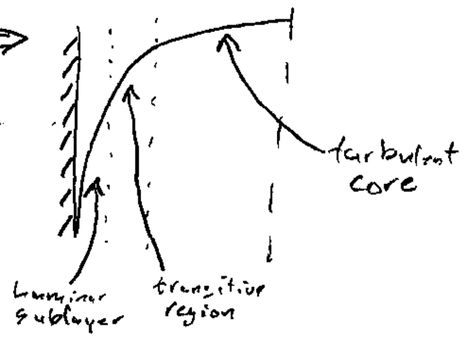
$$\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot\rho\vec{v}\vec{v} = -\nabla p - \nabla\cdot\tau^T + \rho\vec{g} \qquad [\text{Turbulent M.E.}]$$

$$\rho C_v \frac{D\overline{T}}{Dt} = \left[k\nabla^2 T - \nabla\cdot\rho c_v \overline{T'v'}\right] + g' \qquad [\text{Turbulent E.E.}]$$

Velocity Profiles

turbulent $v = v_{max}\left[1 - \frac{v}{v_0}\right]^{1/7}$ $\Longrightarrow$

Laminar $v = v_{max}\left[1 - \left(\frac{v}{v_{max}}\right)^2\right]$



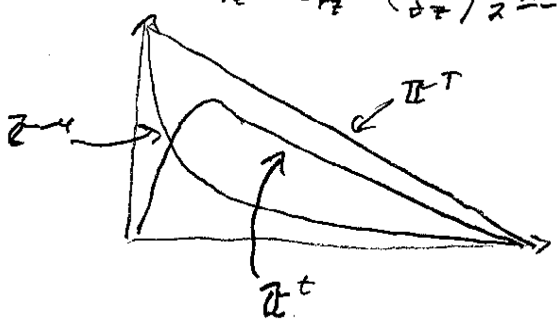
Stress in turbulent pipe flow

steady state
No Gravity
Fully developed
Axisymmetric

$$\rho\left(\frac{\partial v}{\partial t} + \bar{v}_r\frac{\partial v_z}{\partial r} + \frac{\vec{v}_\theta}{r}\frac{\partial v_z}{\partial\theta} + \bar{v}_z\frac{\partial \bar{v}_z}{\partial z}\right) = -\frac{\partial\bar{p}}{\partial t} + \rho\vec{g} + \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}^T) + \frac{1}{r}\frac{\partial\tau_{\theta z}^T}{\partial\theta} + \frac{\partial\tau_{zz}^T}{\partial z}\right]$$

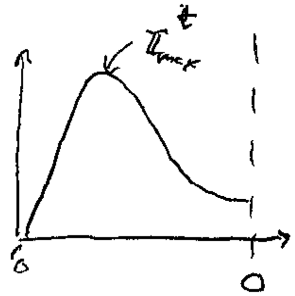
$$\frac{\partial\bar{p}}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}^T)$$

$$\tau_{rz}^T = \tau_{rz}^\mu + \tau_{rz}^t = \left(\frac{\partial p}{\partial z}\right)\frac{r}{2} = -\left(\frac{\Delta p}{L}\right)\frac{r}{2}$$



$\Longrightarrow$

$$I = \frac{\sqrt{\overline{v_z'^2}}}{V_{z,max}}$$



$$\tau^t_{max} \text{ at } \sim\left(\frac{r}{r_0} \sim 0.9\right)$$

Prandtl Turbulence

(1) → Fluid element moves but doesn't transfer any momentum until it has traveled the entire length

(2) momentum x gained by δm

$$\begin{Bmatrix} \text{x-comp} \\ \text{momentum} \\ \text{gained by} \\ \delta m \end{Bmatrix} = \delta m\, \delta v_x \;;\quad \begin{Bmatrix} \text{momentum} \\ \text{transfer} \\ \text{rate} \end{Bmatrix} = \frac{\delta m\, \delta v_x}{\delta t}$$

$$\begin{Bmatrix} \text{shear force} \\ \text{on fluid} \end{Bmatrix} = \mathbb{F} = \frac{\delta m\, \delta v_x}{\delta t}$$

$$\begin{Bmatrix} \text{shear} \\ \text{stress} \end{Bmatrix} = \tau = \frac{\mathbb{F}}{A} = \frac{1}{A}\frac{\delta m}{\delta t}\delta v_x$$

$$\delta v_x = \sim \frac{d\bar{v}_x}{dy}\ell \qquad \Rightarrow \tau^t = -\ell\,|v_y'|\,\frac{d\bar{v}_x}{dy}$$

$$\frac{1}{A}\frac{\delta m}{\delta t} = \rho\,|v_y'|$$

(3) $v_y \propto v_x$

$$|v_y'| = k_1 v_x'$$

$$v_x' = k_2\, \delta v_y = k_2\, \ell\, \frac{d\bar{v}_x}{dy} \Rightarrow \frac{\tau^t}{\rho} = -\ell^2\left|\frac{d\bar{v}_x}{dy}\right|\frac{d\bar{v}_x}{dy}$$

$\ell = ky \longrightarrow$ distance from wall,

$$\frac{\tau^t}{\rho} = -k^2 y^2\left|\frac{d\bar{v}_x}{dy}\right|\frac{d\bar{v}_x}{dy}$$

k → experimentally determined

NEAR wall $\quad \tau^t_{yx} = -\tau_w \Rightarrow \frac{d\bar{v}_x}{dy} = \frac{1}{k}\sqrt{\frac{\tau_w}{\rho}}\,\frac{1}{y} \Rightarrow v_x = \log(y)$

Introduce non dimensionalized parameters

$$v^* = \frac{\bar{v}_x}{\sqrt{\tau_w/\rho}}, \quad y^* = \frac{y\sqrt{\tau_w/\rho}}{\nu}$$

$$\frac{dv^*}{dy^*} = \frac{1}{ky^*} \Rightarrow v^* = -\frac{1}{k}\ln y^* + C_1$$

so 3 regions

1) Laminar sublayer $\quad [\tau^t = 0 \Rightarrow v^* = y^* \quad$ for $y^* < 5]$

2) Buffer Layer $\quad [v^* = -3.05 + 5\ln y^* \quad$ for $5 \le y^* \le 30]$

3) Turbulent core $\quad [v^* = 5.5 + 2.5\ln y^* \quad$ for $y^* > 30]$