NUCL 511 HMWK 3

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Consider a three-group representation of a thermal reactor spectrum composed of a fission spectrum, a $1/E$ spectrum, and a Maxwell spectrum

$$\varphi_1(E) = A_1 \frac{2}{\sqrt{\pi}} \frac{\sqrt{E}}{\left(kT_1\right)^{3/2}} e^{-E/kT_1}$$

with $kT_1 = 1.4\,MeV$ **for** $E_1 = 820.3\,keV < E < E_0 = 20\,MeV$

$$\varphi_2(E) = \frac{A_2}{E}$$

for $E_2 = 0.1\,eV < E \leq E_1$

$$\varphi_3(E)dE = A_3 \frac{E}{(kT_3)^2} e^{-E/kT_3}$$

with $kT_3 = 0.0253\,eV$ **for** $E_3 = 0 \leq E \leq E_2$

Find the normalization constants A_1, A_2, and A_3 such that the spectrum is continuous at the group boundaries and the integration over the whole energy range is unity. Evaluate the group-wise integrals numerically if necessary. We have one integral equation with three boundary conditions for this problem. To ensure the integration over the whole energy range is unity, we must have

$$\int_0^{E_0} \varphi(E)dE = 1$$

Which, because the spectrum is discrete, becomes

$$\int_{E_3}^{E_2} \varphi_3(E)dE + \int_{E_2}^{E_1} \varphi_2(E)dE + \int_{E_1}^{E_0} \varphi_1(E)dE = 1$$

$$\int_0^{0.1\,eV} A_3 \frac{E}{(kT_3)^2} e^{-E/kT_3} dE + \int_{0.1\,eV}^{820.3\,keV} \frac{A_2}{E} dE + \int_{820.3\,keV}^{20\,MeV} A_1 \frac{2}{\sqrt{\pi}} \frac{\sqrt{E}}{\left(kT_1\right)^{3/2}} e^{-E/kT_1} dE = 1$$

To integrate the first term, φ_3, we can use integration by parts

$$\frac{A_3}{\left(kT_3\right)^2} \int_{E_3}^{E_2} E e^{-E/kT_3} dE$$

with

$$u = E,\ v = ?$$

$$du = ?,\ dv = e^{-E/kT_3} dE$$

and we find that $du = dE$ and $v = -kT_3 e^{-E/kT_3}$, so

$$\int u dv = uv - \int v du$$

$$\int Ee^{-E/kT_3} = -EkT_3e^{-E/kT_3} - \int -kT_3e^{-E/kT_3}dE = kT_3Ee^{-E/kT_3} + kT_3\int e^{-E/kT_3}dE = kT_3Ee^{-E/kT_3} - (kT_3)^2 e^{-E/kT_3}$$

so the integral is

$$\frac{A_3}{(kT_3)^2}\left[kT_3Ee^{-E/kT_3} - (kT_3)^2 e^{-E/kT_3}\right]_{E_3}^{E_2}$$

The second integral is simply done, and is

$$A_2\left[\ln(E)\right]_{E_2}^{E_1}$$

And finally, the third integration is

$$\frac{A_1}{(kT_1)^{3/2}}\frac{2}{\sqrt{\pi}}\int_{E_1}^{E_0}\sqrt{E}e^{-E/kT_1}dE$$

The numerical integration included in MATLAB's Symbolic Toolbox was used, which utilizes an adaptive Simpson quadrature method [2]. This method estimates the integral as a series of trapezoids under the curve (called Simpon's Rule), but also calculates the derivative of the function, and increases the amount of trapezoids used in quickly varying regions of the function [1, p. 458, 505-506]. We can determine the integral equation to be equivalent to

$$\left(9.173\times 10^{-10}\right)A_1 + (15.920)\,A_2 + (0.905)\,A_3 = 1$$

We also have the continuity conditions, such that

$$\varphi_3(E_2) = \varphi_2(E_2)$$

$$A_3\frac{E}{(kT_3)^2}e^{-E/kT_3}\bigg|_{E_2} = \frac{A_2}{E}\bigg|_{E_2}$$

$$(3.000)\,A_3 = (10.0)\,A_2$$

and

$$\varphi_2(E_1) = \varphi_1(E_1)$$

$$A_1\frac{2}{\sqrt{\pi}}\frac{\sqrt{E}}{(kT_1)^{3/2}}e^{-E/kT_1}\bigg|_{E_1} = \frac{A_2}{E}\bigg|_{E_1}$$

$$\left(4.146\times 10^{-16}\right)A_1 = \left(1.21\times 10^{-6}\right)A_2$$

These three equations help us solve for A_1, A_2, and A_3. Reiterating

$$\left(9.173\times 10^{-10}\right)A_1 + (15.920)\,A_2 + (0.905)\,A_3 = 1$$

$$(3.000)\,A_3 = (10.0)\,A_2$$

$$\left(4.146\times 10^{-16}\right)A_1 = \left(1.21\times 10^{-6}\right)A_2$$

This can be written as

$$\begin{bmatrix} 9.173\times 10^{-10} & 15.920 & 0.905 \\ 0 & 10 & -3 \\ 4.146\times 10^{-16} & -1.21\times 10^{-6} & 0 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which can be solved any number of ways, including backdivision. Again, MATLAB's symbolic Toolbox allows us to solve this without explicitly writing a numerical method. The toolbox uses Cholesky Factorization [2] to

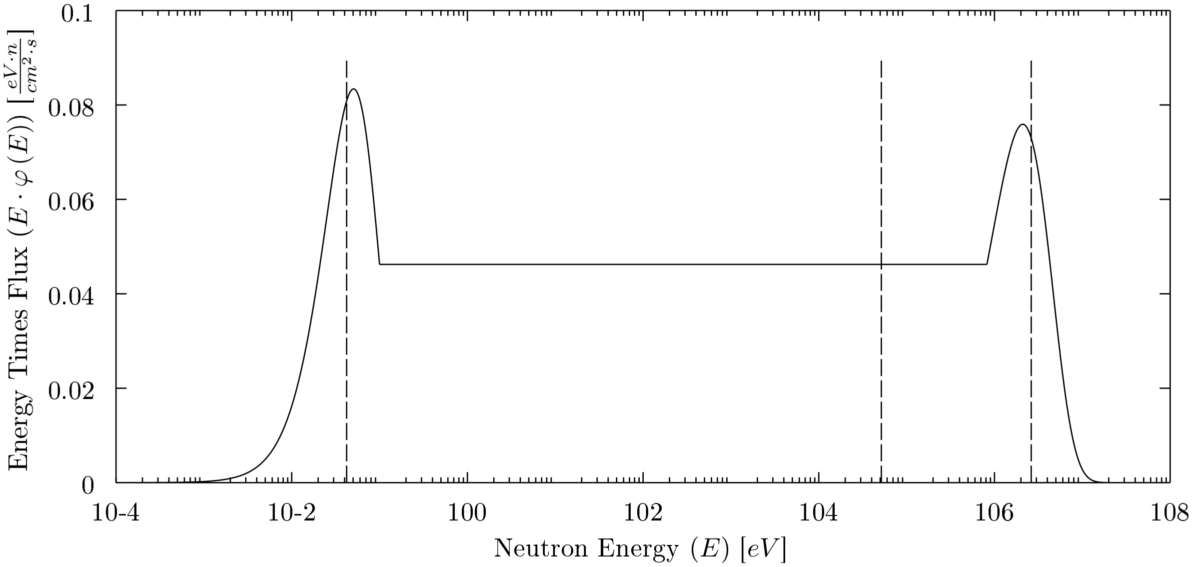

Figure 1: Energy Spectrum given by φ_1, φ_2, and φ_3

invert the matrix and multiply that by the solution vector. The Cholesky factorization decomposes the matrix into a lower triangular matrix and is more efficient than LU decomposition [1, p. 216]. This gives the constants

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0.1641 \\ 0.04623 \\ 0.1541 \end{bmatrix}$$

Using these constants, we can plot the spectrum, as shown in Figure 1. This spectrum shows the typical shape of a fission spectrum, with a fast and thermal hump, and also demonstrates the continuity of the solution.

Find the average velocities in individual groups ($\bar{v_1}$, $\bar{v_2}$, and $\bar{v_3}$). Finding the average energies can be done using

$$\bar{E} = \frac{\int \varphi(E) E dE}{\int \varphi(E) dE}$$

At any energy, the velocity is

$$\bar{v} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2c^2 E}{939.5\,MeV}}$$

The average velocity can be calculated by finding

$$\overline{v} = \frac{\int \varphi(E) v(E) dE}{\int \varphi(E) dE} = \frac{\int \varphi(E) \sqrt{\frac{2c^2 E}{939.5\,MeV}} dE}{\int \varphi(E) dE}$$

When calculated for each group, we get the values

$$\bar{v_1} = 2.15 \times 10^7 \, \frac{m}{s}$$

$$\bar{v_2} = 1.57 \times 10^6 \, \frac{m}{s}$$

$$\bar{v_3} = 2711.89 \, \frac{m}{s}$$

The average energies have been plotted on Figure 1 as dotted lines for visual representation. Note that the average energy does not correspond to the average velocity, because velocity is a function of the square root of energy.

Find the one-group values of $\bar{v}$ and $\overline{1/v}$ using the three-group values $\bar{v_1}$, $\bar{v_2}$, and $\bar{v_3}$. We have the three group velocities from the three group values above, and when stitched together, we get

$$v = \begin{cases} \overline{v_1} & E_1 \leq E < E_0 \\ \overline{v_2} & E_2 \leq E < E_1 \\ \overline{v_3} & E < E_2 \end{cases}$$

and the average one-group velocity is given by

$$\overline{v} = \frac{\overline{v_1} \cdot \Delta v_1 + \overline{v_2} \cdot \Delta v_2 + \overline{v_3} \cdot \Delta v_3}{\Delta v + \Delta v_2 + \Delta v_3}$$

with

$$\Delta v_i = \sqrt{\frac{2c^2 E_{i-1}}{939.5\, MeV}} - \sqrt{\frac{2c^2 E_i}{939.5\, MeV}}$$

this gives the average one group velocity as

$$\overline{v} = 1.74 \times 10^7\, \frac{m}{s} = 1.74 \times 10^9\, \frac{cm}{s}$$

This can also be done for $\overline{1/v}$

$$\overline{1/v} = \overline{v} = \frac{\overline{1/v_1} \cdot \Delta v_1 + \overline{1/v_2} \cdot \Delta v_2 + \overline{1/v_3} \cdot \Delta v_3}{\Delta v + \Delta v_2 + \Delta v_3} = 7.48 \times 10^{-5}\, \frac{s}{m} = 7.48 \times 10^{-7}\, \frac{s}{cm}$$

Define a one-group $\nu\Sigma_f$ based on the three group-values $\nu\Sigma_{f1} = 0.017\, cm^{-1}$, $\nu\Sigma_{f2} = 0.015\, cm^{-1}$, and $\nu\Sigma_{f3} = 0.3\, cm^{-1}$. We can average the fission cross section times the fission neutron yield in the same way that velocity was averaged, because cross sections depend on the velocity of the particle. Thus, the fission cross sections times the fission yield for one group is given as

$$\overline{\nu\Sigma_f} = \frac{\overline{\nu\Sigma_{f1}} \cdot \Delta v_1 + \overline{\nu\Sigma_{f2}} \cdot \Delta v_2 + \overline{\nu\Sigma_{f3}} \cdot \Delta v_3}{\Delta v + \Delta v_2 + \Delta v_3} = 0.0773\, cm^{-1}$$

This seems reasonable because it is skewed upward by the thermal region, but the low yield at most energies keeps it below $0.1\, cm^{-1}$.

Calculate the generation time Λ with $\bar{v}$ and $1\bar{/}v$. Discuss the results. The generation time is given as [3, p. 23]

$$\Lambda = \frac{1}{\overline{v}\nu\Sigma_f}$$

With the values developed so far in this assignment, this could be calculated as either

$$\Lambda = \frac{1}{\overline{v}} \cdot \frac{1}{\overline{\nu\Sigma_f}} = \frac{1}{1.74 \times 10^9\, \frac{cm}{s}} \cdot \frac{1}{0.0773\, cm^{-1}} = 7.4348 \times 10^{-9}\, s$$

or as

$$\Lambda = \overline{1/v} \cdot \frac{1}{\overline{\nu\Sigma_f}} = 7.48 \times 10^{-7}\, \frac{cm}{s} \cdot \frac{1}{0.0773\, cm^{-1}} = 9.68 \times 10^{-6}\, s$$

There are several orders of magnitude difference in the results provided when different averaging is used. The generation time is developed by Ott by finding first the mean free path for fission ($1/\Sigma_f$), and then dividing that by the fission yield (ν). This provides the average distance a neutron travels for each neutron it creates by fission. By then dividing by the velocity (v), this is converted to a time. As such, the first averaging method makes more physical sense. For prompt neutrons, a very short neutron lifetime is expected, so the answer on the order of 10^{-9} is not unreasonable.

References

[1] Brian Bradie. *A Friendly Introduction to Numerical Analysis*. Pearson Prentice Hall, Upper Saddle River, New Jersey, 2006.

[2] Mathworks. Symbolic Toolbox: User's Guide (r2013a). Technical report, 2013.

[3] K Ott and R Neuhold. *Introductory Nuclear Reactor Dynamics*. American Nuclear Society, La Grange Park, Illinois, 1985.