NUCL 510
Nuclear Reactor Theory

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Lecture Note 7

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Two-Group Diffusion Equation

- Two-group diffusion equation

$$\begin{bmatrix} -\nabla\cdot D_1(\vec r)\nabla+\Sigma_{r1}(\vec r) & 0 \\ -\Sigma_{s12}(\vec r) & -\nabla\cdot D_2(\vec r)\nabla+\Sigma_{r2}(\vec r) \end{bmatrix}\begin{bmatrix}\phi_1(\vec r)\\ \phi_2(\vec r)\end{bmatrix}=\lambda\begin{bmatrix}\nu\Sigma_{f1}(\vec r) & \nu\Sigma_{f2}(\vec r)\\ 0 & 0\end{bmatrix}\begin{bmatrix}\phi_1(\vec r)\\ \phi_2(\vec r)\end{bmatrix}$$

$$\Sigma_{r1}=\Sigma_{a1}+\Sigma_{s12},\quad \Sigma_{r2}=\Sigma_{a2}$$

- Two-group diffusion equation with constant cross sections

$$\begin{bmatrix} -D_1\nabla^2+\Sigma_{r1} & 0 \\ -\Sigma_{s12} & -D_2\nabla^2+\Sigma_{a2} \end{bmatrix}\begin{bmatrix}\phi_1(\vec r)\\ \phi_2(\vec r)\end{bmatrix}=\lambda\begin{bmatrix}\nu\Sigma_{f1} & \nu\Sigma_{f2}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\phi_1(\vec r)\\ \phi_2(\vec r)\end{bmatrix}$$

- Fission source in terms of four factors

$$S_f(\vec r)=\nu\Sigma_{f1}\phi_1(\vec r)+\nu\Sigma_{f2}\phi_2(\vec r)=\varepsilon\eta f\,\Sigma_{a2}\phi_2(\vec r)=\frac{k_\infty}{p}\Sigma_{a2}\phi_2(\vec r)$$

$$f=\frac{\Sigma_{a2}^F\phi_2}{\Sigma_{a2}\phi_2}=\frac{\Sigma_{a2}^F}{\Sigma_{a2}},\quad \eta=\frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a2}^F\phi_2}=\frac{\nu\Sigma_{f2}}{\Sigma_{a2}^F},\quad \varepsilon=\frac{\nu\Sigma_{f1}\phi_1+\nu\Sigma_{f2}\phi_2}{\nu\Sigma_{f2}\phi_2}$$

- Slowing-down source from group 1 to group 2

$$\Sigma_{12}\phi_1(\vec r)=\frac{\Sigma_{12}}{\Sigma_{r1}}\Sigma_{r1}\phi_1(\vec r)=p\Sigma_{r1}\phi_1(\vec r)$$

$$\frac{\Sigma_{s12}}{\Sigma_{r1}}=\frac{\Sigma_{s12}}{\Sigma_{a1}+\Sigma_{s12}}=p$$

2-G Diffusion Equation in Terms of Four Factors

- Two-group diffusion equation in terms of four factors

$$\begin{bmatrix} -D_1\nabla^2+\Sigma_{r1} & 0 \\ -p\Sigma_{r1} & -D_2\nabla^2+\Sigma_{a2} \end{bmatrix}\begin{bmatrix} \phi_1(\vec{r}) \\ \phi_2(\vec{r}) \end{bmatrix}=\lambda\begin{bmatrix} 0 & \frac{k_\infty}{p}\Sigma_{a2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(\vec{r}) \\ \phi_2(\vec{r}) \end{bmatrix}$$

$$\begin{bmatrix} -\nabla^2+\frac{\Sigma_{r1}}{D_1} & 0 \\ -p\frac{\Sigma_{r1}}{D_2} & -\nabla^2+\frac{\Sigma_{a2}}{D_2} \end{bmatrix}\begin{bmatrix} \phi_1(\vec{r}) \\ \phi_2(\vec{r}) \end{bmatrix}=\lambda\begin{bmatrix} 0 & \frac{k_\infty}{p}\frac{\Sigma_{a2}}{D_1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(\vec{r}) \\ \phi_2(\vec{r}) \end{bmatrix}$$

$$\begin{bmatrix} -\nabla^2+\frac{1}{L_1^2} & 0 \\ -p\frac{\Sigma_{r1}}{D_2} & -\nabla^2+\frac{1}{L_2^2} \end{bmatrix}\begin{bmatrix} \phi_1(\vec{r}) \\ \phi_2(\vec{r}) \end{bmatrix}=\lambda\begin{bmatrix} 0 & \frac{k_\infty}{p}\frac{\Sigma_{a2}}{D_1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(\vec{r}) \\ \phi_2(\vec{r}) \end{bmatrix}$$

$L_1^2 = D_1/\Sigma_{r1}$ (fast neutron diffusion area $\approx$ Fermi age)

$L_2^2 = D_2/\Sigma_{a2}$ (thermal neutron diffusion area)

Separation of Space and Energy Variables

- Separation of space and energy dependencies

$$\begin{bmatrix} \phi_1(\vec{r}) \\ \phi_2(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \phi(\vec{r})$$

$$\begin{bmatrix} -\nabla^2 + \frac{1}{L_1^2} & -\lambda \frac{k_\infty}{p} \frac{\Sigma_{a2}}{D_1} \\ -p\frac{\Sigma_{r1}}{D_2} & -\nabla^2 + \frac{1}{L_2^2} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \phi(\vec{r}) = 0 \qquad \begin{bmatrix} -\frac{\nabla^2\phi(\vec{r})}{\phi(\vec{r})} + \frac{1}{L_1^2} & -\lambda \frac{k_\infty}{p} \frac{\Sigma_{a2}}{D_1} \\ -p\frac{\Sigma_{r1}}{D_2} & -\frac{\nabla^2\phi(\vec{r})}{\phi(\vec{r})} + \frac{1}{L_2^2} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = 0$$

$$-\frac{\nabla^2\phi(\vec{r})}{\phi(\vec{r})} = B^2 \qquad \nabla^2\phi(\vec{r}) + B^2\phi(\vec{r}) = 0$$

- Spectrum equation

$$\begin{bmatrix} B^2 + \frac{1}{L_1^2} & -\lambda \frac{k_\infty}{p} \frac{\Sigma_{a2}}{D_1} \\ -p\frac{\Sigma_{r1}}{D_2} & B^2 + \frac{1}{L_2^2} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = 0$$

This is a homogeneous equation, and thus has a non-trivial solution if and only if the coefficient matrix is singular.

Separation of Spatial Variables

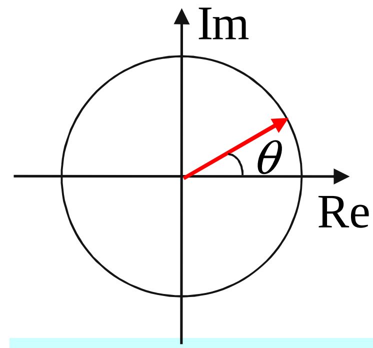
- Spatial eigenvalue equation

$$\nabla^2\phi(\vec{r}) + B^2\phi(\vec{r}) = 0$$

- Separation of variables

$$\phi(\vec{r}) = X(x)Y(y)Z(z)$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + B^2 = 0$$



$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{\partial^2 X}{\partial x^2} + B_x^2 X = 0, \quad \frac{\partial^2 Y}{\partial y^2} + B_y^2 Y = 0, \quad \frac{\partial^2 Z}{\partial z^2} + B_z^2 Z = 0, \quad B^2 = B_x^2 + B_y^2 + B_z^2$$

$$X(x) = A_x e^{\pm iB_x x}, \quad Y(y) = A_y e^{\pm iB_y y}, \quad Z(z) = A_z e^{\pm iB_z z}$$

  – B_x, B_y, and B_z are determined from the imposed boundary conditions
  – Fundamental mode flux shape in a rectangular parallelepiped with width a, length b, and height c

$$\phi(x,y,z) = A\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\cos\frac{\pi z}{c}$$

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Two-Group Criticality Equation (1)

- Condition for non-trivial solution

$$\begin{vmatrix} B^2+\dfrac{1}{L_1^2} & -\lambda\dfrac{k_\infty}{p}\dfrac{\Sigma_{a2}}{D_1} \\ -p\dfrac{\Sigma_{r1}}{D_2} & B^2+\dfrac{1}{L_2^2} \end{vmatrix} = \left(B^2+\frac{1}{L_1^2}\right)\left(B^2+\frac{1}{L_2^2}\right)-\lambda k_\infty\frac{\Sigma_{a2}}{D_1}\frac{\Sigma_{r1}}{D_2}=0$$

$$\left(B^2+\frac{1}{L_1^2}\right)\left(B^2+\frac{1}{L_2^2}\right)-\lambda\frac{k_\infty}{L_1^2L_2^2}=0$$

$$(1+B^2L_1^2)(1+B^2L_2^2)-\lambda k_\infty=0$$

- Multiplication factor for given composition and geometry
  - B^2 is the geometrical buckling determined from the given geometry

$$k=\frac{1}{\lambda}=\frac{k_\infty}{(1+B^2L_1^2)(1+B^2L_2^2)}=k_\infty P_1^{NL}P_2^{NL}$$

$$P_g^{NL}=\frac{1}{1+B^2L_g^2} \quad \text{(non-leakage probability of group } g \text{ neutrons)}$$

Two-Group Criticality Equation (2)

- Fast to thermal flux ratio
  - Eigenvector from the second equation

$$-p\frac{\Sigma_{r1}}{\Sigma_{a2}}\varphi_1+(1+B^2L_2^2)\varphi_2=0 \quad\Rightarrow\quad \frac{\varphi_2}{\varphi_1}=\frac{p}{1+B^2L_2^2}\frac{\Sigma_{r1}}{\Sigma_{a2}}=\frac{1}{1+B^2L_2^2}\frac{\Sigma_{s12}}{\Sigma_{a2}}$$

  - Thermal absorption rate

$$\Sigma_{a2}\phi_2(\vec r)=\frac{\Sigma_{s12}}{1+B^2L_2^2}\phi_1(\vec r)$$

- Non-leakage probability
  - Non-leakage probability during slowing-down (τ: Fermi age for thermal neutrons)

$$e^{-B^2\tau}\simeq 1-B^2\tau\simeq\frac{1}{1+B^2\tau}\quad \text{for } B^2<<1 \quad\Rightarrow\quad \tau\simeq L_1^2$$

  - Combined non-leakage probabilities and migration area

$$P_{NL}=\frac{1}{1+B^2L_1^2}\frac{1}{1+B^2L_2^2}=\frac{1}{1+B^2(L_1^2+L_2^2)+B^4L_1^2L_2^2}\simeq\frac{1}{1+B^2(L_1^2+L_2^2)}=\frac{1}{1+B^2M^2}$$

$$M^2=L_1^2+L_2^2$$

Migration Length and Leakage Effect

- Modified one-group critical equation
  - Diffusion area L^2 is replaced with migration area M^2

$$k = \frac{k_\infty}{1 + M^2 B^2}$$

- Migration length

Type	$L_1 = \sqrt{\tau}, cm$	L_2, cm	M, cm
PWR	7.36	1.96	7.62
CANDU	11.6	15.6	19.4
HTGR	17.1	10.6	20.2

Low Σ_{a2} for CANDU, and high D_1 for HTGR

- Non-leakage probability
  - Right cylinder with $D=H$

$$B^2 = \frac{2.405^2}{(H/2)^2} + \frac{\pi^2}{H^2} \simeq \frac{33}{H^2}$$

$$k = \frac{k_\infty}{1 + 33(M/H)^2}$$

PWR $\quad H = 100cm, \; k = \dfrac{k_\infty}{1 + 33(7.62/100)^2} = \dfrac{k_\infty}{1.19}$

$H = 200cm, \; k = \dfrac{k_\infty}{1.048}$

Material Buckling and Fundamental Spectrum

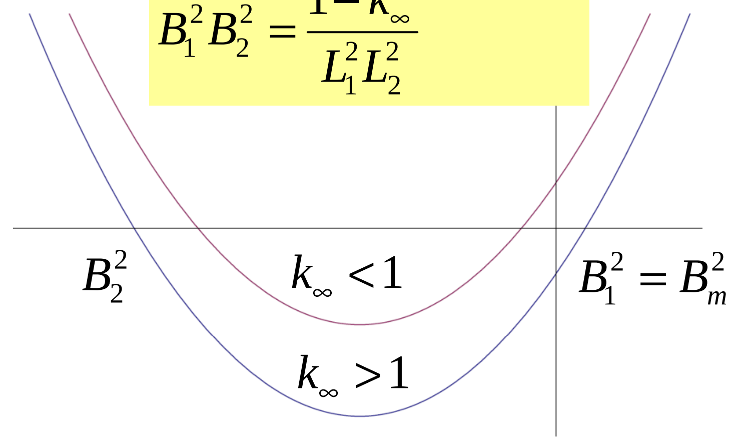
- Condition for non-trivial solution ($\lambda=1$)

$$\begin{vmatrix} B^2+\dfrac{1}{L_1^2} & -\dfrac{k_\infty}{p}\dfrac{\Sigma_{a2}}{D_1} \\ -p\dfrac{\Sigma_{r1}}{D_2} & B^2+\dfrac{1}{L_2^2} \end{vmatrix} = \frac{(1+B^2L_1^2)(1+B^2L_2^2)-k_\infty}{L_1^2L_2^2}=0$$

$$L_1^2L_2^2(B^2)^2+(L_1^2+L_2^2)B^2+1-k_\infty=0$$

$$B_1^2+B_2^2=-\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)$$

$$B_1^2B_2^2=\frac{1-k_\infty}{L_1^2L_2^2}$$



- Material and higher bucklings

$$B^2=\frac{-(L_1^2+L_2^2)\pm\left[(L_1^2+L_2^2)^2-4L_1^2L_2^2(1-k_\infty)\right]^{1/2}}{2L_1^2L_2^2}$$

$$1+B^2L_2^2=\frac{(L_1^2-L_2^2)\pm\left[(L_1^2+L_2^2)^2-4L_1^2L_2^2(1-k_\infty)\right]^{1/2}}{2L_1^2}$$

$$\begin{array}{lll} k_\infty>1, & B_m^2>0 & B_2^2<0 \\ k_\infty<1, & B_m^2<0 & B_2^2<0 \end{array}$$

- Spectra

$$\left.\frac{\varphi_2}{\varphi_1}\right|_m=\frac{1}{1+B_m^2L_2^2}\frac{\Sigma_{s12}}{\Sigma_{a2}}>0 \qquad \left.\frac{\varphi_2}{\varphi_1}\right|_{(2)}=\frac{1}{1+B_2^2L_2^2}\frac{\Sigma_{s12}}{\Sigma_{a2}}<0$$

Space-Dependent Flux Components

- Space-dependent flux components (symmetrical slab problem)

$k_\infty > 1$

$$\frac{d^2}{dx^2}\phi^{(m)}(x) + B_m^2\phi^{(m)}(x) = 0 \qquad \phi^{(m)}(x) = \cos B_m x$$

$$\frac{d^2}{dx^2}\phi^{(2)}(x) - |B_2^2|\phi^{(2)}(x) = 0 \qquad \phi^{(2)}(x) = \cosh|B_2|x$$

$k_\infty < 1$

$$\frac{d^2}{dx^2}\phi^{(m)}(x) - |B_m^2|\phi^{(m)}(x) = 0 \qquad \phi^{(m)}(x) = \cosh|B_m|x$$

$$\frac{d^2}{dx^2}\phi^{(2)}(x) - |B_2^2|\phi^{(2)}(x) = 0 \qquad \phi^{(2)}(x) = \cosh|B_2|x$$

- Asymptotic flux

$$\boldsymbol{\varphi}^{as} = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_m \cos B_m x \quad \text{for } k_\infty > 1, \quad \boldsymbol{\varphi}^{as} = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_m \cosh|B_m|x \quad \text{for } k_\infty < 1$$

  - It is the material buckling (not the geometric buckling) which quantifies the intrinsic neutron balance and thus the corresponding curvature
  - The higher mode solution provides the adjustment needed to satisfy the boundary condition.