NUCL 511
Nuclear Reactor Theory and Kinetics

Lecture Note 7

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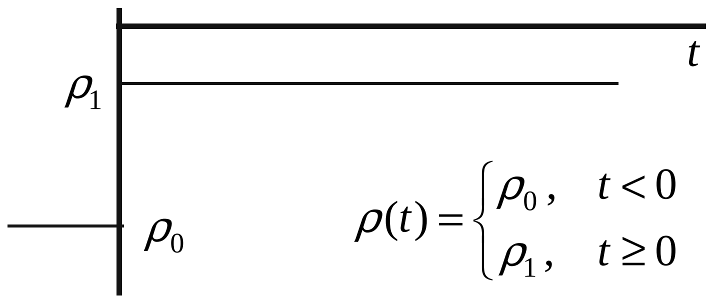
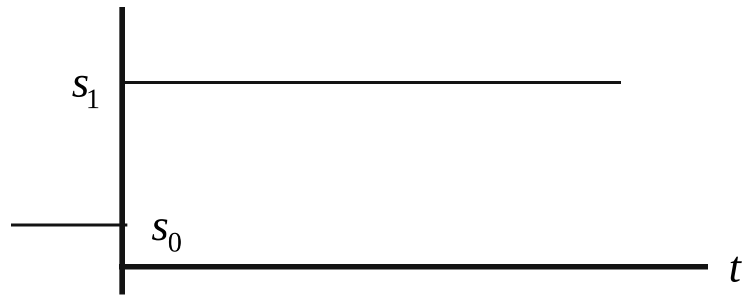
Prompt Jump in Subcritical Reactor

- Transient due to a step change in reactivity and source



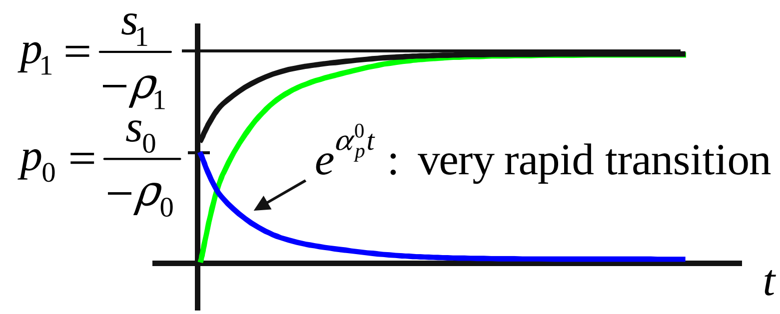
$$\rho(t) = \begin{cases} \rho_0\,, & t < 0 \\ \rho_1\,, & t \geq 0 \end{cases}$$



- Kinetics without delayed neutrons

$$\dot{p} = \frac{\rho_1}{\Lambda} p + \frac{s_1}{\Lambda} = \alpha_p^0 p + \frac{s_1}{\Lambda}, \quad p_0 = \frac{s_0}{-\rho_0}$$

– α_p^0 includes 0 to signify no consideration of delayed neutrons (large negative)



$$\dot{p} - \alpha_p^0 p = \frac{s_1}{\Lambda} \quad \Rightarrow \quad \frac{d}{dt}\left(p e^{-\alpha_p^0 t}\right) = \frac{s_1}{\Lambda} e^{-\alpha_p^0 t} \quad \Rightarrow \quad p(t) e^{-\alpha_p^0 t} - p_0 = \frac{s_1}{\Lambda \alpha_p^0}\left(1 - e^{-\alpha_p^0 t}\right)$$

$$p(t) = p_0 e^{\alpha_p^0 t} + \frac{s_1}{\Lambda(-\alpha_p^0)}\left(1 - e^{\alpha_p^0 t}\right) = p_0 e^{\alpha_p^0 t} + \frac{s_1}{-\rho_1}\left(1 - e^{\alpha_p^0 t}\right)$$

Prompt Jump in Subcritical Reactor

- CDS approximation

$$\dot{p} = \frac{\rho_1 - \beta}{\Lambda} p + \frac{s_{d0} + s_1}{\Lambda} = \alpha_p p + \frac{s_{d0} + s_1}{\Lambda} \quad \Rightarrow \quad p(t) = p_0 e^{\alpha_p t} + \frac{s_{d0} + s_1}{\beta - \rho_1}(1 - e^{\alpha_p t})$$

- Differences from the kinetics without delayed neutrons
  - $\alpha_p \neq \alpha_p^0$ but still very large and negative
  - Source multiplication factor

$$\frac{s_{d0} + s_1}{\beta - \rho_1} \neq \frac{s_1}{-\rho_1}$$

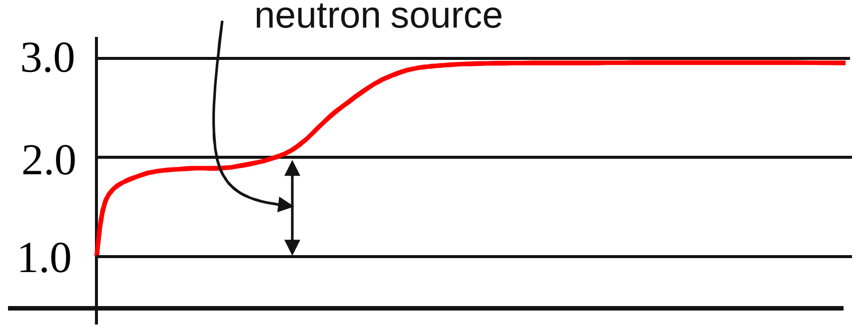
If $s_1 = s_0 = -\rho_0 p_0$, $\quad \dfrac{s_{d0} + s_1}{\beta - \rho_1} = \dfrac{\beta p_0 - \rho_0 p_0}{\beta - \rho_1} = \dfrac{1 - \rho_{0\$}}{1 - \rho_{1\$}} p_0$

For example, consider a transient from $\rho_0 = -3\beta$, $p_0 = 1$ (i.e. $s_0 = 3\beta$) to $\rho_1 = -1\beta$, $s_1 = s_0$

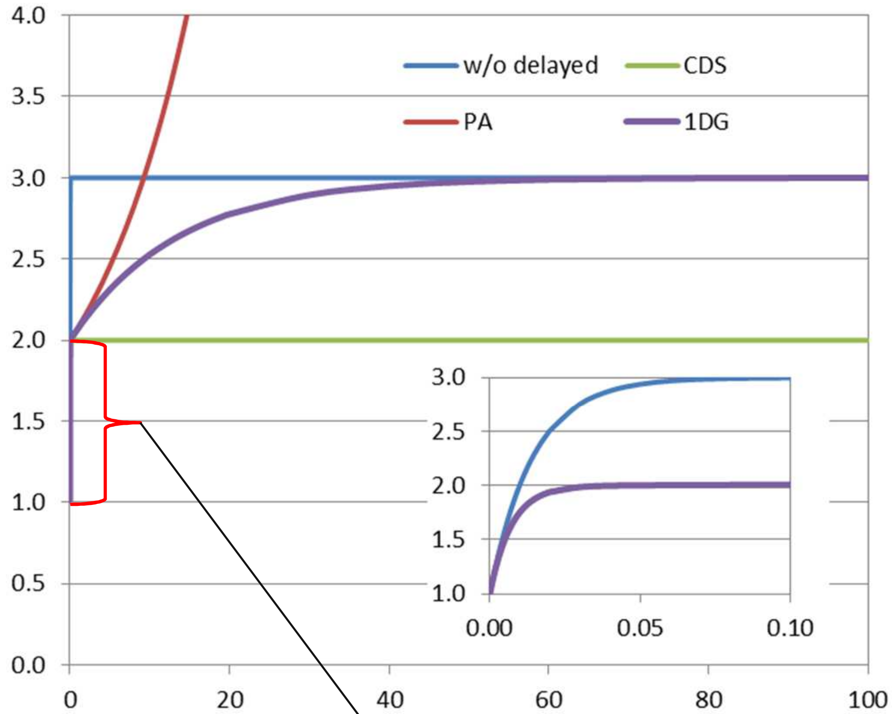
$$\frac{1 - \rho_{0\$}}{1 - \rho_{1\$}} = \frac{1 - (-3)}{1 - (-1)} = 2$$

$$\frac{s_1}{-\rho} = \frac{3\beta}{-1\beta} = 3 \quad \text{(new steady state)}$$

Prompt jump: prompt response of flux (power) to a reactivity or source change with unchanged delayed neutron source

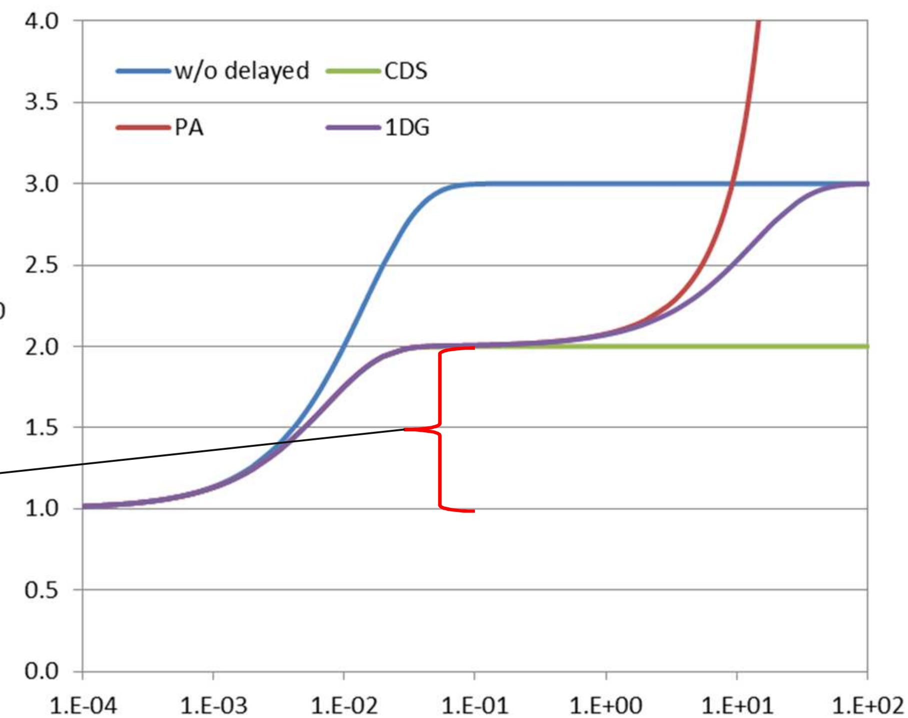


Prompt Jump in Subcritical Reactor



$$\Lambda = 10^{-4}\ \text{s}, \quad \lambda = 0.15\ \text{s}^{-1}$$
$$\rho_0 = -3\beta, \quad s_0 = 3\beta$$
$$\rho_1 = -1\beta, \quad s_1 = s_0$$

Prompt jump: prompt response of flux (power) to a reactivity or source change with unchanged delayed neutron source



Prompt Jump in Critical Reactor

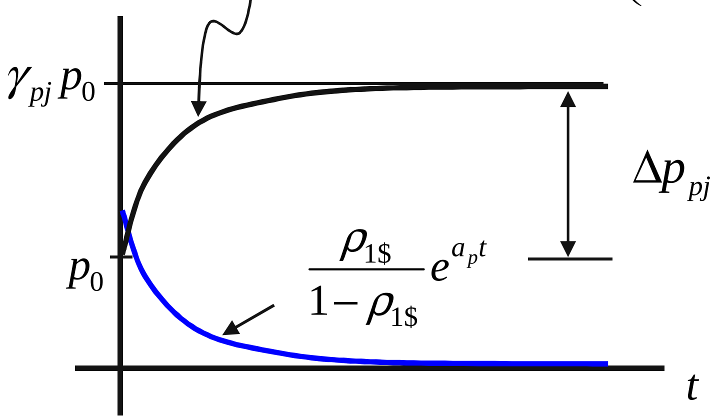
- Step reactivity insertion into an initially critical reactor $\rho_0=0,\ s_0=0,\ \rho_1<\beta$

$$\dot{p}=\frac{\rho_1-\beta}{\Lambda}p+\frac{1}{\Lambda}\sum_{k=1}^{K}\lambda_k\zeta_k \xrightarrow{\text{CDS}} \dot{p}=\alpha_p p+\frac{\beta p_0}{\Lambda}$$

$$p(t)=p_0e^{\alpha_p t}+\frac{\beta p_0}{\Lambda(-\alpha_p)}(1-e^{\alpha_p t})=p_0e^{\alpha_p t}+\frac{\beta p_0}{\beta-\rho_1}(1-e^{\alpha_p t})=p_0e^{\alpha_p t}+p_{pj}(1-e^{\alpha_p t})$$

$$p_{pj}=\frac{\beta}{\beta-\rho_1}p_0=\frac{1}{1-\rho_{1\$}}p_0=\gamma_{pj}p_0$$

$$p(t)=p_0[\gamma_{pj}+(1-\gamma_{pj})e^{\alpha_p t}]=p_0\left(\frac{1}{1-\rho_{1\$}}-\frac{\rho_{1\$}}{1-\rho_{1\$}}e^{\alpha_p t}\right)$$



$$\Delta p_{pj}=(\gamma_{pj}-1)p_0=\frac{\rho_{1\$}}{1-\rho_{1\$}}p_0$$

Kinetics with One-Group Delayed Neutrons

- Reactivity insertion into an initially critical reactor

$$\Lambda \dot{p} = [\rho(t) - \beta)]p(t) + \bar{\lambda}\zeta(t), \quad \dot{\zeta}(t) = \beta p(t) - \bar{\lambda}\zeta(t)$$

- This system of two 1st order ODEs can be reduced to a 2nd order ODE by eliminating the reduced precursor concentration

$$\Lambda \ddot{p} = (\rho - \beta)\dot{p} + \dot{\rho} p + \bar{\lambda}\dot{\zeta} = (\rho - \beta)\dot{p} + \dot{\rho} p + \bar{\lambda}(\beta p - \bar{\lambda}\zeta)$$

$$= (\rho_1 - \beta)\dot{p} + \dot{\rho} p + \bar{\lambda}\beta p - \bar{\lambda}[\Lambda\dot{p} - (\rho_1 - \beta)p]$$

$$\boxed{\Lambda \ddot{p} + (\bar{\lambda}\Lambda + \beta - \rho)\dot{p} - (\bar{\lambda}\rho + \dot{\rho})p = 0}$$

  – Initial conditions

$$p(0) = p_0$$

$$\Lambda \dot{p}(0) = [\rho(0) - \beta]p(0) + \bar{\lambda}\zeta(0) = [\rho(0) - \beta]p_0 + \beta p_0 = \rho(0)p_0$$

- For a step reactivity insertion

$$\boxed{\Lambda \ddot{p} + (\bar{\lambda}\Lambda + \beta - \rho_1)\dot{p} - \bar{\lambda}\rho_1 p = 0}$$

$$p(t) = Ae^{\alpha t} \quad \Rightarrow \quad \alpha^2 + (\bar{\lambda} - \alpha_p)\alpha - \bar{\lambda}\rho_1 / \Lambda = 0$$

Kinetics with One-Group Delayed Neutrons

- Roots of characteristic equation

$$\alpha^2 + (\bar{\lambda} - \alpha_p)\alpha - \frac{\bar{\lambda}\rho_1}{\Lambda} = 0 \qquad \bar{\lambda} - \alpha_p > 0 \quad \& \quad \bar{\lambda} << |\alpha_p|$$

$$\alpha = \frac{1}{2}\left\{(\alpha_p - \bar{\lambda}) \pm \left[(\bar{\lambda} - \alpha_p)^2 + 4\frac{\bar{\lambda}\rho_1}{\Lambda}\right]^{1/2}\right\} = \frac{1}{2}(\alpha_p - \bar{\lambda})\left\{1 \mp \left[1 + 4\frac{\bar{\lambda}\rho_1}{\Lambda(\bar{\lambda} - \alpha_p)^2}\right]^{1/2}\right\}$$

$$\frac{\bar{\lambda}\rho_1}{\Lambda(\bar{\lambda} - \alpha_p)^2} \approx \frac{\bar{\lambda}\rho_1}{\Lambda(\alpha_p)^2} = \frac{\bar{\lambda}\rho_1\Lambda}{(\rho_1 - \beta)^2} << 1 \quad \& \quad (1+\varepsilon)^{1/2} \approx 1 + \frac{1}{2}\varepsilon$$

$$\alpha \approx \frac{1}{2}(\alpha_p - \bar{\lambda})\left\{1 \mp \left[1 + 2\frac{\bar{\lambda}\rho_1}{\Lambda(\bar{\lambda} - \alpha_p)^2}\right]\right\}$$

$$\alpha_1 = \frac{1}{2}(\alpha_p - \bar{\lambda})\frac{(-2)\bar{\lambda}\rho_1}{\Lambda(\bar{\lambda} - \alpha_p)^2} = -\frac{\bar{\lambda}\rho_1}{\Lambda(\alpha_p - \bar{\lambda})} \approx -\frac{\bar{\lambda}\rho_1}{\Lambda\alpha_p} = -\frac{\bar{\lambda}\rho_1}{\rho_1 - \beta} = \frac{\bar{\lambda}\rho_1}{\beta - \rho_1} = \frac{\rho_{1\$}}{1 - \rho_{1\$}}\bar{\lambda}$$

$$\alpha_2 = \frac{1}{2}(\alpha_p - \bar{\lambda})\left[2 + \frac{2\bar{\lambda}\rho_1}{\Lambda(\bar{\lambda} - \alpha_p)^2}\right] = (\alpha_p - \bar{\lambda}) + \frac{\bar{\lambda}\rho_1}{\Lambda(\alpha_p - \bar{\lambda})} \approx \alpha_p + \frac{\bar{\lambda}\rho_1}{\rho_1 - \beta} = \alpha_p - \alpha_1 = \alpha_p$$

Kinetics with One-Group Delayed Neutrons

- Roots of characteristic equation

$$\alpha_1 = \frac{\bar{\lambda}\rho_1}{\beta-\rho_1} = \frac{\rho_{1\$}}{1-\rho_{1\$}}\bar{\lambda} > 0 \quad (\rho_{1\$} < 1\$)$$

$$\alpha_2 = \alpha_p - \alpha_1 \approx \alpha_p$$

- General solution

$$p(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

$$p_0 = A_1 + A_2$$

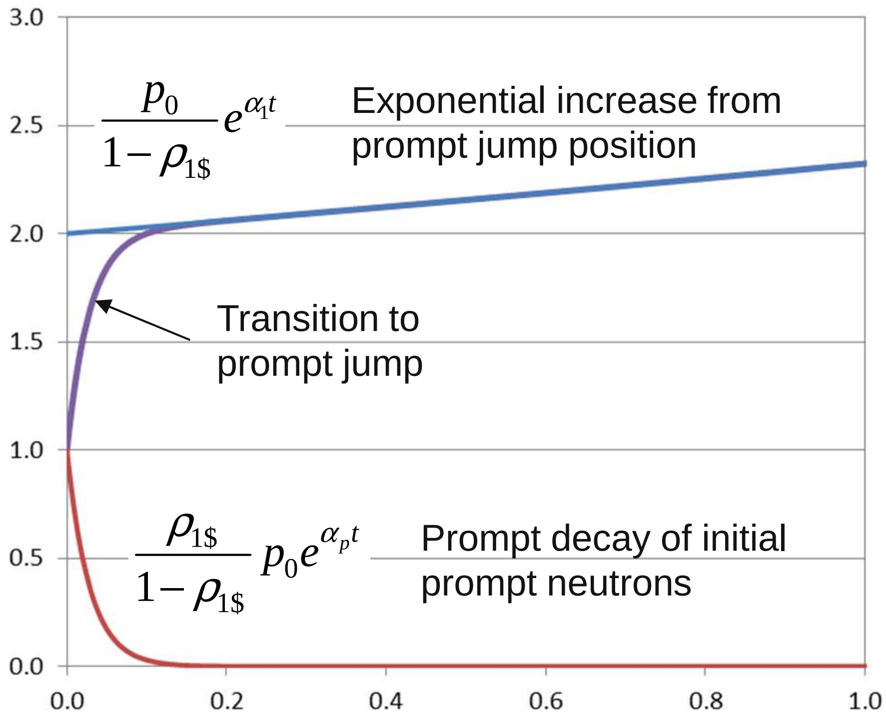
$$\dot{p}_0 = A_1\alpha_1 + A_2\alpha_2 = \frac{\rho_1}{\Lambda}p_0 \quad \Rightarrow \quad A_1\alpha_1 + (p_0 - A_1)\alpha_2 = \frac{\rho_1}{\Lambda}p_0 \quad \Rightarrow \quad (\alpha_1 - \alpha_2)A_1 = \left(\frac{\rho_1}{\Lambda} - \alpha_2\right)p_0$$

$$A_1 = \frac{(\rho_1/\Lambda - \alpha_2)p_0}{\alpha_1 - \alpha_2} = \frac{(\rho_1/\Lambda - \alpha_2)p_0}{-\alpha_p} \approx \frac{(\rho_1/\Lambda - \alpha_p)p_0}{-\alpha_p} = \left(1 - \frac{\rho_1}{\Lambda\alpha_p}\right)p_0 = \frac{\beta}{\beta-\rho_1}p_0 = \frac{1}{1-\rho_{1\$}}p_0$$

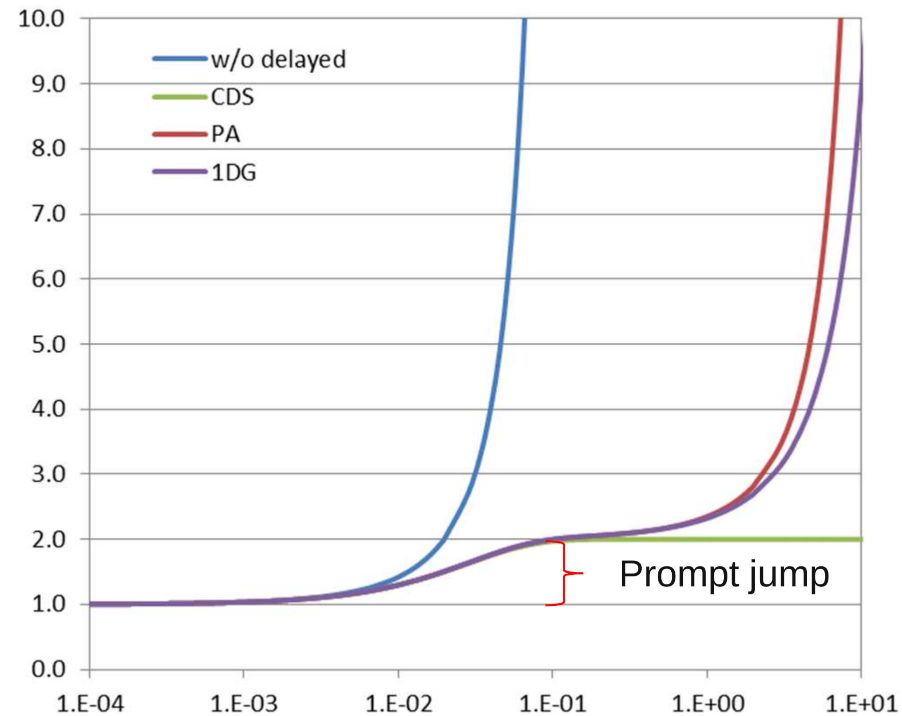
$$A_2 = p_0 - \frac{1}{1-\rho_{1\$}}p_0 = -\frac{\rho_{1\$}}{1-\rho_{1\$}}p_0$$

$$\boxed{p(t) = \frac{p_0}{1-\rho_{1\$}}e^{\alpha_1 t} - \frac{\rho_{1\$}p_0}{1-\rho_{1\$}}e^{\alpha_p t}}$$

Kinetics with One-Group Delayed Neutrons



$$\Lambda = 10^{-4}\ \text{s},\quad \lambda = 0.15\ \text{s}^{-1}$$
$$\rho_1 = 0.5\beta$$



Prompt Jump Approximation (PJA) with One-group

- For modest transients $\Lambda \dot{p} \sim 0$
  - Power change is not rapid $\dot{p} << 1/\Lambda$
  - Normally satisfied when $\rho_1 < 0.9\beta$
- One-group PJA with the limit $\Lambda \to 0$

$$\Lambda \ddot{p} + (\lambda\Lambda + \beta - \rho)\dot{p} - (\lambda\rho + \dot{\rho})p = 0 \qquad \Rightarrow \qquad \frac{\dot{p}}{p} + \frac{(-\dot{\rho})}{\beta - \rho} = \frac{\lambda\rho}{\beta - \rho}$$
$$\Rightarrow \quad (\beta - \rho)\dot{p} - (\lambda\rho + \dot{\rho})p = 0$$

  - This equation can be integrated from 0^+ to t as

$$\ln\frac{p(t)}{p(0^+)} + \ln\frac{\beta - \rho(t)}{\beta - \rho(0^+)} = \int_{0^+}^{t} \frac{\lambda\rho(t')}{\beta - \rho(t')} dt'$$

$$p(t) = p(0^+)\frac{\beta - \rho(0^+)}{\beta - \rho(t)} \exp\left[\int_0^t \frac{\lambda\rho(t')}{\beta - \rho(t')} dt'\right]$$

$$p(0^+) = \begin{cases} p_0, & \text{for gradual reactivity insertion } [\rho(0^+) = 0] \\ p^0 = \beta p_0/(\beta - \rho_1), & \text{for initial reactivity step } [\rho(0^+) = \rho_1 \neq 0] \end{cases}$$

$$\boxed{p(t) = \frac{\beta p_0}{\beta - \rho(t)} \exp\left[\int_0^t \frac{\lambda\rho(t')}{\beta - \rho(t')} dt'\right]} \qquad \text{For } \rho(t) = \rho_1,\ p(t) = \frac{\beta p_0}{\beta - \rho_1} \exp\left[\frac{\lambda\rho_1}{\beta - \rho_1} t\right]$$

Kinetics with Six-Group Delayed Neutrons

- A step reactivity yields a system of ODEs with constant coefficients

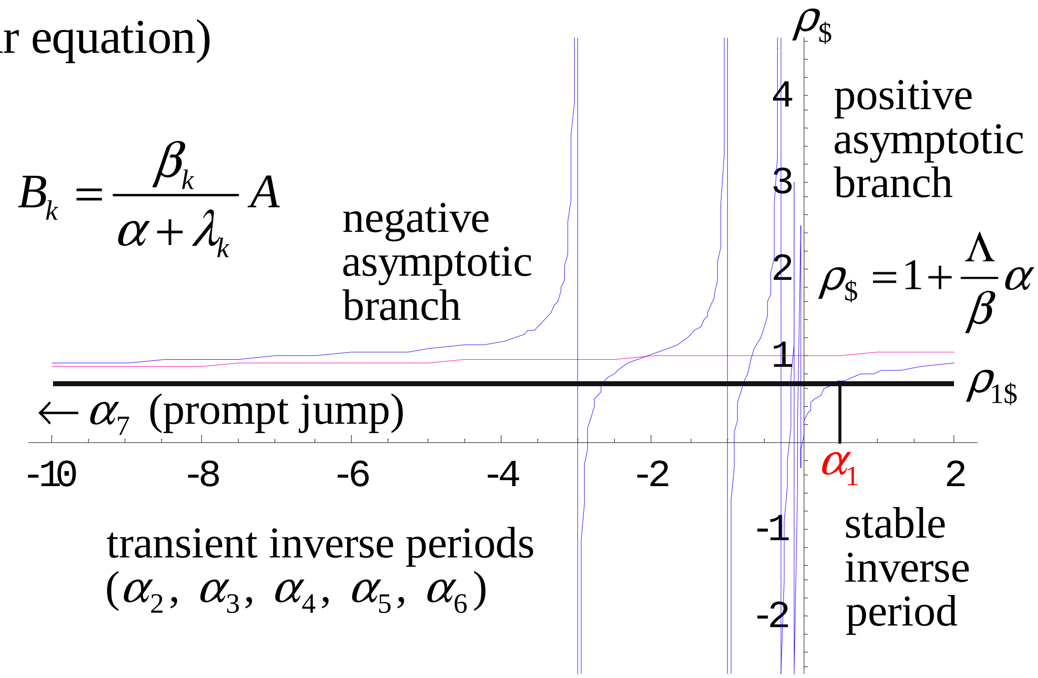
$$p(t) = Ae^{\alpha t},\ \zeta_k(t) = B_k e^{\alpha t} \quad \rightarrow \quad \dot{p} = \frac{\rho_1 - \beta}{\Lambda} p + \frac{1}{\Lambda}\sum_k \lambda_k \zeta_k, \quad \dot{\zeta}_k = \beta_k p - \lambda_k \zeta_k$$

$$\alpha = \frac{\rho_1 - \beta}{\Lambda} + \frac{1}{\Lambda}\sum_k \frac{\lambda_k \beta_k}{\alpha + \lambda_k} \quad \text{(inhour equation)}$$

$$\alpha B_k e^{\alpha t} = -\lambda_k B_k e^{\alpha t} + \beta_k A e^{\alpha t} \quad \Rightarrow \quad B_k = \frac{\beta_k}{\alpha + \lambda_k} A$$

$$\rho_1 = \Lambda \alpha + \beta - \sum_{k=1}^{6} \frac{\lambda_k \beta_k}{\alpha + \lambda_k} \quad \Rightarrow$$

$$\alpha_7 < \alpha_6 < \alpha_5 < \alpha_4 < \alpha_3 < \alpha_2 < 0 < \alpha_1$$

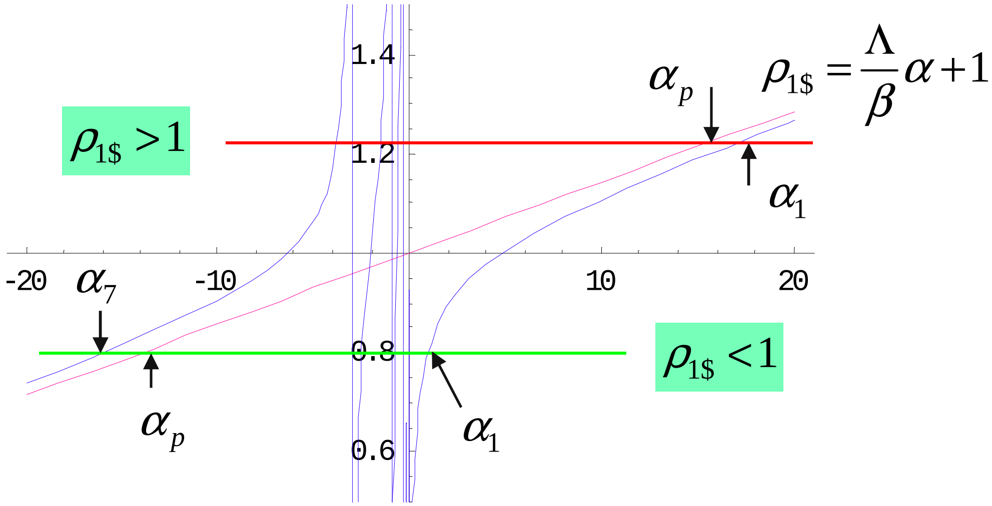


$$p(t) = \sum_{i=1}^{7} A_i e^{\alpha_i t} \quad \rightarrow \quad A_1 e^{\alpha_1 t}$$

$A_7 e^{\alpha_7 t}$ decays most rapidly and $A_1 e^{\alpha_1 t}$ term will be dominant after a while

For $\rho_1 < 0.9\beta$, $\quad \rho_1 = \Lambda \alpha_7 + \beta \quad \Rightarrow \quad \alpha_7 = \dfrac{\rho_1 - \beta}{\Lambda} = \alpha_p \quad \Rightarrow \quad \alpha_p^{6G} = \alpha_p^{1G}$

Prompt and Stable Branches



For $\rho_1 > 1\$$, $0 < \alpha_p = \dfrac{\rho - \beta}{\Lambda} < \alpha_1$

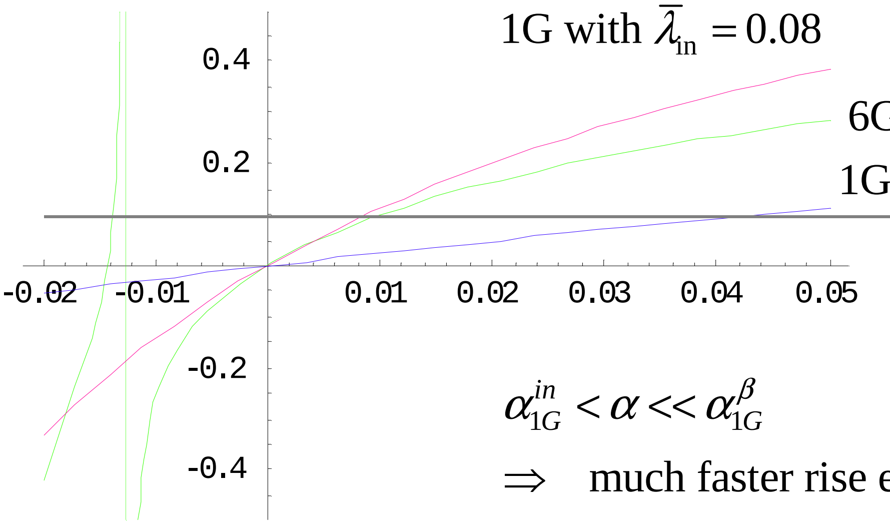
$\Rightarrow$ more rapid rise than α_p

For $\rho_1 < 1\$$, $\alpha_7 < \alpha_p = \dfrac{\rho - \beta}{\Lambda} < 0$

$\Rightarrow$ more rapid decrease than α_p

$$\rho_1 = \Lambda\alpha + \beta - \sum_{k=1}^{6} \frac{\lambda_k \beta_k}{\alpha + \lambda_k}$$

$$\rho_1 = \Lambda\alpha + \beta - \frac{\bar{\lambda}\beta}{\alpha + \bar{\lambda}}$$



$\alpha_{1G}^{in} < \alpha << \alpha_{1G}^{\beta}$

$\Rightarrow$ much faster rise expected for $\bar{\lambda}_{\beta}$

Prompt Jump with Six Group

- Transition to prompt jump

$$p(t) = A_1 e^{\alpha_1 t}\left(1 + \boxed{\sum_{i=2}^{6} \frac{A_i}{A_1} e^{(\alpha_i - \alpha_1)t}} + \boxed{\frac{A_7}{A_1} e^{(\alpha_7 - \alpha_1)t}}\right)$$

Transition terms

Prompt jump (very rapid die out) $\alpha_7 \sim \alpha_p$

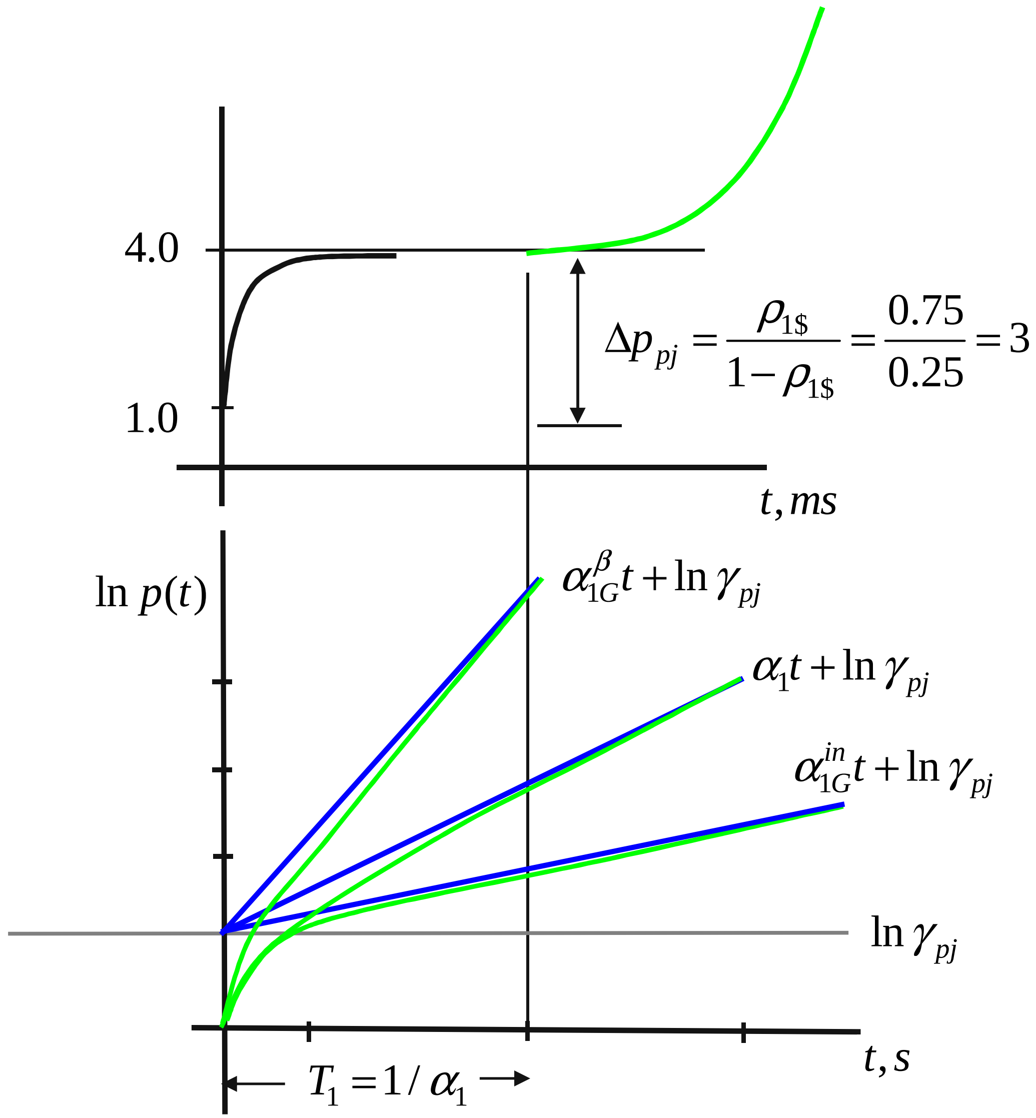
- Delayed adjustment period
  - Time for the adjustment to the asymptotic stable behavior
  - Determined by the longest living term

$$e^{(\alpha_2 - \alpha_1)t} \quad \Rightarrow \quad T_{DA} = \frac{1}{\alpha_1 - \alpha_2} \approx \frac{1}{\alpha_1} \quad \text{if } \alpha_1 > 0.1$$

$$-\frac{1}{80} < \alpha_2 < 0 \text{ (Br-87)} \quad \Rightarrow \quad \alpha_2 \sim 0$$

  - Transition completes in one stable period

Example



$$\rho_1 = 0.75\beta, \quad p_0 = 1$$

$$\gamma_{pj} = \frac{1}{1-0.75} = 4$$

$$\alpha_p = \frac{\rho_1 - \beta}{\Lambda} = (\rho_{1\$} - 1)\frac{\beta}{\Lambda}$$

$$\alpha_p = (0.75-1)\frac{0.007}{10^{-5}} = -\frac{1}{4}700 = -175$$

$$T_p = -\frac{1}{\alpha_p} = \frac{4}{700} = 0.0057$$

$$\alpha_1 = \frac{\rho_{1\$}\bar{\lambda}_{in}}{1-\rho_{1\$}} = \frac{0.75 \bullet 0.08}{1-0.75} = 0.24$$

$$T_1 = \frac{1}{0.24} \sim 4\sec$$

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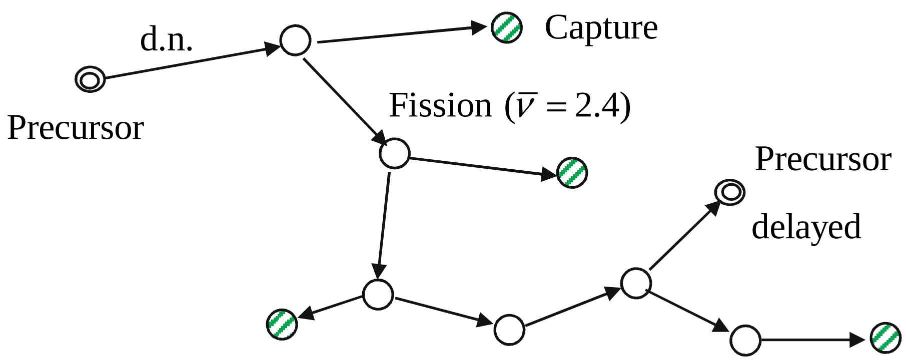
Micro Kinetics

- Fission chain initiated by the emission of a delayed neutron



1 d.n. produces 1 d.n.
in a critical system

- Average time between two delayed neutron emissions

1 fission neutron is produced during a generation time Λ

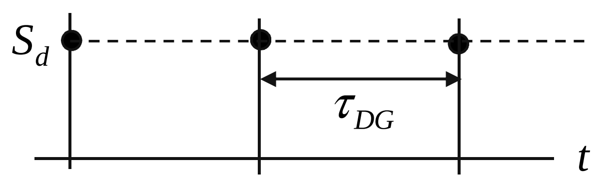
$\Rightarrow$ β delayed neutrons are produced per generation time Λ

$\Rightarrow$ $1/\beta$ generations is needed to produce 1 delayed neuron

$\Rightarrow$ Λ/β sec is needed to produced 1 delayed neutron

$$n_{DG} = \frac{1}{\beta} = \frac{1}{0.007} =\sim 140 \text{ generations} \qquad \tau_{DG} = n_{DG}\Lambda \cong 140 \times 2.5 \times 10^{-5} \text{ sec} = 3.5 \text{ ms}$$

$$S_d(t) = \sum_{n=1}^{\infty} \delta(t - n\tau)$$



Delayed Neutron Multiplication

- Multiplication of a delayed neutron source in a subcritical system

$$k = 1 - \frac{1}{\rho}, \quad \beta_p = 1 - \beta, \quad k\beta_p < 1$$

$$\begin{array}{lccccccccc}
\text{prompt neutrons} & 1 & \rightarrow & k\beta_p & \rightarrow & (k\beta_p)^2 & \rightarrow & (k\beta_p)^3 & \rightarrow & \cdots \\
 & & \searrow & & \searrow & & \searrow & & & \\
\text{D.N. precursors} & & & k\beta & & (k\beta)(k\beta_p) & & (k\beta)(k\beta_p)^2 & & \cdots
\end{array}$$

$$n_p = \frac{k\beta_p}{1 - k\beta_p} = \frac{1-\beta}{1/k - (1-\beta)} = \frac{1-\beta}{\beta - \rho}$$

$$n_d = \frac{k\beta}{1 - k\beta_p} = \frac{\beta}{1/k - (1-\beta)} = \frac{\beta}{\beta - \rho}$$

$$n_t = n_p + n_d = \frac{1}{\beta - \rho} \quad \begin{cases} \text{total number of fission neutrons} \\ \text{i.e., delayed neutron multiplication factor} \end{cases}$$

- Generalized source multiplication factor

$$p(t) = \frac{s_d(t)}{\beta - \rho(t)}$$

PKE Solution for Discrete Delayed Neutron Emission

- Point kinetics equation for discrete delayed neutron source

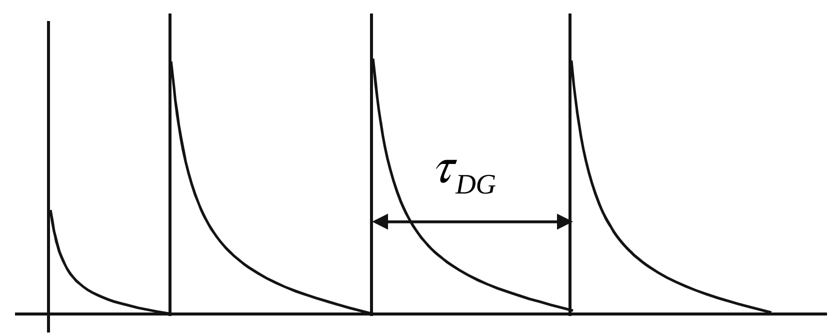
$$\dot{p} = \frac{\rho - \beta}{\Lambda} p + \frac{1}{\Lambda}\sum_{n=1}^{\infty}\delta(t - n\tau) \quad \Rightarrow \quad \dot{p} - \alpha_p p = \frac{1}{\Lambda}\sum_{n=1}^{\infty}\delta(t - n\tau)$$

$$\frac{d}{dt}[p(t)e^{-\alpha_p t}] = \frac{1}{\Lambda}\sum_{n=1}^{\infty}\delta(t - n\tau)e^{-\alpha_p t}$$

$$p(t)e^{-\alpha_p t} - p(0) = \frac{1}{\Lambda}\int_0^t \sum_{n=1}^{\infty}\delta(t' - n\tau)e^{-\alpha_p t'}dt' = \frac{1}{\Lambda}\sum_{n=1}^{N} e^{-\alpha_p n\tau}$$

(N is the largest interger $\ni$ $N\tau \le t$)

$$p(t) = p(0)e^{\alpha_p t} + \frac{1}{\Lambda}\sum_{n=1}^{N} e^{\alpha_p (t - n\tau)}$$



$$\frac{1}{\Lambda}\int_0^{\infty} e^{\alpha_p t} dt = -\frac{1}{\Lambda\alpha_p} = \frac{1}{\beta - \rho}$$

(total number of fission neutrons in an average chain)