NUCL 510
Nuclear Reactor Theory

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Lecture Note 11

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Resonance Effects in Heterogeneous Systems

- Effects of lumping fuel into a distinct heterogeneous form
  - Improvement in resonance escape probability is a prominent factor
    - *Penetration of neutrons into central regions is prohibited by "spatial self-shielding"*
    - *The reduced neutron absorption results in an increase in the resonance escape probability*
  - Reduced absorption in the thermal region decreases the thermal utilization factor
  - Fast fission factor is increased since neutrons born within the fuel have a higher chance for fast fission within the same fuel
  - The net effect is an increase in the medium multiplication factor k_∞
- Heterogeneous fuel cell or bundle calculations
  - In a bundle of fuel, the diffusion theory is inappropriate and transport theory must be used since the medium has many internal interfaces
  - The collision probability and characteristics methods using the integral transport equation are most popular methods
  - Cell or assembly calculations with reflective boundary conditions are typically used with assumed repeated structure

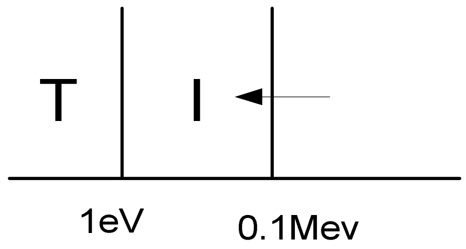
Resonance Escape in Heterogeneous System (1)

- Overall neutron balance
  - Neutrons slowed down to intermediate energy range (below fission source range) by scattering with moderator will be absorbed in the fuel and moderator

$$A_{tot} = V_f\left(\bar{\Sigma}_{aT}^f \phi_T^f + \bar{\Sigma}_{aI}^f \phi_I^f\right) + V_m \bar{\Sigma}_{aT}^m \phi_T^m = q_f V_f + q_m V_m \cong q_m V_m \quad (q_f << q_m)$$

- Resonance escape probability

$$p = \frac{A_T}{A_{tot}} = 1 - \frac{A_I}{A_{tot}}$$



- Absorption fraction in fertile isotope resonances

$$a = \frac{V_f \bar{\Sigma}_{aI}^f \phi_I^f}{q_m V_m} = \frac{V_f}{q_m V_m}\int_I \Sigma_a^{fe}(E)\phi_f(E)dE$$

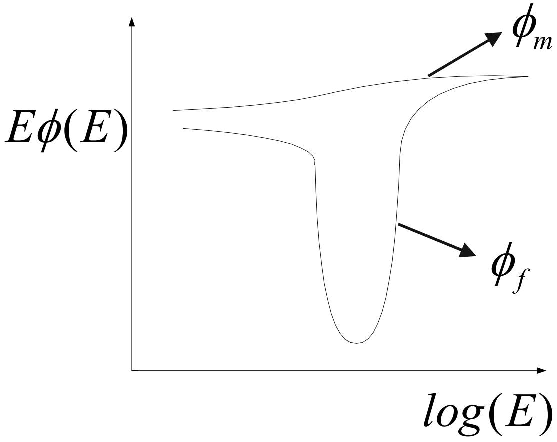
$$= \frac{V_f}{V_m \xi_m \Sigma_s^m E\phi_m(E)}\int_I \Sigma_a^{fe}(E)\phi_f(E)dE \qquad E\phi_m\left(E\right) = \frac{q_m}{\xi_m \Sigma_s^m} = \text{const}$$

$$= \frac{V_f}{V_m}\frac{N^{fe}}{\xi \Sigma_s^m}\int_I \sigma_a^{fe}\left(E\right)\frac{\phi_f\left(E\right)}{\phi_m\left(E\right)}\frac{dE}{E} = \frac{V_f}{V_m}\frac{N^{fe}}{\xi \Sigma_s^m} I$$

Resonance Escape in Heterogeneous System (2)

- Resonance integral for each resonance

$$I_i = \int_{R_i} \sigma_a^{fe}(E)\frac{\phi_f(E)}{\phi_m(E)}\frac{dE}{E}; \quad \frac{\phi_f(E)}{\phi_m(E)} = E\text{-dependent spatial self-shielding}$$



- Resonance escape probability for each resonance

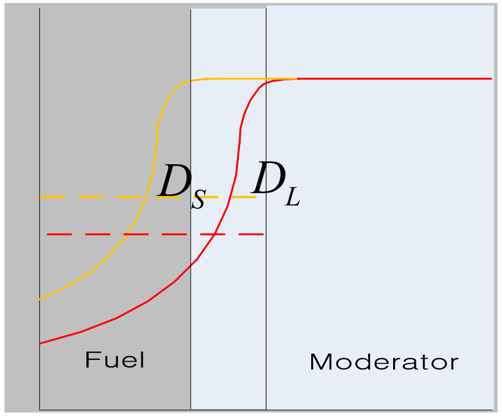
$$p_i = 1 - \frac{N^{fe}}{\xi\Sigma_p^m}\frac{V_f}{V_m}I_i \cong \exp\left[-\frac{N^{fe}}{\xi\Sigma_p^m}\frac{V_f}{V_m}I_i\right]$$

- Total resonance escape probability

$$p = p_i \cdots p_N = \exp\left[-\frac{N^{fe}}{\xi\Sigma_p^m}\frac{V_f}{V_m}\sum_i I_i\right]$$

- Resonance integral of UO_2 fuel rod
  - Larger D → more spatial self-shielding → lower I → higher p

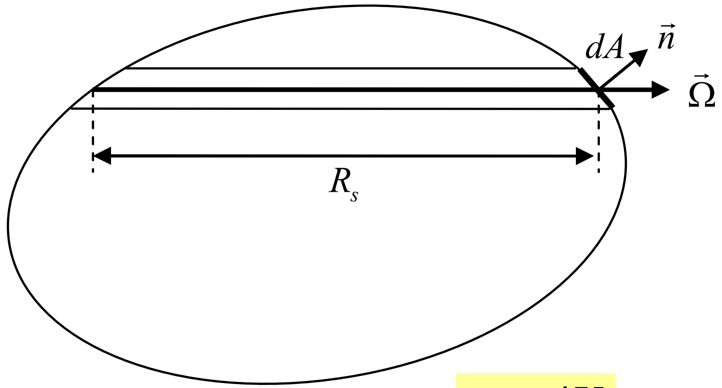
$$I = \sum_i I_i = 4.45 + 26.6\sqrt{\frac{4}{\rho D}} \quad (\rho \text{ in g/cm}^3,\ D \text{ in cm},\ I \text{ in barn})$$



Average Chord Length

- Average chord length of a convex body
  - A chord length of a convex body depends on the direction and surface area element (i.e., its normal vector). The average chord length can be determined by considering the rays of positive directions as

$$\bar{R} = \frac{\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega} R_s(\vec{n},\vec{\Omega})}{\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega}}$$



  - For a fixed direction $\vec{\Omega}$, $R_s(\vec{n}\cdot\vec{\Omega})dA$ is the volume of the cylinder-like element

$$\int_A dA \vec{n}\cdot\vec{\Omega} R_s = 2V$$

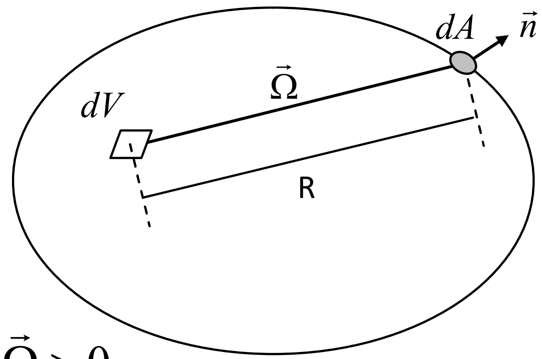
$$\boxed{\bar{R} = \frac{4V}{A}}$$

$$\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega} R_s = \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \int_A dA \vec{n}\cdot\vec{\Omega} R_s = 2V \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega = 4\pi V$$

$$\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega} = \int_A dA \int_0^{2\pi} d\varphi \int_0^1 d\mu \mu = A(2\pi)(1/2) = \pi A$$

First-Flight Escape Probability (1)

- First-flight escape probability of a convex body
  - Probability that a neutron born uniformly and isotropically inside the volume V will escape through its surface A without collision



* Number of neutrons born in dV that leaves an isolated medium through area dA

$$(SdV)e^{-\Sigma_t R}\frac{dA}{4\pi R^2}=\frac{S}{4\pi}e^{-\Sigma_t R}dVd\Omega \quad (\Leftarrow dA = R^2 d\Omega)$$

$$P_0=\frac{1}{4\pi V}\int_V dV\int d\Omega\, e^{-\Sigma_t R} \qquad \Leftarrow dV = dR(dA\vec{n}\cdot\vec{\Omega}) \text{ with } \vec{n}\cdot\vec{\Omega}>0$$

$$P_0=\frac{1}{4\pi V}\int_A dA\int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega\,\vec{n}\cdot\vec{\Omega}\int_0^{R_s} dR\,e^{-\Sigma_t R}=\frac{A}{4V\Sigma_t}\frac{1}{\pi A}\int_A dA\int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega\,\vec{n}\cdot\vec{\Omega}(1-e^{-\Sigma_t R_s})$$

$$P_0=\frac{1}{\bar{R}\Sigma_t}(1-<e^{-\Sigma_t R_s}>)$$

$$P_0\to 1 \text{ as } \Sigma_t\bar{R}\to 0;\quad P_0\to 0 \text{ as } \Sigma_t\bar{R}\to\infty$$

- Wigner rational approximation

$$P_0\approx\frac{1}{1+\Sigma_t\bar{R}}=\frac{\Sigma_e}{\Sigma_t+\Sigma_e} \qquad \Sigma_e=\frac{1}{\bar{R}} \quad \text{(escape cross section)}$$

Escape Probability of Sphere

- Sphere of radius *a*

$$R_s = 2a\vec{n}\cdot\vec{\Omega} = 2a\cos\gamma$$

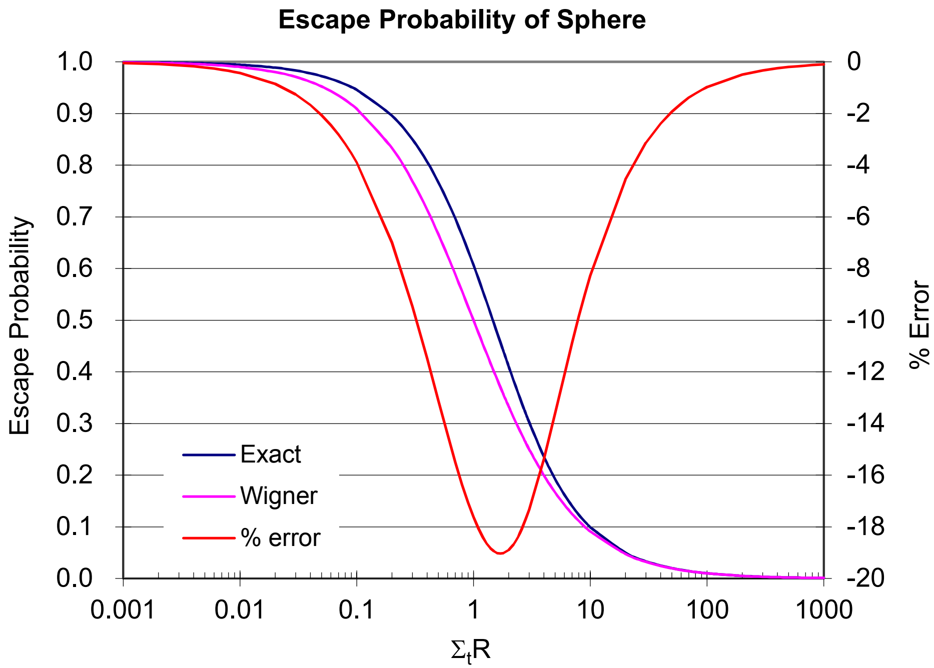
$$< e^{-\Sigma_t R_s} >= \frac{1}{\pi A}\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega\vec{n}\cdot\vec{\Omega}e^{-\Sigma_t R_s}$$

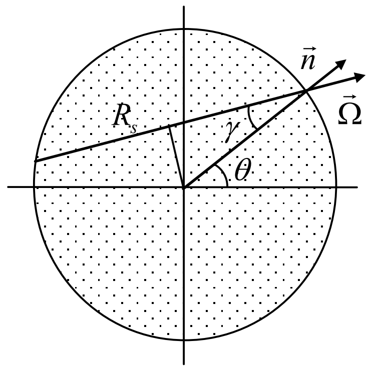
$$= \frac{2\pi A}{\pi A}\int_0^1 d\mu\mu e^{-2a\Sigma_t\mu}$$

$$= \frac{1}{2(a\Sigma_t)^2} - \frac{1}{a\Sigma_t}\left(1+\frac{1}{2a\Sigma_t}\right)e^{-2a\Sigma_t}$$

$$\bar{R} = 4\times\frac{4\pi a^3}{3}\times\frac{1}{4\pi a^2} = \frac{4a}{3}$$

$$P_0 = \frac{1}{\bar{R}\Sigma_t}\left[1-\frac{8}{9(\bar{R}\Sigma_t)^2}+\frac{4}{3\bar{R}\Sigma_t}\left(1+\frac{2}{3\bar{R}\Sigma_t}\right)e^{-3\bar{R}\Sigma_t/2}\right]$$





- Wigner's rational approximation

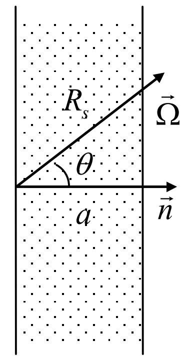
$$P_0 = \frac{1}{1+\bar{R}\Sigma_t}$$

Escape Probability of Slab

- Infinite slab of thickness *a*

$$P_0 = \frac{1}{\bar{R}\Sigma_t}\left[1 - 2E_3\left(\frac{\bar{R}\Sigma_t}{2}\right)\right]; \quad \bar{R} = 2a$$

$$E_n(x) = \int_1^\infty d\tau \tau^{-n} e^{-x\tau} = \int_0^1 d\mu \mu^{n-2} e^{-x/\mu} \text{ (Exponential integral function)}$$



- Wigner's rational approximation

$$P_0 = \frac{1}{1 + \bar{R}\Sigma_t}$$

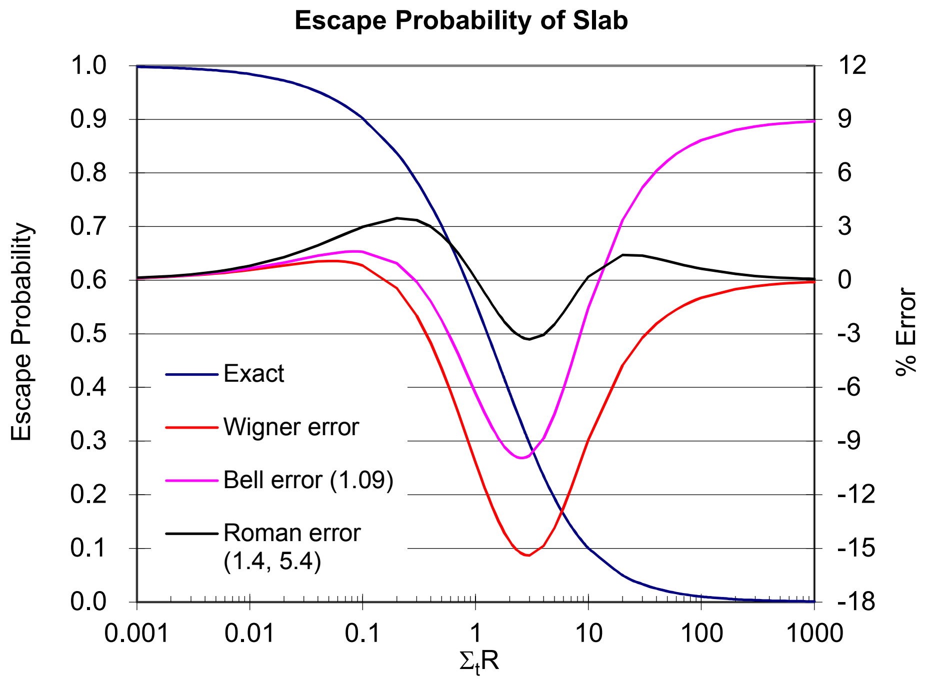
- Bell factor

$$P_0 = \frac{c}{c + \bar{R}\Sigma_t}; \quad c = 1.09$$

- Roman's two-term rational approximation

$$P_0 = \frac{1}{c_1 - c_2}\left[\frac{c_1(1-c_2)}{c_1 + \bar{R}\Sigma_t} - \frac{c_2(1-c_1)}{c_2 + \bar{R}\Sigma_t}\right]$$

$$c_1 = 1.4; \quad c_2 = 5.4$$

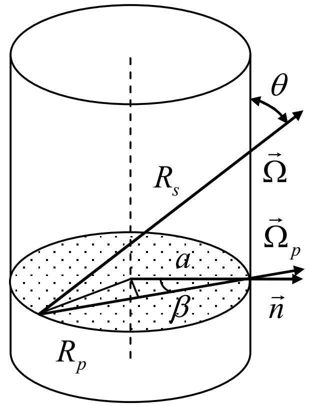


Escape Probability of Cylinder

- Infinite cylinder of radius *a*

$$P_0 = \frac{1}{\bar{R}\Sigma_t}\left[1 - \frac{4}{\pi}\int_0^{\pi/2} d\beta \cos\beta Ki_3(\bar{R}\Sigma_t \cos\beta)\right]; \quad \bar{R} = 2a$$

$$Ki_n(x) = \int_0^{\pi/2} d\theta \sin^{n-1}\theta e^{-x/\sin\theta} = \int_0^{\infty} d\tau \cosh^{-n}\tau e^{-x\cosh\tau} \text{ (Bickley function)}$$



- Wigner's rational approximation

$$P_0 = \frac{1}{1 + \bar{R}\Sigma_t}$$

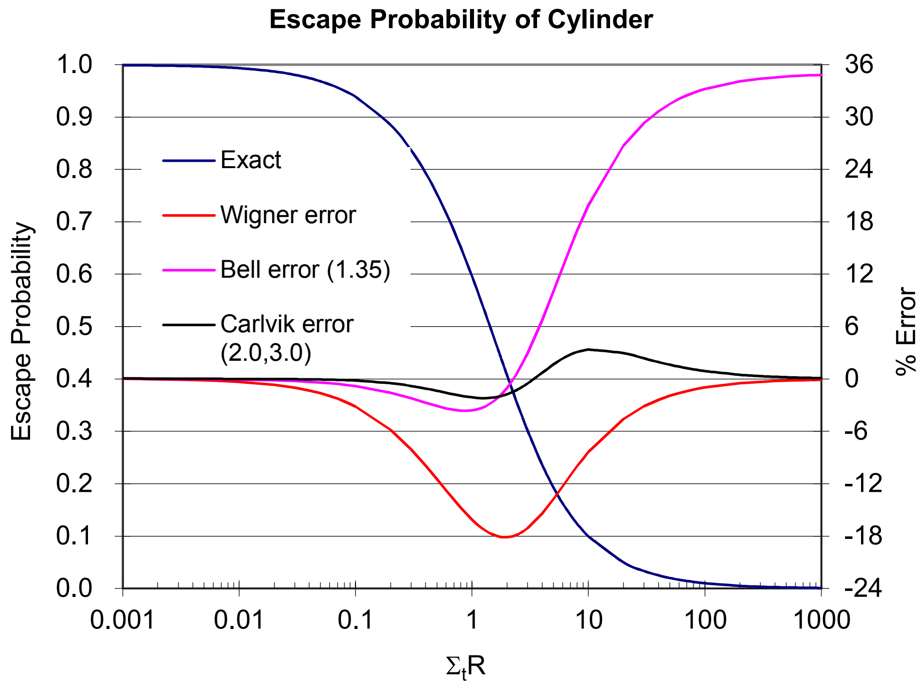
- Bell factor

$$P_0 = \frac{c}{c + \bar{R}\Sigma_t}; \quad c = 1.35$$

- Carlvik's two-term rational approximation

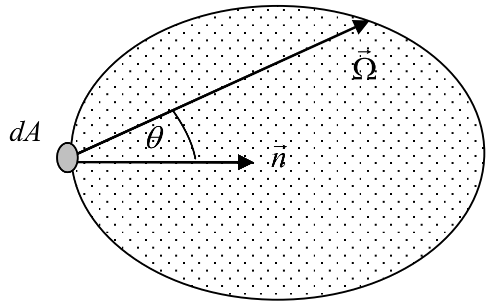
$$P_0 = \frac{1}{c_1 - c_2}\left[\frac{c_1(1-c_2)}{c_1 + \bar{R}\Sigma_t} - \frac{c_2(1-c_1)}{c_2 + \bar{R}\Sigma_t}\right]$$

$$c_1 = 2.0; \quad c_2 = 3.0$$



First-Flight Blackness

- First-flight blackness of a convex body
  - Probability that a neutron impinging on a body uniformly and isotropically through its surface will suffer a collision in that body



* For a convex body which is completely immersed in an isotropic flux of magnitude equal to ϕ, the number of neutrons passing through the area element dA with inward normal vector $\vec{n}$ into a solid angle $d\Omega$ about $\vec{\Omega}$ is given by

$(\phi / 4\pi)(\vec{n}\cdot\vec{\Omega}dA)d\Omega$ (cosine distributed current)

* Number of interactions made by these neutrons within the body

$$\frac{\phi}{4\pi}\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega}(1-e^{-\Sigma_t R_s})$$

* Total number of neutrons entering the body

$$\frac{\phi}{4\pi}\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega} = \frac{\phi A}{4}$$

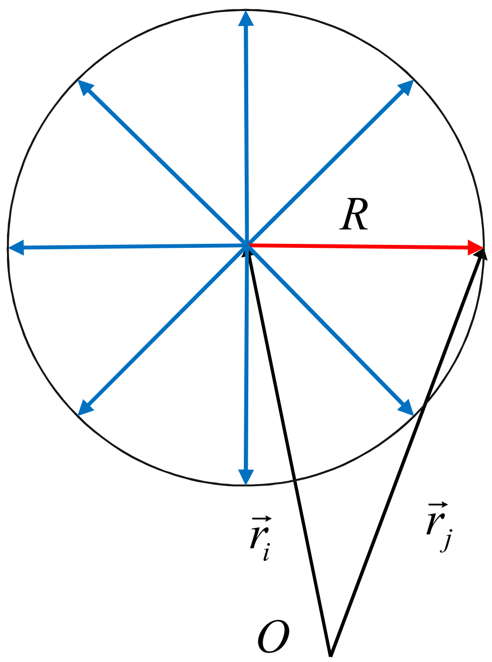
$$G_0 = \left(\frac{\phi A}{4}\right)^{-1}\frac{\phi}{4\pi}\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega}(1-e^{-\Sigma_t R_s}) = \frac{1}{\pi A}\int_A dA \int_{\vec{n}\cdot\vec{\Omega}>0} d\Omega \vec{n}\cdot\vec{\Omega}(1-e^{-\Sigma_t R_s})$$

$$G_0 = 1 - \langle e^{-\Sigma_t R_s} \rangle = \bar{R}\Sigma_t P_0$$

Reciprocity Relation

- Integral transport equation
  - Neutron flux at r_j in region j due to a unit isotropic source at r_i in region i

$$\varphi(\vec{r}_j;\vec{r}_i) = \frac{\exp\left[-\int_0^{|\vec{r}_j-\vec{r}_i|}\Sigma(s)ds\right]}{4\pi|\vec{r}_j-\vec{r}_i|^2}$$



- Collision probability P_{ij}
  - Probability that a neutron originating in region i makes its next collision in region j

$$P_{ij} = \frac{\Sigma_j}{V_i}\int_{V_i} dV_i \int_{V_j} dV_j \varphi(\vec{r}_j;\vec{r}_i)$$

- Reciprocity relation

$$\int_{V_i} dV_i \int_{V_j} dV_j \varphi(\vec{r}_j;\vec{r}_i) = \frac{P_{ij}V_i}{\Sigma_j} = \frac{P_{ji}V_j}{\Sigma_i} \quad \Rightarrow \quad \Sigma_i V_i P_{ij} = \Sigma_j V_j P_{ji}$$

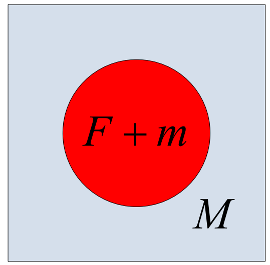
Similarly, between a bounding surface element b and region i $\Rightarrow$ $\frac{A_b P_{bi}}{4} = \Sigma_i V_i P_{ib}$

Heterogeneous Two-Region Cell (1)

- Slowing-down equation in fuel region

$$V_F\Sigma_{tF}(E)\phi_F(E)=P_{MF}V_M\frac{1}{1-\alpha_M}\int_E^{E/\alpha_M}\phi_M(E')\Sigma_{sM}\frac{dE'}{E'}$$

$$+(1-P_{FM})V_F\left[\frac{1}{1-\alpha_F}\int_E^{E/\alpha_F}\phi_F(E')\Sigma_{sF}(E')\frac{dE'}{E'}+\frac{1}{1-\alpha_m}\int_E^{E/\alpha_m}\phi_F(E')\Sigma_{sm}\frac{dE'}{E'}\right]$$

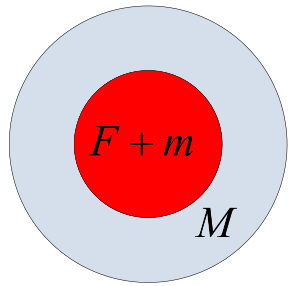


The subscript m denotes the moderator in fuel region

$\Sigma_{sM}V_M P_{MF}=\Sigma_{tF}V_F P_{FM}$ (reciprocity relation)

$$\Sigma_{tF}(E)\phi_F(E)=P_{FM}\Sigma_{tF}(E)\frac{1}{1-\alpha_M}\int_E^{E/\alpha_M}\phi_M(E')\frac{dE'}{E'}$$

$$+(1-P_{FM})\left[\frac{1}{1-\alpha_F}\int_E^{E/\alpha_F}\phi_F(E')\Sigma_{sF}(E')\frac{dE'}{E'}+\frac{1}{1-\alpha_m}\int_E^{E/\alpha_m}\phi_F(E')\Sigma_{sm}\frac{dE'}{E'}\right]$$



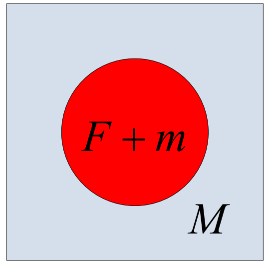
Wigner-Seitz cell

- NR approximation for moderator collisions

$$\Sigma_{tF}(E)\phi_F(E)=(1-P_{FM})\frac{1}{1-\alpha_F}\int_E^{E/\alpha_F}\phi_F(E')\Sigma_{sF}(E')\frac{dE'}{E'}+\left[P_{FM}\Sigma_{tF}+(1-P_{FM})\Sigma_{sm}\right]\frac{1}{E}$$

Heterogeneous Two-Region Cell (2)

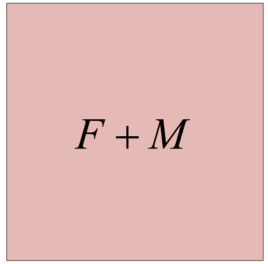
- Rational approximation for escape probability



$$P_{FM} = P_{0F} \approx \frac{\Sigma_{eF}}{\Sigma_{tF} + \Sigma_{eF}}$$

$$(\Sigma_{tF} + \Sigma_{eF})\phi_F(E) = \frac{1}{1-\alpha_F}\int_E^{E/\alpha_F} \phi_F(E')\Sigma_{sF}(E')\frac{dE'}{E'} + (\Sigma_{sm} + \Sigma_{eF})\frac{1}{E}$$

- Slowing-down equation in homogeneous medium (NR)



$$\Sigma_t(E)\phi(E) = \frac{1}{1-\alpha_F}\int_E^{E/\alpha_F} \phi(E')\Sigma_{sF}(E')\frac{dE'}{E'} + \frac{\Sigma_{sM}}{E}$$

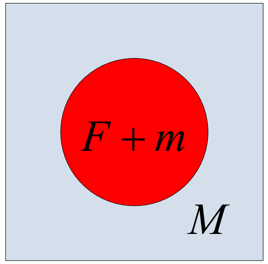
- Equivalence relation
  - The heterogeneous two-region cell is equivalent to a homogeneous mixture under the narrow resonance approximation and the rational approximation for the collision probabilities
  - Escape cross section is added to the total and moderator scattering XS
  - This equivalence relation is the basis for the Bondarenko method
  - The equivalence relation consists in simulating the geometrical escape by the addition of a fictitious scattering cross section (energetical escape)

Heterogeneous Two-Region Cell (3)

- Slowing-down equation in fuel region

$$(\sigma_{tF}+\sigma_{eF})\phi_F(E)=\frac{1}{1-\alpha_F}\int_E^{E/\alpha_F}\phi_F(E')\sigma_{sF}(E')\frac{dE'}{E'}+(\sigma_{mF}+\sigma_{eF})\frac{1}{E}$$

$$\sigma_{eF}=\frac{\Sigma_{eF}}{N_F},\quad \sigma_{mF}=\frac{\Sigma_{sm}}{N_F}$$



- Intermediate resonance approximation

$$(\sigma_{tF}+\sigma_{eF})\phi_F(E)=(1-\lambda)\sigma_{sF}(E)\phi_F(E)+(\lambda\sigma_{pF}+\sigma_{sm}+\sigma_{eF})\frac{1}{E}$$

- IR approximation flux in fuel region

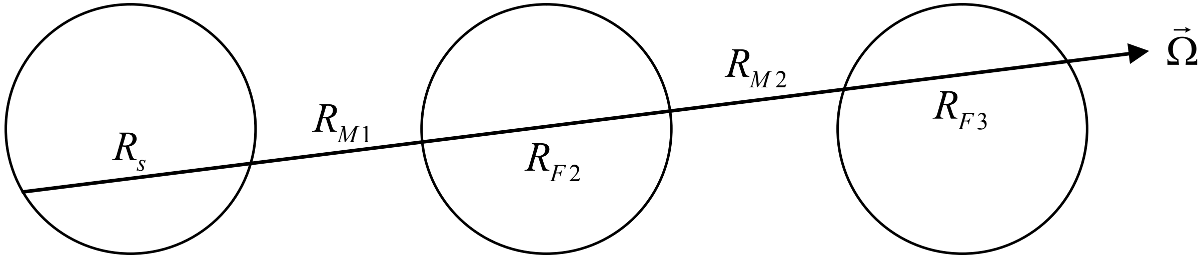
$$\phi_F(E)=\frac{\lambda\sigma_{pF}+\sigma_{mF}+\sigma_{eF}}{\sigma_{aF}(E)+\lambda\sigma_{sF}(E)+\sigma_{mF}+\sigma_{eF}}\frac{1}{E}=\frac{\sigma_b}{\sigma_{aF}(E)+\lambda\sigma_{sF}^r(E)+\sigma_b}\frac{1}{E}$$

$$\phi(E)=\frac{\sigma_b}{\sigma_a(E)+\lambda\sigma_s^r(E)+\sigma_b}\frac{1}{E}$$

$$\sigma_b=\lambda\sigma_{pF}+\sigma_{mF}+\sigma_{eF}$$

Dancoff Factor (1)

- Fuel pin lattice (e.g., a number of fuel pins in a periodic array)



P_M : Probability that a neutron incident on the moderator will collide with the moderator

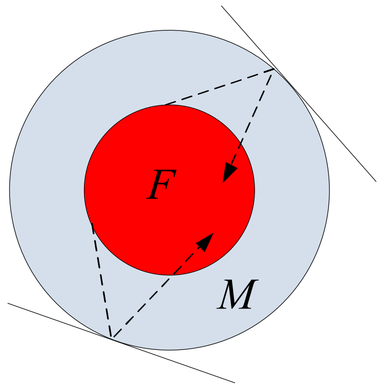
* Probability that a neutron born in the fuel will make its next collision in the moderator

$$P_{FM} = P_0^F P_M + P_0^F(1-P_M)(1-G_0^F)P_M + P_0^F(1-P_M)(1-G_0^F)(1-P_M)(1-G_0^F)P_M + \cdots$$

$$= P_0^F \frac{P_M}{1-(1-P_M)(1-G_0^F)} = P_0^F \frac{1-C}{1-C(1-G_0^F)}$$

$C = 1 - P_M$ (Dancoff factor)

- Unit cell calculation
  - In unit cell calculations, Dancoff factor is the probability in the black fuel limit that a neutron drawn from an isotropic flux at a rod surface will be reabsorbed in the rod after reflection



Dancoff Factor (2)

- Dancoff factor with rational approximation

$$P_0^F = \frac{1}{1+\Sigma_t^F \bar{R}_F} = \frac{\Sigma_e^F}{\Sigma_t^F + \Sigma_e^F} \quad (\Sigma_e^F = 1/\bar{R}_F)$$

$$P_{FM} = P_0^F \frac{1-C}{1-C(1-G_0^F)} = P_0^F \frac{1-C}{1-C(1-\Sigma_t^F \bar{R}_F P_0^F)} = \frac{\Sigma_e^F (1-C)}{\Sigma_t^F + \Sigma_e^F (1-C)} = \frac{\Sigma_e^{F*}}{\Sigma_t^F + \Sigma_e^{F*}}$$

$$\Sigma_e^{F*} = \Sigma_e^F (1-C) = \frac{1}{\bar{R}_F / (1-C)} = \frac{A_F (1-C)}{4V_F}$$

- In the rational approximation, the Dancoff correction is equivalent to increasing the mean chord length or decreasing the surface area (shadowing of fuel surface)
- Dancoff factor is also defined as the relative reduction in current through the surface element of a fuel lump due to the shadowing of source by other fuel lumps
- With the improved rational approximation with Bell factor *a*

$$\Sigma_e^{F*} = \Sigma_e^F \frac{a(1-C)}{1+(a-1)C}$$

Slowing Down Equation in Average Pin Cell

- Neutron balance equation in fuel region
  - IR approximation for scattering by fuel nuclides
  - NR approximation for other scatterings

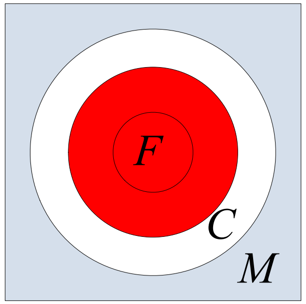
$$V_F \Sigma_F(u)\varphi_F(u) = \sum_{J \neq F} V_J \Sigma_J \cdot 1 \cdot P_{JF} \qquad \boxed{\Leftarrow \quad \varphi_M(u) = \varphi_F(u) = 1 \ , \ u < u_L^k}$$
$$+ V_F \left[\lambda_F \Sigma_{pF} + (1-\lambda_F)\Sigma_{sF}(u)\varphi_F(u) \right] P_{FF}$$

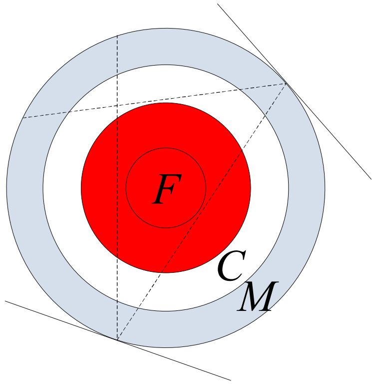
$\Sigma_J P_{JF} = \Sigma_F P_{FJ}$ (reciprocity relation)

$$\Rightarrow \quad \sum_{J \neq F} V_J \Sigma_J P_{JF} = V_F \Sigma_F(u) \sum_{J \neq F} P_{FJ} = V_F \Sigma_F(u)(1 - P_{FF}) = V_F \Sigma_F(u) P_{esc}^F$$

$$V_F \Sigma_F(u)\varphi_F(u) = V_F \Sigma_F(u) P_{esc}^F + V_F \left[\lambda_F \Sigma_{pF} + (1-\lambda_F)\Sigma_{sF}(u)\varphi_F(u) \right] P_{FF}$$

$$V_F \left[\Sigma_F(u) - (1-\lambda_F)\Sigma_{sF}(u) P_{FF} \right] \varphi_F(u) = V_F \Sigma_F(u) P_{esc}^F + V_F \lambda_F \Sigma_{pF} P_{FF}$$





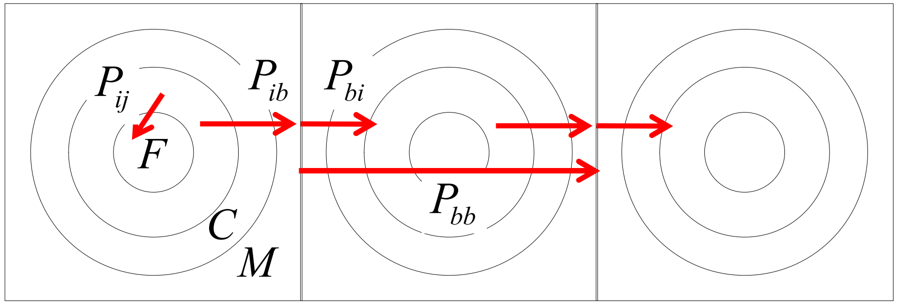
Wigner-Seitz cell

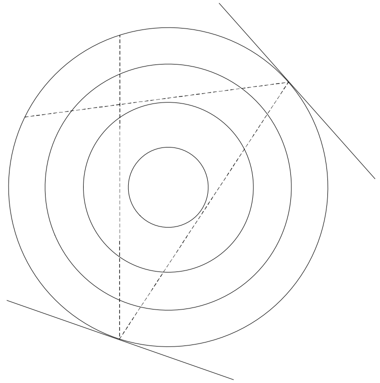
$$\boxed{\begin{array}{l} \Sigma_J = \Sigma_{sJ} = \text{const.} \\ \Sigma_F = \Sigma_{tF}(u) \end{array}}$$

- IR flux in fuel region for known collision probabilities

$$\varphi_F(u) = \frac{\lambda_F \Sigma_{pF} P_{FF} + \Sigma_F(u) P_{esc}^F}{\Sigma_F(u) - (1-\lambda_F)\Sigma_{sF}(u) P_{FF}}$$

Fuel to Fuel Collision Probability in Cell and Lattice





- First-flight probabilities of a pin cell

P_{ij} : Probability for a neutron born in region i to have its first collision in region j

P_{ib} : Probability for a neutron in region i to reach the boundary without collision

P_{bi} : Probability for a neutron entering the boundary isotropically (cosine current) to have its first collision in region i

P_{bb} : Probability for a neutron entering the boundary isotropically (cosine current) to pass through the cell without collision

- Reflection (re-entrance) ratio R (=α): Faction of neutrons escaping from a cell that returns to the cell (1.0 for infinite lattice)
- Lattice collision probability (upper case suffices)

$$P_{IJ} = P_{ij} + P_{ib}RP_{bj} + \underbrace{P_{ib}RP_{bb}RP_{bj}}_{\text{Coll. in 3rd cell}} + P_{ib}R(P_{bb}R)^2 P_{bj} + \cdots = P_{ij} + \frac{P_{ib}RP_{bj}}{1 - RP_{bb}}$$

Lattice FF CP is increased by this amount

First Flight Probabilities for Average Pin Cell

- First-flight probabilities for average pin cell

  P_{ff}: fuel-to-fuel first collision probability $\Rightarrow P_{esc}^{f} = 1 - P_{ff}$

  P_{fb}: probability for a neutron born in the fuel to reach the cell boundary S_b without collision

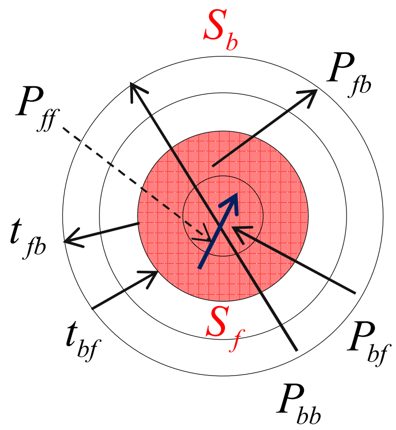
  P_{bf}: probability of neutrons entering S_b with a cosine distribution to collide inside the fuel

  P_{bb}: transmission ptobability from S_b to S_b

  t_{fb} : transmission probability from S_f to S_b

  t_{bf} : transmission probability from S_b to S_f

  γ_f : first flight blackness of fuel; probability of neutrons entering the fuel with a cosine distribution through S_f to collide inside the fuel



- Cell blackness

  γ_f^b : probability for neutrons entering through S_b to collide within the fuel

$$\gamma_f^b = P_{bf} = \frac{4V_f\Sigma_t^f}{S_b}P_{fb}; \quad \gamma_f = \bar{R}_f\Sigma_t^f P_{esc}^f = \frac{4V_f\Sigma_t^f}{S_f}(1-P_{ff}) = x(1-P_{ff})$$

- Reciprocity relations

$$P_{bf} = \frac{4V_f\Sigma_t^f}{S_b}P_{fb} \quad \Rightarrow \quad S_f t_{fb} = S_b t_{bf} \qquad P_{fb} \simeq P_{esc}^f t_{fb} \qquad P_{bf} \simeq t_{bf}\gamma_f$$

Determination of First Flight Probabilities

- Transmission probabilities t_{bf} and t_{fb}
  - Generally determined by collision probability calculation with fuel XS bigger than 5 cm^{-1} (e.g., 5000, black fuel approximation) $\Rightarrow \quad \gamma_f^\infty = 1$

$$t_{bf} = t_{bf}\gamma_f^\infty = \gamma_f^{b,\infty} = \frac{4V_f}{S_b}\Sigma_t^f P_{fb}^\infty = \frac{4}{S_b}\Sigma_t^f V_f (1 - \sum_{j=1}^{n} P_{fj}^\infty)$$

$$t_{fb} = \frac{S_b}{S_f} t_{bf}$$

* Superscript ∞ and 0 denote the limiting values for fuel with $\Sigma_f = \infty$ and $\Sigma_f = 0$

- First flight cell blackness

$$1 - P_{bb} = \sum_{j \neq f} P_{bj}^\infty \Big\lfloor^{a} + t_{bf}\gamma_f \Big\lfloor^{b} + t_{bf}(1-\gamma_f)(1-t_{fb}) \Big\lfloor^{c}$$

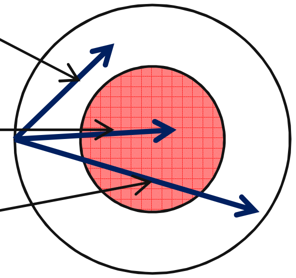
$$= t_{bf}t_{fb}\gamma_f + \sum_{j \neq f} P_{bj}^\infty + t_{bf}(1 - t_{fb})$$

$$= \frac{S_f}{S_b} t_{fb}^2 \underbrace{\bar{R}_f \Sigma_f}_{x} P_{esc}^f + \gamma_b^0 = \frac{S_f}{S_b} t_{fb}^2 x (1 - P_{ff}) + \gamma_b^0$$

$\lfloor^{a}$ collisions of neutrons that have never reached the fuel

$\lfloor^{b}$ collisions inside the fuel

$\lfloor^{c}$ collisions of neutrons that have traversed the fuel



- Cell transmission probability

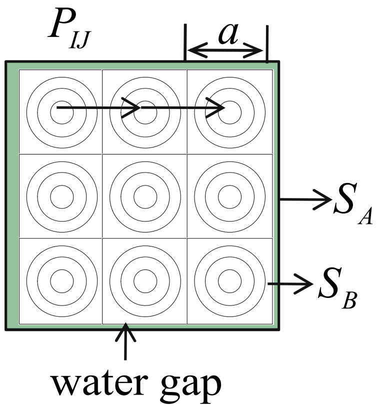
$$P_{bb} = 1 - \frac{S_f}{S_b} t_{fb}^2 x (1 - P_{ff}) - \gamma_b^0$$

First flight cell blackness

$$\gamma_b = \sum_j \gamma_{bj} = 1 - P_{bb}$$

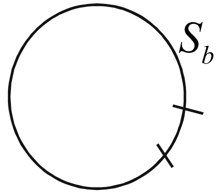
P_{bb} is used rather than γ_b for the use of rational expression that invloves x (contained in γ_f)

Reflection Probability (R) for an Assembly



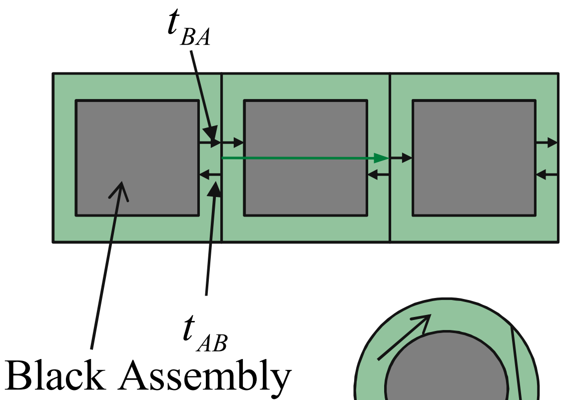
- Ratio of assembly surface area (S_B) to total surface area of $M = n \times n$ cells
  - Fraction of neutrons exiting a cell that escapes through the assembly surface
  - These neutrons encounter the assembly gap

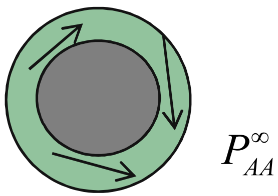
$$f = \frac{S_B}{MS_b} = \frac{4 \cdot na}{n^2 \cdot 4a} = \frac{1}{n} = \frac{1}{\sqrt{M}}$$



$\frac{S_B}{M}$ open to gap

- Fraction of neutrons leaving a FA that returns to a FA without being lost in the FA gap





$$g = t_{BA} t_{AB} + t_{BA} \underbrace{P_{AA}^{\infty}}_{\text{assembly bypass fraction}} t_{AB} + \underbrace{t_{BA} P_{AA}^{\infty\,2} t_{AB}}_{\text{absorbed in the 4th FA}} + \cdots = \boxed{\frac{t_{BA} t_{AB}}{1 - P_{AA}^{\infty}}}$$

- Reflection probability for a neutron leaving a cell to return to the cell in a lattice

$$\boxed{R = (1 - f) + f \cdot g}$$

With no gap, $g = 1 \quad \Rightarrow \quad R = 1.$

Fuel Collision Probability in Assembly

- Lattice enhanced fuel-to-fuel collision probability

$$X_{FF}=\frac{P_{fb}RP_{bf}}{1-RP_{bb}}=\frac{[(1-P_{ff})t_{fb}][t_{bf}\gamma_f]}{\frac{1}{R}-P_{bb}}=\frac{(1-P_{ff})t_{fb}\left[\frac{S_f}{S_b}t_{fb}\right]x(1-P_{ff})}{\frac{1}{R}-\left(1-\frac{S_f}{S_b}t_{fb}^2x(1-P_{ff})-\gamma_b^0\right)}=\frac{x(1-P_{ff})^2\frac{S_f}{S_b}t_{fb}^2}{\frac{1-R}{R}+\frac{S_f}{S_b}t_{fb}^2x(1-P_{ff})+\gamma_b^0}$$

$$=\frac{x(1-P_{ff})^2}{x(1-P_{ff})+\underbrace{\frac{S_b}{S_ft_{fb}^2}\gamma_b^0}_{A}+\underbrace{\frac{S_b}{S_ft_{fb}^2}\frac{f(1-g)}{1-f+f\cdot g}}_{B}}=\frac{x(1-P_{ff})^2}{x(1-P_{ff})+A+B}=\frac{\gamma_f P_{esc}^f}{\gamma_f+A+B}$$

If no gap, $g=1\Rightarrow B=0.$

- Ratio of fuel escape probability of assembly to that of cell

$$P_{FF}>P_{ff}\quad\Rightarrow\quad 1-P_{FF}<1-P_{ff}\quad\Rightarrow\quad P_{esc}^F<P_{esc}^f\quad\Rightarrow\quad P_{esc}^F=DP_{esc}^f\ \text{with}\ D<1$$

$$D=\frac{P_{esc}^F}{P_{esc}^f}=\frac{P_{esc}^f-X_{FF}}{P_{esc}^f}=1-\frac{\gamma_f}{\gamma_f+A+B}$$

$$\lim_{\Sigma_t^f\to\infty}\gamma_f=1\quad\Rightarrow\quad\lim_{\Sigma_t^f\to\infty}D\equiv D_B=1-\frac{1}{1+\underbrace{A+B}_{C}}=1-\frac{1}{1+C}=\frac{C}{1+C}$$

(Black Dancoff factor)

Dancoff Factor and Wigner Approximation

- Various definitions of black Dancoff factor
  - Ratio of fuel escape probability of fuel assembly to that of cell in case of black fuel
  - First flight blackness of all other materials than fuel
  - Probability of a neutron leaving a fuel pin (black fuel pin) to have its first collision in other materials than fuel

$$D_B^\infty = 1 - \frac{1}{1 + A + \cancel{B}} = \frac{A}{1+A} \text{ for infinite cell lattice } (B = 0 \text{ with no gap})$$

- Single-term rational approximation

$$P_{ff} = \frac{x}{x+a} = \frac{\bar{R}_f \Sigma_t^f}{\bar{R}_f \Sigma_t^f + a} = \frac{\Sigma_t^f}{\Sigma_t^f + a\Sigma_e^f} \quad (a: \text{ Bell factor; } \Sigma_e^f = 1/\bar{R}_f: \text{ escape cross section})$$

$$P_{FF} = P_{ff} + \frac{x(1-P_{ff})^2}{x(1-P_{ff})+C} = \frac{x}{x+a} + \frac{x\left(\frac{a}{x+a}\right)^2}{x\frac{a}{x+a}+C} = \frac{x}{x+a}\left(1 + \frac{\frac{a^2}{x+a}}{x\frac{a}{x+a}+C}\right) = \frac{x}{x+a}\left(\frac{a+C}{\frac{ax}{x+a}+C}\right)$$

$$= \frac{x}{x + \underbrace{\frac{aC}{a+C}}_{\alpha}} = \frac{x}{x+\alpha}$$

$$\boxed{\text{Fuel-to-fuel CP of assembly is also given in a rational function form!}}$$

Escape Cross Section

- Black Dancoff factor with rational approximation

$$D_B = \lim_{\Sigma_t^f >>1} \frac{1-P_{FF}}{1-P_{ff}} = \lim_{\Sigma_t^f >>1} \frac{\alpha/(x+\alpha)}{a/(x+\alpha)} = \frac{\alpha}{a} = \frac{aC}{a(a+C)} = \frac{C}{a+C}$$

$\alpha = aD_B$ for normal fuel

(α from Black Dancoff factor)

$$\lim_{\Sigma_t^f \to\infty} a = 1 \quad \Rightarrow \quad D_B = \frac{C}{1+C}$$ (Winger approximation is valid for black fuel)

- Fuel flux in assembly with rational approximation

$$\mathrm{P}_{FF} = \frac{x}{x+\alpha} = \frac{\bar{R}_f \Sigma_t^f}{\bar{R}_f \Sigma_t^f + \alpha} = \frac{\Sigma_t^f}{\Sigma_t^f + \alpha \Sigma_e^f} = \frac{\Sigma_t^f}{\Sigma_t^f + D_B a \Sigma_e^f} = \frac{\Sigma_t^f}{\Sigma_t^f + \Sigma_e^{f*}} \quad \Rightarrow \quad P_{esc}^F = 1 - P_{FF} = \frac{\Sigma_e^{f*}}{\Sigma_t^f + \Sigma_e^{f*}}$$

$$\varphi_F(u) = \frac{\lambda_f \Sigma_p^f \tilde{P}_{FF} + \Sigma_t^f(u) P_{esc}^F}{\Sigma_t^f(u) - (1-\lambda_f)\Sigma_s^f(u)\tilde{P}_{FF}} = \frac{\lambda_f \Sigma_p^f \dfrac{\cancel{\Sigma_t^f}}{\Sigma_t^f + \Sigma_e^{f*}} + \cancel{\Sigma_t^f} \dfrac{\Sigma_e^{f*}}{\Sigma_t^f + \Sigma_e^{f*}}}{\cancel{\Sigma_t^f} - (1-\lambda_f)\Sigma_s^f \dfrac{\cancel{\Sigma_t^f}}{\Sigma_t^f + \Sigma_e^{f*}}} = \frac{\lambda_f \Sigma_p^f + \Sigma_e^{f*}}{\Sigma_t^f + \Sigma_e^{f*} - (1-\lambda_f)\Sigma_s^f}$$

$$= \frac{\lambda_f \Sigma_p^f + \Sigma_e^{f*}}{\Sigma_a^f(u) + \lambda_f \Sigma_s^f(u) + \Sigma_e^{f*}} = \frac{\lambda_f \Sigma_p^f + \Sigma_e^{f*}}{\Sigma_a^f(u) + \lambda_f \Sigma_s^{res}(u) + \lambda_f \Sigma_p^f + \Sigma_e^{f*}} = \frac{\sigma_b}{\sigma_a^f(u) + \lambda_f \sigma_s^{res}(u) + \sigma_b}$$

$$\sigma_b = \frac{1}{N_F}(\lambda_f \Sigma_p^f + \Sigma_e^{f*})$$

Equivalent to homogeneous system
with additional escape cross section!

Two-Term Rational Approximation by Calvik

- Two-term rational approximation by Calvik (1962)

$$P_{ff} = x\left(\frac{b_1}{x+a_1} + \frac{b_2}{x+a_2}\right)$$

$$P_{ff} = 1 \text{ as } x \to \infty \text{ (black boundary condition)} \quad \Rightarrow \quad b_1 + b_2 = 1$$

$$\lim_{x>>1} x(1-P_{ff}) = \lim_{x>>1} xP_{esc}^{f} = \lim_{\Sigma_t^f>>1} \gamma_f = 1 \quad \Rightarrow \quad \lim_{x>>1}(1-P_{ff}) = \frac{1}{x} \quad \Rightarrow \quad \lim_{x>>1} P_{ff} = 1 - \frac{1}{x}$$

$$\Rightarrow \quad \lim_{x>>1} \frac{d}{dx} P_{ff} = \frac{1}{x^2} \quad \Rightarrow \quad \lim_{x>>1}\left[\frac{a_1 b_1}{(x+a_1)^2} + \frac{a_2 b_2}{(x+a_2)^2}\right] = \frac{1}{x^2} \quad \Rightarrow \quad a_1 b_1 + a_2 b_2 = 1$$

$$\lim_{x\to 0} \frac{d}{dx} P_{ff} = \frac{2}{3} \text{ (white boundary condition)} \quad \Rightarrow \quad \frac{b_1}{a_1} + \frac{b_2}{a_2} = \frac{2}{3}$$

$$\lim_{x<<1} P_{ff} = \frac{2}{3}x \text{ (for cylinder by Case, de Hoffman and Placzek)} \quad \Rightarrow \quad \lim_{x<<1}\left[\frac{b_1}{x+a_1} + \frac{b_2}{x+a_2}\right] = \frac{2}{3}$$

$$\Rightarrow \quad \lim_{x<<1} \frac{2a_1(a_1-1) + (2a_1-3)x}{(a_1+x)[3(a_1-1)+(2a_1-3)x]} = \frac{2}{3}$$

$$a_1 = 2, \quad a_2 = 3, \quad b_1 = 2, \quad b_2 = -1$$

$$P_{ff} = x\left(\frac{2}{x+2} - \frac{1}{x+3}\right)$$

N-term Rational Approximation

- Fuel-to-fuel collision probability of assembly

$$P_{ff} = \sum_{n=1}^{N} \frac{b_n x}{x+a_n}; \quad \sum_{n=1}^{N} b_n = 1 \quad (P_{ff} = 1 \text{ as } x \to \infty) \quad \Rightarrow \quad 1 - P_{ff} = \sum_{n=1}^{N} b_n - \sum_{n=1}^{N} \frac{b_n x}{x+a_n} = \sum_{n=1}^{N} \frac{a_n b_n}{x+a_n}$$

$$P_{FF} = P_{ff} + \frac{x(1-P_{ff})^2}{x(1-P_{ff})+C} = \frac{x(1-P_{ff})P_{ff} + x(1-P_{ff})^2 + CP_{ff}}{x(1-P_{ff})+C} = \frac{x(1-P_{ff}) + CP_{ff}}{x(1-P_{ff})+C}$$

$$= \frac{x\left(\sum_{n=1}^{N} \frac{a_n b_n}{x+a_n} + C\sum_{n=1}^{N} \frac{b_n}{x+a_n}\right)}{x\sum_{n=1}^{N} \frac{a_n b_n}{x+a_n} + C} = \frac{x\left(\sum_{n=1}^{N} a_n b_n \prod_{\substack{j=1 \\ j\neq n}}^{N}(x+a_j) + C\sum_{n=1}^{N} b_n \prod_{\substack{j=1 \\ j\neq n}}^{N}(x+a_j)\right)}{x\sum_{n=1}^{N} a_n b_n \prod_{\substack{j=1 \\ j\neq n}}^{N}(x+a_j) + C\prod_{n=1}^{N}(x+a_n)}$$

$$= x\frac{\left(\sum_{n=1}^{N} a_n b_n + C\sum_{n=1}^{N} b_n\right)x^{N-1} + \cdots}{\left(\sum_{n=1}^{N} a_n b_n + C\right)x^{N} + \cdots} = x\frac{x^{N-1} + d_{N-2}x^{N-2} + \cdots}{x^{N} + c_{N-1}x^{N-1} + \cdots} = x\frac{P_{N-1}(x)}{\prod_{n=1}^{N}(x+\alpha_n)} = x\sum_{n=1}^{N} \frac{\beta_n}{x+\alpha_n}$$

by method of partial fraction

$$\boxed{\text{coeff. of } x^{N-1} = 1 \Rightarrow \sum_{n=1}^{N} \beta_n = 1}$$

$$= \sum_{n=1}^{N} \frac{\beta_n \Sigma_t^f}{\Sigma_t^f + \alpha_n \Sigma_e^f} \Leftarrow \quad x = \bar{R}_f \Sigma_t^f = \frac{\Sigma_t^f}{\Sigma_e^f}$$

Flux with Rational Approximation

- Collision and escape probabilities

$$P_{FF}=\sum_{n=1}^{N}\frac{\beta_n\Sigma_t^f}{\Sigma_t^f+\alpha_n\Sigma_e^f}\quad P_{ESC}=1-P_{FF}=\sum_{n=1}^{N}\left(\beta_n-\frac{\beta_n\Sigma_t^f}{\Sigma_t^f+\alpha_n\Sigma_e^f}\right)=\sum_{n=1}^{N}\frac{\alpha_n\beta_n\Sigma_e^f}{\Sigma_t^f+\alpha_n\Sigma_e^f}$$

- IR approximation flux in fuel

$$\varphi_F(u)=\frac{\lambda_f\Sigma_p^f P_{FF}+\Sigma_t^f(u)P_{esc}^F}{\Sigma_t^f(u)-(1-\lambda_f)\Sigma_s^f(u)P_{FF}}$$

$$=\frac{\lambda_f\Sigma_p^f\sum_{n=1}^{N}\frac{\beta_n\Sigma_t^f}{\Sigma_t^f+\alpha_n\Sigma_e^f}+\cancel{\Sigma_t^f}\sum_{n=1}^{N}\frac{\alpha_n\beta_n\Sigma_e^f}{\Sigma_t^f+\alpha_n\Sigma_e^f}}{\cancel{\Sigma_t^f}-(1-\lambda_f)\Sigma_s^f\sum_{n=1}^{N}\frac{\beta_n\cancel{\Sigma_t^f}}{\Sigma_t^f+\alpha_n\Sigma_e^f}}=\frac{\sum_{n=1}^{N}(\lambda_f\Sigma_p^f+\alpha_n\Sigma_e^f)\frac{\beta_n}{\Sigma_t^f+\alpha_n\Sigma_e^f}}{1-(1-\lambda_f)\Sigma_s^f\sum_{n=1}^{N}\frac{\beta_n}{\Sigma_t^f+\alpha_n\Sigma_e^f}}$$

$$\lambda_f=1\quad\Rightarrow$$

- NR approximation flux

$$\varphi_F(u)=\sum_{n=1}^{N}\beta_n\frac{\Sigma_p^f+\alpha_n\Sigma_e^f}{\Sigma_t^f+\alpha_n\Sigma_e^f}=\sum_{n=1}^{N}\beta_n\frac{\Sigma_p^f+\alpha_n\Sigma_e^f}{\Sigma_a(u)+\Sigma_s^{Res}(u)+\Sigma_p^f+\alpha_n\Sigma_e^f}$$

approprimated separate equivalence with NR

$$\varphi_F(u)=\beta\frac{\Sigma_p^f+\alpha_1\Sigma_e^f}{\Sigma_t^f+\alpha_1\Sigma_e^f}+(1-\beta)\frac{\Sigma_p^f+\alpha_2\Sigma_e^f}{\Sigma_t^f+\alpha_2\Sigma_e^f}\quad\text{for two terms}$$

Effective Cross Section with Calvik's Rational Appr.

- NR flux in fuel

$$\varphi_F(u) = \beta\frac{\Sigma_p^f + \alpha_1\Sigma_e^f}{\Sigma_t^f + \alpha_1\Sigma_e^f} + (1-\beta)\frac{\Sigma_p^f + \alpha_2\Sigma_e^f}{\Sigma_t^f + \alpha_2\Sigma_e^f} = \beta\frac{\sigma_p^f + \alpha_1\sigma_e^f}{\sigma_t^f + \alpha_1\sigma_e^f} + (1-\beta)\frac{\sigma_p^f + \alpha_2\sigma_e^f}{\sigma_t^f + \alpha_2\sigma_e^f} \quad \left(\sigma_e^f = \frac{S_f}{4V_f N_f}\right)$$

$$= \beta\frac{\sigma_p^f + \alpha_1\sigma_e^f}{\sigma_a^f(u) + \sigma_s^{res}(u) + \sigma_p^f + \alpha_1\sigma_e^f} + (1-\beta)\frac{\sigma_p^f + \alpha_2\sigma_e^f}{\sigma_a^f(u) + \sigma_s^{res}(u) + \sigma_p^f + \alpha_2\sigma_e^f}$$

$$= \beta\frac{\sigma_{b1}}{\sigma_a(u) + \sigma_{b1}} + (1-\beta)\frac{\sigma_{b2}}{\sigma_a(u) + \sigma_{b2}} \quad \text{with neglect of } \sigma_s^{res} \text{ through adjusted RI's}$$

- Effective cross section

$$\bar{\sigma}_g = \frac{\int_g \left(\beta\frac{\sigma_a(u)\sigma_{b1}}{\sigma_a(u) + \sigma_{b1}} + (1-\beta)\frac{\sigma_a(u)\sigma_{b2}}{\sigma_a(u) + \sigma_{b2}}\right)du}{\int_g \left(\beta\frac{\sigma_{b1}}{\sigma_a(u) + \sigma_{b1}} + (1-\beta)\frac{\sigma_{b2}}{\sigma_a(u) + \sigma_{b2}}\right)du}$$

$$\boxed{\bar{\sigma}_{i,g} = \bar{\sigma}_g \frac{\bar{I}_g(\sigma_p^f + D_i\sigma_e^f)}{\bar{I}_g(\sigma_p^f + D_B^\infty\sigma_e^f)}}$$

with position-dependent Dancoff

$$= \frac{\beta\bar{I}_g(\sigma_{b1}) + (1-\beta)\bar{I}_g(\sigma_{b2})}{1 - \left(\beta\frac{\bar{I}_g(\sigma_{b1})}{\sigma_{b1}} + (1-\beta)\frac{\bar{I}_g(\sigma_{b2})}{\sigma_{b2}}\right)} \Rightarrow \text{Equivalence Theorem used in CASMO}$$