NUCL 510 Nuclear Reactor Theory I
Fall 2011

Midterm Exam

October 6, 2011

Name:

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1. Assuming that the neutron half-life is 12 minutes, calculate the relative probability that a neutron moving in a speed of 2200 m/sec will undergo radioactive decay before being absorbed in an infinite medium with macroscopic absorption cross section of 0.022 cm^{-1}?

Absorption mean free path: $\lambda_a = \dfrac{1}{\Sigma_a}$

Absorption mean lifetime: $t_a = \dfrac{\lambda_a}{v} = \dfrac{1}{\Sigma_a v}$

Decay mean lifetime: $t_d = \dfrac{1}{\lambda} = \dfrac{T_{1/2}}{\ln 2}$

Relative probability to decay before being absorbed

$$P = \frac{t_a}{t_d} = \frac{\ln 2}{\Sigma_a v T_{1/2}} = \frac{\ln 2}{0.022\text{cm}^{-1} \times (2200 \times 100\text{cm/s}) \times 12 \times 60\text{s}} = 1.99 \times 10^{-7}$$

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2. The scattering cross section of H-1 at a certain energy is isotropic in the center of mass system (CMS) and the total scattering cross section is σ_s. Using the relation between the scattering angle (θ_c) in the center of mass system (CMS) and that (θ_s) in the laboratory system (LS) $\cos\theta_s = (1 + A\cos\theta_c)/\sqrt{A^2 + 2A\cos\theta_c + 1}$, (a) show that $\theta_s = \theta_c / 2$, (b) determine the differential scattering cross section in CMS, and (c) determine the differential cross section in LS.

   (a) For $A = 1$, $\cos\theta_s = \sqrt{\dfrac{1+\cos\theta_c}{2}} \quad\Rightarrow\quad 2\cos^2\theta_s - 1 = \cos\theta_c$

   $\Rightarrow \quad \cos(2\theta_s) = \cos\theta_c \quad\Rightarrow\quad 2\theta_s = \theta_c$

   (b) Since the scattering isotropic in CMS

   $$\sigma_s(\theta_c, \varphi_c) = \frac{\sigma_s}{4\pi} \quad\Rightarrow\quad \sigma_s(\mu_c) = \frac{\sigma_s}{2}$$

   (c) $\sigma_s(\mu_s) = \sigma_s(\mu_c)\left|\dfrac{d\mu_c}{d\mu_s}\right|$

   $$\mu_s = \sqrt{\frac{1+\mu_c}{2}} \quad\Rightarrow\quad d\mu_s = \frac{1}{2\sqrt{2(1+\mu_c)}}\,d\mu_c$$

   $$\sigma_s(\mu_s) = \frac{\sigma_s}{2} \times 2\sqrt{2(1+\mu_c)} = \sigma_s\sqrt{2(1+\mu_c)}$$

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3. The angular flux at a position $\vec{r}$ is given by $\psi(\vec{\Omega}) = \psi_0 + \psi_x\Omega_x + \psi_y\Omega_y + \psi_z\Omega_z$, where Ω_x, Ω_y and Ω_z are respectively the x-, y-, and z-directional components of the angular direction vector $\vec{\Omega}$. Determine (a) the scalar flux ϕ, (b) the current density vector (J_x, J_y, J_z), and (c) partial currents J^+ and J^- with respect to the normal vector which is parallel to the x axis.

(a) $\Omega_x = \sin\theta\cos\varphi, \quad \Omega_y = \sin\theta\sin\varphi, \quad \Omega_z = \cos\theta$

$$\phi = \int_{4\pi} d\Omega\,\psi(\vec{\Omega}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta [\psi_0 + \psi_x\Omega_x + \psi_y\Omega_y + \psi_z\Omega_z]$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta = 4\pi$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_x = \int_0^{2\pi} d\varphi \cos\varphi \int_0^{\pi} d\theta \sin^2\theta = 0$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_y = \int_0^{2\pi} d\varphi \sin\varphi \int_0^{\pi} d\theta \sin^2\theta = 0$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_z = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\cos\theta = \pi \int_0^{\pi} d\theta \sin 2\theta = 0$$

$$\therefore\ \phi = 4\pi\psi_0$$

(b) $$J_x = \int_{4\pi} d\Omega\Omega_x\,\psi(\vec{\Omega}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta [\psi_0\Omega_x + \psi_x\Omega_x\Omega_x + \psi_y\Omega_x\Omega_y + \psi_z\Omega_x\Omega_z]$$

$$J_y = \int_{4\pi} d\Omega\Omega_y\,\psi(\vec{\Omega}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta [\psi_0\Omega_y + \psi_x\Omega_y\Omega_x + \psi_y\Omega_y\Omega_y + \psi_z\Omega_y\Omega_z]$$

$$J_z = \int_{4\pi} d\Omega\Omega_z\,\psi(\vec{\Omega}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta [\psi_0\Omega_z + \psi_x\Omega_z\Omega_x + \psi_y\Omega_z\Omega_y + \psi_z\Omega_z\Omega_z]$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_x\Omega_x = \int_0^{2\pi} d\varphi \cos^2\varphi \int_0^{\pi} d\theta \sin^3\theta = \int_0^{2\pi} d\varphi \frac{1+\cos(2\varphi)}{2} \int_{-1}^{1} d\mu (1-\mu^2) = \frac{4\pi}{3}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_y\Omega_y = \int_0^{2\pi} d\varphi \sin^2\varphi \int_0^{\pi} d\theta \sin^3\theta = \int_0^{2\pi} d\varphi \frac{1-\cos(2\varphi)}{2} \int_{-1}^{1} d\mu (1-\mu^2) = \frac{4\pi}{3}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_z\Omega_z = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\cos^2\theta = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\mu\mu^2 = \frac{4\pi}{3}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_x\Omega_y = \int_0^{2\pi} d\varphi \cos\varphi\sin\varphi \int_0^{\pi} d\theta \sin^2\theta = \int_0^{2\pi} d\varphi \frac{\sin(2\varphi)}{2} \int_0^{\pi} d\theta \sin^2\theta = 0$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_y\Omega_z = \int_0^{2\pi} d\varphi \sin\varphi \int_0^{\pi} d\theta \sin^2\theta\cos\theta = 0$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,\Omega_z\Omega_x = \int_0^{2\pi} d\varphi \cos\varphi \int_0^{\pi} d\theta \sin^2\theta\cos\theta = 0$$

$$J_x = \frac{4\pi}{3}\psi_x, \quad J_y = \frac{4\pi}{3}\psi_y, \quad J_z = \frac{4\pi}{3}\psi_z$$

(c) $\vec{\Omega}\cdot\vec{n}=(\Omega_x,\Omega_y,\Omega_z)\cdot(1,0,0)=\Omega_x$

$$J^+=\int_{\vec{\Omega}\cdot\vec{n}>0}d\Omega(\vec{\Omega}\cdot\vec{n})\psi(\vec{\Omega})=\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta[\psi_0\Omega_x+\psi_x\Omega_x\Omega_x+\psi_y\Omega_x\Omega_y+\psi_z\Omega_x\Omega_z]$$

$$\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta\,\Omega_x=\int_{-\pi/2}^{\pi/2}d\varphi\cos\varphi\int_0^{\pi}d\theta\sin^2\theta=\pi$$

$$\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta\,\Omega_x\Omega_x=\int_{-\pi/2}^{\pi/2}d\varphi\cos^2\varphi\int_0^{\pi}d\theta\sin^3\theta=\int_{-\pi/2}^{\pi/2}d\varphi\frac{1+\cos(2\varphi)}{2}\int_{-1}^{1}d\mu(1-\mu^2)=\frac{2\pi}{3}$$

$$\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta\,\Omega_x\Omega_y=\int_{-\pi/2}^{\pi/2}d\varphi\cos\varphi\sin\varphi\int_0^{\pi}d\theta\sin^2\theta=\int_{-\pi/2}^{\pi/2}d\varphi\frac{\sin(2\varphi)}{2}\int_0^{\pi}d\theta\sin^2\theta=0$$

$$\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta\,\Omega_z\Omega_x=\int_{-\pi/2}^{\pi/2}d\varphi\cos\varphi\int_0^{\pi}d\theta\sin^2\theta\cos\theta=0$$

$$J^+=\pi\psi_0+\frac{2\pi}{3}\psi_x$$

$$J^-=\int_{\vec{\Omega}\cdot\vec{n}<0}d\Omega\,|\vec{\Omega}\cdot\vec{n}|\,\psi(\vec{\Omega})=-\int_{\pi/2}^{3\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta[\psi_0\Omega_x+\psi_x\Omega_x\Omega_x+\psi_y\Omega_x\Omega_y+\psi_z\Omega_x\Omega_z]$$

$$\int_{\pi/2}^{3\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta\,\Omega_x=\int_{\pi/2}^{3\pi/2}d\varphi\cos\varphi\int_0^{\pi}d\theta\sin^2\theta=-\pi$$

$$\int_{\pi/2}^{3\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta\,\Omega_x\Omega_x=\int_{\pi/2}^{3\pi/2}d\varphi\cos^2\varphi\int_0^{\pi}d\theta\sin^3\theta=\int_{\pi/2}^{3\pi/2}d\varphi\frac{1+\cos(2\varphi)}{2}\int_{-1}^{1}d\mu(1-\mu^2)=\frac{2\pi}{3}$$

$$\int_{\pi/2}^{3\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta\,\Omega_x\Omega_y=\int_{\pi/2}^{3\pi/2}d\varphi\cos\varphi\sin\varphi\int_0^{\pi}d\theta\sin^2\theta=\int_{\pi/2}^{3\pi/2}d\varphi\frac{\sin(2\varphi)}{2}\int_0^{\pi}d\theta\sin^2\theta=0$$

$$\int_{\pi/2}^{3\pi/2}d\varphi\int_0^{\pi}d\theta\sin\theta\,\Omega_z\Omega_x=\int_{\pi/2}^{3\pi/2}d\varphi\cos\varphi\int_0^{\pi}d\theta\sin^2\theta\cos\theta=0$$

$$J^+=\pi\psi_0-\frac{2\pi}{3}\psi_x$$

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4. The distribution of source neutrons (including fission and scattering neutrons) in a sphere of radius R is known to be $S(r,E)$, where r is the distance from the center of the sphere and E is the neutron energy. For a position-independent, macroscopic total cross section $\Sigma_t(E)$, derive the integral transport equation for the scalar flux at the center of the sphere $\phi(0,E)$.

   Source $S(r,E)$ is independent of angular variables, that is, isotropic. Thus the contribution from the source in a volume element $dV = 4\pi r^2 dr$ around a radius r to the flux at the origin is

   $$d\phi = S(r,E)\frac{e^{-\Sigma_t(E)r}}{4\pi r^2}dV = S(r,E)e^{-\Sigma_t(E)r}dr$$

   Thus the flux at the origin is given by

   $$\phi(0,E) = \int_V dV\,\frac{S(r,E)e^{-\Sigma_t(E)r}}{4\pi r^2} = \int_0^R drS(r,E)e^{-\Sigma_t(E)r}$$

5. For a medium of linearly anisotropic scattering, the Boltzmann equation in the one-dimensional plane geometry can be written as

$$\mu\frac{d}{dx}\psi(x,E,\mu)+\Sigma_t(x,E)\psi(x,E,\mu)=\frac{\lambda}{2}\chi(E)\int_{E'}dE'\nu\Sigma_f(\bar{r},E')\phi(\bar{r},E')$$
$$+\frac{1}{2}\int_{E'}dE'\Sigma_s(x,E'\to E)\phi(x,E')+\frac{3}{2}\mu\int_{E'}dE'\Sigma_{s1}(x,E'\to E)J(x,E')$$

where ϕ is the scalar flux and J is the current in the x direction.

(a) By integrating the transport equation over μ from -1 to 1, derive the neutron balance equation.

(b) By approximating the angular flux with a linear function of angular variable μ as $\psi(x,E,\mu)=\phi(x,E)/2+3J(x,E)\mu/2$ and by integrating the transport equation over μ with a weight function $P_1(\mu)=\mu$, derive the relation between the scalar flux ϕ and the current J, i.e., the P_1 equation.

(c) Under the assumption of no energy loss in anisotropic scattering, derive the diffusion equation.

(a) Balance equation

$$\frac{d}{dx}\int_{-1}^{1}d\mu\mu\psi(x,E,\mu)+\Sigma_t(x,E)\int_{-1}^{1}d\mu\psi(x,E,\mu)=\frac{\lambda}{2}\chi(E)\int_{-1}^{1}d\mu\int_{E'}dE'\nu\Sigma_f(\bar{r},E')\phi(\bar{r},E')$$
$$+\frac{1}{2}\int_{-1}^{1}d\mu\int_{E'}dE'\Sigma_s(x,E'\to E)\phi(x,E')+\frac{3}{2}\int_{-1}^{1}d\mu\mu\int_{E'}dE'\Sigma_{s1}(x,E'\to E)J(x,E')$$

$$\frac{d}{dx}J(x,E)+\Sigma_t(x,E)\phi(x,E)=\lambda\chi(E)\int_{E'}dE'\nu\Sigma_f(\bar{r},E')\phi(\bar{r},E')$$
$$+\int_{E'}dE'\Sigma_s(x,E'\to E)\phi(x,E')$$

(b) P_1 equation

$$\frac{d}{dx}\int_{-1}^{1}d\mu\mu^2\frac{\phi+3\mu J}{2}+\Sigma_t(x,E)\int_{-1}^{1}d\mu\mu\frac{\phi+3\mu J}{2}=\frac{\lambda}{2}\chi(E)\int_{-1}^{1}d\mu\mu\int_{E'}dE'\nu\Sigma_f(\bar{r},E')\phi(\bar{r},E')$$
$$+\frac{1}{2}\int_{-1}^{1}d\mu\mu\int_{E'}dE'\Sigma_s(x,E'\to E)\phi(x,E')+\frac{3}{2}\int_{-1}^{1}d\mu\mu^2\int_{E'}dE'\Sigma_{s1}(x,E'\to E)J(x,E')$$

$$\frac{1}{3}\frac{d}{dx}\phi(x,E)+\Sigma_t(x,E)J(x,E)=\int_{E'}dE'\Sigma_{s1}(x,E'\to E)J(x,E')$$

(c) Under assumption of no energy loss in anisotropic scattering, the P_1 equation becomes

$$\frac{1}{3}\frac{d}{dx}\phi(x,E)+\Sigma_t(x,E)J(x,E)=\Sigma_{s1}(x,E)J(x,E)$$

$$J(x,E) = -\frac{1}{3[\Sigma_t(x,E) - \Sigma_{s1}(x,E)]}\frac{d}{dx}\phi(x,E) = -D(x,E)\frac{d}{dx}\phi(x,E)$$

$$-\frac{d}{dx}D(x,E)\frac{d}{dx}\phi(x,E) + \Sigma_t(x,E)\phi(x,E) = \lambda\chi(E)\int_{E'} dE'\nu\Sigma_f(\vec{r},E')\phi(\vec{r},E')$$
$$+\int_{E'} dE'\Sigma_s(x,E' \to E)\phi(x,E')$$

6. Under the assumption of no up-scattering and $\chi_1 = 1$, the 2-group diffusion equation in a one-region slab reactor can be written as

$$-D_1 \frac{d^2}{dx^2}\phi_1(x) + \Sigma_{r1}\phi_1(x) = \frac{1}{k}[\nu\Sigma_{f1}\phi_1(x) + \nu\Sigma_{f2}\phi_2(x)]$$

$$-D_2 \frac{d^2}{dx^2}\phi_2(x) + \Sigma_{a2}\phi_2(x) = \Sigma_{s1\to 2}\phi_1(x)$$

(a) Assuming that the neutron flux is separable in its space and energy dependence, i.e., $\phi_1(x) = \varphi_1\phi(x)$ and $\phi_2(x) = \varphi_2\phi(x)$, derive the differential equation for spatial flux shape $\phi(x)$ and a matrix equation for the energy dependence vector (φ_1, φ_2).

(b) Determine the multiplication factor k when the slab thickness is a.

(c) Determine the infinite multiplication factor k_∞ when a goes to the infinity.

(a) $\phi_1(x) = \varphi_1\phi(x)$ and $\phi_2(x) = \varphi_2\phi(x)$

$$-D_1\varphi_1 \frac{d^2}{dx^2}\phi(x) + \Sigma_{r1}\varphi_1\phi(x) = \frac{1}{k}[\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2]\phi(x)$$

$$-D_2\varphi_2 \frac{d^2}{dx^2}\phi(x) + \Sigma_{a2}\varphi_2\phi(x) = \Sigma_{s1\to 2}\varphi_1\phi(x)$$

$$-D_1\varphi_1 \frac{\frac{d^2}{dx^2}\phi(x)}{\phi(x)} + \Sigma_{r1}\varphi_1 = \frac{1}{k}[\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2]$$

$$-D_2\varphi_2 \frac{\frac{d^2}{dx^2}\phi(x)}{\phi(x)} + \Sigma_{a2}\varphi_2 = \Sigma_{s1\to 2}\varphi_1$$

$$\frac{\frac{d^2}{dx^2}\phi(x)}{\phi(x)} = -B^2 \quad \Rightarrow \quad \frac{d^2}{dx^2}\phi(x) + B^2\phi(x) = 0$$

$$\begin{cases} (D_1B^2 + \Sigma_{r1})\varphi_1 = \frac{1}{k}[\nu\Sigma_{f1}\varphi_1 + \nu\Sigma_{f2}\varphi_2] \\ (D_2B^2 + \Sigma_{a2})\varphi_2 = \Sigma_{s1\to 2}\varphi_1 \end{cases}$$

(b) $B^2 = \left(\frac{\pi}{a}\right)^2$

$$\begin{bmatrix} D_1B^2 + \Sigma_{r1} - \frac{1}{k}\nu\Sigma_{f1} & -\frac{1}{k}\nu\Sigma_{f2} \\ -\Sigma_{s1\to 2} & D_2B^2 + \Sigma_{a2} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = 0$$

$$\left(D_1B^2+\Sigma_{r1}-\frac{1}{k}\nu\Sigma_{f1}\right)\left(D_2B^2+\Sigma_{a2}\right)-\frac{1}{k}\nu\Sigma_{f2}\Sigma_{s1\to 2}=0$$

$$\frac{1}{k}\left[\nu\Sigma_{f1}\left(D_2B^2+\Sigma_{a2}\right)+\nu\Sigma_{f2}\Sigma_{s1\to 2}\right]=\left(D_1B^2+\Sigma_{r1}\right)\left(D_2B^2+\Sigma_{a2}\right)$$

$$k=\frac{\nu\Sigma_{f1}}{D_1B^2+\Sigma_{r1}}+\frac{\Sigma_{s1\to 2}}{D_1B^2+\Sigma_{r1}}\frac{\nu\Sigma_{f2}}{D_2B^2+\Sigma_{a2}}$$

(c) $B^2=\left(\frac{\pi}{a}\right)^2\to 0$ as $a\to\infty$

$$k=\frac{\nu\Sigma_{f1}}{D_1B^2+\Sigma_{r1}}+\frac{\Sigma_{s1\to 2}}{D_1B^2+\Sigma_{r1}}\frac{\nu\Sigma_{f2}}{D_2B^2+\Sigma_{a2}}\quad\to\quad k_\infty=\frac{\nu\Sigma_{f1}}{\Sigma_{r1}}+\frac{\Sigma_{s1\to 2}}{\Sigma_{r1}}\frac{\nu\Sigma_{f2}}{\Sigma_{a2}}$$