Algebraic Number Theory, a Computational Approach

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Contents

Preface

This book is based on notes for a one-semester undergraduate course on Algebraic Number Theory, which the author taught at Harvard during Spring 2004 and Spring 2005, then at UCSD and at University of Washington many times as a graduate course. This book was mainly inspired by the [SD01, Ch. 1] and Cassels's article *Global Fields* in [Cas67]. Travis Scholl also very carefully read the entire book and made numerous improvements throughout as part of a reading course in 2015.

Please send any typos or corrections to `wstein@gmail.com`.

Acknowledgement: This book closely builds on Swinnerton-Dyer's book [SD01] and Cassels's article [Cas67]. Many of the students of Math 129 at Harvard during Spring 2004 and 2005 made helpful comments: Jennifer Balakrishnan, Peter Behrooz, Jonathan Bloom, David Escott Jayce Getz, Michael Hamburg, Deniz Kural, Danielle Li, Andrew Ostergaard, Gregory Price, Grant Schoenebeck, Jennifer Sinnott, Stephen Walker, Daniel Weissman, and Inna Zakharevich in 2004; Mauro Braunstein, Steven Byrnes, William Fithian, Frank Kelly, Alison Miller, Nizameddin Ordulu, Corina Patrascu, Anatoly Preygel, Emily Riehl, Gary Sivek, Steven Sivek, Kaloyan Slavov, Gregory Valiant, and Yan Zhang in 2005. Also the course assistants Matt Bainbridge and Andrei Jorza made many helpful comments. The mathematical software [S$^+$11], [PAR], and [BCP97] were used in writing this book.

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Chapter 1

Introduction

1.1 Mathematical background

In addition to general mathematical maturity, this book assumes you have the following background:

- Basics of finite group theory
- Commutative rings, ideals, quotient rings
- Some elementary number theory
- Basic Galois theory of fields
- Point set topology
- Basics of topological rings, groups, and measure theory

For example, if you have never worked with finite groups before, you should read another book first. If you haven't seen much elementary ring theory, there is still hope, but you will have to do some additional reading and exercises. We will briefly review the basics of the Galois theory of number fields.

Some of the homework problems involve using a computer, but there are examples which you can build on. We will not assume that you have a programming background or know much about algorithms. Most of the book uses Sage (`http://sagemath.org`), which is free open source mathematical software. The following is an example Sage session:

```
2 + 2
```

```
4
```

```
k.<a> = NumberField(x^2 + 1); k
```

```
Number Field in a with defining polynomial x^2 + 1
```

1.2 What is algebraic number theory?

A number field K is a finite degree extension of the rational numbers $\mathbb{Q}$. The primitive element theorem from field theory asserts that every such extension can be represented as the set of all polynomials of degree less than $d = [K : \mathbb{Q}] = \dim_{\mathbb{Q}} K$ in a single root α of some polynomial with coefficients in $\mathbb{Q}$:

$$K = \mathbb{Q}(\alpha) = \left\{ \sum_{n=0}^{d-1} a_n \alpha^n : a_n \in \mathbb{Q} \right\}.$$

Note that $\mathbb{Q}(\alpha)$ is non-canonically isomorphic to $\mathbb{Q}[x]/(f)$, where f is the minimal polynomial of α over $\mathbb{Q}$. The homomorphism $\mathbb{Q}[x] \to \mathbb{Q}(\alpha)$ that sends x to α has kernel (f), hence it induces an isomorphism between $\mathbb{Q}[x]/(f)$ and $\mathbb{Q}(\alpha)$. It is not canonical, since $\mathbb{Q}(\alpha)$ could have nontrivial automorphisms. For example, if $\alpha = \sqrt{2}$, then $\mathbb{Q}(\sqrt{2})$ is isomorphic as a field to $\mathbb{Q}(-\sqrt{2})$ via $\sqrt{2} \mapsto -\sqrt{2}$. There are two isomorphisms $\mathbb{Q}[x]/(x^2 - 2) \to \mathbb{Q}(\sqrt{2})$.

Algebraic number theory is the study of number fields, their rings of integers, and related objects (e.g., functions fields, elliptic curves, etc.). To gain a deeper understanding of these concepts, one uses techniques from (mostly commutative) algebra and finite group theory. The main objects that we study in this book are number fields, rings of integers of number fields, unit groups, ideal class groups, norms, traces, discriminants, prime ideals, Hilbert and other class fields and associated reciprocity laws, zeta and L-functions, and algorithms for computing with each of the above.

1.2.1 Topics in this book

These are some of the main topics that are discussed in this book:

- Rings of integers of number fields
- Unique factorization of nonzero ideals in Dedekind domains
- Structure of the group of units of the ring of integers
- Fractional ideals and class groups
- Decomposition and inertia groups, Frobenius elements
- Ramification
- Discriminant and different
- Quadratic and biquadratic fields
- Cyclotomic fields (and applications)
- How to use Sage to compute with many of the above objects

We will also touch on elliptic curves and L-functions. However we will not do anything nontrivial with these subjects.

1.3 Some applications of algebraic number theory

The following examples illustrate some of the power, depth, and importance of algebraic number theory.

Integer factorization: The number field sieve is the asymptotically fastest known algorithm for factoring general large integers (that don't have too special of a form). On December 12, 2009, the number field sieve was used to factor the RSA-768 challenge, which is a 232 digit number that is a product of two primes:

```
rsa768 = 123018668453011775513049495838496272077285356959\
533479219732245215172640050726365751874520219978646938\
995647494277406384592519255732630345373154826850791702612\
214291346167042921431160222124047927473779408066535141959\
7459856902143413
n = 334780716989568987860441698482126908177047949837137685\
689124313889828837938780022876147116525317430877378144\
67999489
m = 367460436667995904282446337996279526322791581643430876\
426760322838157396665112792333734171433968102700927987\
36308917
n*m == rsa768
```

```
True
```

This record integer factorization cracked a certain 768-bit public key cryptosystem (see [KAF+10]), thus establishing a lower bound on one's choice of key size:

```
$ man ssh-keygen   # in ubuntu-12.04
...
     -b bits
             Specifies the number of bits in the key to create.
             For RSA keys, the minimum size is 768 bits ...
```

Primality testing: Agrawal and his students Saxena and Kayal found in 2002 the first ever deterministic polynomial-time (in the number of digits) primality test [AKS04]. Their methods involve arithmetic in quotients of $(\mathbb{Z}/n\mathbb{Z})[x]$, which are best understood in the context of algebraic number theory.

Deeper point of view: Some questions in number theory are best viewed from the point of view of algebraic number theory such as:

- Pell's Equation $x^2 - dy^2 = 1$ can be reinterpreted in terms of units in real quadratic fields, which leads to a study of unit groups of number fields.
- Integer factorization is a special case of factoring nonzero ideals in rings of integers of number fields.

- The Riemann hypothesis about the zeros of $\zeta(s)$ generalizes to zeta functions of number fields.
- Reinterpreting Gauss's quadratic reciprocity law in terms of the arithmetic of cyclotomic fields $\mathbb{Q}(e^{2\pi i/n})$ leads to class field theory, which in turn leads to the Langlands program.

Fermat's Last Theorem: This classical theorem says $x^n + y^n = z^n$ has no solutions with x, y, z, n all positive integers and $n \geq 3$. Wiles's proof of Fermat's Last Theorem uses methods from algebraic number theory extensively, in addition to many other deep techniques. Attempts to prove Fermat's Last Theorem long ago were hugely influential in the development of algebraic number theory by Dedekind, Hilbert, Kummer, Kronecker, and others.

Arithmetic geometry: This is a huge field that studies solutions to polynomial equations that lie in arithmetically interesting rings, such as the integers or number fields. A major triumph of arithmetic geometry is Faltings's proof of Mordell's Conjecture.

Theorem 1.3.1 (Faltings). *Let X be a nonsingular plane algebraic curve over a number field K. Assume that the manifold $X(\mathbb{C})$ of complex solutions to X has genus at least 2 (i.e., $X(\mathbb{C})$ is topologically a donut with at least two holes). Then the set $X(K)$ of points on X with coordinates in K is finite.*

For example, Theorem 1.3.1 implies that for any $n \geq 4$ and any number field K, there are only finitely many solutions in K to $x^n + y^n = 1$.

A major open problem in arithmetic geometry is the *Birch and Swinnerton-Dyer conjecture*. An *elliptic curve E* is an algebraic curve with at least one point with coordinates in K such that the set of complex points $E(\mathbb{C})$ is a topological torus (i.e., $E(\mathbb{C})$ is topologically a donut with one hole). The Birch and Swinnerton-Dyer conjecture gives a criterion for whether or not $E(K)$ is infinite in terms of analytic properties of the L-function $L(E, s)$. See `http://www.claymath.org/millennium/Birch_and_Swinnerton-Dyer_Conjecture/`.

Chapter 2

Basic Commutative Algebra

The commutative algebra in this chapter provides a foundation for understanding the more refined number-theoretic structures associated to number fields.

First we prove the structure theorem for finitely generated abelian groups. Then we establish the standard properties of Noetherian rings and modules, including a proof of the Hilbert basis theorem. We also observe that finitely generated abelian groups are Noetherian $\mathbb{Z}$-modules. After establishing properties of Noetherian rings, we consider rings of algebraic integers and discuss some of their properties.

2.1 Finitely Generated Abelian Groups

Finitely generated abelian groups arise all over algebraic number theory. For example, they will appear in this book as class groups, unit groups, and the underlying additive groups of rings of integers, and as Mordell-Weil groups of elliptic curves.

In this section, we prove the structure theorem for finitely generated abelian groups, since it will be crucial for much of what we will do later.

Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ denote the ring of (rational) integers, and for each positive integer n, let $\mathbb{Z}/n\mathbb{Z}$ denote the ring of integers modulo n, which is a cyclic abelian group of order n under addition.

Definition 2.1.1 (Finitely Generated)**.** A group G is *finitely generated* if there exists $g_1, \ldots, g_n \in G$ such that every element of G can be expressed as a finite product (or sum, if we write G additively) of positive or negative powers of the g_i.

For example, the group $\mathbb{Z}$ is finitely generated, since it is generated by 1.

Theorem 2.1.2 (Structure Theorem for Finitely Generated Abelian Groups)**.** *Let G be a finitely generated abelian group. Then there is an isomorphism*

$$G \approx (\mathbb{Z}/n_1\mathbb{Z}) \oplus (\mathbb{Z}/n_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/n_s\mathbb{Z}) \oplus \mathbb{Z}^r,$$

where $r, s \geq 0$, $n_i > 1$ for all i, and $n_1 \mid n_2 \mid \cdots \mid n_s$. Furthermore, the n_i and r are uniquely determined by G.

Exercise 2.1.3. Quick! Guess how many abelian groups there are of order less than 12. Use Theorem 2.1.2 to classify all abelian groups of order less than 12. How many do you think there are? How many are there?

We will prove the theorem as follows. We first remark that any subgroup of a finitely generated free abelian group is finitely generated. Then we see how to represent finitely generated abelian groups as quotients of finite rank free abelian groups, and how to reinterpret such a presentation in terms of matrices over the integers. Next we describe how to use row and column operations over the integers to show that every matrix over the integers is equivalent to one in a canonical diagonal form, called the Smith normal form. We obtain a proof of the theorem by reinterpreting the Smith normal form in terms of groups. Finally, we observe that the representation in the theorem is necessarily unique.

Proposition 2.1.4. *If H is a subgroup of a finitely generated abelian group G, then H is finitely generated.*

The key reason that this is true is that G is a finitely generated module over the principal ideal domain $\mathbb{Z}$. We defer the proof of Proposition 2.1.4 to Section 2.2, where we will give a complete proof of a beautiful generalization in the context of Noetherian rings (the Hilbert basis theorem).

Corollary 2.1.5. *Suppose G is a finitely generated abelian group. Then there are finitely generated free abelian groups F_1 and F_2 and there is a homomorphism $\psi : F_2 \to F_1$ such that $G \approx F_1/\psi(F_2)$.*

Proof. Let $x_1, \ldots, x_m$ be generators for G. Let $F_1 = \mathbb{Z}^m$ and let $\varphi : F_1 \to G$ be the homomorphism that sends the ith generator $(0, 0, \ldots, 1, \ldots, 0)$ of $\mathbb{Z}^m$ to x_i. Then φ is surjective, and by Proposition 2.1.4 the kernel $\ker(\varphi)$ of φ is a finitely generated abelian group. Suppose there are n generators for $\ker(\varphi)$, let $F_2 = \mathbb{Z}^n$ and fix a surjective homomorphism $\psi : F_2 \to \ker(\varphi)$. Then $F_1/\psi(F_2)$ is isomorphic to G. □

An *sequence* of homomorphisms of abelian groups

$$H \xrightarrow{f} G \xrightarrow{g} K$$

is exact if $\mathrm{im}(f) = \ker(g)$. For longer sequences, exactness means every three consecutive terms with two arrows are exact. Given a finitely generated abelian group G, Corollary 2.1.5 provides an exact sequence

$$F_2 \xrightarrow{\psi} F_1 \to G \to 0.$$

Suppose G is a nonzero finitely generated abelian group. By the corollary, there are free abelian groups F_1 and F_2 and there is a homomorphism $\psi : F_2 \to F_1$ such that $G \approx F_1/\psi(F_2)$. Upon choosing a basis for F_1 and F_2, we obtain isomorphisms $F_1 \approx \mathbb{Z}^n$ and $F_2 \approx \mathbb{Z}^m$ for integers n and m. Just as in linear algebra, we view

$\psi : F_2 \to F_1$ as being given by left multiplication by the $n \times m$ matrix A whose columns are the images of the generators of F_2 in $\mathbb{Z}^n$. We visualize this as follows:

$$s\mathbb{Z}^m \xrightarrow{A} \mathbb{Z}^n \to G \to 0$$

The *cokernel* of the homomorphism defined by A is the quotient of $\mathbb{Z}^n$ by the image of A (i.e., the $\mathbb{Z}$-span of the columns of A), and this cokernel is isomorphic to G.

The following proposition implies that we may choose a bases for F_1 and F_2 such that the matrix of A only has nonzero entries along the diagonal, so that the structure of the cokernel of A is trivial to understand.

Proposition 2.1.6 (Smith normal form)**.** *Suppose A is an $n \times m$ integer matrix. Then there exist invertible integer matrices P and Q such that $A' = PAQ$ only has nonzero entries along the diagonal, and these entries are $n_1, n_2, \ldots, n_s, 0, \ldots, 0$, where $s \geq 0$, $n_i \geq 1$ for all i, and $n_1 \mid n_2 \mid \cdots \mid n_s$.*

Remark 2.1.7*.* Note that the matrices P and Q are invertible as integer matrices, so $\det(P)$ and $\det(Q)$ are ± 1. In particular $\det A' = \pm \det A$.

Definition 2.1.8. The matrix A' in Proposition 2.1.6 is called the *Smith normal form* of A.

We will see in the proof of Theorem 2.1.2 that A' is uniquely determined by A. An example of a matrix in Smith normal form is

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof of Proposition 2.1.6. The matrix P will be a product of matrices that define elementary row operations and Q will be a product corresponding to elementary column operations. The elementary row and column operations over $\mathbb{Z}$ are as follows:

Add multiple: Add an integer multiple of one row to another (or a multiple of one column to another).

Swap: Interchange two rows or two columns.

Rescale: Multiply a row by -1.

Each of these operations is given by left or right multiplying by an invertible matrix E with integer entries, where E is the result of applying the given operation to the identity matrix, and E is invertible because each operation can be reversed using another row or column operation over the integers.

To see that the proposition must be true, assume $A \neq 0$ and perform the following steps (compare [Art91, pg. 459]):

1. By permuting rows and columns, move a nonzero entry of A with smallest absolute value to the upper left corner of A. Now "attempt" (as explained in detail below) to make all other entries in the first row and column 0 by adding multiples of the top row or first column to other rows or columns, as follows:

   > Suppose a_{i1} is a nonzero entry in the first column, with $i > 1$. Using the division algorithm, write $a_{i1} = a_{11}q + r$, with $0 \leq r < a_{11}$. Now add $-q$ times the first row to the ith row. If $r > 0$, then go to step 1 (so that an entry with absolute value at most r is the upper left corner).

   If at any point this operation produces a nonzero entry in the matrix with absolute value smaller than $|a_{11}|$, start the process over by permuting rows and columns to move that entry to the upper left corner of A. Since the integers $|a_{11}|$ are a decreasing sequence of positive integers, we will not have to move an entry to the upper left corner infinitely often, so when this step is done the upper left entry of the matrix is nonzero, and all entries in the first row and column are 0.

2. We may now assume that a_{11} is the only nonzero entry in the first row and column. If some entry a_{ij} of A is not divisible by a_{11}, add the column of A containing a_{ij} to the first column, thus producing an entry in the first column that is nonzero. When we perform step 2, the remainder r will be greater than 0. Permuting rows and columns results in a smaller $|a_{11}|$. Since $|a_{11}|$ can only shrink finitely many times, eventually we will get to a point where every a_{ij} is divisible by a_{11}. If a_{11} is negative, multiple the first row by -1.

After performing the above operations, the first row and column of A are zero except for a_{11} which is positive and divides all other entries of A. We repeat the above steps for the matrix B obtained from A by deleting the first row and column. The upper left entry of the resulting matrix will be divisible by a_{11}, since every entry of B is. Repeating the argument inductively proves the proposition. □

Example 2.1.9. The matrix $\begin{pmatrix} -2 & 2 \\ -3 & 4 \end{pmatrix}$ has Smith normal form $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and the matrix $\begin{pmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \\ 49 & 64 & 81 \end{pmatrix}$ has Smith normal form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 72 \end{pmatrix}$. As a double check, note that the determinants of a matrix and its Smith normal form match, up to sign. This is because

$$\det(PAQ) = \det(P)\det(A)\det(Q) = \pm\det(A).$$

We compute each of the above Smith forms using Sage, along with the corresponding transformation matrices. To do this we use the Sage command `matrix`, which takes as input the base ring, the number of rows, and the entries. The output of `matrix` is a matrix object which has the method `smith_form`.

First the 2 × 2 matrix.

```
A = matrix(ZZ, 2, [-2,2, -3,4])
S, P, Q = A.smith_form(); S
```

```
[1 0]
[0 2]
```

```
P*A*Q
```

```
[1 0]
[0 2]
```

```
P
```

```
[0 1]
[1 0]
```

```
Q
```

```
[1 -4]
[1 -3]
```

Next the 3 × 3 matrix.

```
A = matrix(ZZ, 3, [1,4,9, 16,25,36, 49,64,81])
S, P, Q = A.smith_form(); S
```

```
[ 1  0  0]
[ 0  3  0]
[ 0  0 72]
```

```
P*A*Q
```

```
[ 1  0  0]
[ 0  3  0]
[ 0  0 72]
```

```
P
```

```
[  0   0   1]
[  0   1  -1]
[  1 -20 -17]
```

```
Q
```

```
[  47   74   93]
[ -79 -125 -156]
[  34   54   67]
```

For one more example, we compute the Smith form of a 3 × 3 matrix of rank 2:

```
m = matrix(ZZ, 3, [2..10]); m
```

```
[ 2  3  4]
[ 5  6  7]
[ 8  9 10]
```

```
m.smith_form()[0]
```

```
[1 0 0]
[0 3 0]
[0 0 0]
```

Proof of Theorem 2.1.2. Suppose G is a finitely generated abelian group, which we may assume is nonzero. As in the paragraph before Proposition 2.1.6, we use Corollary 2.1.5 to write G as the cokernel of an $n \times m$ integer matrix A. By Proposition 2.1.6 there are isomorphisms $Q : \mathbb{Z}^m \to \mathbb{Z}^m$ and $P : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $A' = PAQ$ has diagonal entries $n_1, n_2, \ldots, n_s, 0, \ldots, 0$, where $n_1 > 1$ and $n_1 \mid n_2 \mid \ldots \mid n_s$. Then G is isomorphic to the cokernel of the diagonal matrix A', so

$$G \cong (\mathbb{Z}/n_1\mathbb{Z}) \oplus (\mathbb{Z}/n_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/n_s\mathbb{Z}) \oplus \mathbb{Z}^r, \tag{2.1.1}$$

as claimed. The n_i are determined by G, because n_i is the smallest positive integer n such that nG requires at most $s + r - i$ generators. We see from the representation (2.1.1) of G as a product that n_i has this property and that no smaller positive integer does. □

Exercise 2.1.10. Recall Smith normal form defined in Proposition 2.1.6. With only minor modifications, then the proposition and proof will work over any principle ideal domain. Find and apply these modifications then find the Smith normal form of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1+i & 2 \\ 0 & 1 & 5 \end{pmatrix}$.

[*Hint*: You can use Sage to verify your answer. However, you will need to make explicitly construct the Gaussian integers in order to input the matrix. You can do this by the following code.]

```
K.<i> = QuadraticField(-1)
R = K.maximal_order()
M = matrix(R, 3, [1,2,3,0,1+i,2,0,1,5]); show(M)
#show(M.smith_form()[0]) #uncomment for the answer
```

Exercise 2.1.11. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

1. Find the Smith normal form of A.

2. Prove that the cokernel of the map $\mathbb{Z}^3 \to \mathbb{Z}^3$ given by multiplication by A is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}$.

2.2 Noetherian Rings and Modules

A module M over a commutative ring R with unit element is much like a vector space, but with more subtle structure. In this book, most of the modules we encounter will be noetherian, which is a generalization of the "finite dimensional" property of vector spaces. This section is about properties of noetherian modules (and rings), which are crucial to much of this book. We thus give complete proofs of these properties, so you will have a solid foundation on which to learn algebraic number theory.

We first define noetherian rings and modules, then introduce several equivalent characterizations of them. We prove that when the base ring is noetherian, a module is finitely generated if and only if it is noetherian. Next we define short exact sequences, and prove that the middle module in a sequence is noetherian if and only if the first and last modules are noetherian. Finally, we prove the Hilbert basis theorem, which asserts that adjoining finitely many elements to a noetherian ring results in a noetherian ring.

Let R be a commutative ring with unity. An *R-module* is an additive abelian group M equipped with a map $R \times M \to M$ such that for all $r, r' \in R$ and all $m, m' \in M$ we have $(rr')m = r(r'm)$, $(r+r')m = rm + r'm$, $r(m+m') = rm + rm'$, and $1m = m$. A *submodule* of M is a subgroup of M that is preserved by the action of R. For example, R is a module over itself, and any ideal I in R is an R-submodule of R.

Example 2.2.1. Abelian groups are the same as $\mathbb{Z}$-modules, and vector spaces over a field K are the same as K-modules.

An R-module M is finitely generated if there are elements $m_1, \ldots, m_n \in M$ such that every element of M is an R-linear combination of the m_i. The noetherian property is stronger than just being finitely generated:

Definition 2.2.2 (Noetherian)**.** An R-module M is *noetherian* if every submodule of M is finitely generated. A ring R is *noetherian* if R is noetherian as a module over itself, i.e., if every ideal of R is finitely generated.

Any submodule M' of a noetherian module M is also noetherian. Indeed, if every submodule of M is finitely generated then so is every submodule of M', since submodules of M' are also submodules of M.

Example 2.2.3. Let $R = M = \mathbb{Q}[x_1, x_2, \ldots]$ be a polynomial ring over $\mathbb{Q}$ in infinitely many indeterminants x_i. Then M is finitely generated *as an R-module* (!), since it is generated by 1. Consider the submodule $I = (x_1, x_2, \ldots)$ of polynomials with 0 constant term, and suppose it is generated by polynomials $f_1, \ldots, f_n$. Let x_i be an indeterminant that does not appear in any f_j, and suppose there are $h_k \in R$ such that $\sum_{k=1}^{n} h_k f_k = x_i$. Setting $x_i = 1$ and all other $x_j = 0$ on both sides of this equation and using that the f_k all vanish (they have 0 constant term), yields $0 = 1$, a contradiction. We conclude that the ideal I is not finitely generated, hence M is not a noetherian R-module, despite being finitely generated.

Definition 2.2.4 (Ascending chain condition)**.** An R-module M satisfies the *ascending chain condition* if every sequence $M_1 \subset M_2 \subset M_3 \subset \cdots$ of submodules of M eventually stabilizes, i.e., there is some n such that $M_n = M_{n+1} = M_{n+2} = \cdots$.

We will use the notion of maximal element below. If $\mathcal{X}$ is a set of subsets of a set S, ordered by inclusion, then a *maximal element* $A \in \mathcal{X}$ is a set such that no superset of A is contained in $\mathcal{X}$. Note that $\mathcal{X}$ may contain many different maximal elements.

Proposition 2.2.5. *If M is an R-module, then the following are equivalent:*

1. *M is noetherian,*
2. *M satisfies the ascending chain condition, and*
3. *Every nonempty set of submodules of M contains at least one maximal element.*

Proof.

($1 \implies 2$): Suppose $M_1 \subset M_2 \subset \cdots$ is a sequence of submodules of M. Then $M_\infty = \cup_{n=1}^{\infty} M_n$ is a submodule of M. Since M is noetherian and M_∞ is a submodule of M, there is a finite set $a_1, \ldots, a_m$ of generators for M_∞. Each a_i must be contained in some M_j, so there is an n such that $a_1, \ldots, a_m \in M_n$. But then $M_k = M_n$ for all $k \geq n$, which proves that the chain of M_i stabilizes, so the ascending chain condition holds for M.

($2 \implies 3$): Suppose 3 were false, so there exists a nonempty set S of submodules of M that does not contain a maximal element. We will use S to construct an infinite ascending chain of submodules of M that does not stabilize. Note that S is infinite, otherwise it would contain a maximal element. Let M_1 be any element of S. Then there is an M_2 in S that strictly contains M_1, otherwise S would contain the maximal element M_1. Continuing inductively in this way we find an M_3 in S that properly contains M_2, etc., and we produce an infinite ascending chain of submodules of M, which contradicts the ascending chain condition.

($3 \implies 1$): Suppose 1 is false, so there is a submodule M' of M that is not finitely generated. We will show that the set S of all finitely generated submodules of M' does not have a maximal element, which will be a contradiction. Suppose S does have a maximal element L. Since L is finitely generated and $L \subset M'$, and M' is not finitely generated, there is an $a \in M'$ such that $a \notin L$. Then $L' = L + Ra$ is an element of S that strictly contains the presumed maximal element L, a contradiction.

$\square$

Definition 2.2.6. A *homomorphism* of R-modules $\varphi : M \to N$ is an abelian group homomorphism such that for any $r \in R$ and $m \in M$ we have $\varphi(rm) = r\varphi(m)$. A sequence

$$L \xrightarrow{f} M \xrightarrow{g} N,$$

where f and g are homomorphisms of R-modules, is *exact* if $\mathrm{im}(f) = \ker(g)$. A *short exact sequence* of R-modules is a sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

that is exact at each point, i.e., f is injective, g is surjective, and $\mathrm{im}(f) = \ker(g)$.

Example 2.2.7. The sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is an exact sequence, where the first map sends 1 to 2, and the second is the natural quotient map.

Lemma 2.2.8. *If*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is a short exact sequence of R-modules, then M is noetherian if and only if both L and N are noetherian.

Proof. First suppose that M is noetherian. Then L is a submodule of M, so L is noetherian. Let N' be a submodule of N; then the inverse image of N' in M is a submodule of M, so it is finitely generated, hence its image N' is also finitely generated. Thus N is noetherian as well.

Next assume nothing about M, but suppose that both L and N are noetherian. Suppose M' is a submodule of M; then $M_0 = f(L) \cap M'$ is isomorphic to a submodule of the noetherian module L, so M_0 is generated by finitely many elements $a_1, \ldots, a_n$. The quotient M'/M_0 is isomorphic (via g) to a submodule of the noetherian module N, so M'/M_0 is generated by finitely many elements $b_1, \ldots, b_m$. For each $i \leq m$, let c_i be a lift of b_i to M', modulo M_0. Then the elements $a_1, \ldots, a_n, c_1, \ldots, c_m$ generate M', for if $x \in M'$, then there is some element $y \in M_0$ such that $x-y$ is an R-linear combination of the c_i, and y is an R-linear combination of the a_i. □

Proposition 2.2.9. *Suppose R is a noetherian ring. Then an R-module M is noetherian if and only if it is finitely generated.*

Proof. If M is noetherian then every submodule of M is finitely generated so M itself is finitely generated. Conversely, suppose M is finitely generated, say by elements $a_1, \ldots, a_n$. Then there is a surjective homomorphism from $R^n = R \oplus \cdots \oplus R$ to M that sends $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the ith factor) to a_i. Using Lemma 2.2.8 and exact sequences of R-modules such as $0 \to R \to R \oplus R \to R \to 0$, we see inductively that R^n is noetherian. Again by Lemma 2.2.8, homomorphic images of noetherian modules are noetherian, so M is noetherian. □

Lemma 2.2.10. *Suppose $\varphi : R \to S$ is a surjective homomorphism of rings and R is noetherian. Then S is noetherian.*

Proof. The kernel of φ is an ideal I in R, and we have an exact sequence

$$0 \to I \to R \to S \to 0$$

with R noetherian. This is an exact sequence of R-modules, where S has the R-module structure induced from φ (if $r \in R$ and $s \in S$, then we define $rs = \varphi(r)s$). By Lemma 2.2.8, it follows that S is a noetherian R-modules. Suppose J is an ideal of S. Since J is an R-submodule of S, if we view J as an R-module, then J is finitely generated. Since R acts on J through S, the R-generators of J are also S-generators of J, so J is finitely generated as an ideal. Thus S is noetherian. □

Theorem 2.2.11 (Hilbert Basis Theorem)**.** *If R is a noetherian ring and S is finitely generated as a ring over R, then S is noetherian. In particular, for any n the polynomial ring $R[x_1, \ldots, x_n]$ and any of its quotients are noetherian.*

Proof. Assume first that we have already shown that for any n the polynomial ring $R[x_1, \ldots, x_n]$ is noetherian. Suppose S is finitely generated as a ring over R, so there are generators $s_1, \ldots, s_n$ for S. Then the map $x_i \mapsto s_i$ extends uniquely to a surjective homomorphism $\pi : R[x_1, \ldots, x_n] \twoheadrightarrow S$, and Lemma 2.2.10 implies that S is noetherian.

The rings $R[x_1, \ldots, x_n]$ and $(R[x_1, \ldots, x_{n-1}])[x_n]$ are isomorphic, so it suffices to prove that if R is noetherian then $R[x]$ is also noetherian. (Our proof follows [Art91, §12.5].) Thus suppose I is an ideal of $R[x]$ and that R is noetherian. We will show that I is finitely generated.

Let A be the set of leading coefficients of polynomials in I. (The leading coefficient of a polynomial is the coefficient of the highest degree monomial, or 0 if the polynomial is 0; thus $3x^7 + 5x^2 - 4$ has leading coefficient 3.) We will first show that A is an ideal of R. Suppose $a, b \in A$ are nonzero with $a + b \neq 0$. Then there are polynomials f and g in I with leading coefficients a and b. If $\deg(f) \leq \deg(g)$, then $a+b$ is the leading coefficient of $x^{\deg(g)-\deg(f)}f + g$, so $a + b \in A$; the argument when $\deg(f) > \deg(g)$ is analogous. Suppose $r \in R$ and $a \in A$ with $ra \neq 0$. Then ra is the leading coefficient of rf, so $ra \in A$. Thus A is an ideal in R.

Since R is noetherian and A is an ideal of R, there exist nonzero $a_1, \ldots, a_n \in A$ that generate A as an ideal. Since A is the set of leading coefficients of elements of I, and the a_j are in A, we can choose for each $j \leq n$ an element $f_j \in I$ with leading coefficient a_j. By multipying the f_j by some power of x, we may assume that the f_j all have the same degree $d \geq 1$.

Let $S_{<d}$ be the set of elements of I that have degree strictly less than d. This set is closed under addition and under multiplication by elements of R, so $S_{<d}$ is a module over R. The module $S_{<d}$ is the submodule of the R-module of polynomials of degree less than n, which is noetherian by Proposition 2.2.9 because it is generated by $1, x, \ldots, x^{n-1}$. Thus $S_{<d}$ is finitely generated, and we may choose generators $h_1, \ldots, h_m$ for $S_{<d}$.

We finish by proving using induction on the degree that every $g \in I$ is an $R[x]$-linear combination of $f_1, \ldots, f_n, h_1, \ldots, h_m$. If $g \in I$ has degree 0, then $g \in S_{<d}$, since $d \geq 1$, so g is a linear combination of $h_1, \ldots, h_m$. Next suppose $g \in I$ has degree e, and that we have proven the statement for all elements of I of degree $< e$. If $e \leq d$, then $g \in S_{<d}$, so g is in the $R[x]$-ideal generated by $h_1, \ldots, h_m$. Next suppose that $e \geq d$. Then the leading coefficient b of g lies in the ideal A of leading coefficients of elements of I, so there exist $r_i \in R$ such that $b = r_1 a_1 + \cdots + r_n a_n$. Since f_i has leading coefficient a_i, the difference $g - x^{e-d} r_i f_i$ has degree less than the degree e of g. By induction $g - x^{e-d} r_i f_i$ is an $R[x]$ linear combination of $f_1, \ldots, f_n, h_1, \ldots, h_m$, so g is also an $R[x]$ linear combination of $f_1, \ldots, f_n, h_1, \ldots, h_m$. Since each f_i and h_j lies in I, it follows that I is generated by $f_1, \ldots, f_n, h_1, \ldots, h_m$, so I is finitely generated, as required. □

2.2.1 The Ring $\mathbb{Z}$ is Noetherian

The ring $\mathbb{Z}$ is noetherian since every ideal of $\mathbb{Z}$ is generated by one element.

Proposition 2.2.12. *Every ideal of the ring $\mathbb{Z}$ is principal.*

Proof. Suppose I is a nonzero ideal in $\mathbb{Z}$. Let d be the least positive element of I. Suppose that $a \in I$ is any nonzero element of I. Using the division algorithm, we write $a = dq + r$, where q is an integer and $0 \leq r < d$. We have $r = a - dq \in I$ and $r < d$, so our assumption that d is minimal implies that $r = 0$, hence $a = dq$ is in the ideal generated by d. Thus I is the principal ideal generated by d. □

Example 2.2.13. Let $I = (12, 18)$ be the ideal of $\mathbb{Z}$ generated by 12 and 18. If $n = 12a + 18b \in I$, with $a, b \in \mathbb{Z}$, then $6 \mid n$, since $6 \mid 12$ and $6 \mid 18$. Also, $6 = 18 - 12 \in I$, so $I = (6)$.

The ring $\mathbb{Z}$ in Sage is ZZ, which is Noetherian.

```
ZZ.is_noetherian()
```

```
True
```

We create the ideal I in Sage as follows, and note that it is principal:

```
I = ideal(12,18); I
```

```
Principal ideal (6) of Integer Ring
```

```
I.is_principal()
```

```
True
```

We could also create I as follows:

```
ZZ.ideal(12,18)
```

```
Principal ideal (6) of Integer Ring
```

Propositions 2.2.9 and 2.2.12 together imply that any finitely generated abelian group is noetherian. This means that subgroups of finitely generated abelian groups are finitely generated, which provides the missing step in our proof of the structure theorem for finitely generated abelian groups.

Exercise 2.2.14. There is another way to show every principle ideal domain (for example $\mathbb{Z}$) is noetherian (contrast to the proof in Section 2.2.1). Let R be a PID and (a) an arbitrary ideal. Use the facts that $(b) \supseteq (a)$ if and only if $b \mid a$ and R is a UFD to show that ascending chain of ideals starting with (a) must stabilize.

2.3 Rings of Algebraic Integers

In this section we introduce the central objects of this book, which are the rings of algebraic integers. These are noetherian rings with an enormous amount of structure. We also introduce a function field analogue of these rings.

An *algebraic number* is a root of some nonzero polynomial $f(x) \in \mathbb{Q}[x]$. For example, $\sqrt{2}$ and $\sqrt{5}$ are both algebraic numbers, being roots of $x^2 - 2$ and $x^2 - 5$, respectively. But is $\sqrt{2} + \sqrt{5}$ necessarily the root of some polynomial in $\mathbb{Q}[x]$? This isn't quite so obvious.

Proposition 2.3.1. *An element α of a field extension of $\mathbb{Q}$ is an algebraic number if and only if the ring $\mathbb{Q}[\alpha]$ generated by α is finite dimensional as a $\mathbb{Q}$ vector space.*

Proof. Suppose α is an algebraic number, so there is a nonzero polynomial $f(x) \in \mathbb{Q}[x]$, so that $f(\alpha) = 0$. The equation $f(\alpha) = 0$ implies that $\alpha^{\deg(f)}$ can be written in terms of smaller powers of α, so $\mathbb{Q}[\alpha]$ is spanned by the finitely many numbers $1, \alpha, \ldots, \alpha^{\deg(f)-1}$, hence finite dimensional. Conversely, suppose $\mathbb{Q}[\alpha]$ is finite dimensional. Then for some $n \geq 1$, we have that α^n is in the $\mathbb{Q}$-vector space spanned by $1, \alpha, \ldots, \alpha^{n-1}$. Thus α satisfies a polynomial $f(x) \in \mathbb{Q}[x]$ of degree n. □

Proposition 2.3.2. *Suppose K is a field and $\alpha, \beta \in K$ are two algebraic numbers. Then $\alpha\beta$ and $\alpha + \beta$ are also algebraic numbers.*

Proof. Let $n = \dim_{\mathbb{Q}} \mathbb{Q}[\alpha]$ and $n = \dim_{\mathbb{Q}} \mathbb{Q}[\beta]$. The subring $\mathbb{Q}[\alpha, \beta] \subset K$ is a $\mathbb{Q}$-vector space that is spanned by the numbers $\alpha^i \beta^j$, where $0 \leq i < n$ and $0 \leq j < m$. Thus $\mathbb{Q}[\alpha, \beta]$ is finite dimensional, and since $\alpha + \beta$ and $\alpha\beta$ are both in $\mathbb{Q}[\alpha, \beta]$, we conclude by Proposition 2.3.1 that both are algebraic numbers. □

Suppose C is a field extension of $\mathbb{Q}$ such that every polynomial $f(x) \in \mathbb{Q}[x]$ factors completely in C. The algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside C is the field generated by all roots in C of polynomials in $\mathbb{Q}[x]$. The fundamental theorem of algebra tells us that $C = \mathbb{C}$ is one choice of field C as above. There are other fields C, e.g., constructed using p-adic numbers. One can show that any two choices of $\overline{\mathbb{Q}}$ are isomorphic; however, there will be *many* isomorphisms between them.

Definition 2.3.3 (Algebraic Integer)**.** An element $\alpha \in \overline{\mathbb{Q}}$ is an *algebraic integer* if it is a root of some monic polynomial with coefficients in $\mathbb{Z}$.

For example, $\sqrt{2}$ is an algebraic integer, since it is a root of the monic integral polynomial $x^2 - 2$. As we will see below, $1/2$ is not an algebraic integer.

The following two propositions are analogous to Propositions 2.3.1–2.3.2 above, with the proofs replacing basic facts about vector spaces with facts we proved above about noetherian rings and modules.

Proposition 2.3.4. *An element $\alpha \in \overline{\mathbb{Q}}$ is an algebraic integer if and only if $\mathbb{Z}[\alpha]$ is finitely generated as a $\mathbb{Z}$-module.*

Proof. Suppose α is integral and let $f \in \mathbb{Z}[x]$ be a monic integral polynomial such that $f(\alpha) = 0$. Then, as a $\mathbb{Z}$-module, $\mathbb{Z}[\alpha]$ is generated by $1, \alpha, \alpha^2, \ldots, \alpha^{d-1}$, where d is the degree of f. Conversely, suppose $\alpha \in \overline{\mathbb{Q}}$ is such that $\mathbb{Z}[\alpha]$ is finitely generated as a module over $\mathbb{Z}$, say by elements $f_1(\alpha), \ldots, f_n(\alpha)$. Let d be any integer bigger than the degrees of all f_i. Then there exist integers a_i such that $\alpha^d = \sum_{i=1}^n a_i f_i(\alpha)$, hence α satisfies the monic polynomial $x^d - \sum_{i=1}^n a_i f_i(x) \in \mathbb{Z}[x]$, so α is an algebraic integer. □

The proof of the following proposition uses repeatedly that any submodule of a finitely generated $\mathbb{Z}$-module is finitely generated, which uses that $\mathbb{Z}$ is noetherian and that finitely generated modules over a noetherian ring are noetherian.

Proposition 2.3.5. *Suppose K is a field and $\alpha, \beta \in K$ are two algebraic integers. Then $\alpha\beta$ and $\alpha + \beta$ are also algebraic integers.*

Proof. Let m, n be the degrees of monic integral polynomials that have α, β as roots, respectively. Then we can write α^m in terms of smaller powers of α and likewise for β^n, so the elements $\alpha^i\beta^j$ for $0 \leq i < m$ and $0 \leq j < n$ span the $\mathbb{Z}$-module $\mathbb{Z}[\alpha, \beta]$. Since $\mathbb{Z}[\alpha + \beta]$ is a submodule of the finitely-generated $\mathbb{Z}$-module $\mathbb{Z}[\alpha, \beta]$, it is finitely generated, so $\alpha + \beta$ is integral. Likewise, $\mathbb{Z}[\alpha\beta]$ is a submodule of $\mathbb{Z}[\alpha, \beta]$, so it is also finitely generated, and $\alpha\beta$ is integral. □

2.3.1 Minimal Polynomials

Definition 2.3.6 (Minimal Polynomial)**.** The *minimal polynomial* of $\alpha \in \overline{\mathbb{Q}}$ is the monic polynomial $f \in \mathbb{Q}[x]$ of least positive degree such that $f(\alpha) = 0$.

It is a consequence of Lemma 2.3.9 below that "the" minimal polynomial of α is unique. The minimal polynomial of $1/2$ is $x - 1/2$, and the minimal polynomial of $\sqrt[3]{2}$ is $x^3 - 2$.

Example 2.3.7. We compute the minimal polynomial of a number expressed in terms of $\sqrt[4]{2}$:

```
k.<a> = NumberField(x^4 - 2)
a^4
```

```
2
```

```
(a^2 + 3).minpoly()
```

```
x^2 - 6*x + 7
```

Exercise 2.3.8. Find the minimal polynomial of $\sqrt{2}+\sqrt{3}$ by hand. Check your result with Sage.

Lemma 2.3.9. *Suppose $\alpha \in \overline{\mathbb{Q}}$. Then the minimal polynomial of α divides any polynomial h such that $h(\alpha) = 0$.*

Proof. Let f be a choice of minimal polynomial of α, as in Definition 2.3.6, and let h be a polynomial with $h(\alpha) = 0$. Use the division algorithm to write $h = qf + r$, where $0 \leq \deg(r) < \deg(f)$. We have

$$r(\alpha) = h(\alpha) - q(\alpha)f(\alpha) = 0,$$

so α is a root of r. However, f is a polynomial of least positive degree with root α, so $r = 0$. □

Exercise 2.3.10. Show that the minimal polynomial of an algebraic number $\alpha \in \overline{\mathbb{Q}}$ is unique.

Lemma 2.3.11. *Suppose $\alpha \in \overline{\mathbb{Q}}$. Then α is an algebraic integer if and only if the minimal polynomial f of α has coefficients in $\mathbb{Z}$.*

Proof.

($\Longleftarrow$): Since $f \in \mathbb{Z}[x]$ is monic (by definition) and $f(\alpha) = 0$, we see immediately that α is an algebraic integer.

($\Longrightarrow$): Since α is an algebraic integer, there is some nonzero monic $g \in \mathbb{Z}[x]$ such that $g(\alpha) = 0$. By Lemma 2.3.9, we have $g = fh$, for some $h \in \mathbb{Q}[x]$, and h is monic because f and g are. If $f \notin \mathbb{Z}[x]$, then some prime p divides the denominator of some coefficient of f. Let p^i be the largest power of p that divides some denominator of some coefficient f, and likewise let p^j be the largest power of p that divides some denominator of a coefficient of h. Then $p^{i+j}g = (p^i f)(p^j h)$, and if we reduce both sides modulo p, then the left hand side is 0 but the right hand side is a product of two nonzero polynomials in $\mathbb{F}_p[x]$, hence nonzero, a contradiction.

□

Exercise 2.3.12. Which of the following numbers are algebraic integers?

(*a*) The number $(1+\sqrt{5})/2$.

(*b*) The number $(2+\sqrt{5})/2$.

(*c*) The value of the infinite sum $\sum_{n=1}^{\infty} 1/n^2$.

(*d*) The number $\alpha/3$, where α is a root of $x^4+54x+243$.

Example 2.3.13. We compute some minimal polynomials in Sage. The minimal polynomial of $1/2$:

```
(1/2).minpoly()
```

```
x - 1/2
```

We construct a root a of x^2-2 and compute its minimal polynomial:

```
k.<a> = NumberField(x^2 - 2)
a^2 - 2
```

```
0
```

```
a.minpoly()
```

```
x^2 - 2
```

Finally we compute the minimal polynomial of $\alpha = \sqrt{2}/2+3$, which is not integral, hence Proposition 2.3.4 implies that α is not an algebraic integer:

```
(a/2 + 3).minpoly()
```

```
x^2 - 6*x + 17/2
```

The only elements of $\mathbb{Q}$ that are algebraic integers are the usual integers $\mathbb{Z}$, since $\mathbb{Z}[1/d]$ is not finitely generated as a $\mathbb{Z}$-module. Watch out since there are elements of $\overline{\mathbb{Q}}$ that seem to *appear* to have denominators when written down, but are still algebraic integers. This is an artifact of how we write them down, e.g., if we wrote our integers as a multiple of $\alpha = 2$, then we would write 1 as $\alpha/2$. For example,

$$\alpha = \frac{1+\sqrt{5}}{2}$$

is an algebraic integer, since it is a root of the monic integral polynomial x^2-x-1. We verify this using Sage below, though of course this is easy to do by hand (you should try much more complicated examples in Sage).

```
k.<a> = QuadraticField(5)
a^2
```

```
5
```

```
alpha = (1 + a)/2
alpha.minpoly()
```

```
x^2 - x - 1
```

```
alpha.is_integral()
```

```
True
```

Since $\sqrt{5}$ can be expressed in terms of radicals, we can also compute this minimal polynomial using the symbolic functionality in Sage.

```
alpha = (1+sqrt(5))/2
alpha.minpoly()
```

```
x^2 - x - 1
```

Here is a more complicated example using a similar approach:

```
alpha = sqrt(2) + 3^(1/4)
alpha.minpoly()
```

```
x^8 - 8*x^6 + 18*x^4 - 104*x^2 + 1
```

Example 2.3.14. We illustrate an example of a sum and product of two algebraic integers being an algebraic integer. We first make the relative number field obtained by adjoining a root of $x^3 - 5$ to the field $\mathbb{Q}(\sqrt{2})$:

```
k.<a, b> = NumberField([x^2 - 2, x^3 - 5])
k
```

```
Number Field in a with defining polynomial x^2 + -2 over its base field
```

Here a and b are roots of $x^2 - 2$ and $x^3 - 5$, respectively.

```
a^2
```

```
2
```

```
b^3
```

```
5
```

We compute the minimal polynomial of the sum and product of $\sqrt[3]{5}$ and $\sqrt{2}$. The command `absolute_minpoly` gives the minimal polynomial of the element over the rational numbers $\mathbb{Q}$.

```
(a+b).absolute_minpoly()
```

```
x^6 - 6*x^4 - 10*x^3 + 12*x^2 - 60*x + 17
```

```
(a*b).absolute_minpoly()
```

```
x^6 - 200
```

The minimal polynomial of the product is $\sqrt[3]{5}\sqrt{2}$ is trivial to compute by hand. In light of the Cayley-Hamilton theorem, we can compute the minimal polynomial of $\alpha = \sqrt[3]{5} + \sqrt{2}$ by hand by computing the determinant of the matrix given by left multiplication by α on the basis

$$1, \sqrt{2}, \sqrt[3]{5}, \sqrt[3]{5}\sqrt{2}, \sqrt[3]{5}^2, \sqrt[3]{5}^2\sqrt{2}.$$

This is a general method which works well for computers. However it can also be done using simple algebra.

Exercise 2.3.15. Compute the minimal polynomial of $\alpha = \sqrt[3]{5}+\sqrt{2}$ by hand without finding the determinate of a 6×6 matrix.

[*Hint*: Let $a^2 = 2$, $b^3 = 5$, and $x = a + b$. Then $(x - a)^3 = b^3 = 5$. Now simplify and use the fact that $a^2 = 2$.]

The following is an alternative, more symbolic way to compute the minimal polynomials above, though it is not provably correct. We compute α to 100 bits precision (via the `n` command), then use the LLL algorithm (via the `algdep` command) to heuristically find a linear relation between the first 6 powers of α (see Section 2.5 below for more about LLL).

```
a = 5^(1/3); b = sqrt(2)
c = a+b; c
```

```
5^(1/3) + sqrt(2)
```

```
(a+b).n(100).algdep(6)
```

```
x^6 - 6*x^4 - 10*x^3 + 12*x^2 - 60*x + 17
```

```
(a*b).n(100).algdep(6)
```

```
x^6 - 200
```

Exercise 2.3.16. Let $\alpha = \sqrt{2} + \frac{1+\sqrt{5}}{2}$.

(*a*) Is α an algebraic integer?

(*b*) Explicitly write down the minimal polynomial of α as an element of $\mathbb{Q}[x]$.

2.3.2 Number fields, rings of integers, and orders

Definition 2.3.17 (Number field)**.** A *number field* is a field K that contains the rational numbers $\mathbb{Q}$ such that the degree $[K:\mathbb{Q}] = \dim_{\mathbb{Q}}(K)$ is finite.

If K is a number field, then by the primitive element theorem there is an $\alpha \in K$ so that $K = \mathbb{Q}(\alpha)$. Let $f(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α. Fix a choice of algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Associated to each of the $\deg(f)$ roots $\alpha' \in \overline{\mathbb{Q}}$ of f, we obtain a field embedding $K \hookrightarrow \overline{\mathbb{Q}}$ that sends α to α'. Thus any number field can be embedded in $[K:\mathbb{Q}] = \deg(f)$ distinct ways in $\overline{\mathbb{Q}}$.

Definition 2.3.18 (Ring of Integers)**.** The *ring of integers* of a number field K is the ring

$$\mathcal{O}_K = \{x \in K : \ x \text{ is an algebraic integer}\}.$$

One of the most basic facts about $\mathcal{O}_K$ is that it is indeed a ring. This fact is important enough to be stated as a separate theorem.

Theorem 2.3.19. *Let K be a number field. The ring of integers $\mathcal{O}_K$ is a ring.*

Proof. This follows directly from Proposition 2.3.5. □

Example 2.3.20. The field $\mathbb{Q}$ of rational numbers is a number field of degree 1, and the ring of integers of $\mathbb{Q}$ is $\mathbb{Z}$. The field $K = \mathbb{Q}(i)$ of Gaussian integers has degree 2 and $\mathcal{O}_K = \mathbb{Z}[i]$.

Example 2.3.21. The golden ratio $\varphi = (1+\sqrt{5})/2$ is in the quadratic number field $K = \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\varphi)$; notice that φ satisfies $x^2 - x - 1$, so $\varphi \in \mathcal{O}_K$. To see that $\mathcal{O}_K = \mathbb{Z}[\varphi]$ directly, we proceed as follows. By Proposition 2.3.4, the algebraic integers K are exactly the elements $a + b\sqrt{5} \in K$, with $a, b \in \mathbb{Q}$ that have integral minimal polynomial. The matrix of $a + b\sqrt{5}$ with respect to the basis $1, \sqrt{5}$ for K is $m = \left(\begin{smallmatrix} a & 5b \\ b & a \end{smallmatrix}\right)$. The characteristic polynomial of m is $f = (x-a)^2 - 5b^2 = x^2 - 2ax + a^2 - 5b^2$, which is in $\mathbb{Z}[x]$ if and only if $2a \in \mathbb{Z}$ and $a^2 - 5b^2 \in \mathbb{Z}$. Thus $a = a'/2$ with $a' \in \mathbb{Z}$, and $(a'/2)^2 - 5b^2 \in \mathbb{Z}$, so $5b^2 \in \frac{1}{4}\mathbb{Z}$, so $b \in \frac{1}{2}\mathbb{Z}$ as well. If a has a denominator of 2, then b must also have a denominator of 2 to ensure that the difference $a^2 - 5b^2$ is an integer. This proves that $\mathcal{O}_K = \mathbb{Z}[\varphi]$.

Example 2.3.22. The ring of integers of $K = \mathbb{Q}(\sqrt[3]{9})$ is $\mathbb{Z}[\sqrt[3]{3}]$, where $\sqrt[3]{3} = \frac{1}{3}(\sqrt[3]{9})^2 \not\in \mathbb{Z}[\sqrt[3]{9}]$. As we will see, in general the problem of computing $\mathcal{O}_K$ given K may be very hard, since it requires factoring a certain potentially large integer.

Exercise 2.3.23. From basic definitions, find the rings of integers of the fields $\mathbb{Q}(\sqrt{11})$ and $\mathbb{Q}(\sqrt{-6})$.

Definition 2.3.24 (Order)**.** An *order* in $\mathcal{O}_K$ is any subring R of $\mathcal{O}_K$ such that the quotient $\mathcal{O}_K/R$ of abelian groups is finite. (By definition R must contain 1 because it is a ring.)

As noted above, $\mathbb{Z}[i]$ is the ring of integers of $\mathbb{Q}(i)$. For every nonzero integer n, the subring $\mathbb{Z}+ni\mathbb{Z}$ of $\mathbb{Z}[i]$ is an order. The subring $\mathbb{Z}$ of $\mathbb{Z}[i]$ is not an order, because $\mathbb{Z}$ does not have finite index in $\mathbb{Z}[i]$. Also the subgroup $2\mathbb{Z}+i\mathbb{Z}$ of $\mathbb{Z}[i]$ is not an order because it is not a ring.

Exercise 2.3.25. Let K be a quadratic extension of $\mathbb{Q}$ and R be any order in $\mathcal{O}_K$. Show that $\mathcal{O}_K/R$ is cyclic as an abelian group and that there is a bijection between orders of $\mathcal{O}_K$ containing R and divisors of $[\mathcal{O}_K : R]$.

Remark 2.3.26. Exercise 2.3.25 is used in elliptic curve cryptography to measure the number of isogenies; this is used in [KKM11, §11.2] for an example.

We define the number field $\mathbb{Q}(i)$ and compute its ring of integers.

```
K.<i> = NumberField(x^2 + 1)
OK = K.ring_of_integers(); OK
```

```
Order with module basis 1, i in Number Field in i with
defining polynomial x^2 + 1
```

Next we compute the order $\mathbb{Z}+3i\mathbb{Z}$.

```
O3 = K.order(3*i); O3
```

```
Order with module basis 1, 3*i in Number Field in i with
defining polynomial x^2 + 1
```

```
O3.gens()
```

```
[1, 3*i]
```

We test whether certain elements are in the order.

```
5 + 9*i in O3
```

```
True
```

```
1 + 2*i in O3
```

```
False
```

We will frequently consider orders because they are often much easier to write down explicitly than $\mathcal{O}_K$. For example, if $K=\mathbb{Q}(\alpha)$ and α is an algebraic integer, then $\mathbb{Z}[\alpha]$ is an order in $\mathcal{O}_K$, but frequently $\mathbb{Z}[\alpha]\neq\mathcal{O}_K$.

Example 2.3.27. In this example $[\mathcal{O}_K : \mathbb{Z}[a]] = 2197$. First we define the number field $K=\mathbb{Q}(a)$ where a is a root of $x^3-15x^2-94x-3674$, then we compute the order $\mathbb{Z}[a]$ generated by a.

```
K.<a> = NumberField(x^3 - 15*x^2 - 94*x - 3674)
Oa = K.order(a); Oa
```

```
Order with module basis 1, a, a^2 in Number Field in a with defining
polynomial x^3 - 15*x^2 - 94*x - 3674
```

```
Oa.basis()
```

```
[1, a, a^2]
```

Next we compute a $\mathbb{Z}$-basis for the maximal order $\mathcal{O}_K$ of K, and compute that the index of $\mathbb{Z}[a]$ in $\mathcal{O}_K$ is $2197 = 13^3$.

```
OK = K.maximal_order()
OK.basis()
```

```
[25/169*a^2 + 10/169*a + 1/169, 5/13*a^2 + 1/13*a, a^2]
```

```
Oa.index_in(OK)
```

```
2197
```

Lemma 2.3.28. *Let $\mathcal{O}_K$ be the ring of integers of a number field. Then $\mathcal{O}_K \cap \mathbb{Q} = \mathbb{Z}$ and $\mathbb{Q}\mathcal{O}_K = K$.*

Proof. Suppose $\alpha \in \mathcal{O}_K \cap \mathbb{Q}$ with $\alpha = a/b \in \mathbb{Q}$ in lowest terms and $b > 0$. Since α is integral, $\mathbb{Z}[a/b]$ is finitely generated as a module, so $b = 1$.

To prove that $\mathbb{Q}\mathcal{O}_K = K$, suppose $\alpha \in K$, and let $f(x) \in \mathbb{Q}[x]$ be the minimal monic polynomial of α. For any positive integer d, the minimal monic polynomial of $d\alpha$ is $d^{\deg(f)} f(x/d)$, i.e., the polynomial obtained from $f(x)$ by multiplying the coefficient of $x^{\deg(f)}$ by 1, multiplying the coefficient of $x^{\deg(f)-1}$ by d, multiplying the coefficient of $x^{\deg(f)-2}$ by d^2, etc. If d is the least common multiple of the denominators of the coefficients of f, then the minimal monic polynomial of $d\alpha$ has integer coefficients, so $d\alpha$ is integral and $d\alpha \in \mathcal{O}_K$. This proves that $\mathbb{Q}\mathcal{O}_K = K$. □

Exercise 2.3.29. Which are the following rings are orders in the given number field, i.e. orders in the ring of integers of the given number field.

(*a*) The ring $R = \mathbb{Z}[i]$ in the number field $\mathbb{Q}(i)$.

(*b*) The ring $R = \mathbb{Z}[i/2]$ in the number field $\mathbb{Q}(i)$.

(*c*) The ring $R = \mathbb{Z}[17i]$ in the number field $\mathbb{Q}(i)$.

(*d*) The ring $R = \mathbb{Z}[i]$ in the number field $\mathbb{Q}(\sqrt[4]{-1})$.

2.3.3 Function fields

Let k be any field. We can also make the same definitions, but with $\mathbb{Q}$ replaced by the field $k(t)$ of rational functions in an indeterminate t, and $\mathbb{Z}$ replaced by $k[t]$. The analogue of a number field is called a *function field*; it is a finite algebraic extension field K of $k(t)$. Elements of K have a unique minimal polynomial as above, and the ring of integers of K consists of those elements whose monic minimal polynomial has coefficients in the polynomial ring $k[t]$.

Geometrically, if $F(x,t) = 0$ is an affine equation that defines (via projective closure) a nonsingular projective curve C, then $K = k(t)[x]/(F(x,t))$ is a function field. We view the field K as the field of all rational functions on the projective closure of the curve C. The ring of integers $\mathcal{O}_K$ is the subring of rational functions that have no poles on the affine curve $F(x,t) = 0$, though they may have poles at infinity, i.e., at the extra points we introduce when passing to the projective closure C. The algebraic arguments we gave above prove that $\mathcal{O}_K$ is a ring. This is also geometrically intuitive, since the sum and product of two functions with no poles also have no poles.

Exercise 2.3.30. Let $k = \mathbb{F}_p$ be the finite field with p elements where p is some prime. Find all automorphisms of $k(t)$. Note that an automorphism is completely characterized by its value on t. How many such automorphisms are there?

[*Hint*: For some people, it is easier to think about the equivalent question: What rational functions $f \in k(t)$ is the map $k(t) \to k(t)$ given by $t \mapsto f(t)$ an automorphism?]

2.4 Norms and Traces

In this section we develop some basic properties of norms, traces, and discriminants, and give more properties of rings of integers in the general context of Dedekind domains.

Before discussing norms and traces we introduce some notation for field extensions. If $K \subset L$ are number fields, we let $[L : K]$ denote the dimension of L viewed as a K-vector space. If K is a number field and $a \in \overline{\mathbb{Q}}$, let $K(a)$ be the extension of K generated by a, which is the smallest number field that contains both K and a. If $a \in \overline{\mathbb{Q}}$ then a has a minimal polynomial $f(x) \in \mathbb{Q}[x]$, and the *Galois conjugates* of a are the roots of f. These are called the Galois conjugates because they are the orbit of a under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Example 2.4.1. The element $\sqrt{2}$ has minimal polynomial $x^2 - 2$ and the Galois conjugates of $\sqrt{2}$ are $\sqrt{2}$ and $-\sqrt{2}$. The cube root $\sqrt[3]{2}$ has minimial polynomial $x^3 - 2$ and three Galois conjugates $\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}$, where ζ_3 is a cube root of unity.

We create the extension $\mathbb{Q}(\zeta_3)(\sqrt[3]{2})$ in Sage.

```
L.<cuberoot2> = CyclotomicField(3).extension(x^3 - 2)
cuberoot2^3
```

```
2
```

Then we list the Galois conjugates of $\sqrt[3]{2}$.

```
cuberoot2.galois_conjugates(L)
```

```
[cuberoot2, (-zeta3 - 1)*cuberoot2, zeta3*cuberoot2]
```

Note that $\zeta_3^2 = -\zeta_3 - 1$:

```
zeta3 = L.base_field().0
zeta3^2
```

```
-zeta3 - 1
```

Suppose $K \subset L$ is an inclusion of number fields and let $a \in L$. Then left multiplication by a defines a K-linear transformation $\ell_a : L \to L$. (The transformation ℓ_a is K-linear because L is commutative.)

Definition 2.4.2 (Norm and Trace)**.** The *norm* and *trace* of a from L to K are

$$\mathrm{Norm}_{L/K}(a) = \det(\ell_a) \quad \text{and} \quad \mathrm{tr}_{L/K}(a) = \mathrm{tr}(\ell_a).$$

We know from linear algebra that determinants are multiplicative and traces are additive, so for $a, b \in L$ we have

$$\mathrm{Norm}_{L/K}(ab) = \mathrm{Norm}_{L/K}(a) \cdot \mathrm{Norm}_{L/K}(b)$$

and

$$\mathrm{tr}_{L/K}(a+b) = \mathrm{tr}_{L/K}(a) + \mathrm{tr}_{L/K}(b).$$

Note that if $f \in \mathbb{Q}[x]$ is the characteristic polynomial of ℓ_a, then the constant term of f is $(-1)^{\deg(f)}\det(\ell_a)$, and the coefficient of $x^{\deg(f)-1}$ is $-\mathrm{tr}(\ell_a)$.

Proposition 2.4.3. *Let $a \in L$ and let $\sigma_1, \ldots, \sigma_d$, where $d = [L : K]$, be the distinct field embeddings $L \hookrightarrow \overline{\mathbb{Q}}$ that fix every element of K. Then*

$$\mathrm{Norm}_{L/K}(a) = \prod_{i=1}^{d} \sigma_i(a) \quad \textit{and} \quad \mathrm{tr}_{L/K}(a) = \sum_{i=1}^{d} \sigma_i(a).$$

Proof. We prove the proposition by computing the characteristic polynomial of a. Let $f \in K[x]$ be the minimal polynomial of a over K, and note that f has distinct roots and is irreducible, since it is the polynomial in $K[x]$ of least degree that is satisfied by a and K has characteristic 0. Since f is irreducible, we have $K(a) \cong K[x]/(f)$, so $[K(a) : K] = \deg(f)$. Also a satisfies a polynomial if and only if ℓ_a

does, so the characteristic polynomial of ℓ_a acting on $K(a)$ is f. Let $b_1, \ldots, b_n$ be a basis for L over $K(a)$ and note that $1, \ldots, a^m$ is a basis for $K(a)/K$, where $m = \deg(f) - 1$. Then $a^i b_j$ is a basis for L over K, and left multiplication by a acts the same way on the span of $b_j, ab_j, \ldots, a^m b_j$ as on the span of $b_k, ab_k, \ldots, a^m b_k$, for any pair $j, k \leq n$. Thus the matrix of ℓ_a on L is a block direct sum of copies of the matrix of ℓ_a acting on $K(a)$, so the characteristic polynomial of ℓ_a on L is $f^{[L:K(a)]}$. The proposition follows because the roots of $f^{[L:K(a)]}$ are exactly the images $\sigma_i(a)$, with multiplicity $[L : K(a)]$, since each embedding of $K(a)$ into $\overline{\mathbb{Q}}$ extends in exactly $[L : K(a)]$ ways to L. □

Warning 2.4.4. It is important in Proposition 2.4.3 that the product and sum be over *all* the images $\sigma_i(a)$, not over just the distinct images. For example, if $a = 1 \in L$, then $\mathrm{Tr}_{L/K}(a) = [L : K]$, whereas the sum of the distinct conjugates of a is 1.

Remark 2.4.5. Let $K \subset L$ be an extension of number fields. If $\alpha \in \mathcal{O}_L$, then the formula of Proposition 2.4.3 implies that the norm and trace down to K of α is an element of $\mathcal{O}_K$, because the sum and product of algebraic integers is an algebraic integer.

The following corollary asserts that the norm and trace behave well in towers.

Corollary 2.4.6. *Suppose $K \subset L \subset M$ is a tower of number fields, and let $a \in M$. Then*

$$\mathrm{Norm}_{M/K}(a) = \mathrm{Norm}_{L/K}(\mathrm{Norm}_{M/L}(a)) \quad \text{and} \quad \mathrm{tr}_{M/K}(a) = \mathrm{tr}_{L/K}(\mathrm{tr}_{M/L}(a)).$$

Proof. The proof uses that every embedding $L \hookrightarrow \overline{\mathbb{Q}}$ extends in exactly $[M : L]$ way to an embedding $M \hookrightarrow \overline{\mathbb{Q}}$. This is clear if we view M as $L[x]/(h(x))$ for some irreducicble polynomial $h(x) \in L[x]$ of degree $[M : L]$, and note that the extensions of $L \hookrightarrow \overline{\mathbb{Q}}$ to M correspond to the roots of h, of which there are $\deg(h)$, since $\overline{\mathbb{Q}}$ is algebraically closed.

For the first equation, both sides are the product of $\sigma_i(a)$, where σ_i runs through the embeddings of M into $\overline{\mathbb{Q}}$ that fix K. To see this, suppose $\sigma : L \to \overline{\mathbb{Q}}$ fixes K. If σ' is an extension of σ to M, and $\tau_1, \ldots, \tau_d$ are the embeddings of M into $\overline{\mathbb{Q}}$ that fix L, then $\sigma'\tau_1, \ldots, \sigma'\tau_d$ are exactly the extensions of σ to M. For the second statement, both sides are the sum of the $\sigma_i(a)$. □

Proposition 2.4.7. *Let K be a number field. The ring of integers $\mathcal{O}_K$ is a lattice in K, i.e., $\mathbb{Q}\mathcal{O}_K = K$ and $\mathcal{O}_K$ is an abelian group of rank $[K : \mathbb{Q}]$.*

Proof. We saw in Lemma 2.3.28 that $\mathbb{Q}\mathcal{O}_K = K$. Thus there exists a basis $a_1, \ldots, a_n$ for K, where each a_i is in $\mathcal{O}_K$. Suppose that as $x = \sum_{i=1}^n c_i a_i \in \mathcal{O}_K$ varies over all elements of $\mathcal{O}_K$ the denominators of the coefficients c_i are not all uniformly bounded. Then subtracting off integer multiples of the a_i, we see that as $x = \sum_{i=1}^n c_i a_i \in \mathcal{O}_K$ varies over elements of $\mathcal{O}_K$ with c_i between 0 and 1, the denominators of the c_i are also arbitrarily large. This implies that there are infinitely many elements of $\mathcal{O}_K$ in the bounded subset

$$S = \{c_1 a_1 + \cdots + c_n a_n : c_i \in \mathbb{Q},\ 0 \leq c_i \leq 1\} \subset K.$$

Thus for any $\varepsilon > 0$, there are elements $a, b \in \mathcal{O}_K$ such that the coefficients of $a - b$ are all less than ε (otherwise the elements of $\mathcal{O}_K$ would all be a "distance" of least ε from each other, so only finitely many of them would fit in S).

As mentioned above, the norms of elements of $\mathcal{O}_K$ are integers. Since the norm of an element is the determinant of left multiplication by that element, the norm is a homogenous polynomial of degree n in the indeterminate coefficients c_i, which is 0 only on the element 0, so the constant term of this polynomial is 0. If the c_i get arbitrarily small for elements of $\mathcal{O}_K$, then the values of the norm polynomial get arbitrarily small, which would imply that there are elements of $\mathcal{O}_K$ with positive norm too small to be in $\mathbb{Z}$, a contradiction. So the set S contains only finitely many elements of $\mathcal{O}_K$. Thus the denominators of the c_i are bounded, so for some d, we have that $\mathcal{O}_K$ has finite index in $A = \frac{1}{d}\mathbb{Z}a_1 + \cdots + \frac{1}{d}\mathbb{Z}a_n$. Since A is isomorphic to $\mathbb{Z}^n$, it follows from the structure theorem for finitely generated abelian groups that $\mathcal{O}_K$ is isomorphic as a $\mathbb{Z}$-module to $\mathbb{Z}^n$, as claimed. □

Corollary 2.4.8. *The ring of integers $\mathcal{O}_K$ of a number field is noetherian.*

Proof. By Proposition 2.4.7, the ring $\mathcal{O}_K$ is finitely generated as a module over $\mathbb{Z}$, so it is certainly finitely generated as a ring over $\mathbb{Z}$. By Theorem 2.2.11, $\mathcal{O}_K$ is noetherian. □

2.5 Recognizing Algebraic Numbers using LLL

Suppose we somehow compute a decimal approximation α to some rational number $\beta \in \mathbb{Q}$ and from this wish to recover β. For concreteness, say

$$\beta = \frac{22}{389} = 0.05655526992287917737789203084832904884318766066838046\ldots$$

and we compute

$$\alpha = 0.056555.$$

Now suppose given only α that you would like to recover β. A standard technique is to use continued fractions, which yields a sequence of good rational approximations for α; by truncating right before a surprisingly big partial quotient (the 23 in the continued fraction `v`), we obtain β:

```
v = continued_fraction(0.056555); v
```

```
[0, 17, 1, 2, 6, 1, 23, 1, 1, 1, 1, 1, 2]
```

```
convergents([0, 17, 1, 2, 6, 1])
```

```
[0, 1/17, 1/18, 3/53, 19/336, 22/389]
```

Generalizing this, suppose next that somehow you numerically approximate an algebraic number, e.g., by evaluating a special function and get a decimal approximation $\alpha \in \mathbb{C}$ to an algebraic number $\beta \in \overline{\mathbb{Q}}$. For concreteness, suppose $\beta = \frac{1}{3} + \sqrt[4]{3}$:

```
N(1/3 + 3^(1/4), digits=50)
```

```
1.6494073462858257941525522351303323884934019235391 6
```

Now suppose you very much want to find the (rescaled) minimal polynomial $f(x) \in \mathbb{Z}[x]$ of β just given this numerical approximation α. This is of great value even without proof, since often in practice once you know a potential minimal polynomial you can verify that it is in fact right. Exactly this situation arises in the explicit construction of class fields (a more advanced topic in number theory) and in the construction of Heegner points on elliptic curves. As we will see, the LLL algorithm provides a polynomial time way to solve this problem, assuming α has been computed to sufficient precision.

2.5.1 LLL Reduced Basis

Given a basis $b_1, \ldots, b_n$ for $\mathbb{R}^n$, the *Gramm-Schmidt orthogonalization* process produces an orthogonal basis $b_1^*, \ldots, b_n^*$ for $\mathbb{R}^n$ as follows. Define inductively

$$b_i^* = b_i - \sum_{j<i} \mu_{i,j} b_j^*$$

where

$$\mu_{i,j} = \frac{b_i \cdot b_j^*}{b_j^* \cdot b_j^*}.$$

Example 2.5.1. We compute the Gramm-Schmidt orthogonal basis of the rows of a matrix. Note that no square roots are introduced in the process; there would be square roots if we constructed an orthonormal basis.

```
A = matrix(ZZ, 2, [1,2, 3,4]); A
```

```
[1 2]
[3 4]
```

```
Bstar, mu = A.gramm_schmidt()
```

The rows of the matrix B^* are obtained from the rows of A by the Gramm-Schmidt procedure.

```
Bstar
```

```
[   1     2]
[ 4/5  -2/5]
```

```
mu
```

```
[   0     0]
[11/5     0]
```

A *lattice* $L \subset \mathbb{R}^n$ is a subgroup that is free of rank n such that $\mathbb{R}L = \mathbb{R}^n$.

Definition 2.5.2 (LLL-reduced basis)**.** The basis $b_1, \ldots, b_n$ for a lattice $L \subset \mathbb{R}^n$ is *LLL reduced* if for all i, j,

$$|\mu_{i,j}| \leq \frac{1}{2}$$

and for each $i \geq 2$,

$$|b_i^*|^2 \geq \left(\frac{3}{4} - \mu_{i,i-1}^2\right) |b_{i-1}^*|^2$$

For example, the basis $b_1 = (1,2)$, $b_2 = (3,4)$ for a lattice L is *not* LLL reduced because $b_1^* = b_1$ and

$$\mu_{2,1} = \frac{b_2 \cdot b_1^*}{b_1^* \cdot b_1^*} = \frac{11}{5} > \frac{1}{2}.$$

However, the basis $b_1 = (1,0)$, $b_2 = (0,2)$ for L is LLL reduced, since

$$\mu_{2,1} = \frac{b_2 \cdot b_1^*}{b_1^* \cdot b_1^*} = 0,$$

and

$$2^2 \geq (3/4) \cdot 1^2.$$

```
A = matrix(ZZ, 2, [1,2, 3,4])
A.LLL()
```

```
[1 0]
[0 2]
```

2.5.2 What LLL really means

The following theorem is not too difficult to prove.

Let $b_1, \ldots, b_n$ be an LLL reduced basis for a lattice $L \subset \mathbb{R}^n$. Let $d(L)$ denote the absolute value of the determinant of any matrix whose rows are basis for L. Then the vectors b_i are "nearly orthogonal" and "short" in the sense of the following theorem:

Theorem 2.5.3. *We have*

1. $d(L) \leq \prod_{i=1}^{n} |b_i| \leq 2^{n(n-1)/4} d(L)$.

2. *For* $1 \leq j \leq i \leq n$, *we have*

$$|b_j| \leq 2^{(i-1)/2} |b_i^*|.$$

3. *The vector* b_1 *is very short in the sense that*

$$|b_1| \leq 2^{(n-1)/4} d(L)^{1/n}$$

and for every nonzero $x \in L$ *we have*

$$|b_1| \leq 2^{(n-1)/2} |x|.$$

4. *More generally, for any linearly independent* $x_1, \ldots, x_t \in L$, *we have*

$$|b_j| \leq 2^{(n-1)/2} \max(|x_1|, \ldots, |x_t|)$$

for $1 \leq j \leq t$.

Perhaps the most amazing thing about the idea of an LLL reduced basis is that there is an algorithm (in fact many) that given a basis for a lattice L produce an LLL reduced basis for L, and do so *quickly*, i.e., in polynomial time in the number of digits of the input. The current optimal implementation (and practically optimal algorithms) for computing LLL reduced basis are due to Damien Stehle, and are included standard in Magma in Sage. Stehle's code is amazing – it can LLL reduce a random lattice in $\mathbb{R}^n$ for $n < 1000$ in a matter of minutes!

```
A = random_matrix(ZZ, 200)
t = cputime()
B = A.LLL()
cputime(t)     # random output
```

```
3.0494159999999999
```

There is even a very fast variant of Stehle's implementation that computes a basis for L that is very likely LLL reduced but may in rare cases fail to be LLL reduced.

```
t = cputime()
B = A.LLL(algorithm="fpLLL:fast")   # not tested
cputime(t)      # random output
```

```
0.96842699999999837
```

2.5.3 Applying LLL

The LLL definition and algorithm has many application in number theory, e.g., to cracking lattice-based cryptosystems, to enumerating all short vectors in a lattice, to finding relations between decimal approximations to complex numbers, to very fast univariate polynomial factorization in $\mathbb{Z}[x]$ and more generally in $K[x]$ where K is a number fields, and to computation of kernels and images of integer matrices. LLL can also be used to solve the problem of recognizing algebraic numbers mentioned at the beginning of Section 2.5.

Suppose as above that α is a decimal approximation to some algebraic number β, and to for simplicity assume that $\alpha \in \mathbb{R}$ (the general case of $\alpha \in \mathbb{C}$ is described in [Coh93]). We finish by explaining how to use LLL to find a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\alpha)$ and its coefficients are small, hence has a shot at being the minimal polynomial of β.

Given a real number decimal approximation α, an integer d (the degree), and an integer K (a function of the precision to which α is known), the following steps produce a polynomial $f(x) \in \mathbb{Z}[x]$ of degree at most d such that $f(\alpha)$ is small.

1. Form the lattice in $\mathbb{R}^{d+2}$ with basis the rows of the matrix A whose first $(d+1) \times (d+1)$ part is the identity matrix, and whose last column has entries

   $$K, \lfloor K\alpha \rfloor, \lfloor K\alpha^2 \rfloor, \ldots, \lfloor K\alpha^d \rfloor. \tag{2.5.1}$$

   (Note this matrix is $(d+1) \times (d+2)$ so the lattice is not of full rank in $\mathbb{R}^{d+2}$, which isn't a problem, since the LLL definition also makes sense for fewer vectors.)

2. Compute an LLL reduced basis for the $\mathbb{Z}$-span of the rows of A, and let B be the corresponding matrix. Let $b_1 = (a_0, a_1, \ldots, a_{d+1})$ be the first row of B and notice that B is obtained from A by left multiplication by an invertible integer matrix. Thus $a_0, \ldots, a_d$ are the linear combination of the (2.5.1) that equals a_{d+1}. Moreover, since B is LLL reduced we expect that a_{d+1} is relatively small.

3. Output $f(x) = a_0 + a_1 x + \cdots + a_d x^d$. We have that $f(\alpha) \sim a_{d+1}/K$, which is small. Thus $f(x)$ may be a very good candidate for the minimal polynomial of β (the algebraic number we are approximating), assuming d was chosen minimally and α was computed to sufficient precision.

The following is a complete implementation of the above algorithm in Sage:

```
def myalgdep(a, d, K=10^6):
    aa = [floor(K*a^i) for i in range(d+1)]
    A = identity_matrix(ZZ, d+1)
    B = matrix(ZZ, d+1, 1, aa)
    A = A.augment(B)
    L = A.LLL()
    v = L[0][:-1].list()
    return ZZ['x'](v)
```

Here is an example of using it:

```
R.<x> = RDF[]
f = 2*x^3 - 3*x^2 + 10*x - 4
a = f.roots()[0][0]; a
myalgdep(a, 3, 10^6)        # not tested
```

```
2*x^3 - 3*x^2 + 10*x - 4
```

Chapter 3

Unique Factorization of Ideals

Unique factorization into irreducible elements frequently fails for rings of integers of number fields. In this chapter we will deduce a central property of the ring of integers $\mathcal{O}_K$ of an algebraic number field, namely that every nonzero *ideal* factors uniquely as a products of prime ideals. Along the way, we will introduce fractional ideals and prove that they form a free abelian group under multiplication. Factorization of *elements* of $\mathcal{O}_K$ (and much more!) is governed by the class group of $\mathcal{O}_K$, which is the quotient of the group of fractional ideals by the principal fractional ideals (see Chapter 7).

3.1 Dedekind Domains

Recall (Corollary 2.4.8) that we proved that the ring of integers $\mathcal{O}_K$ of a number field is noetherian as follows. As we saw before using norms, the ring $\mathcal{O}_K$ is finitely generated as a module over $\mathbb{Z}$, so it is certainly finitely generated as a ring over $\mathbb{Z}$. By the Hilbert Basis Theorem (Theorem 2.2.11), $\mathcal{O}_K$ is noetherian.

If R is an integral domain, the *field of fractions* $\mathrm{Frac}(R)$ of R is the field of all equivalence classes of formal quotients a/b, where $a, b \in R$ with $b \neq 0$, and $a/b \sim c/d$ if $ad = bc$. For example, the field of fractions of $\mathbb{Z}$ is (canonically isomorphic to) $\mathbb{Q}$ and the field of fractions of $\mathbb{Z}[(1+\sqrt{5})/2]$ is $\mathbb{Q}(\sqrt{5})$. The field of fractions of the ring $\mathcal{O}_K$ of integers of a number field K is just the number field K (see Lemma 2.3.28).

Example 3.1.1. We compute the fraction fields mentioned above.

```
Frac(ZZ)
```

```
Rational Field
```

In Sage the `Frac` command usually returns a field canonically isomorphic to the fraction field (not a formal construction).

```
K.<a> = QuadraticField(5)
OK = K.ring_of_integers(); OK
```

```
Maximal Order in Number Field in a with defining polynomial x^2 - 5
```

```
OK.basis()
```

```
[1/2*a + 1/2, a]
```

```
Frac(OK)
```

```
Number Field in a with defining polynomial x^2 - 5
```

Remark 3.1.2. Note that in computers `1/2 * x` means the same as `(1/2)*x`. For more information about the order of operations in programming see `http://en.wikipedia.org/wiki/Order_of_operations`. In Sage the ^ symbol is replaced with python's exponentiation `**` at execution.[1]

The fraction field of an *order* – i.e., a subring of $\mathcal{O}_K$ of finite index – is also the number field again.

```
O2 = K.order(2*a); O2
```

```
Order in Number Field in a with defining polynomial x^2 - 5
```

```
Frac(O2)
```

```
Number Field in a with defining polynomial x^2 - 5
```

Definition 3.1.3 (Integrally Closed)**.** An integral domain R is *integrally closed in its field of fractions* if whenever α is in the field of fractions of R and α satisfies a monic polynomial $f \in R[x]$, then $\alpha \in R$.

For example, every field is integrally closed in its field of fractions, as is the ring $\mathbb{Z}$ of integers. However, $\mathbb{Z}[\sqrt{5}]$ is not integrally closed in its field of fractions, since $(1+\sqrt{5})/2$ is integrally over $\mathbb{Z}$ and lies in $\mathbb{Q}(\sqrt{5})$, but not in $\mathbb{Z}[\sqrt{5}]$

Proposition 3.1.4. *If K is any number field, then $\mathcal{O}_K$ is integrally closed. Also, the ring $\overline{\mathbb{Z}}$ of all algebraic integers (in a fixed choice of $\overline{\mathbb{Q}}$) is integrally closed.*

Proof. We first prove that $\overline{\mathbb{Z}}$ is integrally closed. Suppose $\alpha \in \overline{\mathbb{Q}}$ is integral over $\overline{\mathbb{Z}}$, so there is a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_i \in \overline{\mathbb{Z}}$ and $f(\alpha) = 0$. The a_i all lie in the ring of integers $\mathcal{O}_K$ of the number field $K = \mathbb{Q}(a_0, a_1, \ldots a_{n-1})$, and $\mathcal{O}_K$ is finitely generated as a $\mathbb{Z}$-module, so $\mathbb{Z}[a_0, \ldots, a_{n-1}]$ is finitely generated as a $\mathbb{Z}$-module. Since $f(\alpha) = 0$, we can write α^n

[1] Another source for order of operations specific to python is `https://docs.python.org/2/reference/expressions.html#operator-precedence`.

as a $\mathbb{Z}[a_0,\ldots,a_{n-1}]$-linear combination of α^i for $i<n$, so the ring $\mathbb{Z}[a_0,\ldots,a_{n-1},\alpha]$ is also finitely generated as a $\mathbb{Z}$-module. Thus $\mathbb{Z}[\alpha]$ is finitely generated as a $\mathbb{Z}$-module because it is a submodule of a finitely generated $\mathbb{Z}$-module, which implies that α is integral over $\mathbb{Z}$.

Without loss we may assume that $K\subset\overline{\mathbb{Q}}$, so that $\mathcal{O}_K=\overline{\mathbb{Z}}\cap K$. Suppose $\alpha\in K$ is integral over $\mathcal{O}_K$. Then since $\overline{\mathbb{Z}}$ is integrally closed, α is an element of $\overline{\mathbb{Z}}$, so $\alpha\in K\cap\overline{\mathbb{Z}}=\mathcal{O}_K$, as required. □

Exercise 3.1.5. Prove that $\overline{\mathbb{Z}}$ is not noetherian.

[*Hint*: Consider an ideal generated by fractional powers of a prime.]

Definition 3.1.6 (Dedekind Domain)**.** An integral domain R is a *Dedekind domain* if it is noetherian, integrally closed in its field of fractions, and every nonzero prime ideal of R is maximal.

Exercise 3.1.7. Let K be a field.

(a) Prove that the polynomial ring $K[x]$ is a Dedekind domain.

(b) Is $\mathbb{Z}[x]$ a Dedekind domain?

The ring $\mathbb{Z}\oplus\mathbb{Z}$ is not a Dedekind domain because it is not an integral domain. The ring $\mathbb{Z}[\sqrt{5}]$ is not a Dedekind domain because it is not integrally closed in its field of fractions. The ring $\mathbb{Z}$ is a Dedekind domain, as is any ring of integers $\mathcal{O}_K$ of a number field, as we will see below. Also, any field K is a Dedekind domain, since it is an integral domain, it is trivially integrally closed in itself, and there are no nonzero prime ideals so the condition that they be maximal is empty.

Exercise 3.1.8. In Proposition 3.1.4 we showed that $\overline{\mathbb{Z}}$ is integrally closed in its field of fractions. Prove that and every nonzero prime ideal of $\overline{\mathbb{Z}}$ is maximal. Together with Exercise 3.1.5, this shows $\overline{\mathbb{Z}}$ is not a Dedekind domain only because it is not noetherian.

Exercise* 3.1.9. Show that Dedekind domains are closed under localization. This means the following: given any nonzero prime $\mathfrak{p}$ in R, the *localization* $R_\mathfrak{p}$ of R at $\mathfrak{p}$ is the ring formed by inverting all elements of R not contained in $\mathfrak{p}$. Thus $R_\mathfrak{p}$ is a subring of the field of fractions K of R which contains R. For example, $\mathbb{Z}_{(2)}$ is the localization of $\mathbb{Z}$ at the prime ideal (2). Note $\mathbb{Z}_{(2)}$ contains $\frac{1}{3}$ but not $\frac{1}{2}$. This exercise will show $R_\mathfrak{p}$ is again a Dedekind domain. In general, any element of $R_\mathfrak{p}$ can be written as a quotient $\frac{a}{b}$ for some $a\in R$ and $b\in R\setminus\mathfrak{p}$.

[*Hint*: It is a standard fact of localizations that the set of prime ideals in $R_\mathfrak{p}$ is in bijection with the set of prime ideals of R contained in $\mathfrak{p}$. Use this to show $R_\mathfrak{p}$ is noetherian and all prime ideals of $R_\mathfrak{p}$ are maximal. It remains to show $R_\mathfrak{p}$ is integrally closed. Let $\alpha\in K$ satisfy a monic polynomial with coefficients in $R_\mathfrak{p}$. By clearing denominators show that $s\alpha\in R$ for some $s\in R\setminus\mathfrak{p}$.]

Proposition 3.1.10. *The ring of integers $\mathcal{O}_K$ of a number field is a Dedekind domain.*

Proof. By Proposition 3.1.4, the ring $\mathcal{O}_K$ is integrally closed, and by Proposition 2.4.8 it is noetherian. Suppose that $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_K$. Let $\alpha \in \mathfrak{p}$ be a nonzero element, and let $f(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α. Then

$$f(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0,$$

so $a_0 = -(\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha) \in \mathfrak{p}$. Since f is irreducible, a_0 is a nonzero element of $\mathbb{Z}$ that lies in $\mathfrak{p}$. Every element of the finitely generated abelian group $\mathcal{O}_K/\mathfrak{p}$ is killed by a_0, so $\mathcal{O}_K/\mathfrak{p}$ is a finite set. Since $\mathfrak{p}$ is prime, $\mathcal{O}_K/\mathfrak{p}$ is an integral domain. Every finite integral domain is a field (see Exercise **??**), so $\mathfrak{p}$ is maximal, which completes the proof. □

3.2 Factorization of Ideals

If I and J are ideals in a ring R, the product IJ is the ideal *generated by* all products of elements in I with elements in J:

$$IJ = (ab : a \in I, b \in J) \subset R.$$

Note that the set of all products ab, with $a \in I$ and $b \in J$, need not be an ideal, so it is important to take the ideal generated by that set (see Exercise **??**).

Definition 3.2.1 (Fractional Ideal)**.** A *fractional ideal* is a nonzero $\mathcal{O}_K$-submodule I of K that is finitely generated as an $\mathcal{O}_K$-module.

We will sometimes call a genuine ideal $I \subset \mathcal{O}_K$ an *integral ideal*. The notion of fractional ideal makes sense for an arbitrary Dedekind domain R – it is an R-module $I \subset K = \mathrm{Frac}(R)$ that is finitely generated as an R-module.

Example 3.2.2. We multiply two fractional ideals in Sage:

```
K.<a> = NumberField(x^2 + 23)
I = K.fractional_ideal(2, 1/2*a - 1/2)
J = I^2
I
```

```
Fractional ideal (2, 1/2*a - 1/2)
```

```
J
```

```
Fractional ideal (4, 1/2*a + 3/2)
```

```
I*J
```

```
Fractional ideal (1/2*a + 3/2)
```

Since fractional ideals I are finitely generated, we can clear denominators of a generating set to see that there exists some nonzero $\alpha \in K$ such that

$$\alpha I = J \subset \mathcal{O}_K,$$

with J an integral ideal. Thus dividing by α, we see that every fractional ideal is of the form

$$aJ = \{ab : b \in J\}$$

for some $a \in K$ and integral ideal $J \subset \mathcal{O}_K$.

For example, the set $\frac{1}{2}\mathbb{Z}$ of rational numbers with denominator 1 or 2 is a fractional ideal of $\mathbb{Z}$.

Theorem 3.2.3. *The set of fractional ideals of a Dedekind domain R is an abelian group under ideal multiplication with identity element R.*

Note that fractional ideals are nonzero by definition, so it is not necessary to write "nonzero fractional ideals" in the statement of the theorem. We will *only* prove Theorem 3.2.3 in the case when $R = \mathcal{O}_K$ is the ring of integers of a number field K. The general case can be found in many algebraic number theory books such as [Mar77, Ch. 3]. Before proving Theorem 3.2.3 we prove a lemma. For the rest of this section $\mathcal{O}_K$ is the ring of integers of a number field K.

Definition 3.2.4 (Divides for Ideals)**.** Suppose that I, J are ideals of $\mathcal{O}_K$. Then we say that I *divides* J if $I \supset J$.

To see that this notion of divides is sensible, suppose $K = \mathbb{Q}$, so $\mathcal{O}_K = \mathbb{Z}$. Then $I = (n)$ and $J = (m)$ for some integer n and m, and I divides J means that $(n) \supset (m)$, i.e., that there exists an integer c such that $m = cn$, which exactly means that n divides m, as expected.

Lemma 3.2.5. *Suppose I is a nonzero ideal of $\mathcal{O}_K$. Then there exist prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdots \mathfrak{p}_n \subset I$, i.e., I divides a product of prime ideals.*

Proof. Let S be the set of nonzero ideals of $\mathcal{O}_K$ that do not satisfy the conclusion of the lemma. The key idea is to use that $\mathcal{O}_K$ is noetherian to show that S is the empty set. If S is nonempty, then since $\mathcal{O}_K$ is noetherian, there is an ideal $I \in S$ that is maximal as an element of S. If I were prime, then I would trivially contain a product of primes, so we may assume that I is not prime. Thus there exists $a, b \in \mathcal{O}_K$ such that $ab \in I$ but $a \notin I$ and $b \notin I$. Let $J_1 = I + (a)$ and $J_2 = I + (b)$. Then neither J_1 nor J_2 is in S, since I is maximal, so both J_1 and J_2 contain a product of prime ideals, say $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subset J_1$ and $\mathfrak{q}_1 \cdots \mathfrak{q}_s \subset J_2$. Then

$$\mathfrak{p}_1 \cdots \mathfrak{p}_r \cdot \mathfrak{q}_1 \cdots \mathfrak{q}_s \subset J_1 J_2 = I^2 + I(b) + (a)I + (ab) \subset I,$$

so I contains a product of primes. This is a contradiction, since we assumed $I \in S$. Thus S is empty, which completes the proof. □

We are now ready to prove the theorem.

Proof of Theorem 3.2.3. Note that we will *only* prove Theorem 3.2.3 in the case when $R = \mathcal{O}_K$ is the ring of integers of a number field K.

The product of two fractional ideals is again finitely generated, so it is a fractional ideal, and $I\mathcal{O}_K = I$ for any ideal I, so to prove that the set of fractional ideals under multiplication is a group it suffices to show the existence of inverses. We will first prove that if $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}$ has an inverse, then we will prove that all nonzero integral ideals have inverses, and finally observe that every fractional ideal has an inverse. (Note: Once we know that the set of fractional ideals is a group, it will follows that inverses are unique; until then we will be careful to write "an" instead of "the".)

Suppose $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_K$. We will show that the $\mathcal{O}_K$-module

$$I = \{a \in K : a\mathfrak{p} \subset \mathcal{O}_K\}$$

is a fractional ideal of $\mathcal{O}_K$ such that $I\mathfrak{p} = \mathcal{O}_K$, so that I is an inverse of $\mathfrak{p}$.

For the rest of the proof, fix a nonzero element $b \in \mathfrak{p}$. Since I is an $\mathcal{O}_K$-module, $bI \subset \mathcal{O}_K$ is an $\mathcal{O}_K$ ideal, hence I is a fractional ideal. Since $\mathcal{O}_K \subset I$ we have $\mathfrak{p} \subset I\mathfrak{p} \subset \mathcal{O}_K$, hence since $\mathfrak{p}$ is maximal, either $\mathfrak{p} = I\mathfrak{p}$ or $I\mathfrak{p} = \mathcal{O}_K$. If $I\mathfrak{p} = \mathcal{O}_K$, we are done since then I is an inverse of $\mathfrak{p}$. Thus suppose that $I\mathfrak{p} = \mathfrak{p}$. Our strategy is to show that there is some $d \in I$, with $d \notin \mathcal{O}_K$. Since $I\mathfrak{p} = \mathfrak{p}$, such a d would leave $\mathfrak{p}$ invariant, i.e., $d\mathfrak{p} \subset \mathfrak{p}$. Since $\mathfrak{p}$ is a finitely generated $\mathcal{O}_K$-module we will see that it will follow that $d \in \mathcal{O}_K$, a contradiction.

By Lemma 3.2.5, we can choose a product $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$, with m minimal, with

$$\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_m \subset (b) \subset \mathfrak{p}.$$

If no $\mathfrak{p}_i$ is contained in $\mathfrak{p}$, then we can choose for each i an $a_i \in \mathfrak{p}_i$ with $a_i \notin \mathfrak{p}$; but then $\prod a_i \in \mathfrak{p}$, which contradicts that $\mathfrak{p}$ is a prime ideal. Thus some $\mathfrak{p}_i$, say $\mathfrak{p}_1$, is contained in $\mathfrak{p}$, which implies that $\mathfrak{p}_1 = \mathfrak{p}$ since every nonzero prime ideal is maximal. Because m is minimal, $\mathfrak{p}_2 \cdots \mathfrak{p}_m$ is not a subset of (b), so there exists $c \in \mathfrak{p}_2 \cdots \mathfrak{p}_m$ that does not lie in (b). Then $\mathfrak{p}(c) \subset (b)$, so by definition of I we have $d = c/b \in I$. However, $d \notin \mathcal{O}_K$, since if it were then c would be in (b). We have thus found our element $d \in I$ that does not lie in $\mathcal{O}_K$.

To finish the proof that $\mathfrak{p}$ has an inverse, we observe that d preserves the finitely generated $\mathcal{O}_K$-module $\mathfrak{p}$, and is hence in $\mathcal{O}_K$, a contradiction. More precisely, if $b_1, \ldots, b_n$ is a basis for $\mathfrak{p}$ as a $\mathbb{Z}$-module, then the action of d on $\mathfrak{p}$ is given by a matrix with entries in $\mathbb{Z}$, so the minimal polynomial of d has coefficients in $\mathbb{Z}$ (because d satisfies the minimal polynomial of ℓ_d, by the Cayley-Hamilton theorem – here we also use that $\mathbb{Q} \otimes \mathfrak{p} = K$, since $\mathcal{O}_K/\mathfrak{p}$ is a finite set). This implies that d is integral over $\mathbb{Z}$, so $d \in \mathcal{O}_K$ since $\mathcal{O}_K$ is by definition the set of elements of K that are integral over $\mathbb{Z}$.

So far we have proved that if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$, then

$$\mathfrak{p}^{-1} = \{a \in K : a\mathfrak{p} \subset \mathcal{O}_K\}$$

is the inverse of $\mathfrak{p}$ in the monoid of nonzero fractional ideals of $\mathcal{O}_K$. As mentioned after Definition 3.2.1, every nonzero fractional ideal is of the form aI for $a \in K$

and I an integral ideal, so since (a) has inverse $(1/a)$, it suffices to show that every integral ideal I has an inverse. If not, then there is a nonzero integral ideal I that is maximal among all nonzero integral ideals that do not have an inverse. Every ideal is contained in a maximal ideal, so there is a nonzero prime ideal $\mathfrak{p}$ such that $I \subset \mathfrak{p}$. Multiplying both sides of this inclusion by $\mathfrak{p}^{-1}$ and using that $\mathcal{O}_K \subset \mathfrak{p}^{-1}$, we see that

$$I \subset \mathfrak{p}^{-1} I \subset \mathfrak{p}^{-1}\mathfrak{p} = \mathcal{O}_K.$$

If $I = \mathfrak{p}^{-1}I$, then arguing as in the proof that $\mathfrak{p}^{-1}$ is an inverse of $\mathfrak{p}$, we see that each element of $\mathfrak{p}^{-1}$ preserves the finitely generated $\mathbb{Z}$-module I and is hence integral. But then $\mathfrak{p}^{-1} \subset \mathcal{O}_K$, which, upon multiplying both sides by $\mathfrak{p}$, implies that $\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^{-1} \subset \mathfrak{p}$, a contradiction. Thus $I \neq \mathfrak{p}^{-1}I$. Because I is maximal among ideals that do not have an inverse, the ideal $\mathfrak{p}^{-1}I$ does have an inverse J. Then $\mathfrak{p}^{-1}J$ is an inverse of I, since $(J\mathfrak{p}^{-1})I = J(\mathfrak{p}^{-1}I) = \mathcal{O}_K$. □

We can finally deduce the crucial Theorem 3.2.6, which will allow us to show that any nonzero ideal of a Dedekind domain can be expressed uniquely as a product of primes (up to order). Thus unique factorization holds for ideals in a Dedekind domain, and it is this unique factorization that initially motivated the introduction of ideals to mathematics over a century ago.

Theorem 3.2.6. *Suppose I is a nonzero integral ideal of $\mathcal{O}_K$. Then I can be written as a product*

$$I = \mathfrak{p}_1 \cdots \mathfrak{p}_n$$

of prime ideals of $\mathcal{O}_K$, and this representation is unique up to order.

Proof. Suppose I is an ideal that is maximal among the set of all ideals in $\mathcal{O}_K$ that cannot be written as a product of primes. Every ideal is contained in a maximal ideal, so I is contained in a nonzero prime ideal $\mathfrak{p}$. If $I\mathfrak{p}^{-1} = I$, then by Theorem 3.2.3 we can cancel I from both sides of this equation to see that $\mathfrak{p}^{-1} = \mathcal{O}_K$, a contradiction. Since $\mathcal{O}_K \subset \mathfrak{p}^{-1}$, we have $I \subset I\mathfrak{p}^{-1}$, and by the above observation I is strictly contained in $I\mathfrak{p}^{-1}$. By our maximality assumption on I, there are maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $I\mathfrak{p}^{-1} = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. Then $I = \mathfrak{p} \cdot \mathfrak{p}_1 \cdots \mathfrak{p}_n$, a contradiction. Thus every ideal can be written as a product of primes.

Suppose $\mathfrak{p}_1 \cdots \mathfrak{p}_n = \mathfrak{q}_1 \cdots \mathfrak{q}_m$. If no $\mathfrak{q}_i$ is contained in $\mathfrak{p}_1$, then for each i there is an $a_i \in \mathfrak{q}_i$ such that $a_i \not\in \mathfrak{p}_1$. But the product of the a_i is in $\mathfrak{p}_1 \cdots \mathfrak{p}_n$, which is a subset of $\mathfrak{p}_1$, which contradicts that $\mathfrak{p}_1$ is a prime ideal. Thus $\mathfrak{q}_i = \mathfrak{p}_1$ for some i. We can thus cancel $\mathfrak{q}_i$ and $\mathfrak{p}_1$ from both sides of the equation by multiplying both sides by the inverse. Repeating this argument finishes the proof of uniqueness. □

Theorem 3.2.7. *If I is a fractional ideal of $\mathcal{O}_K$ then there exists prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$, unique up to order, such that*

$$I = (\mathfrak{p}_1 \cdots \mathfrak{p}_n)(\mathfrak{q}_1 \cdots \mathfrak{q}_m)^{-1}.$$

Proof. We have $I = (a/b)J$ for some $a, b \in \mathcal{O}_K$ and integral ideal J. Applying Theorem 3.2.6 to (a), (b), and J gives an expression as claimed. For uniqueness, if one has two such product expressions, multiply through by the denominators and use the uniqueness part of Theorem 3.2.6 □

Example 3.2.8. The ring of integers of $K = \mathbb{Q}(\sqrt{-6})$ is $\mathcal{O}_K = \mathbb{Z}[\sqrt{-6}]$. We have

$$6 = -\sqrt{-6}\sqrt{-6} = 2 \cdot 3.$$

If $ab = \sqrt{-6}$, with $a, b \in \mathcal{O}_K$ and neither a unit, then $\mathrm{Norm}(a)\,\mathrm{Norm}(b) = 6$, so without loss $\mathrm{Norm}(a) = 2$ and $\mathrm{Norm}(b) = 3$. If $a = c + d\sqrt{-6}$, then $\mathrm{Norm}(a) = c^2 + 6d^2$; since the equation $c^2 + 6d^2 = 2$ has no solution with $c, d \in \mathbb{Z}$, there is no element in $\mathcal{O}_K$ with norm 2, so $\sqrt{-6}$ is irreducible. Also, $\sqrt{-6}$ is not a unit times 2 or times 3, since again the norms would not match up. Thus 6 cannot be written uniquely as a product of irreducibles in $\mathcal{O}_K$. Theorem 3.2.7, however, implies that the principal ideal (6) can, however, be written uniquely as a product of prime ideals. An explicit decomposition is

$$(6) = (2, 2 + \sqrt{-6})^2 \cdot (3, 3 + \sqrt{-6})^2, \tag{3.2.1}$$

where each of the ideals $(2, 2 + \sqrt{-6})$ and $(3, 3 + \sqrt{-6})$ is prime. We will discuss algorithms for computing such a decomposition in detail in Chapter 4. The first idea is to write $(6) = (2)(3)$, and hence reduce to the case of writing the (p), for $p \in \mathbb{Z}$ prime, as a product of primes. Next one decomposes the finite (as a set) ring $\mathcal{O}_K/p\mathcal{O}_K$.

The factorization (3.2.1) can be compute using Sage as follows:

```
K.<a> = NumberField(x^2 + 6); K
```

```
Number Field in a with defining polynomial x^2 + 6
```

```
K.factor(6)
```

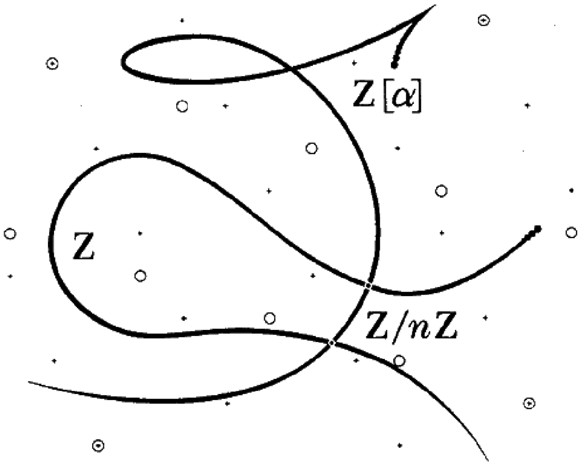
```
(Fractional ideal (2, a))^2 * \
(Fractional ideal (3, a))^2
```

Chapter 4

Factoring Primes

Let p be a prime and $\mathcal{O}_K$ the ring of integers of a number field. This chapter is about how to write $p\mathcal{O}_K$ as a product of prime ideals of $\mathcal{O}_K$. Paradoxically, computing the explicit prime ideal factorization of $p\mathcal{O}_K$ is easier than computing $\mathcal{O}_K$.

4.1 The Problem



A diagram from [LL93].

> "The obvious mathematical breakthrough would be development of an easy way to factor large prime numbers."
> – Bill Gates, *The Road Ahead*, 1st ed., pg 265

Bill Gates meant[1] factoring products of two primes, which would break the RSA cryptosystem (see e.g. [Ste09, §3.2]). However, perhaps Gates is an algebraic number theorist, and he really meant what he said: then we might imagine that he meant factorization of primes of $\mathbb{Z}$ in rings of integers of number fields. For example, $2^{16}+1 = 65537$ is a "large" prime, and in $\mathbb{Z}[i]$ we have

$$(65537) = (65537, 2^8 + i) \cdot (65537, 2^8 - i).$$

4.1.1 Geometric Intuition

Let $K = \mathbb{Q}(\alpha)$ be a number field, and let $\mathcal{O}_K$ be the ring of integers of K. To employ our geometric intuition, as the Lenstras did on the cover of [LL93], it is helpful to view $\mathcal{O}_K$ as a 1-dimensional scheme

$$X = \mathrm{Spec}(\mathcal{O}_K) = \{\text{all prime ideals of } \mathcal{O}_K\}$$

over

$$Y = \mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{p\mathbb{Z} : p \in \mathbb{Z}_{>0} \text{ is prime}\}.$$

There is a natural map $\pi : X \to Y$ that sends a prime ideal $\mathfrak{p} \in X$ to $\mathfrak{p} \cap \mathbb{Z} \in Y$. For example, if

$$\mathfrak{p} = (65537, 2^8 + i) \subset \mathbb{Z}[i],$$

then $\mathfrak{p} \cap \mathbb{Z} = (65537)$. For more on this viewpoint, see [Har77] and [EH00, Ch. 2].

If $p \in \mathbb{Z}$ is a prime number, then the ideal $p\mathcal{O}_K$ of $\mathcal{O}_K$ factors uniquely as a product $\prod \mathfrak{p}_i^{e_i}$, where the $\mathfrak{p}_i$ are maximal ideals of $\mathcal{O}_K$. We may imagine the decomposition of $p\mathcal{O}_K$ into prime ideals geometrically as the fiber $\pi^{-1}(p\mathbb{Z})$, where the exponents e_i are the multiplicities of the fibers. Notice that the elements of $\pi^{-1}(p\mathbb{Z})$ are the prime ideals of $\mathcal{O}_K$ that contain p, i.e., the primes that divide $p\mathcal{O}_K$. This chapter is about how to compute the $\mathfrak{p}_i$ and e_i.

Remark 4.1.1. More technically, in algebraic geometry one defines the inverse image of the point $p\mathbb{Z}$ to be the spectrum of the tensor product $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}/\mathfrak{p}\mathbb{Z}$; by a generalization of the Chinese Remainder Theorem, we have

$$\mathcal{O}_K \otimes_{\mathbb{Z}} (\mathbb{Z}/\mathfrak{p}\mathbb{Z}) \cong \oplus \mathcal{O}_K/\mathfrak{p}_i^{e_i}.$$

Exercise 4.1.2. Which of the following rings have infinitely many prime ideals?

[1] This quote is on page 265 of the first edition. In the second edition, on page 303, this sentence is changed to "The obvious mathematical breakthrough that would defeat our public key encryption would be the development of an easy way to factor large numbers." This is less nonsensical; however, fast factoring is *not* known to break all commonly used public-key cryptosystem. For example, there are cryptosystems based on the difficulty of computing discrete logarithms in $\mathbb{F}_p^*$ and on elliptic curves over $\mathbb{F}_p$, which (presumably) would not be broken even if one could factor large numbers quickly.

- The integers $\mathbb{Z}$.
- The ring $\mathbb{Z}[x]$ of polynomials over $\mathbb{Z}$.
- The quotient ring $\mathbb{C}[x]/(x^{2005}-1)$.
- The ring $(\mathbb{Z}/6\mathbb{Z})[x]$ of polynomials over the ring $\mathbb{Z}/6\mathbb{Z}$.
- The quotient ring $\mathbb{Z}/n\mathbb{Z}$, for a fixed positive integer n.
- The rational numbers $\mathbb{Q}$.
- The polynomial ring $\mathbb{Q}[x,y,z]$ in three variables.

4.1.2 Examples

The following Sage session shows the commands needed to compute the factorization of $p\mathcal{O}_K$ for K the number field defined by a root of $x^5+7x^4+3x^2-x+1$ and $p=2$ and 5. We first create an element $f \in \mathbb{Q}[x]$ in Sage:

```
R.<x> = QQ[]
f = x^5 + 7*x^4 + 3*x^2 - x + 1
```

Then we create the corresponding number field obtained by adjoining a root of f, and find its ring of integers.

```
K.<a> = NumberField(f)
OK = K.ring_of_integers()
OK.basis()
```

```
[1, a, a^2, a^3, a^4]
```

We define the ideal $2\mathcal{O}_K$ and factor – it turns out to be prime.

```
I = K.fractional_ideal(2); I
```

```
Fractional ideal (2)
```

```
I.factor()
```

```
Fractional ideal (2)
```

```
I.is_prime()
```

```
True
```

Finally we factor $5\mathcal{O}_K$, which factors as a product of three primes.

```
I = K.fractional_ideal(5); I
```

```
Fractional ideal (5)
```

```
I.factor()
```

```
(Fractional ideal (5, -2*a^4 - 13*a^3 + 7*a^2 - 6*a + 2)) * \
(Fractional ideal (5, a^4 + 7*a^3 + 3*a + 1)) * \
(Fractional ideal (5, a^4 + 7*a^3 + 3*a - 3))^2
```

Notice that the polynomial f factors in a similar way:

```
f.factor_mod(5)
```

```
(x + 2) * (x + 3)^2 * (x^2 + 4*x + 2)
```

Thus $2\mathcal{O}_K$ is already a prime ideal, and

$$5\mathcal{O}_K = (5, 2+a)\cdot(5, 3+a)^2\cdot(5, 2+4a+a^2).$$

Notice that in this example $\mathcal{O}_K = \mathbb{Z}[a]$. (Warning: There are examples of $\mathcal{O}_K$ such that $\mathcal{O}_K \neq \mathbb{Z}[a]$ for any $a \in \mathcal{O}_K$, as Example 4.3.2 below illustrates.) When $\mathcal{O}_K = \mathbb{Z}[a]$ it is relatively easy to factor $p\mathcal{O}_K$, at least assuming one can factor polynomials in $\mathbb{F}_p[x]$. The following factorization gives a hint as to why:

$$x^5 + 7x^4 + 3x^2 - x + 1 \equiv (x+2)\cdot(x+3)^2\cdot(x^2+4x+2) \pmod{5}.$$

The exponent 2 of $(5, 3+a)^2$ in the factorization of $5\mathcal{O}_K$ above suggests "ramification", in the sense that the cover $X \to Y$ has less points (counting their "size", i.e., their residue class degree) in its fiber over 5 than it has generically. See Figure 4.1.1.

4.2 A Method for Factoring Primes that Often Works

Suppose $a \in \mathcal{O}_K$ is such that $K = \mathbb{Q}(a)$, and let $f(x) \in \mathbb{Z}[x]$ be the minimal polynomial of a. Then $\mathbb{Z}[a] \subset \mathcal{O}_K$, and we have a diagram of schemes

$$\begin{tikzcd}
\bigcup \mathrm{Spec}(\mathcal{O}_K/\mathfrak{p}_i^{e_i}) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(\mathcal{O}_K) \arrow[d] \\
\bigcup \mathrm{Spec}(\mathbb{F}_p[x]/(\overline{f}_i^{e_i})) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(\mathbb{Z}[a]) \arrow[d] \\
\mathrm{Spec}(\mathbb{F}_p) \arrow[r, hook] & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd}$$

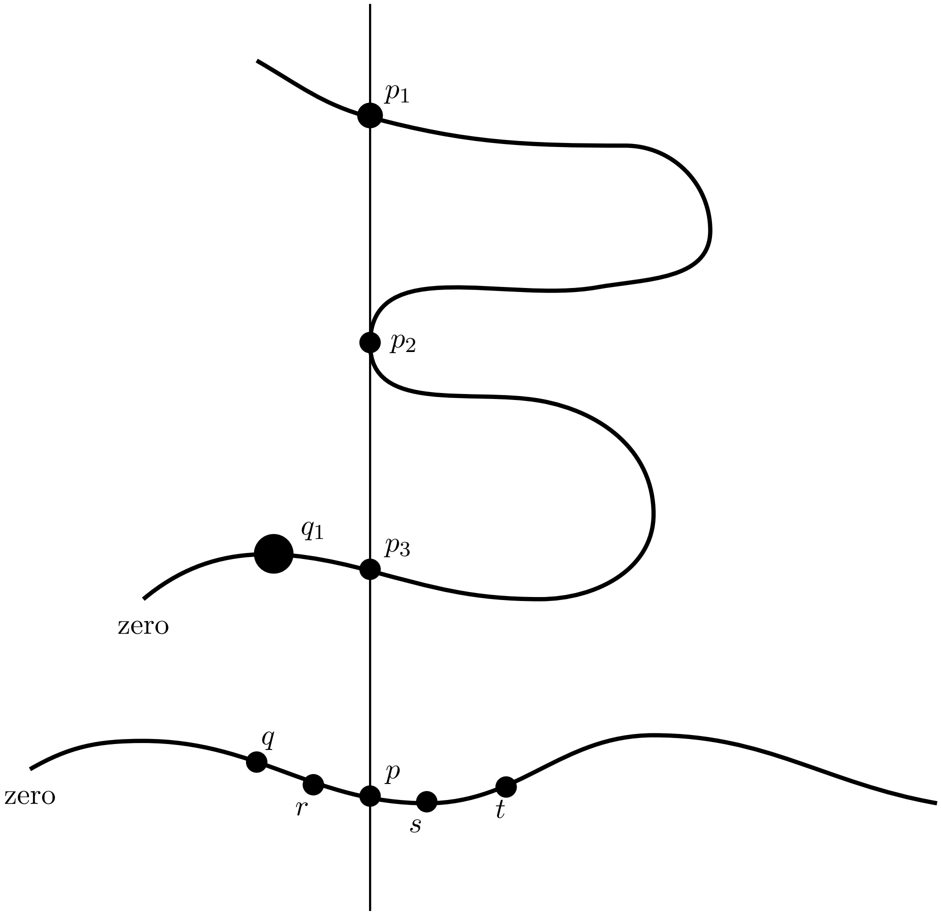


Figure 4.1.1: Diagram of $\mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spec}(\mathbb{Z})$

where $\overline{f} = \prod_i \overline{f}_i^{e_i}$ is the factorization of the image of f in $\mathbb{F}_p[x]$, and $p\mathcal{O}_K = \prod \mathfrak{p}_i^{e_i}$ is the factorization of $p\mathcal{O}_K$ in terms of prime ideals of $\mathcal{O}_K$. On the level of rings, the bottom horizontal map is the quotient map $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$. The middle horizontal map is induced by

$$\mathbb{Z}[x] \to \bigoplus_i \mathbb{F}_p[x]/(\overline{f}_i^{e_i}),$$

and the top horizontal map is induced by

$$\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K \cong \bigoplus \mathcal{O}_K/\mathfrak{p}_i^{e_i},$$

where the isomorphism is by the Chinese Remainder Theorem, which is Theorem 5.1.4 below. The left vertical maps come from the inclusions

$$\mathbb{F}_p \hookrightarrow \mathbb{F}_p[x]/(\overline{f}_i^{e_i}) \hookrightarrow \mathcal{O}_K/\mathfrak{p}_i^{e_i},$$

and the right from the inclusions $\mathbb{Z} \hookrightarrow \mathbb{Z}[a] \hookrightarrow \mathcal{O}_K$.

The cover $\pi : \mathrm{Spec}(\mathbb{Z}[a]) \to \mathrm{Spec}(\mathbb{Z})$ is easy to understand because it is defined by the single equation $f(x)$, in the sense that $\mathbb{Z}[a] \cong \mathbb{Z}[x]/(f(x))$. To give a maximal ideal $\mathfrak{p}$ of $\mathbb{Z}[a]$ such that $\pi(\mathfrak{p}) = p\mathbb{Z}$ is the same as giving a homomorphism $\varphi : \mathbb{Z}[x]/(f) \to \overline{\mathbb{F}}_p$ up to automorphisms of the image, which is in turn the same as giving a root of f in $\overline{\mathbb{F}}_p$ up to automorphism, which is the same as giving an irreducible factor of the reduction of f modulo p.

Lemma 4.2.1. *Suppose the index of $\mathbb{Z}[a]$ in $\mathcal{O}_K$ is coprime to p. Then the primes $\mathfrak{p}_i$ in the factorization of $p\mathbb{Z}[a]$ do not decompose further going from $\mathbb{Z}[a]$ to $\mathcal{O}_K$, so finding the prime ideals of $\mathbb{Z}[a]$ that contain p yields the primes that appear in the factorization of $p\mathcal{O}_K$.*

Proof. Fix a basis for $\mathcal{O}_K$ and for $\mathbb{Z}[a]$ as $\mathbb{Z}$-modules. Form the matrix A whose columns express each basis element of $\mathbb{Z}[a]$ as a $\mathbb{Z}$-linear combination of the basis for $\mathcal{O}_K$. Then

$$\det(A) = \pm[\mathcal{O}_K : \mathbb{Z}[a]]$$

is coprime to p, by hypothesis. Thus the reduction of A modulo p is invertible, so it defines an isomorphism $\mathbb{Z}[a]/p\mathbb{Z}[a] \cong \mathcal{O}_K/p\mathcal{O}_K$.

Let $\overline{\mathbb{F}}_p$ denote a fixed algebraic closure of $\mathbb{F}_p$; thus $\overline{\mathbb{F}}_p$ is an algebraically closed field of characteristic p, over which all polynomials in $\mathbb{F}_p[x]$ factor into linear factors. Any homomorphism $\mathcal{O}_K \to \overline{\mathbb{F}}_p$ sends p to 0, so is the composition of a homomorphism $\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K$ with a homomorphism $\mathcal{O}_K/p\mathcal{O}_K \to \overline{\mathbb{F}}_p$. Since $\mathcal{O}_K/p\mathcal{O}_K \cong \mathbb{Z}[a]/p\mathbb{Z}[a]$, the homomorphisms $\mathcal{O}_K \to \overline{\mathbb{F}}_p$ are in bijection with the homomorphisms $\mathbb{Z}[a] \to \overline{\mathbb{F}}_p$. The homomorphisms $\mathbb{Z}[a] \to \overline{\mathbb{F}}_p$ are in bijection with the roots of the reduction modulo p of the minimal polynomial of a in $\overline{\mathbb{F}}_p$. □

Remark 4.2.2. Here is a "high-brow" proof of Lemma 4.2.1. By hypothesis we have an exact sequence of abelian groups

$$0 \to \mathbb{Z}[a] \to \mathcal{O}_K \to H \to 0,$$

where H is a finite abelian group of order coprime to p. Tensor product is right exact, and there is an exact sequence

$$\mathrm{Tor}_1(H, \mathbb{F}_p) \to \mathbb{Z}[a] \otimes \mathbb{F}_p \to \mathcal{O}_K \otimes \mathbb{F}_p \to H \otimes \mathbb{F}_p \to 0,$$

and $\mathrm{Tor}_1(H, \mathbb{F}_p) = 0$ (since H has no p-torsion), so $\mathbb{Z}[a] \otimes \mathbb{F}_p \cong \mathcal{O}_K \otimes \mathbb{F}_p$.

As suggested in the proof of the lemma, we find all homomorphisms $\mathcal{O}_K \to \overline{\mathbb{F}}_p$ by finding all homomorphism $\mathbb{Z}[a] \to \overline{\mathbb{F}}_p$. In terms of ideals, if $\mathfrak{p} = (f(a), p)\mathbb{Z}[a]$ is a maximal ideal of $\mathbb{Z}[a]$, then the ideal $\mathfrak{p}' = (f(a), p)\mathcal{O}_K$ of $\mathcal{O}_K$ is also maximal, since

$$\mathcal{O}_K/\mathfrak{p}' \cong (\mathcal{O}_K/p\mathcal{O}_K)/(f(\tilde{a})) \cong (\mathbb{Z}[a]/p\mathbb{Z}[a])/(f(\tilde{a})) \subset \overline{\mathbb{F}}_p,$$

where $\tilde{a}$ denotes the image of a in $\mathcal{O}_K/p\mathcal{O}_K$.

We formalize the above discussion in the following theorem (note: we will not prove that the powers are e_i here):

Theorem 4.2.3. *Let $f \in \mathbb{Z}[x]$ be the minimal polynomial of a over $\mathbb{Z}$. Suppose that $p \nmid [\mathcal{O}_K : \mathbb{Z}[a]]$ is a prime. Let*

$$\overline{f} = \prod_{i=1}^{t} \overline{f}_i^{e_i} \in \mathbb{F}_p[x]$$

where the $\overline{f}_i$ are distinct monic irreducible polynomials. Let $\mathfrak{p}_i = (p, f_i(a))$ where $f_i \in \mathbb{Z}[x]$ is a lift of $\overline{f}_i$ in $\mathbb{F}_p[x]$. Then

$$p\mathcal{O}_K = \prod_{i=1}^{t} \mathfrak{p}_i^{e_i}.$$

We return to the example from above, in which $K = \mathbb{Q}(a)$, where a is a root of $f = x^5 + 7x^4 + 3x^2 - x + 1$. The ring of integers $\mathcal{O}_K$ has discriminant $2945785 = 5 \cdot 353 \cdot 1669$, as the following Sage code shows.

```
K.<a> = NumberField(x^5 + 7*x^4 + 3*x^2 - x + 1)
D = K.discriminant(); D
```

```
2945785
```

```
factor(D)
```

```
5 * 353 * 1669
```

The order $\mathbb{Z}[a]$ has the same discriminant as $f(x)$, which is the same as the discriminant of $\mathcal{O}_K$, so $\mathbb{Z}[a] = \mathcal{O}_K$ and we can apply the above theorem. (Here we use that the index of $\mathbb{Z}[a]$ in $\mathcal{O}_K$ is the square of the quotient of their discriminants, a fact we will prove later in Section 6.2.)

```
R.<x> = QQ[]
discriminant(x^5 + 7*x^4 + 3*x^2 - x + 1)
```

```
2945785
```

We have

$$x^5 + 7x^4 + 3x^2 - x + 1 \equiv (x+2) \cdot (x+3)^2 \cdot (x^2 + 4x + 2) \pmod{5},$$

which yields the factorization of $5\mathcal{O}_K$ given before the theorem.

If we replace a by $b = 7a$, then the index of $\mathbb{Z}[b]$ in $\mathcal{O}_K$ will be a power of 7, which is coprime to 5, so the above method will still work.

```
K.<a> = NumberField(x^5 + 7*x^4 + 3*x^2 - x + 1)
f = (7*a).minpoly('x')
f
```

```
x^5 + 49*x^4 + 1029*x^2 - 2401*x + 16807
```

```
f.disc()
```

```
235050861175510968365785
```

```
factor(f.disc() / K.disc())
```

```
7^20
```

```
f.factor_mod(5)
```

```
(x + 4) * (x + 1)^2 * (x^2 + 3*x + 3)
```

Thus 5 factors in $\mathcal{O}_K$ as

$$5\mathcal{O}_K = (5, 7a+1)^2 \cdot (5, 7a+4) \cdot (5, (7a)^2 + 3(7a) + 3).$$

If we replace a by $b = 5a$ and try the above algorithm with $\mathbb{Z}[b]$, then the method fails because the index of $\mathbb{Z}[b]$ in $\mathcal{O}_K$ is divisible by 5.

```
K.<a> = NumberField(x^5 + 7*x^4 + 3*x^2 - x + 1)
f = (5*a).minpoly('x')
f
```

```
x^5 + 35*x^4 + 375*x^2 - 625*x + 3125
```

```
f.factor_mod(5)
```

```
x^5
```

4.3 A General Method

There are numbers fields K such that $\mathcal{O}_K$ is not of the form $\mathbb{Z}[a]$ for any $a \in K$. Even worse, Dedekind found a field K such that $2 \mid [\mathcal{O}_K : \mathbb{Z}[a]]$ for *all* $a \in \mathcal{O}_K$, so there is no choice of a such that Theorem 4.2.3 can be used to factor 2 for K (see Example 4.3.2 below).

4.3.1 Inessential Discriminant Divisors

Definition 4.3.1. A prime p is an *inessential discriminant divisor* if $p \mid [\mathcal{O}_K : \mathbb{Z}[a]]$ for *every* $a \in \mathcal{O}_K$.

See Example 6.2.7 below for why it is called an inessential "discriminant divisor" instead of an inessential "index divisor".

Since $[\mathcal{O}_K : \mathbb{Z}[a]]^2$ is the absolute value of $\mathrm{Disc}(f(x))/\mathrm{Disc}(\mathcal{O}_K)$, where $f(x)$ is the characteristic polynomial of $f(x)$, an inessential discriminant divisor divides the discriminant of the characteristic polynomial of any element of $\mathcal{O}_K$.

Example 4.3.2 (Dedekind). Let $K = \mathbb{Q}(a)$ be the cubic field defined by a root a of the polynomial $f = x^3 + x^2 - 2x + 8$. We will use Sage to show that 2 is an inessential discriminant divisor for K.

```
K.<a> = NumberField(x^3 + x^2 - 2*x + 8); K
```

```
Number Field in a with defining polynomial x^3 + x^2 - 2*x + 8
```

```
K.factor(2)
```

```
(Fractional ideal (1/2*a^2 - 1/2*a + 1)) * \
(Fractional ideal (-a^2 + 2*a - 3)) * \
(Fractional ideal (-3/2*a^2 + 5/2*a - 4))
```

Thus $2\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$, with the $\mathfrak{p}_i$ distinct, and one sees directly from the above expressions that $\mathcal{O}_K/\mathfrak{p}_i \cong \mathbb{F}_2$ for each i. If $\mathcal{O}_K = \mathbb{Z}[a]$ for some $a \in \mathcal{O}_K$ with minimal polynomial f, then $\overline{f}(x) \in \mathbb{F}_2[x]$ must be a product of three *distinct* linear factors, which is impossible, since the only linear polynomials in $\mathbb{F}_2[x]$ are x and $x + 1$.

4.3.2 Remarks on Ideal Factorization in General

Recall (from Definition 2.3.24) that an *order* in $\mathcal{O}_K$ is a subring $\mathcal{O}$ of $\mathcal{O}_K$ that has finite index in $\mathcal{O}_K$. For example, if $\mathcal{O}_K = \mathbb{Z}[i]$, then $\mathcal{O} = \mathbb{Z} + 5\mathbb{Z}[i]$ is an order in $\mathcal{O}_K$, and as an abelian group $\mathcal{O}_K/\mathcal{O}$ is cyclic of order 5.

Most algebraic number theory books do not describe an algorithm for decomposing primes in the general case. Fortunately, Cohen's book [Coh93, Ch. 6] does describe how to solve the general problem, in more than one way. The algorithms are nontrivial, and occupy a substantial part of Chapter 6 of Cohen's book. Our goal for the rest of this section is to give a hint as to what goes into them.

The general solutions to prime ideal factorization are somewhat surprising, since the algorithms are much more sophisticated than the one suggested by Theorem 4.2.3. However, these complicated algorithms all run very quickly in practice, even without assuming the maximal order is already known. In fact, they avoid computing $\mathcal{O}_K$ altogether, and instead compute only an order $\mathcal{O}$ that is *p-maximal*, i.e., is such that $p \nmid [\mathcal{O}_K : \mathcal{O}]$.

For simplicity we consider the following slightly easier problem whose solution illustrates the key ideas needed in the general case.

Problem 4.3.3. Let $\mathcal{O}$ be any order in $\mathcal{O}_K$ and let p be a prime of $\mathbb{Z}$. Find the prime ideals of $\mathcal{O}$ that contain p.

Given a prime p that we wish to factor in $\mathcal{O}_K$, we first find a p-maximal order $\mathcal{O}$. We then use a solution to Problem 4.3.3 to find the prime ideals $\mathfrak{p}$ of $\mathcal{O}$ that contain p. Second, we find the exponents e such that $\mathfrak{p}^e$ exactly divides $p\mathcal{O}$. The resulting factorization in $\mathcal{O}$ completely determines the factorization of $p\mathcal{O}_K$.

A p-maximal order can be found reasonably quickly in practice using algorithms called "round 2" and "round 4". To compute $\mathcal{O}_K$, given an order $\mathbb{Z}[\alpha] \subset \mathcal{O}_K$, one takes a sum of p-maximal orders, one for every p such that p^2 divides $\mathrm{Disc}(\mathbb{Z}[\alpha])$. The time-consuming part of this computation is finding the primes p such that $p^2 \mid \mathrm{Disc}(\mathbb{Z}[\alpha])$, not finding the p-maximal orders. This example illustrates that a fast algorithm for factoring integers would not only break the RSA cryptosystems, but would massively speed up computation of the ring of integers of a number field.

Remark 4.3.4. The MathSciNet review of [BL94] by J. Buhler contains the following:

> A result of Chistov says that finding the ring of integers $\mathcal{O}_K$ in an algebraic number field K is equivalent, under certain polynomial time reductions, to the problem of finding the largest squarefree divisor of a positive integer. No feasible (i.e., polynomial time) algorithm is known for the latter problem, and it is possible that it is no easier than the more general problem of factoring integers.

Thus it appears that computing the ring $\mathcal{O}_K$ is quite hard.

4.3.3 Finding a p-Maximal Order

Before describing the general factorization algorithm, we sketch some of the theory behind the general algorithms for computing a p-maximal order $\mathcal{O}$ in $\mathcal{O}_K$. The main input is the following theorem:

Theorem 4.3.5 (Pohst-Zassenhaus). *Let $\mathcal{O}$ be an order in the ring of integers $\mathcal{O}_K$ of a number field, let $p \in \mathbb{Z}$ be a prime, and let*

$$I_p = \{x \in \mathcal{O} : x^m \in p\mathcal{O} \text{ for some } m \geq 1\} \subset \mathcal{O}$$

be the radical of $p\mathcal{O}$, which is an ideal of $\mathcal{O}$. Let

$$\mathcal{O}' = \{x \in K : xI_p \subset I_p\}.$$

Then $\mathcal{O}'$ is an order and either $\mathcal{O}' = \mathcal{O}$, in which case $\mathcal{O}$ is p-maximal, or $\mathcal{O} \subset \mathcal{O}'$ and p divides $[\mathcal{O}' : \mathcal{O}]$.

Proof. We prove here only that $[\mathcal{O}' : \mathcal{O}] \mid p^n$, where n is the degree of K. We have $p \in I_p$, so if $x \in \mathcal{O}'$, then $xp \in I_p \subset \mathcal{O}$, which implies that $x \in \frac{1}{p}\mathcal{O}$. Since $(\frac{1}{p}\mathcal{O})/\mathcal{O}$ is of order p^n, the claim follows.

To complete the proof, we would show that if $\mathcal{O}' = \mathcal{O}$, then $\mathcal{O}$ is already p-maximal. See [Coh93, §6.1.1] for the rest if this proof. □

After deciding on how to represent elements of K and orders and ideals in K, one can give an efficient algorithm to compute the $\mathcal{O}'$ of the theorem. The algorithm mainly involves linear algebra over finite fields. It is complicated to describe, but efficient in practice, and is conceptually simple—just compute $\mathcal{O}'$. The trick for reducing the computation of $\mathcal{O}'$ to linear algebra is the following lemma:

Lemma 4.3.6. *Define a homomorphism $\psi : \mathcal{O} \hookrightarrow \mathrm{End}(I_p/pI_p)$ given by sending $\alpha \in \mathcal{O}$ to left multiplication by the reduction of α modulo p. Then*

$$\mathcal{O}' = \frac{1}{p}\,\mathrm{Ker}(\psi).$$

Proof. If $x \in \mathcal{O}'$, then $xI_p \subset I_P$, so $\psi(x)$ is the 0 endomorphism. Conversely, if $\psi(x)$ acts as 0 on I_p/pI_p, then clearly $xI_p \subset I_p$. □

Note that to give an algorithm one must also figure out how to explicitly compute I_p/pI_p and the kernel of this map (see the next section for more details).

4.3.4 General Factorization Algorithm of Buchman-Lenstra

We finally give an algorithm to factor $p\mathcal{O}_K$ in general. This is a summary of the algorithm described in more detail in [Coh93, §6.2].

Algorithm 4.3.7 (Factoring a Finite Separable Algebra)**.** Let A be a finite separable algebra over $\mathbb{F}_p$. This algorithm either shows that A is a field or finds a nontrivial idempotent in A, i.e., an $\varepsilon \in A$ such that $\varepsilon^2 = \varepsilon$ with $\varepsilon \neq 0$ and $\varepsilon \neq 1$.

1. The dimension of the kernel V of the map $x \mapsto x^p - x$ is equal to k. This is because abstractly we have that $A \approx A_1 \times \cdots \times A_k$, with each A_i a finite field extension of $\mathbb{F}_p$.

2. If $k = 1$ we are done. Terminate.

3. Otherwise, choose $\alpha \in V$ with $\alpha \notin \mathbb{F}_p$. (Think of $\mathbb{F}_p$ as the diagonal embedding of $\mathbb{F}_p$ in $A_1 \times \cdots \times A_k$). Compute powers of α and find the minimal polynomial $m(X)$ of α.

4. Since $V \approx \mathbb{F}_p \times \cdots \times F_p$ (k factors), the polynomial $m(X)$ is a square-free product of linear factors, that has degree > 1 since $\alpha \notin \mathbb{F}_p$. Thus we can compute a splitting $m(X) = m_1(X) \cdot m_2(X)$, where both $m_i(X)$ have positive degree.

5. Use the Euclidean algorithm in $\mathbb{F}_p[X]$ to find $U_1(X)$ and $U_2(X)$ such that
$$U_1 m_1 + U_2 m_2 = 1.$$

6. Let $\varepsilon = (U_1 m_1)(\alpha)$. Then we have
$$U_1 m_1 U_1 m_1 + U_2 m_2 U_1 m_1 = U_1 m_1,$$
so since $(m_1 m_2)(\alpha) = m(\alpha) = 01$, we have $\varepsilon^2 = \varepsilon$. Also, since $\gcd(U_1, m_2) = \gcd(U_2, m_1) = 1$, we have $\varepsilon \neq 0$ and $\varepsilon \neq 1$.

Given Algorithm 4.3.7, we compute an idempotent $\varepsilon \in A$, and observe that
$$A \cong \mathrm{Ker}(1-\varepsilon) \oplus \mathrm{Ker}(\varepsilon).$$
Since $(1-\varepsilon) + \varepsilon = 1$, we see that $(1-\varepsilon)v + \varepsilon v = v$, so that the sum of the two kernels equals A. Also, if v is in the intersection of the two kernels, then $\varepsilon(v) = 0$ and $(1-\varepsilon)(v) = 0$, so $0 = (1-\varepsilon)(v) = v - \varepsilon(v) = v$, so the sum is direct.

Remark 4.3.8. The beginning of [Coh93, §6.2.4] suggests that one can just randomly find an $\alpha \in A$ such that $A \cong \mathbb{F}_p[x]/(m(x))$ where m is the minimal polynomial of α. This is usually the case, but is *wrong in general*, since there need *not* be an $\alpha \in A$ such that $A \cong \mathbb{F}_p[\alpha]$. For example, let $p = 2$ and K be as in Example 4.3.2. Then $A \cong \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$, which as a ring is not generated by a single element, since there are only 2 distinct linear polynomials over $\mathbb{F}_2[x]$.

Algorithm 4.3.9 (Factoring a General Prime Ideal)**.** Let $K = \mathbb{Q}(a)$ be a number field given by an algebraic integer a as a root of its minimal monic polynomial f of degree n. We assume that an order $\mathcal{O}$ has been given by a basis $w_1, \ldots, w_n$ and that $\mathcal{O}$ that contains $\mathbb{Z}[a]$. For any prime $p \in \mathbb{Z}$, the following algorithm computes the set of maximal ideals of $\mathcal{O}$ that contain p.

1. [Check if easy] If $p \nmid \mathrm{disc}(\mathbb{Z}[a])/\mathrm{disc}(\mathcal{O})$ (so $p \nmid [\mathcal{O} : \mathbb{Z}[a]]$), then using Theorem 4.2.3 we factor $p\mathcal{O}$.

2. [Compute radical] Let I be the *radical* of $p\mathcal{O}$, which is the ideal of elements $x \in \mathcal{O}$ such that $x^m \in p\mathcal{O}$ for some positive integer m. Note that $p\mathcal{O} \subset I$, i.e., $I \mid p\mathcal{O}$; also I is the product of the primes that divide p, without multiplicity. Using linear algebra over the finite field $\mathbb{F}_p$, we compute a basis for $I/p\mathcal{O}$ by computing the abelian subgroup of $\mathcal{O}/p\mathcal{O}$ of all nilpotent elements. This computes I, since $p\mathcal{O} \subset I$.

3. [Compute quotient by radical] Compute an $\mathbb{F}_p$ basis for
$$A = \mathcal{O}/I = (\mathcal{O}/p\mathcal{O})/(I/p\mathcal{O}).$$
The second equality comes from the fact that $p\mathcal{O} \subset I$. Note that $\mathcal{O}/p\mathcal{O}$ is obtained by simply reducing the basis $w_1, \ldots, w_n$ modulo p. Thus this step entirely involves linear algebra modulo p.

4. [Decompose quotient] The ring A is isomorphic to the quotient of $\mathcal{O}$ by a radical ideal, so it decomposes as a product $A \cong A_1 \times \cdots \times A_k$ of finite fields. We find such a decomposition explicitly using Algorithm 4.3.7.

5. [Compute the maximal ideals over p] Each maximal ideal $\mathfrak{p}_i$ lying over p is the kernel of one of the compositions

$$\mathcal{O} \to A \approx A_1 \times \cdots \times A_k \to A_i.$$

Algorithm 4.3.9 finds all primes of $\mathcal{O}$ that contain the radical I of $p\mathcal{O}$. Every such prime clearly contains p, so to see that the algorithm is correct, we prove that the primes $\mathfrak{p}$ of $\mathcal{O}$ that contain p also contain I. If $\mathfrak{p}$ is a prime of $\mathcal{O}$ that contains p, then $p\mathcal{O} \subset \mathfrak{p}$. If $x \in I$ then $x^m \in p\mathcal{O}$ for some m, so $x^m \in \mathfrak{p}$ which implies that $x \in \mathfrak{p}$ by the primality of $\mathfrak{p}$. Thus $\mathfrak{p}$ contains I, as required. Note that we do not find the powers of primes that divide p in Algorithm 4.3.9; that's left to another algorithm that we will not discuss in this book.

Algorithm 4.3.9 was invented by J. Buchmann and H. W. Lenstra, though their paper seems to have never been published; however, the algorithm is described in detail in [Coh93, §6.2.5]. Incidentally, this chapter is based on Chapters 4 and 6 of [Coh93], which is highly recommended, and goes into much more detail about these algorithms.

Chapter 5

The Chinese Remainder Theorem

In this chapter, we prove the Chinese Remainder Theorem (CRT) for arbitrary commutative rings, then apply CRT to prove that every ideal in a Dedekind domain R is generated by at most two elements. We also prove that $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is (noncanonically) isomorphic to $R/\mathfrak{p}$ as an R-module, for any nonzero prime ideal $\mathfrak{p}$ of R. The tools we develop in this chapter will be used frequently to prove other results later.

5.1 The Chinese Remainder Theorem

5.1.1 CRT in the Integers

The classical CRT asserts that if $n_1, \ldots, n_r$ are integers that are coprime in pairs, and $a_1, \ldots, a_r$ are integers, then there exists an integer a such that $a \equiv a_i \pmod{n_i}$ for each $i = 1, \ldots, r$. Here "coprime in pairs" means that $\gcd(n_i, n_j) = 1$ whenever $i \neq j$; it does *not* mean that $\gcd(n_1, \ldots, n_r) = 1$, though it implies this. In terms of rings, CRT asserts that the natural map

$$\mathbb{Z}/(n_1 \cdots n_r)\mathbb{Z} \to (\mathbb{Z}/n_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/n_r\mathbb{Z}) \tag{5.1.1}$$

that sends $a \in \mathbb{Z}$ to its reduction modulo each n_i, is an isomorphism.

This map is *never* an isomorphism if the n_i are not coprime. Indeed, the cardinality of the image of the left hand side of (5.1.1) is $\mathrm{lcm}(n_1, \ldots, n_r)$, since it is the image of a cyclic group and $\mathrm{lcm}(n_1, \ldots, n_r)$ is the largest order of an element of the right hand side, whereas the cardinality of the right hand side is $n_1 \cdots n_r$.

The isomorphism (5.1.1) can alternatively be viewed as asserting that any system of linear congruences

$$x \equiv a_1 \pmod{n_1}, \quad x \equiv a_2 \pmod{n_2}, \quad \ldots, \quad x \equiv a_r \pmod{n_r}$$

with pairwise coprime moduli has a unique solution modulo $n_1 \cdots n_r$.

Before proving the CRT in more generality, we prove (5.1.1). There is a natural map

$$\phi : \mathbb{Z} \to (\mathbb{Z}/n_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/n_r\mathbb{Z})$$

given by projection onto each factor. Its kernel is

$$n_1\mathbb{Z} \cap \cdots \cap n_r\mathbb{Z}.$$

If n and m are integers, then $n\mathbb{Z} \cap m\mathbb{Z}$ is the set of multiples of both n and m, so $n\mathbb{Z} \cap m\mathbb{Z} = \mathrm{lcm}(n, m)\mathbb{Z}$. Since the n_i are coprime,

$$n_1\mathbb{Z} \cap \cdots \cap n_r\mathbb{Z} = n_1 \cdots n_r\mathbb{Z}.$$

Thus we have proved there is an inclusion

$$i : \mathbb{Z}/(n_1 \cdots n_r)\mathbb{Z} \hookrightarrow (\mathbb{Z}/n_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/n_r\mathbb{Z}). \tag{5.1.2}$$

This is half of the CRT; the other half is to prove that this map is surjective. In this case, it is clear that i is also surjective, because i is an injective map between finite sets of the same cardinality. We will, however, give a proof of surjectivity that doesn't use finiteness of the above two sets.

To prove surjectivity of i, note that since the n_i are coprime in pairs,

$$\gcd(n_1, n_2 \cdots n_r) = 1,$$

so there exists integers x, y such that

$$xn_1 + yn_2 \cdots n_r = 1.$$

To complete the proof, observe that $yn_2 \cdots n_r = 1 - xn_1$ is congruent to 1 modulo n_1 and 0 modulo $n_2 \cdots n_r$. Thus $(1, 0, \ldots, 0) = i(yn_2 \cdots n_r)$ is in the image of i. By a similar argument, we see that $(0, 1, \ldots, 0)$ and the other similar elements are all in the image of i, so i is surjective, which proves CRT.

5.1.2 CRT in General

Recall that *all rings in this book are commutative with unity.* Let R be such a ring.

Definition 5.1.1 (Coprime)**.** Ideals I and J of R are *coprime* if $I + J = (1)$.

For example, if I and J are nonzero ideals in a Dedekind domain, then they are coprime precisely when the prime ideals that appear in their two (unique) factorizations are disjoint.

Lemma 5.1.2. *If I and J are coprime ideals in a ring R, then $I \cap J = IJ$.*

Proof. Choose $x \in I$ and $y \in J$ such that $x + y = 1$. If $c \in I \cap J$ then

$$c = c \cdot 1 = c \cdot (x + y) = cx + cy \in IJ + IJ = IJ,$$

so $I \cap J \subset IJ$. The other inclusion is obvious by the definition of an ideal. □

Lemma 5.1.3. *Suppose $I_1, \dots, I_s$ are pairwise coprime ideals. Then I_1 is coprime to the product $I_2 \cdots I_s$.*

Proof. In the special case of a Dedekind domain, we could easily prove this lemma using unique factorization of ideals as products of primes (Theorem 3.2.6); instead, we give a direct general argument.

It suffices to prove the lemma in the case $s = 3$, since the general case then follows from induction. By assumption, there are $x_1 \in I_1, y_2 \in I_2$ and $a_1 \in I_1, b_3 \in I_3$ such

$$x_1 + y_2 = 1 \qquad \text{and} \qquad a_1 + b_3 = 1.$$

Multiplying these two relations yields

$$x_1 a_1 + x_1 b_3 + y_2 a_1 + y_2 b_3 = 1 \cdot 1 = 1.$$

The first three terms are in I_1 and the last term is in $I_2 I_3 = I_2 \cap I_3$ (by Lemma 5.1.2), so I_1 is coprime to $I_2 I_3$. □

Next we prove the general Chinese Remainder Theorem. We will apply this result with $R = \mathcal{O}_K$ in the rest of this chapter.

Theorem 5.1.4 (Chinese Remainder Theorem)**.** *Suppose $I_1, \dots, I_r$ are nonzero ideals of a ring R such I_m and I_n are coprime for any $m \neq n$. Then the natural homomorphism $R \to \bigoplus_{n=1}^r R/I_n$ induces an isomorphism*

$$\psi : R/\prod_{n=1}^{r} I_n \to \bigoplus_{n=1}^{r} R/I_n.$$

Thus given any $a_n \in R$, for $n = 1, \dots, r$, there exists some $a \in R$ such that $a \equiv a_n \pmod{I_n}$ for $n = 1, \dots, r$; moreover, a is unique modulo $\prod_{n=1}^r I_n$.

Proof. Let $\varphi : R \to \bigoplus_{n=1}^r R/I_n$ be the natural map induced by reduction modulo the I_n. An inductive application of Lemma 5.1.2 implies that the kernel $\cap_{n=1}^r I_n$ of φ is equal to $\prod_{n=1}^r I_n$, so the map ψ of the theorem is injective.

Each projection $R \to R/I_n$ is surjective, so to prove that ψ is surjective, it suffices to show that $(1, 0, \dots, 0)$ is in the image of φ, and similarly for the other factors. By Lemma 5.1.3, $J = \prod_{n=2}^r I_n$ is coprime to I_1, so there exists $x \in I_1$ and $y \in J$ such that $x + y = 1$. Then $y = 1 - x$ maps to 1 in R/I_1 and to 0 in R/J, hence to 0 in R/I_n for each $n \geq 2$, since $J \subset I_n$. □

5.2 Structural Applications of the CRT

Let $\mathcal{O}_K$ be the ring of integers of some number field K, and suppose I is a nonzero ideal of $\mathcal{O}_K$. As an abelian group $\mathcal{O}_K$ is free of rank $[K : \mathbb{Q}]$, and I is of finite index in $\mathcal{O}_K$, so I is generated by $[K : \mathbb{Q}]$ generators as an abelian group, so as an R-ideal I requires at most $[K : \mathbb{Q}]$ generators. The main result of this section asserts something better, namely that I can be generated *as an ideal* by at most two elements. Moreover, our result is more general, since it applies to an arbitrary Dedekind domain R. Thus, for the rest of this section, R is any Dedekind domain, e.g., the ring of integers of either a number field or function field. We use CRT to prove that every ideal of R can be generated by two elements.

Remark 5.2.1. Caution – If we replace R by an order in a Dedekind domain, i.e., by a subring of finite index,then there may be ideals that require far more than 2 generators.

Suppose that I is a nonzero integral ideal of R. If $a \in I$, then $(a) \subset I$, so I divides (a) and the quotient $(a)I^{-1}$ is an integral ideal. The following lemma asserts that (a) can be chosen so the quotient $(a)I^{-1}$ is coprime to any given ideal.

Lemma 5.2.2. *If I and J are nonzero integral ideals in R, then there exists an $a \in I$ such that the integral ideal $(a)I^{-1}$ is coprime to J.*

Before we give the proof in general, note that the lemma is trivial when I is principal, since if $I = (b)$, just take $a = b$, and then $(a)I^{-1} = (a)(a^{-1}) = (1)$ is coprime to every ideal.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the prime divisors of J. For each n, let v_n be the largest power of $\mathfrak{p}_n$ that divides I. Since $\mathfrak{p}_n^{v_n} \neq \mathfrak{p}_n^{v_n+1}$, we can choose an element $a_n \in \mathfrak{p}_n^{v_n}$ that is not in $\mathfrak{p}_n^{v_n+1}$. By Theorem 5.1.4 applied to the $r+1$ coprime integral ideals

$$\mathfrak{p}_1^{v_1+1}, \ldots, \mathfrak{p}_r^{v_r+1}, I \cdot \left(\prod \mathfrak{p}_n^{v_n}\right)^{-1},$$

there exists $a \in R$ such that

$$a \equiv a_n \pmod{\mathfrak{p}_n^{v_n+1}}$$

for all $n = 1, \ldots, r$ and also

$$a \equiv 0 \quad \left(\text{mod } I \cdot \left(\prod \mathfrak{p}_n^{v_n}\right)^{-1}\right).$$

To complete the proof we show that $(a)I^{-1}$ is not divisible by any $\mathfrak{p}_n$, or equivalently, that each $\mathfrak{p}_n^{v_n}$ exactly divides (a). First we show that $\mathfrak{p}_n^{v_n}$ divides (a). Because $a \equiv a_n \pmod{\mathfrak{p}_n^{v_n+1}}$, there exists $b \in \mathfrak{p}_n^{v_n+1}$ such that $a = a_n + b$. Since $a_n \in \mathfrak{p}_n^{v_n}$ and $b \in \mathfrak{p}_n^{v_n+1} \subset \mathfrak{p}_n^{v_n}$, it follows that $a \in \mathfrak{p}_n^{v_n}$, so $\mathfrak{p}_n^{v_n}$ divides (a). Now assume for the sake of contradiction that $\mathfrak{p}_n^{v_n+1}$ divides (a); then $a_n = a - b \in \mathfrak{p}_n^{v_n+1}$, which contradicts that we chose $a_n \not\in \mathfrak{p}_n^{v_n+1}$. Thus $\mathfrak{p}_n^{v_n+1}$ does not divide (a), as claimed. □

Proposition 5.2.3. *Suppose I is a fractional ideal in a Dedekind domain R. Then there exist $a, b \in K$ such that $I = (a, b) = \{\alpha a + \beta b : \alpha, \beta \in R\}$.*

Proof. If $I = (0)$, then I is generated by 1 element and we are done. If I is not an integral ideal, then there is an $x \in K$ such that xI is an integral ideal, and the number of generators of xI is the same as the number of generators of I, so we may assume that I is an integral ideal.

Let a be *any* nonzero element of the integral ideal I. We will show that there is some $b \in I$ such that $I = (a, b)$. Let $J = (a)$. By Lemma 5.2.2, there exists $b \in I$ such that $(b)I^{-1}$ is coprime to (a). Since $a, b \in I$, we have $I \mid (a)$ and $I \mid (b)$, so $I \mid (a, b)$. Suppose $\mathfrak{p}^n \mid (a, b)$ with $\mathfrak{p}$ prime and $n \geq 1$. Then $\mathfrak{p}^n \mid (a)$ and $\mathfrak{p}^n \mid (b)$, so $\mathfrak{p} \nmid (b)I^{-1}$, since $(b)I^{-1}$ is coprime to (a). We have $\mathfrak{p}^n \mid (b) = I \cdot (b)I^{-1}$ and $\mathfrak{p} \nmid (b)I^{-1}$, so $\mathfrak{p}^n \mid I$. Thus by unique factorization of ideals in R we have that $(a, b) \mid I$. Since $I \mid (a, b)$ we conclude that $I = (a, b)$, as claimed. □

We can also use Theorem 5.1.4 to determine the R-module structure of $\mathfrak{p}^n/\mathfrak{p}^{n+1}$.

Proposition 5.2.4. *Let $\mathfrak{p}$ be a nonzero prime ideal of R, and let $n \geq 0$ be an integer. Then $\mathfrak{p}^n/\mathfrak{p}^{n+1} \cong R/\mathfrak{p}$ as R-modules.*

Proof[1]. Since $\mathfrak{p}^n \neq \mathfrak{p}^{n+1}$, by unique factorization, there is an element $b \in \mathfrak{p}^n$ such that $b \notin \mathfrak{p}^{n+1}$. Let $\varphi : R \to \mathfrak{p}^n/\mathfrak{p}^{n+1}$ be the R-module morphism defined by $\varphi(a) = ab$. The kernel of φ is $\mathfrak{p}$ since clearly $\varphi(\mathfrak{p}) = 0$ and if $\varphi(a) = 0$ then $ab \in \mathfrak{p}^{n+1}$, so $\mathfrak{p}^{n+1} \mid (a)(b)$, so $\mathfrak{p} \mid (a)$, since $\mathfrak{p}^{n+1}$ does not divide (b). Thus φ induces an injective R-module homomorphism $R/\mathfrak{p} \hookrightarrow \mathfrak{p}^n/\mathfrak{p}^{n+1}$.

It remains to show that φ is surjective, and this is where we will use Theorem 5.1.4. Suppose $c \in \mathfrak{p}^n$. By Theorem 5.1.4 there exists $d \in R$ such that

$$d \equiv c \pmod{\mathfrak{p}^{n+1}} \qquad \text{and} \qquad d \equiv 0 \pmod{(b)/\mathfrak{p}^n}.$$

We have $\mathfrak{p}^n \mid (d)$ since $d \in \mathfrak{p}^n$ and $(b)/\mathfrak{p}^n \mid (d)$ by the second displayed condition, so since $\mathfrak{p} \nmid (b)/\mathfrak{p}^n$, we have $(b) = \mathfrak{p}^n \cdot (b)/\mathfrak{p}^n \mid (d)$, hence $d/b \in R$. Finally

$$\varphi\left(\frac{d}{b}\right) \equiv \frac{d}{b} \cdot b \pmod{\mathfrak{p}^{n+1}} \equiv d \pmod{\mathfrak{p}^{n+1}} \equiv c \pmod{\mathfrak{p}^{n+1}},$$

so φ is surjective. □

Exercise 5.2.5. (See [Mar77, Thm. 22(a)]) Let R be a Dedekind domain and $\mathfrak{p}$ a nonzero prime ideal in R. Show that $\#(R/\mathfrak{p}^m) = \#(R/\mathfrak{p})^m$.

Note: $\#(R/\mathfrak{p})$ is not finite in general! For example, The ring of formal power series $k[[t]]$ for some field k is a Dedekind domain and the residue field at the prime (t) is k.

[*Hint*: Consider the exact sequence

$$0 \to \mathfrak{p}/\mathfrak{p}^m \to R/\mathfrak{p}^m \to R/\mathfrak{p}^{m-1} \to 0$$

and the chain

$$\mathfrak{p}^m \subseteq \mathfrak{p}^{m-1} \subseteq \cdots \subseteq \mathfrak{p}^2 \subseteq \mathfrak{p}.$$

]

Remark 5.2.6. There is one special case of the previous exercise that you probably have seen before: the size of $\mathbb{Z}/4\mathbb{Z}$ is the same as $(\mathbb{Z}/2\mathbb{Z})^2$. In fact you might have seen a proof of the fact that $\mathbb{Z}/n^m\mathbb{Z}$ has the same cardinality as $(\mathbb{Z}/n\mathbb{Z})^m$ in a standard group theory or abstract algebra course.

5.3 Computing Using the CRT

In order to explicitly compute an a as given by Theorem 5.1.4, usually one first precomputes elements $v_1, \dots, v_r \in R$ such that $v_1 \mapsto (1, 0, \dots, 0)$, $v_2 \mapsto (0, 1, \dots, 0)$, etc. Then given any $a_n \in R$, for $n = 1, \dots, r$, we obtain an $a \in R$ with $a_n \equiv a \pmod{I_n}$ by taking

$$a = a_1 v_1 + \cdots + a_r v_r.$$

How to compute the v_i depends on the ring R. It reduces to the following problem: Given coprimes ideals $I, J \subset R$, find $x \in I$ and $y \in J$ such that $x + y = 1$. If R is torsion free and of finite rank as a $\mathbb{Z}$-module, so $R \approx \mathbb{Z}^n$, then I, J can be represented by giving a basis in terms of a basis for R, and finding x, y such that $x+y = 1$ can then be reduced to a problem in linear algebra over $\mathbb{Z}$. More precisely, let A be the matrix whose columns are the concatenation of a basis for I with a basis for J. Suppose $v \in \mathbb{Z}^n$ corresponds to $1 \in \mathbb{Z}^n$. Then finding x, y such that $x + y = 1$ is equivalent to finding a solution $z \in \mathbb{Z}^n$ to the matrix equation $Az = v$. This latter linear algebra problem can be solved using Hermite normal form (see [Coh93, §4.7.1]), which is a generalization over $\mathbb{Z}$ of reduced row echelon form.

5.3.1 Sage

[[TODO]]

5.3.2 Magma

The Magma command `ChineseRemainderTheorem` implements the algorithm suggested by Theorem 5.1.4. In the following example, we compute a prime over (3) and a prime over (5) of the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$, and find an element of $\mathcal{O}_K$ that is congruent to $\sqrt[3]{2}$ modulo one prime and 1 modulo the other.

```
> R<x> := PolynomialRing(RationalField());
> K<a> := NumberField(x^3-2);
> OK := RingOfIntegers(K);
> I := Factorization(3*OK)[1][1];
> J := Factorization(5*OK)[1][1];
> I;
Prime Ideal of OK
Two element generators:
    [3, 0, 0]
    [4, 1, 0]
```

```
> J;
Prime Ideal of OK
Two element generators:
    [5, 0, 0]
    [7, 1, 0]
> b := ChineseRemainderTheorem(I, J, OK!a, OK!1);
> K!b;
-4
> b - a in I;
true
> b - 1 in J;
true
```

5.3.3 PARI

There is also a CRT algorithm for number fields in PARI, but it is more cumbersome to use. First we defined $\mathbb{Q}(\sqrt[3]{2})$ and factor the ideals (3) and (5).

```
? f = x^3 - 2;
? k = nfinit(f);
? i = idealfactor(k,3);
? j = idealfactor(k,5);
```

Next we form matrix whose rows correspond to a product of two primes, one dividing 3 and one dividing 5:

```
? m = matrix(2,2);
? m[1,] = i[1,];
? m[1,2] = 1;
? m[2,] = j[1,];
```

Note that we set `m[1,2] = 1`, so the exponent is 1 instead of 3. We apply the CRT to obtain a lift in terms of the basis for $\mathcal{O}_K$.

```
? ?idealchinese
idealchinese(nf,x,y): x being a prime ideal factorization and y
a vector of elements, gives an element b such that
v_p(b-y_p)>=v_p(x) for all prime ideals p dividing x,
and v_p(b)>=0 for all other p.
? idealchinese(k, m, [x,1])
[0, 0, -1]~
? nfbasis(f)
[1, x, x^2]
```

Thus PARI finds the lift $-(\sqrt[3]{2})^2$, and we finish by verifying that this lift is correct. The `idealval` function returns the number of times a prime appears in the factorization of an ideal. We will use it to check that $-(\sqrt[3]{2})^2 - \sqrt[3]{2}$ is contained in the prime above 3 and $-(\sqrt[3]{2})^2 - 1$ is contained in the prime above 5.

```
? idealval(k,-x^2 - x,i[1,1])
1
? idealval(k,-x^2 - 1,j[1,1])
1
```

Chapter 6

Discriminants and Norms

In this chapter we give a geometric interpretation of the discriminant of an order in a number field. We also define norms of ideals and prove that the norm function is multiplicative. Discriminants of orders and norms of ideals will play a crucial role in our proof of finiteness of the class group in the next chapter.

6.1 Viewing $\mathcal{O}_K$ as a Lattice in a Real Vector Space

Let K be a number field of degree n. By the primitive element theorem, $K = \mathbb{Q}(\alpha)$ for some α, so we can write $K \cong \mathbb{Q}[x]/(f)$, where $f \in \mathbb{Q}[x]$ is the minimal polynomial of α. Because $\mathbb{C}$ is algebraically closed and f is irreducible, it has exactly $n = [K : \mathbb{Q}]$ complex roots. Each of these roots $z \in \mathbb{C}$ induces a homomorphism $\mathbb{Q}[x] \to \mathbb{C}$ given by $x \mapsto z$, whose kernel is the ideal (f). Thus we obtain n embeddings of $K \cong \mathbb{Q}[x]/(f)$ into $\mathbb{C}$:

$$\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{C}.$$

Example 6.1.1. We compute the embeddings listed above for $K = \mathbb{Q}(\sqrt[3]{2})$.

```
K = QQ[2^(1/3)]; K
```

```
Number Field in a with defining polynomial x^3 - 2
```

```
K.complex_embeddings()
```

```
[Ring morphism: ...
  Defn: a |--> -0.629960524947 - 1.09112363597*I,
 Ring morphism: ...
  Defn: a |--> -0.629960524947 + 1.09112363597*I,
 Ring morphism: ...
  Defn: a |--> 1.25992104989]
```

Let $\sigma : K \hookrightarrow \mathbb{C}^n$ be the map $a \mapsto (\sigma_1(a), \ldots, \sigma_n(a))$, and let $V = \mathbb{R}\sigma(K)$ be the $\mathbb{R}$-span of the image $\sigma(K)$ of K inside $\mathbb{C}^n$.

Lemma 6.1.2. *Suppose $L \subset \mathbb{R}^n$ is a subgroup of the vector space $\mathbb{R}^n$. Then the induced topology on L is discrete if and only if for every $H > 0$ the set*

$$X_H = \{v \in L : \max\{|v_1|, \dots, |v_n|\} \leq H\}$$

is finite.

Proof. If L is not discrete, then there is a point $x \in L$ such that for every $\varepsilon > 0$ there is $y \in L$ such that $0 < |x - y| < \varepsilon$. By choosing smaller and smaller ε, we find infinitely many elements $x - y \in L$ all of whose coordinates are smaller than 1. The set X_1 is thus not finite. Thus if the sets X_H are all finite, L must be discrete.

Next assume that L is discrete and let $H > 0$ be any positive number. Then for every $x \in X_H$ there is an open ball B_x that contains x but no other element of L. Since X_H is closed and bounded, the Heine-Borel theorem implies that X_H is compact, so the open covering $\cup B_x$ of X_H has a finite subcover, which implies that X_H is finite, as claimed. □

Lemma 6.1.3. *If L if a free abelian group that is discrete in a finite-dimensional real vector space V and $\mathbb{R}L = V$, then the rank of L equals the dimension of V.*

Proof. Let $x_1, \dots, x_m \in L$ be an $\mathbb{R}$-vector space basis for $\mathbb{R}L$, and consider the $\mathbb{Z}$-submodule $M = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_m$ of L. If the quotient L/M is infinite, then there are infinitely many distinct elements of L that all lie in a fundamental domain for M, so Lemma 6.1.2 implies that L is not discrete. This is a contradiction, so L/M is finite, and the rank of L is $m = \dim(\mathbb{R}L)$, as claimed. □

Proposition 6.1.4. *The $\mathbb{R}$-vector space $V = \mathbb{R}\sigma(K)$ spanned by the image $\sigma(K)$ of K has dimension n.*

Proof. We prove this by showing that the image $\sigma(\mathcal{O}_K)$ is discrete. If $\sigma(\mathcal{O}_K)$ were not discrete it would contain elements all of whose coordinates are simultaneously arbitrarily small. The norm of an element $a \in \mathcal{O}_K$ is the product of the entries of $\sigma(a)$, so the norms of nonzero elements of $\mathcal{O}_K$ would go to 0. This is a contradiction, since the norms of nonzero elements of $\mathcal{O}_K$ are nonzero integers.

Since $\sigma(\mathcal{O}_K)$ is discrete in $\mathbb{C}^n$, Lemma 6.1.3 implies that $\dim(V)$ equals the rank of $\sigma(\mathcal{O}_K)$. Since σ is injective, $\dim(V)$ is the rank of $\mathcal{O}_K$, which equals n by Proposition 2.4.7. □

6.1.1 A Determinant

Suppose $w_1, \dots, w_n$ is a basis for $\mathcal{O}_K$, and let A be the matrix whose ith row is $\sigma(w_i)$. Consider the determinant $\det(A)$.

Example 6.1.5. The ring $\mathcal{O}_K = \mathbb{Z}[i]$ of integers of $K = \mathbb{Q}(i)$ has $\mathbb{Z}$-basis $w_1 = 1$, $w_2 = i$. The map $\sigma : K \to \mathbb{C}^2$ is given by

$$\sigma(a + bi) = (a + bi, a - bi) \in \mathbb{C}^2.$$

The image $\sigma(\mathcal{O}_K)$ is spanned by $(1,1)$ and $(i,-i)$. The determinant is

$$\left|\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\right| = -2i.$$

Let $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$ be the ring of integers of $K = \mathbb{Q}(\sqrt{2})$. The map σ is

$$\sigma(a + b\sqrt{2}) = (a + b\sqrt{2}, a - b\sqrt{2}) \in \mathbb{R}^2,$$

and

$$A = \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix},$$

which has determinant $-2\sqrt{2}$.

As the above example illustrates, the determinant $\det(A)$ most certainly need not be an integer. However, as we will see, it's square is an integer that does not depend on our choice of basis for $\mathcal{O}_K$.

6.2 Discriminants

Suppose $w_1, \ldots, w_n$ is a basis for $\mathcal{O}_K$ as a $\mathbb{Z}$-module, which we view as a $\mathbb{Q}$-vector space. Let $\sigma : K \hookrightarrow \mathbb{C}^n$ be the embedding $\sigma(a) = (\sigma_1(a), \ldots, \sigma_n(a))$, where $\sigma_1, \ldots, \sigma_n$ are the distinct embeddings of K into $\mathbb{C}$. Let A be the matrix whose rows are $\sigma(w_1), \ldots, \sigma(w_n)$.

Changing our choice of basis for $\mathcal{O}_K$ is the same as left multiplying A by an integer matrix U of determinant ± 1, which changes $\det(A)$ by ± 1. This leads us to consider $\det(A)^2$ instead, which does not depend on the choice of basis; moreover, as we will see, $\det(A)^2$ is an integer. Note that

$$\begin{aligned}\det(A)^2 &= \det(AA) = \det(A)\det(A) = \det(A)\det(A^t) = \det(AA^t) \\ &= \det\left(\sum_{k=1,\ldots,n} \sigma_k(w_i)\sigma_k(w_j)\right) = \det\left(\sum_{k=1,\ldots,n} \sigma_k(w_i w_j)\right) \\ &= \det(\mathrm{Tr}(w_i w_j)_{1\le i,j\le n}),\end{aligned}$$

so $\det(A)^2$ can be defined purely in terms of the trace without mentioning the embeddings σ_i. Moreover, if we change basis hence multiplying A by some U with determinant ± 1, then $\det(UA)^2 = \det(U)^2 \det(A)^2 = \det(A)^2$. Because $\det(A)$ is an algebraic integer and $\mathrm{Tr}(w_i w_j) \in \mathbb{Q}$, it follows that $\det(A)^2$ is an algebraic integer in $\mathbb{Q}$. Thus $\det(A)^2 \in \mathbb{Z}$ is well defined as a quantity associated to $\mathcal{O}_K$.

If we view K as a $\mathbb{Q}$-vector space, then $(x,y) \mapsto \mathrm{Tr}(xy)$ defines a bilinear pairing $K \times K \to \mathbb{Q}$ on K, which we call the *trace pairing*. The following lemma asserts that this pairing is nondegenerate, so $\det(\mathrm{Tr}(w_i w_j)) \neq 0$ hence $\det(A) \neq 0$.

Lemma 6.2.1. *The trace pairing is nondegenerate.*

Proof. If the trace pairing is degenerate, then there exists $0 \neq a \in K$ such that for every $b \in K$ we have $\mathrm{Tr}(ab) = 0$. In particularly, taking $b = a^{-1}$ we see that $0 = \mathrm{Tr}(aa^{-1}) = \mathrm{Tr}(1) = [K : \mathbb{Q}] > 0$, which is absurd. □

Definition 6.2.2 (Discriminant)**.** Suppose $a_1, \ldots, a_n$ is any $\mathbb{Q}$-basis of K. The *discriminant* of $a_1, \ldots, a_n$ is

$$\mathrm{Disc}(a_1, \ldots, a_n) = \det(\mathrm{Tr}(a_i a_j)_{1 \leq i,j \leq n}) \in \mathbb{Q}.$$

The *discriminant* $\mathrm{Disc}(\mathcal{O})$ of an order $\mathcal{O}$ in $\mathcal{O}_K$ is the discriminant of any $\mathbb{Z}$-basis for $\mathcal{O}$. The *discriminant* $d_K = \mathrm{Disc}(K)$ of the number field K is the discriminant of $\mathcal{O}_K$. Note that these discriminants are all nonzero by Lemma 6.2.1.

Remark 6.2.3. It is also standard to define the discriminant of a monic polynomial to be the product of the differences of the roots. If $\alpha \in \mathcal{O}_K$ with $\mathbb{Z}[\alpha]$ of finite index in $\mathcal{O}_K$, and f is the minimal polynomial of α, then $\mathrm{Disc}(f) = \mathrm{Disc}(\mathbb{Z}[\alpha])$. To see this, note that if we choose the basis $1, \alpha, \ldots, \alpha^{n-1}$ for $\mathbb{Z}[\alpha]$, then both discriminants are the square of the same Vandermonde determinant.

Remark 6.2.4. If S/R is an extension of Dedekind domains, with S a free R module of finite rank, then the above definition of a *relative* discriminant of S/R does not make sense in general. The problem is that R may have more units than $\{\pm 1\}$, in which case $\det(A^2)$ is not well defined. To generalize the notion of discriminant to arbitrary finite extensions of Dedekind domains, one must instead introduce a discriminant *ideal*.

Example 6.2.5. In Sage, we compute the discriminant of a number field or order using the discriminant command:

```
K.<a> = NumberField(x^2 - 5)
K.discriminant()
```

```
5
```

This also works for orders (notice the square factor below, which will be explained by Proposition 6.2.6):

```
R = K.order([7*a]); R
```

```
Order in Number Field in a with defining polynomial x^2 - 5
```

```
factor(R.discriminant())
```

```
2^2 * 5 * 7^2
```

Warning: In Magma $\mathrm{Disc}(K)$ is defined to be the discriminant of the polynomial you happened to use to define K.

```
> K := NumberField(x^2-5);
> Discriminant(K);
20
```

This is an intentional choice done for efficiency reasons, since computing the maximal order can take a long time. Nonetheless, it conflicts with standard mathematical usage, so beware.

The following proposition asserts that the discriminant of an order $\mathcal{O}$ in $\mathcal{O}_K$ is bigger than $\mathrm{disc}(\mathcal{O}_K)$ by a factor of the square of the index.

Proposition 6.2.6. *Suppose $\mathcal{O}$ is an order in $\mathcal{O}_K$. Then*

$$\mathrm{Disc}(\mathcal{O}) = \mathrm{Disc}(\mathcal{O}_K) \cdot [\mathcal{O}_K : \mathcal{O}]^2.$$

Proof. Let A be a matrix whose rows are the images via σ of a basis for $\mathcal{O}_K$, and let B be a matrix whose rows are the images via σ of a basis for $\mathcal{O}$. Since $\mathcal{O} \subset \mathcal{O}_K$ has finite index, there is an integer matrix C such that $CA = B$, and $|\det(C)| = [\mathcal{O}_K : \mathcal{O}]$. Then

$$\mathrm{Disc}(\mathcal{O}) = \det(B)^2 = \det(CA)^2 = \det(C)^2 \det(A)^2 = [\mathcal{O}_K : \mathcal{O}]^2 \cdot \mathrm{Disc}(\mathcal{O}_K).$$

□

Example 6.2.7. Let K be a number field and consider the quantity

$$D(K) = \gcd\{\mathrm{Disc}(\alpha) : \alpha \in \mathcal{O}_K \text{ and } [\mathcal{O}_K : \mathbb{Z}[\alpha]] < \infty\}.$$

One might hope that $D(K)$ is equal to the discriminant $\mathrm{Disc}(\mathcal{O}_K)$ of K, but this is not the case in general. Recall Example 4.3.2, in which we considered the field K generated by a root of $f = x^3 + x^2 - 2x + 8$. In that example, the discriminant of $\mathcal{O}_K$ is -503 with 503 prime:

```
K.<a> = NumberField(x^3 + x^2 - 2*x + 8)
factor(K.discriminant())
```

```
-1 * 503
```

For every $\alpha \in \mathcal{O}_K$, we have $2 \mid [\mathcal{O}_K : \mathbb{Z}[\alpha]]$, since $\mathcal{O}_K$ fails to be monogenic at 2. By Proposition 6.2.6, the discriminant of $\mathbb{Z}[\alpha]$ is divisible by 4 for all α, so $\mathrm{Disc}(\alpha)$ is also divisible by 4. This is why 2 is called an "inessential *discriminant* divisor".

Proposition 6.2.6 gives an algorithm for computing $\mathcal{O}_K$, albeit a slow one. Given K, find some order $\mathcal{O} \subset K$, and compute $d = \mathrm{Disc}(\mathcal{O})$. Factor d, and use the factorization to write $d = s \cdot f^2$, where f^2 is the largest square that divides d. Then the index of $\mathcal{O}$ in $\mathcal{O}_K$ is a divisor of f, and we (tediously) can enumerate all rings R with $\mathcal{O} \subset R \subset K$ and $[R : \mathcal{O}] \mid f$, until we find the largest one all of whose elements are integral. A much better algorithm is to proceed exactly as just described, except use the ideas of Section 4.3.3 to find a p-maximal order for each prime divisor of f, then add these p-maximal orders together.

Example 6.2.8. Consider the ring $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{5})/2]$ of integers of $K = \mathbb{Q}(\sqrt{5})$. The discriminant of the basis $1, a = (1+\sqrt{5})/2$ is

$$\mathrm{Disc}(\mathcal{O}_K) = \left|\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\right| = 5.$$

Let $\mathcal{O} = \mathbb{Z}[\sqrt{5}]$ be the order generated by $\sqrt{5}$. Then $\mathcal{O}$ has basis $1, \sqrt{5}$, so

$$\mathrm{Disc}(\mathcal{O}) = \left|\begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}\right| = 20 = [\mathcal{O}_K : \mathcal{O}]^2 \cdot 5,$$

hence $[\mathcal{O}_K : \mathcal{O}] = 2$.

Example 6.2.9. Consider the cubic field $K = \mathbb{Q}(\sqrt[3]{2})$, and let $\mathcal{O}$ be the order $\mathbb{Z}[\sqrt[3]{2}]$. Relative to the base $1, \sqrt[3]{2}, (\sqrt[3]{2})^2$ for $\mathcal{O}$, the matrix of the trace pairing is

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix}.$$

Thus

$$\mathrm{disc}(\mathcal{O}) = \det(A) = 108 = 2^2 \cdot 3^3.$$

Suppose we do not know that the ring of integers $\mathcal{O}_K$ is equal to $\mathcal{O}$. By Proposition 6.2.6, we have

$$\mathrm{Disc}(\mathcal{O}_K) \cdot [\mathcal{O}_K : \mathcal{O}]^2 = 2^2 \cdot 3^3,$$

so $3 \mid \mathrm{disc}(\mathcal{O}_K)$, and $[\mathcal{O}_K : \mathcal{O}] \mid 6$. Thus to prove $\mathcal{O} = \mathcal{O}_K$ it suffices to prove that $\mathcal{O}$ is 2-maximal and 3-maximal, which could be accomplished as described in Section 4.3.3.

6.3 Norms of Ideals

In this section we extend the notion of norm to ideals. This will be helpful in the next chapter, where we will prove that the group of fractional ideals modulo principal fractional ideals of a number field is finite by showing that every ideal is equivalent to an ideal with norm at most some bound. This is enough, because as we will see below there are only finitely many ideals of bounded norm.

Definition 6.3.1 (Lattice Index)**.** If L and M are two lattices in a vector space V, then the *lattice index* $[L : M]$ is by definition the absolute value of the determinant of any linear automorphism A of V such that $A(L) = M$.

For example, if $L = 2\mathbb{Z}$ and $M = 10\mathbb{Z}$, then

$$[L : M] = [2\mathbb{Z} : 10\mathbb{Z}] = \det([5]) = 5,$$

since 5 multiplies $2\mathbb{Z}$ onto $10\mathbb{Z}$.

The lattice index has the following properties:

- If $M \subset L$, then $[L : M] = \#(L/M)$.
- If M, L, N are any lattices in V, then
$$[L : N] = [L : M] \cdot [M : N].$$

Definition 6.3.2 (Norm of Fractional Ideal)**.** Suppose I is a fractional ideal of $\mathcal{O}_K$. The *norm* of I is the lattice index
$$\mathrm{Norm}(I) = [\mathcal{O}_K : I] \in \mathbb{Q}_{\geq 0},$$
or 0 if $I = 0$.

Note that if I is an integral ideal, then $\mathrm{Norm}(I) = \#(\mathcal{O}_K/I)$.

Lemma 6.3.3. *Suppose $a \in K$ and I is an integral ideal. Then*
$$\mathrm{Norm}(aI) = \left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right| \mathrm{Norm}(I).$$

Proof. By properties of the lattice index mentioned above we have
$$[\mathcal{O}_K : aI] = [\mathcal{O}_K : I] \cdot [I : aI] = \mathrm{Norm}(I) \cdot \left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right|.$$
Here we have used that $[I : aI] = \left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right|$, which is because left multiplication ℓ_a by a is an automorphism of K that sends I onto aI, so
$$[I : aI] = |\det(\ell_a)| = \left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right|.$$
□

Proposition 6.3.4. *If I and J are fractional ideals, then*
$$\mathrm{Norm}(IJ) = \mathrm{Norm}(I) \cdot \mathrm{Norm}(J).$$

Proof. By Lemma 6.3.3, it suffices to prove this when I and J are integral ideals. If I and J are coprime, then Theorem 5.1.4 (the Chinese Remainder Theorem) implies that $\mathrm{Norm}(IJ) = \mathrm{Norm}(I) \cdot \mathrm{Norm}(J)$. Thus we reduce to the case when $I = \mathfrak{p}^m$ and $J = \mathfrak{p}^k$ for some prime ideal $\mathfrak{p}$ and integers m, k. By Proposition 5.2.4, which is a consequence of CRT, the filtration of $\mathcal{O}_K/\mathfrak{p}^n$ given by powers of $\mathfrak{p}$ has successive quotients isomorphic to $\mathcal{O}_K/\mathfrak{p}$. Thus we see that $\#(\mathcal{O}_K/\mathfrak{p}^n) = \#(\mathcal{O}_K/\mathfrak{p})^n$, which proves that $\mathrm{Norm}(\mathfrak{p}^n) = \mathrm{Norm}(\mathfrak{p})^n$. □

Example 6.3.5. We compute some ideal norms using Sage.

```
K.<a> = NumberField(x^2 - 5)
I = K.fractional_ideal(a)
I.norm()
```

```
5
```

```
J = K.fractional_ideal(17)
J.norm()
```

```
289
```

We can also use functional notation:

```
norm(I*J)
```

```
1445
```

We will use the following proposition in the next chapter when we prove finiteness of class groups.

Proposition 6.3.6. *Fix a number field K. Let B be a positive integer. There are only finitely many integral ideals I of $\mathcal{O}_K$ with norm at most B.*

Proof. An integral ideal I is a subgroup of $\mathcal{O}_K$ of index equal to the norm of I. If G is any finitely generated abelian group, then there are only finitely many subgroups of G of index at most B. This is because the subgroups of index dividing an integer n are all subgroups of G that contain nG, and the group G/nG is finite. □

Chapter 7

Finiteness of the Class Group

Frequently $\mathcal{O}_K$ is not a principal ideal domain. This chapter is about a way to understand how badly $\mathcal{O}_K$ fails to be a principal ideal domain. The class group of $\mathcal{O}_K$ measures this failure. As one sees in a course on Class Field Theory, the class group and its generalizations also yield deep insight into the extensions of K that are Galois with abelian Galois group.

In Section 7.1, we define the class group and state the main theorem of this chapter. We then illustrate the implications of this theorem in detail for the field $\mathbb{Q}(\sqrt{10})$, proving that it has class group of order 2. Next, we prove several geometric lemmas, building very heavily on ours results from Chapter 6. Finally, we close the section by giving a complete proof of finiteness of the class group, but leave an explicit upper bound as an exercise in calculus. In Section 7.2 we very briefly discuss how often number fields have class number 1. Finally, in Section 7.3 we further discuss how to compute class groups, though nothing we do in this book begins to approach the state of the art regarding such computations – for that, see Cohen's books.

7.1 The Class Group

Definition 7.1.1 (Class Group)**.** Let $\mathcal{O}_K$ be the ring of integers of a number field K. The *class group* C_K of K is the group of fractional ideals modulo the subgroup of principal fractional ideals (a), for $a \in K$.

Note that if we let $\mathrm{Div}(\mathcal{O}_K)$ denote the group of fractional ideals, then we have an exact sequence

$$0 \to \mathcal{O}_K^* \to K^* \to \mathrm{Div}(\mathcal{O}_K) \to C_K \to 0.$$

That the class group C_K is finite follows from the first part of the following theorem and that there are only finitely many ideals of norm less than a given integer (Proposition 6.3.6).

Theorem 7.1.2 (Finiteness of the Class Group)**.** *Let K be a number field. There is a constant $C_{r,s}$ that depends only on the number r, s of real and pairs of complex conjugate embeddings of K, respectively, such that every ideal class of $\mathcal{O}_K$ contains an integral ideal of norm at most $C_{r,s}\sqrt{|d_K|}$, where $d_K = \mathrm{Disc}(\mathcal{O}_K)$. Thus by Proposition 6.3.6 the class group C_K of K is finite. In fact, one can take*

$$C_{r,s} = \left(\frac{4}{\pi}\right)^s \frac{n!}{n^n}.$$

The explicit bound in the theorem

$$M_K = \left(\frac{4}{\pi}\right)^s \frac{n!}{n^n} \cdot \sqrt{|d_K|}$$

is called the *Minkowski bound*. There are other better bounds, but they depend on unproven conjectures.

The following two examples illustrate how to apply Theorem 7.1.2 to compute C_K in simple cases.

Example 7.1.3. Let $K = \mathbb{Q}[i]$. Then $n = 2$, $s = 1$, and $|d_K| = 4$, so the Minkowski bound is

$$\sqrt{4} \cdot \left(\frac{4}{\pi}\right)^1 \frac{2!}{2^2} = \frac{4}{\pi} < 2.$$

Thus every fractional ideal is equivalent to an ideal of norm 1. Since (1) is the only ideal of norm 1, every ideal is principal, so C_K is trivial.

Example 7.1.4. Let $K = \mathbb{Q}(\sqrt{10})$. We have $\mathcal{O}_K = \mathbb{Z}[\sqrt{10}]$, so $n = 2$, $s = 0$, $|d_K| = 40$, and the Minkowski bound is

$$\sqrt{40} \cdot \left(\frac{4}{\pi}\right)^0 \cdot \frac{2!}{2^2} = 2 \cdot \sqrt{10} \cdot \frac{1}{2} = \sqrt{10} = 3.162277\ldots.$$

We compute the Minkowski bound in Sage as follows:

```
K = QQ[sqrt(10)]; K
```

```
Number Field in sqrt10 with defining polynomial x^2 - 10
```

```
B = K.minkowski_bound(); B
```

```
sqrt(10)
```

```
B.n()
```

```
3.16227766016838
```

Theorem 7.1.2 implies that every ideal class has a representative that is an integral ideal of norm 1, 2, or 3. The ideal $2\mathcal{O}_K$ is ramified in $\mathcal{O}_K$, so

$$2\mathcal{O}_K = (2, \sqrt{10})^2.$$

If $(2, \sqrt{10})$ were principal, say (α), then $\alpha = a + b\sqrt{10}$ would have norm ± 2. Then the equation

$$x^2 - 10y^2 = \pm 2, \tag{7.1.1}$$

would have an integer solution. But the squares mod 5 are $0, \pm 1$, so (7.1.1) has no solutions. Thus $(2, \sqrt{10})$ defines a nontrivial element of the class group, and it has order 2 since its square is the principal ideal $2\mathcal{O}_K$. Thus $2 \mid \#C_K$.

To find the integral ideals of norm 3, we factor $x^2 - 10$ modulo 3, and see that

$$3\mathcal{O}_K = (3, 2 + \sqrt{10}) \cdot (3, 4 + \sqrt{10}).$$

If either of the prime divisors of $3\mathcal{O}_K$ were principal, then the equation $x^2 - 10y^2 = \pm 3$ would have an integer solution. Since it does not have one mod 5, the prime divisors of $3\mathcal{O}_K$ are both nontrivial elements of the class group. Let

$$\alpha = \frac{4 + \sqrt{10}}{2 + \sqrt{10}} = \frac{1}{3} \cdot (1 + \sqrt{10}).$$

Then

$$(3, 2 + \sqrt{10}) \cdot (\alpha) = (3\alpha, 4 + \sqrt{10}) = (1 + \sqrt{10}, 4 + \sqrt{10}) = (3, 4 + \sqrt{10}),$$

so the classes over 3 are equal.

In summary, we now know that every element of C_K is equivalent to one of

$$(1), \quad (2, \sqrt{10}), \quad \text{or} \quad (3, 2 + \sqrt{10}).$$

Thus the class group is a group of order at most 3 that contains an element of order 2. Thus it must have order 2. We verify this in Sage below, where we also check that $(3, 2 + \sqrt{10})$ generates the class group.

```
K.<sqrt10> = QQ[sqrt(10)]; K
```

```
Number Field in sqrt10 with defining polynomial x^2 - 10
```

```
G = K.class_group(); G
```

```
Class group of order 2 with structure C2 of Number Field ...
```

```
G.0
```

```
Fractional ideal class (3, sqrt10 + 1)
```

```
G.0^2
```

```
Trivial principal fractional ideal class
```

```
G.0 == G( (3, 2 + sqrt10) )
```

```
True
```

Before proving Theorem 7.1.2, we prove a few lemmas. The strategy of the proof is to start with any nonzero ideal I, and prove that there is some nonzero $a \in K$ having very small norm, such that aI is an integral ideal. Then $\mathrm{Norm}(aI) = \mathrm{Norm}_{K/\mathbb{Q}}(a)\,\mathrm{Norm}(I)$ will be small, since $\mathrm{Norm}_{K/\mathbb{Q}}(a)$ is small. The trick is to determine precisely how small an a we can choose subject to the condition that aI is an integral ideal, i.e., that $a \in I^{-1}$.

Let S be a subset of $V = \mathbb{R}^n$. Then S is *convex* if whenever $x, y \in S$ then the line connecting x and y lies entirely in S. We say that S is *symmetric about the origin* if whenever $x \in S$ then $-x \in S$ also. If L is a lattice in the real vector space $V = \mathbb{R}^n$, then the *volume* of V/L is the volume of the compact real manifold V/L, which is the same thing as the absolute value of the determinant of any matrix whose rows form a basis for L.

Lemma 7.1.5 (Blichfeld)**.** *Let L be a lattice in $V = \mathbb{R}^n$, and let S be a bounded closed convex subset of V that is symmetric about the origin. If $\mathrm{Vol}(S) \geq 2^n \,\mathrm{Vol}(V/L)$, then S contains a nonzero element of L.*

Proof. First assume that $\mathrm{Vol}(S) > 2^n \,\mathrm{Vol}(V/L)$. If the map $\pi : \frac{1}{2}S \to V/L$ is injective, then

$$\frac{1}{2^n}\,\mathrm{Vol}(S) = \mathrm{Vol}\left(\frac{1}{2}S\right) \leq \mathrm{Vol}(V/L),$$

a contradiction. Thus π is not injective, so there exist $P_1 \neq P_2 \in \frac{1}{2}S$ such that $P_1 - P_2 \in L$. Because S is symmetric about the origin, $-P_2 \in \frac{1}{2}S$. By convexity, the average $\frac{1}{2}(P_1 - P_2)$ of P_1 and $-P_2$ is also in $\frac{1}{2}S$. Thus $0 \neq P_1 - P_2 \in S \cap L$, as claimed.

Next assume that $\mathrm{Vol}(S) = 2^n \cdot \mathrm{Vol}(V/L)$. Then for all $\varepsilon > 0$ there is $0 \neq Q_\varepsilon \in L \cap (1+\varepsilon)S$, since $\mathrm{Vol}((1+\varepsilon)S) > \mathrm{Vol}(S) = 2^n \cdot \mathrm{Vol}(V/L)$. If $\varepsilon < 1$ then the Q_ε are all in $L \cap 2S$, which is finite since $2S$ is bounded and L is discrete. Hence there exists nonzero $Q = Q_\varepsilon \in L \cap (1+\varepsilon)S$ for arbitrarily small ε. Since S is closed, $Q \in L \cap S$. □

Lemma 7.1.6. *If L_1 and L_2 are lattices in V, then*

$$\mathrm{Vol}(V/L_2) = \mathrm{Vol}(V/L_1) \cdot [L_1 : L_2].$$

Proof. Let A be an automorphism of V such that $A(L_1) = L_2$. Then A defines an isomorphism of real manifolds $V/L_1 \to V/L_2$ that changes volume by a factor of $|\det(A)| = [L_1 : L_2]$. The claimed formula then follows, since $[L_1 : L_2] = |\det(A)|$, by definition. □

Fix a number field K with ring of integers $\mathcal{O}_K$. Let $\sigma_1, \ldots, \sigma_r$ be the real embeddings of K and $\sigma_{r+1}, \ldots, \sigma_{r+s}$ be half the complex embeddings of K, with one representative of each pair of complex conjugate embeddings. Let $\sigma : K \to V = \mathbb{R}^n$ be the embedding

$$\begin{aligned}\sigma(x) = \big(&\sigma_1(x), \sigma_2(x), \ldots, \sigma_r(x),\\ &\mathrm{Re}(\sigma_{r+1}(x)), \ldots, \mathrm{Re}(\sigma_{r+s}(x)), \mathrm{Im}(\sigma_{r+1}(x)), \ldots, \mathrm{Im}(\sigma_{r+s}(x))\big),\end{aligned}$$

Warning 7.1.7. Note that this σ is *not* exactly the same as the one at the beginning of Section 6.2 if $s > 0$.

Lemma 7.1.8. *Let σ be the map described above. Then*

$$\mathrm{Vol}(V/\sigma(\mathcal{O}_K)) = 2^{-s}\sqrt{|d_K|}.$$

Proof. Let $L = \sigma(\mathcal{O}_K)$. From a basis $w_1, \ldots, w_n$ for $\mathcal{O}_K$ we obtain a matrix A whose ith row is

$$(\sigma_1(w_i), \cdots, \sigma_r(w_i), \mathrm{Re}(\sigma_{r+1}(w_i)), \ldots, \mathrm{Re}(\sigma_{r+s}(w_i)), \mathrm{Im}(\sigma_{r+1}(w_i)), \ldots, \mathrm{Im}(\sigma_{r+s}(w_i)))$$

and whose determinant has absolute value equal to the volume of V/L. By doing the following three column operations, we obtain a matrix whose rows are exactly the images of the w_i under *all* embeddings of K into $\mathbb{C}$, which is the matrix that came up when we defined $d_K = \mathrm{Disc}(\mathcal{O}_K)$ in Section 6.2.

1. Add $i = \sqrt{-1}$ times each column with entries $\mathrm{Im}(\sigma_{r+j}(w_i))$ to the column with entries $\mathrm{Re}(\sigma_{r+j}(w_i))$.

2. Multiply all columns with entries $\mathrm{Im}(\sigma_{r+j}(w_i))$ by $-2i$, thus changing the determinant by $(-2i)^s$.

3. Add each column that now has entries $\mathrm{Re}(\sigma_{r+j}(w_i))+i\mathrm{Im}(\sigma_{r+j}(w_i))$ to the column with entries $-2i\mathrm{Im}(\sigma_{r+j}(w_i))$ to obtain columns $\mathrm{Re}(\sigma_{r+j}(w_i))-i\mathrm{Im}(\sigma_{r+j}(w_i))$.

Recalling the definition of discriminant, we see that if B is the matrix constructed by doing the above three operations to A, then $\left|\det(B)^2\right| = |d_K|$. Thus

$$\mathrm{Vol}(V/L) = |\det(A)| = \left|(-2i)^{-s} \cdot \det(B)\right| = 2^{-s}\sqrt{|d_K|}.$$

□

Lemma 7.1.9. *If I is a fractional $\mathcal{O}_K$-ideal, then $\sigma(I)$ is a lattice in V and*

$$\mathrm{Vol}(V/\sigma(I)) = 2^{-s}\sqrt{|d_K|} \cdot \mathrm{Norm}(I).$$

Proof. Since $\sigma(\mathcal{O}_K)$ has rank n as an abelian group, and Lemma 7.1.8 implies that $\sigma(\mathcal{O}_K)$ also spans V, it follows that $\sigma(\mathcal{O}_K)$ is a lattice in V. For some nonzero integer m we have $m\mathcal{O}_K \subset I \subset \frac{1}{m}\mathcal{O}_K$, so $\sigma(I)$ is also a lattice in V. To prove the displayed volume formula, combine Lemmas 7.1.6 and 7.1.8 to get

$$\mathrm{Vol}(V/\sigma(I)) = \mathrm{Vol}(V/\sigma(\mathcal{O}_K)) \cdot [\mathcal{O}_K : I] = 2^{-s}\sqrt{|d_K|}\,\mathrm{Norm}(I).$$

□

Proof of Theorem 7.1.2. Let K be a number field with ring of integers $\mathcal{O}_K$, let $\sigma : K \hookrightarrow V \cong \mathbb{R}^n$ be as above, and let $f : V \to \mathbb{R}$ be the function defined by

$$f(x_1, \ldots, x_n) = |x_1 \cdots x_r \cdot (x_{r+1}^2 + x_{(r+1)+s}^2) \cdots (x_{r+s}^2 + x_n^2)|.$$

Notice that if $x \in K$ then $f(\sigma(x)) = |\operatorname{Norm}_{K/\mathbb{Q}}(x)|$, and for any $a \in \mathbb{R}$,

$$f(ax_1, \ldots, ax_n) = |a|^n f(x_1, \ldots, x_n).$$

Let $S \subset V$ be any fixed choice of closed, bounded, convex, subset with positive volume that is symmetric with respect to the origin. Since S is closed and bounded,

$$M = \max\{f(x) : x \in S\}$$

exists.

Suppose I is any fractional ideal of $\mathcal{O}_K$. Our goal is to prove that there is an integral ideal aI with small norm. We will do this by finding an appropriate $a \in I^{-1}$. By Lemma 7.1.9,

$$c = \operatorname{Vol}(V/\sigma(I^{-1})) = 2^{-s}\sqrt{|d_K|} \cdot \operatorname{Norm}(I)^{-1} = \frac{2^{-s}\sqrt{|d_K|}}{\operatorname{Norm}(I)}.$$

Let $\lambda = 2 \cdot \left(\frac{c}{v}\right)^{1/n}$, where $v = \operatorname{Vol}(S)$. Then

$$\operatorname{Vol}(\lambda S) = \lambda^n \operatorname{Vol}(S) = 2^n \cdot \frac{c}{v} \cdot v = 2^n \cdot c = 2^n \operatorname{Vol}(V/\sigma(I^{-1})),$$

so by Lemma 7.1.5 there exists $0 \neq b \in \sigma(I^{-1}) \cap \lambda S$. Let $a \in I^{-1}$ be such that $\sigma(a) = b$. Since M is the largest norm of an element of S, the largest norm of an element of $\sigma(I^{-1}) \cap \lambda S$ is at most $\lambda^n M$, so

$$\left|\operatorname{Norm}_{K/\mathbb{Q}}(a)\right| \leq \lambda^n M.$$

Since $a \in I^{-1}$, we have $aI \subset \mathcal{O}_K$, so aI is an integral ideal of $\mathcal{O}_K$ that is equivalent to I, and

$$\begin{aligned}
\operatorname{Norm}(aI) &= \left|\operatorname{Norm}_{K/\mathbb{Q}}(a)\right| \cdot \operatorname{Norm}(I) \\
&\leq \lambda^n M \cdot \operatorname{Norm}(I) \\
&\leq 2^n \frac{c}{v} M \cdot \operatorname{Norm}(I) \\
&= 2^n \cdot 2^{-s}\sqrt{|d_K|} \cdot M \cdot v^{-1} \\
&= 2^{r+s}\sqrt{|d_K|} \cdot M \cdot v^{-1}.
\end{aligned}$$

Notice that the right hand side is independent of I. It depends only on r, s, $|d_K|$, and our choice of S. This completes the proof of the theorem, except for the assertion that S can be chosen to give the claim at the end of the theorem which is shown in Exercise 7.1.10. □

Exercise 7.1.10. Show that in the proof of Theorem 7.1.2, S can be chosen so that the final bound matches the statement of the theorem. This means S can be chosen so that

$$\mathrm{Norm}(aI) \leq \left(\frac{4}{\pi}\right)^s \frac{n!}{n^n}\sqrt{|d_K|}.$$

[*Hint*: Consider the subset S of $\mathbb{R}^n$ defined by

$$|x_1| + \cdots + |x_r| + 2\left(\sqrt{x_{r+1}^2 + x_{(r+1)+s}^2} + \cdots + \sqrt{x_{r+s}^2 + x_{(r+s)+s}^2}\right) \leq 1.$$

Suppose $a \in \mathcal{O}_K$ such that $\sigma(a) \in S$. What can you say about $\mathrm{Norm}_{K/\mathbb{Q}}(a)$? What is $\mathrm{Vol}(S)$?]

Corollary 7.1.11. *Suppose that $K \neq \mathbb{Q}$ is a number field. Then $|d_K| > 1$.*

Proof. Applying Theorem 7.1.2 to the unit ideal, we get the bound

$$1 \leq \sqrt{|d_K|} \cdot \left(\frac{4}{\pi}\right)^s \frac{n!}{n^n}.$$

Thus

$$\sqrt{|d_K|} \geq \left(\frac{\pi}{4}\right)^s \frac{n^n}{n!},$$

and the right hand quantity is strictly bigger than 1 for any $s \leq n/2$ and any $n > 1$, see Exercise 7.1.12. □

Exercise 7.1.12. Prove the statement at the end of the proof for Corollary 7.1.11, i.e. suppose $n > 1$ and $s \leq \frac{n}{2}$ as above. Show that $\left(\frac{\pi}{4}\right)^s \frac{n^n}{n!} > 1$.

A prime p ramifies in $\mathcal{O}_K$ if and only if $d \mid d_K$, so the corollary implies that every nontrivial extension of $\mathbb{Q}$ is ramified at some prime.

7.2 Class Number 1

The fields of class number 1 are exactly the fields for which $\mathcal{O}_K$ is a principal ideal domain. How many such number fields are there? We still don't know.

Conjecture 7.2.1. *There are infinitely many number fields K such that the class group of K has order 1.*

For example, if we consider real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, with d positive and square free, many class numbers are probably 1, as suggested by the Sage output below. It looks like 1's will keep appearing infinitely often, and indeed Cohen and Lenstra conjecture that they do ([CL84]).[1]

[1] Specifically, Cohen and Lenstra conjecture that $75.446\ldots\%$ of real quadratic fields with prime discriminant have class number 1.

```
for d in [2..1000]:
    if is_fundamental_discriminant(d):
        h = QuadraticField(d, ’a’).class_number()
        if h == 1:
            print d,
```

```
5 8 12 13 17 21 24 28 29 33 37 41 44 53 56 57 61 69
73 76 77 88 89 92 93 97 101 109 113 124 129 133 137
141 149 152 157 161 172 173 177 181 184 188 193 197
201 209 213 217 233 236 237 241 248 249 253 268 269
277 281 284 293 301 309 313 317 329 332 337 341 344
349 353 373 376 381 389 393 397 409 412 413 417 421
428 433 437 449 453 457 461 472 489 497 501 508 509
517 521 524 536 537 541 553 556 557 569 573 581 589
593 597 601 604 613 617 632 633 641 649 652 653 661
664 668 669 673 677 681 701 709 713 716 717 721 737
749 753 757 764 769 773 781 789 796 797 809 813 821
824 829 844 849 853 856 857 869 877 881 889 893 908
913 917 921 929 933 937 941 953 956 973 977 989 997
```

In contrast, if we look at class numbers of quadratic imaginary fields, only a few at the beginning have class number 1.

```
for d in [-1,-2..-1000]:
    if is_fundamental_discriminant(d):
        h = QuadraticField(d, ’a’).class_number()
        if h == 1:
            print d
```

```
-3 -4 -7 -8 -11 -19 -43 -67 -163
```

It is a theorem that was proved independently and in different ways by Heegner, Stark, and Baker that the above list of 9 fields is the complete list with class number 1. More generally, it is possible, using deep work of Gross, Zagier, and Goldfeld involving zeta functions and elliptic curves, to enumerate all quadratic number fields with a given class number (Mark Watkins has done very substantial work in this direction).

7.3 More About Computing Class Groups

If $\mathfrak{p}$ is a prime of $\mathcal{O}_K$, then the intersection $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. We say that $\mathfrak{p}$ *lies over* $p \in \mathbb{Z}$. Note $\mathfrak{p}$ lies over $p \in \mathbb{Z}$ if and only if $\mathfrak{p}$ is one of the prime factors in the factorization of the ideal $p\mathcal{O}_K$. Geometrically, $\mathfrak{p}$ is a point of $\mathrm{Spec}(\mathcal{O}_K)$ that lies over the point $p\mathbb{Z}$ of $\mathrm{Spec}(\mathbb{Z})$ under the map induced by the inclusion $\mathbb{Z} \hookrightarrow \mathcal{O}_K$ as described in Section 4.1.1.

Lemma 7.3.1. *Let K be a number field with ring of integers $\mathcal{O}_K$. Then the class group $\mathrm{Cl}(K)$ is generated by the prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ lying over primes $p \in \mathbb{Z}$ with $p \leq B_K = \sqrt{|d_K|} \cdot \left(\frac{4}{\pi}\right)^s \cdot \frac{n!}{n^n}$, where s is the number of complex conjugate pairs of embeddings $K \hookrightarrow \mathbb{C}$.*

Proof. Theorem 7.1.2 asserts that every ideal class in $\mathrm{Cl}(K)$ is represented by an ideal I with $\mathrm{Norm}(I) \leq B_K$. Write $I = \prod_{i=1}^m \mathfrak{p}_i^{e_i}$, with each $e_i \geq 1$. Then by multiplicativity of the norm, each $\mathfrak{p}_i$ also satisfies $\mathrm{Norm}(\mathfrak{p}_i) \leq B_K$. If $\mathfrak{p}_i \cap \mathbb{Z} = p\mathbb{Z}$, then $p \mid \mathrm{Norm}(\mathfrak{p}_i)$, since p is the residue characteristic of $\mathcal{O}_K/\mathfrak{p}$, so $p \leq B_K$. Thus I is a product of primes $\mathfrak{p}$ that satisfies the norm bound of the lemma. □

This is a sketch of how to compute $\mathrm{Cl}(K)$:

1. Use the algorithms of Chapter 4 to list all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ that appear in the factorization of a prime $p \in \mathbb{Z}$ with $p \leq B_K$.

2. Find the group generated by the ideal classes $[\mathfrak{p}]$, where the $\mathfrak{p}$ are the prime ideals found in step 1. (In general, this step can become fairly complicated.)

The following three examples illustrate computation of $\mathrm{Cl}(K)$ for $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-6})$.

Example 7.3.2. We compute the class group of $K = \mathbb{Q}(i)$. We have

$$n = 2, \quad r = 0, \quad s = 1, \quad d_K = -4,$$

so

$$B_K = \sqrt{4} \cdot \left(\frac{4}{\pi}\right)^1 \cdot \left(\frac{2!}{2^2}\right) = \frac{8}{\pi} < 3.$$

Thus $\mathrm{Cl}(K)$ is generated by the prime divisors of 2. We have

$$2\mathcal{O}_K = (1+i)^2,$$

so $\mathrm{Cl}(K)$ is generated by the principal prime ideal $\mathfrak{p} = (1+i)$. Thus $\mathrm{Cl}(K) = 0$ is trivial.

Example 7.3.3. We compute the class group of $K = \mathbb{Q}(\sqrt{5})$. We have

$$n = 2, \quad r = 2, \quad s = 0, \quad d_K = 5,$$

so

$$B = \sqrt{5} \cdot \left(\frac{4}{\pi}\right)^0 \cdot \left(\frac{2!}{2^2}\right) < 3.$$

Thus $\mathrm{Cl}(K)$ is generated by the primes that divide 2. We have $\mathcal{O}_K = \mathbb{Z}[\gamma]$, where $\gamma = \frac{1+\sqrt{5}}{2}$ satisfies $x^2 - x - 1$. The polynomial $x^2 - x - 1$ is irreducible mod 2, so $2\mathcal{O}_K$ is prime. Since it is principal, we see that $\mathrm{Cl}(K) = 1$ is trivial.

Example 7.3.4. In this example, we compute the class group of $K = \mathbb{Q}(\sqrt{-6})$. We have

$$n = 2, \quad r = 0, \quad s = 1, \quad d_K = -24,$$

so

$$B = \sqrt{24} \cdot \frac{4}{\pi} \cdot \left(\frac{2!}{2^2}\right) \sim 3.1.$$

Thus $\mathrm{Cl}(K)$ is generated by the prime ideals lying over 2 and 3. We have $\mathcal{O}_K = \mathbb{Z}[\sqrt{-6}]$, and $\sqrt{-6}$ satisfies $x^2+6=0$. Factoring x^2+6 modulo 2 and 3 we see that the class group is generated by the prime ideals

$$\mathfrak{p}_2 = (2, \sqrt{-6}) \qquad \text{and} \qquad \mathfrak{p}_3 = (3, \sqrt{-6}).$$

Also, $\mathfrak{p}_2^2 = 2\mathcal{O}_K$ and $\mathfrak{p}_3^2 = 3\mathcal{O}_K$, so $\mathfrak{p}_2$ and $\mathfrak{p}_3$ define elements of order dividing 2 in $\mathrm{Cl}(K)$.

Is either $\mathfrak{p}_2$ or $\mathfrak{p}_3$ principal? Fortunately, there is an easier norm trick that allows us to decide. Suppose $\mathfrak{p}_2 = (\alpha)$, where $\alpha = a + b\sqrt{-6}$. Then

$$2 = \mathrm{Norm}(\mathfrak{p}_2) = |\mathrm{Norm}(\alpha)| = (a + b\sqrt{-6})(a - b\sqrt{-6}) = a^2 + 6b^2.$$

Trying the first few values of $a, b \in \mathbb{Z}$, we see that this equation has no solutions, so $\mathfrak{p}_2$ can not be principal. By a similar argument, we see that $\mathfrak{p}_3$ is not principal either. Thus $\mathfrak{p}_2$ and $\mathfrak{p}_3$ define elements of order 2 in $\mathrm{Cl}(K)$.

Does the class of $\mathfrak{p}_2$ equal the class of $\mathfrak{p}_3$? Since $\mathfrak{p}_2$ and $\mathfrak{p}_3$ define classes of order 2, we can decide this by finding the class of $\mathfrak{p}_2 \cdot \mathfrak{p}_3$. We have

$$\mathfrak{p}_2 \cdot \mathfrak{p}_3 = (2, \sqrt{-6}) \cdot (3, \sqrt{-6}) = (6, 2\sqrt{-6}, 3\sqrt{-6}) \subset (\sqrt{-6}).$$

The ideals on both sides of the inclusion have norm 6, so by multiplicativity of the norm, they must be the same ideal. Thus $\mathfrak{p}_2 \cdot \mathfrak{p}_3 = (\sqrt{-6})$ is principal, which shows $\mathfrak{p}_3$ is the inverse of $\mathfrak{p}_2$ in $\mathrm{Cl}(K)$. But $\mathfrak{p}_2$ had order 2, so $\mathfrak{p}_2$ and $\mathfrak{p}_3$ represent the same element of $\mathrm{Cl}(K)$. We conclude that

$$\mathrm{Cl}(K) = \langle \mathfrak{p}_2 \rangle = \mathbb{Z}/2\mathbb{Z}.$$

Chapter 8

Dirichlet's Unit Theorem

In this chapter we will prove Dirichlet's unit theorem, which is a structure theorem for the group of units of the ring of integers of a number field. The answer is remarkably simple: if K has r real and s pairs of complex conjugate embeddings, then

$$\mathcal{O}_K^* \approx \mathbb{Z}^{r+s-1} \times T,$$

where T is a finite cyclic group.

Many questions can be encoded as questions about the structure of the group of units. For example, Dirichlet's unit theorem explains the structure of the integer solutions (x, y) to Pell's equation $x^2 - dy^2 = 1$ (see Section 8.2.1).

8.1 The Group of Units

Definition 8.1.1 (Unit Group)**.** The *group of units* U_K associated to a number field K is the group of elements of $\mathcal{O}_K$ that have an inverse in $\mathcal{O}_K$.

Theorem 8.1.2 (Dirichlet)**.** *The group U_K is the product of a finite cyclic group of roots of unity with a free abelian group of rank $r+s-1$, where r is the number of real embeddings of K and s is the number of complex conjugate pairs of embeddings.*

(Note that we will prove a generalization of Theorem 8.1.2 in Section 12.1 below.)

We prove the theorem by defining a map $\varphi : U_K \to \mathbb{R}^{r+s}$, and showing that the kernel of φ is finite and the image of φ is a lattice in a hyperplane in $\mathbb{R}^{r+s}$. The trickiest part of the proof is showing that the image of φ spans a hyperplane, and we do this by a clever application of Blichfeld's Lemma 7.1.5.

Remark 8.1.3. Theorem 8.1.2 is due to Dirichlet who lived 1805–1859. Thomas Hirst described Dirichlet thus:

> He is a rather tall, lanky-looking man, with moustache and beard about to turn grey with a somewhat harsh voice and rather deaf. He was unwashed, with his cup of coffee and cigar. One of his failings is forgetting time, he pulls his watch out, finds it past three, and runs out without even finishing the sentence.

Koch wrote that:

> ... important parts of mathematics were influenced by Dirichlet. His proofs characteristically started with surprisingly simple observations, followed by extremely sharp analysis of the remaining problem.

I think Koch's observation nicely describes the proof we will give of Theorem 8.1.2.

Units have a simple characterization in terms of their norm.

Proposition 8.1.4. *An element $a \in \mathcal{O}_K$ is a unit if and only if* $\mathrm{Norm}_{K/\mathbb{Q}}(a) = \pm 1$.

Proof. Write $\mathrm{Norm} = \mathrm{Norm}_{K/\mathbb{Q}}$. If a is a unit, then a^{-1} is also a unit, and $1 = \mathrm{Norm}(a)\,\mathrm{Norm}(a^{-1})$. Since both $\mathrm{Norm}(a)$ and $\mathrm{Norm}(a^{-1})$ are integers, it follows that $\mathrm{Norm}(a) = \pm 1$. Conversely, if $a \in \mathcal{O}_K$ and $\mathrm{Norm}(a) = \pm 1$, then the equation $aa^{-1} = 1 = \pm\,\mathrm{Norm}(a)$ implies that $a^{-1} = \pm\,\mathrm{Norm}(a)/a$. But $\mathrm{Norm}(a)$ is the product of the images of a in $\mathbb{C}$ by all embeddings of K into $\mathbb{C}$, so $\mathrm{Norm}(a)/a$ is also a product of images of a in $\mathbb{C}$, hence a product of algebraic integers, hence an algebraic integer. Thus $a^{-1} \in K \cap \overline{\mathbb{Z}} = \mathcal{O}_K$, which proves that a is a unit. □

Remark 8.1.5. Proposition 8.1.4 is false if we replace $\mathcal{O}_K$ by K. For example, if α is a root of $x^2 - \frac{1}{2}x + 1$, then α has norm ± 1, but α is not a unit of $\mathcal{O}_K$, since $\alpha \notin \mathcal{O}_K$. To general Proposition 8.1.4 to an arbitrary finite extension R/S of Dedekind domains, we replace ± 1 by "an element of S^*".

Let r be the number of real and s the number of complex conjugate embeddings of K into $\mathbb{C}$, so $n = [K : \mathbb{Q}] = r + 2s$. Define the *log map*

$$\varphi : U_K \to \mathbb{R}^{r+s}$$

by

$$\varphi(a) = (\log |\sigma_1(a)|, \ldots, \log |\sigma_{r+s}(a)|).$$

Here $|z|$ is the usual absolute value of $z = x + iy \in \mathbb{C}$ (so $|z| = \sqrt{x^2 + y^2}$), and the maps σ_i are the same as those described in Lemma 7.1.8. In particular, $\sigma_1, \ldots, \sigma_r$ represent all real embeddings $K \to \mathbb{R}$ and $\sigma_{r+1}, \ldots, \sigma_{r+s}$ represent half of the complex embeddings $K \to \mathbb{C}$, with one representative for each pair of complex conjugate embeddings.

Lemma 8.1.6. *The image of φ lies in the hyperplane*

$$H = \{(x_1, \ldots, x_{r+s}) \in \mathbb{R}^{r+s} : x_1 + \cdots + x_r + 2x_{r+1} + \cdots + 2x_{r+s} = 0\}. \qquad (8.1.1)$$

Proof. If $a \in U_K$, then by Proposition 8.1.4,

$$\left(\prod_{i=1}^{r} |\sigma_i(a)|\right) \cdot \left(\prod_{i=r+1}^{r+s} |\sigma_i(a)|^2\right) = \left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right| = 1.$$

Taking logs of both sides proves the lemma. □

Lemma 8.1.7. *The kernel of φ is finite.*

Proof. We have

$$\begin{aligned} \mathrm{Ker}(\varphi) &\subset \{a \in \mathcal{O}_K \colon |\sigma_i(a)| = 1 \text{ for } i = 1, \ldots, r+s\} \\ \sigma(\mathrm{Ker}(\varphi)) &\subset \sigma(\mathcal{O}_K) \cap X \end{aligned}$$

where $\sigma : \mathcal{O}_K \to \mathbb{C}^{r+s}$ is given by $\sigma(a) = (\sigma_1(a), \ldots, \sigma_{r+s}(a))$ and X is the set $\{(z_1, \ldots, z_{r+s}) \in \mathbb{C}^{r+s} \colon |z_i| \leq 1\}$. Since $\sigma(\mathcal{O}_K)$ is a lattice (see Proposition 2.4.7) and X is compact, the intersection $\sigma(\mathcal{O}_K) \cap X$ is finite. This implies $\mathrm{Ker}(\varphi)$ is finite. □

Lemma 8.1.8. *The kernel of φ is a finite cyclic group.*

Proof. Lemma 8.1.7 implies that $\ker(\varphi)$ is a finite group. It is a general fact that any finite subgroup G of the multiplicative group K^* of a field is cyclic (see Exercise 8.1.9). □

Exercise 8.1.9. Finish the proof of Lemma 8.1.8 by showing that for a field K, every finite subgroup G of the multiplicative group K^* is cyclic.

[*Hint*: Every element in G satisfies a polynomial of the form $x^n - 1$. Recall that a polynomial of degree n over a field has at most n distinct roots. Now consider the orders of the elements of G.]

To prove Theorem 8.1.2, it suffices to prove that $\mathrm{Im}(\varphi)$ is a lattice in the hyperplane H of (8.1.1), which we view as a vector space of dimension $r+s-1$.

Define an embedding

$$\sigma : K \hookrightarrow \mathbb{R}^n \tag{8.1.2}$$

given by $\sigma(x) = (\sigma_1(x), \ldots, \sigma_{r+s}(x))$, where we view $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$ via $a + bi \mapsto (a, b)$. Thus this is the embedding

$$\begin{aligned} x \mapsto \big(&\sigma_1(x), \sigma_2(x), \ldots, \sigma_r(x), \\ &\mathrm{Re}(\sigma_{r+1}(x)), \mathrm{Im}(\sigma_{r+1}(x)), \ldots, \mathrm{Re}(\sigma_{r+s}(x)), \mathrm{Im}(\sigma_{r+s}(x))\big). \end{aligned}$$

Lemma 8.1.10. *The image $\varphi : U_K \to \mathbb{R}^{r+s}$ is discrete.*

Proof. Let X be a bounded subset of $\mathbb{R}^{r+s}$. We will show that the intersection $\varphi(U_K) \cap X$ is finite. Since X is bounded, for any $u \in Y = \varphi^{-1}(X) \subset U_K$ the coordinates of $\sigma(u)$ are bounded, since $|\log(x)|$ is bounded on bounded subsets of $[1, \infty)$. Thus $\sigma(Y)$ is a bounded subset of $\mathbb{R}^n$. Since $\sigma(Y) \subset \sigma(\mathcal{O}_K)$, and $\sigma(\mathcal{O}_K)$ is a lattice in $\mathbb{R}^n$, it follows that $\sigma(Y)$ is finite; moreover, σ is injective, so Y is finite. Thus $\varphi(U_K) \cap X \subset \varphi(Y) \cap X$ is finite. □

We will use the following lemma in our proof of Theorem 8.1.2.

Lemma 8.1.11. *Let $n \geq 2$ be an integer, suppose $w_1, \ldots, w_n \in \mathbb{R}$ are not all equal, and suppose $A, B \in \mathbb{R}$ are positive. Then there exist $d_1, \ldots, d_n \in \mathbb{R}_{>0}$ such that*

$$|w_1 \log(d_1) + \cdots + w_n \log(d_n)| > B$$

and $d_1 \cdots d_n = A$.

Proof. Order the w_i so that $w_1 \neq 0$. By hypothesis there exists a w_j such that $w_j \neq w_1$, and again re-ordering we may assume that $j = 2$. Set $d_3 = \cdots = d_{r+s} = 1$. Suppose d_1, d_2 are any positive real numbers with $d_1 d_2 = A$. Since $\log(1) = 0$,

$$\begin{aligned} \left| \sum_{i=1}^{n} w_i \log(d_i) \right| &= |w_1 \log(d_1) + w_2 \log(d_2)| \\ &= |w_1 \log(d_1) + w_2 \log(A/d_1)| \\ &= |(w_1 - w_2) \log(d_1) + w_2 \log(A)| \end{aligned}$$

Since $w_1 \neq w_2$, we have $|(w_1 - w_2) \log(d_1) + w_2 \log(A)| \to \infty$ as $d_1 \to \infty$. It is thus possible to choose the d_i as in the lemma. □

Proof of Theorem 8.1.2. By Lemma 8.1.10, the image $\varphi(U_K)$ is discrete, so it remains to show that $\varphi(U_K)$ spans H. Let W be the $\mathbb{R}$-span of the image $\varphi(U_K)$, and note that W is a subspace of H, by Lemma 8.1.6. We will show that $W = H$ indirectly by showing that if $v \notin H^{\perp}$, where $\perp$ is the orthogonal complement with respect to the dot product on $\mathbb{R}^{r+s}$, then $v \notin W^{\perp}$. This will show that $W^{\perp} \subset H^{\perp}$, hence that $H \subset W$, as required.

Thus suppose $z = (z_1, \ldots, z_{r+s}) \notin H^{\perp}$. Define a function $f : K^* \to \mathbb{R}$ by

$$f(x) = z_1 \log |\sigma_1(x)| + \cdots + z_{r+s} \log |\sigma_{r+s}(x)|. \tag{8.1.3}$$

Note that $f(U_K) = \{0\}$ if and only if $z \in W^{\perp}$, so to show that $z \notin W^{\perp}$ we show that there exists some $u \in U_K$ with $f(u) \neq 0$.

Let

$$A = \sqrt{|d_K|} \cdot \left(\frac{2}{\pi}\right)^s \in \mathbb{R}_{>0}.$$

Choose any positive real numbers $c_1, \ldots, c_{r+s} \in \mathbb{R}_{>0}$ such that

$$c_1 \cdots c_r \cdot (c_{r+1} \cdots c_{r+s})^2 = A.$$

Let

$$\begin{aligned} S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n :& \\ & |x_i| \leq c_i \text{ for } 1 \leq i \leq r, \\ & |x_i^2 + x_{i+s}^2| \leq c_i^2 \text{ for } r < i \leq r+s\} \subset \mathbb{R}^n. \end{aligned}$$

Then S is closed, bounded, convex, symmetric with respect to the origin, and of dimension $r + 2s$, since S is a product of r intervals and s discs, each of which has these properties. Viewing S as a product of intervals and discs, we see that the volume of S is

$$\mathrm{Vol}(S) = \prod_{i=1}^{r} (2c_i) \cdot \prod_{i=1}^{s} (\pi c_i^2) = 2^r \cdot \pi^s \cdot A = 2^{r+s}\sqrt{|d_K|} = 2^n \cdot 2^{-s}\sqrt{|d_K|}.$$

Recall Blichfeldt's Lemma 7.1.5, which asserts that if L is a lattice and S is closed, bounded, etc., and has volume at least $2^n \cdot \mathrm{Vol}(V/L)$, then $S \cap L$ contains a nonzero element. To apply this lemma, we take $L = \sigma(\mathcal{O}_K) \subset \mathbb{R}^n$, where σ is as in (8.1.2). By Lemma 7.1.8, we have $\mathrm{Vol}(\mathbb{R}^n/L) = 2^{-s}\sqrt{|d_K|}$. To check the hypothesis of Blichfeld's lemma, note that

$$\mathrm{Vol}(S) = 2^n \cdot 2^{-s}\sqrt{|d_K|} = 2^n \,\mathrm{Vol}(\mathbb{R}^n/L).$$

Thus there exists a nonzero element x in $S \cap \sigma(\mathcal{O}_K)$. Let $a \in \mathcal{O}_K$ with $\sigma(a) = x$, then $\sigma(a) \in S$, so $|\sigma_i(a)| \leq c_i$ for $1 \leq i \leq r+s$. We then have

$$\begin{aligned} \left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right| &= \left|\prod_{i=1}^{r+2s} \sigma_i(a)\right| \\ &= \prod_{i=1}^{r} |\sigma_i(a)| \cdot \prod_{i=r+1}^{s} |\sigma_i(a)|^2 \\ &\leq c_1 \cdots c_r \cdot (c_{r+1} \cdots c_{r+s})^2 = A. \end{aligned}$$

Since $a \in \mathcal{O}_K$ is nonzero, we also have

$$\left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right| \geq 1.$$

Moreover, if for any $i \leq r$, we have $|\sigma_i(a)| < \frac{c_i}{A}$, then

$$1 \leq \left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right| < c_1 \cdots \frac{c_i}{A} \cdots c_r \cdot (c_{r+1} \cdots c_{r+s})^2 = \frac{A}{A} = 1,$$

a contradiction, so $|\sigma_i(a)| \geq \frac{c_i}{A}$ for $i = 1, \ldots, r$. Likewise, $|\sigma_i(a)|^2 \geq \frac{c_i^2}{A}$, for $i = r+1, \ldots, r+s$. Rewriting this we have

$$\frac{c_i}{|\sigma_i(a)|} \leq A \quad \text{for } i \leq r \quad \text{and} \quad \left(\frac{c_i}{|\sigma_i(a)|}\right)^2 \leq A \quad \text{for } i = r+1, \ldots, r+s. \quad (8.1.4)$$

Recall that our overall strategy is to use an appropriately chosen a to construct a unit $u \in U_K$ such $f(u) \neq 0$. First, let $b_1, \ldots, b_m$ be representative generators for the finitely many nonzero principal ideals of $\mathcal{O}_K$ of norm at most A. Since $\left|\mathrm{Norm}_{K/\mathbb{Q}}(a)\right| \leq A$, we have $(a) = (b_j)$, for some j, so there is a unit $u \in \mathcal{O}_K$ such that $a = ub_j$.

Let

$$t = t_{c_1,\ldots,c_{r+s}} = z_1 \log(c_1) + \cdots + z_{r+s} \log(c_{r+s}),$$

and recall $f : K^* \to \mathbb{R}$ defined in (8.1.3) above. We have

$$\begin{aligned}
|f(u) - t| &= |f(a) - f(b_j) - t| \\
&\leq |f(b_j)| + |t - f(a)| \\
&= |f(b_j)| + |z_1(\log(c_1) - \log(|\sigma_1(a)|)) + \cdots + z_{r+s}(\log(c_{r+s}) - \log(|\sigma_{r+s}(a)|))| \\
&= |f(b_j)| + |z_1 \cdot \log(c_1/|\sigma_1(a)|) + \cdots + \frac{z_{r+s}}{2} \cdot \log((c_{r+s}/|\sigma_{r+s}(a)|)^2)| \\
&\leq |f(b_j)| + \log(A) \cdot \left(\sum_{i=1}^{r} |z_i| + \frac{1}{2} \cdot \sum_{i=r+1}^{s} |z_i|\right) \stackrel{\text{def}}{=} B_j.
\end{aligned}$$

In the last step we use (8.1.4).

Let $B = \max_j B_j$, and note that B does not depend on the choice of the c_i; in fact, it only depends our *fixed* choice of z and on the field K. Moreover, for any choice of the c_i as above, we have

$$|f(u) - t| \leq B.$$

If we can choose positive real numbers c_i such that

$$\begin{aligned}
c_1 \cdots c_r \cdot (c_{r+1} \cdots c_{r+s})^2 &= A \\
|t_{c_1,\ldots,c_{r+s}}| &> B,
\end{aligned}$$

then the fact that $|f(u) - t| \leq B$ would then imply that $|f(u)| > 0$, which is exactly what we aimed to prove.

If $r+s=1$, then we are trying to prove that $\varphi(U_K)$ is a lattice in $\mathbb{R}^0 = \mathbb{R}^{r+s-1}$, which is automatically true, so assume $r+s>1$. To finish the proof, we explain how to use Lemma 8.1.11 to choose c_i such that $|t|>B$. We have

$$\begin{aligned} t &= z_1\log(c_1)+\cdots+z_{r+s}\log(c_{r+s}) \\ &= z_1\log(c_1)+\cdots+z_r\log(c_r)+\frac{1}{2}\cdot z_{r+1}\log(c_{r+1}^2)+\cdots+\frac{1}{2}\cdot z_{r+s}\log(c_{r+s}^2) \\ &= w_1\log(d_1)+\cdots+w_r\log(d_r)+w_{r+1}\log(d_{r+1})+\cdots+\cdot w_{r+s}\log(d_{r+s}), \end{aligned}$$

where $w_i = z_i$ and $d_i = c_i$ for $i \le r$, and $w_i = \frac{1}{2}z_i$ and $d_i = c_i^2$ for $r < i \le r+s$. The condition that $z \not\in H^\perp$ is that the w_i are not all the same, and in our new coordinates the lemma is equivalent to showing that $|\sum_{i=1}^{r+s} w_i \log(d_i)| > B$, subject to the condition that $\prod_{i=1}^{r+s} d_i = A$. But this is exactly what Lemma 8.1.11 shows. It is thus possible to find a unit u such that $|f(u)|>0$. Thus $z \not\in W^\perp$, so $W^\perp \subset H^\perp$, whence $H \subset W$, which finishes the proof of Theorem 8.1.2. □

8.2 Examples with Sage

8.2.1 Pell's Equation

The so-called "Pell's equation" is $x^2 - dy^2 = 1$ with $d>0$ square free, and we seek integer solutions x, y to this equation. If $x + y\sqrt{d} \in K = \mathbb{Q}(\sqrt{d})$, then

$$\mathrm{Norm}(x+y\sqrt{d}) = (x+y\sqrt{d})(x-y\sqrt{d}) = x^2 - dy^2.$$

Thus if (x,y) are integers such that $x^2 - dy^2 = 1$, then $\alpha = x + \sqrt{d}y \in \mathcal{O}_K$ has norm 1, so by Proposition 8.1.4 we have $\alpha \in U_K$. The integer solutions to Pell's equation thus form a finite-index subgroup of the group of units in the ring of integers of $\mathbb{Q}(\sqrt{d})$. Dirichlet's unit theorem implies that for any d the solutions to Pell's equation with x, y not both negative forms an infinite cyclic group, which is a fact that takes substantial work to prove using only elementary number theory (for example, using continued fractions).

We first solve Pell's equation $x^2 - 5y^2 = 1$ with $d = 5$ by finding the units of the ring of integers of $\mathbb{Q}(\sqrt{5})$ using Sage. Recall from Example 2.3.21 that the ring of integers of $\mathbb{Q}(\sqrt{5})$ is $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$

```
K.<sqrt5> = QuadraticField(5)
G = K.unit_group(); G
```

```
Unit group with structure C2 x Z of Number Field in sqrt5 with
defining polynomial x^2 - 5
```

```
u = G.1.value(); v = G.0.value(); (u, v)
```

```
(1/2*sqrt5 + 1/2, -1)
```

The subgroup of cubes of u gives us the units with integer x, y (not both negative).

```
[u^(3*i) for i in [0..9]]
```

```
[1, sqrt5 + 2, 4*sqrt5 + 9, 17*sqrt5 + 38, 72*sqrt5 + 161, \
305*sqrt5 + 682, 1292*sqrt5 + 2889, 5473*sqrt5 + 12238, \
23184*sqrt5 + 51841, 98209*sqrt5 + 219602]
```

However, the norm of $u = \frac{1+\sqrt{5}}{2}$ is -1. So the 6th powers of u will generate solutions to Pell's Equation. We can also list the coefficients for these powers as follows.

```
[list(u^(6*i)) for i in [0..7]]
```

```
[[1, 0], [9, 4], [161, 72], [2889, 1292], [51841, 23184], \
[930249, 416020], [16692641, 7465176], [299537289, 133957148]]
```

Remark 8.2.1. A great article about Pell's equation is [Len02]. The MathSciNet review begins: "This wonderful article begins with history and some elementary facts and proceeds to greater and greater depth about the existence of solutions to Pell equations and then later the algorithmic issues of finding those solutions. The cattle problem is discussed, as are modern smooth number methods for solving Pell equations and the algorithmic issues of representing very large solutions in a reasonable way."

The simplest solutions to Pell's equation can be huge, even when d is quite small. Read Lenstra's paper for some examples from over two thousand years ago. Here is one example for $d = 10000019$.

```
K.<a> = QuadraticField(next_prime(10^7))
G = K.unit_group(); G.1.value()
```

```
1635802598803463282255922381210946254991426776931429155067472530000\
3400641003657678728904388162492712664239981750303094365756106316339\
2723776016806037958837914778176119741840754457028237899759459100042\
8895693238165048098039*a - \
5172866928858149674701706723683467983036290343735752029750756050558\
7149580808939912744279034480986438365128783512278562690868566790078\
3049793210477650310733452599026227120591649690086336036036403311755\
6634562204182936222240930
```

Exercise 8.2.2. Let U be the group of units of the ring of integers of $K = \mathbb{Q}(\sqrt{5})$.

(a) Prove that the set S of units $x + y\sqrt{5} \in U$ with $x, y \in \mathbb{Z}$ is a subgroup of U. (The main point is to show that the inverse of a unit with $x, y \in \mathbb{Z}$ again has coefficients in $\mathbb{Z}$.)

(b) Let U^3 denote the subgroup of cubes of elements of U. Prove that $S = U^3$ by showing that $U^3 \subset S \subsetneq U$ and that there are no groups H with $U^3 \subsetneq H \subsetneq U$.

8.2.2 Examples with Various Signatures

In this section we give examples for various (r, s) pairs. First we consider $K = \mathbb{Q}(i)$.

```
K.<a> = QuadraticField(-1)
K.signature()
```

```
(0, 1)
```

```
U = K.unit_group(); U
```

```
Unit group with structure C4 of Number Field in a with
defining polynomial x^2 + 1
```

```
U.0.value()
```

```
a
```

The `signature` method returns the number of real and complex conjugate embeddings of K into $\mathbb{C}$. The `unit_group` method, which we used above, returns the unit group U_K as an abstract abelian group and a homomorphism $U_K \to \mathcal{O}_K$.

Next we consider $K = \mathbb{Q}(\sqrt[3]{2})$.

```
K.<a> = NumberField(x^3 - 2)
K.signature()
```

```
(1, 1)
```

```
U = K.unit_group(); U
```

```
Unit group with structure C2 x Z of Number Field in a with
defining polynomial x^3 - 2
```

```
[u.value() for u in U.gens()]
```

```
[-1, a - 1]
```

```
u = U.1.value(); u
```

```
a - 1
```

Below we use the `places` command, which returns the real embeddings and representatives for the complex conjugate embeddings. We use the places to define the log map φ, which plays such a big role in this chapter.

```
S = K.places(prec=53); S
```

```
[Ring morphism:
  From: Number Field in a with defining polynomial x^3 - 2
  To:   Real Double Field
  Defn: a |--> 1.25992104989, \
 Ring morphism:
  From: Number Field in a with defining polynomial x^3 - 2
  To:   Complex Double Field
  Defn: a |--> -0.629960524947 + 1.09112363597*I]
```

```
def phi(z):
    return [log(abs(sigma(z))) for sigma in S]
phi(u)
```

```
[-1.3473773483293832, 0.673688674164692]
```

```
phi(K(-1))
```

```
[0.0, 0.0]
```

Note that $\varphi : U_K \to \mathbb{R}^2$, and the image lands in the 1-dimensional subspace of (x_1, x_2) such that $x_1 + 2x_2 = 0$. Also, note that $\varphi(-1) = (0, 0)$.

Let's try a field such that $r + s - 1 = 2$. First, one with $r = 0$ and $s = 3$:

```
K.<a> = NumberField(x^6 + x + 1)
K.signature()
```

```
(0, 3)
```

```
U = K.unit_group(); U
```

```
Unit group with structure C2 x Z x Z of Number Field in a with
defining polynomial x^6 + x + 1
```

```
u1 = U.1.value(); u1
```

```
a
```

```
u2 = U.2.value(); u2
```

```
a^3 + a
```

```
S = K.places(prec=53)
def phi(z):
    return [log(abs(sigma(z))) for sigma in S]
phi(u1)
```

```
[-0.16741548328589614, 0.04864390975267338, 0.11877157353322298]
```

```
phi(u2)
```

```
[0.30678570892329504, -1.0725146505489758, 0.7657289416256803]
```

```
phi(K(-1))
```

```
[0.0, 0.0, 0.0]
```

```
sum(phi(u1))
```

```
2.220446049250313e-16
```

```
sum(phi(u2))
```

```
-4.440892098500626e-16
```

Notice that the log image of u_1 is clearly not a real multiple of the log image of u_2 (e.g., the scalar would have to be positive because of the first coefficient, but negative because of the second). This illustrates the fact that the log images of u_1 and u_2 span a two-dimensional space.

Next we compute a field with $r = 3$ and $s = 0$. (A field with $s = 0$ is called totally real.)

```
K.<a> = NumberField(x^3 + x^2 - 5*x - 1)
K.signature()
```

```
(3, 0)
```

```
U = K.unit_group(); U
```

```
Unit group with structure C2 x Z x Z of Number Field in a with
defining polynomial x^3 + x^2 - 5*x - 1
```

```
u1 = U.1.value(); u1
```

```
1/2*a^2 + a - 1/2
```

```
u2 = U.2.value(); u2
```

```
a
```

```
S = K.places(prec=53)
def phi(z):
    return [log(abs(sigma(z))) for sigma in S]
phi(u1)
```

```
[-0.7747670223461895, -0.3928487245813982, 1.1676157469275887]
```

```
phi(u2)
```

```
[0.9966812040934553, -1.6402241503223172, 0.6435429462288627]
```

A field with $r = 0$ is called totally complex. For example, the *cyclotomic fields* $\mathbb{Q}(\zeta_n)$ are totally complex, where ζ_n is a primitive nth root of unity. The degree of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ is $\varphi(n)$ and $r = 0$, so $s = \varphi(n)/2$ (assuming $n > 2$). Here φ is the Euler Totient function which on n is defined as the number of integers k such that $0 < k \leq n$ and $\gcd(k, n) = 1$.

```
K.<a> = CyclotomicField(11); K
```

```
Cyclotomic Field of order 11 and degree 10
```

```
K.signature()
```

```
(0, 5)
```

```
U = K.unit_group(); U
```

```
Unit group with structure C22 x Z x Z x Z x Z of \
Cyclotomic Field of order 11 and degree 10
```

```
u = U.1.value(); u
```

```
a^7 + a^6
```

```
S = K.places(prec=20)
def phi(z):
    return [log(abs(sigma(z))) for sigma in S]
phi(u)
```

```
[-1.2566, -0.18533, 0.26982, 0.52028, 0.65180]
```

```
for u in U.gens():
    print phi(u.value())
```

```
[0.00000, 0.00000, 0.00000, -9.5367e-7, 0.00000]
[-1.2566, -0.18533, 0.26982, 0.52028, 0.65180]
[-0.26981, -0.52028, 0.18533, -0.65180, 1.2566]
[0.65180, 0.26981, -1.2566, -0.18533, 0.52029]
[-0.084486, -1.1721, -0.33496, 0.60477, 0.98675]
```

How far can we go computing unit groups of cyclotomic fields directly with Sage?

```
%time U = CyclotomicField(11).unit_group()
```

```
CPU time: 0.01 s, Wall time: 0.01 s
```

```
%time U = CyclotomicField(13).unit_group()
```

```
CPU time: 0.30 s, Wall time: 0.30 s
```

```
%time U = CyclotomicField(17).unit_group()
```

```
CPU time: 1.13 s, Wall time: 1.31 s
```

```
%time U = CyclotomicField(23).unit_group()
```

```
.... I waited a few minutes and gave up....
```

However, if you are willing to assume some conjectures (something related to the Generalized Riemann Hypothesis), you can go further:

```
proof.number_field(False)
%time U = CyclotomicField(11).unit_group()
```

```
CPU time: 0.07 s, Wall time: 0.07 s
```

```
%time U = CyclotomicField(13).unit_group()
```

```
CPU time: 0.03 s, Wall time: 0.03 s
```

```
%time U = CyclotomicField(17).unit_group()
```

```
CPU time: 0.06 s, Wall time: 0.06 s
```

```
%time U = CyclotomicField(23).unit_group()
```

```
CPU time: 0.26 s, Wall time: 0.31 s
```

```
%time U = CyclotomicField(29).unit_group()
```

```
CPU time: 0.60 s, Wall time: 0.62 s
```

The generators of the units for $\mathbb{Q}(\zeta_{29})$ are

$$\begin{aligned}
u_0 &= -\zeta_{29}^3\\
u_1 &= \zeta_{29}^{26} + \zeta_{29}^{25} + \zeta_{29}^{22} + \zeta_{29}^{21} + \zeta_{29}^{19} + \zeta_{29}^{18} + \zeta_{29}^{15} + \zeta_{29}^{14} + \zeta_{29}^{11} + \zeta_{29}^{8} + \zeta_{29}^{7} + \zeta_{29}^{4} + \zeta_{29}^{3} + \zeta_{29} + 1\\
u_2 &= \zeta_{29}^{14} + \zeta_{29}^{3}\\
u_3 &= \zeta_{29}^{3} + 1\\
u_4 &= \zeta_{29}^{26} + \zeta_{29}^{20} + \zeta_{29}^{3}\\
u_5 &= \zeta_{29}^{22} + \zeta_{29}^{11} + \zeta_{29}^{2}\\
u_6 &= \zeta_{29}^{10} + \zeta_{29}^{9} + \zeta_{29}^{8}\\
u_7 &= \zeta_{29}^{23} + \zeta_{29}\\
u_8 &= \zeta_{29}^{17} + \zeta_{29}^{11}\\
u_9 &= \zeta_{29}^{22} + \zeta_{29}^{3}\\
u_{10} &= \zeta_{29}^{24} + \zeta_{29}^{19} + \zeta_{29}^{5} + 1\\
u_{11} &= \zeta_{29}^{19} + \zeta_{29}^{6}\\
u_{12} &= \zeta_{29}^{27} + \zeta_{29}^{19} + \zeta_{29}^{11} + \zeta_{29}^{6} + \zeta_{29}^{3}\\
u_{13} &= \zeta_{29}^{26} + \zeta_{29}^{15} + \zeta_{29}^{4}.
\end{aligned}$$

There are better ways to compute units in cyclotomic fields than to just use general purpose software. For example, there are explicit *cyclotomic units* that can be written down and generate a finite subgroup of U_K. See [Was97, Ch. 8], which would be a great book to read now that you've gone this far in the present book. Also, using ideas explained in that book, it is probably possible to make the `unit_group` command in Sage for cyclotomic fields extremely fast, which would be an interesting project for a reader who also likes to code.

Chapter 9

Decomposition and Inertia Groups

In this chapter we will study extra structure in the case when K is Galois over $\mathbb{Q}$. We will learn about Frobenius elements, the Artin symbol, decomposition groups, and how the Galois group of K is related to Galois groups of residue class fields. These are the basic structures needed to attach L-function to representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which will play a central role in the next few chapters.

9.1 Galois Extensions

In this section we give a survey (no proofs) of the basic facts about Galois extensions of $\mathbb{Q}$ that will be needed in the rest of this chapter.

Definition 9.1.1 (Galois)**.** An extension K/L of number fields is *Galois* if

$$\#\,\mathrm{Aut}(K/L) = [K : L],$$

where $\mathrm{Aut}(K/L)$ is the group of automorphisms of K that fix L. We write

$$\mathrm{Gal}(K/L) = \mathrm{Aut}(K/L).$$

For example, if $K \subset \mathbb{C}$ is a number field embedded in the complex numbers, then K is *Galois* over $\mathbb{Q}$ if every field homomorphism $K \to \mathbb{C}$ has image K. As another example, any quadratic extension K/L is Galois over L, since it is of the form $L(\sqrt{a})$, for some $a \in L$, and the nontrivial automorphism is induced by $\sqrt{a} \mapsto -\sqrt{a}$, so there is always one nontrivial automorphism. If $f \in L[x]$ is an irreducible cubic polynomial, and a is a root of f, then one proves in a course on Galois theory that $L(a)$ is Galois over L if and only if the discriminant of f is a perfect square in L. "Random" number fields of degree bigger than 2 are rarely Galois.

If $K \subset \mathbb{C}$ is a number field, then the *Galois closure* K^{gc} of K in $\mathbb{C}$ is the field generated by all images of K under all embeddings in $\mathbb{C}$ (more generally, if K/L

is an extension, the Galois closure of K over L is the field generated by images of embeddings $K \to \mathbb{C}$ that are the identity map on L). If $K = \mathbb{Q}(a)$, then K^{gc} is the field generated by all of the conjugates of a, and is hence Galois over $\mathbb{Q}$, since the image under an embedding of any polynomial in the conjugates of a is again a polynomial in conjugates of a.

How much bigger can the degree of K^{gc} be as compared to the degree of $K = \mathbb{Q}(a)$? There is an embedding of $\mathrm{Gal}(K^{\mathrm{gc}}/\mathbb{Q})$ into the group of permutations of the conjugates of a. If a has n conjugates, then this is an embedding $\mathrm{Gal}(K^{\mathrm{gc}}/\mathbb{Q}) \hookrightarrow S_n$, where S_n is the symmetric group on n symbols, which has order $n!$. Thus the degree of the K^{gc} over $\mathbb{Q}$ is a divisor of $n!$. Also $\mathrm{Gal}(K^{\mathrm{gc}}/\mathbb{Q})$ is a transitive subgroup of S_n, which constrains the possibilities further. When $n = 2$, we recover the fact that quadratic extensions are Galois. When $n = 3$, we see that the Galois closure of a cubic extension is either the cubic extension or a quadratic extension of the cubic extension. One can show that the Galois closure of a cubic extension is obtained by adjoining the square root of the discriminant, which is why an irreducible cubic defines a Galois extension if and only if the discriminant is a perfect square.

For an extension K of $\mathbb{Q}$ of degree 5, it is "frequently" the case that the Galois closure has degree 120, and in fact it is an interesting problem to enumerate examples of degree 5 extension in which the Galois closure has degree smaller than 120. For example, the only possibilities for the order of a transitive proper subgroup of S_5 are 5, 10, 20, and 60; there are also proper subgroups of S_5 order $2, 3, 4, 6, 8, 12$, and 24, but none are transitive.

Let n be a positive integer. Consider the field $K = \mathbb{Q}(\zeta_n)$, where $\zeta_n = e^{2\pi i/n}$ is a primitive nth root of unity. If $\sigma : K \to \mathbb{C}$ is an embedding, then $\sigma(\zeta_n)$ is also an nth root of unity, and the group of nth roots of unity is cyclic, so $\sigma(\zeta_n) = \zeta_n^m$ for some m which is invertible modulo n. Thus K is Galois and $\mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow (\mathbb{Z}/n\mathbb{Z})^*$. However, $[K : \mathbb{Q}] = \varphi(n)$, so this map is an isomorphism. (Remark: Taking a limit using the maps $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\zeta_{p^r})/\mathbb{Q})$, we obtain a homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^*$, which is called the *p-adic cyclotomic character*.)

Compositums of Galois extensions are Galois. For example, the biquadratic field $K = \mathbb{Q}(\sqrt{5}, \sqrt{-1})$ is a Galois extension of $\mathbb{Q}$ of degree 4, which is the compositum of the Galois extensions $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-1})$ of $\mathbb{Q}$.

Fix a number field K that is Galois over a subfield L. Then the Galois group $G = \mathrm{Gal}(K/L)$ acts on many of the object that we have associated to K.

Exercise 9.1.2. Describe the natural action of G on the following objects:

- The ring of integers $\mathcal{O}_K$
- The group units U_K
- The set of ideals of $\mathcal{O}_K$
- The group of fractional ideals of $\mathcal{O}_K$
- The class group $\mathrm{Cl}(K)$

- The set $S_{\mathfrak{p}}$ of prime ideals lying over a given nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$, i.e., the prime divisors of $\mathfrak{p}\mathcal{O}_K$

In the next section we will be concerned with the action of $\mathrm{Gal}(K/L)$ on $S_{\mathfrak{p}}$, though actions on each of the other objects, especially $\mathrm{Cl}(K)$, are also of great interest. Understanding the action of $\mathrm{Gal}(K/L)$ on $S_{\mathfrak{p}}$ will enable us to associate, in a natural way, a holomorphic L-function to any complex representation $\mathrm{Gal}(K/L) \to \mathrm{GL}_n(\mathbb{C})$.

9.2 Decomposition of Primes: $efg = n$

If $I \subset \mathcal{O}_K$ is any ideal in the ring of integers of a Galois extension K of $\mathbb{Q}$ and $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, then

$$\sigma(I) = \{\sigma(x) : x \in I\}$$

is also an ideal of $\mathcal{O}_K$.

Fix a prime $\mathfrak{p} \subset \mathcal{O}_K$ and write $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}$, so $S_{\mathfrak{p}} = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\}$.

Definition 9.2.1 (Residue class degree)**.** Suppose $\mathfrak{P}$ is a prime of $\mathcal{O}_K$ lying over $\mathfrak{p}$. Then the *residue class degree* of $\mathfrak{P}$ is

$$f_{\mathfrak{P}/\mathfrak{p}} = [\mathcal{O}_K/\mathfrak{P} : \mathcal{O}_L/\mathfrak{p}],$$

i.e., the degree of the extension of residue class fields.

If $M/K/L$ is a tower of field extensions and $\mathfrak{q}$ is a prime of M over $\mathfrak{P}$, then

$$f_{\mathfrak{q}/\mathfrak{p}} = [\mathcal{O}_M/\mathfrak{q} : \mathcal{O}_L/\mathfrak{p}] = [\mathcal{O}_M/\mathfrak{q} : \mathcal{O}_K/\mathfrak{P}] \cdot [\mathcal{O}_K/\mathfrak{P} : \mathcal{O}_L/\mathfrak{p}] = f_{\mathfrak{q}/\mathfrak{P}} \cdot f_{\mathfrak{P}/\mathfrak{p}},$$

so the residue class degree is multiplicative in towers.

Note that if $\sigma \in \mathrm{Gal}(K/L)$ and $\mathfrak{P} \in S_p$, then σ induces an isomorphism of finite fields $\mathcal{O}_K/\mathfrak{P} \to \mathcal{O}_K/\sigma(\mathfrak{P})$ that fixes the common subfield $\mathcal{O}_L/\mathfrak{p}$. Thus the residue class degrees of $\mathfrak{P}$ and $\sigma(\mathfrak{P})$ are the same. In fact, much more is true.

Theorem 9.2.2. *Suppose K/L is a Galois extension of number fields, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_L$. Write $\mathfrak{p}\mathcal{O}_K = \prod_{i=1}^{g} \mathfrak{P}_i^{e_i}$, and let $f_i = f_{\mathfrak{P}_i/\mathfrak{p}}$. Then $G = \mathrm{Gal}(K/L)$ acts transitively on the set $S_{\mathfrak{p}}$ of primes $\mathfrak{P}_i$, and*

$$e_1 = \cdots = e_g, \qquad f_1 = \cdots = f_g.$$

Moreover, if we let e be the common value of the e_i, f the common value of the f_i, and $n = [K : L]$, then

$$efg = n.$$

Proof. For simplicity, we will give the proof only in the case $L = \mathbb{Q}$, but the proof works in general. Suppose $p \in \mathbb{Z}$ and $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$, and $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_g\}$. We will first prove that G acts transitively on S. Let $\mathfrak{p} = \mathfrak{p}_i$ for some i. Recall

Lemma 5.2.2 which we proved long ago using the Chinese Remainder Theorem (Theorem 5.1.4). It showed there exists $a \in \mathfrak{p}$ such that $(a)/\mathfrak{p}$ is an integral ideal that is coprime to $p\mathcal{O}_K$. The product

$$I = \prod_{\sigma\in G} \sigma((a)/\mathfrak{p}) = \prod_{\sigma\in G} \frac{(\sigma(a))\mathcal{O}_K}{\sigma(\mathfrak{p})} = \frac{(\mathrm{Norm}_{K/\mathbb{Q}}(a))\mathcal{O}_K}{\displaystyle\prod_{\sigma\in G}\sigma(\mathfrak{p})} \tag{9.2.1}$$

is a nonzero integral $\mathcal{O}_K$ ideal since it is a product of nonzero integral $\mathcal{O}_K$ ideals. Since $a \in \mathfrak{p}$ we have that $\mathrm{Norm}_{K/\mathbb{Q}}(a) \in \mathfrak{p}\cap\mathbb{Z} = p\mathbb{Z}$. Thus the numerator of the rightmost expression in (9.2.1) is divisible by $p\mathcal{O}_K$. Also, because $(a)/\mathfrak{p}$ is coprime to $p\mathcal{O}_K$, each $\sigma((a)/\mathfrak{p})$ is coprime to $p\mathcal{O}_K$ as well. Thus I is coprime to $p\mathcal{O}_K$. This means the denominator of the rightmost expression in (9.2.1) must also be divisible by $p\mathcal{O}_K$ in order to cancel the $p\mathcal{O}_K$ in the numerator. Thus we have shown that for any i,

$$\prod_{j=1}^{g} \mathfrak{p}_j^{e_j} = p\mathcal{O}_K \;\Big|\; \prod_{\sigma\in G}\sigma(\mathfrak{p}_i).$$

By unique factorization, since every $\mathfrak{p}_j$ appears in the left hand side, we must have that for each j there is a σ with $\sigma(\mathfrak{p}_i) = \mathfrak{p}_j$, i.e., G acts transitively on S.

Choose some j and suppose that $k \neq j$ is another index. Because G acts transitively, there exists $\sigma \in G$ such that $\sigma(\mathfrak{p}_k) = \mathfrak{p}_j$. Applying σ to the factorization $p\mathcal{O}_K = \prod_{i=1}^g \mathfrak{p}_i^{e_i}$, we see that

$$\prod_{i=1}^{g} \mathfrak{p}_i^{e_i} = \prod_{i=1}^{g} \sigma(\mathfrak{p}_i)^{e_i}.$$

Using unique factorization, we get $e_j = e_k$. Thus $e_1 = e_2 = \cdots = e_g$.

As was mentioned right before the statement of the theorem, for any $\sigma \in G$ we have $\mathcal{O}_K/\mathfrak{p}_i \cong \mathcal{O}_K/\sigma(\mathfrak{p}_i)$. Since G acts transitively it follows that $f_1 = f_2 = \cdots = f_g$. We have, upon applying the Chinese Remainder Theorem and noting $\#(\mathcal{O}_K/(\mathfrak{p}^m)) = \#(\mathcal{O}_K/\mathfrak{p})^m$ (see Exercise 5.2.5), that

$$\begin{aligned}[][K:\mathbb{Q}] &= \dim_{\mathbb{Z}} \mathcal{O}_K = \dim_{\mathbb{F}_p} \mathcal{O}_K/p\mathcal{O}_K \\ &= \dim_{\mathbb{F}_p}\left(\bigoplus_{i=1}^{g} \mathcal{O}_K/\mathfrak{p}_i^{e_i}\right) = \sum_{i=1}^{g} e_i f_i = efg,\end{aligned}$$

which completes the proof. □

The rest of this section illustrates the theorem for quadratic fields and a cubic field and its Galois closure.

9.2.1 Special Cases

Quadratic Extensions

Suppose $K/\mathbb{Q}$ is a quadratic field. Then K is Galois, so for each prime $p \in \mathbb{Z}$ we have $2 = efg$. There are exactly three possibilities:

Ramified: $e = 2$, $f = g = 1$: The prime p *ramifies* in $\mathcal{O}_K$, which means $p\mathcal{O}_K = \mathfrak{p}^2$. Let α be a generator for $\mathcal{O}_K$ and $h \in \mathbb{Z}[x]$ a minimal polynomial for α. By Theorem 4.2.3 a prime p is ramified in $\mathcal{O}_K$ if and only if h has a double root modulo p, which is equivalent to p dividing the discriminant of h. This shows there are only finitely many ramified primes. More generally, the ramified primes are exactly the ones that divide the discriminant (see [Mar77, Thm. 24] or [NS99, Cor. III.2.12]).

Inert: $e = 1$, $f = 2$, $g = 1$: The prime p is *inert* in $\mathcal{O}_K$, which means $p\mathcal{O}_K = \mathfrak{p}$ is prime. It is a nontrivial theorem that this happens half of the time, as we will see illustrated below for a particular example.

Split: $e = f = 1$, $g = 2$: The prime p *splits* in $\mathcal{O}_K$, which means $p\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathfrak{p}_1 \neq \mathfrak{p}_2$. This happens the other half of the time.

Example 9.2.3. Let $K = \mathbb{Q}(\sqrt{5})$, so $\mathcal{O}_K = \mathbb{Z}[\gamma]$, where $\gamma = (1+\sqrt{5})/2$. Then $p = 5$ is ramified, since $5\mathcal{O}_K = (\sqrt{5})^2$. More generally, the order $\mathbb{Z}[\sqrt{5}]$ has index 2 in $\mathcal{O}_K$, so for any prime $p \neq 2$ we can determine the factorization of p in $\mathcal{O}_K$ by finding the factorization of the polynomial $x^2 - 5 \in \mathbb{F}_p[x]$. The polynomial $x^2 - 5$ splits as a product of two distinct factors in $\mathbb{F}_p[x]$ if and only if $e = f = 1$ and $g = 2$. For $p \neq 2, 5$ this is the case if and only if 5 is a square in $\mathbb{F}_p$, i.e., if $\left(\frac{5}{p}\right) = 1$, where $\left(\frac{5}{p}\right)$ is $+1$ if 5 is a square mod p and -1 if 5 is not. By quadratic reciprocity,

$$\left(\frac{5}{p}\right) = (-1)^{\frac{5-1}{2}\cdot\frac{p-1}{2}} \cdot \left(\frac{p}{5}\right) = \left(\frac{p}{5}\right) = \begin{cases} +1 & \text{if } p \equiv \pm 1 \pmod 5 \\ -1 & \text{if } p \equiv \pm 2 \pmod 5. \end{cases}$$

Thus whether p splits or is inert in $\mathcal{O}_K$ is determined by the residue class of p modulo 5. It is a theorem of Dirichlet, which was massively generalized by Chebotarev, that $p \equiv \pm 1$ half the time and $p \equiv \pm 2$ the other half the time.[1]

The Cube Root of Two

Suppose $K/\mathbb{Q}$ is not Galois. Then e_i, f_i, and g are defined for each prime $p \in \mathbb{Z}$, but we need not have $e_1 = \cdots = e_g$ or $f_1 = \cdots = f_g$. We do still have that $\sum_{i=1}^{g} e_i f_i = n$, by the Chinese Remainder Theorem. For a proof of this identity, see [Mar77, Thm. 21], or, for a slightly more general version, [NS99, Prop. I.8.2]

Consider the case where $K = \mathbb{Q}(\sqrt[3]{2})$. We know that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{2}]$. Thus $2\mathcal{O}_K = (\sqrt[3]{2})^3$, so for 2 we have $e = 3$ and $f = g = 1$.

[1] For a technical statement and proof of this theorem, see [NS99] Theorem VII.13.4.

Working modulo 5 we have

$$x^3 - 2 = (x+2)(x^2+3x+4) \in \mathbb{F}_5[x],$$

and the quadratic factor is irreducible. Thus

$$5\mathcal{O}_K = (5, \sqrt[3]{2}+2) \cdot (5, \sqrt[3]{2}^2 + 3\sqrt[3]{2} + 4).$$

Thus here $g = 2$, $e_1 = e_2 = 1$, $f_1 = 1$, and $f_2 = 2$. Thus when K is not Galois we need not have that the f_i are all equal.

9.2.2 Definitions and Terminology

In the previous sections we used words like "ramify", "inert", and "split" to describe the decomposition of a prime in an extension. This section will define the generalizations of these concepts which will be used in later sections.

Let K/L be an extension of number fields. Let $\mathcal{O}_K, \mathcal{O}_L$ denote the respective ring of integers and $\mathfrak{q}$ a prime in $\mathcal{O}_L$. By Theorem 3.2.6 we know that the ideal $\mathfrak{q}\mathcal{O}_K$ factors uniquely into a product of primes $\mathfrak{p}_i$ in $\mathcal{O}_K$ given by

$$\mathfrak{q}\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}.$$

Let f_i be the degree of the extension of residue fields, i.e.,

$$f_i = [\mathcal{O}_K/\mathfrak{p}_i : \mathcal{O}_L/\mathfrak{q}].$$

Definition 9.2.4. The prime $\mathfrak{q}$ *ramifies* in L if $e_i > 1$ for some $1 \leq i \leq g$. Otherwise $\mathfrak{q}$ is *unramified.* If $\mathfrak{q}$ is ramified and moreover $f_i = 1$ for all i, then $\mathfrak{q}$ is *totally ramified.*

Definition 9.2.5. The prime $\mathfrak{p}$ is *inert* in L if $\mathfrak{p}\mathcal{O}_L$ is prime. In this case we have $g = 1$, $\mathfrak{q}_1 = \mathfrak{p}\mathcal{O}_L$, and $e_1 = 1$.

Definition 9.2.6. The prime $\mathfrak{p}$ is *split* in L if $g > 1$. If moreover $g = [L : K]$, then $\mathfrak{p}$ *splits completely* or is *totally split.*

It will sometimes be helpful to emphasize which prime we are referring to. To do this we will use the notation $e(\mathfrak{p}/\mathfrak{q})$ to represent the power of $\mathfrak{p}$ appearing in the factorization of $\mathfrak{q}\mathcal{O}_K$. The number $e(\mathfrak{p}/\mathfrak{q})$ is called the *ramification index* of $\mathfrak{p}$ over $\mathfrak{q}$. In this notation we could write $\mathfrak{q}\mathcal{O}_K = \prod \mathfrak{p}^{e(\mathfrak{p}/\mathfrak{q})}$ where the product ranges over all primes $\mathfrak{p}$ in $\mathcal{O}_K$. We will similarly denote $f(\mathfrak{p}/\mathfrak{q})$ to be the degree of the extension of residue fields $[\mathcal{O}_K/\mathfrak{p} : \mathcal{O}_L/\mathfrak{q}]$. The number $f(\mathfrak{p}/\mathfrak{q})$ is called the *inertia degree* of $\mathfrak{p}/\mathfrak{q}$. Because the number of primes over $\mathfrak{q}$ depends on the field K, we sometimes denote g by $g_K(\mathfrak{q})$.

Exercise 9.2.7. The following are some basic properties of decompositions. For each one, compare the result with previous examples we have seen such as Example 9.2.3.

Let $K/L/\mathbb{Q}$ be a tower of number fields. Let p be a prime in $\mathbb{Z}$, $\mathfrak{q}$ a prime in $\mathcal{O}_L$ lying over p, and $\mathfrak{p}$ a prime in $\mathcal{O}_K$ lying over $\mathfrak{q}$.

(a) Show that e is multiplicative, that is $e(\mathfrak{p}/p) = e(\mathfrak{p}/\mathfrak{q}) \cdot e(\mathfrak{q}/p)$.

(b) Show that f is multiplicative, that is $f(\mathfrak{p}/p) = f(\mathfrak{p}/\mathfrak{q}) \cdot f(\mathfrak{q}/p)$.

(c) Let $g_L(p)$ be the number of primes of $\mathcal{O}_L$ lying over p. Show that $g_K(p) = \sum_{\mathfrak{q} \text{ lies over } p} g_L(\mathfrak{q})$.

Exercise 9.2.8 (See [Mar77, Ch. 4, Exercise 24])**.** Continue the notation from the previous exercise.

(a) If p it totally ramified in K then it is totally ramified in L.

(b) Let K' be another extension of L. If $\mathfrak{p}$ is totally ramified in K and unramified in K' then $K \cap K' = L$.

9.3 The Decomposition Group

Suppose K is a number field that is Galois over $\mathbb{Q}$ with group $G = \mathrm{Gal}(K/\mathbb{Q})$. Fix a prime $\mathfrak{p} \subset \mathcal{O}_K$ lying over $p \in \mathbb{Z}$.

Definition 9.3.1 (Decomposition group)**.** The *decomposition group* of $\mathfrak{p}$ is the subgroup

$$D_{\mathfrak{p}} = \{\sigma \in G : \sigma(\mathfrak{p}) = \mathfrak{p}\} \subset G.$$

Note that $D_{\mathfrak{p}}$ is the stabilizer of $\mathfrak{p}$ for the action of G on the set of primes lying over p.

It also makes sense to define decomposition groups for relative extensions K/L, but for simplicity and to fix ideas in this section we only define decomposition groups for a Galois extension $K/\mathbb{Q}$.

Let $k_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}$ denote the residue class field of $\mathfrak{p}$. In this section we will prove that there is an exact sequence

$$1 \to I_{\mathfrak{p}} \to D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p) \to 1,$$

where $I_{\mathfrak{p}}$ is the *inertia subgroup* of $D_{\mathfrak{p}}$, and $\#I_{\mathfrak{p}} = e = e(\mathfrak{p}/p)$. The most interesting part of the proof is showing that the natural map $D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p)$ is surjective. We will also discuss the structure of $D_{\mathfrak{p}}$ and introduce Frobenius elements, which play a crucial role in understanding Galois representations.

Recall from Theorem 9.2.2 that G acts transitively on the set of primes $\mathfrak{p}$ lying over p. The orbit-stabilizer theorem implies that $[G : D_{\mathfrak{p}}]$ equals the cardinality of the orbit of $\mathfrak{p}$, which by Theorem 9.2.2 equals the number g of primes lying over p, so $[G : D_{\mathfrak{p}}] = g$.

Lemma 9.3.2. *The decomposition subgroups $D_{\mathfrak{p}}$ corresponding to primes $\mathfrak{p}$ lying over a given p are all conjugate as subgroups of G.*

Proof. See Exercise 9.3.3. □

Exercise 9.3.3. Prove Lemma 9.3.2.

[*Hint*: For $\sigma, \tau \in G$ you need to show $\tau D_{\mathfrak{p}} \tau^{-1} = D_{\tau\mathfrak{p}}$. Start by writing down what it means for $\sigma \in D_{\mathfrak{p}}$ and $\tau\sigma\tau^{-1} \in D_{\tau\mathfrak{p}}$.]

The decomposition group is useful because it allows us to refine the extension $K/\mathbb{Q}$ into a tower of extensions, such that at each step in the tower we understand the splitting behavior of the primes lying over p.

Recall the correspondence between subgroups of the Galois group G and subfields of K. The fixed fields corresponding to the decomposition and inertia subgroups have an important description in terms of the splitting behavior of the prime $\mathfrak{p}$. We characterize the fixed field of $D = D_{\mathfrak{p}}$ as follows.

Proposition 9.3.4. *The fixed field*

$$K^D = \{a \in K : \sigma(a) = a \textit{ for all } \sigma \in D\}$$

of D is the smallest subfield $L \subset K$ such that the prime ideal $\mathfrak{q} = \mathfrak{p} \cap \mathcal{O}_L$ has $g_K(\mathfrak{q}) = 1$, i.e., there is a unique prime of $\mathcal{O}_K$ lying over $\mathfrak{q}$.

Proof. First suppose $L = K^D$, and note that by Galois theory $\mathrm{Gal}(K/L) \cong D$, and by Theorem 9.2.2, the group D acts transitively on the primes of K lying over $\mathfrak{q}$. One of these primes is $\mathfrak{p}$, and D fixes $\mathfrak{p}$ by definition, so there is only one prime of K lying over $\mathfrak{q}$, that is $g = 1$. Conversely, if $L \subset K$ is such that $\mathfrak{q}$ has $g = 1$, then $\mathrm{Gal}(K/L)$ fixes $\mathfrak{p}$ (since it is the only prime over $\mathfrak{q}$), so $\mathrm{Gal}(K/L) \subset D$, hence $K^D \subset L$. □

Thus p does not split in going from K^D to K—it does some combination of ramifying and staying inert. To fill in more of the picture, the following proposition asserts that p splits completely and does not ramify in $K^D/\mathbb{Q}$.

Proposition 9.3.5. *Fix a finite Galois extension K of $\mathbb{Q}$, let $\mathfrak{p}$ be a prime lying over p with decomposition group D, and set $L = K^D$ and $\mathfrak{q} = \mathfrak{p} \cap \mathcal{O}_L$. Then $e(\mathfrak{q}/p) = f(\mathfrak{q}/p) = 1$, $g_L(p) = [L : \mathbb{Q}]$, $e(\mathfrak{p}/p) = e(\mathfrak{p}/\mathfrak{q})$ and $f(\mathfrak{p}/p) = f(\mathfrak{p}/\mathfrak{q})$.*

Proof. As mentioned right after Definition 9.3.1, the orbit-stabilizer theorem implies that $g_K(p) = [G : D]$, and by Galois theory $[G : D] = [L : \mathbb{Q}]$, so $g_K(p) = [L : \mathbb{Q}]$. By Proposition 9.3.4, we have $g_K(\mathfrak{q}) = 1$ so by Theorem 9.2.2,

$$\begin{aligned} e(\mathfrak{p}/\mathfrak{q}) \cdot f(\mathfrak{p}/\mathfrak{q}) = [K : L] &= \frac{[K : \mathbb{Q}]}{[L : \mathbb{Q}]} \\ &= \frac{e(\mathfrak{p}/p) \cdot f(\mathfrak{p}/p) \cdot g_K(p)}{[L : \mathbb{Q}]} \\ &= e(\mathfrak{p}/p) \cdot f(\mathfrak{p}/p). \end{aligned}$$

Now $e(\mathfrak{p}/\mathfrak{q}) \leq e(\mathfrak{p}/p)$ and $f(\mathfrak{p}/\mathfrak{q}) \leq f(\mathfrak{p}/p)$, so we must have $e(\mathfrak{p}/\mathfrak{q}) = e(\mathfrak{p}/p)$ and $f(\mathfrak{p}/\mathfrak{q}) = f(\mathfrak{p}/p)$. Since from Exercise 9.2.7 we have $e(\mathfrak{p}/p) = e(\mathfrak{p}/\mathfrak{q}) \cdot e(\mathfrak{q}/p)$ and $f(\mathfrak{p}/q) = f(\mathfrak{p}/\mathfrak{q}) \cdot f(\mathfrak{q}/p)$, it follows that $e(\mathfrak{q}/p) = f(\mathfrak{q}/p) = 1$. □

We summarize the results of the decomposition of a prime in the tower $K \supseteq L = K^D \supseteq \mathbb{Q}$ in Table 9.3.1. This table shows the ramification indices, inertia degrees, and the number of primes at each step of the tower.

Ramification (e)	Inertia (f)	Splitting (g)	Primes	Fields
			$\mathfrak{p}$	K
$e(\mathfrak{p}/p)$	$f(\mathfrak{p}/p)$	1	$\vert$	$\vert$
			$\mathfrak{q}$	L
1	1	$[L:\mathbb{Q}]$	$\vert$	$\vert$
			p	$\mathbb{Q}$

Table 9.3.1: Decomposition in the fixed field $L = K^D$.

9.3.1 Galois groups of finite fields

Each $\sigma \in D = D_{\mathfrak{p}}$ acts in a well-defined way on the finite field $k_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}$, so we obtain a homomorphism

$$\varphi : D_{\mathfrak{p}} \to \mathrm{Aut}(k_{\mathfrak{p}}/\mathbb{F}_p).$$

We pause for a moment and review a few basic properties of extensions of finite fields. In particular, they turn out to be Galois so the map φ above is actually a map $D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p)$. The properties in this section are general properties of Galois groups for finite fields.

Definition 9.3.6. Let k be any field of characteristic p. Define $\mathrm{Frob}_p : k \to k$ to be the homomorphism given by $a \mapsto a^p$. The map Frob_p is called the *Frobenius* homomorphism.

Exercise 9.3.7.

(a) Show the map Frob_p is in fact a field homomorphism, that is $\mathrm{Frob}_p(a+b) = \mathrm{Frob}_p(a) + \mathrm{Frob}_p(b)$ and $\mathrm{Frob}_p(ab) = \mathrm{Frob}_p(a)\,\mathrm{Frob}_p(b)$.

(b) Suppose $k = \mathbb{F}_p$. Then show $\mathrm{Frob}_p = id$, i.e., $a^p = a$ for any $a \in \mathbb{F}_p$.

(c) Suppose $k = \mathbb{F}_q$ where $q = p^f$ for some $f \geq 1$. Show that $\mathrm{Frob}_p : k \to k$ is an automorphism.

(d) Continuing part (c), note that by Exercise 8.1.9 k^* is cyclic. Let $a \in k$ be a generator for k^*, so a has multiplicative order $p^f - 1$ and $k = \mathbb{F}_p(a)$. Show that

$$\mathrm{Frob}_p^n(a) = a^{p^n} = a \quad \Leftrightarrow \quad (p^f - 1) \mid p^n - 1 \quad \Leftrightarrow \quad f \mid n$$

Remark 9.3.8. Exercise 9.3.7 shows that all finite fields are *perfect*. For more on perfect fields see a standard abstract algebra text such as [DF04].

By Exercise 9.3.7(b,c) the map Frob_p is an automorphism of $k_{\mathfrak{p}}$ fixing $\mathbb{F}_p$ and hence defines an element in $\mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p)$. Let $f = f_{\mathfrak{p}/p}$ be the residue degree of $\mathfrak{p}$, i.e., $f = [k_{\mathfrak{p}} : \mathbb{F}_p]$. Exercise 9.3.7(d) shows the order of Frob_p is f. Since the order of the automorphism group of a field extension is at most the degree of the extension, we conclude that $\mathrm{Aut}(k_{\mathfrak{p}}/\mathbb{F}_p)$ is generated by Frob_p. This shows $\mathrm{Aut}(k_{\mathfrak{p}}/\mathbb{F}_p)$ has order equal to the degree $[k_{\mathfrak{p}}/\mathbb{F}_p]$ so we conclude that $k_{\mathfrak{p}}/\mathbb{F}_p$ is Galois. We summarize the discussion into the following theorem.

Theorem 9.3.9. *The extension $k_{\mathfrak{p}}/\mathbb{F}_p$ is Galois and moreover,* $\mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p)$ *is generated by the Frobenius map* Frob_p *defined by $a \mapsto a^p$.*

Exercise 9.3.10. Prove that up to isomorphism there is exactly one finite field of each degree.

[*Hint*: By Theorem 9.3.9 all elements in a finite field satisfy an equation of the form $x^{p^f} - x$ where p is the characteristic and f is the degree over the field $\mathbb{F}_p$.]

9.3.2 The Exact Sequence

Because $D_{\mathfrak{p}}$ preserves $\mathfrak{p}$, there is a natural reduction homomorphism

$$\varphi : D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p).$$

Theorem 9.3.11. *The homomorphism φ is surjective.*

Proof. Let $D = D_{\mathfrak{p}}$ and $\tilde{a} \in k_{\mathfrak{p}}$ be an element such that $k_{\mathfrak{p}} = \mathbb{F}_p(\tilde{a})$. Lift $\tilde{a}$ to an algebraic integer $a \in \mathcal{O}_K$, and let $h = \prod_{\sigma \in D}(x - \sigma(a)) \in K^D[x]$. Let $\tilde{h}$ be the reduction of h modulo $\mathfrak{p}$. Note that $h(a) = 0$ so $\tilde{h}(\tilde{a}) = 0$.

Note that the coefficients of h lie in $\mathcal{O}_{K^D}$. By Proposition 9.3.5, the residue field of $\mathcal{O}_{K^D}$ is $\mathbb{F}_p$ so $\tilde{h} \in \mathbb{F}_p[x]$. Therefore $\tilde{h}$ is a multiple of the minimal polynomial of $\tilde{a}$ over $\mathbb{F}_p$. In particular, $\mathrm{Frob}_p(\tilde{a})$ must also be a root of $\tilde{h}$. Since the roots of $\tilde{h}$ are of the form $\widetilde{\sigma(a)}$ this shows that $\widetilde{\sigma(a)} = \mathrm{Frob}(\tilde{a})$ for some $\sigma \in D$. Hence $\varphi(\sigma)(\tilde{a}) = \mathrm{Frob}(\tilde{a})$. Since elements of $\mathrm{Gal}(K_{\mathfrak{p}}/\mathbb{F}_p)$ are determined by their action on $\tilde{a}$ by choice of $\tilde{a}$, it follows that $\varphi(\sigma) = \mathrm{Frob}$ and hence φ is surjective because Frob_p generates $\mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p)$. □

Definition 9.3.12 (Inertia Group)**.** The *inertia group associated to* $\mathfrak{p}$ is the kernel $I_{\mathfrak{p}}$ of $D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p)$.

We have an exact sequence of groups

$$1 \to I_{\mathfrak{p}} \to D_{\mathfrak{p}} \to \mathrm{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p) \to 1. \qquad (9.3.1)$$

The inertia group is a measure of how p ramifies in K.

Corollary 9.3.13. *We have $\#I_{\mathfrak{p}} = e = e(\mathfrak{p}/p)$.*

Proof. The exact sequence (9.3.1) implies that $\#I_\mathfrak{p} = \#D_\mathfrak{p}/f$ where $f = f(\mathfrak{p}/p) = [k_\mathfrak{p} : \mathbb{F}_p]$. Applying Propositions 9.3.4 and 9.3.5, we have

$$\#D_\mathfrak{p} = [K : L] = \frac{[K : \mathbb{Q}]}{g} = \frac{efg}{g} = ef.$$

Dividing both sides by f proves the corollary. □

We have the following characterization of $I_\mathfrak{p}$.

Proposition 9.3.14. *Let $K/\mathbb{Q}$ be a Galois extension with group G, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ lying over a prime p. Then*

$$I_\mathfrak{p} = \{\sigma \in G : \sigma(a) \equiv a \pmod{\mathfrak{p}} \text{ for all } a \in \mathcal{O}_K\}.$$

Proof. By definition $I_\mathfrak{p} = \{\sigma \in D_\mathfrak{p} : \sigma(a) \equiv a \pmod{\mathfrak{p}} \text{ for all } a \in \mathcal{O}_K\}$, so it suffices to show that if $\sigma \notin D_\mathfrak{p}$, then there exists $a \in \mathcal{O}_K$ such that $\sigma(a) \not\equiv a \pmod{\mathfrak{p}}$. If $\sigma \notin D_\mathfrak{p}$, then $\sigma^{-1} \notin D_\mathfrak{p}$, so $\sigma^{-1}(\mathfrak{p}) \neq \mathfrak{p}$. Since both are maximal ideals, there exists $a \in \mathfrak{p}$ with $a \notin \sigma^{-1}(\mathfrak{p})$, i.e., $\sigma(a) \notin \mathfrak{p}$. Thus $\sigma(a) \not\equiv a \pmod{\mathfrak{p}}$. □

9.4 Frobenius Elements

Suppose that $K/\mathbb{Q}$ is a finite Galois extension with group G and p is a prime such that $e = 1$ (i.e., an unramified prime). Then $I = I_\mathfrak{p} = 1$ for any $\mathfrak{p} \mid p$, so the map φ of Theorem 9.3.11 is a canonical isomorphism $D_\mathfrak{p} \cong \mathrm{Gal}(k_\mathfrak{p}/\mathbb{F}_p)$. By Section 9.3.1, the group $\mathrm{Gal}(k_\mathfrak{p}/\mathbb{F}_p)$ is cyclic with canonical generator Frob_p. The *Frobenius element* corresponding to $\mathfrak{p}$ is $\mathrm{Frob}_\mathfrak{p} \in D_\mathfrak{p}$. It is the unique (see Exercise 9.4.1) element of G such that for all $a \in \mathcal{O}_K$ we have

$$\mathrm{Frob}_\mathfrak{p}(a) \equiv a^p \pmod{\mathfrak{p}}.$$

Exercise 9.4.1. With the notation above, prove that $\mathrm{Frob}_\mathfrak{p}$ is unique. That is, if σ satisfies $\sigma(a) \equiv a^p \pmod{\mathfrak{p}}$ for all $a \in \mathcal{O}_K$ then $\sigma = \mathrm{Frob}_\mathfrak{p}$.

[*Hint*: First show $\sigma \in D_\mathfrak{p}$, then argue as in the proof of Proposition 9.3.14.]

Just as the primes $\mathfrak{p}$ and decomposition groups $D_\mathfrak{p}$ are all conjugate, the Frobenius elements corresponding to primes $\mathfrak{p} \mid p$ are all conjugate as elements of G.

Proposition 9.4.2. *For each $\sigma \in G$, we have*

$$\mathrm{Frob}_{\sigma\mathfrak{p}} = \sigma \,\mathrm{Frob}_\mathfrak{p}\, \sigma^{-1}.$$

In particular, the Frobenius elements lying over a given prime are all conjugate.

Proof. Fix $\sigma \in G$. For any $a \in \mathcal{O}_K$ we have $\mathrm{Frob}_\mathfrak{p}(\sigma^{-1}(a)) - \sigma^{-1}(a)^p \in \mathfrak{p}$. Applying σ to both sides, we see that $\sigma \,\mathrm{Frob}_\mathfrak{p}(\sigma^{-1}(a)) - a^p \in \sigma\mathfrak{p}$, so $\sigma \,\mathrm{Frob}_\mathfrak{p}\, \sigma^{-1} = \mathrm{Frob}_{\sigma\mathfrak{p}}$. □

Thus the conjugacy class of $\mathrm{Frob}_{\mathfrak{p}}$ in G is a well-defined function of p. For example, if G is abelian, then $\mathrm{Frob}_{\mathfrak{p}}$ does not depend on the choice of $\mathfrak{p}$ lying over p and we obtain a well defined symbol $\left(\frac{K/\mathbb{Q}}{p}\right) = \mathrm{Frob}_{\mathfrak{p}} \in G$ called the *Artin symbol*. It extends to a homomorphism from the free abelian group on unramified primes p to G. Class field theory (for $\mathbb{Q}$) sets up a natural bijection between abelian Galois extensions of $\mathbb{Q}$ and certain maps from certain subgroups of the group of fractional ideals for $\mathbb{Z}$ (i.e., $\mathbb{Q}^*$). We have just described one direction of this bijection, which associates to an abelian extension the Artin symbol (which is a homomorphism). The Kronecker-Weber theorem asserts that the abelian extensions of $\mathbb{Q}$ are exactly the subfields of the fields $\mathbb{Q}(\zeta_n)$, as n varies over all positive integers. By Galois theory there is a correspondence between the subfields of the field $\mathbb{Q}(\zeta_n)$, which has Galois group $(\mathbb{Z}/n\mathbb{Z})^*$, and the subgroups of $(\mathbb{Z}/n\mathbb{Z})^*$. If $H \subseteq (\mathbb{Z}/n\mathbb{Z})^*$ is the subgroup corresponding to $K \subset \mathbb{Q}(\zeta_n)$ then the Artin reciprocity map $p \mapsto \left(\frac{K/\mathbb{Q}}{p}\right)$ is given by $p \mapsto [p] \in (\mathbb{Z}/n\mathbb{Z})^*/H$.

Remark 9.4.3. Notice above that the n used is not unique. That is, if K is an abelian extension of $\mathbb{Q}$ then it lies in some $\mathbb{Q}(\zeta_n)$. But then it also lies inside of $\mathbb{Q}(\zeta_{dn})$ for any positive integer d. However, a different choice of n would mean a different choice of H. Note that the quotient $(\mathbb{Z}/n\mathbb{Z})^*/H$ used is not dependent on n since it is isomorphic to the Galois group of $K/\mathbb{Q}$.

9.5 The Artin Conjecture

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is an object of central importance in number theory, and we can interpret much of number theory as the study of this group. A good way to study a group is to study how it acts on various objects, that is, to study its representations.

Endow $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the topology which has as a basis of open neighborhoods of the origin the subgroups $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, where K varies over finite Galois extensions of $\mathbb{Q}$. Fix a positive integer n and let $\mathrm{GL}_n(\mathbb{C})$ be the group of $n \times n$ invertible matrices over $\mathbb{C}$ with the discrete topology.

Warning 9.5.1. The topology on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is **not** the topology induced by taking as a basis of open neighborhoods around the origin the collection of finite-index normal subgroups of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, see [Mil14, Ch. 7] or Exercise 9.5.5. In particular, there exist nonopen normal subgroups of finite index which do not correspond to subgroups $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ for some finite Galois extension $K/\mathbb{Q}$.

Definition 9.5.2. A *complex n-dimensional representation* of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a continuous homomorphism

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C}).$$

For ρ to be continuous means that if K is the fixed field of $\mathrm{Ker}(\rho)$, then $K/\mathbb{Q}$ is a finite Galois extension. We have a diagram

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{GL}_n(\mathbb{C}) \\
& \mathrm{Gal}(K/\mathbb{Q}) \arrow[ur, hook, "\rho'"'] &
\end{tikzcd}$$

Exercise 9.5.3. Suppose $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$ is continuous. Show that the image is finite.

Remark 9.5.4. The converse to Exercise 9.5.3 is **false** in general (see Exercise 9.5.5). This is essentially the same warning as Warning 9.5.1, however it is worth pointing out to avoid mistakes.[2]

Exercise 9.5.5. Find a nonopen subgroup of index 2 in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Note this is also an example of a non-continuous homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$ with finite image.

[*Hint*: Use Zorn's lemma to show that there are homomorphisms $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \{\pm 1\}$ with finite image that are not continuous, since they do not factor through the Galois group of any finite Galois extension.]

[*Hint*: The extension $\mathbb{Q}(\sqrt{d}, d \in \mathbb{Q}^*/(\mathbb{Q}^*)^2)$ is an extension of $\mathbb{Q}$ with Galois group $X \approx \prod \mathbb{F}_2$. The index-two open subgroups of X correspond to the quadratic extensions of $\mathbb{Q}$. However, Zorn's lemma implies that X contains many index-two subgroups that do not correspond to quadratic extensions of $\mathbb{Q}$.]

Fix a Galois representation ρ and let K be the fixed field of $\ker(\rho)$, so ρ factors through $\mathrm{Gal}(K/\mathbb{Q})$. For each prime $p \in \mathbb{Z}$ that is not ramified in K, there is an element $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(K/\mathbb{Q})$ that is well-defined up to conjugation by elements of $\mathrm{Gal}(K/\mathbb{Q})$. This means that $\rho'(\mathrm{Frob}_p) \in \mathrm{GL}_n(\mathbb{C})$ is well-defined up to conjugation. Thus the characteristic polynomial $F_p(x) \in \mathbb{C}[x]$ of $\rho'(\mathrm{Frob}_p)$ is a well-defined invariant of p and ρ. Let

$$R_p(x) = x^{\deg(F_p)} \cdot F_p(1/x) = 1 + \cdots + \det(\mathrm{Frob}_p) \cdot x^{\deg(F_p)}$$

be the polynomial obtain by reversing the order of the coefficients of F_p. Following E. Artin [Art23, Art30], set

$$L(\rho, s) = \prod_{p \text{ unramified}} \frac{1}{R_p(p^{-s})}. \tag{9.5.1}$$

We view $L(\rho, s)$ as a function of a single complex variable s. One can prove that $L(\rho, s)$ is holomorphic on some right half plane, and extends to a meromorphic function on all $\mathbb{C}$.

[2] See [Kim94, pg. 1].

Conjecture 9.5.6 (Artin). *The L-function of any continuous representation*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$$

is an entire function on all $\mathbb{C}$, except possibly at 1.

This conjecture asserts that there is some way to analytically continue $L(\rho, s)$ to the whole complex plane, except possibly at 1. (A standard fact from complex analysis is that this analytic continuation must be unique.) The simple pole at $s = 1$ corresponds to the trivial representation (the Riemann zeta function), and if $n \geq 2$ and ρ is irreducible, then the conjecture is that ρ extends to a holomorphic function on all $\mathbb{C}$.

The conjecture is known when $n = 1$. Assume for the rest of this paragraph that ρ is odd, i.e., if $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is complex conjugation, then $\det(\rho(c)) = -1$. When $n = 2$ and the image of ρ in $\mathrm{PGL}_2(\mathbb{C})$ is a solvable group, the conjecture is known, and is a deep theorem of Langlands and others (see [Lan80]), which played a crucial roll in Wiles's proof of Fermat's Last Theorem. When $n = 2$ and the image of ρ in $\mathrm{PGL}_2(\mathbb{C})$ is not solvable, the only possibility is that the projective image is isomorphic to the alternating group A_5. Because A_5 is the symmetry group of the icosahedron, these representations are called *icosahedral*. In this case, Joe Buhler's Harvard Ph.D. thesis [Buh78] gave the first example in which ρ was shown to satisfy Conjecture 9.5.6. There is a book [Fre94], which proves Artin's conjecture for 7 icosahedral representation (none of which are twists of each other). Kevin Buzzard and the author proved the conjecture for 8 more examples [BS02]. Subsequently, Richard Taylor, Kevin Buzzard, Nick Shepherd-Barron, and Mark Dickinson proved the conjecture for an infinite class of icosahedral Galois representations (disjoint from the examples) [BDSBT01]. The general problem for $n = 2$ is in fact now completely solved, due to recent work of Khare and Wintenberger [KW08] that proves Serre's conjecture.

Chapter 10

Elliptic Curves, Galois Representations, and L-functions

This chapter is about elliptic curves and the central role they play in algebraic number theory. Our approach will be less systematic and more a survey than most of the rest of this book. The goal is to give you a glimpse of the forefront of research by assuming many basic facts that can be found in other books (see, e.g., [Sil92]).

10.1 Groups Attached to Elliptic Curves

Definition 10.1.1 (Elliptic Curve)**.** An *elliptic curve* over a field K is a genus one curve E defined over K equipped with a distinguished point $\mathcal{O} \in E(K)$. Here $E(K)$ is the set of all points on E defined over K.

We will not define *genus* in this book, except to note that a nonsingular curve over K has genus one if and only if over $\overline{K}$ it can be realized as a nonsingular plane cubic curve.[1] Moreover, one can show (using the Riemann-Roch formula) that over any field a genus one curve with a rational point can always be defined by a projective cubic equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

In this form the distinguished point $\mathcal{O}$ is $(X : Y : Z) = (0 : 1 : 0)$. Note that $\mathcal{O}$ is the only point on the curve with $Z = 0$. So we can consider the rest of the curve in the affine coordinates by projecting onto the affine plane defined by $Z \neq 0$. This gives the equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \qquad (10.1.1)$$

[1] For a detailed and technical explanation of genus see [Har77, Ch. II.8] or [LE06, Ch. 7.3]

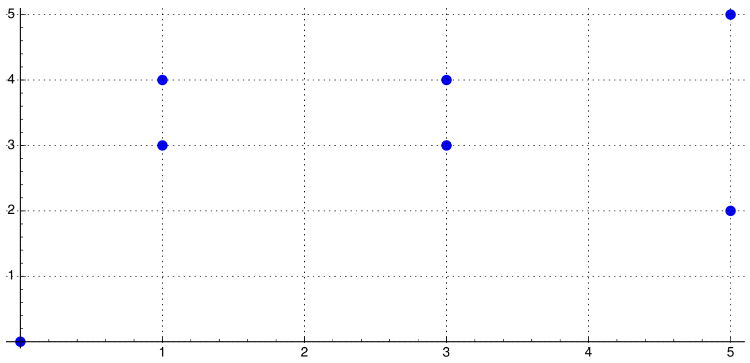
Thus one often presents an elliptic curve by giving a *Weierstrass equation* (10.1.1), though there are significant computational advantages to other equations for curves (e.g., Edwards coordinates – see work of Bernstein and Lange in [BL07]).

Using Sage we plot an elliptic curve over the finite field $\mathbb{F}_7$ and an elliptic curve defined over $\mathbb{Q}$.

```
E = EllipticCurve(GF(7), [1,0])
E
```

```
Elliptic Curve defined by y^2 = x^3 + x over
    Finite Field of size 7
```

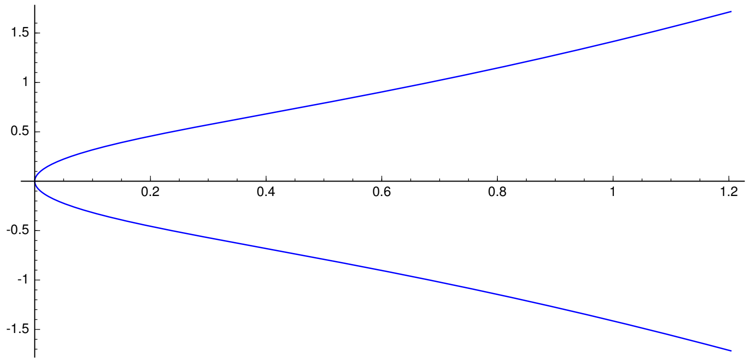
```
E.plot(pointsize=60, gridlines=True)
```



```
E = EllipticCurve([1,0])
E
```

```
Elliptic Curve defined by y^2 = x^3 + x over
    Rational Field
```

```
E.plot()
```



Note that both plots above are of the affine equation $y^2 = x^3 + x$, and do not include the distinguished point $\mathcal{O}$, which lies at infinity.

Remark 10.1.2. The command `EllipticCurve` in Sage can take as input a list `[a4,a6]` of coefficients and returns an elliptic curve given by a Weirstrass equation with $a_1 = a_2 = a_3 = 0$ and a_4, a_6 as specified.

10.1.1 Abelian Groups Attached to Elliptic Curves

If E is an elliptic curve over K, then we give the set $E(K)$ of all K-rational points on E the structure of abelian group with identity element $\mathcal{O}$.[2] If we embed E in the projective plane, then this group is determined by the condition that three points sum to the zero element $\mathcal{O}$ if and only if they lie on a common line (some care needs to be taken when the points are not distinct). In our affine picture, a line will intersect the point at infinity if it is vertical, or equivalently if it of the form $x = a$ for some fixed $a \in K$.

Example 10.1.3. On the curve $y^2 = x^3 - 5x + 4$, we have $(0, 2) + (1, 0) = (3, 4)$. This is because $(0, 2)$, $(1, 0)$, and $(3, -4)$ are on a common line (given by the equation $y = 2 - 2x$) hence they sum to zero:

$$(0, 2) + (1, 0) + (3, -4) = \mathcal{O}.$$

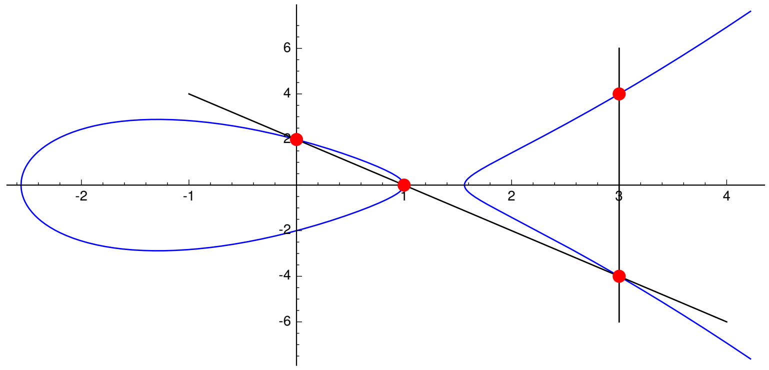
Notice $(3, 4)$, $(3, -4)$, and $\mathcal{O}$ (the point at infinity on the curve) are also on a common line (given by $x = 3$), so $(3, 4) = -(3, -4)$. We can illustration this in Sage:

```
E = EllipticCurve([-5,4])
E(0,2) + E(1,0)
```

```
(3 : 4 : 1)
```

[2] As a reminder, we will not give rigorous proofs of any facts in this section. For a more detailed and technical explanation of the group structure for elliptic curves see [Sil92, Ch. III.2].

```
G = E.plot()
G += points ([(0,2) , (1,0) , (3,4) , (3,-4)],
    pointsize=90 , color='red', zorder=10)
G += line ([(-1,4) , (4,-6)] , color='black')
G += line ([(3,-6) , (3,6)] , color='black')
G.show()
```



Iterating the group operation often leads quickly to very complicated points:

```
7*E(0,2)
```

```
(14100601873051200/48437552041038241 :
-17087004418706677845235922/10660394576906522772066289 :
 1)
```

Remark 10.1.4. In the previous example we saw that iterating the group operation led to points which used a lot of digits to write down. This notion can be made formal and is called the *height* of the point. The height function is used to prove the general Mordell-Weil theorem, see [Sil92, Ch. VIII.4]

Exercise 10.1.5. Let E be an elliptic curve given by a Weirstrass equation such as (10.1.1) with $a_1 = a_3 = 0$. Show that the points of order two are exactly the points on E with y-coordinate equal to 0.

[*Hint*: Recall that a point P has order 2 if $P + P + \mathcal{O} = \mathcal{O}$, which means the tangent line at P goes through the point at infinity.]

That the above condition—three points on a line sum to zero—defines an abelian group structure on $E(K)$ is not obvious. Depending on your perspective, the trickiest part is seeing that the operation satisfies the associative axiom. The best way to understand the group operation on $E(K)$ is to view $E(K)$ as being related to a class group. As a first observation, note that the ring

$$R = K[x, y]/(y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6))$$

is a Dedekind domain, so $\mathrm{Cl}(R)$ is defined, and every nonzero fractional ideal can be written uniquely in terms of prime ideals. When K is a perfect field, the prime ideals correspond to the Galois orbits of affine points of $E(\overline{K})$. Note that these do not include the point at infinity.

Let $\mathrm{Div}(E/K)$ be the free abelian group on the Galois orbits of points of $E(\overline{K})$, which as explained above is analogous to the group of fractional ideals of a number field (here we *do* include the point at infinity). We call the elements of $\mathrm{Div}(E/K)$ *divisors*. Let $\mathrm{Pic}(E/K)$ be the quotient of $\mathrm{Div}(E/K)$ by the *principal divisors*, i.e., the divisors associated to rational functions $f \in K(E)^*$ via

$$f \mapsto (f) = \sum_P \mathrm{ord}_P(f)[P].$$

Here $K(E)$ is the fraction field of the ring R defined above. Note that the principal divisor associated to f is analogous to the principal fractional ideal associated to a nonzero element of a number field. The definition of $\mathrm{ord}_P(f)$ is analogous to the "power of P that divides the principal ideal generated by f". Define the *class group* $\mathrm{Pic}(E/K)$ to be the quotient of the divisors by the principal divisors, so we have an exact sequence:

$$1 \to K(E)^*/K^* \to \mathrm{Div}(E/K) \to \mathrm{Pic}(E/K) \to 0.$$

A key difference between elliptic curves and algebraic number fields is that the principal divisors in the context of elliptic curves all have degree 0, i.e., the sum of the coefficients of the divisor (f) is always 0. This might be a familiar fact to you: the number of zeros of a nonzero rational function on a projective curve equals the number of poles, counted with multiplicity. If we let $\mathrm{Div}^0(E/K)$ denote the subgroup of divisors of degree 0, then we have an exact sequence

$$1 \to K(E)^*/K^* \to \mathrm{Div}^0(E/K) \to \mathrm{Pic}^0(E/K) \to 0.$$

To connect this with the group law on $E(K)$, note that there is a natural map

$$E(K) \to \mathrm{Pic}^0(E/K), \qquad P \mapsto [P - \mathcal{O}].$$

Using the Riemann-Roch theorem, one can prove that this map is a bijection, which is moreover an isomorphism of abelian groups. Thus really when we discuss the group of K-rational points on an E, we are talking about the class group $\mathrm{Pic}^0(E/K)$.

Recall that we proved (Theorem 7.1.2) that the class group $\mathrm{Cl}(\mathcal{O}_K)$ of a number field is finite. The group $\mathrm{Pic}^0(E/K) = E(K)$ of an elliptic curve can be either finite (e.g., for $y^2 + y = x^3 - x + 1$) or infinite (e.g., for $y^2 + y = x^3 - x$), and determining which is the case for any particular curve is one of the central unsolved problems in number theory.

The Mordell-Weil theorem (see Chapter 12) asserts that if E is an elliptic curve over a number field K, then there is a nonnegative integer r, referred to as the *algebraic rank of* E, such that

$$E(\mathbb{Q}) \approx \mathbb{Z}^r \oplus T, \tag{10.1.2}$$

where T is a finite group. This is similar to Dirichlet's unit theorem, which gives the structure of the unit group of the ring of integers of a number field. The main difference is that T need not be cyclic, and computing r appears to be much more difficult than just finding the number of real and complex roots of a polynomial!

Example 10.1.6. Sage has algorithms which can compute this rank for us. For example we can compute the ranks of the curves $y^2+y = x^3-x+1$ and $y^2+y = x^3-x$ respectively.

```
EllipticCurve([0,0,1,-1,1]).rank()
```

```
0
```

```
EllipticCurve([0,0,1,-1,0]).rank()
```

```
1
```

Also, if L/K is an arbitrary extension of fields, and E is an elliptic curve over K, then there is a natural inclusion homomorphism $E(K) \hookrightarrow E(L)$. Thus instead of just obtaining one group attached to an elliptic curve, we obtain a whole collection, one for each extension of L. Even more generally, if S/K is an arbitrary scheme, then $E(S)$ is a group, and the association $S \mapsto E(S)$ defines a functor from the category of schemes to the category of groups. Thus each elliptic curve gives rise to map:

$$\{\text{Schemes over } K\} \longrightarrow \{\text{Abelian Groups}\}$$

Remark 10.1.7. Elliptic curves are not the only objects that induce a functor from schemes to groups. *Abelian varieties* are a larger class of schemes, which includes elliptic curves, that also induce such a functor. For more on Abelian varieties see [Mil86].

10.1.2 A Formula for Adding Points

We close this section with an explicit formula for adding two points in $E(K)$. If E is an elliptic curve over a field K, given by an equation $y^2 = x^3 + ax + b$, then we can compute the group addition using the following algorithm.

Algorithm 10.1.8 (Elliptic Curve Group Law)**.** Given $P_1, P_2 \in E(K)$, this algorithm computes the sum $R = P_1 + P_2 \in E(K)$.

1. [One Point $\mathcal{O}$] If $P_1 = \mathcal{O}$ set $R = P_2$ or if $P_2 = \mathcal{O}$ set $R = P_1$ and terminate. Otherwise write $P_i = (x_i, y_i)$.

2. [Negatives] If $x_1 = x_2$ and $y_1 = -y_2$, set $R = \mathcal{O}$ and terminate.

3. [Compute λ] Set $\lambda = \begin{cases} (3x_1^2 + a)/(2y_1) & \text{if } P_1 = P_2, \\ (y_1 - y_2)/(x_1 - x_2) & \text{otherwise.} \end{cases}$
   Note: If $y_1 = 0$ and $P_1 = P_2$, output $\mathcal{O}$ and terminate.

4. [Compute Sum] Then $R = \left(\lambda^2 - x_1 - x_2, -\lambda x_3 - \nu\right)$, where $\nu = y_1 - \lambda x_1$ and x_3 is the x coordinate of R.

10.1.3 Other Groups

There are other abelian groups attached to elliptic curves, such as the torsion subgroup $E(K)_{\mathrm{tor}}$ of elements of $E(K)$ of finite order. The torsion subgroup is (isomorphic to) the group T that appeared in Equation (10.1.2) above). When K is a number field, there is a group called the Shafarevich-Tate group $\text{Ш}(E/K)$ attached to E, which plays a role similar to that of the class group of a number field (though it is an open problem to prove that $\text{Ш}(E/K)$ is finite in general). The definition of $\text{Ш}(E/K)$ involves Galois cohomology, so we wait until Chapter 11 to define it. There are also component groups attached to E, one for each prime of $\mathcal{O}_K$. These groups all come together in the Birch and Swinnerton-Dyer conjecture (see `http://wstein.org/books/bsd/`).

10.2 Galois Representations Attached to Elliptic Curves

Let E be an elliptic curve over a number field K. In this section we attach representations of $G_K = \mathrm{Gal}(\overline{K}/K)$ to E, and use them to define an L-function $L(E, s)$. This L-function is yet another generalization of the Riemann Zeta function, that is different from the L-functions attached to complex representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$, which we encountered before in Section 9.5.

There is a natural action of G_K on the points of $E(\overline{K})$. Given a point $P = (a, b) \in E(\overline{K})$ we define $\sigma(P)$ to be the point $(\sigma(a), \sigma(b))$. Since E is defined over K the point $\sigma(P)$ will again lie on E so the action is well defined. Note that the group structure on E is defined by algebraic formulas with coefficients in K. It follows that the action commutes with point addition meaning that $\sigma(P + Q) = \sigma(P) + \sigma(Q)$. Now fix an integer n. From what we have seen, the subgroup

$$E[n] = \{P \in E(\overline{K}) : nP = \mathcal{O}\}$$

is invariant under the action of G_K. We thus obtain a homomorphism

$$\overline{\rho}_{E,n} : G_K \to \mathrm{Aut}(E[n]).$$

Warning 10.2.1. Though the action of G_K leaves the group $E[n]$ fixed, it may act non-trivially on individual elements! Otherwise $\overline{\rho}_{E,n}$ would not be very interesting.

For any positive integer n, the group $E[n]$ is isomorphic as an abstract abelian group to $(\mathbb{Z}/n\mathbb{Z})^2$. There are various related ways to see why this is true. One is to use the Weierstrass $\wp$-theory to parametrize $E(\mathbb{C})$ by the complex numbers, i.e., to find an isomorphism $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, where Λ is a lattice in $\mathbb{C}$ and the isomorphism is given by $z \mapsto (\wp(z), \wp'(z))$ with respect to an appropriate choice of coordinates on $E(\mathbb{C})$. It is then an easy exercise to verify that $(\mathbb{C}/\Lambda)[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$. For a detailed and rigorous walk through of this method see [DS05, Ch. 1.4].

Another way to understand $E[n]$ is to use the fact that $E(\mathbb{C})_{\text{tor}}$ is isomorphic to the quotient

$$\mathrm{H}_1(E(\mathbb{C}), \mathbb{Q}) / \mathrm{H}_1(E(\mathbb{C}), \mathbb{Z})$$

of homology groups and that the homology of a curve of genus g is isomorphic to $\mathbb{Z}^{2g}$. Then we have a non-canonical isomorphism

$$E[n] \approx (\mathbb{Q}/\mathbb{Z})^2[n] = (\mathbb{Z}/n\mathbb{Z})^2.$$

Technically the previous arguments have shown $E(\mathbb{C})[n] \approx (\mathbb{Z}/n\mathbb{Z})^2$. However, our definition of $E[n]$ used points in $E(\overline{K})$. So we need to show the points $E(\mathbb{C})[n]$ are actually defined over $\overline{K}$. Note that $E(\mathbb{C})[n]$ is finite and invariant under $\mathrm{Aut}(\mathbb{C}/\overline{K})$ for the same reason as $E[n]$ was invariant under $\mathrm{Gal}(\overline{K}/K)$ (point addition is defined by algebraic formulas with coefficients in K). It follows that $E(\mathbb{C})[n]$ is indeed defined over $E(\overline{K})$ so the arguments above show that $E[n] \approx (\mathbb{Z}/n\mathbb{Z})^2$.

Remark 10.2.2. Notice that the arguments above used many analytic facts about geometry over $\mathbb{C}$ (e.g. homology, analytic structure) in order to prove algebraic facts (e.g. the number of torsion points) about $E(\overline{K})$. This is part of a more general concept called the *Lefschetz principle* which generally relates geometry over an algebraically closed field of characteristic 0 to geometry over $\mathbb{C}$. For more on this see [Sil92, Ch. VI.6].

Remark 10.2.3. In fact, if p is a prime that does not divide n then $E[n] \approx (\mathbb{Z}/n\mathbb{Z})^2$ over fields of characteristic p. However, the methods we used above do not apply to the case of positive characteristic. Another method is to show the multiplication by n map is separable and has degree n^2. For a detailed proof see [Sil92, Cor. III.6.4].

Exercise 10.2.4. Let E be an elliptic curve defined over a number field K. Fix an integer n and consider the extension of K given by

$$K(E[n]) = K(\{a, b : (a, b) \in E[n]\}).$$

Show that $K(E[n])/K$ is a finite Galois extension.

Hint: By the arguments above $\#E[n] = n^2$ which shows the extension is finite. Next recall that $E[n]$ is left invariant by the action of $\mathrm{Gal}(\overline{K}/K)$. What can you say about the embeddings from $K(E[n])$ into $\overline{K}$ which leave K fixed?

Example 10.2.5. Consider the case when $n = 2$. From Exercise 10.1.5 we know that the points in $E[2]$ are exactly the points with y-coordinate 0. Let E be the elliptic curve given by $E : y^2 = x^3 + x + 1$. If $y = 0$ then x has to be a root of the polynomial $x^3 + x + 1$, so the points in $E[2]$ are defined over the splitting field of $x^3 + x + 1$. We can compute these points in Sage.

```
E = EllipticCurve([1,1]); E
```

```
Elliptic Curve defined by y^2 = x^3 + x + 1 over
    Rational Field
```

```
R.<x> = QQ[]; R
```

```
Univariate Polynomial Ring in x over Rational Field
```

```
f = x^3 + x + 1
K.<a> = NumberField(f)
M.<b> = K.galois_closure(); M
```

```
Number Field in b with defining polynomial
    x^6 + 6*x^4 + 9*x^2 + 31
```

```
F = E.change_ring(M)
T = F.torsion_subgroup(); T
```

```
Torsion Subgroup isomorphic to Z/2 + Z/2 associated
    to the Elliptic Curve defined by y^2 = x^3 + x + 1
    over Number Field in b with defining polynomial
    x^6 + 6*x^4 + 9*x^2 + 31
```

```
T.gens()
```

```
((1/18*b^4 + 5/18*b^2 + 1/2*b + 2/9 : 0 : 1),
    (1/18*b^4 + 5/18*b^2 - 1/2*b + 2/9 : 0 : 1))
```

Note that this matches with what we expected: we computed two generators for $E[2]$ (the output of the last cell) corresponding to two generators of $(\mathbb{Z}/2\mathbb{Z})^2$.

If $n = p$ is a prime, then upon chosing a basis for the two-dimensional $\mathbb{F}_p$-vector space $E[p]$, we obtain an isomorphism $\mathrm{Aut}(E[p]) \cong \mathrm{GL}_2(\mathbb{F}_p)$. We thus obtain a mod p Galois representation

$$\overline{\rho}_{E,p} : G_K \to \mathrm{GL}_2(\mathbb{F}_p).$$

This representation $\overline{\rho}_{E,p}$ is continuous if $\mathrm{GL}_2(\mathbb{F}_p)$ is endowed with the discrete topology, because the field $K(E[p])$ is a Galois extension of K of finite degree by Exercise 10.2.4.

In order to attach an L-function to E, one could try to embed $\mathrm{GL}_2(\mathbb{F}_p)$ into $\mathrm{GL}_2(\mathbb{C})$ and use the construction of Artin L-functions from Section 9.5. Unfortunately, this approach is doomed in general, since $\mathrm{GL}_2(\mathbb{F}_p)$ frequently does not embed in $\mathrm{GL}_2(\mathbb{C})$. The following Sage session shows that for $p = 5, 7$, there are no 2-dimensional irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$, so $\mathrm{GL}_2(\mathbb{F}_p)$ does not embed in $\mathrm{GL}_2(\mathbb{C})$. The notation in the output below is `[degree of rep, number of times it occurs]`.

```
GL(2,GF(2)).gap().CharacterTable().CharacterDegrees()
```

```
[ [ 1, 2 ], [ 2, 1 ] ]
```

```
GL(2,GF(3)).gap().CharacterTable().CharacterDegrees()
```

```
[ [ 1, 2 ], [ 2, 3 ], [ 3, 2 ], [ 4, 1 ] ]
```

```
GL(2,GF(5)).gap().CharacterTable().CharacterDegrees()
```

```
[ [ 1, 4 ], [ 4, 10 ], [ 5, 4 ], [ 6, 6 ] ]
```

```
GL(2,GF(7)).gap().CharacterTable().CharacterDegrees()
```

```
[ [ 1, 6 ], [ 6, 21 ], [ 7, 6 ], [ 8, 15 ] ]
```

Instead of using the complex numbers, we use the *p-adic numbers*[3], as follows. For each power p^m of p, we have defined a homomorphism

$$\overline{\rho}_{E,p^m} : G_K \to \mathrm{Aut}(E[p^m]) \approx \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z}).$$

We combine together all of these representations (for all $m \geq 1$) using the inverse limit. Recall that the p-adic numbers are

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^m\mathbb{Z},$$

which is the set of all compatible choices of integers modulo p^m for all m. We obtain a (continuous) homomorphism

$$\rho_{E,p} : G_K \to \mathrm{Aut}(\varprojlim E[p^m]) \cong \mathrm{GL}_2(\mathbb{Z}_p),$$

where $\mathbb{Z}_p$ is the ring of p-adic integers. The composition of this homomorphism with the reduction map $\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$ is the representation $\overline{\rho}_{E,p}$, which we defined above, which is why we denoted it by $\overline{\rho}_{E,p}$. We next try to mimic the construction of $L(\rho, s)$ from Section 9.5 in the context of a p-adic Galois representation $\rho_{E,p}$.

Definition 10.2.6 (Tate module)**.** The *p-adic Tate module of E* is

$$T_p(E) = \varprojlim E[p^n].$$

Let M be the fixed field of $\ker(\rho_{E,p})$. The image of $\rho_{E,p}$ is infinite, so M is an infinite extension of K. Fortunately, one can prove that M is ramified at only finitely many primes (the primes of *bad reduction* for E and p—see [ST68]). If ℓ is a prime of K, let D_ℓ be a choice of decomposition group for some prime $\mathfrak{p}$ of M lying over ℓ, and let I_ℓ be the inertia group. We haven't defined inertia and decomposition groups for infinite Galois extensions, but the definitions are almost

[3] For a review of p-adic numbers and p-adic analysis see [Kob96].

the same: choose a prime of $\mathcal{O}_M$ over ℓ, and let D_ℓ be the subgroup of $\mathrm{Gal}(M/K)$ that leaves $\mathfrak{p}$ invariant. Then the submodule $T_p(E)^{I_\ell}$ of inertia invariants is a module for D_ℓ and the characteristic polynomial $F_\ell(x)$ of Frob_ℓ on $T_p(E)^{I_\ell}$ is well defined (since inertia acts trivially). Let $R_\ell(x)$ be the polynomial obtained by reversing the coefficients of $F_\ell(x)$. One can prove that $R_\ell(x) \in \mathbb{Z}[x]$ and that $R_\ell(x)$, for $\ell \neq p$ does not depend on the choice of p. Define $R_\ell(x)$ for $\ell = p$ using a different prime $q \neq p$, so the definition of $R_\ell(x)$ does not depend on the choice of p.

Definition 10.2.7. The L-series of E is

$$L(E,s) = \prod_\ell \frac{1}{R_\ell(\ell^{-s})}.$$

A prime $\mathfrak{p}$ of $\mathcal{O}_K$ is a prime of *good reduction* for E if there is an equation for E such that $E \mod \mathfrak{p}$ is an elliptic curve over the field $\mathcal{O}_K/\mathfrak{p}$. If $K = \mathbb{Q}$ and ℓ is a prime of good reduction for E, then one can show that that $R_\ell(\ell^{-s}) = 1 - a_\ell \ell^{-s} + \ell^{1-2s}$, where $a_\ell = \ell + 1 - \#\tilde{E}(\mathbb{F}_\ell)$ and $\tilde{E}$ is the reduction of a local minimal model for E modulo ℓ. (There is a similar statement for $K \neq \mathbb{Q}$.)

One can prove using fairly general techniques that the product expression for $L(E,s)$ defines a holomorphic function in some right half plane of $\mathbb{C}$, i.e., the product converges for all s with $\mathrm{Re}(s) > \alpha$, for some real number α.

Recall that the Artin L-function from Section 9.5 (see Equation 9.5.1) extended to meromorphic function on the entire complex plane and Artin conjectured that the L-function of any continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$ also extends to a meromorphic function on $\mathbb{C}$. We could ask the same question for the L-functions attached to elliptic curves. However, we will instead ask for something stronger:

Does the L-function $L(E,s)$ attached to an elliptic curve E extends to a holomorphic function on $\mathbb{C}$?

This question was one of the central topics in number theory in the late 1990s and early 2000s. An amazing fact is that the question has been answered in the affirmative.

Theorem 10.2.8. *The function $L(E,s)$ extends to a holomorphic function on all $\mathbb{C}$.*

This is a corollary to the modularity theorem described in the next section, see Corollary 10.2.10.

10.2.1 Modularity of Elliptic Curves over $\mathbb{Q}$

Fix an elliptic curve E over $\mathbb{Q}$. In this section we will explain what it means for E to be modular, and note the connection with Conjecture 10.2.8 from the previous section.

First, we give the general definition of modular form (of weight 2). The complex *upper half plane* is $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. A *cuspidal modular form f* of level N

(of weight 2) is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that $\lim_{z \to i\infty} f(z) = 0$ and for every integer matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant 1 and $c \equiv 0 \pmod{N}$, we have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{-2} f(z).$$

For each prime number ℓ of good reduction, let $a_\ell = \ell + 1 - \#\tilde{E}(\mathbb{F}_\ell)$. If ℓ is a prime of bad reduction let $a_\ell = 0, 1, -1$, depending on how singular the reduction $\tilde{E}$ of E is over $\mathbb{F}_\ell$. If $\tilde{E}$ has a cusp, then $a_\ell = 0$, and $a_\ell = 1$ or -1 if $\tilde{E}$ has a node; in particular, let $a_\ell = 1$ if and only if the tangents at the cusp are defined over $\mathbb{F}_\ell$.

Extend the definition of the a_ℓ to a_n for all positive integers n as follows. If $\gcd(n, m) = 1$ let $a_{nm} = a_n \cdot a_m$. If p^r is a power of a prime p of good reduction, let

$$a_{p^r} = a_{p^{r-1}} \cdot a_p \; - \; p \cdot a_{p^{r-2}}.$$

If p is a prime of bad reduction let $a_{p^r} = (a_p)^r$.

Attach to E the function

$$f_E(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i z}.$$

It is an extremely deep theorem that $f_E(z)$ is actually a cuspidal modular form, and not just some random function.

The following theorem is called the modularity theorem for elliptic curves over $\mathbb{Q}$. Before it was proved it was known as the Taniyama-Shimura-Weil conjecture.

Theorem 10.2.9 (Wiles, Brueil, Conrad, Diamond, Taylor)**.** *Every elliptic curve over $\mathbb{Q}$ is modular, i.e, the function $f_E(z)$ is a cuspidal modular form.*

Corollary 10.2.10 (Hecke)**.** *If E is an elliptic curve over $\mathbb{Q}$, then the L-function $L(E, s)$ has an analytic continuous to the whole complex plane.*

Chapter 11

Galois Cohomology

Let G be a group and suppose G acts on an abelian group A (defined below). In this chapter we will study abelian groups attached to the action of G on A. These are called *cohomology groups* and denoted by $\mathrm{H}^n(G, A)$. The theory of these groups is referred to as *group cohomology*. In the later sections G will represent the Galois group of a field extension. This is called *Galois cohomology*. Studying Galois cohomology helps us understand the structure of Galois groups such as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

11.1 Group Rings and Modules

In this section we define group modules, which are analogous to modules over a ring. For a review of the theory of modules over a ring see [DF04, Ch. 10].

Definition 11.1.1. Let G be any group. The *group ring* $\mathbb{Z}[G]$ of G is the free abelian group (equivalently the free $\mathbb{Z}$-module) on the elements of G equipped with multiplication given by the group structure on G. Note that $\mathbb{Z}[G]$ is a commutative ring if and only if G is abelian.

Example 11.1.2. For example, the group ring of the cyclic group $C_n = \langle a \rangle$ of order n is the free $\mathbb{Z}$-module on $1, a, \ldots, a^{n-1}$, and the multiplication is induced by $a^i a^j = a^{i+j} = a^{i+j \pmod{n}}$ extended linearly. For example, in $\mathbb{Z}[C_3]$ we have

$$(1+2a)(1-a^2) = 1 - a^2 + 2a - 2a^3 = 1 + 2a - a^2 - 2 = -1 + 2a - a^2.$$

Since $a^3 = 1$ you might think that $\mathbb{Z}[C_3]$ is isomorphic to the ring $\mathbb{Z}[\zeta_3]$ of integers of $\mathbb{Q}(\zeta_3)$, but you would be wrong, since the ring of integers is isomorphic to $\mathbb{Z}^2$ as an abelian group, but $\mathbb{Z}[C_3]$ is isomorphic to $\mathbb{Z}^3$ as abelian group. Note that $\mathbb{Q}(\zeta_3)$ is a quadratic extension of $\mathbb{Q}$.

Exercise 11.1.3. Is $\mathbb{Z}[\zeta_3]$ isomorphic to the group ring of some group?

Hint: Note that the rank of the group ring as a $\mathbb{Z}$-module is equal to the size of the group. If $\mathbb{Z}[\zeta_3]$ was a group ring then it would have to be isomorphic to $\mathbb{Z}[C_2]$.

Exercise 11.1.4.

(a) Write down an two elements of $\mathbb{Z}[\mathbb{Z}]$ and multiply them. This is not hard, but is good practice with the concept of a group ring.

(b) Show $\mathbb{Z}[\mathbb{Z}]$ is isomorphic to $\mathbb{Z}\left[x, \frac{1}{x}\right]$.

Definition 11.1.5. Let G be a finite group. A *G-module* is an abelian group A equipped with a left action of G, i.e., a group homomorphism $G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ denotes the group of group isomorphisms $A \to A$ with the operation of function composition.

Exercise 11.1.6. Fix an abelian group A. Show the following are equivalent sets of data. Specifically, given any one of the following objects, there is a natural way to construct another.

(a) A group homomorphism $G \to \mathrm{Aut}(A)$.

(b) A map $\rho : G \times A \to A$ such that for all $g, h \in G$ and $a, b \in A$,

(i) $\rho(g, a+b) = \rho(g, a) + \rho(g, b)$

(ii) $\rho(e, a) = a$ where e is the identity in G.

(iii) $\rho(gh, a) = \rho(g, \rho(h, a))$

(c) A ring homomorphism $\mathbb{Z}[G] \to \mathrm{End}(A)$.

(d) A map $\rho : \mathbb{Z}[G] \times A \to A$ with the same properties listed in (b).

Remark 11.1.7. In Exercise 11.1.6, part (a) is our definition of a G-module and parts (c) and (d) are the data of a $\mathbb{Z}[G]$-module. This shows that a G-module in the above sense is the same as a $\mathbb{Z}[G]$-module in the usual module sense.

Example 11.1.8. If G is any finite group and A any abelian group then we can always make A into a G-module by giving it the trivial action. In particular, $\mathbb{Z}$ with the trivial action is a module over any group G, as is $\mathbb{Z}/m\mathbb{Z}$ for any positive integer m. Another example is $G = (\mathbb{Z}/n\mathbb{Z})^*$, which acts via multiplication on $A = \mathbb{Z}/n\mathbb{Z}$.

Remark 11.1.9. The construction $\mathbb{Z}[G]$ from G is natural, in the sense that it defines a functor between categories. Moreover, $\mathbb{Z}[G]$ is the most natural way to construct a ring from a group in the sense that the group ring functor is a left adjoint to the forgetful functor from rings to groups. These types of functors are sometimes called "free" functors. If you are interested in free objects, see if you can come up with a natural way to add structure to other objects. Could you make a set into a group? How about a vector space?

11.2 Group Cohomology

Let G be a finite group and A a G-module. For each integer $n \geq 0$ there is an abelian group $\mathrm{H}^n(G, A)$ called the *nth cohomology group of G acting on A*. The general definition is somewhat complicated, but the definition for $n \leq 1$ is fairly concrete. For example, the 0*th cohomology group*

$$\mathrm{H}^0(G, A) = \{x \in A : \sigma x = x \text{ for all } \sigma \in G\} = G^A$$

is the subgroup of elements of A that are fixed by every element of G.

The *first cohomology group*

$$\mathrm{H}^1(G, A) = C^1(G, A)/B^1(G, A)$$

is the group C^1 of 1-*cocycles* modulo the group B^1 of 1-*coboundaries*, where

$$C^1(G, A) = \{f : G \to A \text{ such that } f(\sigma\tau) = f(\sigma) + \sigma f(\tau)\}$$

where the maps $f : G \to A$ range over all set-theoretic maps. If we let $f_a : G \to A$ denote the set-theoretic map $f_a(\sigma) = \sigma(a) - a$, then

$$B^1(G, A) = \{f_a : a \in A\}.$$

There are also explicit, and increasingly complicated, definitions of $\mathrm{H}^n(G, A)$ for each $n \geq 2$ in terms of *crossed homomorphisms*, which are certain maps $G \times \cdots \times G \to A$ modulo a subgroup. We will not need these maps, but for more information about them see [Cp86, Ch. IV.2].

Exercise 11.2.1. Suppose G acts trivially on A. Show that $B^1(G, A) = 0$ and $C^1(G, A) \cong \mathrm{Hom}(G, A)$. In particular, this shows $\mathrm{H}^1(G, A) \cong \mathrm{Hom}(G, A)$. Deduce that if $A = \mathbb{Z}$ then $\mathrm{H}^1(G, \mathbb{Z}) = 0$. Here $\mathrm{Hom}(G, A)$ represents the set of group homomorphisms from G to A. It comes with a natural group structure given by $(f_1 + f_2)(a) = f_1(a) + f_2(a)$.

[*Hint*: For any $\sigma \in G$ we have $f_a(\sigma) = \sigma(a) - a = a - a = 0$. Also for any finite group G, show that $\mathrm{Hom}(G, \mathbb{Z}) = 0$.]

Example 11.2.2. The groups $H^n(G, \mathbb{Z})$ and $H^n(G, \mathbb{Z}/p\mathbb{Z})$ (where p is a prime) are computable in Sage. For example we can compute $H^{10}(A_5, \mathbb{Z})$ and $H^7(A_5, \mathbb{Z}/5\mathbb{Z})$ where A_5 is the alternating group of order 120 and $\mathbb{Z}/5\mathbb{Z}$ is given the trivial A_5-module structure.

```
G = AlternatingGroup(5); G
```

```
Alternating group of order 5!/2 as a permutation group
```

```
G.cohomology(10)
```

```
Multiplicative Abelian group isomorphic to C2 x C2
```

```
G.cohomology(7,5)
```

```
Multiplicative Abelian group isomorphic to C5
```

11.2.1 The Main Theorem

Definition 11.2.3. If X is any abelian group, then $A = \mathrm{Hom}(\mathbb{Z}[G], X)$ is a G-module, see Exercise 11.2.4. We call a module constructed in this way *coinduced.*

Exercise 11.2.4. Let X be any abelian group. Show that $A = \mathrm{Hom}(\mathbb{Z}[G], X)$ is a G-module with the action induced by $(g \cdot f)(h) = f(hg)$ for all $g \in G$, $f \in \mathrm{Hom}(\mathbb{Z}[G], X)$, and $h \in \mathbb{Z}[G]$.

The following theorem gives three properties of group cohomology, which uniquely determine group cohomology.

Theorem 11.2.5. *Suppose G is a finite group. Then*

1. *We have $\mathrm{H}^0(G, A) = A^G$.*
2. *If A is a coinduced G-module, then $\mathrm{H}^n(G, A) = 0$ for all $n \geq 1$.*
3. *If $0 \to A \to B \to C \to 0$ is any exact sequence of G-modules, then there is a long exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(G,A) \arrow[r] & \mathrm{H}^0(G,B) \arrow[r] & \mathrm{H}^0(G,C) \arrow[dll, out=-10, in=170] & \\
& \mathrm{H}^1(G,A) \arrow[r] & \mathrm{H}^1(G,B) \arrow[r] & \mathrm{H}^1(G,C) \arrow[dl, out=-10, in=170] & \\
& & \cdots \arrow[dl, out=-10, in=170] & & \\
& \mathrm{H}^n(G,A) \arrow[r] & \mathrm{H}^n(G,B) \arrow[r] & \mathrm{H}^n(G,C) \arrow[dll, out=-10, in=170] & \\
& \mathrm{H}^{n+1}(G,A) \arrow[r] & \mathrm{H}^{n+1}(G,B) \arrow[r] & \mathrm{H}^{n+1}(G,C) \arrow[r] & \cdots
\end{tikzcd}$$

Moreover, the functor $\mathrm{H}^n(G, -)$ is uniquely determined by these three properties.

We will not prove this theorem. For proofs see [Cp86, Atiyah-Wall] and [Ser79, Ch. 7]. The properties of the theorem uniquely determine group cohomology, so one should in theory be able to use them to deduce anything that can be deduced about cohomology groups. Indeed, in practice one frequently proves results about higher cohomology groups $\mathrm{H}^n(G, A)$ by writing down appropriate exact sequences, using explicit knowledge of H^0, and chasing diagrams.

Remark 11.2.6. Alternatively, we could view the defining properties of the theorem as the definition of group cohomology, and could state a theorem that asserts that group cohomology exists.

Remark 11.2.7. For those familiar with commutative and homological algebra, we have

$$\mathrm{H}^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A),$$

where $\mathbb{Z}$ is the trivial G-module.

Remark 11.2.8. One can interpret $\mathrm{H}^2(G, A)$ as the group of equivalence classes of extensions of G by A, where an extension is an exact sequence

$$0 \to A \to M \to G \to 1$$

such that the induced conjugation action of G on A is the given action of G on A. (Note that G acts by conjugation, as A is a normal subgroup since it is the kernel of a homomorphism.)

11.2.2 Example Application of the Theorem

For example, let's see what we get from the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0,$$

where m is a positive integer, and $\mathbb{Z}$ has the structure of trivial G module. By definition we have $\mathrm{H}^0(G, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{H}^0(G, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$. The long exact sequence begins

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "m"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/m\mathbb{Z} \arrow[dll, out=-10, in=170] & \\
& \mathrm{H}^1(G,\mathbb{Z}) \arrow[r, "{[m]}"] & \mathrm{H}^1(G,\mathbb{Z}) \arrow[r] & \mathrm{H}^1(G,\mathbb{Z}/m\mathbb{Z}) \arrow[dll, out=-10, in=170] & \\
& \mathrm{H}^2(G,\mathbb{Z}) \arrow[r, "{[m]}"] & \mathrm{H}^2(G,\mathbb{Z}) \arrow[r] & \mathrm{H}^2(G,\mathbb{Z}/m\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

From the first few terms of the sequence and the fact that $\mathbb{Z}$ surjects onto $\mathbb{Z}/m\mathbb{Z}$, we see that $[m] : \mathrm{H}^1(G, \mathbb{Z}) \to \mathrm{H}^1(G, \mathbb{Z})$ is injective. This is consistent with Exercise 11.2.1 above that showed $\mathrm{H}^1(G, \mathbb{Z}) = 0$. Using this vanishing and the right side of the exact sequence we obtain an isomorphism

$$\mathrm{H}^1(G, \mathbb{Z}/m\mathbb{Z}) \cong \mathrm{H}^2(G, \mathbb{Z})[m]$$

where $\mathrm{H}^2(G, \mathbb{Z})[m]$ is the kernel of the map $[m] : \mathrm{H}^2(G, \mathbb{Z}) \to \mathrm{H}^2(G, \mathbb{Z})$. By Exercise 11.2.1, when a group acts trivially the H^1 is Hom, so

$$\mathrm{H}^2(G, \mathbb{Z})[m] \cong \mathrm{Hom}(G, \mathbb{Z}/m\mathbb{Z}). \tag{11.2.1}$$

One can prove that for any $n > 0$ and any module A that the group $\mathrm{H}^n(G, A)$ has exponent dividing $\#G$ (see Remark 11.3.5). Thus (11.2.1) allows us to understand $\mathrm{H}^2(G, \mathbb{Z})$, and this comprehension arose naturally from the properties in Theorem 11.2.5 that determine the cohomology groups H^n.

11.3 Inflation and Restriction

Suppose H is a subgroup of a finite group G and A is a G-module.

For each $n \geq 0$, there is a natural map

$$\mathrm{res}_H : \mathrm{H}^n(G, A) \to \mathrm{H}^n(H, A)$$

called *restriction*. Elements of $\mathrm{H}^n(G, A)$ can be viewed as classes of n-cocycles, which are certain maps $G \times \cdots \times G \to A$. From this perspective res_H takes a map to its restriction $H \times \cdots \times H \to A$. This is equivalent to precomposing with the natural inclusion $H \times \cdots \times H \to G \times \cdots \times G$.

If H is a normal subgroup of G, there is also an *inflation* map

$$\mathrm{inf}_H : \mathrm{H}^n(G/H, A^H) \to \mathrm{H}^n(G, A),$$

given by taking a cocycle $f : G/H \times \cdots \times G/H \to A^H$ and precomposing with the quotient map $G \to G/H$ to obtain a cocycle for G.

Exercise 11.3.1. Show that if A is a G-module then A^H is naturally a G/H-module for any normal subgroup H. Then give an example in which G acts non-trivially on A but the only action of G/H on A is trivial.

The following proposition will be useful when proving the weak Mordell-Weil theorem (see Theorem 12.2.3).

Proposition 11.3.2. *Suppose H is a normal subgroup of G. Then there is an exact sequence*

$$0 \to \mathrm{H}^1(G/H, A^H) \xrightarrow{\mathrm{inf}_H} \mathrm{H}^1(G, A) \xrightarrow{\mathrm{res}_H} \mathrm{H}^1(H, A).$$

Proof. Our proof follows [Ser79, pg. 117] closely.

We see that $\mathrm{res} \circ \mathrm{inf} = 0$ since on cocycles the composition is defined by precomposing with $H \to G \to G/H$, which gives the trivial map. It remains to prove that inf_H is injective and that the image of inf_H contains the kernel of res_H.

1. *(That* inf_H *is injective):* Suppose $f : G/H \to A^H$ is a cocycle whose image in $\mathrm{H}^1(G, A)$ is equivalent to 0 modulo coboundaries. Then there is an $a \in A$ such that $f(\sigma) = \sigma a - a$, where we identify f with the map $G \to A$ that is constant on the cosets of H. But f depends only on the coset of σ modulo H, so $\sigma a - a = \sigma\tau a - a$ for all $\tau \in H$, i.e., $\tau a = a$ (as we see by adding a to both sides and multiplying by σ^{-1}). Thus $a \in A^H$, so f is equivalent to 0 in $\mathrm{H}^1(G/H, A^H)$.

2. *(The image of* inf_H *contains the kernel of* res_H*):* Suppose $f : G \to A$ is a cocycle whose restriction to H is a coboundary, i.e., there is $a \in A$ such that $f(\tau) = \tau a - a$ for all $\tau \in H$. Subtracting the coboundary $g(\sigma) = \sigma a - a$ for $\sigma \in G$ from f, we may assume $f(\tau) = 0$ for all $\tau \in H$. Examing the equation

$f(\sigma\tau) = f(\sigma) + \sigma f(\tau)$ with $\tau \in H$ shows that f is constant on the cosets of H. Again using this formula, but with $\sigma \in H$ and $\tau \in G$, we see that

$$f(\tau) = f(\sigma\tau) = f(\sigma) + \sigma f(\tau) = \sigma f(\tau),$$

so the image of f is contained in A^H. Thus f defines a cocycle $G/H \to A^H$, i.e., is in the image of $\inf_H$.

□

Example 11.3.3. The sequence of Proposition 11.3.2 need not be surjective on the right. For example, suppose $H = A_3 \subset S_3$, and let S_3 act trivially on the group $\mathbb{Z}/3\mathbb{Z}$. Using the Hom interpretation of H^1, we see that $\mathrm{H}^1(S_3/A_3, \mathbb{Z}/3\mathbb{Z}) = \mathrm{H}^1(S_3, \mathbb{Z}/3\mathbb{Z}) = 0$, but $\mathrm{H}^1(A_3, \mathbb{Z}/3\mathbb{Z})$ has order 3. We can compute this example in Sage as follows.

```
S3 = SymmetricGroup(3); S3
```

```
Symmetric group of order 3! as a permutation group
```

```
S3.cohomology(1,3)
```

```
Trivial Abelian group
```

```
A3 = AlternatingGroup(3); A3
```

```
Alternating group of order 3!/2 as a permutation group
```

```
A3.cohomology(1,3)
```

```
Multiplicative Abelian group isomorphic to C3
```

Remark 11.3.4. One generalization of Proposition 11.3.2 is to a more complicated exact sequence involving the "transgression map" tr:

$$0 \to \mathrm{H}^1(G/H, A^H) \xrightarrow{\inf_H} \mathrm{H}^1(G, A) \xrightarrow{\mathrm{res}_H} \mathrm{H}^1(H, A)^{G/H} \xrightarrow{\mathrm{tr}} \mathrm{H}^2(G/H, A^H) \to \mathrm{H}^2(G, A).$$

Another generalization of Proposition 11.3.2 is that if $\mathrm{H}^m(H, A) = 0$ for $1 \leq m < n$, then there is an exact sequence

$$0 \to \mathrm{H}^n(G/H, A^H) \xrightarrow{\inf_H} \mathrm{H}^n(G, A) \xrightarrow{\mathrm{res}_H} \mathrm{H}^n(H, A).$$

For more information see [Ser79, Ch. VII.6].

Remark 11.3.5. If H is a not-necessarily-normal subgroup of G, there are also maps

$$\mathrm{cores}_H : \mathrm{H}^n(H, A) \to \mathrm{H}^n(G, A)$$

for each n. For $n = 0$ this is the trace map $a \mapsto \sum_{\sigma \in G/H} \sigma a$, but the definition for $n \geq 1$ is more involved. One has $\mathrm{cores}_H \circ \mathrm{res}_H = [\#(G/H)]$. Taking $H = 1$ this implies that for each $n \geq 1$ the group $\mathrm{H}^n(G, A)$ is annihilated by $[\#G]$.

11.4 Galois Cohomology

Suppose L/K is a finite Galois extension of fields (recall that Galois here means is normal and separable), and A is a $\mathrm{Gal}(L/K)$-module. Put

$$\mathrm{H}^n(L/K, A) = \mathrm{H}^n(\mathrm{Gal}(L/K), A).$$

Following Section 9.5, we can put a topology on $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ by taking as a basis of the origin, subgroups of the form $\mathrm{Gal}(K^{\mathrm{sep}}/L)$ where L/K is a finite Galois extension.

Exercise 11.4.1. Let H be a subgroup of $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Show that H is open if and only if H is closed and has finite index in G.

[*Hint*: If H is open then it contains a basis element N. By definition of the basis described above, N is finite index in G. What does this say about the index of H in G? What about the complement of H?]

Definition 11.4.2. Let A be a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module. We say that A is a *continuous* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module if the map $\mathrm{Gal}(K^{\mathrm{sep}}/K) \times A \to A$ (see Exercise 11.1.6) is continuous when A has the discrete topology.

Exercise 11.4.3. Let $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and A be a G-module. Show that A is a continuous G-module if and only if the subgroup $G_a = \{\sigma \in G : \sigma(a) = a\}$ is open for every $a \in A$.

Now let A be a continuous $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module. Let

$$A(L) = A^{\mathrm{Gal}(K^{\mathrm{sep}}/L)} = \{x \in A : \sigma(x) = x \text{ for all } \sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/L)\}.$$

and define

$$\mathrm{H}^n(K, A) = \varinjlim_{L/K} \mathrm{H}^n(L/K, A(L)),$$

where the limit is taken over all finite Galois extensions L/K.

It is not obvious that the groups $\mathrm{H}^n(K, A)$ are actually cohomology groups, i.e., they satisfy the conclusion of Theorem 11.2.5. However one can show they have analogous properties; see [Ser79, Ch. X.3] for references.

Remark 11.4.4. Those familiar with algebraic geometry should compare the groups $\mathrm{H}^n(K, A)$ with the Čech cohomology groups on the étale site over $\mathrm{Spec}\, K$. One can show that Čech cohomology agrees with the derived functor groups of $A \mapsto A^G$, see [Mil80, Ch. 10]. Therefore $\mathrm{H}^n(K, A)$ do indeed define a cohomology theory.

Example 11.4.5. The following are examples of continuous $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules:

$$\overline{\mathbb{Q}}, \quad \overline{\mathbb{Q}}^*, \quad \overline{\mathbb{Z}}, \quad \overline{\mathbb{Z}}^*, \quad E(\overline{\mathbb{Q}}), \quad E(\overline{\mathbb{Q}})[n], \quad \mathrm{Tate}_\ell(E),$$

where E is an elliptic curve over $\mathbb{Q}$. Can you identify the action for each module A? What about $A(L)$ for any finite Galois extension $L/\mathbb{Q}$. It is important to notice that $\overline{\mathbb{Q}}^*(L) = L^*$.

Theorem 11.4.6 (Hilbert 90). *We have* $\mathrm{H}^1(K, \overline{K}^*) = 0$.

Proof. Our proof follows [Ser79, pg. 150] closely.

Because $\mathrm{H}^1(K, \overline{K}^*) = \varinjlim_{L/K} \mathrm{H}^1(L/K, L^*)$ It suffices to prove $\mathrm{H}^1(L/K, L^*) = 0$ for every finite Galois extension L/K. Let $G = \mathrm{Gal}(L/K)$ and f be a 1-cocycle so that $f : G \to L^*$ such that $f(\sigma\tau) = f(\sigma) \cdot \sigma(f(\tau))$. Here " $\cdot$ " represents multiplication in L^*. A standard fact from Galois theory is that the elements of G are L linearly independent. Hence we can find some $c \in L$ such that

$$b = \sum_{\tau \in G} f(\tau) \cdot \tau(c) \neq 0.$$

Now apply σ to both sides to get

$$\begin{aligned}
\sigma(b) &= \sum_{\tau \in G} \sigma(f(\tau)) \cdot \sigma\tau(c) \\
&= \sum_{\tau \in G} f(\sigma)^{-1} \cdot f(\sigma\tau) \cdot \sigma\tau(c) \\
&= f(\sigma)^{-1} \cdot \sum_{\tau \in G} f(\sigma\tau) \cdot (\sigma\tau)(c) \\
&= f(\sigma)^{-1} \cdot b.
\end{aligned}$$

This shows f is a coboundary. Specifically, it shows $f = f_{b^{-1}}$ in the notation we used to define coboundaries above. □

Chapter 12

The Weak Mordell-Weil Theorem

12.1 Kummer Theory of Number Fields

Suppose K is a number field and fix a positive integer n. Let μ_n denote the nth roots of unity in $\overline{K}$ as a group under multiplication. Consider the exact sequence

$$1 \to \mu_n \to \overline{K}^* \xrightarrow{n} \overline{K}^* \to 1,$$

where n denotes the map $a \mapsto a^n$.

The corresponding long exact sequence from Theorem 11.2.5 is

$$1 \to \mu_n(K) \to K^* \xrightarrow{n} K^* \to \mathrm{H}^1(K, \mu_n) \to \mathrm{H}^1(K, \overline{K}^*) = 0,$$

where $\mu_n(K)$ is the nth roots of unity contained in K. The last equality follows from Theorem 11.4.6.

Assume now that the group μ_n is contained in K. Using Galois cohomology we obtain a relatively simple classification of all abelian extensions of K with cyclic Galois group of order dividing n. Moreover, since the action of $\mathrm{Gal}(\overline{K}/K)$ on μ_n is trivial, by our hypothesis that $\mu_n \subset K$, Exercise 11.2.1 implies

$$\mathrm{H}^1(K, \mu_n) = \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mu_n).$$

Thus we obtain an exact sequence

$$1 \to \mu_n \to K^* \xrightarrow{n} K^* \to \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mu_n) \to 1,$$

or equivalently, an isomorphism

$$K^*/(K^*)^n \cong \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mu_n).$$

By Galois theory, homomorphisms $\mathrm{Gal}(\overline{K}/K) \to \mu_n$ (up to automorphisms of μ_n) correspond to cyclic abelian extensions of K with Galois group a subgroup of the cyclic group μ_n. Unwinding the definitions, this says that every cyclic abelian extension of K of degree dividing n is of the form $K(a^{1/n})$ for some element $a \in K$.

One can prove via calculations that $K(a^{1/n})$ is unramified outside n and the primes that divide $\mathrm{Norm}(a)$. Moreover, and this is a much bigger result, one can combine this with facts about class groups and unit groups to prove the following theorem:

Theorem 12.1.1. *Suppose K is a number field with $\mu_n \subset K$, where n is a positive integer. Let L be the maximal extension of K such that*

(i) $\mathrm{Gal}(L/K)$ *is abelian,*

(ii) $n \cdot \mathrm{Gal}(L/K) = 0$*, and*

(iii) L is unramified outside a finite set S of primes.

Then L/K is of finite degree.

Sketch of Proof. Note that we may enlarge S as needed. To see why, choose a finite set $S' \supseteq S$ and let L' the maximal extension with respect to S' as in the statement of the theorem. Because L is unramified outside of S, it is certainly unramified outside of S'. By maximality of L' this implies $L \subseteq L'$. Therefore it's sufficient to show the larger extension L'/K is finite.

We first argue that we can enlarge S so that the ring

$$\mathcal{O}_{K,S} = \{a \in K^* \colon \mathrm{ord}_{\mathfrak{p}}(a\mathcal{O}_K) \geq 0 \text{ for all } \mathfrak{p} \notin S\} \cup \{0\}$$

is a principal ideal domain. One can show that for any S, the ring $\mathcal{O}_{K,S}$ is a Dedekind domain. The condition $\mathrm{ord}_{\mathfrak{p}}(a\mathcal{O}_K) \geq 0$ means that in the prime ideal factorization of the fractional ideal $a\mathcal{O}_K$, we have that $\mathfrak{p}$ occurs to a nonnegative power. Thus we are allowing denominators at the primes in S. Since the class group of $\mathcal{O}_K$ is finite, there are primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ that generate the class group as a group (for example, take all primes with norm up to the Minkowski bound). Enlarge S to contain the primes $\mathfrak{p}_i$.

Note that we have used that *the class group of $\mathcal{O}_K$ is finite.*

Next we want to show $\mathfrak{p}_i\mathcal{O}_{K,S}$ is the unit ideal. To see this, let m be the order of $\mathfrak{p}_i$ in the class group of $\mathcal{O}_K$ so that $\mathfrak{p}_i^m = (\alpha)$ for some $\alpha \in \mathcal{O}_K$. Note the factorization of $\frac{1}{\alpha}\mathcal{O}_K$ is $\mathfrak{p}_i^{-m}$ so by construction $\frac{1}{\alpha} \in \mathcal{O}_{K,S}$. Since $\alpha \in (\mathfrak{p}_i\mathcal{O}_{K,S})^m$ this shows $(\mathfrak{p}_i\mathcal{O}_{K,S})^m$ is the unit ideal. It follows from the unique factorization of ideals in the Dedekind domain $\mathcal{O}_{K,S}$ that $\mathfrak{p}_i\mathcal{O}_{K,S}$ is the unit ideal.

Now we can show $\mathcal{O}_{K,S}$ is a principal ideal domain. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_{K,S}$. Since the $\mathfrak{p}_i$ generate the class group of $\mathcal{O}_K$, the restriction of $\mathfrak{P}$ to $\mathcal{O}_K$ is equivalent modulo a principal ideal to a product of the primes $\mathfrak{p}_i$. Therefore $\mathfrak{P}$ is equivalent modulo a principal ideal to a product of ideals of the form $\mathfrak{p}_i\mathcal{O}_{K,S}$. Because we showed $\mathfrak{p}_i\mathcal{O}_{K,S}$ was the unit ideal, this means $\mathfrak{P}$ is principal.

Next enlarge S so that all primes over $n\mathcal{O}_K$ are in S. Note that $\mathcal{O}_{K,S}$ is still a PID. Let

$$K(S,n) = \{a \in K^*/(K^*)^n \colon n \mid \mathrm{ord}_{\mathfrak{p}}(a) \text{ for all } \mathfrak{p} \notin S\}.$$

Then a refinement of the arguments at the beginning of this section show that L is generated by all nth roots of the elements of $K(S,n)$ (specifically, their representatives in K). Thus it suffices to prove that $K(S,n)$ is finite.

If $a \in \mathcal{O}_{K,S}^*$ then $\mathrm{ord}_{\mathfrak{p}}(a) = 0$ for all $\mathfrak{p} \notin S$. So there is a natural map

$$\phi : \mathcal{O}_{K,S}^* \to K(S,n)$$

sending a to it's residue class in $K^*/(K^*)^n$. Suppose $a \in K^*$ is a representative of an element in $K(S,n)$. The ideal $a\mathcal{O}_{K,S}$ has a factorization which is a product of nth powers, so it is an nth power of an ideal. Since $\mathcal{O}_{K,S}$ is a PID, there is $b \in \mathcal{O}_{K,S}$ and $u \in \mathcal{O}_{K,S}^*$ such that

$$a = b^n \cdot u.$$

Thus $u \in \mathcal{O}_{K,S}^*$ maps to $[a] \in K(S,n)$. This shows ϕ is surjective.

Recall *Dirichlet's unit theorem* (Theorem 8.1.2), which asserts that the group $\mathcal{O}_K^*$ is a finitely generated abelian group of rank $r+s-1$. More generally, we now show that $\mathcal{O}_{K,S}^*$ is a finitely generated abelian group of rank $r+s+\#S-1$. Because we showed ϕ is surjective this would imply $K(S,n)$ is finitely generated. Since $K(S,n)$ is also a torsion group it must be finite which proves the theorem.

The fact that $\mathcal{O}_{K,S}^*$ has rank $r+s-1+\#S$ is sometimes referred to as the *S-unit theorem* or the *Dirichlet S-unit theorem.* To prove this theorem, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the primes in S and define a map $\phi : \mathcal{O}_{K,S}^* \to \mathbb{Z}^m$ by

$$\phi(u) = (\mathrm{ord}_{\mathfrak{p}_1}(u), \ldots, \mathrm{ord}_{\mathfrak{p}_m}(u)).$$

First we show that $\mathrm{Ker}(\phi) = \mathcal{O}_K^*$. We have that $u \in \mathrm{Ker}(\phi)$ if and only if $u \in \mathcal{O}_{K,S}^*$ and $\mathrm{ord}_{\mathfrak{p}_i}(u) = 0$ for all i; but the latter condition implies that u is a unit at each prime in S. But $u \in \mathcal{O}_{K,S}^*$ implies $\mathrm{ord}_{\mathfrak{p}}(u) = 0$ for all $\mathfrak{p} \notin S$, so it follows that $\mathrm{ord}_{\mathfrak{p}}(u) = 0$ for all primes $\mathfrak{p}$ in $\mathcal{O}_K$ and therefore $u \in \mathcal{O}_K^*$. Thus we have an exact sequence

$$1 \to \mathcal{O}_K^* \to \mathcal{O}_{K,S}^* \xrightarrow{\phi} \mathbb{Z}^m.$$

Next we show that the image of ϕ has finite index in $\mathbb{Z}^m$. Let h be the class number of $\mathcal{O}_K$. For each i there exists $\alpha_i \in \mathcal{O}_K$ such that $\mathfrak{p}_i^h = (\alpha_i)$. But $\alpha_i \in \mathcal{O}_{K,S}^*$ since $\mathrm{ord}_{\mathfrak{p}}(\alpha_i) = 0$ for all $\mathfrak{p} \notin S$ (by unique factorization). Then

$$\phi(\alpha_i) = (0, \ldots, 0, h, 0, \ldots, 0).$$

It follows that $(h\mathbb{Z})^m \subset \mathrm{Im}(\phi)$, so the image of ϕ has finite index in $\mathbb{Z}^m$. It follows that $\mathcal{O}_{K,S}^*$ has rank equal to $r+s-1+\#S$. □

12.2 Proof of the Weak Mordell-Weil Theorem

Suppose E is an elliptic curve over a number field K, and fix a positive integer n. Just as with number fields, we have an exact sequence

$$0 \to E[n] \to E \xrightarrow{n} E \to 0.$$

Then we have an exact sequence

$$0 \to E[n](K) \to E(K) \xrightarrow{n} E(K) \to \mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(K, E)[n] \to 0.$$

Note the last term comes from replacing the codomain of $\mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(K, E)$ by the kernel of $\mathrm{H}^1(K, E) \xrightarrow{n} \mathrm{H}^1(K, E)$. From this we obtain a short exact sequence

$$0 \to E(K)/nE(K) \to \mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(K, E)[n] \to 0. \qquad (12.2.1)$$

Now assume, in analogy with Section 12.1, that $E[n] \subset E(K)$, i.e., all n-torsion points are defined over K. Then the Galois action on $E[n]$ is trivial so by exercise 11.2.1 we have

$$\mathrm{H}^1(K, E[n]) = \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), E[n]) \cong \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), (\mathbb{Z}/n\mathbb{Z})^2),$$

and the sequence (12.2.1) induces an inclusion

$$E(K)/nE(K) \hookrightarrow \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), (\mathbb{Z}/n\mathbb{Z})^2). \qquad (12.2.2)$$

Explicitly, this homomorphism sends a point P to the homomorphism defined as follows: Choose $Q \in E(\overline{K})$ such that $nQ = P$; then send each $\sigma \in \mathrm{Gal}(\overline{K}/K)$ to $\sigma(Q) - Q \in E[n]$.

Exercise 12.2.1. Consider the map $E(K) \to \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), E[n])$ defined above. First show this map is well defined, i.e., $\sigma(Q) - Q \in E[n]$ for every $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Then show it does not depend on the choice of P modulo $nE(K)$ so it indeed descends to a homomorphism on $E(K)/nE(K)$.

Because $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$, given a point $P \in E(K)$, we obtain a homomorphism $\varphi : \mathrm{Gal}(\overline{K}/K) \to (\mathbb{Z}/n\mathbb{Z})^2$, whose kernel defines an abelian extension L of K that has exponent n. The amazing fact is that L can be ramified only at the primes of bad reduction for E and the primes that divide n. Thus we can apply theorem 12.1.1 to see that there are only finitely many such L.

Theorem 12.2.2. *Let $P \in E(K)$ and L be the field obtained by adjoining the coordinates of all points $Q \in E(\overline{K})$ such that $nQ = P$. Then L/K is unramified outside the set of primes dividing n and primes of bad reduction for E.*

Sketch of Proof. This sketch closely follows [Sil92, Prop. VIII.1.5b].

Fix a prime $\mathfrak{p}$ of K such that $\mathfrak{p} \nmid n$ and E has good reduction at $\mathfrak{p}$. Let $\mathfrak{q}$ be a prime of L lying over $\mathfrak{p}$. Note that $\mathfrak{q}$ is again a prime of good reduction for E since we may use the same Weierstrass equation for E as an elliptic curve over L.

First one proves that for any extension K'/K and any prime $\mathfrak{p}'$ of K' such that $\mathfrak{p}' \nmid n$ and $\mathfrak{p}'$ is a prime of good reduction for E/K', the natural reduction map $\pi : E(K')[n] \to \tilde{E}(\mathcal{O}_{K'}/\mathfrak{p}')$ is injective. The argument that π is injective uses *formal groups*, whose development is outside the scope of this course.[1]

Next, fix some $Q \in E[n]$ such that $nQ = P$. From Exercise 12.2.1 we have that $\sigma(Q) - Q \in E[n]$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Let $I_\mathfrak{q} \subset \mathrm{Gal}(L/K)$ be the inertia group for $\mathfrak{q}/\mathfrak{p}$. The action of $I_\mathfrak{q}$ is trivial on $\tilde{E}(\mathcal{O}_L/\mathfrak{q})$ so for each $\sigma \in I_\mathfrak{q}$ we have

$$\pi(\sigma(Q) - Q) = \sigma(\pi(Q)) - \pi(Q) = \pi(Q) - \pi(Q) = 0.$$

Since π is injective, it follows that $\sigma(Q) = Q$ for $\sigma \in I_\mathfrak{q}$, i.e., that Q is fixed under $I_\mathfrak{q}$. Repeating this argument for each Q implies $I_\mathfrak{q}$ is trivial and hence $\mathfrak{q}/\mathfrak{p}$ is unramified. □

Theorem 12.2.3 (Weak Mordell-Weil)**.** *Let E be an elliptic curve over a number field K, and let n be any positive integer. Then $E(K)/nE(K)$ is finitely generated.*

Proof. First suppose all elements of $E[n]$ have coordinates in K. Then the homomorphism (12.2.2) provides an injection of $E(K)/nE(K)$ into

$$\mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), (\mathbb{Z}/n\mathbb{Z})^2).$$

By Theorem 12.2.2, the image consists of homomorphisms whose kernels cut out an abelian extension of K unramified outside n and primes of bad reduction for E. Since this is a finite set of primes, Theorem 12.1.1 implies that the homomorphisms all factor through a finite quotient $\mathrm{Gal}(L/K)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Thus there can be only finitely many such homomorphisms, so the image of $E(K)/nE(K)$ is finite. Thus $E(K)/nE(K)$ itself is finite, which proves the theorem in this case.

Next suppose E is an elliptic curve over a number field, but do *not* make the hypothesis that the elements of $E[n]$ have coordinates in K. Since the group $E[n](\mathbb{C})$ is finite and its elements are defined over $\overline{\mathbb{Q}}$, the extension L of K got by adjoining to K all coordinates of elements of $E[n](\mathbb{C})$ is a finite extension. It is also Galois, as we saw when constructing Galois representations attached to elliptic curves. By Proposition 11.3.2, we have an exact sequence

$$0 \to \mathrm{H}^1(L/K, E[n](L)) \to \mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(L, E[n]).$$

The kernel of the restriction map $\mathrm{H}^1(K, E[n]) \to \mathrm{H}^1(L, E[n])$ is finite, since it is isomorphic to the finite cohomology group $\mathrm{H}^1(L/K, E[n](L))$. By the argument of the previous paragraph, the image of $E(K)/nE(K)$ in $\mathrm{H}^1(L, E[n])$ under

$$E(K)/nE(K) \hookrightarrow \mathrm{H}^1(K, E[n]) \xrightarrow{\mathrm{res}} \mathrm{H}^1(L, E[n])$$

is finite, since it is contained in the image of $E(L)/nE(L)$. Thus $E(K)/nE(K)$ is finite, since we just proved the kernel of res is finite. □

[1] For a proof using formal groups see [Sil92, Prop. VII.3.1b].

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