Worksheet: Linear Systems and the Language of Solutions

Group Name:

Part 0: Assign Roles

Before beginning, please assign the following roles to the members of your group.

The Timer: Sets the timer for silent work or reading sections.

The Leader: Gently asks a new person to speak first when the timer rings.

The Writer: Carefully records the group's final answers and submits this worksheet.

Role	**Student Name**
Timer	
Leader	
Writer	

GROUP INSTRUCTIONS: **Timer:** Set a timer for **4 minutes**.
All: Silently read Sections 1.1, 1.2, and 1.3 below.
Stop reading when you reach the "Group Discussion" box.

1 The Simplest Case: One Equation

1.1 One equation, one variable

This is the most basic problem: $2x = 6$. The solution is, of course, $x = 3$. The set of all possible solutions (the *solution space*) is just a single point on the number line: $\{3\}$.

1.2 One equation, two variables

Consider the equation $x + 2y = 5$. We cannot find a *unique* solution. For any y we choose, we can find a corresponding x. This is our first example of *free variables.*

Let's "solve" for x in terms of y: $x = 5 - 2y$. We can choose any value for y at all, and this choice will completely determine x. We call y a **free variable**. Let's set $y = t$, where t can be any real number ($t \in \mathbb{R}$). The full solution set is $(x, y) = (5 - 2t, t)$.

This is a great place to introduce vector notation. We can write the solution as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 - 2t \\ t \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

This is a powerful way to express the solution. It says that every solution is the sum of:

- A *particular solution*, $\vec{p} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ (which is the solution when $t = 0$).

- A value t multiplied by a *direction vector*, $\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

The set of all solutions forms a line in the xy-plane. Because we have **one free variable**, we say the solution space has a **dimension of one**. The vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is a **basis** (a single vector, in this case) for the solution space of the related *homogeneous* equation $x + 2y = 0$. Using **set-builder** notation, we can write the set of infinitely many solutions (one for every value of $t \in \mathbb{R}$) as:

$$\left\{ \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \ : \ t \in \mathbb{R} \right\}.$$

1.3 One equation, three variables

Let's look at $x + 2y - z = 4$. Now, we can "solve" for x: $x = 4 - 2y + z$. To find a solution, we must make *two* choices. We need to choose values for both y and z. This means we have **two free variables**.

Let $y = s$ and $z = t$, where $s, t \in \mathbb{R}$. Then $x = 4 - 2s + t$. The solution set is $(x, y, z) = (4 - 2s + t, s, t)$.

In vector notation, this looks even better:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2s + t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Again, this has the form $\vec{x} = \vec{p} + s\vec{v}_1 + t\vec{v}_2$, which you should imagine as a 2D plane passing through the point $\vec{p}$, and you can move around in that plane using the vectors $\vec{v}_1$ and $\vec{v}_2$.

- $\vec{p} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ is one particular solution (found by setting $s = 0, t = 0$).

- $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ is the solution we get by setting $s = 1$ and $t = 0$ (ignoring $\vec{p}$).

- $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is the solution we get by setting $s = 0$ and $t = 1$ (ignoring $\vec{p}$).

The set of all solutions forms a plane in 3D space. Because we have **two free variables**, we say the solution space has a **dimension of two**. The set of vectors $\{\vec{v}_1, \vec{v}_2\}$ forms a **basis** for the 2D solution space of the homogeneous equation $x + 2y - z = 0$.

GROUP INSTRUCTIONS: **Leader:** Begin a group discussion about the reading above. What were the "Key Takeaways" or any "Tricky Points"?
Writer: Record the summary of your discussion below.
Timer: When writing is complete, set a new timer for **5 minutes** for the exercise in Section 1.4.

Group Notes (Key Takeaways / Tricky Points):

1.4 Exercise: One equation, three variables

Task: Work together to find the solution space for the following equation:

$$2x - y + 3z = 6$$

Follow the method used in Section 1.3. Express your final answer in the vector form $\vec{p} + s\vec{v}_1 + t\vec{v}_2$.

Writer: Record the group's work below.

Number of free variables: ________
Dimension of solution space: ________

GROUP INSTRUCTIONS: **Timer:** Set a timer for **4 minutes**.
All: Silently read Sections 2.0 through 2.4.
Stop reading when you reach the next "Group Discussion" box.

2 Systems of Two Equations: Introducing Elimination

Things get more interesting when equations interact. The key tool you've used before is *substitution* (which is one way to perform *elimination*).

2.1 Two equations, one variable

This is an *overdetermined* system.

$$\begin{aligned} 2x &= 6 \\ 3x &= 7 \end{aligned}$$

The first equation demands $x = 3$. The second demands $x = 7/3$. These are contradictory. There is no value of x that can satisfy both. The **solution space is empty**. As a set, we write the empty set as $\{\}$ or sometimes $\emptyset$.

What if the system was $2x = 6$ and $4x = 12$? The second equation ($4x = 12$) is a multiple of the first. It's *redundant* and provides no new information. The system is equivalent to the 1x1 case $2x = 6$, and the solution space is just a single point $\{3\}$ on the number line.

2.2 Two equations, two variables

This is the high school classic. Three things can happen.

- **Case 1: Unique Solution**

$$\begin{aligned} x + y &= 3 \quad (E_1) \\ x - y &= 1 \quad (E_2) \end{aligned}$$

We can use *substitution*. From Equation (E_1), we solve for x: $x = 3 - y$. Now substitute this into Equation (E_2):

$$(3 - y) - y = 1 \implies 3 - 2y = 1$$

This gives $2y = 2$, so $y = 1$. Substituting $y = 1$ back into $x = 3 - y$ gives $x = 3 - 1 = 2$. The solution is a unique point, $(2, 1)$ in the plane. The solution space has **zero free variables** so we say its **dimension is zero**, even though it is nonempty.

- **Case 2: No Solution**

$$\begin{aligned} x + y &= 3 \quad (E_1) \\ x + y &= 1 \quad (E_2) \end{aligned}$$

Let's use substitution. From (E_1), $x = 3 - y$. Substitute this into (E_2):

$$(3 - y) + y = 1 \implies 3 = 1$$

This is a contradiction. The equations are inconsistent. The **solution space is empty**. We don't give it a dimension. As a set the empty set is $\emptyset$ or $\{\}$. (Geometrically, these are two parallel, non-intersecting lines).

- **Case 3: Infinite Solutions**

$$\begin{aligned} x + y &= 3 \quad (E_1) \\ 2x + 2y &= 6 \quad (E_2) \end{aligned}$$

Let's use substitution. From (E_1), $x = 3 - y$. Substitute this into (E_2):

$$2(3 - y) + 2y = 6 \implies 6 - 2y + 2y = 6 \implies 6 = 6$$

This identity $6 = 6$ (or $0 = 0$) tells us that E_2 was redundant all along (it's just E_1 in disguise). We are really back in the 1x2 case, $x + y = 3$. We have **one free variable**. Let $y = t$. Then $x = 3 - t$. The solution space is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 - t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This is a line, and the solution space has **dimension one**.

2.3 Two equations, three variables

We have more variables than equations, so we expect free variables.

$$\begin{aligned} x + y + z &= 5 \quad (E_1) \\ x - y + 2z &= 2 \quad (E_2) \end{aligned}$$

Let's systematically *eliminate* x using substitution. From Equation (E_1), we can express x as $x = 5 - y - z$. Now, we substitute this expression into Equation (E_2):

$$(5 - y - z) - y + 2z = 2$$

Simplifying this gives:

$$5 - 2y + z = 2 \implies 3 = 2y - z \quad \text{or} \quad 2y - z = 3$$

We have reduced the 2x3 system to a single equation with y and z. We know how to solve this! This is a 1x2 problem. Let z be our free variable. Let $z = t$. Then $2y = 3 + z = 3 + t \implies y = \frac{3}{2} + \frac{1}{2}t$. Now, substitute both back into E_1 to find x: $x = 5 - y - z = 5 - \left(\frac{3}{2} + \frac{1}{2}t\right) - t = \frac{7}{2} - \frac{3}{2}t$. We have **one free variable** (t), so the **dimension is one**. The solution is a line in 3D space:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7/2 - (3/2)t \\ 3/2 + (1/2)t \\ t \end{pmatrix} = \begin{pmatrix} 7/2 \\ 3/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3/2 \\ 1/2 \\ 1 \end{pmatrix}$$

2.4 Two equations, four variables

This is just an extension.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4 \quad (E_1) \\ x_1 + 2x_2 - x_3 &= 2 \quad (E_2) \end{aligned}$$

Let's eliminate x_1 using substitution. From (E_1), we have $x_1 = 4 - x_2 - x_3 - x_4$. We substitute this into (E_2):

$$(4 - x_2 - x_3 - x_4) + 2x_2 - x_3 = 2$$

Simplifying this gives:

$$4 + x_2 - 2x_3 - x_4 = 2 \implies x_2 = 2x_3 + x_4 - 2$$

We are left with one equation that connects x_2, x_3, and x_4. This means we'll have **two free variables**. Let $x_3 = s$ and $x_4 = t$. Now, solve for x_2: $x_2 = 2s + t - 2$. Finally, substitute back into our expression for x_1 (or into E_2, which is simpler): $x_1 = 2 - 2x_2 + x_3 = 2 - 2(2s + t - 2) + s = 2 - 4s - 2t + 4 + s = 6 - 3s - 2t$. The solution space has **dimension two**. In vector notation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 - 3s - 2t \\ -2 + 2s + t \\ s \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 6 \\ -2 \\ 0 \\ 0 \end{pmatrix}}_{\vec{p}} + s \underbrace{\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_1} + t \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_2}$$

GROUP INSTRUCTIONS: Leader: Begin a group discussion about Sections 2.0 to 2.4. What were the "Key Takeaways" or any "Tricky Points"?
Writer: Record the summary of your discussion below.
Timer: When writing is complete, set a new timer for **3 minutes** for the exercise in Section 2.5.

Group Notes (Key Takeaways / Tricky Points):

2.5 Exercise: Two equations, four variables

Task: Find the solution space for the following system.

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

Use elimination (substitution) to remove x_1 from the second equation. Express x_2 in terms of free variables, then go back and find x_1. Express your final answer as a vector sum.

Writer: Record the group's work below.

The free variables are: ____________________
Dimension of solution space: ____________________

GROUP INSTRUCTIONS: Leader: Begin a discussion to verify the solution to Section 2.5 above.
Writer: Ensure the final vector solution is clearly boxed in the space above.

3 Looking Ahead

In these notes, we have informally seen all the key ideas:

- A system can have a unique solution (0-dim space), no solution (empty space), or infinitely many solutions (a k-dim space).

- **Elimination** is the systematic process of simplifying a system.
- The number of **free variables** you must choose determines the **dimension** of the solution space.
- The general solution $\vec{x}$ can always be written as $\vec{x} = \vec{p} + \vec{x}_h$, where $\vec{p}$ is one particular solution and $\vec{x}_h$ is the general solution to the corresponding homogeneous system.

A Solutions to Group Exercises

Solution to 1.4: One equation, three variables

Equation: $2x - y + 3z = 6$.

We choose to solve for y because it has a coefficient of -1, which avoids creating fractions.

$$y = 2x + 3z - 6$$

Since y is determined by x and z, the other two variables are free. Let $x = s$ and $z = t$. Then $y = 2s + 3t - 6$.

Vector Form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ 2s + 3t - 6 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

- **Number of free variables:** 2

- **Dimension of solution space:** 2

Solution to 2.5: Two equations, four variables

System:

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 2 \quad (E_1) \\ x_1 + 2x_2 + x_3 &= 5 \quad (E_2) \end{aligned}$$

Step 1: Elimination. Subtract (E_1) from (E_2) to eliminate x_1:

$$(x_1 + 2x_2 + x_3) - (x_1 + x_2 - x_3 + x_4) = 5 - 2$$

$$x_2 + 2x_3 - x_4 = 3$$

Step 2: Solve for pivot variables. From the new equation, solve for x_2:

$$x_2 = 3 - 2x_3 + x_4$$

Substitute this back into (E_1) to find x_1:

$$x_1 + (3 - 2x_3 + x_4) - x_3 + x_4 = 2$$

$$x_1 + 3 - 3x_3 + 2x_4 = 2$$

$$x_1 = -1 + 3x_3 - 2x_4$$

Step 3: Free variables. The variables x_3 and x_4 appear on the right side, so they are free. Let $x_3 = s$ and $x_4 = t$.

Vector Form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 + 3s - 2t \\ 3 - 2s + t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

- **Number of free variables:** 2

- **Dimension of solution space:** 2