Modal Logics from Categories of Coalgebras

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Overview

- A Brief Review of Coalgebras
- Coalgebraic Semantics of Modal Logic
 (Stone Coalgebras, with C. Kupke and Y. Venema, Amsterdam)
- Modal Logics from Categories of Coalgebras (with M. Bonsangue, Leiden)

Part I: A Brief Review of Coalgebras

$\mathsf{Coalg}(T)$

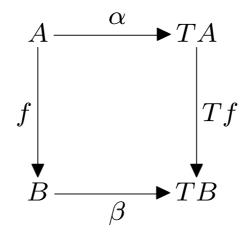
Functor

$$T: \mathcal{X} \to \mathcal{X}$$

Coalgebra $A = (A, \alpha)$

$$A \xrightarrow{\alpha} TA$$

Morphism $(A, \alpha) \xrightarrow{f} (B, \beta)$



Example: Kripke Frames

Functor

$$T: \mathsf{Set} \to \mathsf{Set}$$

$$X \mapsto \mathcal{P}X$$

Kripke frame

$$X \xrightarrow{\xi} \mathcal{P}X$$

for
$$f: X \to Y$$

$$Tf: \mathcal{P}X \to \mathcal{P}Y$$
 , $X \supseteq Z \mapsto f(Z)$

Example: Kripke Models

A set of propositional variables Prop, a two-element set $2 = \{0, 1\}$

Functor

$$TX = \mathcal{P}X \times \prod_{\mathsf{Prop}} 2$$

Coalgebra

$$X \xrightarrow{\langle \xi, v \rangle} \mathcal{P}X imes \prod_{\mathsf{Prop}} 2$$
 $\xi: X o \mathcal{P}X$ $v(x)_p \in 2 \quad \text{for } x \in X, \; p \in \mathsf{Prop}$

Example: Labelled Transitions Systems

$$T:\mathsf{Set}\to\mathsf{Set}$$

$$TX = \mathcal{P}(Act \times X)$$

Process [Milner, 1983]: initial state, equality is bisimulation.

Processes are precisely the elements in the final coalgebra [Aczel, 1988].

Behavioural Equivalence

Given a functor $T: \mathsf{Set} \to \mathsf{Set}$, behavioural equivalence (bisimulation) for T-coalgebras is defined as follows.

Definition: Let \sim be the smallest equivalence relation such that for all morphisms $f: A \to B$

$$(\mathsf{A},a) \sim (\mathsf{B},f(a)).$$

If $(A, a) \sim (B, b)$ we say (A, a) and (B, b) are behaviourally equivalent.

Final Coalgebras

Definition: Z is the **final** T-coalgebra iff for all T-coalgebras. A there is a unique morphism

$$!_A:A \rightarrow Z.$$

Fact: (A, a), (B, b) are behaviourally equivalent iff

$$!_{\mathsf{A}}(a) = !_{\mathsf{B}}(b).$$

Examples:

| Coalgebras | Carrier Z of the Final Coalgebra |
|---|---|
| $A \longrightarrow O \times A$ | $O^{\mathbb{N}}$ |
| $A \longrightarrow 2 \times A^I$ | all languages $L\subseteq I^*$ |
| $A \longrightarrow \mathcal{P}A$ | Aczel's universe of non-well founded sets |
| $A \longrightarrow \mathcal{P}(Act \times A)$ | equivalence classes of processes up to bisimulation |

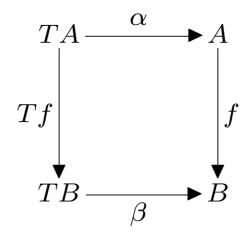
Functor

$$T: \mathcal{X} \to \mathcal{X}$$

Algebra $A = (A, \alpha)$

$$TA \xrightarrow{\alpha} A$$

Morphism $(A, \alpha) \xrightarrow{f} (B, \beta)$



Duality of Algebras and Coalgebras

- $Coalg(T) \rightarrow Set$ is dual to $Alg(T^{op}) \rightarrow Set^{op}$.
- Why algebras for signatures and coalgebras for functors?
 - For $T: \mathsf{Set} \to \mathsf{Set}$, $\mathsf{Alg}(T)$ can be described by operations and equations.
 - Every category of coalgebras can be described by a signature of co-operations and equations, but these descriptions seem less natural.
 For example, even for polynomial functors one may need a proper class of operations and equations.
 - Technically: Finite sets are dense in Set but they are codense in Set only if there do not exist arbitrarily large measurable cardinals.
- Modal logic is dual to equational logic.

Part II: Coalgebraic Semantics of Modal Logic

or

Stone Coalgebras

(with C. Kupke and Y. Venema, CMCS'03)

Modal Logic

The modal logic K is given as follows.

Syntax:
$$\varphi ::= \top \mid \neg \varphi \mid \varphi \wedge \varphi \mid \Box \varphi \mid p$$
 , $p \in \mathsf{Prop}$

Axioms and Rules:

propositional logic

$$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$$

from φ derive $\Box \varphi$

Modal Algebras

 $(A, \top, \neg, \wedge, \Box)$ is a modal algebra if $(A, 1, \neg, \wedge)$ is a Boolean algebra and

$$\Box(a \wedge b) = \Box a \wedge \Box b$$
$$\Box \top = \top$$

Modal formulae and algebraic equations correspond via

$$\varphi \qquad \qquad \varphi = \top$$

$$t \leftrightarrow t' \qquad \qquad t = t'$$

$$\vdash_{\mathbf{K}} \varphi \quad \Leftrightarrow \quad \models_{\mathsf{MA}} \varphi = \top$$
 $\vdash_{\mathbf{K}} t \leftrightarrow t' \quad \Leftrightarrow \quad \models_{\mathsf{MA}} t = t'$

Kripke Semantics

Kripke model: (X, R, v) where $R \subseteq X \times X$, $v : X \to \prod_{\mathsf{Prop}} 2$

$$X, R, v, x \models p \quad \Leftrightarrow \quad v(x)_p = 1$$

 $X, R, v, x \models \Box \varphi \quad \Leftrightarrow \quad \forall y \in X . \ xRy \Rightarrow X, R, v, y \models \varphi$

Kripke frame: (X, R)

$$(X,R) \models \varphi \Leftrightarrow X,R,v,x \models \varphi \text{ for all } v,x$$

Kripke Semantics is Incomplete

There is a consistent normal modal logic \mathcal{L} whose class of Kripke frames is empty, that is,

$$\forall (X,R) . (X,R) \not\models \mathcal{L}$$

General Frames

(X, R, A) is a general frame

if $A \subseteq \mathcal{P}X$ contains X and A is closed under Boolean operations and

$$\Box_R : \mathcal{P}X \longrightarrow \mathcal{P}X$$

$$a \mapsto \{x \in X \mid \forall y . xRy \Rightarrow y \in a\}$$

Note: $(A, X, (\cdot)^c, \cap, \square_R)$ is a modal algebra

From Modal Algebras to General Frames

Let $(A, \top, \neg, \wedge, \square)$ be a modal algebra.

For a Boolean algebra A there is (X, \widehat{A}) with $\widehat{A} \subseteq \mathcal{P}X$ such that

$$\begin{array}{ccc} A & \stackrel{\cong}{\longrightarrow} & \widehat{A} \\ a & \mapsto & \widehat{a} \end{array}$$

Define

$$xR_Ay \Leftrightarrow [\forall a \in A . x \in \widehat{\Box a} \Rightarrow y \in \widehat{a}].$$

Then (X, R_A, \widehat{A}) is a general frame.

Descriptive General Frames

(X, R, A) is a descriptive general frame if

- $\bullet \ x \neq y \Rightarrow \exists \text{ disjoint } a, b \in A . x \in a \text{ and } y \in b$
- If $B \subseteq A$ and for all finite $B' \subseteq B$ holds $\bigcap B' \neq \emptyset$ then $\bigcap B \neq \emptyset$.
- $R[x] = \bigcap \{a \mid x \in \square_R a\}$

Coalgebras over Stone Spaces

Stone space: Hausdorff, compact, basis of clopens

The analog of the powerset for Stone spaces is the *Vietoris functor*.

$$\mathbb{V}: \mathsf{Stone} \longrightarrow \mathsf{Stone}$$

$$(X, \tau) \mapsto (\{W \subseteq X \mid W \; \mathsf{closed}\}, \mathbb{V}\tau)$$

where $V\tau$ is the topology generated by the sets

$$\{F \mid F \text{ closed}, F \subseteq U\}$$
 $\{F \mid F \text{ closed}, F \cap U \neq \emptyset\}$

for all opens $U \in \tau$.

Descriptive General Frames are Coalgebras

Theorem: $DGF \cong Coalg(V)$

Proof:

$$(X,R,A) \qquad \Longleftrightarrow \qquad (X,\tau) \xrightarrow{\xi} (\mathbb{V}X,\mathbb{V}\tau)$$

$$A \text{ is the set of clopens of } \tau \qquad \tau \text{ is the topology generated by } A$$

$$R[x] \qquad = \qquad \xi(x)$$

$$A \text{ is closed under } \square_R \qquad \Leftrightarrow \qquad \xi \text{ is continuous}$$

$$f: X \to Y \text{ general frame morphism} \qquad \Leftrightarrow \qquad f: X \to Y \text{ coalgebra morphism}$$

Descriptive General Frames are Coalgebras (cont'd)

For a coalgebra (X, ξ) a valuation is a function $v: X \to \prod_{\mathsf{Prop}} 2$.

Prop: Let *X* be a Stone space.

 $v: X \to \prod_{\mathsf{Prop}} 2$ is continuous iff $\{x \in X \mid v(x)_p = 1\}$ is clopen for all p.

Proof: $\{x \in X \mid v(x)_p = 1\} = v^{-1}(\pi_p^{-1}(\{1\})).$

Positive Modal Logic

$$\textit{Syntax:} \qquad \varphi ::= \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box \varphi \mid \Diamond \varphi \mid p \quad , \quad p \in \mathsf{Prop}$$

Duality: DistLat^{op}
$$\simeq$$
 Spectral (\cong Priestley)

Vietoris Functor:
$$\mathbb{W}$$
: Priestley \longrightarrow Priestley $(X,\tau) \mapsto (\{W \subseteq X \mid W \text{ closed and convex}\}, \mathbb{V}\tau)$

Theorem (Palmigiano, CMCS'03)

Coalg(W) is dual to the category of positive modal algebras.

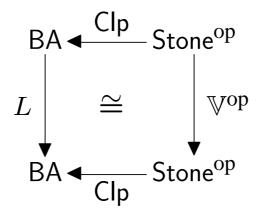
Part III: Modal Logics from Categories of Coalgebras

Dualising the Vietoris Functor on Stone Spaces

Coalg(\mathbb{V}) is dual to Alg(\mathbb{V}^{op}).

Can we describe \mathbb{V}^{op} 'duality-free' (algebraic, localic)?

That is, we want to describe $L: BA \rightarrow BA$ such that



Note that then $Alg(L) \simeq Alg(\mathbb{V}^{op})$.

Dualising the Vietoris Functor on Stone Spaces (cont'd)

$$L:\mathsf{BA}\to\mathsf{BA}$$

LA is the free Boolean algebra generated by

$$\Box a, a \in A$$

and satisfying the relations

$$\Box(a \wedge b) = \Box a \wedge \Box b$$
$$\Box \top = \top$$

The General Picture

Given: concrete duality $\Omega: \mathcal{C}^{op} \xrightarrow{\simeq} \mathcal{D}$, functor $T: \mathcal{C} \to \mathcal{C}$.

Describe the dual $L: \mathcal{D} \to \mathcal{D}$ of T by generators and relations.

$$X \xrightarrow{!_X} Z$$
 $\Omega Z \xleftarrow{i} I_L \xleftarrow{[\cdot]_{\equiv}} I_F$

where Z final in $\operatorname{Coalg}(T)$, I_L initial in $\operatorname{Alg}(L)$, I_F the algebra of all formulae

For a T-coalgebra (X,ξ) and x in X and φ in I_F define

$$x \models \varphi \quad \Leftrightarrow \quad !_X(x) \in i([\varphi]_{\equiv})$$

Summary

I want to emphasise the following two points:

- The modal logic for T-coalgebras is given by the Stone dual L of T.
 Soundness, completeness, expressivity and invariance of formulas under bisimulation are guaranteed by construction.
- For example, taking the Vietoris functor on Stone spaces we obtain classical modal logic, on spectral spaces we obtain positive modal logic.
- The duality of modal algebras and descriptive general frames is an instance of the duality of algebras and coalgebras for a functor.

Acknowledgements

- Peter Johnstone. Vietoris locales and localic semilattices. In R.-E. Hoffmann and K.H. Hofmann, editors, Continuous Lattices and their Applications, volume 101 of Lecture Notes in Pure and Applied Mathematics, pages 155–180. Marcel Dekker, 1985.
- Samson Abramsky. Domain theory in logical form. *Annals of Pure and Applied Logic*, 51:1–77, 1991.