Modal Logics from Categories of Coalgebras

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Overview

- A Brief Review of Coalgebras
- Coalgebraic Semantics of Modal Logic

  (Stone Coalgebras, with C. Kupke and Y. Venema, Amsterdam)
- Modal Logics from Categories of Coalgebras

  (with M. Bonsangue, Leiden)

Part I: A Brief Review of Coalgebras

Coalg(T)

Functor

$$T : \mathcal{X} \to \mathcal{X}$$

Coalgebra $\quad \mathsf{A} = (A, \alpha)$

$$A \xrightarrow{\alpha} TA$$

Morphism $\quad (A, \alpha) \xrightarrow{f} (B, \beta)$

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f"'] & TA \arrow[d, "Tf"] \\
B \arrow[r, "\beta"'] & TB
\end{tikzcd}$$

Example: Kripke Frames

Functor

$$\begin{aligned} T : \mathsf{Set} &\longrightarrow \mathsf{Set} \\ X &\mapsto \mathcal{P}X \end{aligned}$$

Kripke frame

$$X \xrightarrow{\xi} \mathcal{P}X$$

for $f : X \to Y$

$$Tf : \mathcal{P}X \to \mathcal{P}Y \quad , \quad X \supseteq Z \mapsto f(Z)$$

Example: Kripke Models

A set of propositional variables Prop, a two-element set $2 = \{0, 1\}$

Functor

$$TX = \mathcal{P}X \times \prod_{\mathsf{Prop}} 2$$

Coalgebra

$$X \xrightarrow{\langle \xi, v \rangle} \mathcal{P}X \times \prod_{\mathsf{Prop}} 2$$

$$\xi : X \to \mathcal{P}X$$

$$v(x)_p \in 2 \quad \text{for } x \in X, \;\; p \in \mathsf{Prop}$$

Example: Labelled Transitions Systems

$$T : \mathsf{Set} \longrightarrow \mathsf{Set}$$

$$TX = \mathcal{P}(Act \times X)$$

Process [Milner, 1983]: initial state, equality is bisimulation.

Processes are precisely the elements in the final coalgebra [Aczel, 1988].

Behavioural Equivalence

Given a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, behavioural equivalence (bisimulation) for T-coalgebras is defined as follows.

Definition: Let $\sim$ be the smallest equivalence relation such that for all morphisms $f : \mathsf{A} \longrightarrow \mathsf{B}$

$$(\mathsf{A}, a) \sim (\mathsf{B}, f(a)).$$

If $(\mathsf{A}, a) \sim (\mathsf{B}, b)$ we say (A, a) and (B, b) are **behaviourally equivalent**.

Final Coalgebras

Definition: Z is the **final** T-coalgebra iff for all T-coalgebras A there is a unique morphism

$$!_{\mathsf{A}} : \mathsf{A} \to \mathsf{Z}.$$

Fact: (A, a), (B, b) are behaviourally equivalent iff

$$!_{\mathsf{A}}(a) = !_{\mathsf{B}}(b).$$

Examples:

Coalgebras	Carrier Z of the Final Coalgebra
$A \longrightarrow O \times A$	$O^{\mathbb{N}}$
$A \longrightarrow 2 \times A^I$	all languages $L \subseteq I^*$
$A \longrightarrow \mathcal{P}A$	Aczel's universe of non-well founded sets
$A \longrightarrow \mathcal{P}(Act \times A)$	equivalence classes of processes up to bisimulation

Alg(T)

Functor

$$T : \mathcal{X} \to \mathcal{X}$$

Algebra A $= (A, \alpha)$

$$TA \xrightarrow{\alpha} A$$

Morphism $(A, \alpha) \xrightarrow{f} (B, \beta)$

$$\begin{tikzcd}
TA \arrow[r, "\alpha"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\
TB \arrow[r, "\beta"'] & B
\end{tikzcd}$$

Duality of Algebras and Coalgebras

- $\mathsf{Coalg}(T) \to \mathsf{Set}$ is dual to $\mathsf{Alg}(T^{\mathrm{op}}) \to \mathsf{Set}^{\mathrm{op}}$.

- Why algebras for *signatures* and coalgebras for *functors* ?
  - For $T : \mathsf{Set} \to \mathsf{Set}$, $\mathsf{Alg}(T)$ can be described by operations and equations.
  - Every category of coalgebras can be described by a signature of co-operations and equations, but these descriptions seem less natural. For example, even for polynomial functors one may need a proper class of operations and equations.
  - Technically: Finite sets are dense in Set but they are codense in Set only if there do not exist arbitrarily large measurable cardinals.

- Modal logic is dual to equational logic.

Part II: Coalgebraic Semantics of Modal Logic

or

Stone Coalgebras

(with C. Kupke and Y. Venema, CMCS'03)

Modal Logic

The modal logic $\mathbf{K}$ is given as follows.

Syntax: $\varphi ::= \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid p \quad , \quad p \in \mathsf{Prop}$

Axioms and Rules:

propositional logic

$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$

from φ derive $\Box\varphi$

Modal Algebras

$(A, \top, \neg, \wedge, \Box)$ is a modal algebra if $(A, 1, \neg, \wedge)$ is a Boolean algebra and

$$\begin{aligned} \Box(a \wedge b) &= \Box a \wedge \Box b \\ \Box\top &= \top \end{aligned}$$

Modal formulae and algebraic equations correspond via

$$\begin{array}{ccc} \varphi & & \varphi = \top \\ t \leftrightarrow t' & & t = t' \end{array}$$

Theorem:

$$\begin{array}{ccc} \vdash_{\mathbf{K}} \varphi & \Leftrightarrow & \models_{\mathsf{MA}} \varphi = \top \\ \vdash_{\mathbf{K}} t \leftrightarrow t' & \Leftrightarrow & \models_{\mathsf{MA}} t = t' \end{array}$$

Kripke Semantics

Kripke model: (X, R, v) where $R \subseteq X \times X, \quad v : X \to \prod_{\mathsf{Prop}} 2$

$$X, R, v, x \models p \quad \Leftrightarrow \quad v(x)_p = 1$$
$$X, R, v, x \models \Box\varphi \quad \Leftrightarrow \quad \forall y \in X \,.\, xRy \;\Rightarrow\; X, R, v, y \models \varphi$$

Kripke frame: (X, R)

$$(X, R) \models \varphi \quad \Leftrightarrow \quad X, R, v, x \models \varphi \quad \text{for all } v, x$$

Kripke Semantics is Incomplete

There is a consistent normal modal logic $\mathcal{L}$ whose class of Kripke frames is empty, that is,

$$\forall\,(X, R)\;.\;(X, R) \not\models \mathcal{L}$$

General Frames

(X, R, A) is a **general frame**

if $A \subseteq \mathcal{P}X$ contains X and A is closed under Boolean operations and

$$\begin{array}{rcl} \Box_R : \mathcal{P}X & \longrightarrow & \mathcal{P}X \\ a & \mapsto & \{x \in X \mid \forall y \,.\, xRy \;\Rightarrow\; y \in a\} \end{array}$$

Note: $(A, X, (\cdot)^c, \cap, \Box_R)$ is a modal algebra

From Modal Algebras to General Frames

Let $(A, \top, \neg, \wedge, \Box)$ be a modal algebra.

For a Boolean algebra A there is $(X, \widehat{A})$ with $\widehat{A} \subseteq \mathcal{P}X$ such that

$$\begin{array}{ccc} A & \xrightarrow{\cong} & \widehat{A} \\ a & \mapsto & \widehat{a} \end{array}$$

Define $$xR_A y \;\Leftrightarrow\; [\, \forall a \in A \,.\, x \in \widehat{\Box a} \;\Rightarrow\; y \in \widehat{a} \,].$$

Then $(X, R_A, \widehat{A})$ is a general frame.

Descriptive General Frames

(X, R, A) is a **descriptive general frame** if

- $x \neq y \;\Rightarrow\; \exists$ disjoint $a, b \in A \,.\, x \in a$ and $y \in b$
- If $B \subseteq A$ and for all finite $B' \subseteq B$ holds $\bigcap B' \neq \emptyset$ then $\bigcap B \neq \emptyset$.
- $R[x] = \bigcap\{a \mid x \in \Box_R \, a\}$

Coalgebras over Stone Spaces

Stone space: Hausdorff, compact, basis of clopens

The analog of the powerset for Stone spaces is the *Vietoris functor*.

$$\begin{aligned} \mathbb{V} : \mathsf{Stone} &\longrightarrow \mathsf{Stone} \\ (X, \tau) &\mapsto (\{W \subseteq X \mid W \text{ closed}\}, \mathbb{V}\tau) \end{aligned}$$

where $\mathbb{V}\tau$ is the topology generated by the sets

$$\{F \mid F \text{ closed}, F \subseteq U\} \qquad \{F \mid F \text{ closed}, F \cap U \neq \emptyset\}$$

for all opens $U \in \tau$.

Descriptive General Frames are Coalgebras

Theorem: $\mathsf{DGF} \cong \mathsf{Coalg}(\mathbb{V})$

Proof:

$$\frac{(X, R, A) \qquad\qquad \leftrightsquigarrow \qquad\qquad (X, \tau) \xrightarrow{\xi} (\mathbb{V}X, \mathbb{V}\tau)}{\begin{array}{ccc} A \text{ is the set of clopens of } \tau & & \tau \text{ is the topology generated by } A \\ R[x] & = & \xi(x) \\ A \text{ is closed under } \Box_R & \Leftrightarrow & \xi \text{ is continuous} \\ f : X \to Y \text{ general frame morphism} & \Leftrightarrow & f : X \to Y \text{ coalgebra morphism} \end{array}}$$

Descriptive General Frames are Coalgebras (cont'd)

For a coalgebra (X, ξ) a valuation is a function $v : X \to \prod_{\mathsf{Prop}} 2$.

Prop: Let X be a Stone space.

$v : X \to \prod_{\mathsf{Prop}} 2$ is continuous iff $\{x \in X \mid v(x)_p = 1\}$ is clopen for all p.

Proof: $\{x \in X \mid v(x)_p = 1\} = v^{-1}(\pi_p^{-1}(\{1\}))$.

Positive Modal Logic

Syntax: $\varphi ::= \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid p \quad , \quad p \in \mathsf{Prop}$

Duality: $\mathsf{DistLat}^{\mathrm{op}} \simeq \mathsf{Spectral} \; (\cong \mathsf{Priestley})$

Vietoris Functor:
$$\begin{aligned} \mathbb{W} : \mathsf{Priestley} &\longrightarrow \mathsf{Priestley} \\ (X, \tau) &\mapsto (\{W \subseteq X \mid W \text{ closed and convex}\}, \mathbb{V}\tau) \end{aligned}$$

Theorem (Palmigiano, CMCS'03)

$\mathsf{Coalg}(\mathbb{W})$ is dual to the category of positive modal algebras.

Part III: Modal Logics from Categories of Coalgebras

Dualising the Vietoris Functor on Stone Spaces

$\mathsf{Coalg}(\mathbb{V})$ is dual to $\mathsf{Alg}(\mathbb{V}^{\mathrm{op}})$.

Can we describe $\mathbb{V}^{\mathrm{op}}$ ‘duality-free’ (algebraic, localic)?

That is, we want to describe $L : \mathsf{BA} \to \mathsf{BA}$ such that

$$\begin{tikzcd}
\mathsf{BA} \arrow[d, "L"'] \arrow[dr, phantom, "\cong"] & \mathsf{Stone}^{\mathrm{op}} \arrow[l, "\mathsf{Clp}"'] \arrow[d, "\mathbb{V}^{\mathrm{op}}"] \\
\mathsf{BA} & \mathsf{Stone}^{\mathrm{op}} \arrow[l, "\mathsf{Clp}"]
\end{tikzcd}$$

Note that then $\mathsf{Alg}(L) \simeq \mathsf{Alg}(\mathbb{V}^{\mathrm{op}})$.

Dualising the Vietoris Functor on Stone Spaces (cont'd)

$$L : \mathsf{BA} \to \mathsf{BA}$$

LA is the free Boolean algebra generated by

$$\Box a, a \in A$$

and satisfying the relations

$$\begin{aligned} \Box(a \wedge b) &= \Box a \wedge \Box b \\ \Box\top &= \top \end{aligned}$$

The General Picture

Given: concrete duality $\Omega : \mathcal{C}^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{D}$, functor $T : \mathcal{C} \to \mathcal{C}$.

Describe the dual $L : \mathcal{D} \to \mathcal{D}$ of T by generators and relations.

$$X \xrightarrow{\;!_X\;} Z \qquad\qquad \Omega Z \xleftarrow{\;i\;} I_L \xleftarrow{\;[\cdot]_\equiv\;} I_F$$

where Z final in $\mathsf{Coalg}(T)$, I_L initial in $\mathsf{Alg}(L)$, I_F the algebra of all formulae

For a T-coalgebra (X, ξ) and x in X and φ in I_F define

$$x \models \varphi \quad \Leftrightarrow \quad !_X(x) \in i([\varphi]_\equiv)$$

Summary

I want to emphasise the following two points:

- The modal logic for T-coalgebras is given by the Stone dual L of T. Soundness, completeness, expressivity and invariance of formulas under bisimulation are guaranteed by construction.

- For example, taking the Vietoris functor on Stone spaces we obtain classical modal logic, on spectral spaces we obtain positive modal logic.

- The duality of modal algebras and descriptive general frames is an instance of the duality of algebras and coalgebras for a functor.

Acknowledgements

- Peter Johnstone. Vietoris locales and localic semilattices. In R.-E. Hoffmann and K.H. Hofmann, editors, *Continuous Lattices and their Applications*, volume 101 of *Lecture Notes in Pure and Applied Mathematics*, pages 155–180. Marcel Dekker, 1985.

- Samson Abramsky. Domain theory in logical form. *Annals of Pure and Applied Logic*, 51:1–77, 1991.