Towards Universal Algebra over Nominal Sets

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1 Introduction

Nominal sets were introduced by Gabbay and Pitts [8]. This paper describes a step towards universal algebra over nominal sets. There has been some work in this direction, most notably by M.J. Gabbay [7]. The originality of our approach is that we don't redo universal algebra from scratch, replacing the category Set by the category Nom of nominal sets. As shown in [7], this is possible, but it requires ingenuity and adhoc constructions. For example, the logic of [7] is not simple equational logic and the construction of free algebras has to be revisited.

In contrast, we want to explore how far one can get by using completely standard universal algebra. This is based on the well-known observation that Nom is a full reflective subcategory of the presheaf category $\mathsf{Set}^{\mathbb{I}}$ where $\mathbb{I}$ is the category of finite ordinals with injective maps.

$$\begin{tikzcd} \mathsf{Nom} \arrow[rr, bend right] \arrow[rr, phantom, "\perp"] & & \mathsf{Set}^{\mathbb{I}} \arrow[ll, bend right] \end{tikzcd} \tag{1}$$

Moreover, like any presheaf category, $\mathsf{Set}^{\mathbb{I}}$ is a category of many-sorted algebras, the sorts being the objects of $\mathbb{I}$, the (unary) operations being the arrows in $\mathbb{I}$, and the equations being the commuting diagrams of $\mathbb{I}$.

To move from the underlying categories Nom and $\mathsf{Set}^{\mathbb{I}}$ to algebras, we need a notion of signature. This is provided by endo-functors $\tilde{L} : \mathsf{Nom} \to \mathsf{Nom}$ and $L : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$. The intention here is that the two functors $\tilde{L}$ and L are "doing the same", or, more precisely they commute with the left adjoint. Then the adjunction (1) lifts to algebras

$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathsf{Alg}(\tilde{L}) \arrow[r, bend right, "\overline{I}"'] \arrow[r, phantom, "\perp"] \arrow[d, "\tilde{U}"'] & \mathsf{Alg}(L) \arrow[l, bend right, "\overline{R}"'] \arrow[d, "U"] \\
\mathsf{Nom} \arrow[loop left, "\tilde{L}"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathbb{I}} \arrow[l, bend right, "R"'] \arrow[loop right, "L"]
\end{tikzcd} \tag{2}
$$

At this point, we are encouraged by a result of [11], stating that all ‘universal-algebraic’ endo-functors L have a presentation. In particular, it follows that $\mathsf{Alg}(L)$ is an equationally defined class (variety) of many-sorted algebras *in the standard sense of universal algebra*. For example, we have an equational calculus at hand, which we can use to reason about algebras in $\mathsf{Alg}(L)$ and, hence, also to reason about algebras in the full subcategory $\mathsf{Alg}(\tilde{L})$.

To test our approach, we decided to prove an HSP-theorem for algebras over Nom. We have Birkhoff's HSP-theorem for $\mathsf{Alg}(L)$ stating that a class of algebras is equationally definable if it is closed under homomorphic images, subalgebras, and products. An inspection of its proof shows that it goes through the adjunction, yielding an HSP-theorem for $\mathsf{Alg}(\tilde{L})$.

This shows that we have to qualify the statement in [7] “An attempt to directly transfer proofs of the HSP theorem to the nominal setting fails”: Using the category theoretic formulation of Birkhoff's proof (which is standard, see eg [1, Exercise 16G]), we rather would say that “it is straightforward to transfer to the nominal setting, through the adjunction (1), the standard category theoretic proof of the HSP-theorem ”. This is carried out in Section 3.

Central to our approach is the flexibility provided by the notion of presentation of functor (introduced in [3] and further developed in [11, 10]). We expect that much of universal algebra over Nom can be formulated and proved on an abstract level, for arbitrary signature functors $\tilde{L}, L$, possibly satisfying some mild conditions. An example of this is our development in Section 3. On the other hand, everybody who has worked with Nom and $\mathsf{Set}^{\mathbb{I}}$ using concrete syntax knows that these categories are very different. This difference can be made precise now, as it arises from different presentations of equivalent categories and functors.

This leads us to Section 4 where we give two different presentations of the shift (or abstraction) functor δ. First, a presentation of $\mathsf{Set}^{\mathbb{I}}$ and $\delta : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$ is given. Then, to bring us closer to the syntax of nominal sets, we give a presentation of δ on an equivalent category $\mathsf{Set}^{\mathbb{J}}$. Interestingly, the syntax suggested by this approach is not based on permutations but on 'substitutions' $(b/a)_S : S \cup \{a\} \to S \cup \{b\}$ fixing the elements of a set S, but substituting b for a (where $a, b \notin S$).

Section 5 brings together the abstract results of Section 3 with the concrete syntax of Section 4. We show that δ satisfies the conditions set out in Section 3 and obtain and HSP-theorem for algebras with binders. Due to the presentations from Section 4, these algebras are algebras for operations and equations in the usual sense of many-sorted universal algebra. We illustrate the concrete syntax obtained with an axiomatisation of the λ-calculus. Although the many-sorted setting means that the axioms that capture $\alpha\beta\eta$-equivalence are parameterized over an infinite set of sorts, we can see that they are 'uniform' in a sense that will be made precise in Section 6.

A natural question to ask is how our HSP theorem relates to that of [7]. First we should notice that Gabbay's theorem is stronger, in the sense that in his framework the equationally definable classes have to be closed not only under HSP, but also under 'atom-abstraction'. We can call his result the HSPA theorem. Our logic is more expressive in the sense that we allow more general nominal signatures. However, in Section 6 we show that if we restrict our attention to uniform terms, we recover Gabbay's additional closure operator. We also give a syntactic description of uniform terms and give an axiomatisation of the λ-calculus in uniform equations.

2 Preliminaries

2.1 Nominal Sets

For the remainder of this article we denote by $\mathcal{N}$ an infinite countable set of names. We recall the definition of a nominal set. Intuitively this is a set equipped with an additional structure which allows well-behaved name swapping in elements of the set.

A left action of the group $\mathsf{S}(\mathcal{N})$ of all finitely supported permutations of $\mathcal{N}$ is a pair $(|X|, \cdot)$ consisting of a set $|X|$ and a function $\cdot : \mathsf{S}(\mathcal{N}) \times |X| \to |X|$

satisfying:

$$\begin{aligned} \mathsf{id}_{\mathcal{N}} \cdot x &= x \\ (\sigma\tau) \cdot x &= \sigma \cdot (\tau \cdot x) \end{aligned} \tag{3}$$

for all $x \in |X|$ and $\sigma,\tau \in \mathsf{S}(\mathcal{N})$. Let x be an element of $|X|$. We say that a subset $S \subseteq \mathcal{N}$ *supports* x if and only if for all a, $b \in \mathcal{N} \setminus S$ we have that $(a, b) \cdot x = x$, where (a, b) denotes the transposition which swaps a and b. The element x is said to be *finitely supported* if and only if there exists a finite set S which supports x.

Definition 2.1. *A nominal set is a left* $\mathsf{S}(\mathcal{N})$*-action* $(|X|, \cdot)$ *such that each element of* x *is finitely supported.*

One can check that for each element x of a nominal set there exists a smallest set, in the sense of inclusion, which supports x. This set is called the *support* of x and will be denoted by $\mathsf{supp}(x)$. We say that $a \in \mathcal{N}$ is *fresh* for x, if $a \notin \mathsf{supp}(x)$. A morphism of nominal sets $f : (|X|, \cdot) \to (|Y|, \circ)$ is an *equivariant* function between the carrier sets, meaning that f behaves well with respect to permutations of names: $f(\sigma \cdot x) = \sigma \circ f(x)$ for all $x \in |X|$.

We will denote by Nom the category of nominal sets and equivariant maps.

Example 2.2. *The set* $\mathcal{N}$ *equipped with the action given by evaluation,* $\sigma \cdot a = \sigma(a)$*, is a nominal set.*

Remark 2.3. *If* $\mathcal{N}$ *and* $|X|$ *are equipped with the discrete topology, then* $\mathsf{S}(\mathcal{N})$ *can be equipped with the topology induced by the product topology on* $\mathcal{N}^{\mathcal{N}}$*. Then a nominal set* $(|X|, \cdot)$ *is just a continuous* $\mathsf{S}(\mathcal{N})$*-action, i.e.* $\cdot$ *is a continuous function.*

Using this observation, one can see that the category of nominal sets is equivalent to a well-known topos, sometimes called the *Schanuel topos*. We will denote by $\mathbb{I}$ the category whose objects are finite ordinals $[n]$ for $n \in \mathbb{N}$ and morphisms are injective maps. The Schanuel topos $\mathrm{Sh}(\mathbb{I}^{op})$ has several characterizations in the literature. One of these, [9], is the full subcategory of $\mathsf{Set}^{\mathbb{I}}$ consisting of functors preserving monomorphisms and pullbacks. Moreover $\mathrm{Sh}(\mathbb{I}^{op})$ is a reflective subcategory of $\mathsf{Set}^{\mathbb{I}}$, and the left adjoint R, of the inclusion functor, preserves finite limits, [12].

The equivalence between the nominal sets and the Schanuel topos is a corollary of [12, Theorem 1, pp.128]. The inclusion functor $\mathbb{I} \hookrightarrow \mathsf{Set}$ corresponds precisely to the nominal set $(\mathcal{N}, \cdot)$ described in Example 2.2.

2.2 Finitary Presentations for Functors on Many-sorted Varieties

Let S be a set of sorts. A *signature* Σ is a set of operation symbols together with an arity map. To each signature corresponds an endofunctor on Set^S, also denoted by Σ for simplicity. The algebras for a signature Σ are precisely the algebras for the corresponding endofunctor, and form the category denoted by $\mathsf{Alg}(\Sigma)$. The terms over an S-sorted set of variables X are defined in the standard manner and an equation consists of a pair (τ_1, τ_2) of terms of the same sort, usually denoted $\tau_1 = \tau_2$. A Σ-algebra A satisfies this equation if and only if, for any interpretation of the variables of X, we obtain equality in A. A full subcategory $\mathcal{A}$ of Σ-algebras is called an *equational class* if there exists a set of equations E such that an algebra lies in $\mathcal{A}$ if and only if it satisfies all the equations of E. Such an equational class will be denoted by $\mathsf{Alg}(\Sigma, E)$.

The notion of functors with finitary presentations has been introduced in [3] for applications to coalgebraic logic. Let $\mathcal{A} = \mathsf{Alg}(\Sigma, E)$ be an equational class. Denote by $U : \mathcal{A} \to \mathsf{Set}^S$ the forgetful functor and by F its left adjoint F. Intuitively an endofunctor L on $\mathcal{A}$ has a finitary presentation by operations Σ and equations E, if for each object A, we have that LA is uniformly isomorphic to a quotient of $F\Sigma UA$ by the equations E, in a sense that will be made precise in the next definition:

Definition 2.4. *A presentation for an endofunctor L on $\mathcal{A}$ is a pair $\langle \Sigma, E\rangle$ where Σ is a signature and E is a set of equations defined as follows: for any S-sorted set of variables V, E_V is a subset of $(UF\Sigma UFV)^2$ and E is the union of E_V taken over all finite sets of variables V. The functor L is presented by $\langle \Sigma, E\rangle$, if for any $A \in \mathcal{A}$ the algebra LA is the joint coequalizer:*

$$FE_V \underset{\pi_2^\sharp}{\overset{\pi_1^\sharp}{\rightrightarrows}} F\Sigma UFV \xrightarrow{F\Sigma U v^\sharp} F\Sigma UA \xrightarrow{q_A} LA \tag{4}$$

taken over all finite sets of S-sorted variables V and all valuations $v : V \to UA$. Here $v^\sharp$ denotes the adjoint transpose of a valuation v.

Functors with finitary presentations are characterized in [11] as exactly those functors which preserve sifted colimits. Recall that sifted colimits are exactly those colimits which commute with finite products in Set. Examples include filtered colimits and reflexive coequalizers.

For an endofunctor L on a category $\mathcal{A}$, we consider the category of L-*algebras*, denoted by $\mathsf{Alg}(L)$, whose objects are defined as pairs (A, α) such that $\alpha : LA \to A$ is a morphism in $\mathcal{A}$. A morphism of L-algebras $f : (A, \alpha) \to (A', \alpha')$ is a morphism $f : A \to A'$ of $\mathcal{A}$ such that $f \circ \alpha = \alpha' \circ Lf$.

If the functor L on an equational class has a finitary presentation, then $\mathsf{Alg}(L)$ is an equational class as well, and its presentation can be obtained in a modular way. The proof for the next theorem is in [10].

Theorem 2.5. *Let $\mathcal{A} = \mathsf{Alg}(\Sigma_{\mathcal{A}}, E_{\mathcal{A}})$ be an S-sorted equational class and let $L : \mathcal{A} \to \mathcal{A}$ be a functor presented by operations Σ_L and equations E_L. Then $\mathsf{Alg}(L)$ is concretely isomorphic to $\mathsf{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L)$.*

3 HSP Theorems

In this section we will prove an HSP-theorem for nominal sets and nominal algebras in a systematic way.

In the first subsection we prove general results using categorical techniques. To set the scene we outline a categorical proof of Birkhoff's HSP theorem. Then, in Theorem 3.4, we show how to obtain an HSP-theorem for a full reflective subcategory $\mathcal{A}$ of a category of algebras $\mathcal{C}$, if some additional conditions are met. Essentially, this is achieved by 'pushing the proof of the general HSP-theorem through the adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend right] \arrow[rr, phantom, "\perp"] & & \mathcal{C} \arrow[ll, bend right] \end{tikzcd}$$

This result is interesting because $\mathcal{A}$ might not be a variety.

We also prove a general result, Proposition 3.5, concerning a lifting property of an adjunction to categories of algebras for certain functors.

Secondly, in Section 3.2, we apply these results to the nominal setting. Consider diagram (2), where L is an endofunctor on $\mathsf{Set}^{\mathbb{I}}$, that preserves sifted colimits, and $\tilde{L}$ is the corresponding endofunctor on Nom. The idea is that nominal algebras correspond to $\tilde{L}$-algebras. Using Proposition 3.5 we prove that the adjunction between Nom and $\mathsf{Set}^{\mathbb{I}}$ can be lifted to an adjunction between $\mathsf{Alg}(\tilde{L})$ and $\mathsf{Alg}(L)$. On the right hand side of this diagram we have categories monadic over $\mathsf{Set}^{\mathbb{N}}$, for which the classical HSP-theorem holds. We derive HSP theorems for both Nom and $\mathsf{Alg}(\tilde{L})$, by applying Theorem 3.4 on both levels of this diagram.

3.1 HSP theorems for full reflective subcategories

Birkhoff's HSP Theorem. Given a category of algebras $\mathcal{C}$, a full subcategory $\mathcal{B} \subseteq \mathcal{C}$ is closed under quotients (H for homomorphic images), subobjects (S), and products (P) iff $\mathcal{B}$ is definable by equations.

This theorem can be proved at different levels of generality. We assume here that we have a forgetful functor $U : \mathcal{C} \to \mathsf{Set}^\kappa$, for some cardinal κ, with a left-adjoint F. Then we can identify a set of equations Φ in variables X with quotients $FX \to Q$. Indeed, given Φ we let Q be the quotient FX/Φ and, conversely, given $FX \to Q$ we let Φ be the kernel of $FX \to Q$. Further, an algebra $A \in \mathcal{C}$ satisfies the equations iff all $FX \to A$ factor through $FX \to Q$, in a picture

$$A \models \Phi \quad \Leftrightarrow \quad \begin{tikzcd} FX \arrow[rr] \arrow[dr, "\forall"'] & & FX/\Phi \arrow[dl, dotted, "\exists"] \\ & A & \end{tikzcd} \tag{5}$$

Proof of 'if'. We want to show that a subcategory $\mathcal{B}$ defined by equations Φ is closed under HSP. Closure under subobjects $A' \to A$ follows since quotients and subobjects form a factorisation system (see eg [1, 14.1]). Indeed, according to (5), to show $A \models \Phi \Rightarrow A' \models \Phi$ one has to find the dotted arrow in

$$\begin{tikzcd} FX \arrow[r] \arrow[d] & FX/\Phi \arrow[d] \arrow[dl, dotted] \\ A' \arrow[r] & A \end{tikzcd}$$

which exists because of the diagonal fill-in property of factorisation systems. A similar argument works for products (because of their universal property) and for quotients (using that free algebras are projective [1, 9.27]).

Proof of 'only if'. Given $\mathcal{B} \subseteq \mathcal{C}$, we first need to find the equations. Since $\mathcal{B}$ is closed under SP, $\mathcal{B}$ is a full reflective subcategory, that is, the inclusion $\mathcal{B} \to \mathcal{C}$ has a left-adjoint H and, moreover, the unit $A \to HA$ is a quotient.[1] We take as equations all $FX \to HFX$. That all $A \in \mathcal{B}$ satisfy these equations, again using (5), follows immediately from the the

[1] To construct HA given A, consider all arrows $f : A \to B_f$ with codomain in $\mathcal{B}$; factor $f = A \xrightarrow{q_f} \bar{B}_f \xrightarrow{i_f} B$; up to isomorphism, there is a only a proper set of different q_f; now factor $A \xrightarrow{\langle q_f \rangle} \prod_f \bar{B}_f$ as $A \to HA \to \prod_f \bar{B}_f$ to obtain the unit $A \to HA$.

universal property of the left-adjoint H. Conversely, suppose that A satisfies all equations. Consider the equations $q : FUA \to HFUA$. Because of (5), the counit $e : FUA \to A$ must factor as $e = f \circ q$. Since e and q are quotients, so is f. Hence A is a quotient of $HFUA$, which is in $\mathcal{B}$.

Remark 3.1. *Notice that in the proof above we allow quotients $FX \to HFX$ for arbitrary κ-sorted sets X. If the set X is infinite, we allow equations involving infinitely many variables. Therefore we no longer reason within finitary logic. If we impose that the equations involve only finitely many variables, then the HSP-theorem is* not *true for arbitrary many-sorted varieties. Indeed, in the many-sorted case, closure under homomorphic images, sub-algebras and products is no longer enough to deduce equational definability (see [2, Example 10.14.2]). One needs an additional constraint, namely closure under directed unions, [2, Theorem 10.12]. But in the motivating examples of $\mathsf{Set}^{\mathbb{I}}$ and $\mathsf{Set}^{\mathbb{J}}$, we will prove that closure under HSP implies closure under directed unions.*

In the following, we show that it is possible to obtain an HSP-theorem for certain subcategories of varieties, by pushing the argument above through an adjunction. But first let us say what we mean in this context by **equationally definable** and **closed under HSP**. We will work in the following setting.

Definition 3.2. *Let $\mathcal{C}$ be a category monadic over Set^{κ} for some cardinal κ, with U denoting the forgetful functor and F its left adjoint. Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$, which has a factorisation system (E, M) such that the inclusion functor I has a left adjoint which preserves the regular factorisation system of $\mathcal{C}$.*

We say that $\mathcal{B} \hookrightarrow \mathcal{A}$ is ***equationally definable*** *if there exists a set of equations Φ in $\mathcal{C}$, such that an object A of $\mathcal{A}$ lies in $\mathcal{B}$ iff $IA \models \Phi$ (where Φ and $\models$ are as in (5)).*

We say that $\mathcal{B}$ is ***closed under HSP*** *if and only if*

1. *For all morphisms $e : B \to B'$ such that $e \in E$ and Ie is a quotient, we have that $B \in \mathcal{B}$ implies $B' \in \mathcal{B}$.*
2. *For all morphisms $m : B \to B'$ such that $m \in M$ we have $B' \in \mathcal{B}$ implies $B \in \mathcal{B}$.*
3. *If B_i are in $\mathcal{B}$ then their product in $\mathcal{A}$ is an object of $\mathcal{B}$.*

Remark 3.3. *In general, the inclusion functor I does not preserve epimorphisms. We will assume that the arrows in M are monomorphisms. Being a right adjoint, I preserves products and monomorphisms, but we cannot infer from $B' \to IB$ being a monomorphism in $\mathcal{C}$ that B' is (isomorphic to an object) in $\mathcal{A}$.*

Theorem 3.4. *Let $\mathcal{C}$ be a category monadic over Set^κ for some cardinal κ, with U denoting the forgetful functor and F its left adjoint. Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$, which has a factorisation system (E, M) such that the morphisms in M are monomorphisms and the inclusion functor I has a left adjoint R which preserves the regular factorisation system of $\mathcal{C}$. Then $\mathcal{B} \subseteq \mathcal{A}$ is closed under HSP if and only if $\mathcal{B}$ is equationally definable.*

Proof. The 'if' part can be proved in the same fashion as the usual HSP theorem, described at the beginning of the section. Now let us prove that closure under HSP implies equational definability.

Step 1. First let us construct the equations defining $\mathcal{B}$. Let C be an arbitrary object of $\mathcal{C}$ and $f_i : RC \to B_i$ a morphism in $\mathcal{A}$ such that B_i is in $\mathcal{B}$. The corresponding morphism in $\mathcal{C}$, $f_i^\sharp : C \to IB_i$ factors in $\mathcal{C}$:

$$\begin{tikzcd} C \arrow[dd, "f_i^\sharp"'] \arrow[dr, "e_i"] & \\ & \overline{B_i} \arrow[dl, "m_i"] \\ IB_i & \end{tikzcd}$$

We will denote by η and ϵ the unit, respectively the co-unit of the adjunction $R \dashv I$. One can easily show that the following diagram commutes:

$$\begin{tikzcd} RC \arrow[dd, "f_i"'] \arrow[dr, "Re_i"] & \\ & R\overline{B_i} \arrow[dl, "\epsilon_{B_i} \circ Rm_i"] \\ B_i & \end{tikzcd} \tag{6}$$

Since R preserves the factorisation system, and ϵ_{B_i} is an isomorphism (I is full and faithful), we have that (6) is a factorisation of f_i in $\mathcal{A}$. But $\mathcal{B}$ is closed under subobjects, hence $R\overline{B_i}$ is actually an object of $\mathcal{B}$. We consider the product P of the objects of the form $R\overline{B_i}$, obtained as above. Notice that here we use the fact that $\mathcal{C}$ is co-wellpowered, and therefore so is $\mathcal{A}$. P is again an object of $\mathcal{B}$, and we have a morphism $\alpha : RC \to P$, uniquely determined by the Re_i-s. We consider a factorisation in $\mathcal{C}$ of the adjoint map $\alpha^\sharp : C \to IP$:

$$
\begin{tikzcd}
C \arrow[dd, "\alpha^\sharp"'] \arrow[dr, "e"] & \\
& Q_C \arrow[dl, "m"] \\
IP &
\end{tikzcd}
\tag{7}
$$

Using a similar argument as above we deduce that RQ_C is an object of $\mathcal{B}$ and the following diagram commutes:

$$
\begin{tikzcd}
RC \arrow[rr, "Re"] \arrow[d, "f_i"] & & RQ_C \arrow[d, "\epsilon_P \circ Rm"] \\
B_i & R\overline{B_i} \arrow[l, "\epsilon_B \circ Rm_i"] & P \arrow[l, "\pi_i"]
\end{tikzcd}
\tag{8}
$$

We consider the class of equations of the form $FX \twoheadrightarrow Q_{FX}$ for all sets X, and denote by $\mathcal{B}'$ the subcategory of $\mathcal{C}$ defined by these equations.

Step 2. Next we show that if an object B of $\mathcal{A}$ lies in $\mathcal{B}$, then IB satisfies these equations. Let B be an object of $\mathcal{B}$, and let $u : FX \to IB$ be an arbitrary morphism. For the adjoint morphism $u_\sharp : RFX \to B$, one can construct a morphism $g : RQ_{FX} \to B$, obtained as in diagram (8), such that $g \circ Re = u_\sharp$.
It is easy to see that $g^\sharp : Q_{FX} \to IB$ makes the following diagram commutative. This shows that IB satisfies the equation $e : FX \twoheadrightarrow O_{FX}$:

$$
\begin{tikzcd}
FX \arrow[r, "e"] \arrow[d, "u"'] & Q_{FX} \arrow[dl, "g^\sharp"] \\
IB &
\end{tikzcd}
\tag{9}
$$

Step 3. Conversely, if IB satisfies the equation $FUIB \twoheadrightarrow Q_{FUIB}$, so there exists $v : Q_{FUIB} \to IB$ such that $v \circ e = \epsilon'_{IB}$, where ϵ' is the co-unit of the adjunction $F \dashv U$. Since ϵ' is a regular epi, then v is also a regular epi. We have that the composition $\epsilon_B \circ Rv : RQ_{FUIB} \to B$ is a morphism in E. One can check that IRv is also a regular epi, therefore so is $I(\epsilon_B \circ Rv)$. Using the fact that $\mathcal{B}$ is closed under H, and that RQ_{FUIB} is already in $\mathcal{B}$, we can conclude that $B \in \mathcal{B}$. $\square$

Proposition 3.5. *Let $\langle R, I, \eta, \epsilon \rangle : \mathcal{A} \rightharpoonup \mathcal{B}$ be an adjunction such that I is full and faithful. Let K and L be endofunctors on $\mathcal{A}$ and $\mathcal{B}$, respectively. Suppose there exist natural transformations: $\kappa : RK \to LR$ and $\lambda : LI \to IK$ making the following diagrams commute:*

$$
\begin{tikzcd}
LIR \arrow[r, "\lambda R"] & IKR \arrow[r, "I\kappa"] & IRL \\
& L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
KRI \arrow[r, "\kappa I"] \arrow[dr, "K\epsilon"'] & RLI \arrow[r, "R\lambda"] & RIK \arrow[dl, "\epsilon K"] \\
& K &
\end{tikzcd}
\tag{10}
$$

(C_1) (C_2)

Then there exists an adjunction $\langle \overline{R}, \overline{I}, \overline{\eta}, \overline{\epsilon} \rangle : \mathsf{Alg}(K) \rightharpoonup \mathsf{Alg}(L)$, such that $U_{\mathcal{A}}\overline{R} = RU_{\mathcal{B}}$ and $IU_{\mathcal{A}} = U_{\mathcal{B}}\overline{I}$, where $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ denote the forgetful functors as in the next diagram:

$$
\begin{tikzcd}
\mathsf{Alg}(K) \arrow[rr, bend right, "\overline{I}"'] \arrow[rr, phantom, "\perp"] \arrow[d, "U_{\mathcal{A}}"'] & & \mathsf{Alg}(L) \arrow[ll, bend right, "\overline{R}"'] \arrow[d, "U_{\mathcal{B}}"] \\
\mathcal{A} \arrow[loop left, "K"] \arrow[rr, bend right, "I"'] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend right, "R"'] \arrow[loop right, "L"]
\end{tikzcd}
\tag{11}
$$

Proof. First let us define the functor $\overline{I}$. Let $f : KA \to A$ be a K-algebra. We define $\overline{I}(A, f) := (IA, If \circ \lambda_A)$. For an arbitrary morphism of K-algebras $u : (A, f) \to (A', f')$, we define $\overline{I}(u) = Iu$. We need to check that Iu is a morphism of L-algebras. This amounts to proving that the outer square of the next diagram commutes:

$$
\begin{tikzcd}
LIA \arrow[r, "\lambda_A"] \arrow[d, "LIu"'] & IKA \arrow[r, "If"] \arrow[d, "IKu"'] & IA \arrow[d, "Iu"'] \\
LIA' \arrow[r, "\lambda'_A"] & IKA' \arrow[r, "If'"] & IA'
\end{tikzcd}
\tag{12}
$$

But the small squares commute, the former because λ is a natural transformation, and the latter because u is a K-algebra morphism. It is obvious that $\overline{I}$ is a functor and that $IU_{\mathcal{A}} = U_{\mathcal{B}}\overline{I}$. The functor $\overline{R}$ is defined similarly: if $g : LB \to B$ is a L-algebra, we define $\overline{R}(B, g) = (RB, Rg \circ \kappa_B)$. If $v : (B, g) \to (B', g')$ is a L - algebra morphism, we define $\overline{R}(v) = Rv$. The fact that Rv is indeed a K-algebra morphism is verified easily, using the naturality of κ and the fact that v is a L-algebra morphism.

In order to prove that $\overline{R}$ is left adjoint to $\overline{I}$, we will show that the unit η and the co-unit ϵ of the adjunction $R \dashv I$ are L-algebra and K-algebra morphisms respectively. This is equivalent to proving that diagrams (13) are commutative. But this follows from the hypothesis (10) and the naturality of η and ϵ respectively.

$$
\begin{tikzcd}
LB \arrow[r, "L\eta_B"] \arrow[dddd, "g"'] \arrow[ddrr, "\eta_{LB}"] & & LIRB \arrow[d, "\lambda_{RB}"] \\
& & IKRB \arrow[d, "I\kappa_B"] \\
& & IRLB \arrow[d, "IRg"] \\
B \arrow[rr, "\eta_B"] & & IRB
\end{tikzcd}
\qquad
\begin{tikzcd}
KRIA \arrow[rr, "K\epsilon_A"] \arrow[d, "\kappa_{IA}"] & & KA \arrow[ddd, "f"] \\
RLIA \arrow[d, "R\lambda_A"] & & \\
RIKA \arrow[uurr, "\epsilon_{KA}"] \arrow[d, "RIf"] & & \\
RIA \arrow[rr, "\epsilon_A"] & & A
\end{tikzcd}
\tag{13}
$$

Once this is achieved, η can be lifted to natural transformations $\overline{\eta} : \mathrm{id} \to \overline{IR}$, and similarly ϵ can be lifted to a natural transformation $\overline{\epsilon} : \overline{RI} \to \mathrm{id}$. But η and ϵ are the unit and the counit of the adjunction $R \dashv I$, therefore they satisfy the usual triangular equalities. Therefore, $\overline{\eta}$ and $\overline{\epsilon}$ satisfy the triangular equalities for $\overline{R}$ and $\overline{I}$.

□

Remark 3.6. *The proposition has some useful special cases. For each of them, the commutativity of the diagrams (10) is straightforward to verify, using that the co-unit ϵ is iso.*

1. *Suppose L is given and we want to find an appropriate K. Then it follows from the theorem that we can do this, provided there is a natural transformation $\alpha : LIR \to IRL$ such that the diagram below commutes:*

$$
\begin{tikzcd}
LIR \arrow[rr, "\alpha"] & & IRL \\
& L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] &
\end{tikzcd}
\tag{14}
$$

If this is the case, one defines $K = RLI$, $\kappa : KR = RLIR \to RL$ as the composition $\epsilon_{RL} \circ R\alpha$ and $\lambda : LI \to IRLI = IK$ as η_{LI}. Moreover, we have that $LIR \to IRLIR$ is iso ('L preserves sheaves') and that $R\alpha$ and hence $KR \to RL$ are iso. (If we don't find such an α we might think of replacing L by $IRLIR$ since we always have $(IRLIR)IR \to IR(IRLIR)$; we still have $K \cong RLI$.)

2. *More generally, suppose we have given an iso $\kappa : KR \to RL$. Then we define $\lambda = IK\epsilon \circ I\kappa^{-1}I \circ \eta LI$. (Given K we can always find such a κ: Let $L = IKR$ and $\kappa = (\epsilon KR)^{-1} : KR \cong RIKR$.)*

Let Σ be a polynomial functor $\Sigma X = \coprod_{i\in I} X^{n_i}$. If both $\mathcal{A}$ and $\mathcal{B}$ have (co)products, then Σ is defined on both categories, so it makes sense to write $R\Sigma \cong \Sigma R$. The following corollary says that for polynomial functors Σ the adjunction always lifts from the base categories to the categories of Σ-algebras.

Corollary 3.7. *Let $\langle R, I, \eta, \epsilon\rangle : \mathcal{A} \rightharpoonup \mathcal{B}$ be an adjunction such that I is full and faithful. Further assume that both categories have coproducts and finite products and that R preserves finite products. Consider an endofunctor $\Sigma X = \coprod_{i\in I} X^{n_i}$ on $\mathcal{A}$ and on $\mathcal{B}$. Then the adjunction lifts to an adjunction $\langle \tilde{R}, \tilde{I}, \eta, \epsilon\rangle : Alg(\Sigma) \rightharpoonup Alg(\Sigma)$.*

Proof. We use item 2 of the remark above and calculate $R\Sigma A = R(\coprod_{i\in I} A^{n_i}) \cong \coprod_{i\in I} (RA)^{n_i} = \Sigma RA$. □

Remark 3.8. *$\overline{R}$ preserves finite limits whenever R does.*

Proof. Assume $(B, g) = \varprojlim(B_i, g_i)$ is a finite limit in $\mathsf{Alg}(L)$. Since $U_{\mathcal{B}}$ preserves all limits, we have that $B = \varprojlim(B_i)$ in $\mathcal{B}$, and therefore $RB = \varprojlim(RB_i)$. Denote $\overline{R}(B_i, g_i)$ by (RB_i, f_i) and by $\pi_i : RB \to RB_i$ the morphisms of the limiting cone. For each index i we have a map $p_i : KRB \to RB_i$ obtained as the composition $f_i \circ K\pi_i$. From the universal property, we obtain a map $f : KRB \to RB$, such that each π_i ia a K-algebra morphism from (RB, f) to (RB_i, f_i). We prove next that $(RB, f) = \varprojlim(RB_i, f_i)$ in $\mathsf{Alg}(K)$. Assume that we have a cone $q_i : (C, h) \to (RB_i, g_i)$. Since RB is a limit in $\mathcal{A}$ we get a unique map $k : C \to RB$, such that $\pi_i \circ k = q_i$. We need to show that k is a K-algebra morphism. To this end we will use the uniqueness of a morphism from $KC \to RB$ which makes the relevant diagrams commutative. □

3.2 HSP-theorem for nominal sets and nominal algebras

First let us see what it means for subcategory $\mathcal{B}$ of Nom to be closed under HSP, in the sense of the Definition 3.2. We consider the functor $I : \mathsf{Nom} \to \mathsf{Set}^{\mathbb{I}}$. The first condition translates to e being onto and preserving the support of elements. The second condition is the usual closure under subobjects, but note that we cannot infer from $B' \to IB$ being a mono in $\mathsf{Set}^{\mathbb{I}}$ that B' is (isomorphic to an object) in Nom. Products in Nom are the same as in $\mathsf{Set}^{\mathbb{I}}$.

Theorem 3.9. ***HSP-theorem for 'nominal sets'.*** *Let $\mathcal{B}$ be a full subcategory of $\mathrm{Sh}(\mathbb{I}^{op})$. Then $\mathcal{B}$ is closed under HSP if and only if $\mathcal{B}$ is equationally definable.*

Proof. $\mathsf{Set}^{\mathbb{I}}$ is monadic over $\mathsf{Set}^{\mathbb{N}}$, hence it has a regular factorisation system. Since $\mathrm{Sh}(\mathbb{I}^{op})$ is a Grothendieck topos, it has a epi-mono factorisation system. We do know that the reflection functor $R : \mathsf{Set}^{\mathbb{I}} \to \mathrm{Sh}(\mathbb{I}^{op})$ preserves finite limits, see [12]. Being a left adjoint, R preserves all colimits. Therefore it will preserve epis and monos, and also the factorisation system on $\mathsf{Set}^{\mathbb{I}}$. Now we can apply Theorem 3.4. □

Remark 3.10. *Notice that in the case of $\mathsf{Set}^{\mathbb{I}}$, we can assume that the equations defining a subcategory closed under HSP involve only finitely many variables. Indeed if $\mathcal{A} \subseteq \mathsf{Set}^{\mathbb{I}}$ is closed under HSP then $\mathcal{A}$ is also closed under directed unions.*

Proof. Let $(X_i)_{i \in I}$ be a directed family of elements of $\mathcal{A}$. Let us denote by X the directed union of this family. We need to show that $X \in \mathcal{A}$.

If X_i is the empty presheaf for all $i \in I$, then X is also the empty presheaf, thus is in $\mathcal{A}$.

Otherwise, consider the minimum natural number n such that there exists $k \in I$ such that $X_k([n])$ is non-empty. Notice that for all $m \geq n$ and for all $i \in I$ with $i \geq k$ we have that $X_i([m]) \neq \emptyset$. Since $\mathcal{A}$ is closed under products, we have that $\prod\limits_{i \geq k} X_i \in \mathcal{A}$. Also notice that $\prod\limits_{i \geq k} X_i$ is non-empty. We consider the subalgebra Y of $\prod\limits_{i \geq k} X_i$, such that for a natural number m, $Y([m])$ consists of tuples $(x_i)_{i \geq k}$ with the property that $x_i \in X_i$ for all $i \geq k$ and there exists i_0 such that if $i_0 \leq j$ then $x_j = x_{i_0}$. We have that $Y \in \mathcal{A}$ because $\mathcal{A}$ is closed under subalgebras. We finalize the proof by constructing an epimorphism $Y \twoheadrightarrow X$ which maps a family $(x_i)_{i \geq k} \in Y([m])$ like above to x_{i_0}. We conclude that $X \in \mathcal{A}$, since $\mathcal{A}$ is closed under homomorphic images. $\square$

Theorem 3.11. ***HSP-theorem for 'nominal algebras'.*** *In the situation of Diagram (2), let L be an endofunctor with a finitary presentation on $\mathsf{Set}^{\mathbb{I}}$ and let $\tilde{L}$ be such that $\tilde{L}R \cong RL$ and $\tilde{L}$ preserves epis. Then a full subcategory of $\mathsf{Alg}(\tilde{L})$ is closed under HSP if and only if it is equationally definable.*

Proof. Applying Proposition 3.5, we can lift the adjunction $R \dashv I$ to an adjunction $\overline{R} \dashv \overline{I}$ between the categories of $\tilde{L}$-algebras and L-algebras. However in order to be able to apply Theorem 3.4, we need to have a factorisation system on $\mathsf{Alg}(\tilde{L})$ and moreover $\overline{R}$ should preserve it. The following lemma shows that the former condition is met, under the assumption that $\tilde{L}$ preserves epis.

Lemma 3.12. *Assume that $\tilde{L} : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$ preserves epis. Then $\mathsf{Alg}(\tilde{L})$ has a factorisation system (E, M), where E consists of those epis e such that $\tilde{U}(e)$ is epi, and M consists of those monos such that $\tilde{U}(m)$ is mono.*

Next we prove that $\overline{R}$ preserves the factorisation system. We will use the following lemma:

Lemma 3.13. *Assuming L preserves sifted colimits, the forgetful functor $U : \mathsf{Alg}(L) \to \mathsf{Set}^{\mathbb{I}}$ preserves regular epimorphisms.*

Assume e is a regular epi in $\mathsf{Alg}(L)$. By Lemma 3.13 we have that Ue is a regular epi in $\mathsf{Set}^{\mathbb{I}}$ and since R preserves epis we conclude that RUe is an epimorphism in $\mathrm{Sh}(\mathbb{I}^{op})$. But $\tilde{U}\overline{R}e = RUe$, hence $\overline{R}e$ is a quotient in the factorisation system of $\mathsf{Alg}(\tilde{L})$ by Lemma 3.12. Notice that $\overline{R}$, R and U preserve monos, because U is a right adjoint and $\overline{R}$, R preserve finite limits. It follows that $\overline{R}m \in M$ for all monomorphisms m in $\mathsf{Alg}(L)$. So we can apply Theorem 3.4 in this case. $\square$

Proof of Lemma 3.12 Let $f : (A, u) \to (B, v)$ be a $\tilde{L}$-algebra homomorphism. Then $\tilde{U}(f)$ has a epi-mono factorisation in $\mathrm{Sh}(\mathbb{I}^{op})$. So we can factor $f = me$ where $e : A \to C$ is an epi and $m : C \to B$ is a mono in $\mathrm{Sh}(\mathbb{I}^{op})$. We first show that we have a $\tilde{L}$-algebra $w : \tilde{L}C \to C$ such that e and m are $\tilde{L}$-algebra homomorphisms. Notice that $\tilde{L}e : \tilde{L}A \to \tilde{L}C$ is an epi, and that $m : C \to B$ is mono, therefore from the orthogonality property of the factorisation systems it follows that there exists a unique $w : \tilde{L}C \to C$ such that the diagram below commutes:

$$\begin{tikzcd}
\tilde{L}A \arrow[rr, "u"] \arrow[dr, "\tilde{L}e"] \arrow[dd, "\tilde{L}f"'] & & A \arrow[dr, "e"] \arrow[dd, "f"] & \\
& \tilde{L}C \arrow[rr, "w"] \arrow[dl, "\tilde{L}m"] & & C \arrow[dl, "m"] \\
\tilde{L}B \arrow[rr, "v"] & & B &
\end{tikzcd} \tag{15}$$

So we have proved the existence of the factorisation. Now let us prove that (E, M) satisfies the orthogonality property. Assume we have the following commutative diagram of $\tilde{L}$-algebras, such that $e \in E$, $m \in M$ and u, v are $\tilde{L}$-algebras homomorphisms.

$$\begin{tikzcd}
& A \arrow[rr, "e"] \arrow[dd, "u"] & & B \arrow[dd, "v"] \\
\tilde{L}A \arrow[ur, "f_A"] \arrow[rr, "\tilde{L}e"] \arrow[dd, "\tilde{L}u"'] & & \tilde{L}B \arrow[ur, "f_B"] \arrow[dd, "\tilde{L}v"] & \\
& C \arrow[rr, "m"] & & D \\
\tilde{L}C \arrow[ur, "f_C"] \arrow[rr, "\tilde{L}m"] & & \tilde{L}D \arrow[ur, "f_D"] &
\end{tikzcd} \tag{16}$$

From the orthogonality property of the factorisation system on $\mathrm{Sh}(\mathbb{I}^{op})$ we know that there exists a unique $w : B \to C$ such that $we = u$ and $mw = v$. It remains to prove the fact that w is a $\tilde{L}$-algebra homomorphism. To this end notice that there exists a unique morphism from $t : \tilde{L}B \to C$ such that $t\tilde{L}e = uf_A$ and $mt = vf_B$. But both $f_C\tilde{L}w$ and wf_B have this property, therefore we must have that $f_C\tilde{L}w = wf_B$. $\square$

Proof of Lemma 3.13 L preserves sifted colimits, therefore $\mathsf{Alg}(L)$ is a many sorted variety. Denote by $U_1 : \mathsf{Alg}(L) \to \mathsf{Set}^{\mathbb{N}}$ and by $U_2 : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{N}}$ the forgetful functors. In $\mathsf{Set}^{\mathbb{N}}$ the regular epimorphisms are retractions.

This implies that U_1 and U_2 are regularly monadic, so by [1, Proposition 20.30] if follows that they preserve and reflect regular epimorphisms. The fact that U preserves regular epis follows now from the fact that $U_1 = U_2 U$. $\square$

4 Presentations for Presheaves and Algebras of Presheaves

Our motivation comes from the fact that nominal algebras correspond to algebras of sheaves, but the topological feature of sheaves cannot be captured algebraically. However, one can still give equational theories for the category of algebras of presheaves. In the first part of this section we give presentations for the category $\mathsf{Set}^{\mathbb{I}}$ of presheaves on $\mathbb{I}^{op}$, and for $\mathsf{Set}^{\mathbb{J}}$, where $\mathbb{J}$ is the category with objects finite subsets of the set of names $\mathcal{N}$, and morphisms injective maps between them. Next we illustrate by example, how one can obtain equational theories for categories of algebras on $\mathsf{Set}^{\mathbb{I}}$, respectively on $\mathsf{Set}^{\mathbb{J}}$. In particular, in Section 4.4 we define a 'shift' functor on $\mathsf{Set}^{\mathbb{J}}$, that corresponds to the abstraction operator of [8].

4.1 A presentation for $\mathsf{Set}^{\mathbb{I}}$

As is the case for any functor category, $\mathsf{Set}^{\mathbb{I}}$ can be regarded as a many-sorted equational class: the set of sorts is the set of natural numbers $\mathbb{N}$ (the objects of $\mathbb{I}$); all the operations symbols are unary and correspond to the morphisms of $\mathbb{I}$, while the equations are given by the commutative diagrams in $\mathbb{I}$. However we can give a much more efficient axiomatisation for this category, such that the set of equations for each sort is finite. In order to find an equational presentation for $\mathsf{Set}^{\mathbb{I}}$ we consider the following set $\Sigma_{\mathbb{I}}$ of operation symbols:

$$\Sigma_{\mathbb{I}} =^{def} \{\sigma_n^{(i)} |\ \ 1 < n,\ 1 \leq i < n\} \cup \{w_n\ \ |\ \ n \geq 0\} \cup \{\sigma_0\} \qquad (17)$$

The symbol $\sigma_n^{(i)}$ has arity $n \to n$ and corresponds to the transposition $\sigma_n^{(i)} = (i, i+1)$ of the set $[n]$. The symbol w_n has arity $n \to n+1$ and corresponds to the inclusion w_n of $[n]$ into $[n+1]$, while $\sigma_0 : 0 \to 0$ corresponds to the empty map $\emptyset : \emptyset \to \emptyset$. One can see that each injective map in $\mathbb{I}$ can be expressed as a composition of these morphisms, that obviously satisfy the set of equations $E_{\mathbb{I}}$ given below:

$$
\begin{array}{lll}
\sigma_{n+1}^{(i)} w_n = w_n \sigma_n^{(i)} & 1 \le i < n & (\mathrm{E}_1) \\
\sigma_{n+2}^{(n+1)} w_{n+1} w_n = w_{n+1} w_n & & (\mathrm{E}_2) \\
w_0 \sigma_0 = w_0 & & (\mathrm{E}_3) \\
(\sigma_n^{(i)})^2 = id_n & 1 \le i < n & (\mathrm{E}_4) \\
\sigma_n^{(i)} \sigma_n^{(j)} = \sigma_n^{(j)} \sigma_n^{(i)} & j \ne i \pm 1; 1 \le i, j < n & (\mathrm{E}_5) \\
(\sigma_n^{(i)} \sigma_n^{(i+1)})^3 = id_n & 1 \le i < n-1 & (\mathrm{E}_6)
\end{array}
$$

Notice that the equations (E_4)-(E_6) come from the well-known presentation of the symmetric groups.

In [10] we gave such a presentation for the functor category $\mathsf{Set}^{\mathbb{F}}$, where $\mathbb{F}$ is the category of finite ordinals and all maps. Since $\mathbb{I}$ is a subcategory of $\mathbb{F}$, we can regard $\mathsf{Set}^{\mathbb{F}}$ as a subcategory of $\mathsf{Set}^{\mathbb{I}}$. However the equational theory for $\mathsf{Set}^{\mathbb{F}}$ requires more operation symbols to capture non-injective functions, and obviously additional equations involving these symbols.

In order to prove that $(\Sigma_{\mathbb{I}}, E_{\mathbb{I}})$ is a presentation for $\mathsf{Set}^{\mathbb{I}}$ it remains to show that for any two representations of an injective map $i : [n] \to [m]$ in terms of σ-s and w-s, one can prove that the two representations are equivalent via the equations (E_1)-(E_6). To this end, notice that using equations (E_1) and (E_3) one can reduce each representation of i to one of the form $\sigma_m^{(i_1)} \dots \sigma_m^{(i_k)} w_{m-1} \dots w_n$. For this reason it only remains to prove that:

Lemma 4.1. *If the equality $\sigma_m^{(i_1)} \dots \sigma_m^{(i_k)} w_{m-1} \dots w_n = w_{m-1} \dots w_n$ holds in $\mathbb{I}$, then the equality can be deduced from $E_{\mathbb{I}}$.*

Proof. Since the equality $\sigma_m^{(i_1)} \dots \sigma_m^{(i_k)} w_{m-1} \dots w_n = w_{m-1} \dots w_n$ holds in $\mathbb{I}$, we deduce that the permutation $\sigma_m^{(i_1)} \dots \sigma_m^{(i_k)}$ leaves invariant the elements $1, \dots, n$, therefore, using (E_4)-(E_6) if needed, we may assume that $i_1, \dots, i_k$ are greater than n. The proof is by induction on k. It is obvious for $k = 0$. Assume the statement of the lemma holds for $k-1$, and let us prove that it also holds for k. Assume that $\sigma_m^{(i_1)} \dots \sigma_m^{(i_k)} w_{m-1} \dots w_n = w_{m-1} \dots w_n$ holds in $\mathbb{I}$, where $n < i_1, \dots, i_k < m$. Applying repeatedly (E_1), we can deduce that $\sigma_m^{(i_1)} \dots \sigma_m^{(i_k)} w_{m-1} \dots w_n = \sigma_m^{(i_1)} \dots \sigma_m^{(i_{k-1})} w_{m-1} \dots \sigma_{i_k+1}^{(i_k)} w_{i_k} w_{i_k-1} \dots w_n$. Since $i_k > n$ and $\sigma_{i_k+1}^{(i_k)} w_{i_k} w_{i_k-1} = w_{i_k} w_{i_k-1}$ by (E_2), we obtain that

$$
\sigma_m^{(i_1)} \dots \sigma_m^{(i_k)} w_{m-1} \dots w_n = \sigma_m^{(i_1)} \dots \sigma_m^{(i_{k-1})} w_{m-1} \dots w_n \tag{18}
$$

The conclusion follows from the induction hypothesis. $\square$

4.2 A presentation for algebras on $\mathsf{Set}^{\mathbb{I}}$

So far we have proved that $\mathsf{Set}^{\mathbb{I}} \simeq \mathsf{Alg}(\Sigma_{\mathbb{I}}, E_{\mathbb{I}})$. Suppose L is an endofunctor on $\mathsf{Set}^{\mathbb{I}}$ which has a finitary presentation by operations and equations as defined in Section 2.2. Then one can obtain an equational theory for $\mathsf{Alg}(L)$ in a modular way using Theorem 2.5. Let us consider an example. We endow $\mathbb{I}$ with the coproduct structure:

$$\begin{tikzcd} & {[1]} \arrow[d, "new"] \\ {[n]} \arrow[r, "i"] & {[n+1]} \end{tikzcd} \tag{19}$$

where i is the inclusion and $new(1) = n + 1$. We define a 'shift' functor $\delta : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$, as in [6]

$$\delta(A)(f) = A(f + id_1) \tag{20}$$

for all $A \in \mathsf{Set}^{\mathbb{I}}$ and for all maps f in $\mathbb{I}$. Let $L : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$ be the functor given by

$$LX = \delta X + X \times X \tag{21}$$

In [10] we have given a presentation for the functor on $\mathsf{Set}^{\mathbb{F}}$ defined by the same formula, $\delta X + X \times X$. If we disregard the equations in which appear operation symbols corresponding to 'contraction' maps (the additional operation symbols in the presentation of $\mathbb{F}$), we can obtain without difficulty a presentation for the functor L. For each non-negative integer n we need operation symbols $\mathsf{lam}_{n+1}, \mathsf{app}_n$ corresponding semantically to λ-abstraction and application. The equations are of the following form:

$$\begin{array}{ll} \sigma_n^{(i)}(\mathsf{lam}_{n+1}, id_{n+1}t) = id_n(\mathsf{lam}_{n+1}, \sigma_{n+1}^{(i)}t) & [t] \\ w_n(\mathsf{lam}_{n+1}, id_{n+1}t) = id_{n+1}(\mathsf{lam}_{n+2}, \sigma_{n+2}^{(n+1)}w_{n+1}t) & [t] \\ \sigma_n^{(i)}(\mathsf{app}_n, id_n t_1, id_n t_2) = id_n(\mathsf{app}_n, \sigma_n^{(i)}t_1, \sigma_n^{(i)}t_2) & [t_1, t_2] \\ w_n(\mathsf{app}_n, id_n t_1, id_n t_2) = id_{n+1}(\mathsf{app}_{n+1}, w_n t_1, w_n t_2) & [t_1, t_2] \end{array} \tag{22}$$

where t is a variable of sort $n + 1$ and t_1, t_2 are variables of sort n.

4.3 A presentation for $\mathsf{Set}^{\mathbb{J}}$

To make the **connection with nominal sets** more explicit one can replace the category $\mathbb{I}$ by the category $\mathbb{J}$ with objects finite subsets of the set of names

$\mathcal{N}$, and morphisms injective maps between them. Similarly, $\mathsf{Set}^{\mathbb{J}}$ is a many-sorted variety; there exists a sort for each finite subset of $\mathcal{N}$. Although it is equivalent to $\mathsf{Set}^{\mathbb{I}}$, when trying to find a presentation for $\mathsf{Set}^{\mathbb{J}}$ one encounters significant syntactical differences. For example one would be tempted to use operation symbols of the form $(a,b)_S$, which correspond to swapping the names a and b of a set S, and $w_{S,a}$, which correspond to inclusions of S into $S \cup \{a\}$. However swappings and inclusions fail to generate all the bijections in $\mathbb{J}$. For example, if $a \neq b$ and $a, b \notin S$, they cannot generate a bijection from $S \cup \{a\}$ to $S \cup \{b\}$ which maps a to b and acts as identity on the remaining elements of S.

This example suggests the following set $\Sigma_{\mathbb{J}}$ of operation symbols with specified arity.

$$\begin{array}{ll} (b/a)_S : S \cup \{a\} \to S \cup \{b\} & a \neq b,\ a \notin S,\ b \notin S \\ w_{S,a} : S \to S \cup \{a\} & a \notin S \end{array} \tag{23}$$

We will refer to operation symbols of the form $(b/a)_S$ as 'substitutions' and to operations symbols of the form $w_{S,a}$ as 'inclusions'. When the arity can be inferred from the context, or is irrelevant, we will omit S from the subscript.

We consider the set $E_{\mathbb{J}}$ of equations of the form:

$$\begin{array}{lll} (a/b)_S(b/a)_S = id & : S \cup \{a\} & (\mathrm{E}_1) \\ (b/a)_{S\cup\{d\}}(d/c)_{S\cup\{a\}} = (d/c)_{S\cup\{b\}}(b/a)_{S\cup\{c\}} & : S \cup \{b,d\} & (\mathrm{E}_2) \\ (c/b)_S(b/a)_S = (c/a)_S & : S \cup \{c\} & (\mathrm{E}_3) \\ (b/a)_{S\cup\{c\}}w_{S\cup\{a\},c} = w_{S\cup\{b\},c}(b/a)_S & : S \cup \{c,b\} & (\mathrm{E}_4) \\ (b/a)_S w_{S,a} = w_{S,b} & : S \cup \{b\} & (\mathrm{E}_5) \\ w_{S\cup\{b\},a}w_{S,b} = w_{S\cup\{a\},b}w_{S,a} & : S \cup \{a,b\} & (\mathrm{E}_6) \end{array} \tag{24}$$

Theorem 4.2. $(\Sigma_{\mathbb{J}}, E_{\mathbb{J}})$ *is a presentation for* $\mathsf{Set}^{\mathbb{J}}$.

Proof. (a) First we show that $\mathsf{Set}^{\mathbb{J}}$ is a category of $(\Sigma_{\mathbb{J}}, E_{\mathbb{J}})$-algebras. The intended interpretation of the operation symbols above is the following: If $a, b \notin S$ then $(b/a)_S$ corresponds to the bijective map from $S \cup \{a\}$ to $S \cup \{b\}$ which substitutes b for a. The symbol $w_{S,a}$ corresponds to the inclusion of S into $S \cup \{a\}$. It easy to check that these morphisms satisfy the equations listed above. We have to check that each morphism in $\mathbb{J}$ can be written as a composition of such inclusions and substitutions. First notice that the swapping of elements a, b of a set $S \cup \{a, b\}$, is obtained as the composition

$$\sigma_{a,b} = (b/c)_{S\cup\{a\}}(a/b)_{S\cup\{c\}}(c/a)_{S\cup\{b\}} \tag{25}$$

where $c \notin S \cup \{a, b\}$. Therefore all bijections are generated by substitutions. If the cardinality of a subset S of $\mathcal{N}$ is less or equal than the cardinality of a finite subset T of $\mathcal{N}$, then one can construct an injective map $i : S \to T$, by enlarging S with elements of $T \setminus S$ (using the inclusions) until it reaches the cardinality of T, and then by substituting the remaining elements of $T \setminus S$ for those of $S \setminus T$. Now any other map $j : S \to T$ is obtained by composing i with a bijection on T.

(b) Conversely, it is enough to check that different representations of an injective map $\iota : S \to T$ in $\mathbb{J}$ as composition of inclusions and substitutions are equivalent via the equations $E_{\mathbb{J}}$. Using (E_4) one can prove that each representation of ι can be reduced to one of the form $s_1 \dots s_k w_{a_1} \dots w_{a_l}$ where the s_i-s stand for substitutions. Using (E_1) one can reduce the problem to showing that if the equality $s_1 \dots s_k w_{a_1} \dots w_{a_l} = w_{b_1} \dots w_{b_h}$ holds in $\mathbb{J}$ then it can be derived from $E_{\mathbb{J}}$. For cardinality reasons we must have $l = h$. Notice that using (E_5) and (E_6) we can reduce this to the simpler problem in which $\{a_1, \dots, a_l\} = \{b_1, \dots, b_l\}$. Assume that w_{a_l} has arity $S \to S \cup \{a_l\}$. The arities for the rest of the w's can be now deduced. Because the equality holds in $\mathbb{J}$ we have that $s_1 \dots s_k$ is a permutation on $S \cup \{a_1, \dots, a_l\}$ which is the identity when restricted to S.

We finalize the proof using the well know presentation of the symmetric groups. Firstly, Lemma 4.4 asserts that the equations which are enough to give a presentation for the symmetric group are satisfied by $\sigma_{a,b}$-s, where $\sigma_{a,b}$ is the the abbreviation introduced in (25). Secondly, Lemma 4.3 asserts that a sequence of substitutions whose interpretation is a permutation, can be reduced to a sequence of $\sigma_{a,b}$-s. Then the sequence $s_1 \dots s_k$ can be rewritten as a sequence of $\sigma_{a,b}$-s with the a and b only from the set $\{a_1, \dots, a_l\}$. To finish, notice that it is straightforward to derive from $E_{\mathbb{J}}$ that $\sigma_{a_i,a_l} w_{a_i} w_{a_l} = w_{a_i} w_{a_l}$. □

Lemma 4.3. *Let $s_1, \dots, s_k$ be a sequence of substitutions, such that the composition $s_1 \dots s_k$ of their interpretation in $\mathbb{J}$ is possible and moreover it is a permutation. Then we can reduce $s_1 \dots s_k$ to a sequence of σ_{a_i,b_i}'s, where each σ_{a_i,b_i} is a sequence of 3 substitutions as defined in (25).*

Proof. The proof is by induction on k. If $k = 0$ we have nothing to prove. Assume the statement of the lemma has been proved for $k - 1$, let us prove it for k. Assume $s_k = (c/a)$. This means that $s_1 \dots s_k$ is a permutation on a set which contains a. In particular a is in the image of $s_1 \dots s_k$. Therefore there exists i such that $1 \le i < k$ and s_i is of the form $(a/y)_T$ for some atom y and some set T. Consider the i maximal with this property. The idea is to rewrite the sequence using the equations such that the rightmost substitution of the form $(a/y)_T$ can be moved to position $k-1$. If $i < k-1$ we know that the substitutions $s_{i+1}, \dots, s_{k-1}$ do not involve a. We have two cases:

1. y does not appear in the substitutions $s_{i+1}, \dots, s_{k-1}$. We can use (E_2) to prove that the sequence $s_1 \dots s_k$ can be rewritten to a sequence $s'_1 \dots s'_{k-1} s_k$ such that $s'_{k-1} = (a/y)$. If $y = c$ we can reduce the sequence to a shorter one, via (E_1) and apply the induction hypothesis. If $y \neq c$, then, by (E_1), we know that $s'_1 \dots s'_{k-1} s_k = s'1 \dots s'_{k-2}(c/y)(y/c)(a/y)(c/a)$. But this is equal to $s'_1 \dots s'_{k-2}(c/y)\sigma_{a,y}$. By the induction hypothesis $s'_1 \dots s'_{k-2}(c/y)$, which is of length $k-1$ can be reduced to a sequence of transpositions.

2. y does appear in the substitutions $s_{i+1}, \dots, s_{k-1}$. Because of (E_2), we may assume without loss of generality that $s_{i+1} = (y/w)$. But now we can use (E_3) to reduce the sequence $s_i s_{i+1}$ to (a/w). The resulting sequence is shorter and the conclusion follows by the induction hypothesis.

□

Lemma 4.4. *The following can be derived from the equations $E_{\mathbb{J}}$:*

1. *For all pairwise distinct names $a, b, x, y \notin S$ we have that*

$$(b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}} = (b/y)_{S\cup\{a\}}(a/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}}$$

   In what follows we will abbreviate $(b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}}$ by $\sigma_{a,b}$.

2. *If $a \neq b$ then $\sigma^2_{a,b} = \mathrm{id}$*

3. *If a, b, c, d are pairwise distinct names then $\sigma_{a,b}\sigma_{c,d} = \sigma_{c,d}\sigma_{a,b}$.*

4. *If a, b, c are pairwise distinct names then $(\sigma_{a,b}\sigma_{b,c})^3 = \mathrm{id}$.*

Proof. 1. From the equations we can derive:

$$\begin{aligned}
&(b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}} \\
&\quad =^{(E_1)} (b/x)_{S\cup\{a\}}(x/y)_{S\cup\{a\}}(y/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}} \\
&\quad =^{(E_3)} (b/y)_{S\cup\{a\}}(y/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}} \\
&\quad =^{(E_2)} (b/y)_{S\cup\{a\}}(a/b)_{S\cup\{y\}}(y/x)_{S\cup\{b\}}(x/a)_{S\cup\{b\}} \\
&\quad =^{(E_3)} (b/y)_{S\cup\{a\}}(a/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}}
\end{aligned} \tag{26}$$

2. Choose x, y distinct from a, b. Then

$$\begin{aligned}
&\sigma^2_{a,b} \\
&\quad = (b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}}(b/y)_{S\cup\{a\}}(a/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}} \\
&\quad =^{(\mathrm{E}_2)} (b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(b/y)_{S\cup\{x\}}(x/a)_{S\cup\{y\}}(a/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}} \\
&\quad =^{(\mathrm{E}_3)} (b/x)_{S\cup\{a\}}(a/y)_{S\cup\{x\}}(x/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}} \\
&\quad =^{(\mathrm{E}_2)} (b/x)_{S\cup\{a\}}(x/b)_{S\cup\{a\}}(a/y)_{S\cup\{b\}}(y/a)_{S\cup\{b\}} \\
&\quad =^{(\mathrm{E}_1)} \mathrm{id}_{S\cup\{a,b\}}
\end{aligned} \tag{27}$$

3. This follows easily from point 1. above and (E_2).

4. This can be proved in the same spirit as point 2. above. The key is to show that $\sigma_{a,b}\sigma_{b,c} = (c/y)(a/c)(b/a)(y/b)$ for some $y \notin \{a, b, c\}$.

$\square$

4.4 A presentation for the 'shift' functor on $\mathsf{Set}^{\mathbb{J}}$

We will define a 'shift' functor δ on $\mathsf{Set}^{\mathbb{J}}$. Assume $P : \mathbb{J} \to \mathsf{Set}$ is a presheaf and $S \subseteq \mathcal{N}$ is a finite set of names. We define an equivalence relation $\equiv$ on $\coprod_{a\notin S} P(S \cup \{a\})$. If $a, b \notin S$, $x \in P(S \cup \{a\})$ and $y \in P(S \cup \{b\})$ we will say that x and y are equivalent if and only if $P((b/a)_S)(x) = y$. We define $(\delta P)(S)$ as the set of equivalence classes of the elements of $\coprod_{a\notin S} P(S \cup \{a\})$. If $x \in P(S \cup \{a\})$ we denote the equivalence class of x by $\overline{x}^{S,a}$.

If $f : S \to T$ is a morphism in $\mathbb{J}$ and $a \notin S \cup T$ we will denote by $f + a : S \cup \{a\} \to T \cup \{a\}$ the function which restricted to S is f and which maps a to a. We define $(\delta A)(f)(\overline{x}^{S,a}) = \overline{A(f + a)(x)}^{T,a}$ for some $a \notin S$. One can easily check that $(\delta A)(f)$ is well defined and that δ is a functor.

Next, we give a presentation for δ. We will denote by $\mathcal{P}_f(\mathcal{N})$ the set of finite subsets of $\mathcal{N}$. For each finite subset of names and for each $a \notin S$ we consider an operation symbol $[a]_S : S \cup \{a\} \to S$, and we will denote by Σ_δ the corresponding functor on $\mathsf{Set}^{\mathcal{P}_f(\mathcal{N})}$. This is given by $(\Sigma_\delta X)_S = \coprod_{a\notin S} \{[a]_S\} \times X_{S\cup\{a\}}$. We denote by $U : \mathsf{Set}^{\mathbb{J}} \to \mathsf{Set}^{\mathcal{P}_f(\mathcal{N})}$ the forgetful functor and by F its left adjoint. By $\mathcal{P}_f(\mathcal{N})$ we have denoted the set of finite subsets of $\mathcal{N}$. For any functor $P : \mathbb{J} \to \mathsf{Set}$ we can give an interpretation of these operation symbols, captured by a natural transformation $\rho_P : \Sigma_\delta UP \to U\delta P$ defined by:

$$\forall \alpha \in P(S \cup \{a\}) \qquad ([a]_S, \alpha) \mapsto \overline{\alpha}^{S,a} \in (U\delta P)(S)$$

The equations should correspond to the kernel pair of the adjoint transpose $\rho_P^\sharp : F\Sigma_\delta UP \to \delta P$. We will use the fact that for any $X = (X_S)_{S\in\mathcal{P}_f\mathcal{N}}$ we have $(FX)_S = \coprod_{T\in\mathcal{P}_f(\mathcal{N})} X_T \cdot \hom(T,S)$, where $\cdot$ is the copower. For $f : T \to S$ and $x \in X_T$ we denote by fx the element of $(FX)_S$ which is the copy of f corresponding to x. The equations E_δ will have the form:

$$
\begin{array}{lll}
(c/b)_S([a]_{S\cup\{b\}}, t) = \mathrm{id}_{S\cup\{c\}}([a]_{S\cup\{c\}}, (c/b)_{S\cup\{a\}}t) & t : S\cup\{a,b\} & (\mathrm{E}_1)\\
([a]_S, t) = ([b]_S, (b/a)_S t) & t : S\cup\{a\} & (\mathrm{E}_2)\\
w_{S,b}([a]_S, t) = \mathrm{id}_{S\cup\{b\}}([a]_{S\cup\{b\}}, w_{S\cup\{a\},b}t) & t : S\cup\{a\} & (\mathrm{E}_3)
\end{array}
$$

Theorem 4.5. *$(\Sigma_\delta, E_\delta)$ is a presentation for δ.*

Proof. One has to check that $\rho_P^\sharp : F\Sigma_\delta UP \to \delta P$ is a joint coequalizer as in (4).

First we can check that $\rho_P^\sharp$ makes the diagram commutative. Notice that $\rho_P^\sharp(f([a]_S, \alpha)) = (\delta P)(f)(\overline{\alpha}^{S,a})$ for all $\alpha \in P(S\cup\{a\})$ and $f : S \to T$. Let t be a variable of sort $S\cup\{a,b\}$ and let $v : V \to UP$ be a valuation such that $v(t) = \alpha \in (UP)(S\cup\{a,b\})$. On one hand we have:

$$
\begin{aligned}
\rho_P^\sharp(F\Sigma_\delta Uv^\sharp(&(c/b)_S([a]_{S\cup\{b\}}, id_{S\cup\{a,b\}}t)))\\
&= \rho_P^\sharp((c/b)_S([a]_{S\cup\{b\}}, \alpha))\\
&= \delta(P)((c/b)_S)(\overline{\alpha}^{S\cup\{b\},a})\\
&= \overline{P((c/b)_{S\cup\{a\}})(\alpha)}^{S,c}
\end{aligned}
\tag{28}
$$

On the other hand,

$$
\begin{aligned}
\rho_P^\sharp(F\Sigma_\delta Uv^\sharp(&\mathrm{id}_{S\cup\{c\}}([a]_{S\cup\{c\}}, (c/b)_{S\cup\{a\}}t)))\\
&= \rho_P^\sharp(\mathrm{id}_{S\cup\{c\}}([a]_{S\cup\{c\}}, P((c/b)_{S\cup\{a\}})(\alpha)))\\
&= \delta(P)(\mathrm{id}_{S\cup\{c\}})(\overline{P((c/b)_{S\cup\{a\}})(\alpha)}^{S\cup\{b\},a})\\
&= \overline{P((c/b)_{S\cup\{a\}})(\alpha)}^{S,c}
\end{aligned}
\tag{29}
$$

This proves that equations of the form (E_1) are satisfied. The verifications for (E_2) and (E_3) are similar and will be left to the reader.

Conversely, suppose that $\xi : F\Sigma_\delta UA \to D$ is another morphism which makes diagram (4) commutative. We need to construct a morphism $\xi' : \delta P \to D$ such that $\xi'\rho_P^\sharp = \xi$. It is enough to show that if

$$
\rho_P^\sharp(f([a]_S, \alpha)) = \rho_P^\sharp(g([b]_T, \beta))
\tag{30}
$$

for some $f : S \to Q$ and $g : T \to W$, $\alpha \in P(S \cup \{a\})$ and $\beta \in P(T \cup \{b\})$ then we also have that

$$\xi(f([a]_S, \alpha)) = \xi(g([b]_T, \beta)) \tag{31}$$

But (30) is equivalent to $(\delta P)(f)(\overline{\alpha}) = (\delta P)(g)(\overline{\beta})$. From this we obtain that $Q = W$ and that there exists $x \in \mathcal{N} \setminus (S \cup T \cup W)$ such that

$$P(f + x)P((x/a)_S)(\alpha) = P(g + x)P((x/b)_S)(\beta) \tag{32}$$

Claim. $(f([a]_S, t), id_W([x]_W, (f + x)(x/a)_S t))$ belongs to the congruence relation generated by the equations E_δ in $F\Sigma_\delta UFV$ for an arbitrary variable t of sort $S \cup \{a\}$ and $x \notin S \cup W \cup \{a\}$. The statement is obvious if f is the identity, a substitution or a basic inclusion. In general this can be proved by induction on the structure of f.

Fix V a set of variables, and chose $t \in V$ a variable of sort $S \cup \{a\}$ and $s \in V$ of type $S \cup \{b\}$. Consider an instantiation of this variables $v : V \to UA$ such that $v(t) = \alpha$ and $v(s) = \beta$. We have that

$$F\Sigma_\delta Uv^\sharp(\mathrm{id}_{W\cup\{x\}}([x]_W, (f+x)(x/a)_S t)) = \mathrm{id}_{W\cup\{x\}}([x]_W, P(f+x)P((x/a)_S)(\alpha))$$

and analogously

$$F\Sigma_\delta Uv^\sharp(\mathrm{id}_{W\cup\{x\}}([x]_W, (g+x)(x/b)_S s)) = \mathrm{id}_{W\cup\{x\}}([x]_W, P(g+x)P((x/b)_S)(\beta))$$

Using (32) and the claim above applied twice for f, respectively g, and $x \in \mathcal{N} \setminus (S \cup T \cup W)$, we obtain that

$$\xi(F\Sigma_\delta Uv^\sharp(f([a]_S, t))) = \xi(F\Sigma_\delta Uv^\sharp(g([a]_T, s)))$$

and this is equivalent to (31). $\square$

Notation 4.6. *In what follows we will denote an element* $\overline{x}^{S,a} \in (\delta P)(S)$ *by* $\{[a]_S x\}_{\delta P}$.

Remark 4.7. *From Lemma 5.12 it follows that for* $x \in X(S)$ *with* $a, b \in S$ *we have that* $\{[a]_{S\setminus\{a\}} x\}_{\delta P}$ *and* $\{[b]_{S\setminus\{b\}} P(\sigma_{a,b})(x)\}_{\delta P}$ *are identified if they are regarded as elements of* $\delta(P)(S)$.

5 Concrete syntax for λ-calculus

In this section we combine the results of Section 3 and Section 4. We illustrate the concrete syntax obtained in our setting, by giving a theory for λ-calculus. Let us consider an endofunctor $L : \mathsf{Set}^{\mathbb{J}} \to \mathsf{Set}^{\mathbb{J}}$ defined by

$$LX = \mathcal{N} + \delta X + X \times X \tag{33}$$

where δ is defined as in Section 4.

The sheaves for the atomic topology on $\mathbb{J}^{op}$ form a full reflective subcategory $\mathrm{Sh}(\mathbb{J}^{op})$ of $\mathsf{Set}^{\mathbb{J}}$, which is equivalent to $\mathrm{Sh}(\mathbb{I}^{op})$, and therefore also to Nom. We will consider an endofunctor $\tilde{L}$ on $\mathrm{Sh}(\mathbb{J}^{op})$ corresponding to L. Theorem 3.11 is also valid if we replace $\mathbb{I}$ by $\mathbb{J}$, because the proof uses general properties of Grothendieck toposes. In Section 5.2 we establish that the HSP-theorem holds for $\tilde{L}$-algebras. In Section 5.3 we prove that the initial algebra for $\tilde{L}$ is just the sheaf of λ-terms up to α-equivalence, and we give the equations that characterize the subalgebra of λ-terms modulo $\alpha\beta\eta$-equivalence.

5.1 The toposes $\mathrm{Sh}(\mathbb{I}^{op})$ and $\mathrm{Sh}(\mathbb{J}^{op})$

The results of this subsection will only be used for proving some technical lemmas in the next subsection.

The categories $\mathbb{I}$ and $\mathbb{J}$ are equivalent. In order to prove this we will choose an order of the names; assume $\mathcal{N} = \{a_1, a_2, \dots\}$. We can consider a functor $j : \mathbb{I} \to \mathbb{J}$ defined by $j([n]) = \{a_1, \dots, a_n\}$ for any object $[n]$ of $\mathbb{I}$ and $j(f)(a_i) = a_{f(i)}$ for any $f : [n] \to [m]$.

Define $c : \mathbb{J} \to \mathbb{I}$ by $c(S) = [n]$ if and only if n is the cardinal of the set S. Assume $S = \{a_{i_1}, \dots, a_{i_n}\}$ such that $i_1 < \dots < i_n$. We can consider the bijection $\phi_S : S \to [n]$ which maps a_{i_k} to $k \in [n]$. Now we can define c on arrows. If $f : S \to T$ we define $c(f)$ as the composition $\phi_T f \phi_S^{-1}$.

We can see immediately that $cj = \mathrm{id}_{\mathbb{I}}$ and that $jc \simeq \mathrm{id}_{\mathbb{J}}$. This equivalence can be lifted to the presheaf categories, if we define $J- = - \circ j$ and $C- = - \circ c$.

$$\begin{tikzcd} \mathsf{Set}^{\mathbb{I}} \arrow[rr, bend right, "C"'] \arrow[rr, phantom, "\simeq"] & & \mathsf{Set}^{\mathbb{J}} \arrow[ll, bend right, "J"'] \end{tikzcd} \tag{34}$$

It is well known that the category of nominal sets is equivalent to the Schanuel topos $\mathrm{Sh}(\mathbb{I}^{op})$, which consists of the full subcategory of $\mathsf{Set}^{\mathbb{I}}$ of functors which preserve pull-backs and monomorphisms. However we will need some technical details regarding the characterization of $\mathrm{Sh}(\mathbb{I}^{op})$ as a Grothendieck topos of sheaves on $\mathbb{I}^{op}$ for the atomic topology. We refer the reader to [12, Chapter III] for the basic definitions of notions such as Grothendieck topology, sieve, matching family, amalgamation, sheaf or atomic topology. However we shall make these definitions explicit in the case of $\mathbb{I}^{op}$ and $\mathbb{J}^{op}$.

Firstly let us notice that a sieve of $\mathbb{I}^{op}$ is a family of morphisms in $\mathbb{I}$ with common domain. If $n \in \mathbb{N}$ and $k \geq n$ then the set of all morphism $\{f : [n] \to [l] \mid l \geq k\}$ is a sieve on $[n]$, denoted by S_n^k.

Claim 5.1. *For any non-empty sieve $\mathcal{S}$ on $[n]$ there exists a unique $k \in \mathbb{N}$ such that $\mathcal{S} = S_n^k$.*

Proof. We have to consider the smallest k such that there exists $i : [n] \to [k]$ in S. For any $l \geq k$ and any injective map $h : [n] \to [l]$ there exists a morphism in $\mathbb{I}$ $h' : [k] \to [l]$ such that $h = h' \circ i$. The fact that $h \in S$ follows from the fact that S is a sieve on $\mathbb{I}^{op}$. $\square$

Similarly one can characterize the non-empty sieves on an object T of $\mathbb{J}$. Assume n is the cardinal of T. For any non-empty sieve $\mathcal{S}$ on T there exists a natural number k, such that the $\mathcal{S}$ is the set $\mathcal{S}_T^k$ of all morphisms $f : T \to S$, for an arbitrary set S with cardinality larger or equal than k.

The atomic topology on either $\mathbb{I}^{op}$ or $\mathbb{J}^{op}$ is obtained by considering for each object the collection of all non-empty sieves on it. As one might expect the equivalence between $\mathsf{Set}^{\mathbb{I}}$ and $\mathsf{Set}^{\mathbb{J}}$ restricts to an equivalence between the toposes $\mathrm{Sh}(\mathbb{I}^{op})$ and $\mathrm{Sh}(\mathbb{J}^{op})$.

As mentioned in Section 2, the inclusion functor $I : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathsf{Set}^{\mathbb{I}}$ has a left adjoint R which preserves finite limits. The same applies for the inclusion functor $I' : \mathrm{Sh}(\mathbb{J}^{op}) \to \mathsf{Set}^{\mathbb{J}}$, which has a left adjoint R', as described in the diagram below.

$$
\begin{tikzcd}
\mathsf{Set}^{\mathbb{I}} \arrow[rr, bend right, "C"'] \arrow[d, bend right, "R"'] \arrow[rr, phantom, "\simeq"] \arrow[d, phantom, "\dashv"] & & \mathsf{Set}^{\mathbb{J}} \arrow[ll, bend right, "J"'] \arrow[d, bend right, "R'"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{Sh}(\mathbb{I}^{op}) \arrow[u, bend right, "I"'] & & \mathrm{Sh}(\mathbb{J}^{op}) \arrow[u, bend right, "I'"']
\end{tikzcd}
$$

We will now take a closer look at how the 'sheafification' functors R and R' are defined. In both cases, the 'sheafification' of a presheaf P is obtained in two steps. Starting with a presheaf P, one can construct a presheaf P^+: for any object C, $P^+(C)$ is defined as the set of equivalence classes of matching families on C. Two matching families for sieves $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be equivalent if they coincide on a common refinement of the two sieves. For a morphism $i : C \to D$ we define $P^+(i) : P^+(C) \to P^+(D)$ as follows: The equivalence class of a matching family $\{x_f \mid f \in \mathcal{S}\}$ is mapped to the equivalence class of $\{x_{g\circ i} \mid g \in i^*\mathcal{S}\}$, where $i^*\mathcal{S}$ is the sieve containing all the morphisms g such that $gi \in \mathcal{S}$. Notice that in general $(\cdot)^+$ is a functor on the presheaf category. We will denote by $(\cdot)^+$ the corresponding functor on $\mathsf{Set}^{\mathbb{I}}$, and by $(\cdot)^\oplus$ the corresponding functor on $\mathsf{Set}^{\mathbb{J}}$. The sheafification of P is obtained as $R(P) = (P^+)^+$. The unit of the adjunction $R \dashv I$ is obtained in two steps as well: $\eta : P \to R(P)$ is the composition $\nu_{P^+} \circ \nu_P$ where $\nu_P : P \to P^+$ maps an element of $x \in P(n)$ to the equivalence class of the matching $\{P(i)(x) \mid i \in S^n_n\}$. Similarly we have that $R'(P) = (P^\oplus)^\oplus$. We may now prove

Lemma 5.2. *We have that $CIRJ \simeq I'R'$.*

Proof. It is enough to show that $J(\cdot)^\oplus C = (\cdot)^+$. Indeed, this implies that $(\cdot)^\oplus \simeq C(\cdot)^+J$, because $CJ \simeq \mathrm{id}_{\mathbb{J}}$. So we have that $(\cdot)^\oplus(\cdot)^\oplus \simeq C(\cdot)^+JC(\cdot)^+J = C(\cdot)^+(\cdot)^+J$, which means that $CIRJ \simeq I'R'$.

But $(J(\cdot)^\oplus C)(P)([n]) = (CP)^\oplus(\{a_1, \ldots, a_n\})$. An element of this set is just an equivalence class of a matching family $\{x_f | f \in \mathcal{S}^k_{\{a_1,\ldots,a_n\}}\}$. If $f : \{a_1, \ldots, a_n\} \to \{a_{i_1}, \ldots, a_{i_l}\}$, then $x_f \in (CP)(\{a_{i_1}, \ldots, a_{i_l}\})$, so $x_f \in P([l])$. This shows that the matching family $\{x_f | f \in \mathcal{S}^k_{\{a_1,\ldots,a_n\}}\}$ actually corresponds to a matching family $\{y_g | g \in \mathcal{S}^k_n\}$, obtained as follows: For each $g \in \mathcal{S}^k_n$ there exists a unique $f \in \mathcal{S}^k_{\{a_1,\ldots,a_n\}}$, such that $g = c(f)$. We define $y_g = x_f$. One can now verify that $(CP)^\oplus(\{a_1, \ldots, a_n\}) = (P)^+([n])$. $\square$

Lemma 5.3. *Let $f : P \to Q$ be a natural transformation between two sheaves. Then f is an epimorphism in $\mathrm{Sh}(\mathbb{I}^{op})$ if and only if for all natural numbers n and for all $y \in Q([n])$ there exists a morphism $l : [n] \to [m]$ in $\mathbb{I}$ and $x \in P([m])$ such that $f_m(x) = Q(l)(y)$.*

Proof. Let f be a natural transformation satisfying the property stated in the lemma. We will prove that f is indeed an epimorphism. Assume $g, h : Q \to R$ are two morphisms in $\mathrm{Sh}(\mathbb{I}^{op})$ such that $gf = hf$. Let $y \in Q([n])$ for some natural number n. There exists $l : [n] \to [m]$ and $x \in P([m])$ for which $f_m(x) = Q(l)(y)$. This implies that $R(l)(g_n(y)) = g_m f_m(x)$. Similarly $R(l)(h_n(y)) = h_m f_m(x)$. Since $gf = hf$, we get that $R(l)(g_n(y)) = R(l)(h_n(y))$. But R is a sheaf, so it preserves monomorphisms. We conclude that $g_n(y) = h_n(y)$. We obtain that $g = h$, hence f is epimorphism.

Conversely, assume that there exists a natural number n and an element $y \in Q([n])$ such that for all morphisms $l : [n] \to [m]$ and for all $x \in P([m])$ we have $f_m(x) \neq Q(l)(y)$. To complete the proof we will construct a sheaf $R : \mathbb{I} \to \mathsf{Set}$ and two distinct sheaf morphisms $g, h : Q \to R$ satisfying $gf = hf$. We define $R([m]) = Q([m]) \cup \{*\}$ for all naturals m. If $l : [m] \to [m']$ is a morphism in $\mathbb{I}$ we define $R(l) : R([m]) \to R([m'])$ by

$$R(l)(q) = \begin{cases} Q(l)(q) & \text{if } q \in Q([m]) \\ * & \text{if } q = * \end{cases} \tag{35}$$

It is easy to check that R is a presheaf on $\mathbb{I}^{op}$. Since Q is a sheaf, we can also check that R preserves monomorphisms and pullbacks, thus it is a sheaf. We construct a natural transformation $g : Q \to R$, defining $g_n : Q([n]) \to R([n])$ to be the inclusion map.

Next we define a set

$$A_y = \bigcup_{m \in \mathbb{N}} \{x \in Q([m]) \mid \exists u : [m] \to [p], \exists v : [n] \to [p] \ : Q(u)(x) = Q(v)(y)\}$$

Remark 5.4. *For any morphism $l : [m] \to [m']$ and $x \in Q([m])$ we have that $x \in A_y$ if and only if $Q(l)(x) \in A_y$. For the first implication, we have the existence of some morphisms $u : [m] \to [p]$ and $v : [n] \to [p]$ such that $Q(u)(x) = Q(v)(y)$. Note that the diagram below can be completed to a* *commutative square in $\mathbb{I}$ (notice that this is not a push-out):*

$$\begin{tikzcd} [m] \arrow[r, "u"] \arrow[d, "l"] & [p] \arrow[d, "l'"] \\ [m'] \arrow[r, "u'"] & [p'] \end{tikzcd} \qquad (36)$$

So we have that $Q(u')(Q(l)(x)) = Q(l'v)(y)$ which implies that $Q(l)(x) \in A_y$. Conversely if $Q(l)(x) \in A_y$ then there exists u, v such that $Q(u)(Q(l)(x)) = Q(v)(y)$ or equivalently $Q(ul)(x) = Q(v)(y)$, so $x \in A_y$.

Now we define $h_m : Q([m]) \to R([m])$ by $h_m(z) = z$ if $z \notin A_y$, and $h_m(z) = *$ otherwise. h defined as such is a sheaf morphism. This follows easily from Remark 5.4. It remains to prove that $gf = hf$. This is equivalent to $f_m(P([m])) \cap A_y = \emptyset$. If we assume that there exists $x \in P([m])$ such that $f_m(x) \in A_y$, then $Q(u)(f_m(x)) = Q(v)(y)$ for some maps $u : [m] \to [p]$ and $v : [n] \to [p]$, or equivalently $f_p(P(u)(x)) = Q(v)(y)$, which contradicts the hypothesis on y. □

Corollary 5.5. *Assume $f : P \to Q$ is a morphism in $\mathrm{Sh}(\mathbb{J}^{op})$. Then f is an epimorphism if and only if for all finite sets of names S and all $y \in Q(S)$ there exists an inclusion $l : S \to T$ in $\mathsf{Set}^{\mathbb{J}}$ and $x \in P(T)$ such that $f_T(x) = Q(l)(y)$.*

Proof. This follows from the observation that $f : P \to Q$ is an epimorphism in $\mathrm{Sh}(\mathbb{J}^{op})$ if and only if $Jf : JP \to JQ$ is an epimorphism in $\mathrm{Sh}(\mathbb{I}^{op})$. First assume that Jf is an epimorphism in $\mathrm{Sh}(\mathbb{I}^{op})$, and that $y \in Q(S)$ for some finite set of names S. There exists a natural number n such that we have an inclusion $w : S \to \{a_1, \ldots, a_n\}$. This means that $Q(w)(y) \in Q(\{a_1, \ldots, a_n\}) = JQ([n])$. Since Jf is epimorphism, there exists an injective map $i : [n] \to [k]$ and $x \in JP([k])$ such that $Jf_{[k]}(x) = JQ(i)(Q(w)(y))$, or equivalently $f_{\{a_1,\ldots,a_k\}}(x) = Q(j(i)w)(y)$. Set $T = \{a_1, \ldots, a_k\}$ and $l = j(i)w$. The converse can be proved in the same fashion, and is left to the reader. □

5.2 HSP-theorem for $\tilde{L}$-algebras

We consider the endofunctor L on $\mathsf{Set}^{\mathbb{J}}$, defined by (33) and the endofunctor $\tilde{L}$ on $\mathrm{Sh}(\mathbb{J}^{op})$ defined as $R'LI'$. In order to obtain an HSP-theorem for $\tilde{L}$-algebras, we will apply Theorem 3.11 (with $\mathbb{I}$ replaced by $\mathbb{J}$). Using Remark 3.6 it is enough to show that

1. there exists a natural transformation $\alpha : LI'R' \to I'R'L$ such that the diagram below commutes:

$$\begin{tikzcd} LI'R' \arrow[rr, "\alpha"] & & I'R'L \\ & L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] & \end{tikzcd} \tag{37}$$

2. $\tilde{L}$ preserves epimorphisms.

In order to prove the former, notice that both R' and I' preserve coproducts. Indeed R' preserves all colimits and one can easily check that I' has this property, using the characterization of sheaves as pull-back and monomorphism preserving functors. Therefore it will be enough to prove that there exists natural transformations making the diagram (37) commutative for $\mathcal{N}$, δ and $- \times -$. We can check this easily for $\mathcal{N}$ and the product, because $\mathcal{N}$ is already a sheaf and both R' and I' commute with finite products. It only remains to prove

Lemma 5.6. *There exists a natural transformation $\delta I'R' \to I'R'\delta$ such that the diagram below commutes*

$$\begin{tikzcd} \delta I'R' \arrow[rr] & & I'R'\delta \\ & \delta \arrow[ul, "\delta\eta'"] \arrow[ur, "\eta'\delta"'] & \end{tikzcd} \tag{38}$$

It is easier to work with the 'shift' functor $\delta_{\mathbb{I}} : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$ as defined in Section 4. As the next lemma shows the two 'shift' functors are equivalent in the sense that:

Lemma 5.7. *We have that $\delta_{\mathbb{I}} = J\delta C$.*

Proof. Indeed for $P \in \mathsf{Set}^{\mathbb{I}}$, we have

$$\begin{aligned} (J\delta C)(P)([n]) \quad &= (\delta C)(P)(\{a_1, \ldots, a_n\}) \\ &= \coprod C(P)(\{a_1, \ldots, a_n\})/\equiv \\ &= \coprod P([n+1])/\equiv \\ &= P([n+1]) \\ &= \delta_{\mathbb{I}}(P)([n]) \end{aligned} \tag{39}$$

$\square$

Lemma 5.8. *There exists a natural transformation $\mu : \delta_{\mathbb{I}} IR \to IR\delta_{\mathbb{I}}$ such that $\mu \circ \delta_{\mathbb{I}}\eta = \eta\delta_{\mathbb{I}}$.*

Sketch. It is enough to show that there exists a natural transformation β, with $\beta_P : \delta(P^+) \to (\delta P)^+$ for any presheaf $P \in \mathsf{Set}^{\mathbb{I}}$ such that $\beta \circ \delta\nu = \nu\delta$. We define $\beta_n : P^+([n+1]) \to (\delta P)^+([n])$ as follows: the equivalence class of a matching family $\{x_f \mid f \in S^m_{n+1}\}$ is mapped to the equivalence class of $\{y_g \mid g \in S^{m-1}_n\}$ where $y_g = x_{g+1}$. Notice that this is a well defined map. One can easily verify that $\beta \circ \delta\nu = \nu\delta$. The natural transformation μ is obtained now as the composition: $\mu_P = (\beta P)^+ \circ \beta_{P^+}$. $\square$

Proof of Lemma 5.6. Using Lemma 5.8 and the fact that $JC = \mathrm{id}_{\mathsf{Set}^{\mathbb{I}}}$ we have a commutative diagram:

$$\begin{tikzcd}
C\delta_{\mathbb{I}}JCIRJ \arrow[rr] & & CIRJC\delta_{\mathbb{I}}J \\
& C\delta_{\mathbb{I}}J \arrow[ul, "C\delta_{\mathbb{I}}\eta J"] \arrow[ur, "C\eta\delta_{\mathbb{I}}J"'] &
\end{tikzcd} \tag{40}$$

By Lemma 5.2 we know that $CIRJ \simeq I'R'$ and Lemma 5.7 implies that $C\delta_{\mathbb{I}}J \simeq \delta$. $\square$.

Remark 5.9. *Notice that L preserves sheaves. Indeed, using Remark 3.6, we have that $LIR \simeq IRLIR$.*

Lemma 5.10. *$\tilde{L}$ preserves sheaf epimorphisms.*

Proof. We use the characterization of sheaf epimorphisms given in Corollary 5.5 and the fact that L preserves sheaves. Assume that $e : X \to Y$ is a sheaf epimorphism and consider $y \in (LIY)(S)$ for some finite set S of names. We prove that there exists an inclusion $w : S \to T$ and $x \in LIX(T)$ such that $LIY(w)(y) = (LIe)_T(x)$. This is easily verified if $y \in \mathcal{N}(S)$ or if $y \in (IY \times IY)(S)$, so we will give the proof only for the case $y \in (\delta IY)(S)$. Consider a name $a \notin S$ and $z \in IY(S \cup \{a\}) = Y(S \cup \{a\})$ such that $y = \{[a]_S z\}_{\delta A}$. Because $e : X \to Y$ is a sheaf epimorphism, there exists an inclusion $i : S \cup \{a\} \to T \cup \{a\}$ and $x \in X(T \cup \{a\})$ such that $a \notin T$ and $Y(i)(z) = e_{T\cup\{a\}}(x)$. If we denote by i' the inclusion of S into T, we can derive that $(\delta Ie)_T(\{[a]_T x\}_{\delta A}) = (\delta IY)(i')(y)$. $\square$

5.3 Axioms for the λ-calculus

The α-equivalence classes of λ-terms over $\mathcal{N}$ form a sheaf Λ_α in $\mathrm{Sh}(\mathbb{J}^{op})$. Indeed we can define $\Lambda_\alpha(S)$ as the set of α-equivalence classes of λ-terms with free variables in S. On functions Λ_α acts by renaming the free variables. We will show that Λ_α is isomorphic to the initial algebra $\mathcal{I}_{\tilde{L}}$ for $\tilde{L}$.

First let us notice that the underlying presheaf of $\mathcal{I}_{\tilde{L}}$ is the initial algebra $\mathcal{I}_L$ for L. Indeed, one can prove

Lemma 5.11. *We have $I'\mathcal{I}_{\tilde{L}} = \mathcal{I}_L$.*

Proof. The initial algebra $\mathcal{I}_{\tilde{L}}$ is computed as the colimit of the sequence

$$\tilde{0} \to \tilde{L}\tilde{0} \to \tilde{L}^2\tilde{0} \to \cdots \to \mathcal{I}_{\tilde{L}} \tag{41}$$

where $\tilde{0}$ is just the empty sheaf. Let us denote by 0 the empty presheaf. Similarly $\mathcal{I}_L$ is the colimit of the initial sequence for L:

$$0 \to L0 \to L^2 0 \to \cdots \to \mathcal{I}_L \tag{42}$$

Using Remark 5.9 and the fact that $I\tilde{0} = 0$ we can easily verify that $I'\tilde{L}^n\tilde{0} \simeq L^n 0$ for all natural numbers n. But I' preserves filtered colimits, so we have that $I'\mathcal{I}_{\tilde{L}} \simeq \mathcal{I}_L$. $\square$

We consider a functor $\Sigma : \mathsf{Set}^{\mathbb{J}} \to \mathsf{Set}^{\mathbb{J}}$, defined by

$$\Sigma X = \mathcal{N} + \mathcal{N} \times X + X \times X \tag{43}$$

Notice that Σ preserves sheaves and that the initial algebra for Σ, let us denote it by $\mathcal{I}_\Sigma$, is just the presheaf of all λ-terms. Using a similar argument as above we can see that $I'\mathcal{I}_{\tilde{\Sigma}} = \mathcal{I}_\Sigma$. $\mathcal{I}_{\tilde{\Sigma}}$ is the sheaf of all λ-terms. We will prove the isomorphism between Λ_α and $\mathcal{I}_{\tilde{L}}$ by constructing an epimorphism $\mathcal{I}_{\tilde{\Sigma}} \twoheadrightarrow \mathcal{I}_{\tilde{L}}$ in $\mathrm{Sh}(\mathbb{J}^{op})$ that identifies exactly α-equivalent terms.

First let us prove the next lemma:

Lemma 5.12. *There exists a natural transformation $\theta : \mathcal{N} \times - \to \delta$ such that for any sheaf $X \in \mathrm{Sh}(\mathbb{J}^{op})$ and any finite set of names $S \subseteq \mathcal{N}$ we have that $\theta(a, x) = \theta(b, y)$ for some $a, b \in S$ and $x, y \in X(S)$ iff $X(\sigma_{a,b})(x) = y$. Moreover if X is a sheaf, then θ_X is a sheaf epimorphism.*

Proof. We define $\theta_X(S) : \mathcal{N}(S) \times X(S) \to (\delta X)(S)$ by

$$(a, x) \mapsto (\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) \tag{44}$$

where $w_{S\setminus\{a\},a}$ is the inclusion of $S \setminus \{a\}$ into S. It is not difficult to check that this is indeed a natural transformation. Assume now that (a, x), $(b, y) \in \mathcal{N}(S) \times X(S)$ and $c \in \mathcal{N} \setminus S$. We have that $\theta(a, x) = \theta(b, x)$ is equivalent to

$$(\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) = (\delta X)(w_{S\setminus\{b\},b})(\{[b]_{S\setminus\{b\}}y\}_{\delta X}) \tag{45}$$

But

$$\begin{aligned}(\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) &= (\delta X)(w_{S\setminus\{a\},a})(\{[c]_{S\setminus\{a\}}(c/a)_{S\setminus\{a\}}x\}_{\delta X})\\ &= \{[c]_S X(w_{S\setminus\{a\},a} + c)((c/a)_{S\setminus\{a\}}(x))\}_{\delta X}\\ &= \{[c]_S X(w_{S\setminus\{a\}\cup\{c\},a})((c/a)_{S\setminus\{a\}}(x))\}_{\delta X}\end{aligned} \tag{46}$$

Similarly, $(\delta X)(w_{S\setminus\{b\},b})(\{[b]_{S\setminus\{b\}}y\}_{\delta X}) = \{[c]_S X(w_{S\setminus\{b\}\cup\{c\},b})((c/b)_{S\setminus\{b\}}(y))\}_{\delta X}$
Therefore (45) is equivalent to

$$\{[c]_S X(w_{S\setminus\{a\}\cup\{c\},a})((c/a)_{S\setminus\{a\}}(x))\}_{\delta X} = \{[c]_S X(w_{S\setminus\{b\}\cup\{c\},b})((c/b)_{S\setminus\{b\}}(y))\}_{\delta X}$$

Using that $w_{S\setminus\{a\}\cup\{c\},a} = w_{S\setminus\{b\}\cup\{c\},b}(a/b)_{S\setminus\{a,b\}\cup\{c\}}$, we can derive that (45) is equivalent to

$$X(w_{S\setminus\{b\}\cup\{c\},b})X((a/b)_{S\setminus\{a,b\}\cup\{c\}}(c/a)_{S\setminus\{a\}})(x) = X(w_{S\setminus\{b\}\cup\{c\},b})X((c/b)_{S\setminus\{b\}})(y))$$

X is a sheaf, and therefore it preserves monomorphisms, hence we have that

$$X((a/b)_{S\setminus\{a,b\}\cup\{c\}}(c/a)_{S\setminus\{a\}})(x) = X((c/b)_{S\setminus\{b\}})(y))$$

or equivalently,

$$X((b/c)_{S\setminus\{b\}}(a/b)_{S\setminus\{a,b\}\cup\{c\}}(c/a)_{S\setminus\{a\}})(x) = y$$

which means that $X(\sigma_{a,b})(x) = y$.
In order to prove the last statement of the lemma, we use the characterization of sheaf epimorphisms given in Corollary 5.5. Let $\{[c]_S y\}_{\delta X}$ be an arbitrary element of $(\delta X)(S)$. We have that $c \notin S$ and $y \in X(S \cup \{c\})$. The conclusion follows from the fact that $\theta_X(S\cup\{c\})(c, y) = (\delta X)(w_{S,c})(\{[c]_S y\}_{\delta X})$.
□

Proposition 5.13. *The sheaf of α-equivalence classes of λ-terms is isomorphic to the initial $\tilde{L}$-algebra $\mathcal{I}_{\tilde{L}}$.*

Proof. This can be obtained by an inductive argument on the structure of the λ-terms. Using the natural transformation θ defined above we can construct a natural transformation $\vartheta : \Sigma \to L$ defined as $\vartheta_X = \mathrm{id}_{\mathcal{N}} + \theta + \mathrm{id}_X \times \mathrm{id}_X$. Now we can define inductively, a natural transformation $\zeta^{(n)} : \Sigma^n \to L^n$. Explicitly $\zeta^{(0)} = \mathrm{id}_0$ and $\zeta^{(n+1)}_X = L^n(\vartheta_X)\zeta^{(n)}_{\Sigma X}$. We have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\zeta^{(0)}"] & \Sigma 0 \arrow[r] \arrow[d, "\zeta^{(1)}"] & \Sigma^2 0 \arrow[r] \arrow[d, "\zeta^{(2)}"] & \cdots \arrow[r] & \mathcal{I}_\Sigma \arrow[d, "\zeta"] \\ 0 \arrow[r] & L0 \arrow[r] & L^2 0 \arrow[r] & \cdots \arrow[r] & \mathcal{I}_L \end{tikzcd} \tag{47}$$

where ζ is obtained by taking the colimit. As seen above, $\mathcal{I}_L$ and $\mathcal{I}_\Sigma$ are the underlying presheaves for the sheaves $\mathcal{I}_{\tilde{L}}$ and $\mathcal{I}_{\tilde{\Sigma}}$, respectively.

Using Lemma 5.12, we can argue inductively that $\zeta^{(n)}_X$ is a sheaf epimorphism for all n and for all sheaves X. One can verify that this implies that ζ is a sheaf epimorphism. If two terms in $\mathcal{I}_\Sigma$ are identified by ζ, then they must be identified at some stage n by $\zeta^{(n)}$. Using Lemma 5.12 again, we can show by induction that two terms are in the kernel of $\zeta^{(n)}$ if and only if they are α-equivalent. $\square$

Proposition 5.14. *The endofunctor L is presented by a set of operation symbols*

$$\begin{aligned} a_S &: S \cup \{a\} \\ \mathsf{app}_S &: S \times S \to S \\ [a]_S &: S \cup \{a\} \to S \end{aligned}$$

where S is a finite set of names and $a \notin S$, and the following set of equations:

$$\begin{array}{llr} (b/a)_S a_S = b_S & & (E_0) \\ w_{S\cup\{a\},b} a_S = a_{S\cup\{b\}} & & (E_1) \\ (c/b)_S([a]_{S\cup\{b\}}, t) = \mathrm{id}_{S\cup\{c\}}([a]_{S\cup\{c\}}, (c/b)_{S\cup\{a\}} t) & t : S \cup \{a, b\} & (E_2) \\ ([a]_S, t) = ([b]_S, (b/a)_S t) & t : S \cup \{a\} & (E_3) \\ w_{S,b}([a]_S, t) = \mathrm{id}_{S\cup\{b\}}([a]_{S\cup\{b\}}, w_{S\cup\{a\},b} t) & t : S \cup \{a\} & (E_4) \\ w_{S,a}\mathsf{app}_S(t_1, t_2) = \mathsf{app}_{S\cup\{a\}}(w_{S,a} t_1, w_{S,a} t_2) & t1, t_2 : S & (E_5) \\ (b/a)_S \mathsf{app}_{S\cup\{a\}}(t_1, t_2) = \mathsf{app}_{S\cup\{b\}}((b/a)_S t_1, (b/a)_S t_2) & t1, t_2 : S \cup \{a\} & (E_6) \end{array}$$

The non-trivial part of the proof has been dealt with in the previous subsection.

Example 5.15. *We can give the following equations to capture $\alpha\beta\eta$-equivalence:*

$$
\begin{array}{ll}
X : S \ \vdash & \mathsf{app}_S([a]_S(a_S), X) = X \ \ : S \\
X : S;\ Y : S \ \vdash & \mathsf{app}_S([a]_S(w_a Y), X) = Y \ \ : S \\
X, Y : S \cup \{a\};\ Z : S \ \vdash & \mathsf{app}_S([a]_S(\mathsf{app}_{S\cup\{a\}}(X, Y)), Z) = \\
& \mathsf{app}_S(\mathsf{app}_S([a]_S(X), Z), \mathsf{app}_S([a]_S(Y), Z)) \ \ : S \\
X : S \cup \{a, b\};\ Y : S \ \vdash & \mathsf{app}_S([a]_S([b]_{S\cup\{a\}}(X)), Y) = \\
& [b]_S(\mathsf{app}_{S\cup\{b\}}([a]_{S\cup\{b\}}(X), w_a Y)) \ \ : S \\
X : S \cup \{a\} \ \vdash & \mathsf{app}_{S\cup\{b\}}(w_b [a]_S(X), b_S) = (b/a)X \ \ : S \cup \{b\} \\
X : S \ \vdash & [a]_S(\mathsf{app}_{S\cup\{a\}}(w_a X, a_S)) = X \ \ : S
\end{array}
$$

6 A comparison with Gabbay's HSPA-theorem

Gabbay [7] proves an HSP-theorem for nominal algebras, or rather an HSPA-theorem: A class of nominal algebras is definable by a theory of nominal algebra iff it is closed under HSP and under atoms-abstraction.

Our equational logic is more expressive than Gabbay's in the sense that more classes are equationally definable, namely all those closed under HSP where H refers not to closure under all quotients as in [7], but to the weaker property of closure under support-preserving quotients (ie quotients in the presheaf category). Of course, it is a question whether this additional expressivity is wanted. This section investigates a fragment of our logic that is still strong enough to treat nominal algebras in the sense of [7] such as the λ-calculus. We show that classes definable in this fragment are closed under atoms-abstraction and quotients.

We start with the observation that the theory of the λ-calculus up to $\alpha\beta\eta$ (Example 5.15) uses only particular operations: names (atoms in [7]), abstraction, and operations $f_S : A^n(S) \to A(S)$ uniform in S. This is captured by the following

Definition 6.1. *A nominal signature is a polynomial functor $\Sigma A = \coprod_{i\in I} A^{n_i}$ on $\mathsf{Set}^{\mathbb{J}}$. A nominal algebra $\mathbb{A}$ is a sheaf A together with a natural transformation $\mathcal{N} + \delta(A) + \Sigma(A) \to A$, where $\mathcal{N}$ is $\mathbb{J} \hookrightarrow \mathsf{Set}$.*

We denote elements of $\mathcal{N}$ and $\delta\mathbb{A}$ using the presentations as in Proposition 5.14. In subsequent proofs and calculations, we will use the following notation:

- $\mathcal{N} \to A$ maps a_S to $(a_S)^{\mathbb{A}}$;
- $[\cdot]^{\mathbb{A}} : \delta A \to A$, where we abbreviate $[\cdot]^{\mathbb{A}}_S(\{[a]_S x\}_{\delta A})$ as $[a]^{\mathbb{A}}_S(x)$;

$$
\frac{}{a : \{a\}} \qquad \frac{t : T \uplus \{a\}}{[a]t : T} \qquad \frac{t_1 : T, \ \dots \ t_n : T}{f(t_1, \dots t_n) : T}
$$

$$
\frac{t : T}{w_a t : T \uplus \{a\}} \qquad \frac{t : T \uplus \{a\}}{(b/a)t : T \uplus \{b\}} \qquad \frac{}{X_T : T}
$$

Figure 1: Uniform terms

- for $f \in \Sigma$, $A^n \to A$ maps $(x_1, \dots x_n) \in A^n(S)$ to $f^{\mathbb{A}}_S(x_1, \dots x_n)$.

Remark 6.2. *Given the presentations of the functors $\mathcal{N}$ and δ, each signature Σ gives rise to a many-sorted equational logic for nominal algebras. For each algebra $\mathbb{A}$ and each valuation v sending variables $X : T_X$ to elements of $A(T_X)$, a term t of sort (or type) T evaluates to an element $[\![t]\!]_{\mathbb{A},v,T}$ in $A(T)$.*

Whereas our general framework allows as signatures arbitrary endofunctors, the nominal signatures above correspond to the nominal signatures of [7]. In particular, the signature for λ-calculus is given by $\Sigma(A) = A \times A$, giving rise to a family of operation symbols app_S.

Looking at the theory of the λ-calculus in Example 5.15 we find that all operations are equivariant and that the equations are uniform in S. To capture the uniformity of an equational specification we first describe uniform terms, given by the set of rules in Figure 1.

In Figure 1, there are 6 schemas of rules: Three for the operations of a nominal algebra (names, abstraction, operations), two for the operations in J (weakenings, substitutions), and one for variables. Each rule can be instantiated in an infinite number of ways: T ranges over finite sets of names and a, b over names. The notation $T \uplus \{a\}$ implies that an instantiation of the schema is only allowed for those sets T and those atoms a where $a \notin T$.

Remark 6.3. *The rule for operations $f(t_1, \dots t_n)$ requires all arguments to be of the same type. This can be achieved by applying weakenings.*

Definition 6.4. *Consider a nominal signature as in Definition 6.1. A uniform term $t : T$ is a term t of type T formed according to the rules in Figure 1. A uniform term $t : T$ gives rise to a class of terms $t_S : S \cup T$ where S ranges of the finite subsets of $\mathcal{N}$ with $S \cap T = \emptyset$. Explicitly, for $t : T$ and S we have its translation into standard many-sorted algebra syntax*

$tr_S(t : T)$ *given via*

$$\begin{aligned}
tr_S\,(a : \{a\}) \quad &= a_S : S \cup \{a\} \\
tr_S\,([a]t : T) \quad &= [a]_{S\cup T}\; tr_S\,(t : T \uplus \{a\}) \\
tr_S\,(f(t_1, \ldots t_n) : T) \quad &= f_{S\cup T}(tr_S\,(t_1 : T), \ldots, tr_S\,(t_1 : T)) \\
tr_S\,(w_a t : T \uplus \{a\}) \quad &= w_{S\cup T,a}\; tr_S\,(t : T) \\
tr_S\,((b/a)t : T \uplus \{b\}) \quad &= (b/a)_{S\cup T}\; tr_S\,(t : T \uplus \{a\}) \\
tr_S\,(X_T : T) \quad &= X_{S\cup T}
\end{aligned}$$

Definition 6.5. *Consider a nominal signature as in Definition 6.1.*

1. *A uniform equation* $t = u : T$ *is a pair of uniform terms of type* T*. A uniform equation gives rise to a class of equations* $t_S = u_S : S \cup T$*,* $S \cap T = \emptyset$ *(see Definition 6.4).*

2. *An algebra* $\mathbb{A}$ *satisfies the uniform equation* $t = u : T$ *iff for all* $S \cap T = \emptyset$ *and all valuations* v *of variables, we have that* $\mathbb{A}, v \models t_S = u_S$*, that is,* $t_S^{\mathbb{A},v}$ *and* $u_S^{\mathbb{A},v}$ *denote the same element of* $A(S \cup T)$*.*

3. *A uniform theory for a nominal signature consists of a set of uniform equations and of further equations forcing the operations to be equivariant.*

The equivariance requirement in the last item corresponds to the fact that the structure $\mathcal{N} + \delta(A) + \Sigma(A) \to A$ of a nominal algebra is required to be a natural transformation.

Example 6.6. *We define* λ*-algebras as arising from a nominal signature with one binary operation symbol* app *satisfying the equations stating that* app *is equivariant.* λ*-algebras up to* $\alpha\beta\eta$ *are then those satisfying additionally the following 6 uniform equations.*

$$\begin{aligned}
\mathsf{app}([a]a, X) \quad &= X \;\; : \emptyset \\
\mathsf{app}([a](w_a Y), X) \quad &= Y \;\; : \emptyset \\
\mathsf{app}([a](\mathsf{app}(X, Y)), Z) \quad &= \mathsf{app}(\mathsf{app}([a]X, Z), \mathsf{app}([a]Y, Z)) \;\; : \emptyset \\
\mathsf{app}([a]([b]X), Y) \quad &= [b](\mathsf{app}([a]X, w_b Y)) \;\; : \emptyset \\
\mathsf{app}(w_b[a]X, b) \quad &= (b/a)X \;\; : \{b\} \\
[a](\mathsf{app}(w_a X, a)) \quad &= X \;\; : \emptyset
\end{aligned}$$

In the remainder of the section, we are going to show that classes defined by uniform equations are closed under abstraction and quotients, as in Gabbay's [7]. Abstraction (or atoms-abstraction [7]) maps an algebra with carrier A to an algebra with carrier δA. To describe this notion, due to [7, Def 8.14], in our setting, we need recall the definition of δ from Section 4.4. For $c \notin S$, there is an isomorphism

$$\begin{array}{rcl} A(S \cup \{c\}) & \to & \delta A(S) \\ x & \mapsto & \{[c]_S x\}_{\delta A} \end{array} \tag{48}$$

Definition 6.7. *Given a nominal algebra $\mathbb{A}$ with structure $\mathcal{N} + \delta(A) + \Sigma(A) \to A$ its abstraction $\delta\mathbb{A}$ with structure $\mathcal{N} + \delta(\delta A) + \Sigma(\delta A) \xrightarrow{\alpha} \delta A$ is given by*

1. $(a_S)^{\delta\mathbb{A}} = \{[c]_{S\cup\{a\}} a^{\mathbb{A}}_{S\cup\{c\}}\}_{\delta A} \qquad a \neq c \quad a, c \notin S$
2. $[a]^{\delta\mathbb{A}}_S \{[c]_{S\cup\{a\}} x\}_{\delta A} = \{[c]_S\, [a]^{\mathbb{A}}_{S\cup\{c\}} x\}_{\delta A} \qquad c \neq a \quad a, c \notin S$
3. $f^{\delta\mathbb{A}}_S(\{[c]x_1\}_{\delta\mathbb{A}}, \ldots \{[c]x_n\}_{\delta\mathbb{A}}) = \{[c] f^{\mathbb{A}}_S(x^{\mathbb{A}}_1, \ldots x^{\mathbb{A}}_n)\}_{\delta\mathbb{A}} \qquad c \notin S$

Dropping types and curly brackets, the definition above exactly matches [7, Def 8.14].

Remark 6.8. *1.) arises from $\mathcal{N} \to \delta\mathcal{N} \to \delta A$ where $\mathcal{N} \to \delta\mathcal{N}$ maps $a_S \in \mathcal{N}(S \cup \{a\})$ to $\{[c]_{S\cup\{a\}}\, a_{S\cup\{c\}}\}_{\delta\mathcal{N}}$, which is mapped by $\delta((-)^{\mathbb{A}}) : \delta\mathcal{N} \to \delta A$ to $\{[c]_{S\cup\{a\}}\, a^{\mathbb{A}}_{S\cup\{c\}}\}_{\delta A}$. 2.) arises from applying δ to $[\cdot]^{\mathbb{A}} : \delta A \to A$ and pre-composing with a swap-operation $\delta\delta \cong \delta\delta$. 3.) arises from $\Sigma\delta A \cong \delta\Sigma A \to \delta A$.*

The next lemma allows us to compute a uniform term t in δA at S by evaluating it in $\mathbb{A}$ at $S \cup \{c\}$, $c \notin S$. Note that this is possible for uniform terms, but not for terms.[2]

Lemma 6.9. *Consider a uniform term $t : T$ with variables $X_{T_X} : T_X$, a finite set S of names disjoint from T, a valuation v in $\delta\mathbb{A}$ such that $v(X_{T_X}) \in \delta A(S \cup T_X)$, and a name c not in S, T or one of the T_X. Let $v_{\mathbb{A}}$ be the valuation in $\mathbb{A}$ obtained from composing v with the isomorphisms $\delta A(S \cup T_X) \cong A(S \cup T_X \cup \{c\})$ given by (48). Then*

$$[\![t_S]\!]_{\delta\mathbb{A}, v, S\cup T} = \{[c]_{S\cup T}\, [\![t_{S\cup\{c\}}]\!]_{\mathbb{A}, v_{\mathbb{A}}, S\cup T\cup\{c\}}\}_{\delta A}$$

[2] Recall from Definition 6.4 that a uniform term is not a term (in the sense of set-based universal algebra) but a family of terms.

Proof. The proof is by induction on the structure of t. For instance, we calculate

$$\begin{aligned}
[\![([a]t)_S]\!]_{\delta\mathbb{A},v,S\cup T} &= [\![[a]_{S\cup T}\, t_S]\!]_{\delta\mathbb{A},v,S\cup T} \\
&= [a]^{\delta\mathbb{A}}_{S\cup T}([\![t_S]\!]_{\delta\mathbb{A},v,S\cup T\cup\{a\}}) \\
&= [a]^{\delta\mathbb{A}}_{S\cup T}\{[c]_{S\cup T\cup\{a\}}\, [\![t_{S\cup\{c\}}]\!]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{a,c\}}\}_{\delta A} \\
&= \{[c]_{S\cup T}\, [a]^{\mathbb{A}}_{S\cup T\cup\{c\}}([\![t_{S\cup\{c\}}]\!]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{a,c\}})\}_{\delta A} \\
&= \{[c]_{S\cup T}\, [\![([a]t)_{S\cup\{c\}}]\!]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{c\}}\}_{\delta A}
\end{aligned}$$

The case of weakenings and substitutions rely on E_{1-3} of Section 4.4. The case of variables follows by definition of $v_{\mathbb{A}}$. $\square$

It follows that definable classes are closed under abstraction:

Proposition 6.10. *If a class $\mathcal{B}$ of nominal algebras is defined by a uniform set of equations, then $\mathcal{B}$ is closed under abstraction.*

Proof. Assume that the nominal algebra $\mathbb{A}$ satisfies a uniform equation $t = u : T$. Consider a valuation v of the variables of t, u in the algebra $\delta\mathbb{A}$. We need to show $\delta\mathbb{A}, v \models t_S = u_S$ for all finite sets S of names disjoint from T. Consider the valuation $v_{\mathbb{A}}$ as in Lemma 6.9. Since $\mathbb{A}$ satisfies $t = u : T$, we have $[\![t_{S\cup\{a\}}]\!]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{a\}} = [\![u_{S\cup\{a\}}]\!]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{a\}}$. $[\![t_S]\!]_{\delta\mathbb{A},v,S\cup T} = [\![u_S]\!]_{\delta\mathbb{A},v,S\cup T}$ now follows from Lemma 6.9. $\square$

Proposition 6.11. *If a class $\mathcal{B}$ of nominal algebras is defined by a uniform set of equations, then $\mathcal{B}$ is closed under quotients.*

Proof. Consider a quotient of sheaves $f : \mathbb{A} \to \mathbb{B}$, that is, a presheaf morphism such that for all $x \in A(S)$ there is S' and $y \in B(S')$ such that $w(f_S(x)) = f_{S'}(y)$, where w denotes the inclusion $S \to S'$, see Corollary 5.5 . Consider the uniform equation $t = u : T$ and choose S and a valuation v in $\mathbb{B}$. We have to show $[\![t_S]\!]_{\mathbb{B},v,S\cup T} = [\![u_S]\!]_{\mathbb{B},v,S\cup T}$. For this, we find an S' and inclusions $w_X : S\cup T_X \to S'\cup T_X$ and a valuation $v_{\mathbb{A}}$ such that $v_{\mathbb{A}}(X) \in A(S'\cup T_X)$ and $f_{S'\cup T_X}(v_{\mathbb{A}}(X)) = w_X(v(X))$. From this, we prove by induction on the structure of t that $f_{S'\cup T}([\![t_{S'}]\!]_{\mathbb{A},v_{\mathbb{A}},S'\cup T}) = B(w)([\![t_S]\!]_{\mathbb{B},v,S\cup T})$. Since $B(w)$ is injective, this concludes the proof. $\square$

7 Conclusion and further work

An obvious criticism of the many-sorted universal algebra approach to nominal sets is that it leads to syntax which is parameterised by an infinite set of sorts. So let us emphasize that universal algebra is proposed here as a means to prove results about nominal algebras and not to replace nominal sets. On the other hand, making the sorts explicit can also have its own advantages, for example if one wants to study calculi with bounded resources, such as the n-variable fragment of the λ-calculus or of first-order logic.

Our work can be extended in different ways.

To make the transition between nominal sets and universal algebra smooth, we will give precise syntactic translations between nominal algebra syntax in the sense of Gabbay [7] and the uniform fragment of the many-sorted equational logic presented here. We will also consider possible connections with the permissive nominal syntax developed in [?]. One should also look at how the nominal logic of Pitts [13] and the nominal equational logic of Clouston and Pitts [4] relate to our approach.

Further support of our approach will then come from making available other results of universal algebra such as the quasi-variety theorem characterising classes definable by implications as those being closed under SP and filtered colimits.

Our general results are for endofunctors on Nom, so we want to work out the syntactic details for other type constructors apart from δ and polynomial ones, such as exponentiation with a constant or the Day tensor (see eg Fiore [5] for a discussion of this construction on $\mathsf{Set}^{\mathbb{I}}$).

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and Concrete Categories*. John Wiley & Sons, 1990.

[2] J. Adámek, J. Rosický, and E. M. Vitale. *Varieties*. Draft available at `http://www.math.muni.cz/~rosicky`.

[3] M. Bonsangue and A. Kurz. Presenting functors by operations and equations. In *FoSSaCS'06*.

[4] R. Clouston and A. Pitts. Nominal equational logic. In *Computation, Meaning and Logic, Articles dedicated to Gordon Plotkin*, volume 172 of *Electronic Notes in Theoretical Computer Science*. 2007.

[5] M. Fiore. Mathematical models of computational and combinatorial structures. In *FoSSaCS'05*.

[6] M. Fiore, G. Plotkin, and D. Turi. Abstract syntax and variable binding. In *LICS'99*.

[7] M. Gabbay. Nominal algebra and the HSP theorem. *J. Logic Computation*, 2008. doi:10.1093/logcom/exn055.

[8] M. Gabbay and A. Pitts. A new approach to abstract syntax involving binders. In *LICS'99*.

[9] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium, vol. 1 (Oxford Logic Guides, 43)*. Oxford University Press, USA, 2002.

[10] A. Kurz and D. Petrişan. Functorial coalgebraic logic: The case of many-sorted varieties. In *CMCS'08*.

[11] A. Kurz and J. Rosický. Strongly complete logics for coalgebras. July 2006. Submitted.

[12] S. Maclane and I. Moerdijk. *Sheaves in Geometry and Logic : A First Introduction to Topos Theory (Universitext)*. Springer, October 1994.

[13] A. Pitts. Nominal logic, a first order theory of names and binding. *Information and Computation*, 186, 2003.