EQUATIONAL PRESENTATIONS OF FUNCTORS AND MONADS

JIŘÍ VELEBIL AND ALEXANDER KURZ

Abstract. We study equational presentations of functors and monads, defined on a category $\mathscr{K}$ that is equipped by an adjunction $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ of descent type. We detect a class of functors/monads that admit such an equational presentation that involves finitary signatures in $\mathscr{X}$.

We apply these results to description of categories of modal algebras for coalgebraic modal logic.

1. Introduction

In categorical universal algebra, it is well-known that a *finitary equational presentation* of algebras in a finitary variety $\mathscr{K}$ amounts to the existence of a coequalizer of a pair of morphisms between finitely generated free algebras.

The essence of a *universal-algebraic flavour* of a finitary functor $L : \mathscr{K} \longrightarrow \mathscr{K}$ is that L is determined by its behaviour on finitely generated free algebras. In fact, such functors admit an equational presentation as well, this time the coequalizer is more complex but it again involves functors freely generated from Set-functors.

Recently, such functors L appeared naturally in the study of *modal algebras* for coalgebraic modal logic, see, e.g., [BKu] or [KuR]. In fact, for the case of one-sorted varieties $\mathscr{K}$, functors L that are determined by their values on finitely generated free algebras are exactly the functors preserving a class of colimits called *sifted* or, equivalently, they are exactly the class of functors admitting an equational presentation, see [KuR].

In the present paper we study presentations of functors/monads on $\mathscr{K}$ that are determined by finitely generated free algebras but we want to have the requirements on $\mathscr{K}$ as relaxed as possible, in view of [KuPV]. Hence we again study functors/monads $L : \mathscr{K} \longrightarrow \mathscr{K}$ but $\mathscr{K}$ is now required to be only a full subcategory of a variety, but $\mathscr{K}$ must still contain free objects on finitely many generators.

Thus, the initial setting is now given by a finitary adjunction $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ that is of descent type, i.e., such that $\mathscr{K}$ embeds fully into the Eilenberg-Moore category $\mathscr{X}^{\mathbb{T}}$ where $\mathbb{T}$ is the finitary monad on $\mathscr{X}$ generated by $F \dashv U$. In order to be able to speak of finitely generated free algebras we further require both $\mathscr{K}$ and $\mathscr{X}$ to be locally finitely presentable. We call functors/monads on $\mathscr{K}$ that are detemined by their values on finitely generated free objects finitary *based*.

Our main results are the following:

1. We prove that finitary based functors/monads on $\mathscr{K}$ can be equationally presented using finitary signatures on $\mathscr{X}$. This is the contents of Theorems 3.16 and 4.4 below.
2. In Theorem 4.1 below we prove that algebras for finitary based monads on a monadic category is a monadic category again.

Since we expect applications in coalgebraic modal logic in enriched category theory, we state and prove all results in the enriched setting.

Organization of the Paper. Necessary definitions and notational conventions of enriched category theory are gathered in Section 2. The notion of *finitary based* functors is introduced in Section 3. We prove the presentation result in Theorem 3.16. In Section 4 we prove the presentation result for finitary based *monads*. Finally, in Section 5 we show how this result can be applied to the theory of modal algebras.

2. Preliminaries

We recall the necessary notions from enriched category theory we will use in the following text. Our standard reference to enriched category theory is Max Kelly's book [K_1] and for the details on locally finitely presentable categories enriched in $\mathscr{V}$ we refer to [K_2].

We assume that $\mathscr{V} = (\mathscr{V}_o, \otimes, I)$ is a symmetric monoidal closed category that is locally presentable as a closed category. The latter means that the (ordinary) category $\mathscr{V}_o$ is locally finitely presentable and that I

Date: 23 April 2010.

The first author acknowledges the support of the grant MSM6840770014 of the Ministry of Education of the Czech Republic.

is a finitely presentable object in $\mathscr{V}_o$ and that the tensor product $X \otimes Y$ is a finitely presentable object in $\mathscr{V}_o$ whenever X and Y are. See [GU] or [AR] for details on locally finitely presentable categories.

Our assumptions on $\mathscr{V}$ allow us to develop category theory enriched in $\mathscr{V}$ and in what follows we work with $\mathscr{V}$-categories, $\mathscr{V}$-functors, etc. We will frequently omit the prefix $\mathscr{V}$- and speak just about catgeories, functors, etc. Whenever we want to work with $\mathscr{V} = \mathsf{Set}$, we call the categories, functors, etc., *ordinary*.

In enriched category theory we need to consider weighted colimits as the basic colimit concept. Recall that a *colimit* of a diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$ *weighted* by $W : \mathscr{D}^{op} \longrightarrow \mathscr{V}$ is an object $W * D$ in $\mathscr{X}$ together with an isomorphism

$$\mathscr{X}(W * D, X) \cong [\mathscr{D}^{op}, \mathscr{V}](W, \mathscr{X}(D-, X))$$

natural in X.

The dual notion is that of a *limit* of $D : \mathscr{D} \longrightarrow \mathscr{X}$ *weighted* by $W : \mathscr{D} \longrightarrow \mathscr{V}$ which is an object $\{W, D\}$ in $\mathscr{X}$ together with an isomorphism

$$\mathscr{X}(X, \{W, D\}) \cong [\mathscr{D}, \mathscr{V}](W, \mathscr{X}(X, D-))$$

natural in X.

In fact, the assumptions on $\mathscr{V}$ allow us to speak about *locally finitely presentable* (l.f.p.) categories in the enriched sense. For that one only needs to define the concept of being a *filtered* colimit in the enriched setting and then proceeds as in the classical ordinary case. We define *filtered colimits* as those weighted by *flat* weights, where a weight $W : \mathscr{D}^{op} \longrightarrow \mathscr{V}$ is flat whenever the functor

$$W * (-) : [\mathscr{D}, \mathscr{V}] \longrightarrow \mathscr{V}$$

preserves *finite* limits. In enriched category theory a limit $\{K, C\}$ is finite if $K : \mathscr{C} \longrightarrow \mathscr{V}$ is a finite weight. The last assertion means that $\mathscr{C}$ has finitely many objects, and that every hom-object $\mathscr{C}(c, c')$ and every value Kc are finitely presentable in $\mathscr{V}_o$.

Then one defines the notions of being finitely presentable object, finitary functor, etc. in the usual manner. See [K$_2$] for more details.

3. Finitary Based Functors and their Presentations

In this section we introduce the class of finitary functors that are fully determined by their values on "finitely generated" free algebras and call them finitary based (Definition 3.8 below). We show on examples in Section 5 below that such functors arise naturally and that they enjoy nice properties: for example one can give their equational presentation using only operations and equations coming from the category from which we pick the generators of algebras. This is the main result of this section, Theorem 3.16 below.

The idea of being determined by values on free algebras suggests that we need to work relative to a certain fixed adjunction.

Assumption 3.1. We fix a *finitary adjunction*

$$F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$$

between locally finitely presentable categories and we assume that the adjunction is of *descent type*.

Remark 3.2. Being of descent type means that the *comparison functor* $K : \mathscr{K} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is fully faithful, or, equivalently, that every commutative diagram

$$FUFUA \underset{FU\varepsilon A}{\overset{\varepsilon FUA}{\rightrightarrows}} FUA \xrightarrow{\varepsilon A} A \tag{3.1}$$

is a coequalizer, where ε denotes the counit of $F \dashv U$. We will call the above coequalizer the *canonical resolution* of A.

Any adjunction of descent type can be considered as the existence of an *equational presentation* as it will become clearer in Theorems 3.16 and 4.4 below. Namely, the parallel pair in a coequalizer (3.1) can be considered as a system of equations that the object A satisfies. See [KP] for more details on equational presentations and on more properties characterizing adjunctions of descent type.

Notation 3.3. Denote by

$$J : \mathscr{A} \longrightarrow \mathscr{K}$$

the embedding of the full subcategory spanned by objects of the form Fn, n finitely presentable in $\mathscr{X}$.

Remark 3.4. The category $\mathscr{A}$ consists of finitely presentable objects in $\mathscr{K}$. Indeed, due to the adjunction $F \dashv U$ we have that, for every finitely presentable n, the isomorphism

$$\mathscr{K}(Fn, -) \cong \mathscr{X}(n, U-) \cong \mathscr{X}(n, -) \cdot U$$

holds. The latter functor preserves filtered colimits since both U and $\mathscr{X}(n, -)$ do.

We prove now that objects in $\mathscr{A}$ suffice to reconstruct all objects of $\mathscr{K}$. More precisely, we prove that the inclusion J of $\mathscr{A}$ in $\mathscr{K}$ is a *dense* functor. Recall that a functor $J : \mathscr{A} \longrightarrow \mathscr{K}$ is dense, if the left Kan extension of J along itself is (isomorphic to) the identity functor on $\mathscr{K}$. This statement means that every object X of $\mathscr{K}$ can be expressed as a *canonical* colimit

$$\mathscr{K}(J-, X) * J$$

see Chapter 5 of [K$_1$].

Lemma 3.5. *The functor $J : \mathscr{A} \longrightarrow \mathscr{K}$ is dense.*

Proof. The idea of the proof is essentially contained in Theorem 6.9 of [Bi]. Since J is fully faithful, we can use Theorem 5.19(v) of [K$_1$]: J is dense, if there is a class of colimits such that the category $\mathscr{K}$ is the closure of $\mathscr{A}$ under these colimits, and every such colimit is preserved by the functor

$$\widetilde{J} : \mathscr{K} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}], \quad A \mapsto \mathscr{K}(J-, A)$$

We will exhibit the required class of colimits in two steps:

(1) For every object of the form FX do the following:
Consider the canonical colimit

$$\mathscr{X}(E-, X) * E$$

expressing X as a filtered colimit of finitely presentable objects in $\mathscr{X}$, where $E : \mathscr{X}_{fp} \longrightarrow \mathscr{X}$ denotes the full embedding of the subcategory representing all finitely presentable objects of $\mathscr{X}$.
Then FX can be expressed as a colimit

$$\mathscr{X}(E-, X) * FE$$

since F is a left adjoint, hence it preserves (all) colimits.
It remains to prove that the colimit $\mathscr{X}(E-, X) * FE$ is preserved by $\widetilde{J} : \mathscr{K} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$. But this is clear: $\widetilde{J}$ preserves filtered colimits (=colimits weighted by flat weights), since objects of $\mathscr{A}$ are finitely presentable. And the weight $\mathscr{X}(E-, X)$ is flat, since $\mathscr{X}_{fp}$ consists of finitely presentable objects.

(2) For every object A of $\mathscr{K}$ do the following:
Express A as a coequalizer (3.1). This can be done, since the adjunction $F \dashv U$ is assumed to be of descent type.
We claim that the coequalizer (3.1) is preserved by $\widetilde{J}$. First observe that for every finitely presentable object n in $\mathscr{X}$ we have

$$\mathscr{K}(Fn, -) \cong \mathscr{X}(n, U-) = \mathscr{X}(n, -) \cdot U$$

Since the coequalizer (3.1) is U-absolute (it is well-known to be a U-split coequalizer), the image of it under any $\mathscr{X}(n, -) \cdot U$ is a coequalizer again. Hence the image of (3.1) under every $\mathscr{K}(Fn, -)$ is a coequalizer, thus, $\widetilde{J}$ preserves the coequalizer (3.1).

□

Remark 3.6. In fact, in the above result we obtained the *density presentation* of $J : \mathscr{A} \longrightarrow \mathscr{K}$. The density presentation is (see the comments right before Proposition 5.20 of [K$_1$]) a family

$$\Phi = \langle W_\gamma : \mathscr{D}_\gamma^{op} \longrightarrow \mathscr{V}, D_\gamma : \mathscr{D}_\gamma \longrightarrow \mathscr{K} \mid \gamma \in \Gamma \rangle$$

such that each colimit $W_\gamma * D_\gamma$ exists, is preserved by $\widetilde{J}$ and $\mathscr{K}$ is the closure of $\mathscr{A}$ under these colimits.

From the above remark and from the fact that $\mathscr{A}$ consists of finitely presentable objects, we immediately obtain from Theorem 5.29 of [K$_1$] the following:

Corollary 3.7. *For $L : \mathscr{K} \longrightarrow \mathscr{K}$, the following are equivalent:*

(1) *L is of the form* $\mathrm{Lan}_J LJ$.
(2) *L is finitary and preserves all canonical resolutions* (3.1).

Definition 3.8. A finitary endofunctor of $\mathscr{K}$ that preserves canonical resolutions (3.1) will be called *finitary based*. The category of all finitary based functors is denoted by $\mathsf{FinB}(\mathscr{K},\mathscr{K})$.

Remark 3.9. By Remark 3.6 we know that there is an equivalence

$$\mathsf{FinB}(\mathscr{K},\mathscr{K}) \simeq [\mathscr{A},\mathscr{K}]$$

of categories that we will often use below. Hence the category $\mathsf{FinB}(\mathscr{K},\mathscr{K})$ is locally finitely presentable, in particular it is complete and cocomplete.

Example 3.10. Suppose that $\mathscr{K}$ is a finitary, one-sorted variety. Thus we work within ordinary category theory, i.e., $\mathscr{V}$ is the category Set of sets and mappings.

That $\mathscr{K}$ is a finitary one-sorted variety is equivalent to the existence of a finitary monadic adjunction $F \dashv U : \mathscr{K} \longrightarrow \mathsf{Set}$. Suppose that $L : \mathscr{K} \longrightarrow \mathscr{K}$ has the form

$$LA = \coprod_n \mathsf{Set}(n, UA) \bullet \Delta n$$

where the coproduct is taken over all finite ordinals, $n \mapsto \Delta n$ is the assignment of an object of $\mathscr{K}$ to each finite ordinal, and $\mathsf{Set}(n, UA) \bullet \Delta n$ is the coproduct in $\mathscr{K}$ of $\mathsf{Set}(n, UA)$-many copies of Δn. We show that L is finitary based.

It is clear that L preserves filtered colimits, since every $\mathsf{Set}(n, U-)$ does and colimits commute with colimits. To prove that L preserves canonical resolutions, observe that each $\mathsf{Set}(n, U-)$ preserves canonical resolutions, since they are U-absolute coequalizers.

Functors of the above form arise in coalgebraic modal logic, see Section 5 below.

Every finitary endofunctor has a finitary based coreflection.

Lemma 3.11. *The full embedding $\mathsf{FinB}(\mathscr{K},\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K},\mathscr{K})$ has a right adjoint. In particular, the category $\mathsf{FinB}(\mathscr{K},\mathscr{K})$ is closed in $\mathsf{Fin}(\mathscr{K},\mathscr{K})$ under colimits.*

Proof. Denote by $J' : \mathscr{A} \longrightarrow \mathscr{K}_{fp}$ the full embedding of $\mathscr{A}$ into the category representing all finitely presentable objects. Then we have an adjunction

$$\mathrm{Lan}_{J'}(-) \dashv [J', \mathscr{K}] : [\mathscr{K}_{fp}, \mathscr{K}] \longrightarrow [\mathscr{A}, \mathscr{K}]$$

Since, using that $\mathsf{Fin}(\mathscr{K},\mathscr{K}) \cong [\mathscr{K}_{fp}, \mathscr{K}]$ and $\mathsf{FinB}(\mathscr{K},\mathscr{K}) \simeq [\mathscr{A},\mathscr{K}]$, the full embedding $\mathsf{FinB}(\mathscr{K},\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K},\mathscr{K})$ is given by left Kan extension along J', the result follows. □

In proving the presentation results on finitary based functors in Proposition 3.13 and Theorem 3.16 below, we will employ the following technical lemma.

Lemma 3.12. *Suppose $S : \mathscr{B} \longrightarrow \mathscr{C}$ is a functor surjective on objects, where the categories $\mathscr{B}$ and $\mathscr{C}$ are small. Then the composite*

$$[\mathscr{C}, \mathscr{K}] \xrightarrow{[S,\mathscr{K}]} [\mathscr{B}, \mathscr{K}] \xrightarrow{[\mathscr{B},U]} [\mathscr{B}, \mathscr{X}]$$

has a left adjoint and the resulting adjunction is of descent type.

Proof. The functor $[S,\mathscr{K}]$ is monadic. This follows from the fact that S is surjective on objects. For then $[S,\mathscr{K}]$ is faithful and reflects isomorphisms. Since $[S,\mathscr{K}]$ has both left and right adjoints given by Kan extensions, we conclude that $[S,\mathscr{K}]$ is monadic by Beck's theorem.

The functor $[\mathscr{B}, U]$ has a left adjoint $[\mathscr{B}, F]$, this adjunction being the image of the adjunction $F \dashv U$ under the 2-functor $[\mathscr{B}, -]$. Moreover, $[\mathscr{B}, F] \dashv [\mathscr{B}, U]$ is an adjunction of descent type, since $F \dashv U$ is.

Since $[S,\mathscr{K}]$ sends coequalizers to epimorphisms (in fact, it preserves coequalizers, being a left adjoint), the composite $[S, U] = [\mathscr{B}, U] \cdot [S, \mathscr{K}]$ is of descent type by Proposition 3.5 of [KP]. □

We will establish now the first presentation result concerning finitary based functors and endofunctors of the "base" category $\mathscr{X}$.

Proposition 3.13. *The functor*

$$[F, U] : \mathsf{FinB}(\mathscr{K},\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{X},\mathscr{X}), \quad L \mapsto ULF$$

has a left adjoint $H \mapsto \widehat{H}$ and the adjunction is of descent type.

Proof. Using the identifications $\mathsf{FinB}(\mathscr{K},\mathscr{K}) \simeq [\mathscr{A},\mathscr{K}]$ and $\mathsf{Fin}(\mathscr{X},\mathscr{X}) \simeq [\mathscr{X}_{fp},\mathscr{X}]$, observe that the forgetful functor $[F,U] : \mathsf{FinB}(\mathscr{K},\mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{X},\mathscr{X})$ can be written as the composite

$$[F',U] \equiv [\mathscr{A},\mathscr{K}] \xrightarrow{[F',\mathscr{K}]} [\mathscr{X}_{fp},\mathscr{K}] \xrightarrow{[\mathscr{X}_{fp},U]} [\mathscr{X}_{fp},\mathscr{X}] \tag{3.2}$$

where we denote by $F' : \mathscr{X}_{fp} \longrightarrow \mathscr{A}$ the restriction of $F : \mathscr{X} \longrightarrow \mathscr{K}$. Now use Lemma 3.12 with $S = F'$. $\square$

Remark 3.14. The finitary based functor $\widehat{H} : \mathscr{A} \longrightarrow \mathscr{K}$, free on H, is given explicitly, at $F'm$ in $\mathscr{A}$, by the formula

$$\begin{aligned}
\widehat{H}(F'm) &= (\mathrm{Lan}_{F'}(FH))(F'm) \\
&\cong \int^n \mathscr{A}(F'n, F'm) \bullet FHn \\
&\cong \int^n \mathscr{K}(Fn, Fm) \bullet FHn \\
&\cong \int^n \mathscr{X}(n, UFm) \bullet FHn \\
&\cong FHU(Fm)
\end{aligned}$$

where the last isomorphism is by Yoneda Lemma.

Remark 3.15. As always, in the presence of an adjunction of descent type, we have a presentation result: Proposition 3.13 states that every finitary based $L : \mathscr{K} \longrightarrow \mathscr{K}$ can be expressed as a coequalizer

$$\widehat{H_1} \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \widehat{H_2} \xrightarrow{\gamma} L$$

for some suitable finitary functors $H_1, H_2 : \mathscr{X} \longrightarrow \mathscr{X}$. In what follows we want to improve this coequalizer presentation to involve finitary signatures rather than endofunctors.

Before we state our main presentation result recall the monadic adjunction

$$\mathrm{Lan}_E(-) \dashv [E,\mathscr{X}] : \mathsf{Fin}(\mathscr{X},\mathscr{X}) \longrightarrow [|\mathscr{X}_{fp}|,\mathscr{X}] \tag{3.3}$$

where $E : |\mathscr{X}_{fp}| \longrightarrow \mathscr{X}$ is the inclusion of the discrete underlying category of $\mathscr{X}_{fp}$ into $\mathscr{X}$.

The functor category $[|\mathscr{X}_{fp}|,\mathscr{X}]$ is best perceived as the *category of finitary signatures* on $\mathscr{X}$ and we denote it by

$$\mathsf{Sig}_{fin}(\mathscr{X})$$

Such a signature Σ is a collection (Σn) indexed by finitely presentable objects of $\mathscr{X}$. The object Σn is then an object of n-ary *operations*, for each finitely presentable object n in $\mathscr{X}$.

The left adjoint in (3.3) sends each signature Σ to its corresponding *polynomial endofunctor*

$$H_\Sigma X = \coprod_n \mathscr{X}(n,X) \bullet \Sigma n$$

where the coproduct is taken over objects in $|\mathscr{X}_{fp}|$ and $\mathscr{X}(n,X) \bullet \Sigma n$ is the $\mathscr{X}(n,X)$-th *tensor* of Σn in $\mathscr{X}$, defined by the isomorphism

$$\mathscr{X}(\mathscr{X}(n,X) \bullet \Sigma n, X') \cong \mathscr{V}(\mathscr{X}(n,X), \mathscr{X}(\Sigma n, X'))$$

natural in X'.

Theorem 3.16. *The composite*

$$\mathsf{FinB}(\mathscr{K},\mathscr{K}) \xrightarrow{[F,U]} \mathsf{Fin}(\mathscr{X},\mathscr{X}) \xrightarrow{[E,\mathscr{X}]} \mathsf{Sig}_{fin}(\mathscr{X}) \tag{3.4}$$

has a left adjoint and the resulting adjunction is of descent type.

Proof. Recall that $F' : \mathscr{X}_{fp} \longrightarrow \mathscr{A}$ denotes the restriction of $F : \mathscr{X} \longrightarrow \mathscr{K}$ and denote by $E' : |\mathscr{X}_{fp}| \longrightarrow \mathscr{X}_{fp}$ the restriction of the inclusion $E : |\mathscr{X}_{fp}| \longrightarrow \mathscr{X}$. Then, using identifications $\mathsf{FinB}(\mathscr{K},\mathscr{K}) \simeq [\mathscr{A},\mathscr{K}]$ and $\mathsf{Fin}(\mathscr{X},\mathscr{X}) \simeq [\mathscr{X}_{fp},\mathscr{X}]$, the composite (3.4) can be written as the composite

$$[\mathscr{A},\mathscr{K}] \xrightarrow{[F'E',\mathscr{K}]} [|\mathscr{X}_{fp}|,\mathscr{K}] \xrightarrow{[|\mathscr{X}_{fp}|,U]} [|\mathscr{X}_{fp}|,\mathscr{X}]$$

Putting $S = F' \cdot E'$ in Lemma 3.12 finishes the proof. $\square$

The above theorem states that every finitary based functor $L : \mathscr{K} \longrightarrow \mathscr{K}$ can be expressed as a coequalizer in the spirit of [BKu] and it generalizes results of [KuR] from finitary varieties (over Set) to finitary adjunctions of descent type (over an arbitrary l.f.p. category $\mathscr{X}$, enriched in $\mathscr{V}$).

Indeed, Theorem 3.16 states that every finitary based endofunctor $L : \mathscr{K} \longrightarrow \mathscr{K}$ can be written as a coequalizer

$$\widehat{H_\Gamma} \overset{\lambda}{\underset{\rho}{\rightrightarrows}} \widehat{H_\Sigma} \xrightarrow{\gamma} L \tag{3.5}$$

for some suitable finitary signatures Σ and Γ. In fact, the pair λ, ρ in (3.5) can equivalently be given by a parallel pair

$$\Gamma \overset{\lambda^\flat}{\underset{\rho^\flat}{\rightrightarrows}} U\widehat{H_\Sigma}F$$

of signature morphisms. In fact, by Remark 3.14, we have that

$$U\widehat{H_\Sigma}Fm \cong UFH_\Sigma UFm$$

holds and hence the purpose of Γ and the pair $\lambda^\flat$, $\rho^\flat$ is to pick up, for every m, Γm-many pairs of "terms" in $UFH_\Sigma UFm$ to be equal. This is exactly the type of equations treated in [BKu] and [KuR], see Section 5 below for more details.

4. Presentations of Finitary Based Monads

A monad $\mathbb{M} = (M, \eta, \mu)$ on $\mathscr{K}$ is called *finitary based*, provided that its underlying functor $M : \mathscr{K} \longrightarrow \mathscr{K}$ is finitary based. The category of all finitary based monads on $\mathscr{K}$ is denoted by

$$\mathsf{Mnd}_{finb}(\mathscr{K})$$

Before we turn to equational presentations of finitary based monads, let us observe that such monads fulfill the "monadic composition" property. More precisely, the following result holds:

Theorem 4.1. *Suppose $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ is monadic. Consider a finitary based monad $\mathbb{M} = (M, \eta, \mu)$ on $\mathscr{K}$. Then the composite*

$$\mathscr{K}^{\mathbb{M}} \xrightarrow{U^{\mathbb{M}}} \mathscr{K} \xrightarrow{U} \mathscr{X}$$

is monadic, where $U^{\mathbb{M}}$ is the forgetful functor from the category of Eilenberg-Moore algebras for $\mathbb{M}$.

Proof. By Beck's Theorem (see, e.g., Exercise 2, Chapter VI.7 of [McL]), we need to prove that $\mathscr{K}^{\mathbb{M}}$ has and $UU^{\mathbb{M}}$ preserves and reflects $UU^{\mathbb{M}}$-absolute coequalizers.

Since the monad $\mathbb{M}$ is finitary, the category $\mathscr{K}^{\mathbb{M}}$ is locally finitely presentable, hence it has coequalizers.

Take a pair

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \tag{4.1}$$

in $\mathscr{K}^{\mathbb{M}}$ that has a $UU^{\mathbb{M}}$-absolute coequalizer. Consider its image under $U^{\mathbb{M}}$:

$$U^{\mathbb{M}}A \overset{U^{\mathbb{M}}f}{\underset{U^{\mathbb{M}}g}{\rightrightarrows}} U^{\mathbb{M}}B \tag{4.2}$$

The above pair (4.2) in $\mathscr{K}$ has an U-absolute coequalizer

$$UU^{\mathbb{M}}A \overset{UU^{\mathbb{M}}f}{\underset{UU^{\mathbb{M}}g}{\rightrightarrows}} UU^{\mathbb{M}}B \xrightarrow{q} Z \tag{4.3}$$

in $\mathscr{X}$. Since U is assumed to be monadic, there is a coequalizer

$$U^{\mathbb{M}}A \overset{U^{\mathbb{M}}f}{\underset{U^{\mathbb{M}}g}{\rightrightarrows}} U^{\mathbb{M}}B \xrightarrow{c} X \tag{4.4}$$

with $Uc \cong q$.

To finish the proof we only need to endow X in (4.4) with the structure of an $\mathbb{M}$-algebra such that $c : U^{\mathbb{M}}B \longrightarrow X$ is an $\mathbb{M}$-algebra homomorphism. To this end, it suffices to prove that both M and MM preserve the coequalizer (4.4).

Consider the following 3×3 scheme:

$$\begin{tikzcd}
FUFUU^{\mathbb{M}}A \arrow[r, shift left, "FUFUU^{\mathbb{M}}f"] \arrow[r, shift right, "FUFUU^{\mathbb{M}}g"'] \arrow[d, shift right, "\varepsilon FUU^{\mathbb{M}}A"'] \arrow[d, shift left, "FU\varepsilon U^{\mathbb{M}}A"] & FUFUU^{\mathbb{M}}B \arrow[r, "FUFUc"] \arrow[d, shift right, "\varepsilon FUU^{\mathbb{M}}B"'] \arrow[d, shift left, "FU\varepsilon U^{\mathbb{M}}B"] & FUFUX \arrow[d, shift right, "\varepsilon FUX"'] \arrow[d, shift left, "FU\varepsilon X"] \\
FUU^{\mathbb{M}}A \arrow[r, shift left, "FUU^{\mathbb{M}}f"] \arrow[r, shift right, "FUU^{\mathbb{M}}g"'] \arrow[d, "\varepsilon U^{\mathbb{M}}A"'] & FUU^{\mathbb{M}}B \arrow[r, "FUc"] \arrow[d, "\varepsilon U^{\mathbb{M}}B"'] & FUX \arrow[d, "\varepsilon X"'] \\
U^{\mathbb{M}}A \arrow[r, shift left, "U^{\mathbb{M}}f"] \arrow[r, shift right, "U^{\mathbb{M}}g"'] & U^{\mathbb{M}}B \arrow[r, "c"] & X
\end{tikzcd} \tag{4.5}$$

Observe that, by assumption, the first two "rows" are absolute coequalizers and all three "columns" are coequalizers preserved by M.

Hence, by applying M to (4.5) we obtain a 3×3 scheme where all "columns" and first two "rows" are coequalizers. Therefore the bottom "row" must be a coequalizer, which is exactly what we wanted.

The argument for MM is analogous.

Having proved that M and MM preserves the coequalizer (4.4), we define the $\mathbb{M}$-algebra structure $x : MX \longrightarrow X$ on X as the unique mediating morphism in

$$\begin{tikzcd}
MU^{\mathbb{M}}A \arrow[r, shift left, "MU^{\mathbb{M}}f"] \arrow[r, shift right, "MU^{\mathbb{M}}g"'] \arrow[d, "a"] & MU^{\mathbb{M}}B \arrow[r, "Mc"] \arrow[d, "b"] & MX \arrow[d, dotted, "x"] \\
U^{\mathbb{M}}A \arrow[r, shift left, "U^{\mathbb{M}}f"] \arrow[r, shift right, "U^{\mathbb{M}}g"'] & U^{\mathbb{M}}B \arrow[r, "c"] & X
\end{tikzcd} \tag{4.6}$$

It is now standard (see Proposition 3, page 67 of [L]) to prove that $x : MX \longrightarrow X$ is indeed an $\mathbb{M}$-algebra structure (for this one needs that MM preserves (4.4)) and that $c : U^{\mathbb{M}}B \longrightarrow X$ is a coequalizer in the category of $\mathbb{M}$-algebras. $\square$

Remark 4.2. Observe that the assumption that U and $U^{\mathbb{M}}$ be finitary are irrelevant in the above theorem. Indeed, the theorem holds more generally:

> Suppose that $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ is an *arbitrary* monadic adjunction. Suppose further that $\mathbb{M} = (M, \eta, \mu)$ is an *arbitrary* monad on $\mathscr{K}$ such that the functor M preserves canonical resolutions (3.1). Then the composite $UU^{\mathbb{M}} : \mathscr{K}^{\mathbb{M}} \longrightarrow \mathscr{X}$ is monadic.

Proposition 4.3. *The forgetful functor* $\mathsf{Mnd}_{\mathit{finb}}(\mathscr{K}) \longrightarrow \mathsf{FinB}(\mathscr{K}, \mathscr{K})$ *is monadic.*

Proof. It suffices to prove that the free finitary monad $\mathbb{F}_L$ on a finitary based functor L is finitary based. Recall from [A] that the underlying finitary functor F_L of $\mathbb{F}_L$ is given by a colimit of the countable chain

$$W_0^L \xrightarrow{w_{0,1}^L} W_1^L \xrightarrow{w_{1,2}^L} \cdots$$

where $W_0^L = Id$ and $W_{k+1}^L = L \cdot W_k^L + Id$, $w_{0,1}^L = \mathsf{inr}$ and $w_{k+1,k+2}^L = Lw_{k,k+1}^L + id$.

To prove that F_L is finitary based, observe that each W_k is and then use the fact that finitary based functors are closed in the category of finitary functors under colimits (see Lemma 3.11). $\square$

We prove now that finitary based monads on $\mathscr{K}$ can be presented using finitary signatures on $\mathscr{X}$.

Theorem 4.4. *The functor* $\mathsf{Mnd}_{\mathit{finb}}(\mathscr{K}) \longrightarrow \mathsf{Sig}_{\mathit{fin}}(\mathscr{X})$*, sending* $\mathbb{M} = (M, \eta, \mu)$ *to the signature* $n \mapsto UMFn$ *has a left adjoint and the resulting adjunction is of descent type.*

Proof. For the purpose of this proof, let us introduce the following notation:

1. The forgetful monadic functor $\mathsf{Mnd}_{\mathit{finb}}(\mathscr{K}) \longrightarrow \mathsf{FinB}(\mathscr{K}, \mathscr{K})$ is denoted by W. Its left adjoint is denoted by $\mathbb{F}$.
2. The forgetful functor $\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \longrightarrow \mathsf{Sig}_{\mathit{fin}}(\mathscr{X})$ of Theorem 3.16 is denoted by V. By the same theorem, V has a left adjoint, denoted by G, and the adjunction $G \dashv V$ is of descent type.

We need to prove that the composite adjunction $\mathbb{F}G \dashv VW$ is of descent type. We will closely follow the proof of Theorem 5.1 of [KP].

Denote by α the counit of the adjunction $\mathbb{F}G \dashv VW$. We will prove that $\alpha_{\mathbb{T}}$ is W-final, for every finitary based monad $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$. This amounts to proving the following:

> For every finitary based monad $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ and every natural transformation $\tau : T \longrightarrow S$, such that the composite $\tau \cdot \alpha_{\mathbb{T}} : \mathbb{F}GVW(\mathbb{T}) \longrightarrow \mathbb{S}$ is a monad morphism, τ is a monad morphism.

Therefore we assume that the perimeters of the following two diagrams

$$
\begin{tikzcd}
F_{GV(T)} \cdot F_{GV(T)} \arrow[r, "\alpha_{\mathbb{T}} * \alpha_{\mathbb{T}}"] \arrow[d, "\mu^{\mathbb{F}_{GV(T)}}"'] & T \cdot T \arrow[r, "\tau * \tau"] \arrow[d, "\mu^{\mathbb{T}}"] & S \cdot S \arrow[d, "\mu^{\mathbb{S}}"] \\
F_{GV(T)} \arrow[r, "\alpha_{\mathbb{T}}"'] & T \arrow[r, "\tau"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{GV(T)} \arrow[r, "\alpha_{\mathbb{T}}"] & T \arrow[r, "\tau"] & S \\
 & Id \arrow[ul, "\eta^{\mathbb{F}_{GV(T)}}"] \arrow[u, "\eta^{\mathbb{T}}"'] \arrow[ur, "\eta^{\mathbb{S}}"'] &
\end{tikzcd}
$$

commute.

There is nothing to prove about the right-hand triangle: the equality $\tau \cdot \eta^{\mathbb{T}} = \eta^{\mathbb{S}}$ clearly holds. To prove that the right-hand rectangle commutes, we will show that $\alpha_{\mathbb{T}} * \alpha_{\mathbb{T}}$, or more precisely, $W(\alpha_{\mathbb{T}}) * W(\alpha_{\mathbb{T}})$ is an epimorphism in $\mathsf{FinB}(\mathscr{K}, \mathscr{K})$.

We will prove that both

$$W(\alpha_{\mathbb{T}}) * id_{W(\mathbb{F}_{GV(T)})} : W(\mathbb{F}_{GV(T)}) \cdot W(\mathbb{F}_{GV(T)}) \longrightarrow T \cdot W(\mathbb{F}_{GV(T)})$$

and

$$id_{W(\mathbb{T})} * W(\alpha_{\mathbb{T}}) : T \cdot W(\mathbb{F}_{GV(T)}) \longrightarrow T \cdot T$$

are epimorphic, the transformation $W(\alpha_{\mathbb{T}}) * W(\alpha_{\mathbb{T}})$ will then be a composition of two epimorphisms.

(1) The natural transformation $W(\alpha_{\mathbb{T}}) * id_{W(\mathbb{F}_{GV(T)})}$ is an epimorphism.
   To prove it, observe that $VW(\alpha_{\mathbb{T}})$ is a split epimorphism in $\mathsf{Sig}_{\mathit{fin}}(\mathscr{X})$ by triangle equality for $\mathbb{F}G \dashv VW$. Since V is faithful (being of descent type by Theorem 3.16), $W(\alpha_{\mathbb{T}})$ is an epimorphism. Now, the functor $- \cdot W(\mathbb{F}_{GV(T)})$ is a left adjoint, hence it preserves epimorphisms. Therefore $W(\alpha_{\mathbb{T}}) * id_{W(\mathbb{F}_{GV(T)})}$ is an epimorphism.

(2) The natural transformation $id_{W(\mathbb{T})} * W(\alpha_{\mathbb{T}})$ is an epimorphism.
   Again use that $VW(\alpha_{\mathbb{T}})$ is a split epimorphism in $\mathsf{Sig}_{\mathit{fin}}(\mathscr{X})$. Then by Lemma 5.2 of [KP] it follows that $id_{W(\mathbb{T})} * W(\alpha_{\mathbb{T}})$ is an epimorphism.

□

Theorem 4.4 states that, for every finitary based monad $\mathbb{M}$, there is a coequalizer of the form

$$\mathbb{F}_\Gamma \overset{\lambda}{\underset{\rho}{\rightrightarrows}} \mathbb{F}_\Sigma \xrightarrow{\gamma} \mathbb{M}$$

in the category $\mathsf{Mnd}_{\mathit{finb}}(\mathscr{K})$, for some suitable finitary signatures Γ and Σ on the category $\mathscr{X}$. The above coequalizer then represents an equational presentation of $\mathbb{M}$ in the same manner as discussed for functors at the end of the previous section.

5. Applications to Modal Algebras

In coalgebraic modal logic, one is interested in finding a modal logic that is used for description of coalgebras for a given functor $T : \mathsf{Spa} \longrightarrow \mathsf{Spa}$, where Spa is the category of "spaces". The technique involves choosing a category Alg of "algebras" that encodes the propositional part of the logic. Categories Spa and Alg should then be connected by a contravariant adjunction of a special kind, called a *logical connection*.

Given all the above data, one can canonically construct a functor $L : \mathsf{Alg} \longrightarrow \mathsf{Alg}$ such that the category Alg^L of L-algebras forms precisely the "algebras with modal operators".

For example, the study of coalgebras for the covariant powerset functor $\mathcal{P} : \mathsf{Set} \longrightarrow \mathsf{Set}$ is the study of Kripke frames. Choosing the variety BA of Boolean algebras for Alg gives rise to modal logics with classical propositional background. The logical connection of $\mathsf{Spa} = \mathsf{Set}$ and $\mathsf{Alg} = \mathsf{BA}$ is given by homming into the two-element set 2 (considered, when necessary, as a Boolean algebra).

In fact, the functor $L : \mathsf{BA} \longrightarrow \mathsf{BA}$ is then given by the formula

$$LA = \coprod_n \mathsf{Set}(n, UA) \bullet \Delta n$$

where $U : \mathsf{BA} \longrightarrow \mathsf{Set}$ denotes the monadic forgetful functor, $\Delta n = [\mathcal{P}(2^n), 2]$ is the Boolean algebra of n-ary modalities of the resulting modal language and the coproduct is taken over finite ordinals.

It is obvious that L is a finitary based functor, see Example 3.10 above, and L can be presented by a single unary operation $\Box$ subject to equations $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box\top = \top$, a result that goes back at least to [Ab].

Thus BA^L is indeed the variety of Boolean algebras with operators in the sense of classical modal logic. Moreover, the equational presentation is exactly of the form given by Theorem 3.16. Indeed, define the finitary signature

$$\Sigma 1 = \{\Box\}, \quad \Sigma n = \emptyset \text{ otherwise}$$

The corresponding finitary polynomial functor on Set is

$$H_\Sigma X = \coprod_n \mathsf{Set}(n, X) \bullet \Sigma n = X \times \{\Box\}$$

since the components of the coproduct are, in reality, nonempty only for $n = 1$.

By Remark 3.14, the finitary based functor $\widehat{H_\Sigma}$, free on H_Σ, is given by the formula

$$\widehat{H_\Sigma}(Fm) = FH_\Sigma UFm = F(UFm \times \{\Box\})$$

Therefore the underlying finitary signature $m \mapsto U\widehat{H_\Sigma}Fm$ of $\widehat{H_\Sigma}$ is,

$$m \mapsto UF(UFm \times \{\Box\})$$

Define the signature Γ as follows:

$$\begin{aligned} \Gamma 0 &= 1, \text{ (one equation in no variables)} \\ \Gamma 2 &= 1, \text{ (one equation in two variables)} \\ \Gamma m &= \emptyset, \text{ otherwise (no other equations)} \end{aligned}$$

It is clear how to define the pair $\lambda_0^\flat, \rho_0^\flat : \Gamma 0 \longrightarrow UF(UF0 \times \{\Box\})$ to encode the equation $\Box(\top) = \top$ and the pair $\lambda_2^\flat, \rho_2^\flat : \Gamma 2 \longrightarrow UF(UF2 \times \{\Box\})$ to encode the equation $\Box(a \wedge b) = \Box(a) \wedge \Box(b)$.

Not only L is equationally presentable, it is known that the category BA^L of modal algebras is equationally definable.

In [KuR] it is proved that Alg^L is always a finitary variety, provided that $L : \mathsf{Alg} \longrightarrow \mathsf{Alg}$ is finitary based and Alg is a variety, yielding thus a proof system for modal logic. We have achieved a generalization of this result. Namely, it is shown in [KuV] that logical connections can be treated in a many-sorted and enriched way. Thus, one can choose Alg to be a "variety" over a presheaf category $[\mathscr{C}, \mathscr{V}]$, where $\mathscr{C}$ is a small category of "sorts". More precisely, one can choose a finitary monadic adjunction $F \dashv U : \mathsf{Alg} \longrightarrow [\mathscr{C}, \mathscr{V}]$. Such approach seems to be very natural when the original functor T is defined inductively and one takes (the opposite of) the category of ingredients of T for the category $\mathscr{C}$ of sorts. See, for example, [Kup].

The move from $\mathscr{V} = \mathsf{Set}$ to other base categories when studying coalgebraic modal logic makes also perfect sense in applications. For example, $\mathscr{V}$ is the category of posets in [KaKuV].

In full generality, there is a canonical process to obtain a finitary based functor $L : \mathsf{Alg} \longrightarrow \mathsf{Alg}$ that captures the modalities of the logic. Then by the results in the current paper the following holds:

> The functor L is equationally presentable over $[\mathscr{C}, \mathscr{V}]$ and the category Alg^L of modal algebras is monadic over $[\mathscr{C}, \mathscr{V}]$.

The first assertion follows immediately from Theorem 3.16 and the second assertion follows from Theorem 4.1 applied to the well-know fact that the obvious functor $\mathsf{Alg}^L \longrightarrow \mathsf{Alg}$ is monadic and the monad $\mathbb{L}$ on Alg of this adjunction is a finitary based monad ($\mathbb{L}$ is the monad free on L and it is finitary based by Proposition 4.3).

References

[Ab] S. Abramsky, A Cook's tour of the finitary non-well founded sets, in: *We will show them: Essays in honour of Dov Gabbay*, College Publications, 1005, 1–18

[A] J. Adámek, Free algebras and automata realizations in the language of categories, *Comment. Math. Univ. Carolin.* 15 (1974), 589–602

[AR] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, Cambridge University Press, 1994

[Bi] G. J. Bird, *Limits in 2-categories of locally-presented categories*, PhD Thesis, The University of Sydney, 1984

[BKu] M. M. Bonsangue and A. Kurz, Presenting functors by operations and equations, *Proceedings of FOSSACS 2006*, LNCS 3921, 172–186

[GU] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics 221, Springer 1971

[KaKuV] K. Kapulkin, A. Kurz and J. Velebil, Expressivity of coalgebraic logic over posets, to appear in *Short contributions CALCO 2010*

[K$_1$] G. M. Kelly, *Basic concepts of enriched category theory*, London Math. Soc. Lecture Notes Series 64, Cambridge Univ. Press, 1982, also available as *TAC* reprint via `http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html`

[K$_2$] G. M. Kelly, Structures defined by finite limits in the enriched context I, *Cahiers de Top. et Géom. Diff.* XXIII.1 (1982), 3–42

[KP] G. M. Kelly and A. J. Power, Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads, *J. Pure Appl. Algebra* 89 (1993), 163–179

[Kup] C. A. Kupke, *Finitsary coalgebraic logics*, PhD Thesis, University of Amsterdam, 2006

[KuP] A. Kurz and D. Petrişan, On universal algebra over nominal sets, to appear in *Math. Structures Comput. Sci.*

[KuPV] A. Kurz, D. Petrişan and J. Velebil, Algebraic theories over nominal sets, *manuscript*, March 2010

[KuR] A. Kurz and J. Rosický, Strongly complete logics for coalgebras, *submitted*

[KuV] A. Kurz and J. Velebil, Enriched logical connections, *manuscript*, March 2010

[L] F. E. J. Linton, Coequalizers in categories of algebras, in *Seminar on triples and categorical homology theory*, Lecture Notes in Mathematics 80, Springer 1969, 61–72, also available as *TAC* reprint via `http://www.tac.mta.ca/tac/reprints/articles/18/tr18abs.html`

[McL] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Springer 1998

Faculty of Electrical Engineering, Czech Technical University, Prague, Czech Republic
E-mail address: `velebil@math.feld.cvut.cz`

Department of Computer Science, University of Leicester, United Kingdom
E-mail address: `kurz@mcs.le.ac.uk`