Expressivity of Coalgebraic Logic over Posets

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Abstract

We study how to move coalgebraic logic from sets to posets. In particular, we give conditions under which coalgebraic logic is expressive.

1 Introduction

Most of coalgebraic logic so far has focused on set-coalgebras and their Boolean logics. Exceptions include [9], [6], [2] and the work on coalgebras over measurable spaces [3]. Here we start a systematic investigation of coalgebras over posets and set ourselves the modest aim to show how to transfer to Pos the result of Schröder [12] and Klin [5], namely that the logic of all predicate liftings is expressive.

The basic ingredients of Schröder's argument [12] are still available in Pos, if we replace the discrete two-element set 2 by the two-element linear order $\mathbb{2}$. This means that formulas now correspond to upsets and expressivity means not only that non-bisimilar points can be separated, but also that the order between non-bisimilar points can be recovered from the logical information.

Although we do not emphasise this point of view, the proper framework in which to develop this generalisation is that of enriched category theory. Since categories enriched over Pos are simply categories with ordered hom-sets we do not need to go into the technicalities of enriched category theory. It only appears in justifying the crucial use of the Yoneda lemma and the assumption that our functors $T : \mathsf{Pos} \to \mathsf{Pos}$ are locally monotone (i.e., they are enriched functors).

The point where, compared to [12], a new phenomenon enters the picture is that Set but not Pos has the special property that every monomorphism with a non-empty domain is preserved by any functor. Monomorphisms of course play

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a special role, because expressivity is about separating (non-bisimilar) points and the crucial technical step in an expressivity argument (see eg [6] and [4]) typically involves a class of monos preserved by T. In our case, it will be embeddings, ie order-reflecting maps, since we want that the semantic order on coalgebras is characterised by the logical order. Thus, the expressivity result will need that functors $T : \mathsf{Pos} \to \mathsf{Pos}$ preserve embeddings.

We believe that logics of coalgebras over Pos are interesting for several reasons. First, from the point of view of coalgebraic logic, Pos is one of the obvious choices to consider, if one is interested in investigating the inherent generality in the notion of a coalgebra over a base category. Second, from the point of view of Stone duality, the adjunction between Pos and DL (distributive lattices) shares all the basic properties with the adjunction Set and BA (Boolean algebras), which promises that many further results can be obtained in analogy with the Set-based setting. Third, we see coalgebras over Pos as a bridge between set-coalgebras and domain theory (i.e., coalgebras over domains), thus being a step towards a more systematic connection between the two areas. Last but not least, many of the techniques used in the Stone duality approach to coalgebraic logic are available in the setting of enriched category theory, a line of research that will be pursued in the future.

2 Preliminaries

2.1 Logical Connections

The basic ingredient of set-based coalgebraic logic is the adjunction

$$\begin{tikzcd} \mathsf{Set}^{op} \arrow[r, bend right, "Pred"'] \arrow[r, phantom, "\perp"] & \mathsf{BA} \arrow[l, bend right, "Stone"'] \end{tikzcd} \tag{1}$$

where *Pred* and *Stone* are the “predicate” and “Stone” functors, respectively. The functor *Pred* endows the powerset with the natural structure of a Boolean algebra and *Stone* takes the set of ultrafilters on a given Boolean algebra. Many nice properties of the above adjunction follow from the fact that it is given by a two-element set 2, that acts as a *schizophrenic object* in the sense of [11]. We will refer to the above adjunction as (an instance of) a *logical connection*.

Stone’s representation theorem states that the unit $\eta_A : A \to Pred Stone A$ of the above adjunction is injective, which is a way of proving the completeness theorem of classical propositional logic.

By choosing the categories Set and BA we also have made a choice of over which category we will consider the coalgebras (here over Set), and where we will compute with the formulas of the relevant logic (here in BA).

Recall that, given a functor $T : \mathsf{Set} \to \mathsf{Set}$, a T*-coalgebra* (notation: (X, ξ) or just ξ) is a map $\xi : X \to TX$. A morphism $f : \xi \to \xi'$ is a map $f : X \to X'$ such that $Tf \circ \xi = \xi' \circ f$.

The rest of the set-based coalgebraic logic is therefore determined by a choice of a “behaviour” functor $T : \mathsf{Set} \to \mathsf{Set}$ and a functor $L : \mathsf{BA} \to \mathsf{BA}$ that captures the "logic" of coalgebras for T. The choice of T is made first and the functor L is subsequently computed to encode the modal operators and axioms describing T.

Thus, the full picture of set-based coalgebraic modal logic can be conveniently described by the following diagram [7]

$$\begin{tikzcd} \mathsf{Set}^{op} \arrow[loop left, "T^{op}"] \arrow[r, bend right, "Pred"'] \arrow[r, phantom, "\perp"] & \mathsf{BA} \arrow[l, bend right, "Stone"'] \arrow[loop right, "L"] \end{tikzcd} \tag{2}$$

The syntax and proof system of the induced modal logic are given by (a presentation) of the initial L-algebra[1] in BA and the semantics by a natural transformation

$$\delta_X : LPredX \to PredT^{op}X \tag{3}$$

Explicitly, δ associates to any coalgebra (X, ξ) the L-algebra $(PredX, Pred\xi \circ \delta)$ and the map from the initial L-algebra to $PredX$ gives the semantics of the formulas of the logic.

2.2 Posets

We are interested in coalgebras over the category Pos of posets and monotone maps.

Definition 2.1. *An* embedding $X \rightarrowtail Y$ *in* Pos *is a monotone and order-reflecting map.*

Proposition 2.2. *A morphism $f : X \rightarrowtail Y$ is an embedding in* Pos *if and only if it is an equalizer.*

Notation 2.3. $\mathbb{2}$ *denotes the linear order* $0 < 1$. *Given posets X,Y we write $[X, Y]$ for the poset of monotone maps, ordered pointwise.*

In order to be able to use the (enriched) Yoneda lemma, we need

Assumption 1. *From now on, we assume that all functors T :* $\mathsf{Pos} \to \mathsf{Pos}$ *are* locally monotone*, that is, $f \leq g$ implies $Tf \leq Tg$.*

Given a locally monotone $T : \mathsf{Pos} \to \mathsf{Pos}$, we will study T*-coalgebras*, one such being a monotone map $\xi : X \to TX$. A *coalgebra homomorphism* $f : \xi \to \xi'$ is a monotone map $f : X \to X'$ such that $Tf \circ \xi = \xi' \circ f$ holds.

Remark 2.4. *Of course, given a locally monotone T, it is natural to consider the category of T-coalgebras as* enriched *over* Pos*, i.e., we consider coalgebra homomorphisms to be ordered pointwise.*

[1]An *L-algebra* (notation: (A, α) or just α) is an arrow $\alpha : LA \to A$ in BA. A morphism $f : \alpha \to \alpha'$ is an arrow $f : A \to A'$ in BA such that $f \circ \alpha = \alpha' \circ Lf$.

3 Logic for Coalgebras over Posets

Technically, in this paper, we replace the adjunction between Set and BA of diagram (1) by an adjunction

$$\begin{tikzcd} \mathsf{Pos}^{op} \arrow[r, bend right, "Pred"'] \arrow[r, phantom, "\perp"] & \mathsf{DL} \arrow[l, bend right, "Stone"'] \end{tikzcd} \tag{4}$$

between Pos and DL, the category of distributive lattices. The above adjunction is to be considered as an adjunction in the *enriched sense*. This means that both the predicate functor $Pred$ and the Stone functor $Stone$ are locally monotone and that there is an isomorphism

$$\mathsf{Pos}^{op}(Stone A, X) \cong \mathsf{DL}(A, Pred X)$$

of *posets*, natural in X and A.

The predicate functor $Pred$ assigns the poset $\mathsf{Pos}(X, \mathbb{2})$ endowed with the canonical structure of a distributive lattice to a poset X. Observe that it means that the following diagram

$$\begin{tikzcd} \mathsf{Pos}^{op} \arrow[rr, "Pred"] \arrow[dr, "\mathsf{Pos}(-,\mathbb{2})"'] & & \mathsf{DL} \arrow[dl, "U"] \\ & \mathsf{Pos} & \end{tikzcd} \tag{5}$$

commutes up to isomorphism, where U denotes the obvious (locally monotone!) forgetful functor from DL to Pos. Observe that, in elementary terms, $Pred X$ is the distributive lattice of upper-sets in X.

The Stone functor $Stone : \mathsf{DL} \to \mathsf{Pos}^{op}$ assigns the poset $\mathsf{DL}(A, \mathbb{2})$ of prime filters on A to the distributive lattice A.

Notice that the adjunction of diagram (4) is built in the same way as the one of diagram (1), with $\mathbb{2}$ instead of 2. This "sameness" can be stated precisely by introducing schizophrenic objects and adjunctions they generate in the enriched setting. We postpone the development of the theory along the lines of [11] to future work. Let us just comment that basic ideas of [11] carry over to enrichment over Pos without any difficulties.

The whole picture of poset-based coalgebraic logic will therefore be given by the diagram

$$\begin{tikzcd} \mathsf{Pos}^{op} \arrow[loop left, "T^{op}"] \arrow[r, bend right, "Pred"'] \arrow[r, phantom, "\perp"] & \mathsf{DL} \arrow[l, bend right, "Stone"'] \arrow[loop right, "L"] \end{tikzcd} \tag{6}$$

As before the syntax and axioms of the logic will be given by a functor L and the semantics will be given by a natural transformation

$$\delta_X : L Pred X \to Pred T^{op} X \tag{7}$$

Definition 3.1. *We call a pair (L, δ) as in (6) and (7) a* logic for T.

3.1 Expressiveness

We discuss the expressiveness of coalgebraic logics following Klin [6, Theorem4.2] (see also Jacobs and Sokolova [4, Theorem 4]).

First, note that the natural transformation δ induces its *mate* $\tau : T^{op}\,Stone \to Stone L$ given explicitly by the pasting

$$\begin{tikzcd}
& \mathsf{Pos}^{op} \arrow[rr, "T^{op}"] \arrow[dr, "Pred"] \arrow[d, phantom, "\uparrow\eta"] & & \mathsf{Pos}^{op} \arrow[rr, equal] \arrow[dr, "Pred"'] & & \mathsf{Pos}^{op} \\
\mathsf{DL} \arrow[ur, "Stone"] \arrow[rr, equal] & & \mathsf{DL} \arrow[rr, "L"'] \arrow[ur, phantom, "\uparrow\delta"] & & \mathsf{DL} \arrow[ur, "Stone"'] \arrow[u, phantom, "\uparrow\varepsilon"] &
\end{tikzcd} \tag{8}$$

where η and ε are the unit and the counit of the adjunction $Stone \dashv Pred$, see (6).

The following result connects expressivity with properties of δ (or τ).

Theorem 3.2. *Consider a locally monotone functor $T : \mathsf{Pos} \to \mathsf{Pos}$ preserving embeddings and a logic (L, δ) for T. If the τ_X are embeddings, then the logic is expressive.*

Proof. We have the following diagram in Pos:

$$\begin{tikzcd}
Stone\mathbb{2} = 1 \arrow[r] & Stone L\mathbb{2} \arrow[r] & Stone L^2\mathbb{2} \arrow[r] & \cdots \arrow[r] & Stone L^k\mathbb{2} \arrow[r] & {} \\
1 \arrow[u, "\theta_0"] \arrow[r] & T1 \arrow[u, "\theta_1"] \arrow[r] & T^2 1 \arrow[u, "\theta_2"] \arrow[r] & \cdots \arrow[r] & T^k 1 \arrow[u, "\theta_k"] \arrow[r] & {}
\end{tikzcd}$$

We define θ_0 to be the identity on 1 and

$$\theta_{k+1} = \tau_{L^k\mathbb{2}} \circ T(\theta_k).$$

Since T preserves embeddings, θ_k is an embedding for all $k \in \mathbb{N}$. Note that θ_k maps an element of $t \in T^k 1$ to its theory, ie, to the set of formulas satisfied by t. It follows that (L, δ) is expressive in the sense that all states which are not ω-bisimilar[2], are separated by some formula. If T is finitary, then we can drop the "ω-" in the previous sentence.[3] $\square$

Remark 3.3. *The proof can be seen as terminal sequence version of the proof of Klin [6, Theorem4.2], which in turn can be seen as a category theoretic analysis of Schröder [12].*

[2] What we call ω-bisimilarity here was called finite-step equivalence in [8].

[3] Actually, since we insist on embeddings as opposed to injections, we get the stronger property that the order on the final coalgebra is recovered from the logic.

3.2 Predicate Liftings

As in the Set-based case L and δ can be described more explicitly by predicate liftings, which have been introduced by Pattinson [10] and further studied by Schröder [12] and Klin [5].

To transport their approach to our setting, we only need to generalise the n-th power (n is a finite set) of a set S to the "n-th power of a poset S", where n can now be any finite poset. It turns out that the universal property (natural in X)

$$\mathsf{Set}(X, S^n) \cong \mathsf{Set}(n, [X, S])$$

of the n-fold power of a set S can be taken verbatim to define the desired (enriched) notion for posets.

Definition 3.4. *Suppose S is a poset and n is a finite poset. The n-*fold cotensor *of S is a poset $n \pitchfork S$ together with an isomorphism*

$$\mathsf{Pos}(X, n \pitchfork S) \cong \mathsf{Pos}(n, [X, S])$$

natural in X.

It is easy to verify that $n \pitchfork S$ is the poset $[n, S]$ of all monotone maps from n to S, ordered pointwise. The following definition transfers the notion of predicate liftings from Set to Pos.

Definition 3.5. *An n-ary* predicate lifting *for a functor T is a natural transformation λ given by components*

$$\lambda_X : \mathsf{Pos}(X, n \pitchfork \mathbb{2}) \to \mathsf{Pos}(TX, \mathbb{2}) \tag{9}$$

where n can be any finite poset.

As opposed to the Set-based case, all predicate liftings are monotone since each λ_X is an arrow in Pos and hence a monotone map.

Remark 3.6. *It follows from the (enriched) Yoneda lemma that the poset of all predicate liftings is in bijection to*

$$\mathsf{Pos}(T(n \pitchfork \mathbb{2}), \mathbb{2}). \tag{10}$$

Recall also that, for every finite poset n, the poset $\mathsf{Pos}(T(n \pitchfork \mathbb{2}), \mathbb{2})$ is an underlying poset of a distributive lattice that we denote by

$$\Lambda n$$

(where Λ should remind of lifting). The nature of the adjunction (4) and diagram (5) allows us to infer the isomorphism

$$\Lambda n \cong \mathit{Pred}\, T^{op}\, \mathit{Stone}\, Fn$$

where Fn denotes the free DL over a finite poset n. The assignment $n \mapsto \Lambda n$ can be viewed as a finitary signature in the enriched sense: arities are finite posets and n-ary operations (i.e., n-ary predicate liftings) form a distributive lattice.

The following remark explains how predicate liftings fit into diagram (6). Recall that $U : \mathsf{DL} \to \mathsf{Pos}$ denotes the forgetful functor.

Remark 3.7. *Predicate liftings induce a functor $L : \mathsf{DL} \to \mathsf{DL}$. The idea is that LA is the distributive lattice of "modal formulas of depth at most one, labelled in elements of a distributive lattice A". Hence L is the polynomial functor corresponding to the above signature Λ and the precise formula is given by*

$$LA = \coprod_{n \text{ finite poset}} \mathsf{Pos}(n, UA) \otimes \Lambda n \tag{11}$$

where by $\otimes$ we denote the $\mathsf{Pos}(n, UA)$-fold tensor *of the distributive lattice Λn. In general, P-fold tensor of a distributive lattice A is a distributive lattice $P \otimes A$ together with an isomorphism*

$$\mathsf{DL}(P \otimes A, B) \cong \mathsf{Pos}(P, \mathsf{DL}(A, B))$$

natural in B. Since (11) is a left Kan extension formula (in the appropriate enriched sense), L comes equipped for canonical reasons with a natural transformation

$$\delta : LPred \to PredT^{op}$$

given by:

$$\lambda(a_1, \ldots, a_n) \mapsto \ \lambda \circ Ta' : TX \to T(n \pitchfork \mathbb{2}) \to \mathbb{2}, \tag{12}$$

where $a' : X \to n \pitchfork \mathbb{2}$ is the transpose[4] *of $a : n \to UPredX = [X, \mathbb{2}]$, see Definition 3.4.*

It follows immediately from the definition of L that:

Proposition 3.8. *The functor L preserves surjective homomorphisms of distributive lattices.*

Remark 3.9. *All of the above will work if we replace the distributive lattice Λn of* all *n-ary predicate liftings by any* sublattice *$\Lambda' n$.*

3.3 The Logic of all Predicate Liftings is Expressive

The next theorem shows that the logic of all predicate liftings, or also the logic of all predicate liftings with discrete arities, is expressive. The key observations are contained in the following two lemmas.

Lemma 3.10. *Every finite poset X can be embedded into a poset $n \pitchfork \mathbb{2}$ for some finite poset n.*

Proof. It is standard to verify that the function being a transpose of

$$X \times [X, \mathbb{2}] \to \mathbb{2}, \quad (x, f) \mapsto f(x)$$

is an embedding. □

[4]Jiri, a reviewer asked: "In which sense is a' the transpose of a?" ... isnt transpose the right word? I added a reference to Definition 3.4 .

Lemma 3.11. *Every finite poset X can be embedded into a poset $n \pitchfork \mathbb{2}$ for some finite discrete poset n.*

Proof. Let n be a discrete poset with $|X|$ elements. We will construct an embedding $i : X \rightarrowtail n \pitchfork \mathbb{2}$.

Let $f : X \rightarrowtail \{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}$ be any bijection. We define:

$$i(x) = (\max_{y\le x} \pi_1(f(y)), \max_{y\le x} \pi_2(f(y)), \dots, \max_{y\le x} \pi_n(f(y))).$$

It is standard to verify that such i is an appropriate embedding. □

The next lemma requires embeddings as opposed to only injections. It is also the place where it comes in that DL has finite meets and joins. Finally, we note for future generalisations that the lemma makes use of the following: (i) every poset is a filtered colimit of finite posets where we can take the cone to consist of embeddings (reg monos); (ii) if X is finite and $X \to SA$ is reg mono then the transpose $A \to PX$ is reg epi; (iii) L preserves reg epis.

Lemma 3.12. *Consider a finite poset X and an embedding $c : X \to \mathit{Stone}A$. Denote by $c^\sharp : A \to \mathit{Pred}X$ the transpose of c. Then $\mathit{Stone}Lc^\sharp : \mathit{Stone}L\mathit{Pred}X \to \mathit{Stone}LA$ is a regular epi in Pos^{op}, hence an embedding in Pos.*

Proof. We first prove that $c^\sharp$ is a surjection. To this end, it is convenient to identify $\mathit{Pred}X$ with the set of downsets of X (ordered by inclusion), $\mathit{Stone}A$ with the set of prime filters (ordered by reverse conclusion), and to abbreviate $a \in c(x)$ by $x \Vdash a$. In this notation $c(x) = \{a \in A \mid x \Vdash a\}$ and $c^\sharp(a) = \{x \in X \mid x \Vdash a\}$. Moreover, that c is an embedding means that $y \not\le x$ iff there is $a_{xy} \in A$ such that $x \Vdash a_{xy}$ and $y \nVdash a_{xy}$. Now consider a 'principal' downset $\downarrow x \in \mathit{Pred}X$. Since X is finite we find $a_x = \bigwedge\{a_{xy} \mid y \in X \text{ and } y \not\le x\}$ in A. It follows $c^\sharp(a_x) = {\downarrow}x$. Since every downset in $\mathit{Pred}X$ is a finite join of principal downsets, $c^\sharp$ is onto.

Now that $c^\sharp$ is a surjection, by Proposition 3.8, we have that $Lc^\sharp$ is a surjection as well. But in all equational classes of algebras all epimorphisms are regular, so $Lc^\sharp$ is a regular epi in DL. Finally, $\mathit{Stone}Lc^\sharp$ is a regular epi in Pos^{op} since it is an image of a regular epi under a left adjoint. □

Theorem 3.13. *If T is finitary and Λ consists of all predicate liftings with finite arities, then τ is an embedding.*

Proof. (For reasons of type setting, we write $S = \mathit{Stone}$ and $P = \mathit{Pred}$ inside this proof.) By the definition of τ, we need to show that the composite

$$SL\varepsilon_A \circ S\delta_{SA} \circ \eta_{TSA} : TSA \to SPTSA \to SLPSA \to SLA$$

is an embedding. Consider $c : X \to SA$ and denote by $c^\sharp : A \to PX$ its transpose. Note that the diagram

$$\begin{tikzcd} TSA \arrow[r, "\eta_{TSA}"] & SPTSA \arrow[r, "S\delta_{SA}"] & SLPSA \arrow[r, "SL\varepsilon_A"] & SLA \\ TX \arrow[u, "c"] \arrow[r, "\eta_{TX}"'] & SPTX \arrow[r, "S\delta_X"] & SLPX \arrow[ur, "SLc^\sharp"'] & \end{tikzcd} \tag{13}$$

commutes because of the naturality of η and δ. Further, Pos being a locally finitley presentable category [1], SA is a filtered colimit $c_i : X_i \to SA$ where we can take the X_i to be finite and, moreover, the c_i to be embeddings. Therefore, since T preserves filtered colimits, it suffices to prove that the lower composite of (13)

$$SLc^\sharp \circ S\delta_X \circ \eta_{TX} : TX \to SPTX \to SLPX \to SLA$$

is an embedding for finite X and injections $c : X \to SA$. By Lemma 3.12 we need to show that

$$\alpha = S\delta_X \circ \eta_{TX} : TX \to SPTX \to SLPX$$

is an embedding. According to (12), α maps a point $t \in TX$ together with a formula $\lambda(a)$ to its truth value via

$$\lambda \circ Ta : TX \to T(n \pitchfork \mathbb{2}) \to \mathbb{2}. \tag{14}$$

To show that α is order-reflecting, consider $t_1, t_2 \in TX$, $t_1 \not\leq t_2$. Due to Lemma 3.11 and T preserving embeddings, there is $a : X \to n \pitchfork \mathbb{2}$ such that $Ta(t_1) \not\leq Ta(t_2)$. Define $\lambda : T(n \pitchfork \mathbb{2}) \to \mathbb{2}$ as $\lambda(x) = 1 \Leftrightarrow Ta(t_1) \leq x$, which is monotone. We now have $\lambda(Ta(t_1)) \not\leq \lambda(Ta(t_2))$, hence α is order-reflecting. □

Corollary 3.14. *Let $T : \mathsf{Pos} \to \mathsf{Pos}$ be a finitary and locally monotone functor preserving embeddings. The logic of all predicate liftings (with discrete arities) is expressive.*

Remark 3.15. *The same proof, with Pos replaced by Set also shows the expressivity of coalgebraic logic over Set. On the algebraic side, finite joins and meets are needed in Lemma 3.12.*

3.4 Separating Sets of Predicate Liftings

In concrete examples, we are interested not in the logic of all predicate liftings, but in logics of just enough predicate liftings so that expressivity is guaranteed. Going back to (12) and (14), we see that it is enough to require that the set of predicate liftings is separating [10], [12].

Definition 3.16. *A set of predicate liftings Λ is* separating *if the family*

$$(\widehat{\lambda}_X : TX \to [[X, n \pitchfork \mathbb{2}], \mathbb{2}])_{\lambda \in \Lambda n}$$

is jointly order-reflecting, that is, for all finite X and all $t_1, t_2 \in TX$ with $t_1 \not\leq t_2$ there are a finite poset n and $a : X \to n \pitchfork \mathbb{2}$ and $\lambda \in \Lambda_n$ such that $\lambda \circ Ta(t_1) = 1$ and $\lambda \circ Ta(t_2) = 0$.

An inspection of the proof of Theorem 3.13 shows

Proposition 3.17. *A set of predicate liftings is separating only if the transpose $T\mathit{Stone} \to \mathit{Stone}L$ of δ is injective.*

Corollary 3.18. *If Λ is separating then the logic given by Λ is expressive.*

We obtain all the usual examples of functors on Pos. Let us detail one.

Example 3.19. *Let $T = \mathcal{U}p_\omega : \mathsf{Pos} \to \mathsf{Pos}$ be the covariant functor which maps a poset to the set of all finitely generated up-sets ordered by reverse inclusion. LA is generated by $\Box a, a \in A$. (7) is given by*

$$\Box a \quad \mapsto \quad \mathtt{lambda}\, b.\mathtt{if}\ b \subseteq a\ \mathtt{then}\ 1\ \mathtt{else}\ 0 : \mathcal{U}pX \to \mathbb{2}$$

which we can also read as defining a predicate lifting in the form of (9) (with $a \in \mathsf{Pos}(X, \mathbb{2})$ and $b \in \mathcal{U}p(X)$). In the form of (10) it is a function

$$\mathcal{U}p(\mathbb{2}) \to \mathbb{2}$$

mapping $\{0,1\}$ to 0 and $\{1\}$ and $\emptyset$ to 1.

An example of a functor that does not preserve embeddings is the one which maps a poset to the discrete poset of its connected components.

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A Notation

Denote by $D : \mathsf{Set} \to \mathsf{Pos}$ the left-adjoint to the forgetful functor $V : \mathsf{Pos} \to \mathsf{Set}$. A set X is mapped by D to the discrete poset X.

Denote by $\mathsf{Cnct} : \mathsf{Pos} \to \mathsf{Pos}$ the functor that maps a poset P to the (discrete) poset of its connected components. The functor Cnct is locally monotone, but does preserve neither embeddings nor injections. This can be seen from the embedding of 2 into a span $\wedge$.

Claim 0. *A claim is something I believe to be true after possibly only a short amount of thinking and of which I didn't have time to attempt a proof.*

B Extending functors from Set to Pos

Given a set X and a preorder $\vdash$, we may consider $(X, \vdash)$ as a poset $X/{\vdash}$ by quotienting to obtain an anti-symmetric relation.

Any finitary set functor T has a presentation by an onto natural transformation

$$e_X : \coprod_{n<\omega} \Sigma_n \times X^n \to TX$$

Let X be an infinite, countable set (of variables). Then we can consider the kernel $\ker(e_X)$ of e_X as a set of equations in variables from X. And T is completely discribed by the Σ_n plus these equations.

We can use this to extend functors from Set to Pos.

Definition B.1. *Let T be a finitary set functor and consider an onto natural transformation $e_X : \coprod_{n<\omega} \Sigma_n \times X^n \to TX$. The extension $\bar{T} : \mathsf{Pos} \to \mathsf{Pos}$ is given on a poset $(X, \leq)$ by quotienting*

$$\coprod_{n<\omega} \Sigma_n \times X^n$$

wrt to the relation $\vdash \; = \ker(e_X) \cup \mathrm{congr}(\leq)$ where $\mathrm{congr}(\leq)$ is the smallest relation $\lesssim$ on $\coprod_{n<\omega} \Sigma_n \times X^n$ induced by

$$\frac{x_1 \leq y_1 \;\; \ldots \;\; x_n \leq y_n}{\sigma(x_1, \ldots x_n) \lesssim \sigma(y_1, \ldots y_n)} \tag{15}$$

for all $\sigma \in \Sigma_n$, $n < \omega$.

Claim 1.
1. *The definition of $\bar{T}$ induces indeed a functor on Pos. $\bar{T}$ is independent of a choice of presentation of T.*
2. *$\bar{T}$ is locally monotone.*
3. *$\bar{T}$ is the smallest quotient of DTV that is locally monotone.*

4. $\bar{\mathcal{P}}_\omega$ *is the convex powerset, mapping a poset* P *to the collection of its finitely generated convex subsets, where* $X \subseteq P$ *is convex if* $x, y \in X, z \in P$ *only if* $z \in X$. *The order on* $\bar{\mathcal{P}}_\omega(X)$ *is known as the Egli-Milner order, explicitely, for* $A, B \in \bar{\mathcal{P}}_\omega(X)$ *we have* $A \leq B$ *iff*

$$\forall x \in A \,.\, \exists y \in B \,.\, x \leq y \ \wedge \ \forall y \in B \,.\, \exists x \in A \,.\, x \leq y,$$

   which follows from (15).

5. $\bar{T}$ *preserves embeddings. This also shows that* Cnct *is not the extension of a* Set*-functor.*

6. *The predicate liftings of* $\bar{T}$, *given by* $\mathsf{Pos}(\bar{T}(Dn \pitchfork \mathbb{2}), \mathbb{2})$ *as in (10), are in one-to-one correspondence with the monotone predicate liftings of* T, *given by* $\mathsf{Set}(TV(Dn \pitchfork \mathbb{2}), 2)$.

7. *The logic of all monotone predicate liftings is expressive for all finitary set-functors* T. *(Proofsketch: it is expressive for* $\bar{T}$ *and hence* T.*)*

I think 6 and 7 could be a good selling point for coalgebraic logic people. According to Achim Jung, 4 is known in a sense to domain theorists, but the construction above is much more general (they only consider particular monads, whereas we have a result for arbitrary set-functors (if the above is true)).[5]

Question 1. *Jiri, you had another definition of lifting functors from* Set *to* Pos*. Any comments?*

C Equational presentation of Pos-functors

As reviewed at the beginning of the previous section, finitary Set-functors have equational presentations. Here we present the analogous notion for Pos-functors.

Every finitary Pos-functor T can be presented as $\int^p \mathsf{Pos}(p, P) \otimes Tp$, where p ranges over finite posets.

This suggests to define the following notion of a *proof-theoretic presentation of a* Pos*-functor* given by the following data.

For each finite poset p we have a collection Σ_p of Vp-ary operation symbols. If $\sigma \in \Sigma_p$ and $t_1, \ldots t_{Vp}$ are terms, then we are allowed to form $\sigma(t_1, \ldots t_{Vp})$ if we can prove that the t_i satisfy the order specified by p.[6] [7] Thus, the order of p puts a constraint on which terms/elements we are allowed to generate. On the other hand, the order on Σ_p, allows us to specify an order relation on

[5] I recommend to check 4, it is beautiful to see how the convexity and the Egli-Milner arises from the proof-theoretic calculus.

[6] Does 'satisfy the order specified by' refer to $\leq$ or to $<$? Note that poset-maps preserve $\leq$ but not $<$.

[7] The calculus below does not treat composition of terms because it only deals with functors, not monads. Of course, for the general picture, we also need a calculus for general algebraic theories over poset, that is, monads.

terms/elements. All order comes from there because all other logical formulas are equations.

Op For each finite poset $p = (\{x_1, \ldots x_n\}, \leq)$, we describe a poset Σ_p of operations by axioms

$$\frac{}{\sigma(x_1, \ldots\ x_n) \leq \tau(x_1, \ldots\ x_n)}$$

where the x_i are required to be pairwise distinct.

Ax We may put axioms of the form

$$\frac{x_i \leq x_j}{\sigma(x_1, \ldots\ x_n) = \tau(x_{n+1}, \ldots\ x_{n+m})}$$

where the premises $x_i \leq x_j$ specifiy the order $\leq$ of a finite poset $(\{x_1, \ldots x_{n+m}\}, \leq$ $)$ of (not necessarily distinct) 'variables'.[8]

Obviously, the arguments of $\sigma \in \Sigma_p$ and $\tau \in \Sigma_q$ have to satisfy the order stipulated by p and q, respectively.

Cg The congruence rule

$$\frac{x_1 \leq y_1, \ldots x_n \leq y_n}{\sigma(x_1, \ldots\ x_n) \leq \sigma(y_1, \ldots\ y_n)}$$

Order The usual rules of order and equality.

Remark C.1. *The last item above hides some details that need to be worked out. I believe there should be two versions of the proof-theoretic presentation. One is as close as possible to the category theory and formulated in type-theoretic style. For example, we have a proof-system to show that two operations/terms are ordered and a separate proof-system for equality. We then need an explicit rule relating order and equality.*

The other proof-theoretic presentation should be as close as possible to what people do who work with ordered algebras. Maybe here we treat equality as a special case of order.

The first one will have the advantage that it is easier to relate to the category theoretic semantics. And that it is easier to generalise. Eg, it could be of interest to replace posets by categories.

The second one will have the advantage that it is closer to existing mathematical practice and will make it easier to compare our results. It also suggests to develop some categorical universal algebra over posets, such as HSP-theorems.

Remark C.2. *The proof-theoretic presentations indicated above can easily be adapted to enrichement over preorders. This could be of interest as well.*

[8] Can we choose any $(X, \leq)$ such that $\leq$ satisfies the premises, that is, do we allow X and $\leq$ to contain elements not appearing the rule?

Claim 2. *1. Each collection of operations and axioms as above specifies a finitary locally monotone*[9] *functor* $\mathsf{Pos} \to \mathsf{Pos}$*.*

2. *Every finitary locally monotone functor* $T : \mathsf{Pos} \to \mathsf{Pos}$ *has a 'canonical representation' given by* $\Sigma_p = Tp$ *for each finite poset* p *and by equations*

$$Tf(\sigma)(x_1, \ldots x_n) = \sigma(x_{f(1)}, \ldots x_{f(m)}) \tag{16}$$

where $f : p \to q$, $p = (\{1, \ldots m\}, \leq_p)$, $q = (\{1, \ldots n\}, \leq_q)$, $\sigma \in Tp$, *and variables* $(\{x_1, \ldots x_n\}, \leq) \cong q$, *the iso being given by* $(x_1, \ldots x_n) : q \to \{x_1, \ldots x_n\}$.[10]

3. *The functors having a presentation consisting only of discrete aritites, operations, and variables are precisely those which come from set-functors as discussed in Section B.*

Question 2. *Can we characterise the embedding-preserving functors via their representations? My first guess is: if a functor has a presentation where each axiom uses only a discrete poset of variables, then it preserves embeddings.*

Now let us look at some examples. All set-functors are examples of poset-functors and they have 'discrete presentations'. So we will focus on non-discrete presentations below.

The following example has discrete arities and a discrete poset of operations, but a non-discrete poset of variables.

Example C.3. *Consider* $\mathsf{Cnct} : \mathsf{Pos} \to \mathsf{Pos}$*, the functor mapping a poset to its connected components.*

- *arities: 1*
- *operations:* Σ_1 *is the (po)set* $\{\Box\}$*.*
- *equations:* $x_1 \leq x_2$ *Rightarrow* $\Box(x_1) = \Box(x_2)$*. (The poset of variables is* $(\{x_1, x_2\}, \{x_1 < x_2\}$*.)*

Instead of using 'non-discrete equations', one can also use non-discrete arities (this encoding does not depend on the particular example of Cnct).

Example C.4. *Consider* $\mathsf{Cnct} : \mathsf{Pos} \to \mathsf{Pos}$*, the functor mapping a poset to its connected components.*

[9] locally monotone should come from the congruence rule

[10] Should these equations be called non-discrete? No: Obviously, the set of variables has the same order as q which may not be discrete. Yes: The order is implicit already in the terms and does not need to be stated explicitely. Semantically, I tend to 'No', syntactically to 'Yes', that is, it would be nicer to write the equations as we did in (16) instead of

$$\frac{x_i \leq x_j}{Tf(\sigma)(x_1, \ldots x_n) = \sigma(x_{f(1)}, \ldots x_{f(m)})} \tag{17}$$

where $x_i \leq x_j$ specifies the order $\leq$ of q. This is also related to footnote 8 and Example C.4.

- *arities:* $1, \mathbb{2}$
- *operations:* Σ_1 *is the (po)set* $\{\Box\}$. $\Sigma_{\mathbb{2}}$ *is the (po)set* $\{\Diamond\}$
- *equations:* $\Box(x_1) = \Diamond(x_1, x_2), \Box(x_2) = \Diamond(x_1, x_2)$.

There is a question whether the equations should be called discrete, but at the moment we take the view that one does not need to assume $x_1 \leq x_2$ for the equations to be correct, since it is implicit in being able to write $\Diamond(x_1, x_2)$. This is related to footnote 8 and 10.

The following example has a non-discrete poset of operations, but discrete arities and a discrete poset of variables. Recall that $\mathbb{2} = (\{0, 1\}, \{0 < 1\})$.

Example C.5. *Consider* $T : \mathsf{Pos} \to \mathsf{Pos}$*,* $TX = \mathbb{2} \times X$*.*

- *arities: 1*
- *operations:* Σ_1 *is the poset* $\{l, r\}$ *with* $l < r$*.*
- *no equations*

The following example has again non-discrete operations and discrete arities and variables[11].

Example C.6. *Consider* $T : \mathsf{Pos} \to \mathsf{Pos}$*,* $TX = \mathbb{2} \ltimes X$*, where* $\ltimes$ *refers to the lexicographic ordering.*[12] *In other words,* TX *makes two copies of* X *and each element in the left copy is smaller then any element in the right copy. The first idea is two have to unary generators* l, r *as above plus the inequation* $l(x) < r(y)$*, but this is not possible as our calculus only has (generalised) equations and no inequations (other than those given by the order on the poset of operations). But there is an easy trick we can employ:*

- *arities: 2*
- *operations:* $\Sigma_2 = (\{l, r\}, \{l < r\})$
- *equations:* $l(x_1, x_2) = l(x_1, x_3)$*,* $r(x_1, x_2) = r(x_3, x_2)$

The following example has a non-discrete arity and discrete operations and variables.

Example C.7. *Consider* $T : \mathsf{Pos} \to \mathsf{Pos}$*,* $TX = X^{\mathbb{2}}$*, that is,* X *is mapped to the poset of pairs* (x_1, x_2) *with* $x_1 \leq x_2, x_1, x_2 \in X$*.*

- *arity:* $\mathbb{2}$
- *operation:* $\Sigma_{\mathbb{2}} = (\{\Box\})$

[11] I am tempted to say: it has (non-discrete operations and discrete arities and) *discrete equations* ... or one might use the terminology: set-equations and poset-equations

[12] More generally, $Y \ltimes X$ has carrier $Y \times X$ and the order is given by $(y, x) < (y', x') \Leftrightarrow (y < y' \vee (y = y' \wedge x < x'))$.

- *no equation*

With the presentation above a pair (x_1, x_2) *appears as* $\Box(x_1, x_2)$.

Example C.8. *Let* $\mathcal{U}p_\omega : \mathsf{Pos} \to \mathsf{Pos}$ *be the covariant functor which maps a poset to the set of all finitely generated upper sets ordered by reverse inclusion. To present* $\mathcal{U}p_\omega$ *we take the (discrete) presentation of* $\mathcal{P}_\omega$ *(or* $\bar{\mathcal{P}}_\omega$*) and add one (non-discrete) equation. Explicitely,*

- *arities: all discrete finite posets* n
- *operations: for each* n *one operation* σ_n
- *equations:*

$$\sigma_n(x_i) = \sigma_m(y_j) \quad \text{if} \quad \{x_i\} = \{y_j\}$$

$$\sigma_{n+1}(x_1, x_1, \dots x_n) = \sigma_{n+1}(x, x_1 \dots x_n) \quad \text{if} \quad x_1 \leq x$$

Example C.9. *Recent work by Allwein and Harrison advertised the use of partially-ordered modalities. This idea falls under the scope of the current work: If* A *is an ordered set and* $T : \mathsf{Pos} \to \mathsf{Pos}$ *a functor, then* $[A, T]$ *is a functor which has* A*-indexed* T*-modalities. In particular, if we choose* $\mathcal{P}$ *or* $\mathcal{U}p$ *for* T *we move the approach of Allwein and Harrison to the situation where not only modalities, but also carriers of coalgebras may be partially ordered.*

D Further Work

Work out the details sketched above.

The monograph of Adamek and Trnkova (and more recent work by Adamek, Milius, Schwenke, Velebil and possibly others) has lots of results on presentations of set-functors. There should be much to explore here on presentations of poset-functors.

Work out the calculus corresponding to monads instead of functors.

Treat enrichment over preorders (should be a minor variation). Another variation could be to drop reflexivity (partly because of a possible connection with the work of Jung and Moshier on strong proximity lattices). The latter would force us to replace $\leq$ by $<$, but what are the consequences of this?

Develop some universal algebra over posets, such as an HSP theorem.

Find interesting examples of universal algebras over poset where the order is not definable equationally (basically this excludes everything based on semi-lattices, distributive lattices, etc). A possibility could be to look into substructural logics (linear logic, quantales, etc). In these settings it is important to specify adjoints (eg between $\otimes$ and $\to$). Such adjointness is not equational in our sense above. So we would have to look at generalisations of our equational logic to quasi-equational (implicational) logic. As this is standard in universal algebra (over sets) it would be of interest anyway to understand how quasi-equations enrich. The category theroetic properties of quasi-equationally definable classes are not so good, so one could also look at l.f.p. categories instead. On the other hand, there the logic is more complicated ... lots of things to think about ...

What would be the category theory of inequational logic where basic propositions are of the form $t \leq s$? Any hope of generalising known stuff such as monads to this setting?