Unification

Definition 1. Let t and u be any terms. A substitution $\sigma = [s_1/X_1, \ldots, s_m/X_m]$ is a *unifier* for t and u if

$$t\sigma = u\sigma$$

Definition 2. If σ and τ are substitutions then their *composition* $\sigma \circ \tau$ is the substitution given by

$$t(\sigma \circ \tau) = (t\sigma)\tau$$

for all terms t.

If $\sigma = [s_1/X_1, \ldots, s_m/X_m]$ and $\tau = [r_1/Y_1, \ldots, r_k/Y_k]$ then

$$\sigma \circ \tau = [s_1\tau/X_1, \ldots, s_m\tau/X_m, r_1/Y_1, \ldots, r_k/Y_k]$$

E.g., if $\sigma = [\ f(U,V)/X,\ \ g(V)/Y\ ]$ and $\tau = [\ h(Z)/U,\ \ b/V\ ]$ then

$$\sigma \circ \tau = [\ f(h(Z),b)/X,\ \ g(b)/Y,\ \ h(Z)/U,\ \ b/V\ ]$$

Definition 3. If σ_1 and σ_2 are unifiers for t and u then σ_1 is *more general than* σ_2 if there exists a subsitution τ such that

$$\sigma_2 = \sigma_1 \circ \tau$$

A substitution σ is a *most general unifier* (MGU) for the terms t and u if it is a unifier and it is more general than any other unifier for t and u, i.e.

- $t\sigma = u\sigma$, and
- whenever $t\sigma' = u\sigma'$ then $\sigma' = \sigma \circ \tau$ for some substitution τ.

Unification Algorithm

The algorithm takes two terms t and u and either succeeds, if they can be unified, and returns a MGU; or fails otherwise.

- If $t = X$ and $u = Y$ then succeed with $\sigma = [Y/X]$.
- If $t = X$ and $u = g(u_1, \ldots, u_m)$ then succeed with $\sigma = [u/X]$ if X does not occur in u and fail otherwise.
- If $t = f(t_1, \ldots, t_n)$ and $u = Y$ then succeed with $\sigma = [t/Y]$ if Y does not occur in t and fail otherwise.
- If $t = f(t_1, \ldots, t_n)$ and $u = g(u_1, \ldots, u_m)$ where $f \neq g$ then fail.
- If $t = f(t_1, \ldots, t_n)$ and $u = f(u_1, \ldots, u_n)$ then use the algorithm recursively to unify the lists $[t_1, \ldots, t_n]$ and $[u_1, \ldots, u_n]$.

If l and k are lists of terms then the unification of l and k proceeds as follows.

- If $l = [\,]$ and $k = [\,]$ then succeed with the identity substitution.
- If $l = [\,]$ and $k = (u : k')$ then fail.
- If $l = (t : l')$ and $k = [\,]$ then fail.
- If $l = (t : l')$ and $k = (u : k')$ then:
  1. apply unification to t and u, if that succeeds yielding σ_1 then
  2. apply unification recursively to the lists $l'\sigma_1$ and $k'\sigma_1$, if that also succeeds yielding σ_2 then
  3. succeed with $\sigma = \sigma_1 \circ \sigma_2$; otherwise fail.

Theorem 4. For any terms t and u,

- if t and u have a unifier then the algorithm succeeds and returns a MGU for t and u,
- if no unifier exists then the algorithm fails after finitely many steps.

Proof. Omitted. $\square$

Example 5.

$$h(\ f(U,V),\ U,\ g(V)\ )\ \stackrel{?}{=}\ h(\ X,\ g(Z),\ Z\ )$$

1. $f(U,V) \stackrel{?}{=} X$

   $\sigma_1 = [f(U,V)/X]$

2. $U \stackrel{?}{=} g(Z)$

   $\sigma_2 = [g(Z)/U]$

3. $g(V) \stackrel{?}{=} Z$

   $\sigma_3 = [g(V)/Z]$

$$\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3 = [\,f(g(g(V)),V)/X\ ,\ g(g(V))/U\ ,\ g(V)/Z\,]$$

Example 6.

$$h(\ f(U,V),\ g(Y),\ X\ ) \stackrel{?}{=} h(\ X,\ g(Z),\ Z\ )$$

1. $f(U,V) \stackrel{?}{=} X$

   $\sigma_1 = [f(U,V)/X]$

2. $g(Y) \stackrel{?}{=} g(Z)$

   3. $Y \stackrel{?}{=} Z$

      $\sigma_3 = [Y/Z]$

   $\sigma_2 = \sigma_3 = [Y/Z]$

4. $X(\sigma_1 \circ \sigma_2) = f(U,V),\ Z(\sigma_1 \circ \sigma_2) = Y$

   $f(U,V) \stackrel{?}{=} Y$

   $\sigma_4 = [f(U,V)/Y]$

$$\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_4 = [\ f(U,V)/X,\ f(U,V)/Z,\ f(U,V)/Y\ ]$$

Example 7.

$$f(\ f(U,V),\ W\ ) \stackrel{?}{=} f(\ W,\ f(g(V),x)\ )$$

1. $f(U,V) \stackrel{?}{=} W$

   $\sigma_1 = [f(U,V)/W]$

2. $W\sigma_1 = f(U,V)$

   $f(U,V) \stackrel{?}{=} f(g(V),x)$

   3. $U \stackrel{?}{=} g(V)$

      $\sigma_3 = [g(V)/U]$

   4. $V \stackrel{?}{=} x$

      $\sigma_4 = [x/V]$

   $\sigma_2 = \sigma_3 \circ \sigma_4 = [g(x)/U,\ x/V]$

$$\sigma = \sigma_1 \circ \sigma_2 = [\ f(g(x),x)/W,\ g(x)/U,\ x/V\ ]$$

Example 8.

$$f(X, X) \stackrel{?}{=} f(g(Y), Y)$$

$$\begin{aligned}
1.\quad & X \stackrel{?}{=} g(Y) \\
& \sigma_1 = [g(Y)/X] \\
2.\quad & X\sigma_1 = g(Y) \\
& g(Y) \stackrel{?}{=} Y
\end{aligned}$$

Fail (occurs check)

Example 9.

$$h(f(U,V),U,X) \stackrel{?}{=} h(X,g(Z),U)$$

1. $f(U,V) \stackrel{?}{=} X$

   $\sigma_1 = [f(U,V)/X]$

2. $U \stackrel{?}{=} g(Z)$

   $\sigma_2 = [g(Z)/U]$

3. $X(\sigma_1 \circ \sigma_2) = f(g(Z),V),\ U(\sigma_1 \circ \sigma_2) = g(Z)$

   $f(g(Z),V) \stackrel{?}{=} g(Z)$

   Fail