# Unification: Solutions

1. Let $\sigma = [g(Y)/X]$ and $\tau = [a/Y, g(X)/Z]$ be two substitutions and let $t = f(X, h(X, Y), Z)$ be a term. Calculate the following:

   (a) $t\sigma = f(g(Y), h(g(Y), Y), Z)$

   (b) $t\tau = f(X, h(X, a), g(X))$

   (c) $(\sigma \circ \tau) = [g(a)/X, a/Y, g(X)/Z]$
   $(\tau \circ \sigma) = [a/Y, g(g(Y))/Z, g(Y)/X]$

   (d) $t(\sigma \circ \tau) = f(g(a), h(g(a), a), g(X))$
   $t(\tau \circ \sigma) = f(g(Y), h(g(Y), a), g(g(Y)))$.

2. Write down the steps taken by the unification algorithm for each of the following pairs of terms. If the algorithm succeeds then write down the MGU and the corresponding common instance.

   (a) $g(h(X)) \stackrel{?}{=} g(Y)$. We use Case 5 of the algorithm and apply the algorithm recursively.

      i. $h(X) \stackrel{?}{=} Y$. We use Case 3 and the unification of $h(X)$ and $Y$ succeeds with the substitution $\sigma_i = [h(X)/Y]$.

      **The unification of $g(h(X))$ and $g(Y)$ succeeds with substitution $\sigma = \sigma_i = [h(X)/Y]$.**

   (b) $g(h(X)) \stackrel{?}{=} g(a)$. We use Case 5 of the algorithm and apply the algorithm recursively.

      i. $h(X) \stackrel{?}{=} a$. We use Case 4. Since $h \neq a$, the unification of $h(X)$ and $a$ fails.

      **The unification of $g(h(X))$ and $g(a)$ fails.**

   (c) $f(X, a) \stackrel{?}{=} f(b, Y)$. We use Case 5 of the algorithm and apply twice the algorithm recursively.

      i. $X \stackrel{?}{=} b$. We use Case 2. This succeeds with $\sigma_i = [b/X]$.

      ii. $a \stackrel{?}{=} Y$. We use Case 3. This succeeds with $\sigma_{ii} = [a/Y]$.

      **The unification of $f(X, a)$ and $f(b, Y)$ succeeds with substitution $\sigma = \sigma_i \circ \sigma_{ii} = [b/X, a/Y]$.**

(d) $f(X,X) \stackrel{?}{=} f(a,g(Y))$. We use Case 5 of the algorithm and apply twice the algorithm recursively. The second time we have to take into account the substitution $\sigma_i$ obtained from unifying $X$ and $a$.

i. $X \stackrel{?}{=} a$. We use Case 2. This succeeds with $\sigma_i = [a/X]$.

ii. $X\sigma_i \stackrel{?}{=} g(Y)\sigma_i$ which is $a \stackrel{?}{=} g(Y)$. We use Case 3 and since $a \neq g$, this unification fails.

**The unification of $f(X,X)$ and $f(a,g(Y))$ fails.**

(e) $f(X,a) \stackrel{?}{=} f(g(X),a)$. We use Case 5 of the algorithm and apply the algorithm recursively.

i. $X \stackrel{?}{=} g(X)$. We use Case 2. Since $X$ occurs in $g(X)$ the unification of $X$ and $g(X)$ fails.

ii. $a \stackrel{?}{=} a$.

**The unification of $f(X,a)$ and $f(g(X),a)$ fails.**

(f) $f(f(h(X),a),h(Z)) \stackrel{?}{=} f(f(Y,Z),X)$. We use Case 5 of the algorithm and apply the algorithm recursively.

i. $f(h(X),a) \stackrel{?}{=} f(Y,Z)$. We use Case 5 again and apply the algorithm recursively.

A. $h(X) \stackrel{?}{=} Y$. We use Case 3 and this unification succeeds with $\sigma_A = [h(X)/Y]$.

B. $a \stackrel{?}{=} Z$. We use Case 3 and this unification succeeds with $\sigma_B = [a/Z]$.

The unification of $f(h(X),a)$ and $f(Y,Z)$ succeeds with substitution $\sigma_i = \sigma_A \circ \sigma_B = [h(X)/Y, a/Z]$

ii. $h(Z)\sigma_i \stackrel{?}{=} X\sigma_i$ which is $h(a) \stackrel{?}{=} X$. The unification of $h(a)$ and $X$ succeeds with substitution $\sigma_{ii} = [h(a)/X]$.

**The unification of $f(f(h(X),a),h(Z))$ and $f(f(Y,Z),X)$ succeeds with substitution $\sigma = \sigma_i \circ \sigma_{ii} = [h(h(a))/Y, a/Z, h(a)/X]$ .**

(g) $l(V,k(a,V),W) \stackrel{?}{=} l(g(Y,a),k(Y,X),V)$. We use Case 5 of the algorithm and apply the algorithm recursively.

i. $V \stackrel{?}{=} g(Y,a)$ succeeds with $\sigma_i = [g(Y,a)/V]$.

ii. $k(a,V)\sigma_i \stackrel{?}{=} k(Y,X)\sigma_i$ which is $k(a,g(Y,a)) \stackrel{?}{=} k(Y,X)$. We apply the algorithm recursively.

A. $a \stackrel{?}{=} Y$ succeeds with $\sigma_A = [a/Y]$.

B. $g(Y,a)\sigma_A \stackrel{?}{=} X\sigma_A$ which is $g(a,a) \stackrel{?}{=} X$. This succeeds with $\sigma_B = [g(a,a)/X]$.

The unification of $k(a, g(Y, a))$ and $k(Y, X)$ succeeds with substitution $\sigma_{ii} = \sigma_A \circ \sigma_B = [a/Y, g(a, a)/X]$.

iii. $(W\sigma_i)\sigma_{ii} \stackrel{?}{=} (V\sigma_i)\sigma_{ii}$ which is $W \stackrel{?}{=} g(a, a)$. The unification of $W$ and $g(a, a)$ succeeds with $\sigma_{iii} = [g(a, a)/W]$.

**The unification of $l(V, k(a, V), W)$ and $l(g(Y, a), k(Y, X), V)$ succeeds with substitution:**

$$\sigma = \sigma_i \circ \sigma_{ii} \circ \sigma_{iii} = [g(a, a)/V, a/Y, g(a, a)/X, g(a, a)/W]$$

(h) $k(W, k(h(V), W)) \stackrel{?}{=} k(f(X), k(Y, Y))$. We use Case 5 of the algorithm and apply the algorithm recursively.

i. $W \stackrel{?}{=} f(X)$ succeeds with $\sigma_i = [f(X)/W]$.

ii. $k(h(V), W)\sigma_i \stackrel{?}{=} k(Y, Y)\sigma_i$ which is $k(h(V), f(X)) \stackrel{?}{=} k(Y, Y)$. We apply the algorithm recursively.

A. $h(V) \stackrel{?}{=} Y$ succeeds with $\sigma_A = [h(V)/Y]$.

B. $f(X)\sigma_A \stackrel{?}{=} Y\sigma_A$ which is $f(X) \stackrel{?}{=} h(V)$. The unification of $f(X)$ and $h(V)$ fails since $f \neq h$.

The unification of $k(h(V), f(X))$ and $k(Y, Y)$ fails.

**The unification of $k(W, k(h(V), W))$ and $k(f(X), k(Y, Y))$ fails.**