Essential Spectrum of the Laplace and Multiplication operators

Alex H. Room

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Definition. ***(Essential spectrum)*** *The essential spectrum*[1] *of an operator T on a Hilbert space H is defined as the set of all λ such that a* ***Weyl sequence*** *u_n exists for T and λ , i.e. a sequence with the properties:*

- $\|u_n\| = 1 \quad \forall n \in \mathbb{N}$*;*
- $u_n \rightharpoonup 0$ *(where $\rightharpoonup$ denotes weak convergence: $u_n \rightharpoonup u \Leftrightarrow (u_n, g) \to (u, g) \quad \forall g \in H$);*
- $\lim_{n\to\infty} \|(T - \lambda)u_n\| \to 0$.

We can loosen the definition of weak convergence to just require convergence in a dense subspace of H:

Lemma 1. *A bounded sequence u_n, $\|u_n\| \leq C$, is weakly convergent to u in H if and only if it is weakly convergent to u in L where L is a dense subspace of H.*

Proof. Weak convergence in H implying the same in L is obvious by the definition. Conversely, take $g \in H$. For any $\varepsilon > 0$, we have $\|g - \varphi\| < \varepsilon$ for $\varphi \in L$; furthermore by the weak convergence of u_n in L we have $N \in \mathbb{N}$ such that $(u_n - u, \varphi) < \varepsilon$ for $n \geq N$. Then:

$$(u_n - u, g) = (u_n - u, g - \varphi + \varphi) = (u_n - u, g - \varphi) + (u_n - u, \varphi) < \|u_n - u\|\|g - \varphi\| + \varepsilon < \varepsilon(C + 1) \to 0.$$

□

We will now construct a Weyl sequence for the essential spectrum of the 'Laplacian' or 'free Schrödinger' operator $T = -\Delta$ on $L^2(\mathbb{R})$, where Δ is the operator $\Delta f(x) = \frac{d^2}{dx^2} f(x)$

Proposition 1. *The essential spectrum of the operator $T = -\Delta$ is the closed half-axis $[0, +\infty)$.*

Proof. First, note that for the exponential function we have

$$T(\exp)(i\omega x) = \omega^2 \exp(i\omega x). \tag{1}$$

This gives much of the intuition for this proof; the function $\exp_{i\omega} : x \mapsto \exp(i\omega x)$ is not an eigenvector as it is not in $L^2(\mathbb{R})$, but it satisifes the eigenvalue equation for T and so any number $\lambda = \omega^2$ - and thus any $\lambda \in [0, +\infty)$ - is 'almost' an eigenvalue for T.

We take advantage of this by choosing some smooth bump function $\rho \in C_c^\infty(\mathbb{R})$ with $\|\rho\|_2 = 1$. We then define $\rho_n = \frac{1}{\sqrt{n}}\rho(x/n)$. ρ_n has some nice properties: by a substitution of variables and direct calculation we have $\|\rho_n\|_2 = \|\rho\|_2$, and furthermore any k'th derivative $\rho_n^{(k)}$ of ρ_n converges to 0 in L^2. Indeed:

$$\|\rho_n^{(k)}\|_2 = \frac{1}{n^k}\|\frac{1}{\sqrt{n}}\rho^{(k)}(x/n)\|_2 = \frac{\|\rho^{(k)}\|_2}{n^k} \to 0 \tag{2}$$

where one can see $\|\frac{1}{\sqrt{n}}\rho^{(k)}(x/n)\|_2 = \|\rho^{(k)}\|_2$ by the same calculation as $\|\rho_n\|_2 = \|\rho\|_2$.

Now, let our candidate Weyl sequence be $u_n : x \mapsto \rho_n(x)\exp(i\omega x)$, which truncates $\exp(i\omega x)$ to $\text{supp}\rho_n$; this means u_n is in $L^2(\mathbb{R})$. $\|u_n\| = \|\rho_n\|_2 = \|\rho\|_2 = 1$ by direct calculation, and $u_n \rightharpoonup 0$: we can bound u_n

[1] The essential spectrum has several definitions, the most popular usually denoted $\text{Spec}_{e,i}$ for $i \in \{1, 2, 3, 4, 5\}$ in order of size. For most well-behaved operators the definitions are equivalent. This particular definition is Weyl's criterion, $\text{Spec}_{e,2}$.

by $\frac{1}{\sqrt{n}}M\mathbf{1}_{(\mathrm{supp} u_n)}$, where $\mathbf{1}_A$ is the characteristic function of the set A and M is the maximum value of ρ. Then by Lemma **??**, we can simply show weak convergence for any $\varphi \in C_0^\infty$, which is dense in L^2:

$$\begin{aligned}(u_n, \varphi) &= \int_{\mathbb{R}} u_n \varphi \\ &\leq \int_{\mathbb{R}} \frac{1}{\sqrt{n}} M\mathbf{1}_{(\mathrm{supp} u_n)}\varphi \\ &\leq \int_{\mathrm{supp}\varphi} \frac{1}{\sqrt{n}} M\varphi \\ &= \frac{M}{\sqrt{n}} \int_{\mathrm{supp}\varphi} \varphi \to 0, \qquad \text{as the integral of } \varphi \text{ is finite and independent of } n.\end{aligned}$$

Finally, we show that $\lim_{n\to\infty} \|(T-\lambda)u_n\|_2 \to 0$ for $\lambda = \omega^2$:

$$\begin{aligned}\|(T-\lambda)u_n\|_2 &= \|(T(\exp_{i\omega}\rho_n) - \omega^2(\exp_{i\omega}\rho_n)\|_2 \\ &= \|(T(\exp_{i\omega}\rho_n) - T(\exp_{i\omega})\rho_n\|_2 && \textit{(by equation (\textbf{??}))} \\ &= \|\exp_{i\omega} T\rho_n - 2\omega \exp_{i\omega}\frac{d}{dx}\rho_n\|_2 && \textit{(by the product rule)} \\ &= \|T\rho_n - 2\omega\frac{d}{dx}\rho_n\|_2 && \textit{(see } \|\exp_{i\omega}\phi\|_2 = \|\phi\|_2 \textit{ for any } \phi \in L^2) \\ &\leq \|-\frac{d^2}{dx^2}\rho_n\|_2 + 2\omega\|\frac{d}{dx}\rho_n\|_2 \to 0,\end{aligned}$$

converging by equation (**??**). Thus u_n forms a Weyl singular sequence for T and $\lambda \in [0, +\infty)$, as required. □

We can use a similar idea for another example to find the essential spectrum of the multiplication operator:

Proposition 2. *The essential spectrum of the operator M_f on $L^2(0,1)$, where $M_f u(x) = f(x)u(x)$, is the range of f.*

Proof. Similar to before, our initial idea comes from an 'almost-eigenvector'. In this case, if λ is in the range of f with $f(x_0) = \lambda$, we see that $M_f\delta_{x_0} = \lambda\delta_{x_0}$, where δ_{x_0} is the Dirac delta centred at x_0. Again, δ_{x_0} is not an eigenfunction of M_f as it is not in the correct domain - this time, it isn't even strictly a function (it is a distribution).

Now consider a Friedrichs mollifier ρ. This is a function in $C_0^\infty(\mathbb{R})$ with the property that $\sqrt{n}\rho(ny) \to \delta_0$ as $n \to \infty$; we renormalise it such that $\|\rho\|_2 = 1$, and take the sequence

$$u_n : x \mapsto \begin{cases} \sqrt{n}\rho(n(x-x_0)) & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

thus u_n "converges to δ_{x_0}" in the sense of distributions. Note that $\|u_n\|_2 = \|\rho\|_2 = 1$ for all n, and this sequence converges weakly to 0:

$$\begin{aligned}|(u_n, g)| &= \int_{\mathrm{supp} u_n} \sqrt{n}\rho(n(x-x_0))g(x) && \textit{(for any } g \in L^2(0,1)) \\ &\leq \|u_n\|_2 \sqrt{\int_{\mathrm{supp} u_n} |g(x)|^2} && \textit{(by Hölder's inequality)} \\ &= \sqrt{\int_{\mathrm{supp} u_n} |g(x)|^2} \to 0, \quad \text{as } \mathrm{supp}(u_n) \text{ decreases to } 0.\end{aligned}$$

Then we see $\|(M_f - \lambda)u_n\|_2$ converges to zero by similar reasoning:

$$\begin{aligned}
\|(M_f - \lambda)u_n\|_2^2 &= \int_{\text{supp}\rho_n} |(f(x) - f(x_0)\sqrt{n}\rho(n(x - x_0))|^2 && \textit{(using that } \lambda = f(x_0)\textit{)} \\
&= \|(f(x) - f(x_0))^2\|_{L^\infty(\text{supp}\rho_n)}\|\rho_n^2\|_1 && \textit{(by Hölder's inequality)} \\
&= \sup_{x \in \text{supp}\rho_n} \|(f(x) - f(x_0))^2\| \to 0 && \textit{(note } \|\rho_n^2\|_{L^1} = \|\rho_n\|_2 = 1\textit{)}
\end{aligned}$$

converging to zero as suppρ_n shrinks around x_0 by the continuity of f.

$\square$