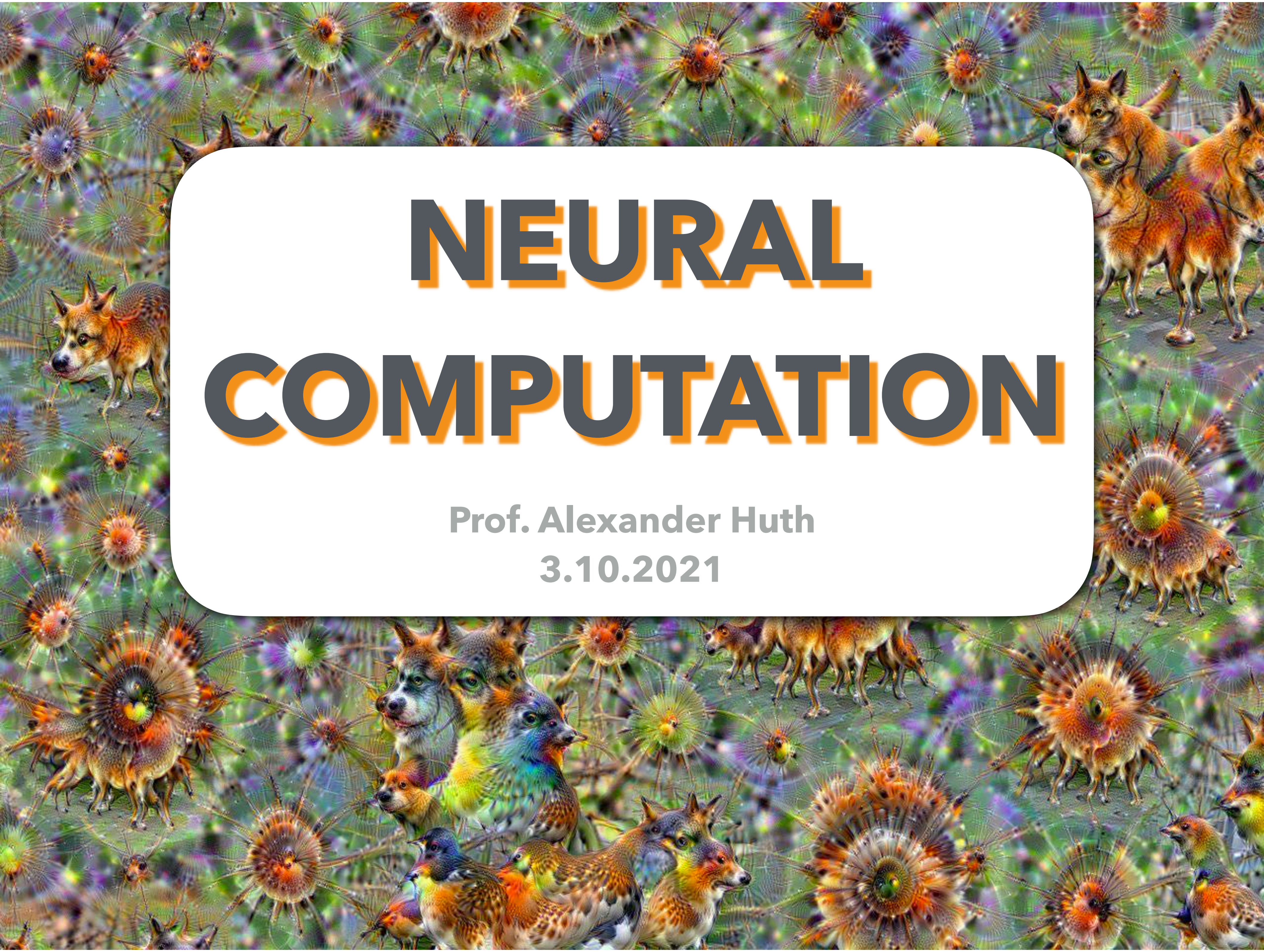


NEURAL COMPUTATION

Prof. Alexander Huth
3.10.2021

RECAP

* Linear regression!
* Ordinary least squares (OLS)
* Regularized regression
  * Priors on weights
  * Penalties on weights
  * Ad hoc metrics (early stopping)

RECAP

* L2 penalty = Gaussian prior = ridge

* L1 penalty = Laplacian prior = LASSO

TODAY

* Analytic solutions to L2-regularized regression problems:

  * Ridge regression

  * Tikhonov regression

RIDGE REGRESSION

* Multivariate normal (MVN) prior on beta

* L2 penalty on beta

* Gradient descent w/ early stopping

RIDGE REGRESSION

$$Y = X\beta + \epsilon$$

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \left[\underbrace{||Y - X\beta||_2^2}_{\text{ERROR or LOSS}} + \underbrace{\lambda ||\beta||_2^2}_{\text{PENALTY}} \right]$$

RIDGE REGRESSION

$$\hat{\beta} = (X^\top X + \lambda I)^{-1} X^\top Y$$

$$\hat{\beta} = X^{+}_{ridge} Y$$

RIDGE REGRESSION

* Efficient solution with SVD

$$\hat{\beta} = (X^\top X + \lambda I)^{-1} X^\top Y$$

(SVD) $X = USV^\top$ $\qquad D = \dfrac{S}{S^2 + \lambda^2}$

$$\hat{\beta} = VDU^\top Y$$

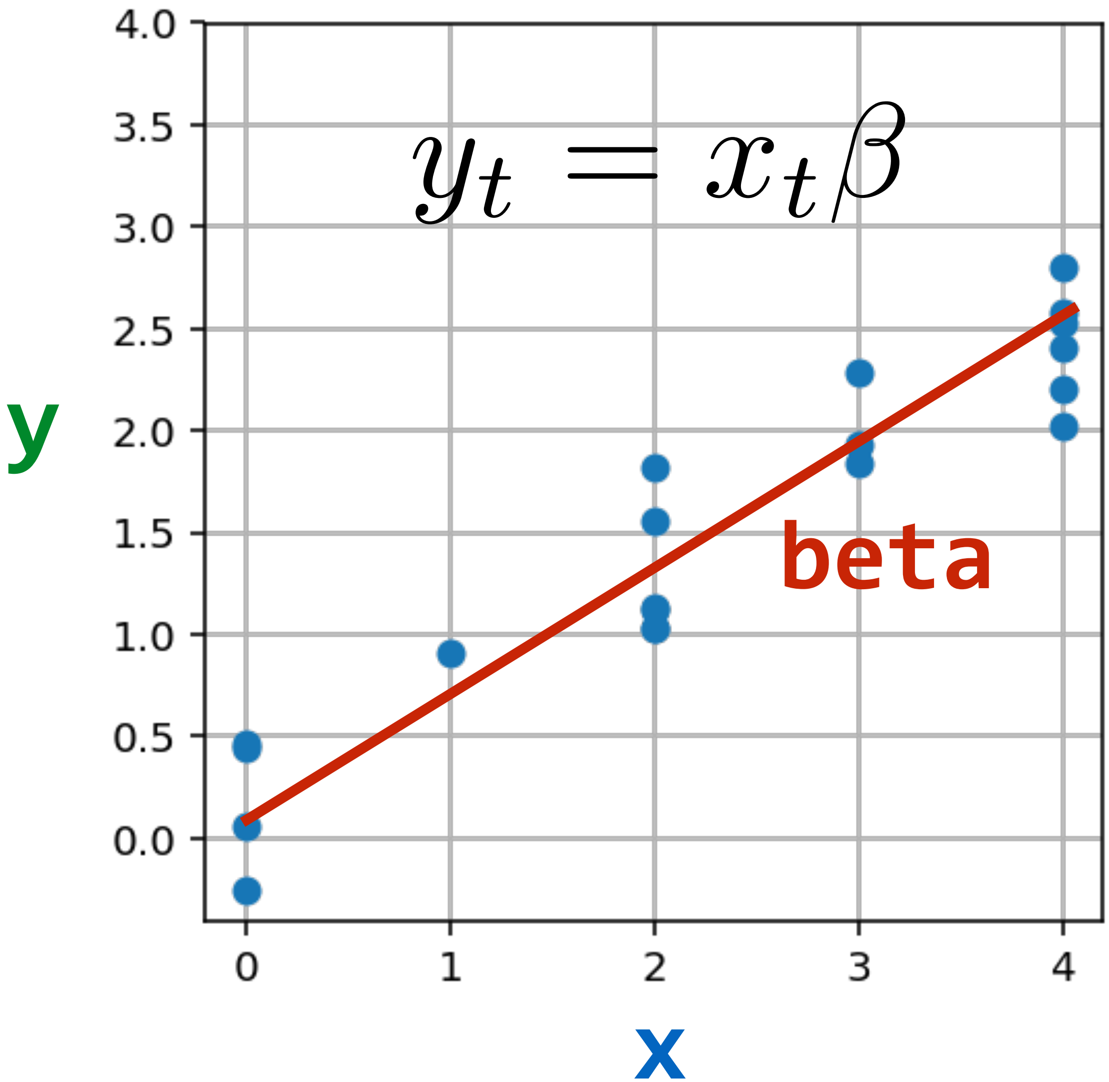
RIDGE REGRESSION

* How to choose lambda?

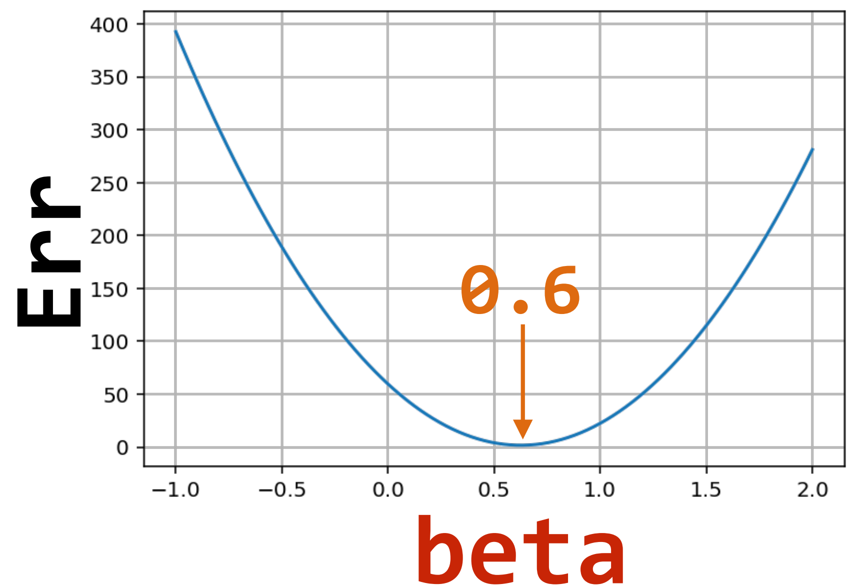
* GCV - Generalized Cross Validation 👎

* Block-wise cross-validation 👍

1D EXAMPLE



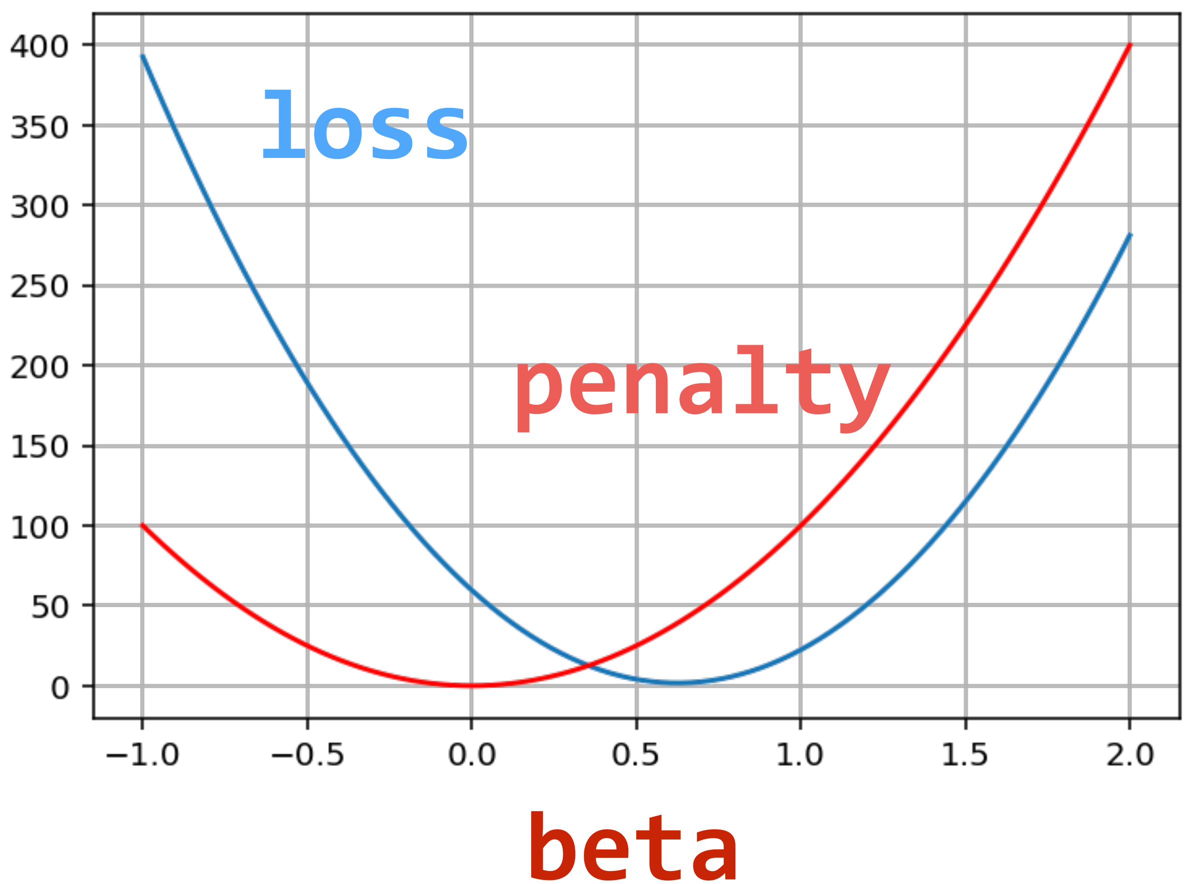
$$Err(\beta) = \sum_{t=1}^{T} (y_t - x_t\beta)^2$$



1D EXAMPLE

L2 Regularization:
(as penalty)

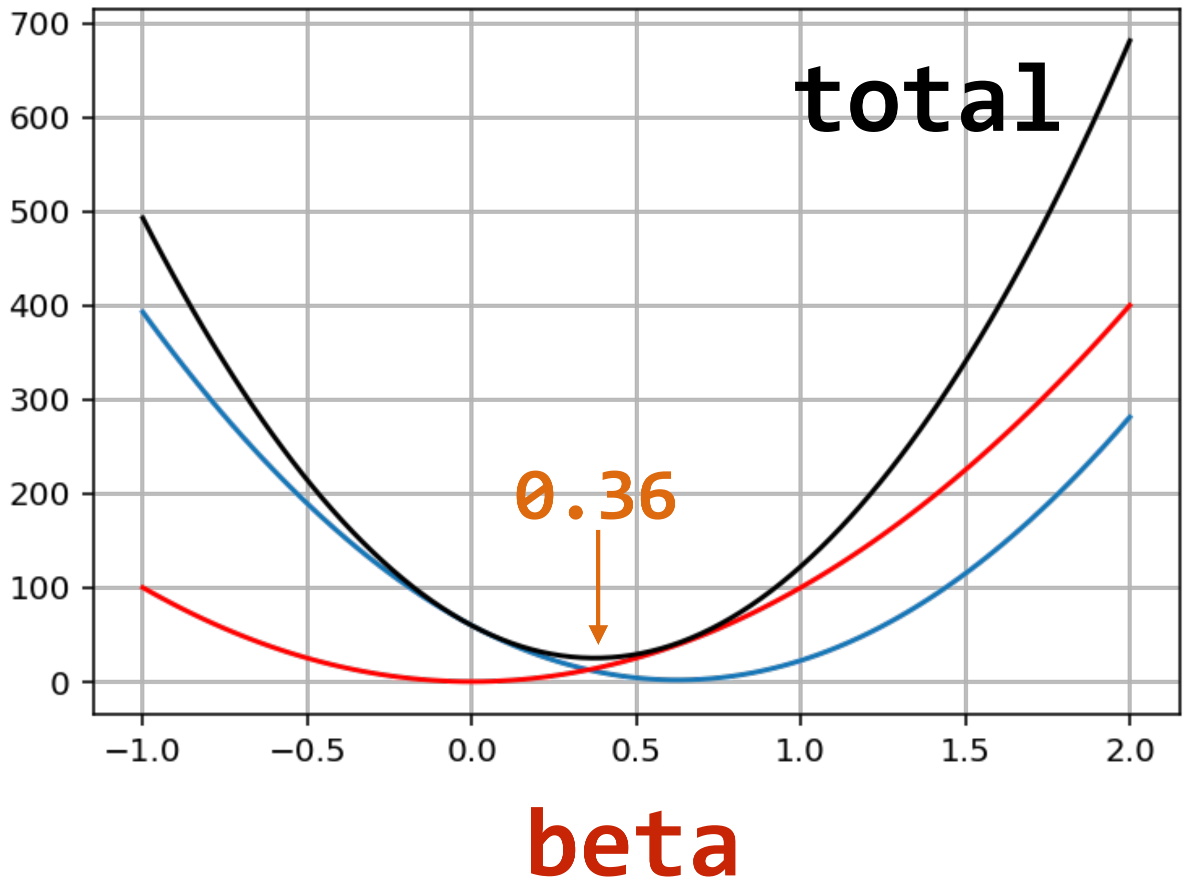
$$Err(\beta) = \boxed{\sum_{t=1}^{T}(y_t - x_t\beta)^2} + \boxed{\lambda\beta^2}$$



1D EXAMPLE

L2 Regularization: (as penalty)

$$Err(\beta) = \boxed{\sum_{t=1}^{T}(y_t - x_t\beta)^2} + \boxed{\lambda\beta^2}$$



COVARIANCE

* A measure of the “joint variability” of two variables

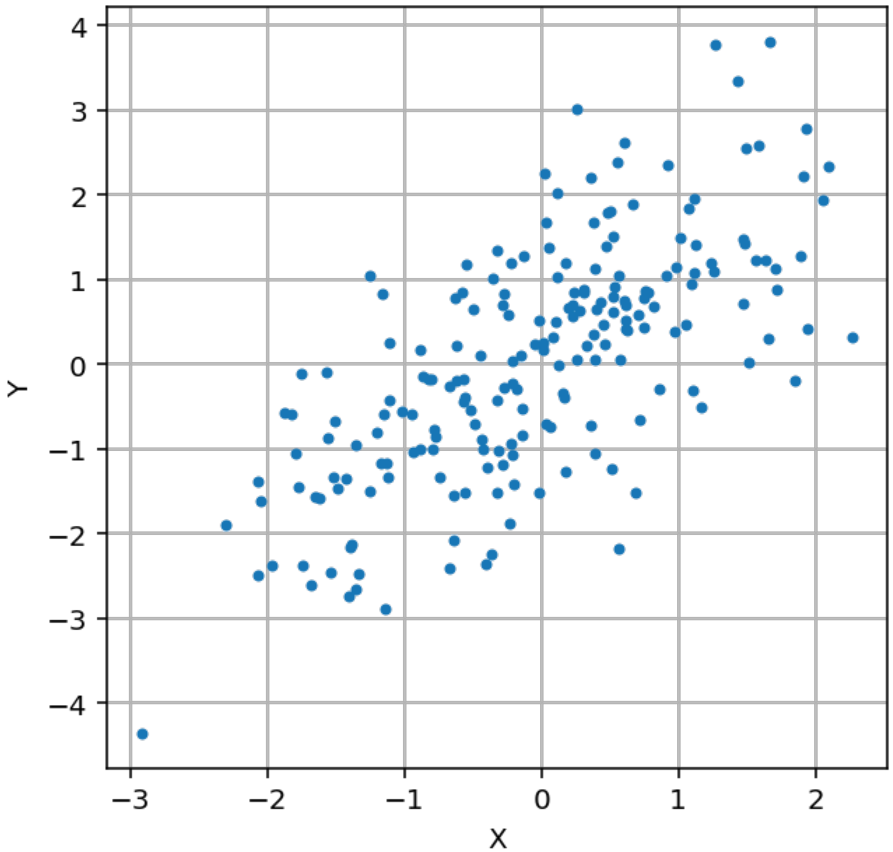
COVARIANCE

$$\mathrm{cov}(\vec{x}, \vec{y}) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

$$\mathrm{cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$\mathrm{cov}(X, X) = E[(X - E[X])(X - E[X])] = E[(X - E[X])^2] = \mathrm{var}(X)$$

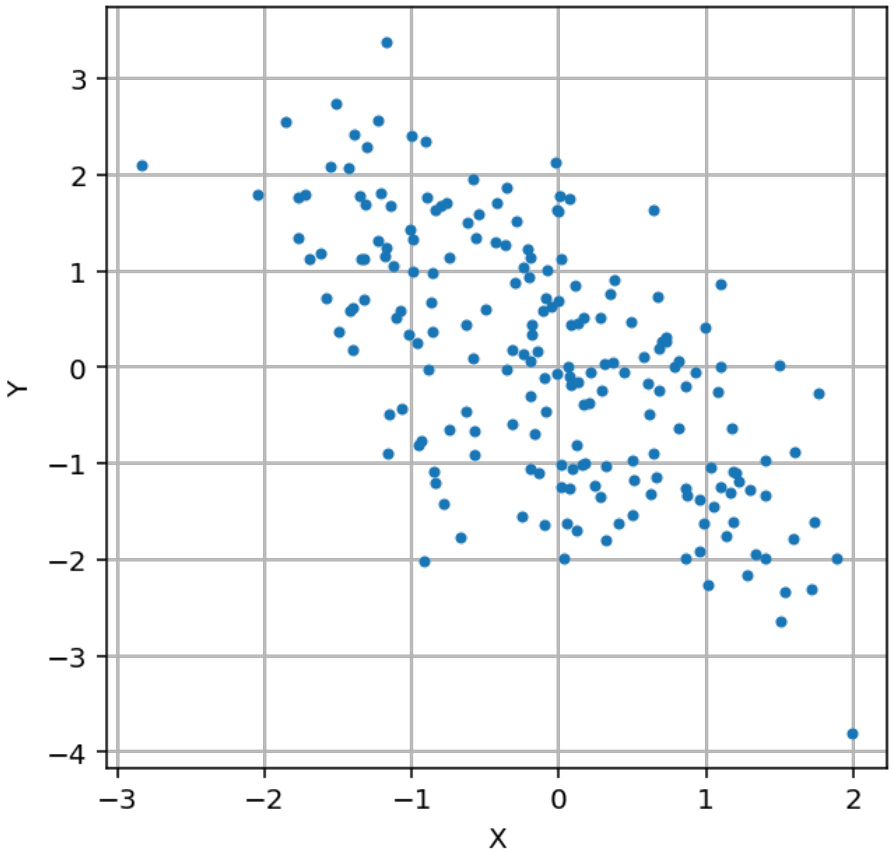
COVARIANCE



cov(x,y) > 0?

cov(x,y) < 0?

COVARIANCE

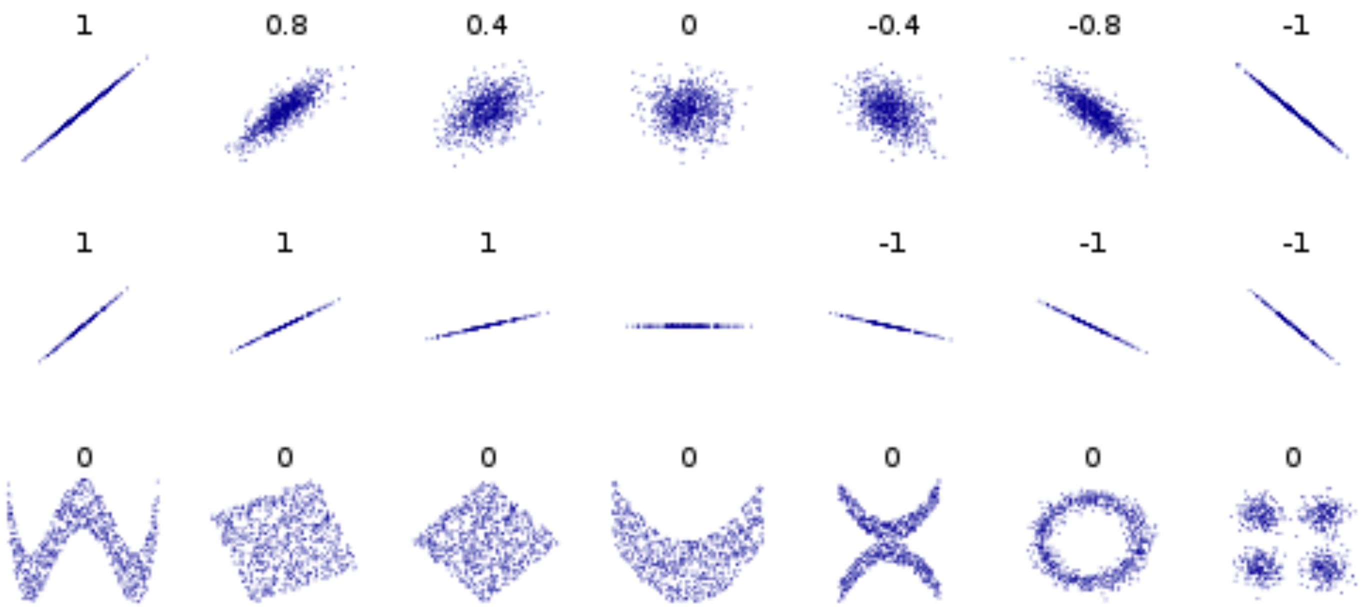


cov(x,y) > 0?

cov(x,y) < 0?

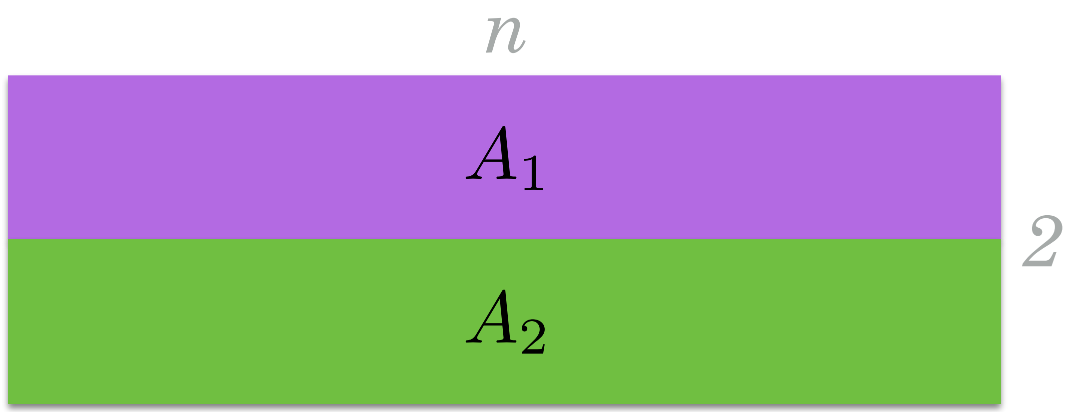
COVARIANCE

(Well, correlation…)



COVARIANCE MATRIX

$A =$



$$\mathrm{cov}(A) = \begin{bmatrix} \mathrm{var}(A_1) & \mathrm{cov}(A_1, A_2) \\ \mathrm{cov}(A_1, A_2) & \mathrm{var}(A_2) \end{bmatrix}$$

(assuming A is mean 0)

$$\mathrm{cov}(A) = \left(\frac{1}{n}\right) AA^T$$

TIKHONOV REGRESSION

$$Y = X\beta + \epsilon$$

* **RIDGE REGRESSION**

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \left[||Y - X\beta||_2^2 + \lambda ||\beta||_2^2 \right]$$

ERROR or LOSS **PENALTY**

* **TIKHONOV REGRESSION**

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \left[||Y - X\beta||_2^2 + \lambda ||C\beta||_2^2 \right]$$

PENALTY MATRIX

TIKHONOV REGRESSION

* **RIDGE REGRESSION** is a special case of **TIKHONOV REGRESSION**

* **TIKHONOV REGRESSION** puts a **ZERO-MEAN MULTIVARIATE NORMAL PRIOR** on the weights

* in **RIDGE REGRESSION** the covariance matrix of the prior has a constant diagonal

  * i.e. the prior is a **SPHERE**

* in **TIKHONOV REGRESSION** the covariance matrix can be ****ANYTHING****

TIKHONOV REGRESSION

* the multivariate normal prior given by **TIKHONOV REGRESSION**

$$\beta \sim N(0, \sigma^2 (C^T C)^{-1})$$

TIKHONOV REGRESSION

* any **TIKHONOV** problem can be converted into a **RIDGE** problem

$$A = XC^{-1}$$

1. CHANGE OF BASIS

$$\hat{\beta}_A = \underset{\beta}{\operatorname{argmin}} \left[||Y - A\beta||_2^2 + \lambda ||\beta||_2^2 \right]$$

2. RIDGE REGRESSION

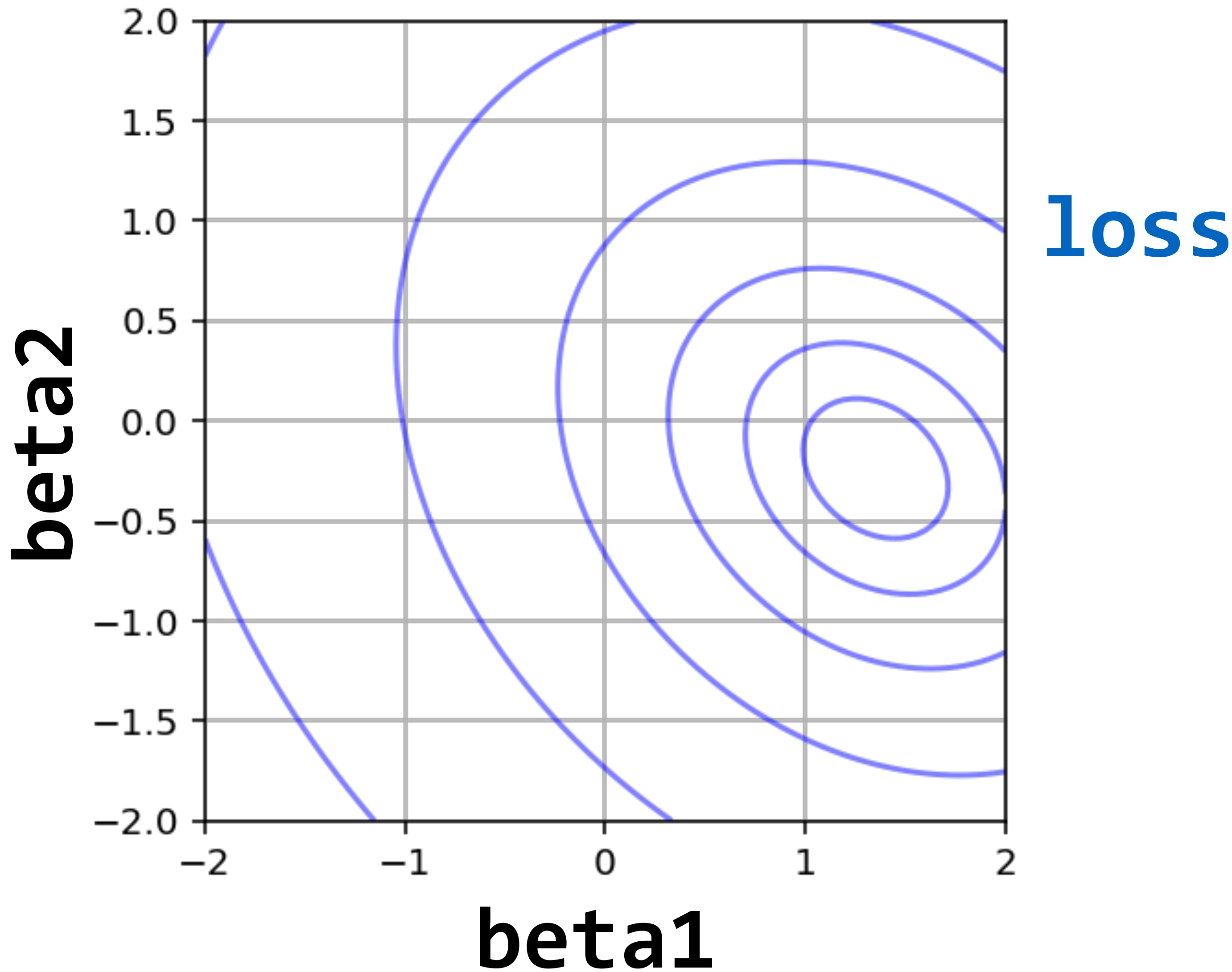
$$\hat{\beta} = C^{-1}\hat{\beta}_A$$

3. CHANGE BASIS AGAIN

Rank-Deficient and Discrete Ill-Posed Problem: Numerical Aspects of Linear Inversion (1998; Per Christian Hansen)

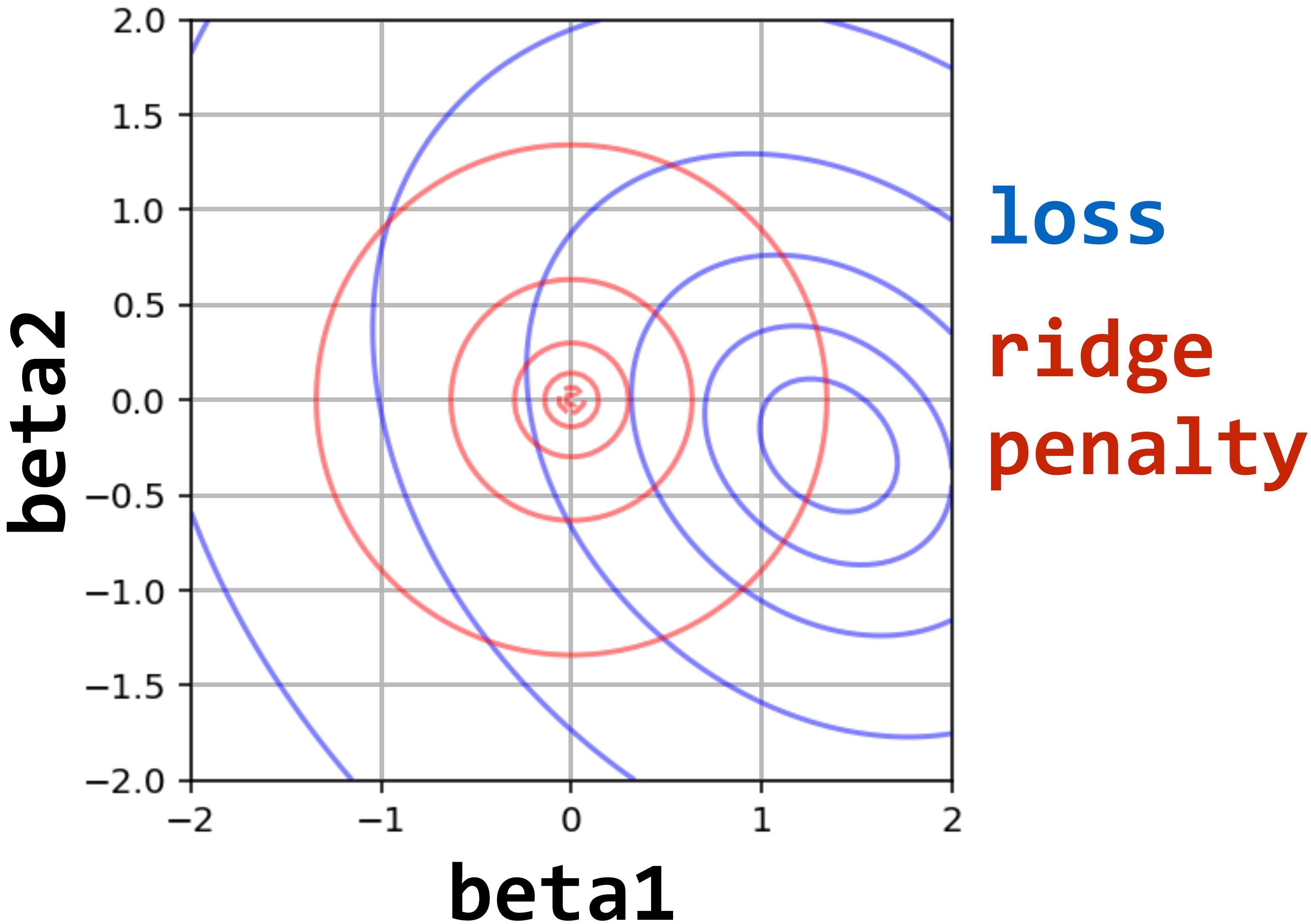
EXAMPLE

$$y_t = x_{1,t}\beta_1 + x_{2,t}\beta_2$$



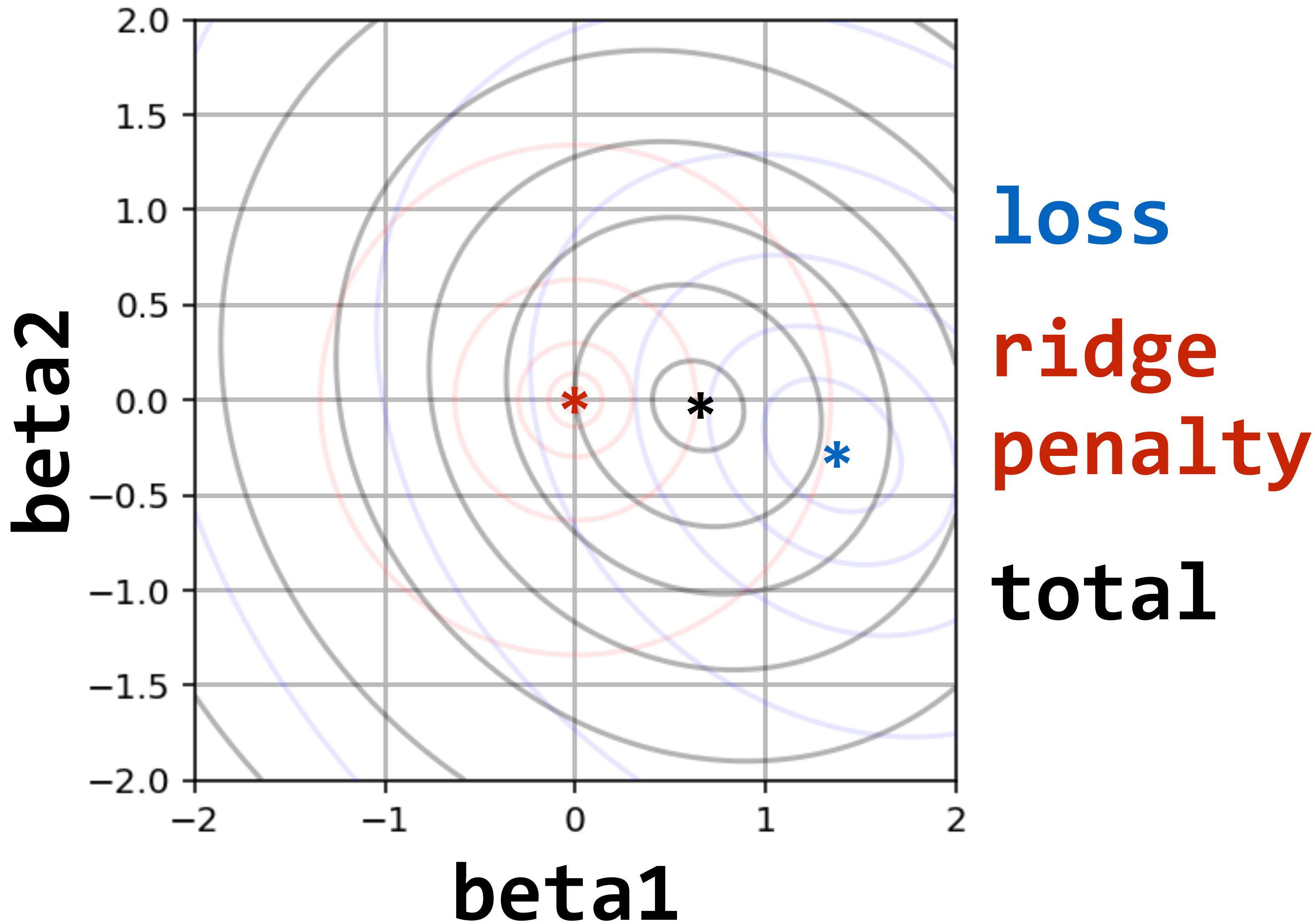
EXAMPLE

$$y_t = x_{1,t}\beta_1 + x_{2,t}\beta_2$$



EXAMPLE

$$y_t = x_{1,t}\beta_1 + x_{2,t}\beta_2$$

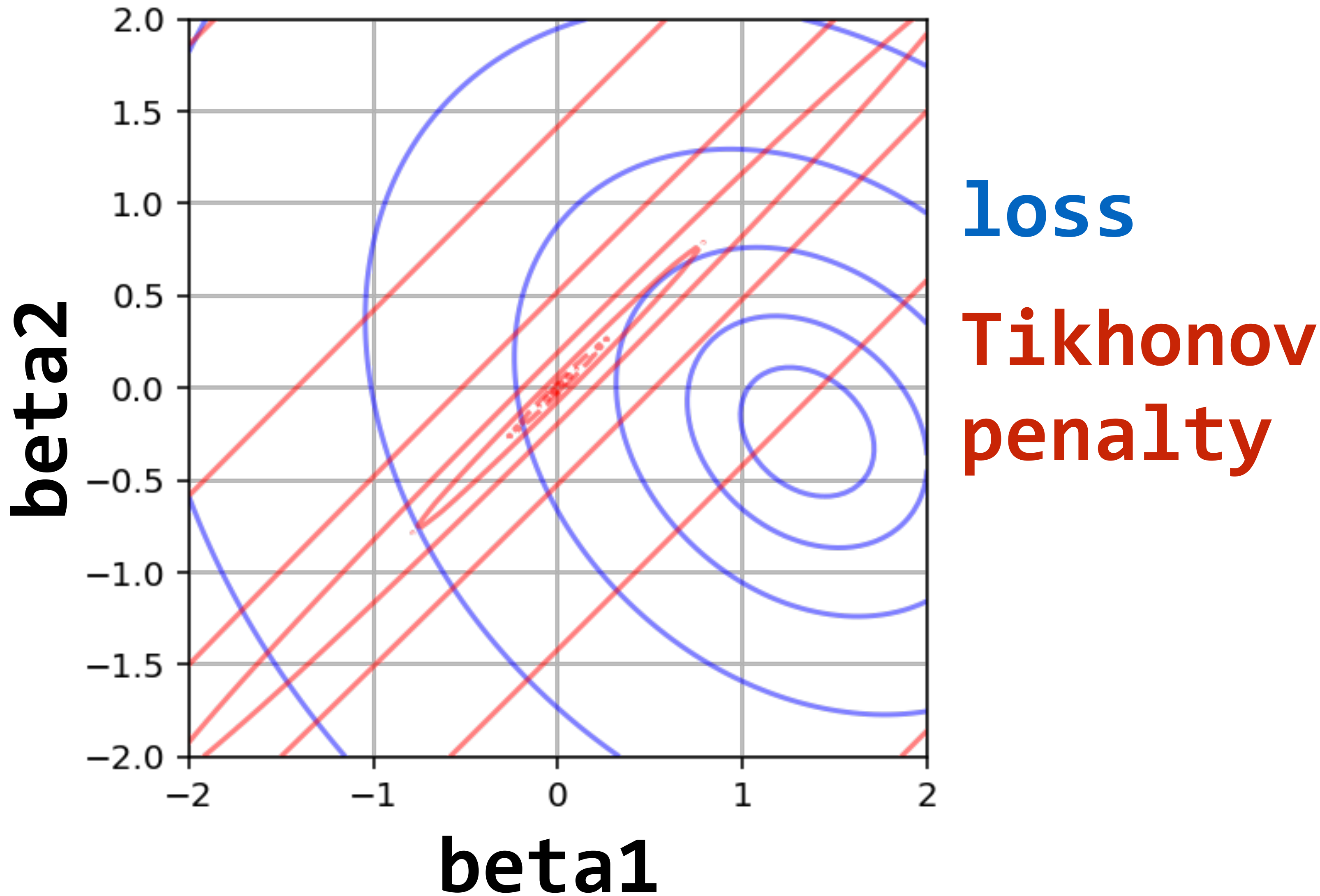


EXAMPLE

* Suppose we strongly suspect that **beta1** and **beta2** should be similar

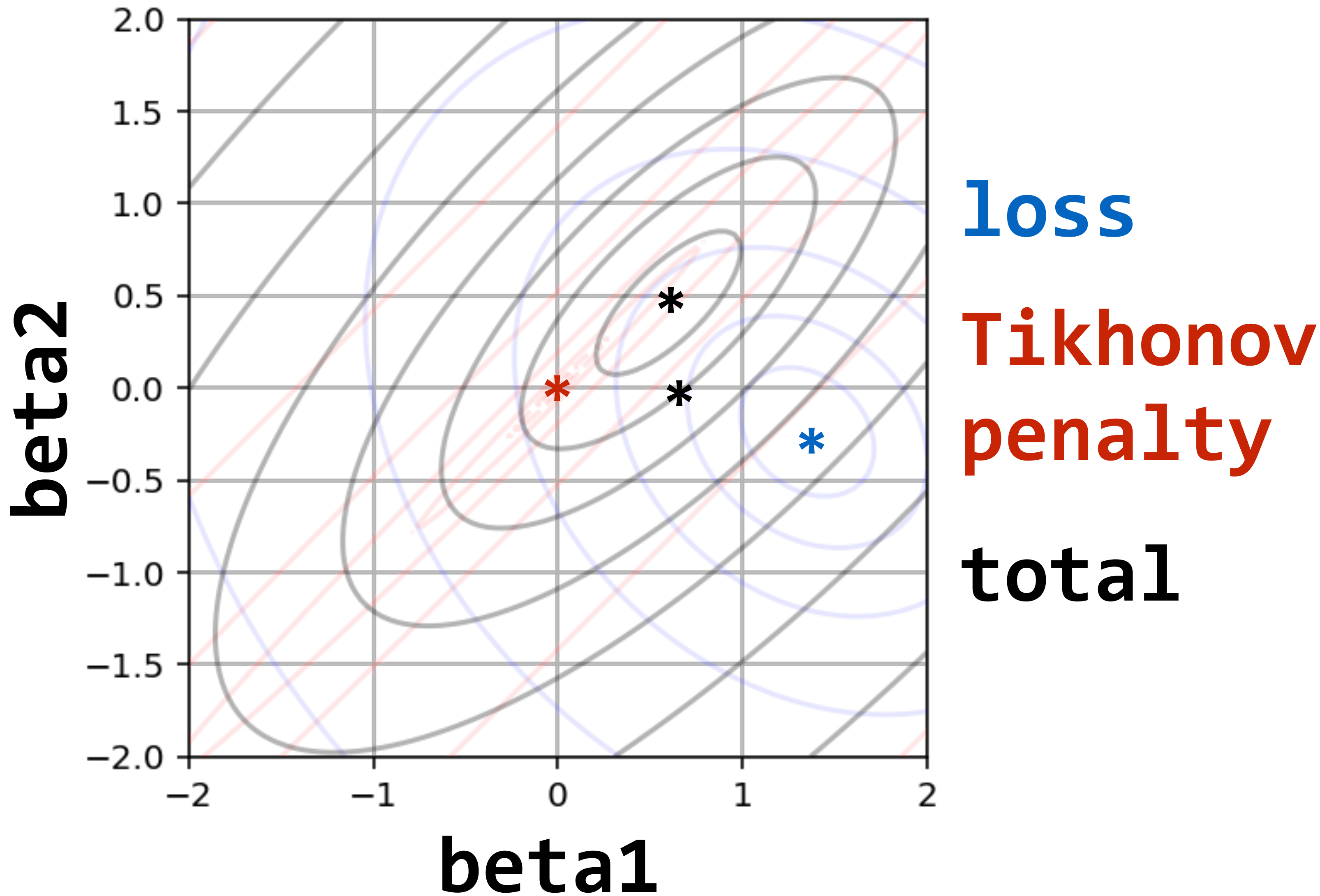
EXAMPLE

$$y_t = x_{1,t}\beta_1 + x_{2,t}\beta_2$$



EXAMPLE

$$y_t = x_{1,t}\beta_1 + x_{2,t}\beta_2$$



TIKHONOV REGRESSION

* any **TIKHONOV** problem can be converted into a **RIDGE** problem by a **LINEAR TRANSFORMATION**

* conversely, **ANY LINEAR TRANSFORMATION** of X followed by **RIDGE REGRESSION** is equivalent to some **TIKHONOV REGRESSION** problem

TIKHONOV REGRESSION

* WORD EMBEDDING MODELS

* think of stimulus matrix as **WORDS** over time projected onto WORD EMBEDDING

WORDS EMBEDDING

$$\underset{t \times n}{X} = \underset{t \times w}{W} \; \underset{w \times n}{E}$$

TIME WORDS EMBEDDING DIMENSIONS

TIKHONOV REGRESSION

* this is equivalent to TIKHONOV REGRESSION on the **WORDS** with a prior determined by the WORD EMBEDDING

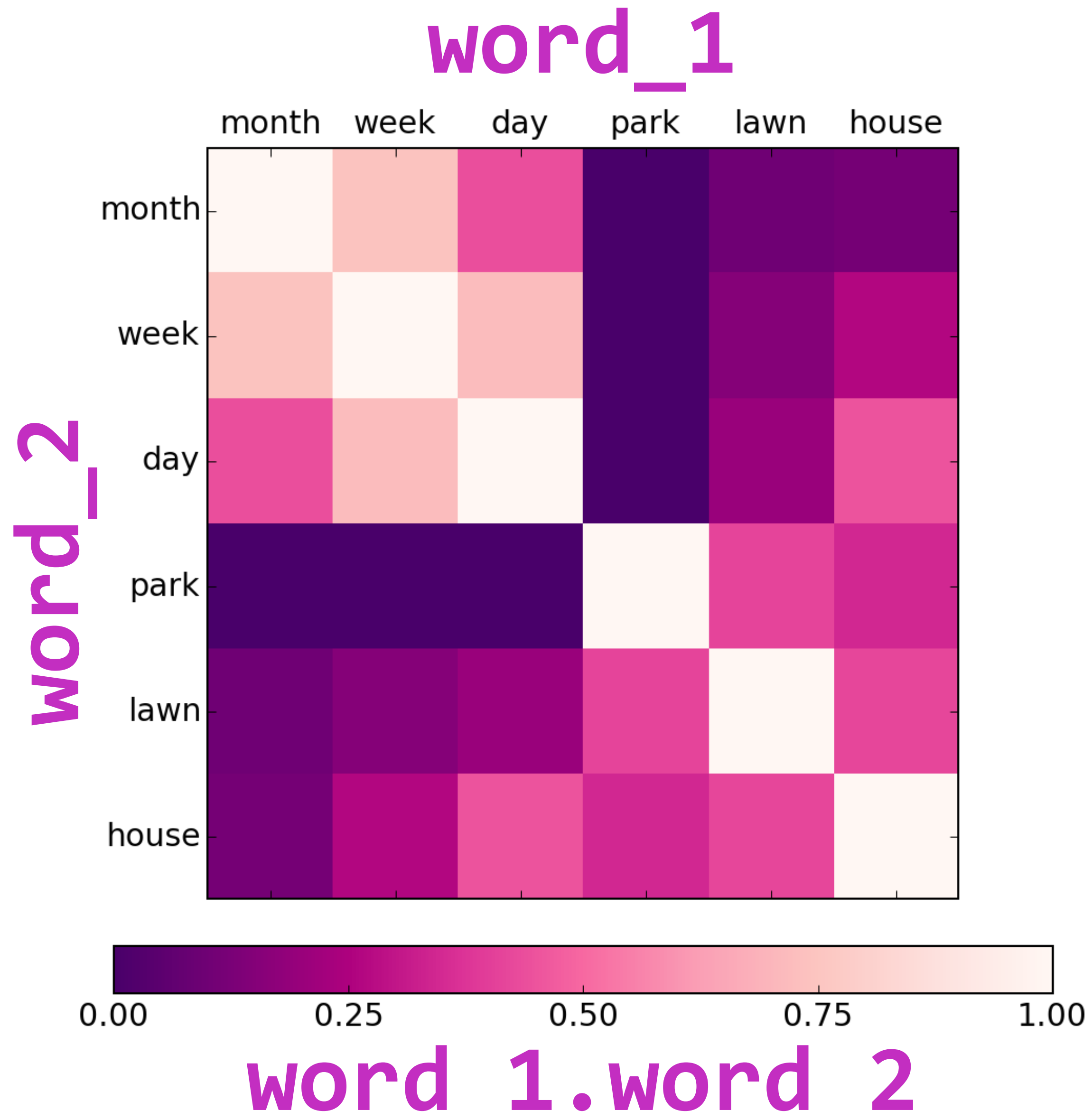
$$\underbrace{\frac{1}{\sigma^2}\Sigma_\beta}_{\text{PRIOR COVARIANCE}} = \underbrace{(C^T C)^{-1}}_{\text{INVERSE OF PENALTY INNER PRODUCT}} = \underbrace{E^T E}_{\text{EMBEDDING INNER PRODUCT}}$$

* **i.e. the prior covariance between two words' weights is equal to the dot product of their embedding vectors**

TIKHONOV REGRESSION

$$E^T E =$$

EMBEDDING INNER PRODUCT, *english1000*



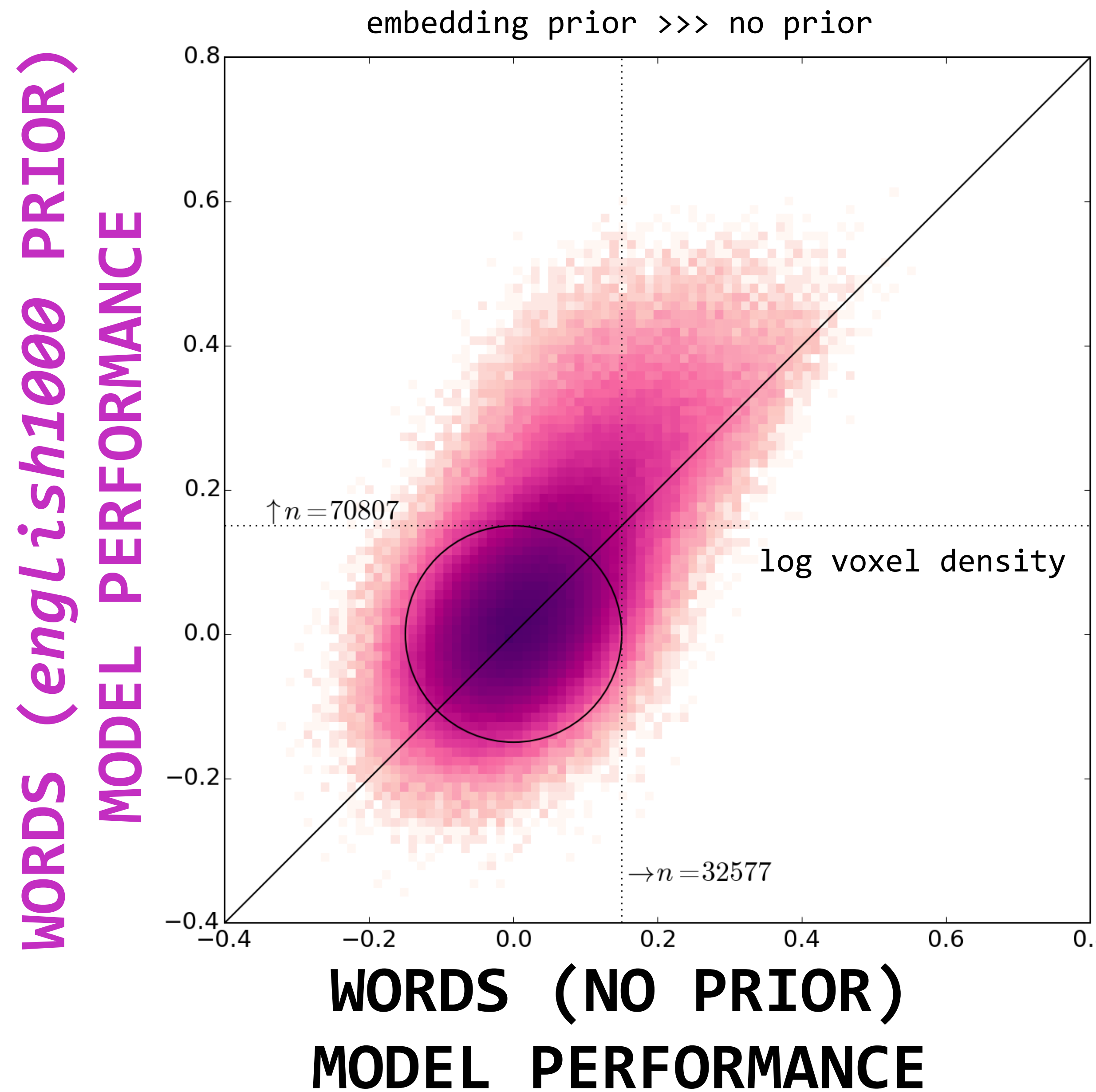
TIKHONOV REGRESSION

* to get **WEIGHTS ON WORDS** we just project onto the EMBEDDING

$$\underset{w \times v}{\overset{\text{WEIGHTS IN WORD SPACE}}{\hat{\beta}_W}} = \underset{w \times n}{\overset{\text{EMBEDDING}}{E}} \; \underset{n \times v}{\overset{\text{WEIGHTS IN EMBEDDING SPACE}}{\hat{\beta}_X}}$$

* (this is equivalent to simulating responses to single words)

TIKHONOV REGRESSION



NEXT TIME

* Data quality!